ON THE NUMBER OF PRIMES

LESS THAN A GIVEN MAGNITUDE

by

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This is to certify that I have examined this copy of a master's thesis

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ABSTRACT

In this study, we give different proofs of the Prime Number Theorem which gives an estimate on the number of primes not exceeding x, for a given real number x. We first prove the theorem with elementary methods in which we do not get use of any complex function theory. This proof does not give an error term but rather gives an asymptotic to the function counting primes up to x. In the second part we give two analytic proofs of the PNT with exponential error terms, the last one providing a better error term. Many of the properties of the Riemann zeta-function are studied since we use them frequently along the way. Finally we give the PNT for arithmetic progressions which gives an estimate on the number of primes not exceeding x belonging to a certain arithmetic progression. The role of the Dirichlet L-functions serves as an analogue to Riemann zeta-function's role in the proof of PNT. So we study the properties of the Dirichlet L-functions as well.

ÖZET

Bu çalışmada, herhangi bir x sayısına kadar olan asal sayıların kaç tane olduğuna dair sonuçlar veren Asal Sayı Teoreminin farklı ispatları verilmiştir. Öncelikle Asal Sayı Teoremini kompleks fonksiyon teorisi kullanmadan, elementer metotlarla ispatlıyoruz. Bu ispatta x' e kadar olan asalları sayan fonksiyona, hata terimi içermeyen bir asimptotik buluyoruz. İkinci kısımda ise, Asal Sayı Teoremine üstel hata terimi de içeren iki farklı analitik ispat veriyoruz. İspatlarda Riemann zeta-fonksiyonun özellikleri sıkça kullanıldığından, bu fonksiyonun birçok özelliği de çalışılmıştır. Son olarak aritmetik dizilerde Asal Sayı Teoremini ele alıyoruz. Bu teorem ise belirli bir aritmetik diziye ait, x' ten büyük olmayan asalların sayısına bir asimptotik vermektedir. Riemann zeta-fonksiyonunun Asal Sayı Teoreminin ispatında oynadığı rolü, bu kısımda Dirichlet L-fonksiyonları üstlenmiştir. Bu sebeple son bölümde Dirichlet L-fonksiyonunun özellikleri çalışılmıştır.

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Contents

LIST OF SYMBOLS/ABBREVIATIONS

$\chi(n)$	A Dirichlet character.
$\chi_0(n)$	The principal character.
$\tau(\chi)$	The Gaussian sum of the character $\chi(n)$.
$L(s,\chi)$	A Dirichlet L-function.
$\zeta(s)$	The Riemann zeta-function.
$li(x)$	$\int_2^x \frac{du}{\log u}$; the logarithmic integral.
$\Re s$	The real part of the complex number s.
$\Im s$	The imaginary part of the complex number s.
$\Gamma(s)$	The Gamma function.
$\Lambda(n)$	The von-Mangoldt function.
$\phi(n)$	The Euler's totient function.
$\psi(x)$	$\sum_{n\leq x}\Lambda(n)$; the Chebyshev's ψ function.
$\vartheta(x)$	$\sum_{p\leq x}\log p$; the Chebyshev's ϑ function.
β_1	An exceptional zero of $L(s,\chi)$.
$\pi(x)$	The number of primes $\leq x$.
$\pi(x;q,a)$	The number of primes $\leq x$ which are $\equiv a \pmod{q}$.
$\psi(x;q,a)$	The sum of $\Lambda(n)$ over $n\leq x$ which are $\equiv a \pmod{q}$.
E	The Euler's constant.
$[x]$	The integer part of x.
x	$x-[x]$; the fractional part of x.
$\mu(n)$	The Möbius mu function.
$f(x)=O(g(x))$	$\lvert f(x)\rvert \leq Cg(x)$ where C is an absolute constant.
$f(x)=o(g(x))$	$\lim f(x)/g(x)=0$.
$f(x)\ll g(x)$	$f(x)=O(g(x))$.
$f(x)\sim g(x)$	$\lim f(x)/g(x)=1$.
PNT	The Prime Number Theorem.

1 PRELIMINARIES

This chapter includes the basic information needed to understand the text as we frequently will refer in the following chapters. It consists of four main sections and in each of them, we will present the functions and some of their properties that we are going to deal with. We also will introduce some main formulas and tools that are widely used in Analytic Number Theory. All these are will be given briefly, without proof, since detailed arguments can be found in [3] or [2].

1.1 Arithmetic Functions

Definition 1. *A real- or complex-valued function defined on the positive integers is called an arithmetic function.*

We introduce some arithmetic functions which play an important role on distribution of primes.

1. The Möbius function μ is defined as follows:
   $\mu(1) = 1$;
   If $n > 1$, write $n = p_1^{a_1} \cdots p_k^{a_k}$. Then

$$\mu(n) = \begin{cases} (-1)^k & \text{if } a_1 = a_2 = \cdots a_k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

2. If $n > 1$ the Euler totient $\phi(n)$ is defined to be the number of positive integers not exceeding n which are relatively prime to n; i.e.,

$$\phi(n) = \sum_{\substack{m=1 \\ (m,n)=1}}^{n} 1.$$

3. The Von Mangoldt function $\Lambda(n)$ is defined as:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and some integer } m \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f(n)$ be an arithmetic function. We usually denote by $F(x)$, the summatory function of $f(n)$

$$F(x) = \sum_{n \leq x} f(n).$$

In analytic number theory, we estimate the averages $\frac{F(x)}{x}$ of arithmetic functions because they are expected to behave more regularly for large x whereas an arithmetic function may behave beyond prediction when n is large. So we are interested in tools for evaluating the averages.

Now let us give the partial summation formula which is one of the most powerful methods for estimating the summatory of arithmetic functions.

Theorem 1.1 (The Partial Summation Formula)**.** *Let x and y be real numbers with $0 < y < x$. Let $f(n)$ be an arithmetic function with summatory function $F(x)$ and $g(t)$ be a function with a continuous derivative on $[y, x]$. Then,*

$$\sum_{y<n\leq x} f(n)g(n) = F(x)g(x) - F(y)g(y) - \int_y^x F(t)g'(t)dt. \tag{1.1}$$

In particular, if $x \geq 2$ and $g(t)$ is continuously differentiable on $[1, x]$, then

$$\sum_{n\leq x} f(n)g(n) = F(x)g(x) - \int_1^x F(t)g'(t)dt. \tag{1.2}$$

This theorem, applied to the functions $f(n) = 1$ and $g(t) = 1/t$ gives

$$\sum_{n\leq x} \frac{1}{n} = \log x + E + r(x) \quad \text{where} \quad |r(x)| < \frac{1}{x}. \tag{1.3}$$

The number E in (1.3) is called the *Euler's constant.* Another application of partial summation gives the following estimate which will be used in the next chapter:

$$\sum_{n\leq x} \log^2 \frac{x}{n} = 2x + O(\log^2 x). \tag{1.4}$$

1.2 Distribution of Primes

Let us first introduce Chebyshev's functions $\psi(x)$ and $\vartheta(x)$ which are of great importance in the study of distribution of primes.

Definition 2. *We define Chebyshev's $\psi(x)$ function to be the summatory function of $\Lambda(n)$ by*

$$\psi(x) = \sum_{n\leq x} \Lambda(n).$$

Definition 3. *We define Chebyshev's $\vartheta(x)$ function by*

$$\psi(x) = \sum_{p\leq x} \log p,$$

where p runs over primes $\leq x$.

Chebyshev has showed that the functions $\psi(x)$, $\vartheta(x)$ and $\pi(x)\log x$ are all of order $O(x)$ and their relation gives equivalent forms of the PNT. He has proved:

Theorem 1.2. *There exists positive constants A and B such that for $x \geq 2$*

$$Ax \leq \vartheta(x) \leq \psi(x) \leq \pi(x)\log x \leq Bx. \tag{1.5}$$

Moreover,

$$\liminf_{x\to\infty} \frac{\vartheta(x)}{x} = \liminf_{x\to\infty} \frac{\psi(x)}{x} = \liminf_{x\to\infty} \frac{\pi(x)\log x}{x} \geq \log 2, \tag{1.6}$$

and

$$\limsup_{x\to\infty} \frac{\vartheta(x)}{x} = \limsup_{x\to\infty} \frac{\psi(x)}{x} = \limsup_{x\to\infty} \frac{\pi(x)\log x}{x} \leq \log 4. \tag{1.7}$$

The following theorem states three equivalent forms of PNT (without error term).

Theorem 1.3. *The following relations are equivalent:*

$$\pi(x) \sim \frac{x}{\log x}. \tag{1.8}$$

$$\vartheta(x) \sim x. \tag{1.9}$$

$$\psi(x) \sim x. \tag{1.10}$$

Another substantial progress was made by Mertens. He has showed:

$$\sum_{n\leq x} \frac{\Lambda(n)}{n} = \log x + O(1), \tag{1.11}$$

$$\sum_{p\leq x} \frac{\log p}{p} = \log x + O(1), \tag{1.12}$$

$$\sum_{p\leq x} \frac{1}{p} = \log\log x + A + O(\frac{1}{\log x}), \tag{1.13}$$

where A is a constant. These estimates are used frequently in the elementary proof of PNT.

1.3 Dirichlet Series

Given an arithmetic function $f(n)$, we define the Dirichlet series associated by f as

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

A Dirichlet series can be regarded as a function of the complex variable s, defined in the region in which the series converges. We write the variable s as

$$s = \sigma + it, \quad \text{where} \quad \sigma = \Re s, t = \Im s,$$

and we will use this notation throughout the text.

An important result about Dirichlet series is the Euler product identity when applied to the Dirichlet series.

Theorem 1.4 (Euler Product Identity)**.** *Let f be a multiplicative arithmetic function with Dirichlet series $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$. Assume $F(s)$ converges absolutely for $\sigma > \sigma_a$, then we have*

$$F(s) = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \dots\right) \quad \textit{for } \sigma > \sigma_a. \tag{1.14}$$

If f is completely multiplicative, then

$$F(s) = \prod_p \left(1 + \frac{f(p)}{p^s}\right)^{-1} \quad \textit{for } \sigma > \sigma_a. \tag{1.15}$$

The most famous Dirichlet series is the one associated with the function $f(n) = 1$, so-called the Riemann zeta function $\zeta(s)$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \qquad (\sigma > 1)$$

We initially define $\zeta(s)$ for $\sigma > 1$ but it has an analytic continuation to the half-plane $\sigma > 0$:

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} \{u\} u^{-s-1} du. \tag{1.16}$$

Moreover, by the Euler product identity (1.15), we have

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \quad (\sigma > 1). \tag{1.17}$$

Logarithmic derivative of the identity (1.17) gives the Dirichlet series for $-\frac{\zeta'(s)}{\zeta(s)}$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \sum_{n=1}^{\infty} \frac{\log p}{p^{ms}} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \prod_p \left(1 + \frac{1}{p^s}\right)^{-1} \quad (\sigma > 1). \tag{1.18}$$

Another important property of Dirichlet series is that we can relate them to the summatory functions of arithmetic functions. Not to cause a confusion of notation, let us denote by $M(f,x) = \sum_{n \le x} f(n)$ the summatory function (which was denoted by $F(x)$ previously) of $f(n)$. There are inversion formulas relating $M(f,x)$ to $F(s)$ and we will give them in the next two theorems.

Theorem 1.5 (Mellin Transform Representation of Dirichlet Series)**.** *Let $f(n)$ be an arithmetic function with the summatory function $M(f,x)$ and the associated Dirichlet series $F(s)$ with finite abscissa of convergence σ_c. Then we have*

$$F(s) = s\int_1^{\infty} M(f,x)x^{-s-1}dx \quad \sigma > \max(0, \sigma_c). \tag{1.19}$$

Next, we give Perron's formula which serves as a converse of Mellin's transform in the sense that we express $M(f,x)$ in terms of $F(s)$.

Theorem 1.6 (Perron's Formula)**.** *Let $f(n)$ be an arithmetic function with the associated Dirichlet series $F(s)$ with finite abscissa of absolute convergence σ_a. Then we have for any $c > \max(0, \sigma_a)$,*

$$\sum_{n \le x}{}' f(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\frac{x^s}{s}dx \quad . \tag{1.20}$$

Here, $\sum'$ indicates that we take the term $f(x)$ to be halved in the case when x is an integer.

1.4 Dirichlet Characters and L-functions

Definition 4. *An arithmetic function $\chi(n)$ is called a Dirichlet character modulo q if it satisfies*
(i) $\chi(n) = 0$ for $(n,q) > 1$,
(ii) $\chi(1) \neq 0$,
(iii) $\chi(n)\chi(m) = \chi(nm)$ for all integers m, n,
(iv) $\chi(n) = \chi(m)$ whenever $n \equiv m$ (mod q), i.e. $\chi(n)$ is q-periodic.

Multiplicativity entails $\chi(1) = 1$, and consequently $\chi(n)$ must be a $(\phi(q))$-th root of unity for $(n,q) = 1$. Also, there are $\phi(q)$ characters to the modulus q. One of them takes the value 1 for all integers relatively prime to q and 0 otherwise, this is called the principal character and denoted by $\chi_0(n)$.

A character $\chi(n)$ modulo q satisfies the following relations:

$$\frac{1}{\phi(q)} \sum_{\chi (\mathrm{mod}\ q)} \chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{q}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.21}$$

and

$$\frac{1}{\phi(q)} \sum_{n (\mathrm{mod}\ q)} \chi(n) = \begin{cases} 1 & \text{if } \chi = \chi_1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.22}$$

From the relation (1.21), it is possible to deduce a relation which will be useful when we aim at working on integers belonging to a certain residue class modulo q. If $(n, q) = 1$, then for any m we have

$$\frac{1}{\phi(q)} \sum_{\chi (\mathrm{mod}\ q)} \chi(m)\bar{\chi}(n) = \begin{cases} 1 & \text{if } m \equiv n \pmod{q}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.23}$$

Dirichlet also defined L-functions denoted by $L(s, \chi)$ to be the Dirichlet series of $\chi(n)$ for $\sigma > 1$,

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

By the Euler product identity we have

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \quad (\sigma > 1). \tag{1.24}$$

As for the function $\zeta(s)$, logarithmic differentiation gives that the Dirichlet series for $-\frac{L'(s,\chi)}{L(s,\chi)}$

$$-\frac{L'(s, \chi)}{L(s, \chi)} = \sum_p \sum_{n=1}^{\infty} \frac{\chi(p) \log p}{p^{ms}} = \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^s} \quad (\sigma > 1). \tag{1.25}$$

A character $\chi(n)$ modulo q, when restricted to the values of n with $(n, q) = 1$, may have a least period less than q, say q_1. Then we say $\chi(n)$ is *imprimitive* , otherwise if q is the least period itself, we say that $\chi(n)$ is *primitive*. We leave the principal character unclassified. In the case when $\chi(n)$ is imprimitive, there is a primitive character $\chi_1(n)$ modulo q_1 with $\chi_1(n) = \chi(n)$ for $(n, q) = 1$ and we say that $\chi_1(n)$ induces $\chi(n)$. Moreover there is a relation between $L(s, \chi_1)$ and $L(s, \chi)$ which follows from the Euler product formula

$$L(s,\chi) = L(s,\chi_1)\prod_{p|q}\left(1-\frac{\chi_1(p)}{p^s}\right)^{-1} \quad (\sigma > 1). \tag{1.26}$$

Definition 5. *Let $\chi(n)$ be a character modulo q. The Gaussian sum $\tau(\chi)$ is defined by*

$$\tau(\chi) = \sum_{n=1}^{q} \chi(n)e^{-\frac{2\pi i n}{q}}. \tag{1.27}$$

For a primitive character $\chi(n)$ modulo q we have

$$\chi(n)\frac{1}{\tau(\bar{\chi})}\sum_{m=1}^{q}\bar{\chi}(m)e^{-\frac{2\pi i m n}{q}}. \tag{1.28}$$

We end this part with a theorem on the Gaussian sum $\tau(\chi)$. We will use it when we construct the functional equation for $L(s,\chi)$.

Theorem 1.7. *Let $\chi(n)$ be a primitive character modulo q. Then we have*

$$|\tau(\chi)| = \sqrt{q}. \tag{1.29}$$

Moreover,

$$\tau(\chi)\tau(\bar{\chi}) = \begin{cases} q & \text{if } \chi(-1) = 1\ , \\ -q & \text{if } \chi(-1) = -1. \end{cases} \tag{1.30}$$

2 ELEMENTARY PROOF OF THE PNT

The study of distribution of primes has been excited mathematicians since ancient times. The infinitude of the primes was first shown by Euclid. After about 2000 years, Euler gave a proof of infinitude of primes showing that the series $\sum_p \frac{1}{p}$ diverges. His proof, though comes after a very long time, was essential in the sense that it involved analytical arguments. We will discuss the analytical aspects of this study on primes in the next chapter. If we denote by $\pi(x)$, the number of primes not exceeding x, it was conjectured first by Gauss and Legendre around 1800 that

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty,$$

which is known as the PNT.

Chebyshev experienced the difficulty of working with the function $\pi(x)$ and introduced his well-known functions $\psi(x)$ and $\vartheta(x)$ which are counting primes with weights. These functions provided equivalent forms of the PNT given in Theorem 1.3 and later mathematicians preferred to use these forms to prove the PNT.

The PNT was first proved by Hadamard and de la Vallée Poussin indepently in 1896. But these proofs had an analytic approach and hence, the number theorists were still seeking for a proof that was purely elemantary, means a proof based on number theoretic methods. It was expected that the elementary proof, if exists, would produce exciting innovations in number theory. In 1948, Selberg and Erdös indepently gave elementary proofs of the PNT. The proofs were found to be clever with delicate arguments, but they were very far form meeting the expectations. They did not bring anything innovative to the theory. But still, the desire for an elementary proof was satisfied. For historical remarks on the development of the PNT, see [6].

We will prove the PNT of the form $\vartheta(x) \sim x$. Equivalently, we will prove $R(x) = o(x)$ where $R(x) = \vartheta(x) - x$. The proof is based on Selberg's formula.

Theorem 2.1 (Selberg's Formula)**.** *For $x \geq 1$, we have*

$$\sum_{n \leq x} \Lambda_2(n) = 2x \log x + O(x), \tag{2.1}$$

where

$$\Lambda_2(n) = \Lambda(n) \log n + \sum_{dk=n} \Lambda(d)\Lambda(k).$$

Proof. First we note that

$$\begin{aligned}\sum_{d|n} \Lambda_2(d) &= \sum_{d|n} \Lambda(d)\log d + \sum_{d|n}\sum_{kl=d} \Lambda(k)\Lambda(l)\\ &= \sum_{d|n} \Lambda(d)\log d + \sum_{k|n} \Lambda(k) \sum_{l|n/k} \Lambda(l)\\ &= \sum_{d|n} \Lambda(d)\log d + \sum_{k|n} \Lambda(k)\log n/k\\ &= \log n \sum_{d|n} \Lambda(d)\\ &= \log^2(n).\end{aligned}$$

Hence by Möbius inversion we get

$$\Lambda_2(n) = \sum_{dk=n} \mu(d)\log^2(k).$$

Now we use the above equation and the estimates $\sum_{n\le x} \frac{1}{n}$ and $\sum_{n\le x} \log^2 \frac{x}{n}$ which were given by (1.3) and (1.4).

$$\begin{aligned}\sum_{n\le x} \Lambda_2(n) &= \sum_{n\le x}\sum_{dk=n} \mu(d)\log^2(k)\\ &= \sum_{dk\le x} \mu(d)\log^2(k)\\ &= \sum_{d\le x} \mu(d) \sum_{k\le x/d} \log^2 k\\ &= \sum_{d\le x} \mu(d)\Big(\frac{x}{d}\log^2\frac{x}{d} - \frac{2x}{d}\log\frac{x}{d} + \frac{2x}{d} + O\Big(\log^2\frac{x}{d}\Big)\Big)\\ &= x\sum_{d\le x} \frac{\mu(d)}{d}\log\frac{x}{d}\Big(\log\frac{x}{d} - 2\Big) + 2x\sum_{d\le x}\frac{\mu(d)}{d} + O\left(\sum_{d\le x}\log^2\frac{x}{d}\right)\\ &= x\sum_{d\le x} \frac{\mu(d)}{d}\log\frac{x}{d}\Big(\log\frac{x}{d} - 2\Big) + O(x)\\ &= x\sum_{d\le x} \frac{\mu(d)}{d}\log\frac{x}{d}\left(\sum_{m\le x/d}\frac{1}{m} - E - 2 + O\left(\frac{d}{x}\right)\right) + O(x)\\ &= x\sum_{d\le x} \frac{\mu(d)}{d}\log\frac{x}{d}\sum_{m\le x/d}\frac{1}{m} - (E+2)x\sum_{d\le x}\frac{\mu(d)}{d}\log\frac{x}{d} + O(x).\end{aligned}$$

We will estimate the two sums on the right separately. From the first one we obtain the main term:

$$\begin{aligned}
x\sum_{d\le x}\frac{\mu(d)}{d}\log\frac{x}{d}\sum_{m\le x/d}\frac{1}{m} &= x\sum_{dm\le x}\frac{\mu(d)}{dm}\log\frac{x}{d}\\
&= x\sum_{n\le x}\frac{1}{n}\sum_{d|n}\mu(d)\log\frac{x}{d}\\
&= x\log x\sum_{n\le x}\frac{1}{n}\sum_{d|n}\mu(d) - x\sum_{n\le x}\frac{1}{n}\sum_{d|n}\mu(d)\log d\\
&= x\log x + x\sum_{n\le x}\frac{\Lambda(n)}{n}\\
&= 2x\log x + O(x)
\end{aligned}$$

by (1.11)
For the second sum we again use the estimate (1.3) for $\sum_{n\le x}\frac{1}{n}$.

$$\begin{aligned}
\sum_{d\le x}\frac{\mu(d)}{d}\log\frac{x}{d} &= \sum_{d\le x}\frac{\mu(d)}{d}\left(\sum_{m\le x/d}\frac{1}{m} - E + O\left(\frac{d}{x}\right)\right)\\
&= \sum_{dm\le x}\frac{\mu(d)}{dm} - E\sum_{d\le x}\frac{\mu(d)}{d} + O(1)\\
&= \sum_{n\le x}\frac{1}{n}\sum_{d|n}\mu(d) + O(1)\\
&= O(1).
\end{aligned}$$

Combining these two estimates we get the result.

□

In the proof of the PNT, we will use several equivalent forms of Selberg's formula. Each of them can be deduced from Selberg's formula (2.1) easily. We state these forms as the following lemma:

Lemma 2.2 (Selberg's Formulae)**.**

$$\vartheta(x)\log x + \sum_{p\le x}\log p\,\vartheta\left(\frac{x}{p}\right) = 2x\log x + O(x). \tag{2.2}$$

$$\sum_{p\le x}\log^2 p + \sum_{pq\le x}\log p\log q = 2x\log x + O(x). \tag{2.3}$$

$$\sum_{p\le x}\log p+\sum_{pq\le x}\frac{\log p\log q}{\log pq}=2x+O\left(\frac{x}{1+\log x}\right). \tag{2.4}$$

Lemma 2.3.

$$\sum_{pq\le x}\frac{\log p\log q}{pq\log pq}=\log x+O(\log\log x).$$

Proof. Let $l(n)=\begin{cases}\log n & \text{if } n \text{ is prime,}\\ 0 & \text{otherwise}\end{cases}$

We will use partial summation with $f(n)=\frac{(l*l)(n)}{n}$, $g(t)=\frac{1}{\log t}$. First we estimate $\sum_{n\le x}f(n)$.

$$\begin{aligned}F(x)=\sum_{n\le x}f(n)&=\sum_{pq\le x}\frac{\log p\log q}{pq}\\&=\sum_{p\le x}\frac{\log p}{p}\sum_{q\le x/p}\frac{\log q}{q}\\&=\sum_{p\le x}\frac{\log p}{p}\left(\log\frac{x}{p}+O(1)\right)\\&=\log x\sum_{p\le x}\frac{\log p}{p}-\sum_{p\le x}\frac{\log^2 p}{p}+O(\log x).\end{aligned}$$

Here we use Mertens' estimate (1.12), $\sum_{p\le x}\frac{\log p}{p}=\log x+O(1)$, for the first sum. For the second sum we use partial summation.

$$\begin{aligned}\sum_{p\le x}\frac{\log^2 p}{p}&=\sum_{n\le x}\frac{l(n)\log n}{n}\\&=\left(\sum_{p\le x}\frac{\log p}{p}\right)\log x-\int_1^x\frac{\log t+O(1)}{t}dt\\&=\log^2 x+O(\log x)-\frac{1}{2}\log^2 x+O(\log x)\\&=\frac{1}{2}\log^2 x+O(\log x).\end{aligned}$$

So $F(x)=\frac{1}{2}\log^2 x+O(\log x)$. Applying partial summation suggested at the beginning we get

$$
\begin{aligned}
\sum_{pq\le x}\frac{\log p\log q}{pq} &= \left(\frac{1}{2}\log^2 x + O(\log x)\right)\frac{1}{\log x} + \int_2^x \frac{1/2\log^2 t + O(\log t)}{t\log^2 t}\\
&= \frac{1}{2}\log x + O(1) + \frac{1}{2}\int_2^x \frac{1}{t}dt + O\left(\int_2^x \frac{1}{t\log t}dt\right)\\
&= \log x + O(\log\log x).
\end{aligned}
$$

□

Lemma 2.4. *For $x > e$,*

$$\sum_{p\le x}\frac{\log p}{p\big(1+\log\frac{x}{p}\big)} = O(\log\log x).$$

Proof. Observe that

$$\sum_{p\le x}\frac{\log p}{p\big(1+\log\frac{x}{p}\big)} = \sum_{j=1}^{[\log x]+1}\ \sum_{\frac{x}{e^j}<p\le\frac{x}{e^{j-1}}}\frac{\log p}{p\big(1+\log\frac{x}{p}\big)}.$$

For every positive integer j, we have by Mertens' estimate

$$\sum_{\frac{x}{e^j}<p\le\frac{x}{e^{j-1}}}\frac{\log p}{p} = \log\frac{x}{e^{j-1}} - \log\frac{x}{e^j} + O(1) = j-(j-1)+O(1) = O(1).$$

Morover, if $\frac{x}{e^j} < p \le \frac{x}{e^{j-1}}$, then $\frac{1}{e^j} < \frac{p}{x} \le \frac{1}{e^{j-1}}$. Hence $\frac{1}{\left(1+\log\frac{x}{p}\right)} \le \frac{1}{j}$ and we have

$$\sum_{\frac{x}{e^j}<p\le\frac{x}{e^{j-1}}}\frac{\log p}{p\big(1+\log\frac{x}{p}\big)} \le \frac{1}{j}\sum_{\frac{x}{e^j}<p\le\frac{x}{e^{j-1}}}\frac{\log p}{p} = O(\frac{1}{j}).$$

Hence

$$\sum_{p\le x}\frac{\log p}{p\big(1+\log\frac{x}{p}\big)} = \sum_{j=1}^{(\log x)+1}\frac{1}{j} = O(\log\log x).$$

□

Theorem 2.5. *For $x \ge 1$,*

$$|R(x)| \le \frac{1}{\log x}\sum_{n\le x}\left|R\Big(\frac{x}{n}\Big)\right| + O\left(\frac{x\log\log x}{\log x}\right).$$

Proof. By Selberg's formula (2.2) we have,

$$\vartheta(x)\log x + \sum_{p\le x}\vartheta\Big(\frac{x}{p}\Big)\log p = 2x\log x + O(x).$$

Now replace $\vartheta(x)$ by $R(x)+x$ above to get

$$R(x)\log x + x\log x + x\sum_{p\le x}\frac{\log p}{p} + \sum_{p\le x}R\Big(\frac{x}{p}\Big)\log p = 2x\log x + O(x),$$

Mertens' estimate in the above equation gives

$$R(x)\log x = -\sum_{p\le x}R\Big(\frac{x}{p}\Big)\log p + O(x). \tag{2.5}$$

For $p \le x$, by Selberg's Formula (2.4) we have

$$\sum_{q\le \frac{x}{p}}\log q + \sum_{qr\le \frac{x}{p}}\frac{\log q\log r}{\log qr} = \frac{2x}{p} + O\left(\frac{x}{p(1+\log\frac{x}{p})}\right).$$

Then

$$\begin{aligned}\sum_{p\le x}\log p\vartheta\Big(\frac{x}{p}\Big) &= \sum_{p\le x}\log p\sum_{q\le \frac{x}{p}}\log q\\ &= 2x\sum_{p\le x}\frac{\log p}{p} - \sum_{p\le x}\log p\sum_{qr\le\frac{x}{p}}\frac{\log q\log r}{\log qr} + O\left(\sum_{p\le x}\frac{x\log p}{p(1+\log\frac{x}{p})}\right)\\ &= 2x(\log x + O(1)) - \sum_{qr\le x}\frac{\log q\log r}{\log qr}\sum_{p\le\frac{x}{qr}}\log p + O(x\log\log x)\\ &= 2x\log x - \sum_{qr\le x}\frac{\log q\log r}{\log qr}\vartheta\Big(\frac{x}{pq}\Big) + O(x\log\log x), \qquad (2.6)\end{aligned}$$

where in the error term we use the estimate given by Lemma 2.4. Now we substitute the RHS of (2.6) in Selberg's formula (2.2) to obtain

$$\vartheta(x)\log x + 2x\log x - \sum_{pq\le x}\frac{\log p\log q}{\log pq}\vartheta\Big(\frac{x}{pq}\Big) + O(x\log\log x) = 2x\log x + O(x).$$

Again writing $R(x)+x$ in place of $\vartheta(x)$ above we have

$$R(x)\log x + x\log x = \sum_{pq\le x}\frac{\log p\log q}{\log pq}R\Big(\frac{x}{pq}\Big) + x\sum_{pq\le x}\frac{\log p\log q}{pq\log pq} + O(x\log\log x).$$

If we use the estimate in Lemma 2.3 for the second sum $\sum_{pq\le x}\frac{\log p\log q}{pq\log pq}$, we have

$$R(x)\log x = \sum_{pq\le x} \frac{\log p \log q}{\log pq} R\Big(\frac{x}{pq}\Big) + O(x\log\log x). \tag{2.7}$$

Adding (2.5) and (2.7) we obtain

$$\begin{aligned}
2\,|R(x)|\log x &\le \sum_{p\le x}\log p \left|R\Big(\frac{x}{p}\Big)\right| + \sum_{pq\le x}\frac{\log p\log q}{\log pq}\left|R\Big(\frac{x}{pq}\Big)\right| + O(x\log\log x)\\
&= \sum_{n\le x} l(n)\left|R\Big(\frac{x}{n}\Big)\right| + \sum_{n\le x}\frac{l*l(n)}{\log n}\left|R\Big(\frac{x}{n}\Big)\right| + O(x\log\log x)\\
&= \sum_{n\le x}\left(l(n)+\frac{l*l(n)}{\log n}\right)\left|R\Big(\frac{x}{n}\Big)\right| + O(x\log\log x).
\end{aligned} \tag{2.8}$$

We deduce the following from the partial summation formula.

$$\sum_{n\le x} f(n)g(n) = \sum_{n\le x-1} F(n)(g(n)-g(n+1)) + F(x)g([x]).$$

Here $F(x)=\sum_{n\le x} f(n)$ is the summatory function of f. We use this formula with $f(n)=l(n)+\frac{l*l(n)}{\log n}$ and $g(n)=\left|R(\frac{x}{n})\right|$.

$$F(n) = \sum_{n\le x} l(n) + \frac{l*l(n)}{\log n} = \sum_{p\le x}\log p + \sum_{pq\le x}\log pq = 2x + O\left(\frac{x}{1+\log x}\right),$$

by Selberg's formula (2.4). And $g([x]) = \left|R(\frac{x}{[x]})\right| = \left|\vartheta(\frac{x}{[x]}) - \frac{x}{[x]}\right| = O(1)$.

So we obtain

$$\sum_{n\le x}\left(l(n)+\frac{l*l(n)}{\log n}\right)\left|R\left(\frac{x}{n}\right)\right|$$

$$\begin{aligned}
&= \sum_{n\le x-1}\left(2n+O\left(\frac{n}{1+\log n}\right)\right)\left(\left|R\left(\frac{x}{n}\right)\right| - \left|R\Big(\frac{x}{n+1}\Big)\right|\right) + O(x)\\
&= \underbrace{2\sum_{n\le x-1} n\left(\left|R\left(\frac{x}{n}\right)\right| - \left|R\Big(\frac{x}{n+1}\Big)\right|\right)}_{A}\\
&\quad + O\Bigg(\underbrace{\sum_{n\le x-1}\left(\frac{n}{1+\log n}\right)\left(\left|R\left(\frac{x}{n}\right)\right| - \left|R\Big(\frac{x}{n+1}\Big)\right|\right)}_{B}\Bigg).
\end{aligned} \tag{2.9}$$

We evaluate the terms A and B of the above equation seperately.

$$\begin{aligned}
A &= 2\sum_{n\le x-1} n\left|R\left(\frac{x}{n}\right)\right| - 2\sum_{n\le x-1} n\left|R\Big(\frac{x}{n+1}\Big)\right| \\
&= 2\sum_{n\le x-1} n\left|R\left(\frac{x}{n}\right)\right| - 2\sum_{2\le x}(n-1)\left|R\left(\frac{x}{n}\right)\right| \\
&= 2\sum_{n\le x} x\left|R\left(\frac{x}{n}\right)\right| - 2[x]\left|R\Big(\frac{x}{[x]}\Big)\right| \\
&= 2\sum_{n\le x} x\left|R\left(\frac{x}{n}\right)\right| + O(x).
\end{aligned}$$

To eveluate B, first observe

$$\begin{aligned}
\left|R\Big(\frac{x}{n}\Big)\right| - \left|R\Big(\frac{x}{n+1}\Big)\right| &= \left|\vartheta\left(\frac{x}{n}\right) - \frac{x}{n}\right| - \left|\vartheta\Big(\frac{x}{n+1}\Big) - \frac{x}{n+1}\right| \\
&\le \left|\vartheta\left(\frac{x}{n}\right) - \vartheta\Big(\frac{x}{n+1}\Big) - \frac{x}{n} - \frac{x}{n-1}\right| \\
&\le \vartheta\left(\frac{x}{n}\right) - \vartheta\Big(\frac{x}{n+1}\Big) + \frac{x}{n} - \frac{x}{n-1} \\
&\le \vartheta\left(\frac{x}{n}\right) - \vartheta\Big(\frac{x}{n+1}\Big) + \frac{x}{n^2}. \qquad (2.10)
\end{aligned}$$

Now if we use (2.10) in the expression for B,

$$B \le \underbrace{\sum_{n\le x-1}\frac{n}{1+\log n}\left(\vartheta\left(\frac{x}{n}\right) - \vartheta\Big(\frac{x}{n+1}\Big)\right)}_{C} + \underbrace{x\sum_{n\le x-1}\frac{1}{n(1+\log n)}}_{D}.$$

$$\begin{aligned}
C &= \sum_{n\le x-1}\frac{n}{1+\log n}\vartheta\left(\frac{x}{n}\right) - \sum_{2\le n\le x}\frac{n-1}{1+\log(n-1)}\vartheta\left(\frac{x}{n}\right) \\
&= \vartheta(x) + \sum_{2\le n\le x-1}\left(\frac{n}{1+\log n} - \frac{n-1}{1+\log(n-1)}\right)\vartheta\left(\frac{x}{n}\right) \\
&\le \vartheta(x) + \sum_{2\le n\le x-1}\frac{1}{1+\log n}\vartheta\left(\frac{x}{n}\right) \\
&= O(x) + O\bigg(x\sum_{2\le n\le x-1}\frac{1}{n(1+\log n)}\bigg).
\end{aligned}$$

But

$$\sum_{n\leq x}\frac{1}{n(1+\log n)}=O\left(\int_1^x\frac{1}{t(1+\log t)}dt\right)=O(\log\log x).$$

So $D=C=O(x\log\log x)$ and $B=O(x\log\log x)$ as well.

Now, substituting our estimates on A and B in (2.9) and then subsequently in (2.8), we obtain

$$\begin{aligned}2|R(x)|\log x &\leq \sum_{n\leq x}\left(l(n)+\frac{l*l(n)}{\log n}\right)\left|R\left(\frac{x}{n}\right)\right|+O(x\log\log x)\\ &=A+O(B)\\ &=2\sum_{n\leq x}\left|R\left(\frac{x}{n}\right)\right|+O(x\log\log x).\end{aligned}$$

Hence the result follows. □

Lemma 2.6. *Let $0<\delta<1$. There exist $c_0\geq 1$ and $x_1(\delta)\geq 4$ such that if $x\geq x_1(\delta)$, then there exists an integer $n\in(x,e^{c_0/\delta}x]$ such that*

$$|R(n)|<\delta n$$

and the constant c_0 does not depend on δ.

Proof. We have

$$\sum_{n\leq x}\frac{1}{n}=\log x+E+r(x)\qquad\text{where}\quad |r(x)|\leq\frac{1}{x}$$

If $1\leq x\leq x'$, then

$$\sum_{x<n\leq x'}=\log\frac{x'}{x}+r'(x)\qquad\text{where}\quad |r'(x)|<\frac{2}{x}\tag{2.11}$$

Now, as can be deduced from Mertens' estimate (1.12), $\sum_{n\leq x}\frac{\vartheta(n)}{n^2}=\log x+O(1)$ and this implies

$$\sum_{n\leq x}\frac{R(n)}{n^2}=\sum_{n\leq x}\frac{\vartheta(n)-n}{n^2}=O(1).$$

So choose $c_0\geq 1$ such that

$$\Big|\sum_{x<n\leq x'}\frac{R(n)}{n^2}\Big|<\frac{c_0}{2}\qquad\text{for all } x,x' \text{ such that } 1\leq x\leq x'.\tag{2.12}$$

Let $0 < \delta < 1$ and $\rho = e^{c_0/\delta}$. Then $\rho x > ex$ since $c_0/\delta > 1$. Choose $x_1(\delta) \geq 4$ such that

$$\log x < \delta x \qquad \text{for all } x \geq x_1(\delta). \tag{2.13}$$

Our claim is that if $x \geq x_1(\delta)$, then there exists an integer $n \in (x, \rho x]$ with $|R(n)| < \delta n$.

There are two cases:

Case 1: Either $R(n) \geq 0$ for all $n \in (x, \rho x]$ or $R(n) \leq 0$ for all $n \in (x, \rho x]$. Then

$$\Big|\sum_{x<n\leq\rho x} \frac{R(n)}{n^2}\Big| = \sum_{x<n\leq\rho x} \frac{|R(n)|}{n^2} = \sum_{x<n\leq\rho x} \left(\frac{|R(n)|}{n}\right)\frac{1}{n}.$$

Let $m^* = \min\left\{\frac{|R(n)|}{n} : n \in (x, \rho x]\right\}$. Then

$$\begin{aligned}
\frac{c_0}{2} > \sum_{x<n\leq\rho x} \frac{|R(n)|}{n}\frac{1}{n} &\geq m^* \sum_{x<n\leq\rho x} \frac{1}{n} \\
&> m^*\left(\log\frac{\rho x}{x} - \frac{2}{x}\right) \\
&\geq m^*\left(\frac{c_0}{\delta} - \frac{1}{2}\right) \\
&\geq \frac{c_0 m^*}{2\delta},
\end{aligned}$$

which implies $0 \leq m^* < \delta$. In the second line of the above inequalities, we used (2.11).

Hence in this case there exists $n \in (x, \rho x]$ with $\frac{|R(n)|}{n} = m^*$. i.e., the inequality $|R(n|) < \delta n$ is satisfied.

Case 2: There exist integers $n-1$ and n in the interval $(x, \rho x]$ such that $R(n-1)R(n) \leq 0$. Moreover, $n - 1 > x \geq x_1(\delta) \geq 4$ implies $n \geq 6$.

Now, for every integer $n \geq 2$ we have

$$R(n) - R(n-1) = \vartheta(n) - \vartheta(n-1) - 1 = \begin{cases} \log n - 1 & \text{if n is a prime,} \\ -1 & \text{if n is not prime.} \end{cases}$$

$i)$ $R(n) < R(n-1)$. Then $R(n) \leq 0 \leq R(n-1)$ and $R(n) - R(n-1) = -1$ and hence $|R(n)| \leq 1 < \log n \leq \delta n$ by (2.13).

ii) $R(n-1) < R(n)$. Then $R(n-1) \leq 0 \leq R(n)$ and $0 \leq R(n) \leq R(n) - R(n-1) = \log n - 1 < \log n < \delta n$ and hence again by (2.13) we have $|R(n)| < \delta n$

In all cases, there exists $n \in (x, \rho x]$ such that $|R(n)| < \delta n$ and the lemma is proved.

□

Lemma 2.7. *Let $c_0 \geq 1$ be the number in Lemma 2.6 and let $0 < \delta < 1$. There exists a number $x_2(\delta)$ such that if $x \geq x_2(\delta)$, then the interval $(x, e^{c_0/\delta}x]$ contains a subinterval $(y, e^{\delta/2}y]$ such that*

$$|R(t)| < 4\delta t \quad \textit{for all } t \in (y, e^{\delta/2}y].$$

Proof. We begin with the Selberg's formula (2.4)

$$\sum_{p \leq x} \log p + \sum_{pq \leq x} \frac{\log p \log q}{\log pq} = 2x + O\left(\frac{x}{1 + \log x}\right).$$

For $1 < u \leq t$, we have

$$\begin{aligned} 0 \leq \sum_{u < p \leq t} \log p &\leq \sum_{u < p \leq t} \log p + \sum_{u < pq \leq t} \frac{\log p \log q}{\log pq} \\ &= 2(t-u) + O\left(\frac{t}{1+\log t}\right) + \left(\frac{u}{1+\log u}\right) \\ &= 2(t-u) + O\left(\frac{t}{1+\log t}\right). \end{aligned} \tag{2.14}$$

Also,

$$\begin{aligned} 0 \leq \sum_{u < p \leq t} \log p &= \vartheta(t) - \vartheta(u) \\ &= (t-u) + R(t) - R(u). \end{aligned}$$

(2.14) and (2.15) gives

$$|R(t) - R(u)| \leq t - u + O\left(\frac{1}{\log t}\right).$$

If $1 < t \le u \le 2t$, then

$$\begin{aligned}|R(t) - R(u)| &\le u - t + O\left(\frac{u}{1 + \log u}\right) \\ &\le |t - u| + O\left(\frac{2t}{1 + \log 2t}\right) \\ &\le |t - u| + O\left(\frac{t}{1 + \log t}\right).\end{aligned}$$

So, in particular, if $u > 4$ and $t/2 \le u \le 2t$, then

$$|R(t)| \le |R(u)| + |t - u| + O\left(\frac{t}{\log t}\right). \tag{2.15}$$

By Lemma 2.6, there exists number $c_0 \ge 1$ such that if $0 < \delta < 1$ and $x \ge x_1(\delta) \ge 4$ there exists an integer $n \in (x, e^{c_0/\delta} x]$ such that $|R(n)| < \delta n$.

If $t \in (n/2, 2n]$, then $t/2 \le n \le 2t$ and since $n > x \ge 4$ we have

$$\log t \ge \log(n/2) > \log(x/2) > \frac{\log x}{2}, \tag{2.16}$$

and by (2.16)

$$\begin{aligned}|R(t)| &\le |R(n)| + |t - n| + O\left(\frac{t}{\log t}\right) \\ &< t\left(\frac{\delta n}{t} + \left|\frac{n}{t} - 1\right| + O\left(\frac{1}{\log t}\right)\right) \\ &\le t\left(2\delta + \left|\frac{n}{t} - 1\right| + \frac{c_2}{\log x}\right),\end{aligned}$$

for some $c_2 > 0$. If $x \ge x_2(\delta) = max(x_1(\delta), e^{c_2/\delta})$, then

$$|R(t)| < t\left(3\delta + \left|\frac{n}{t} - 1\right|\right). \tag{2.17}$$

Choose t in the interval $e^{-\delta/2} n \le t \le e^{\delta/2} n$.

Since $e^{\delta/2} < e^{1/2} < 2$, we have $t \in (n/2, 2n)$. Then (2.17) holds and there are two cases:

If $\frac{t}{n} \ge 1$, then $\left|\frac{n}{t} - 1\right| = 1 - \frac{n}{t} \le 1 - e^{-\delta/2} < e^{\delta/2} - 1 < \delta$ since $e^{\delta/2} < 1 + \delta$ for $0 < \delta < 1$.

If $\frac{t}{n} < 1$, then $\left|\frac{n}{t} - 1\right| = \frac{n}{t} - 1 \le e^{\delta/2} - 1 < \delta$.

In both cases from (2.17) we obtain

$$|R(t)| < 4\delta t \qquad \text{for all} \qquad t \in [e^{-\delta/2}n, e^{\delta/2}n]. \tag{2.18}$$

To complete the proof, we must find a subinterval in which the bound in (2.18) holds. We define y as follows:

If $e^{\delta/2}n \leq e^{c_0/\delta}x$, let $y = n$ then $(y, e^{\delta/2}y] = (n, e^{\delta/2}n] \subseteq (x, e^{c_0/\delta}x]$.

If $e^{\delta/2}n \geq e^{c_0/\delta}x$, let $y = e^{-\delta/2}n$. Then $y = e^{-\delta/2}n > e^{-\delta}e^{c_0/\delta}x = e^{c_0/\delta-\delta}x > x$ and hence $(y, e^{\delta/2}y] = (e^{-\delta/2}n, n] \subseteq (x, e^{c_0/\delta}x]$.

□

Theorem 2.8 (The Prime Number Theorem)**.** *For Chebyshev's function* $\vartheta(x)$*,*

$$\vartheta(x) \sim x \qquad \text{as} \qquad x \to \infty$$

Equivalently,

$$R(x) = o(x) \quad \text{as} \qquad x \to \infty.$$

Proof. By (1.6) and (1.7) we have

$$\limsup_{x\to\infty} \frac{R(x)}{x} = \limsup_{x\to\infty} \frac{\vartheta(x)}{x} - 1 \leq \log 4 - 1 < 0.4,$$
$$\liminf_{x\to\infty} \frac{R(x)}{x} = \liminf_{x\to\infty} \frac{\vartheta(x)}{x} - 1 \geq \log 2 - 1 > -0.4.$$

Hence there exist numbers M and u_1 such that

$$|R(x)| < Mx \quad \forall x \geq 1 \tag{2.19}$$
$$|R(x)| < \delta_1 x \quad \forall x \geq u_1 \quad \text{where} \quad \delta_1 = 0.4 \tag{2.20}$$

Our aim is to contruct sequences of positive real numbers $\{\delta_m\}_1^\infty$ and $\{\epsilon_m\}_1^\infty$ such that $\delta_1 > \delta_2 > \cdots$ and $\lim_{m\to\infty} \epsilon_m = 0$.

We inductively define δ_m and u_m as follows. Let $m \geq 1$ and suppose we have constructed δ_m. If $c_0 > 1$ is the constant in Lemma 2.6, choose ϵ_m such that

$$0 < \epsilon_m < \tfrac{1}{m} \quad \text{and} \quad (1+\epsilon_m)\left(1 - \tfrac{\delta_m^2}{256c_0}\right) < 1,$$

and define

$$\delta_{m+1} = (1+\epsilon_m)\left(1 - \frac{\delta_m^2}{256c_0}\right)\delta_m. \tag{2.21}$$

Then $0 < \delta_{m+1} < \delta_m$ for all m and $\lim_{m\to\infty} \epsilon_m = 0$.

We shall prove for all m, there exists u_m such that

$$|R(x)| < \delta_m x \quad \text{for all} \quad x \geq u_m. \tag{2.22}$$

Our claim is that this sufficient is to show PNT.

The sequence $\{\delta_m\}$ is decreasing, so it converges to some nonnegative number $\delta < 1$. Then if we take $m \to \infty$ in (2.21), we obtain

$$\delta = \left(1 - \frac{\delta^2}{256c_0}\right)\delta.$$

But we know $\left(1 - \frac{\delta^2}{256c_0}\right) < 1$. Hence δ must be equal to 0. So (2.22) implies that $R(x) = o(x)$.

Let us construct u_m inductively. We know by (2.20) there exists u_1 such that

$$|R(x)| < \delta_1 x \quad \text{for all } x \geq u_1.$$

Suppose that we determined u_m. We shall prove that there exists u_{m+1} satisfying

$$|R(x)| < \delta_{m+1} x \quad \text{for all } x \geq u_{m+1}. \tag{2.23}$$

Define

$$\begin{aligned}
\delta'_m &= \frac{\delta_m}{8}, \\
\rho &= e^{c_0/\delta'_m}, \\
x_3(m) &= \max(x_2(\delta_m), u_m),
\end{aligned}$$

where $x_2(\delta_m)$ is the number in Lemma 2.7 Then by Lemma 2.7 each interval $(x, \rho x]$ contains a subinterval $(y, e^{\delta'_m/2}y]$ such that

$$|R(t)| < 4\delta'_m t \quad \text{for all } t \in (y, e^{\delta'_m/2}y]\ .$$

Now let k be the greatest integer such satisfying $\rho^k \leq \frac{x}{x_3(m)}$. Then

$$k \leq \frac{\log(x/x_3(m))}{\log \rho} < k + 1,$$

so that

$$\begin{aligned} k &= \frac{\log(x/x_3(m))}{\log \rho} + O(1) \\ &= \frac{\delta'_m}{c_0} \log x + O(1) \end{aligned} \tag{2.24}$$

By Theorem 2.5,

$$\begin{aligned} |R(x)| &\le \frac{1}{\log x} \sum_{n \le x} \left| R\left(\frac{x}{n}\right)\right| + O\left(\frac{x \log\log x}{\log x}\right) \\ &= \frac{1}{\log x} \sum_{n \le \rho^k} \left| R\left(\frac{x}{n}\right)\right| + \frac{1}{\log x} \sum_{\rho^k < n \le x} \left| R\left(\frac{x}{n}\right)\right| + o(x) \\ &\le \frac{1}{\log x} \sum_{n \le \rho^k} \left| R\left(\frac{x}{n}\right)\right| + \frac{Mx}{\log x} \sum_{\rho^k < n \le x} \frac{1}{n} + o(x) \\ &= \frac{1}{\log x} \sum_{n \le \rho^k} \left| R\left(\frac{x}{n}\right)\right| + o(x) \end{aligned} \tag{2.25}$$

where we use (2.9) in the third line and the last line (2.25) follows by noting $\frac{x}{\rho x_3(m)} < \rho^k$ and by

$$\sum_{\frac{x}{\rho x_3(m)} < n \le x} \frac{1}{n} \le \sum_{\rho^k < n \le x} \frac{1}{n} = \log(\rho x_3(m)) + O\left(\frac{1}{x}\right) = O(1).$$

Now if $1 < n \le \rho^k$, then $\frac{x}{n} \ge \frac{x}{\rho^k} \ge x_3(m) \ge u_m$ and by the definition of u_m we have

$$\left| R\left(\frac{x}{n}\right)\right| \le \frac{\delta_m x}{n}$$

For $j = 1, 2, \ldots, k$, we have

$$\frac{x}{\rho^j} \ge \frac{x}{\rho^k} \ge x_3(m) \ge x_2(\delta'_m),$$

so that Lemma 2.7 applies. Hence each interval $\left(\frac{x}{\rho^j}, \frac{x}{\rho^{j-1}}\right]$ contains a subinterval $I_j = (y_j, e^{\delta'_m/2} y_j]$ such that

$$|R(t)| < 4\delta'_m t = \frac{\delta_m t}{2} \quad \text{for all } t \in I_j.$$

Therefore,

$$\begin{aligned}
\sum_{n\in(\rho^{j-1},\rho^j]} \left|R\left(\frac{x}{n}\right)\right| &= \sum_{n\in(\rho^{j-1},\rho^j]\setminus I_j} \left|R\left(\frac{x}{n}\right)\right| + \sum_{n\in I_j} \left|R\left(\frac{x}{n}\right)\right| \\
&< \delta_m x \sum_{n\in(\rho^{j-1},\rho^j]\setminus I_j} \frac{1}{n} + \frac{\delta_m x}{2} \sum_{n\in I_j} \frac{1}{n} \\
&= \delta_m x \sum_{n\in(\rho^{j-1},\rho^j]} \frac{1}{n} - \frac{\delta_m x}{2} \sum_{n\in I_j} \frac{1}{n}. \qquad (2.26)
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{n\le\rho^k} \left|R\left(\frac{x}{n}\right)\right| &= R(x) + \sum_{j=1}^{k} \sum_{n\in(\rho^{j-1},\rho^j]} \left|R\left(\frac{x}{n}\right)\right| \\
&\le \delta_m x + \sum_{j=1}^{k} \left(\delta_m x \sum_{n\in(\rho^{j-1},\rho^j]} \frac{1}{n} - \frac{\delta_m x}{2} \sum_{n\in I_j} \frac{1}{n}\right) \\
&= \delta_m x \sum_{n\le\rho^k} \frac{1}{n} - \frac{\delta_m x}{2} \sum_{j=1}^{k} \sum_{n\in I_j} \frac{1}{n}. \qquad (2.27)
\end{aligned}$$

For the first sum on the RHS of the above inequality we have

$$\begin{aligned}
\delta_m x \sum_{n\le\rho^k} \frac{1}{n} &= \delta_m x \left(k\log\rho + O\left(\frac{1}{\rho^k}\right)\right) \\
&= \delta_m x \log x + O(x), \qquad (2.28)
\end{aligned}$$

since $k\log\rho \le \log(x/x_3(m)) = \log x + O(1)$.

For the second sum, we first note that

$$\sum_{n\in I_j} \frac{1}{n} = \sum_{n\in(y_j, e^{\delta'_m/2}y_j]} \frac{1}{n} = \frac{\delta'_m}{2} + O\left(\frac{1}{y_j}\right) = \frac{\delta'_m}{2} + O\left(\frac{\rho^j}{x}\right).$$

Therefore, if we use the expression (2.24) in place of k and recall the definition of δ'_m, we have

$$\sum_{j=1}^{k}\sum_{n\in I_j}\frac{1}{n} = \frac{\delta'_m k}{2} + O\left(\sum_{j=1}^{k}\frac{\rho^j}{x}\right)$$
$$= \frac{\delta'_m k}{2}\left(\frac{\delta'_m}{c_0}\log x + O(1)\right) + O(1)$$
$$= \frac{\delta_m^2}{128c_0}\log x + O(1),$$

since

$$\sum_{j=1}^{k}\frac{\rho^j}{x} = \frac{\rho(\rho^k-1)}{x(\rho-1)} < \frac{2\rho^k}{x} \le \frac{2}{x_3(m)} = O(1).$$

Thus we obtain

$$\frac{\delta_m x}{2}\sum_{j=1}^{k}\sum_{n\in I_j}\frac{1}{n} = \frac{\delta_m^3}{256c_0}x\log x + O(x). \tag{2.29}$$

Combining the estimates (2.28) and (2.29) we obtain from (2.27), for $x \ge x_3(m)$,

$$\sum_{n\le\rho^k}\left|R\left(\frac{x}{n}\right)\right| = \left(1-\frac{\delta_m^2}{256c_0}\right)\delta_m x\log x + O(x). \tag{2.30}$$

Now we substitute (2.30) in (2.25) to get

$$|R(x)| \le \frac{1}{\log x}\sum_{n\le\rho^k}\left|R\left(\frac{x}{n}\right)\right| + o(x)$$
$$= \left(1-\frac{\delta_m^2}{256c_0}\right)\delta_m x + o(x).$$

We choose u_{m+1} sufficiently large that for all $x \ge u_{m+1}$ we have

$$o(x) < \epsilon_m\left(1-\frac{\delta_m^2}{256c_0}\right)\delta_m x.$$

Then,

$$|R(x)| < (1+\epsilon_m)\left(1-\frac{\delta_m^2}{256c_0}\right)\delta_m x = \delta_{m+1}x,$$

so that we found u_{m+1} such that for all $x \ge u_{m+1}$,

$$|R(x)| < \delta_{m+1}x$$

which means that (2.23) is satisfied and we have proved the PNT.

□

3 ANALYTIC PROOF OF THE PNT

The PNT was first proved independently, and simultaneously, by Jacques Hadamard and Charles de la Vallée Poussin at the end of 19th century. The proofs of Hadamard and de la Vallée Poussin both used an analytic approach that was originated by the work of Riemann in 1859. We will give an outline of the original proof of the PNT with classical error term in the second part of this chapter. In the first part, we will give an analytic, but less technical proof which requires complex analysis in a modest level.

3.1 An Analytic Method for the PNT with Exponential Error Term

Theorem 3.1 (The Prime Number Theorem)**.** *For $x \geq 2$ we have*

$$\psi(x) = x + O(x \exp(-c(\log x)^{\alpha})), \tag{3.1}$$

where c is a positive constant and $\alpha = 1/10$.

Before we pass to the proof of the PNT, we will establish results on the Riemann zeta function $\zeta(s)$. The following theorems and lemmas include several results on $\zeta(s)$.

Theorem 3.2 (Upper bounds for $\zeta(s)$ and $\zeta'(s)$)**.**

(i) $\quad |\zeta(s)| \leq \dfrac{4|t|^{1-\sigma_0}}{1-\sigma_0} \quad (|t| \geq 2,\ 1/2 \leq \sigma_0 < 1,\ \sigma \geq \sigma_0)$

(ii) $\quad |\zeta(s)| \leq A_1 \log |t| \quad (|t| \geq 2,\ \sigma \geq 1 - \frac{1}{4\log|t|})$

(iii) $\quad |\zeta'(s)| \leq A_2 \log^2 |t| \quad (|t| \geq 2,\ \sigma \geq 1 - \frac{1}{12\log|t|})$

where A_i denote positive constants.

To prove this theorem we need some lemmas.

Lemma 3.3. *For $\sigma > 0$ and $N \in \mathbb{N}$,*

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} - s\int_N^{\infty} \frac{\{u\}}{u^{s+1}} du. \tag{3.2}$$

Proof. Given $N \in \mathbb{N}$, we apply Mellin transform (1.19) of Theorem 1.5 to the Dirichlet series

$$F(s) = \sum_{n=N+1}^{\infty} \frac{1}{n^s}$$

This is the Dirichlet series of the function $f(n) = \begin{cases} 1 & \text{if } n > N, \\ 0 & \text{otherwise.} \end{cases}$ where the summatory function is given by

$$M(f,x) = \sum_{n \le x} f(n) = \begin{cases} [x] - N & \text{if } x \ge N, \\ 0 & \text{otherwise} \end{cases}$$

Hence the Mellin transform gives for $\sigma > 1$

$$\begin{aligned} F(s) = s \int_1^\infty \frac{M(f,x)}{x^{s+1}} dx &= s \int_N^\infty \frac{[x] - N}{x^{s+1}} dx \\ &= s \int_N^\infty \frac{[x]}{x^{s+1}} dx - sN \int_N^\infty \frac{1}{x^{s+1}} dx \\ &= s \int_N^\infty \frac{1}{x^s} dx - sN \int_N^\infty \frac{1}{x^{s+1}} dx - s \int_N^\infty \frac{\{x\}}{x^{s+1}} dx \\ &= -\frac{N^{1-s}}{1-s} - s \int_N^\infty \frac{\{x\}}{x^{s+1}} dx. \end{aligned}$$

$\zeta(s) = F(s) + \sum_{n=1}^N \frac{1}{n^s}$, so that (3.2) is satisfied in the half-plane $\sigma > 1$.

The integral $\int_N^\infty \frac{\{x\}}{x^{s+1}} dx$ is uniformly convergent, hence analytic in any half-plane $\sigma \ge \delta$ with $\delta > 0$. Hence the RHS of (3.2) is analytic in $\sigma > 0$ except at the simple pole at $s = 1$. So (3.2) remains valid in $\sigma > 0$ as well. □

Lemma 3.4. *For $\sigma > 0, N \in \mathbb{N}$, and $t \neq 0$ we have*

$$|\zeta(s)| \le \sum_{n=1}^N \frac{1}{n^\sigma} - \frac{N^{1-\sigma}}{|t|} + \frac{|s|}{\sigma} N^{-\sigma}. \tag{3.3}$$

Proof. For $\sigma > 0$,

$$\left| s \int_N^\infty \frac{\{x\}}{x^{s+1}} dx \right| \le |s| \int_N^\infty \frac{1}{x^{\sigma+1}} dx = \frac{|s|}{\sigma} N^{-\sigma}.$$

Also,

$$\left| \frac{N^{1-s}}{1-s} \right| \le \frac{N^{1-\sigma}}{|t|}$$

and

$$\left| \sum_{n=1}^N \frac{1}{n^s} \right| \le \sum_{n=1}^N \left| \frac{1}{n^s} \right| = \sum_{n=1}^N \frac{1}{n^\sigma}.$$

Hence (3.3) follows from (3.2) taking the absolute values of each term and using the bounds given above. □

Lemma 3.5. *For $N \in \mathbb{N}$, $1/2 < \sigma_0 < 1$, $\sigma \geq \sigma_0 > 0$, and $t \neq 0$ we have*

$$|\zeta(s)| \leq \frac{N^{1-\sigma_0}}{1-\sigma_0} + \frac{N^{1-\sigma_0}}{|t|} + \left(1 + \frac{|t|}{\sigma_0}\right) N^{-\sigma_0}. \tag{3.4}$$

Proof. We begin with (3.3) and show the three terms on RHS are bounded with the corresponding terms in (3.4).

$$\sum_{n=1}^{N} \frac{1}{n^{\sigma}} \leq \sum_{n=1}^{N} \frac{1}{n^{\sigma_0}} \leq 1 + \int_1^N \frac{1}{x^{\sigma_0}} dx = 1 + \frac{N^{1-\sigma_0} - 1}{1-\sigma_0} \leq \frac{N^{1-\sigma_0}}{1-\sigma_0},$$

since $\sigma \geq \sigma_0$ and $\sigma_0 < 1$. Moreover $N^{1-\sigma} \leq N^{1-\sigma_0}$ and

$$\frac{|s|N^{-\sigma}}{\sigma} \leq \frac{(1+|t|)N^{-\sigma}}{\sigma} \leq \left(1 + \frac{|t|}{\sigma_0}\right) N^{-\sigma_0}.$$

□

Proof of Theorem 3.2.

(i) We apply Lemma 3.5 with $N = [|t|]$. Since $0 < \sigma_0 < 1$ we have $N^{1-\sigma} \leq |t|^{1-\sigma_0}$. So Lemma 3.5 gives

$$|\zeta(s)| \leq \frac{|t|^{1-\sigma_0}}{1-\sigma_0} \left(1 + \frac{1-\sigma_0}{|t|} + \frac{1-\sigma_0}{[|t|]} + \frac{(1-\sigma_0)|t|}{\sigma_0[|t|]}\right).$$

For $|t| \geq 2$ and $1/2 \leq \sigma_0 < 1$,

$$\frac{1-\sigma_0}{|t|} \leq \frac{1-\sigma_0}{[|t|]} \leq \frac{1}{4} \quad \text{and} \quad \frac{(1-\sigma_0)|t|}{\sigma_0[|t|]} \leq 2,$$

so that

$$\left(1 + \frac{1-\sigma_0}{|t|} + \frac{1-\sigma_0}{[|t|]} + \frac{(1-\sigma_0)|t|}{\sigma_0[|t|]}\right) \leq 1 + \frac{1}{4} + \frac{1}{4} + 2 < 4,$$

and thus we obtain *(i)*.

(ii) Set $\sigma_0 = 1 - \frac{1}{4\log|t|}$. Since $|t| \geq 2$, we obtain $1/2 \leq \sigma_0 < 1$ and we can apply part *(i)* to have

$$|\zeta(s)| \leq 4\frac{|t|^{1-\sigma_0}}{1-\sigma_0} = \frac{4e^{1/4}}{1/(4\log|t|)} = 16e^{1/4}\log|t|,$$

which is the desired result with $A_1 = 16e^{1/4}$.

(iii) For $\sigma \geq 2$ the Dirichlet series of $\zeta'(s)$ gives

$$|\zeta'(s)| = \left|\sum_{n=1}^{\infty} \frac{\log n}{n^s}\right| \le \sum_{n=1}^{\infty} \frac{\log n}{n^2}.$$

So the asserted bound holds for $\sigma \ge 2$. $\zeta(s)$ is analytic in the region $\{s : \sigma > 0, s \ne 1\}$. Hence $|\zeta'(s)|$ is uniformly bounded in any compact rectangle contained in this region. Thus the bound also holds in the range $2 \le |t| \le 3$, $\sigma \ge 1/2$.

It remains to show that the bound holds when $|t| \ge 3$. Let s be given in the range $\sigma \ge 1 - \delta$ where $\delta = \frac{1}{12 \log |t|}$. Since $|t| > e$ we have

$$\sigma > 1 - \frac{1}{12} \quad \text{and} \quad 0 < \delta < \frac{1}{12}.$$

So the disk $\{s' \in \mathbb{C} : |s - s'| \le \delta\}$ is contained in the region of analyticity of $\zeta(s)$. By Cauchy's theorem

$$|\zeta'(s)| = \left|\frac{1}{2\pi i} \oint_{|s'-s|=\delta} \frac{\zeta(s')}{(s'-s)^2} ds'\right| \le \frac{1}{\delta} \max_{|s'-s|=\delta} |\zeta(s')|.$$

To estimate further we show that for $|s' - s| \le \delta$, $\zeta(s')$ is bounded by a constant multiple of $\log |t|$.

Let $s' = \sigma' + it'$ with $|s' - s| \le \delta$. By hypothesis we have $|t| \ge 3$ and $\sigma \ge 1 - \delta$ and these imply

$$\begin{aligned}
|t'| &\ge |t| - \delta \ge |t| - \frac{1}{12} > 2, \\
|t'| &\le |t| + \delta \le |t| + \frac{1}{12} \le \frac{13}{12}|t| \le |t|^{3/2}.
\end{aligned}$$

Thus,

$$\sigma' \ge \sigma - \delta \ge 1 - \frac{1}{6 \log |t|} \ge 1 - \frac{1}{6 \log |t'|^{2/3}} = 1 - \frac{1}{4 \log |t'|}$$

which implies that s' satisfies the conditions of *(ii)* so that we obtain

$$|\zeta(s')| \le A_1 \log |t'|.$$

Hence

$$|\zeta'(s)| \le \frac{1}{\delta} \max_{|s'-s|=\delta} |\zeta(s')| \le 12 \log |t| A_1 \log |t'| \le 12.\frac{3}{2} A_1 \log^2 |t| = A_2 \log^2 |t|.$$

□

Theorem 3.6 (Zero-free region for $\zeta(s)$ and upper bound for $1/\zeta(s)$)**.** *(i) $\zeta(s)$ has no zeros in the closed half-plane $\sigma \geq 1$.*

(ii) There exist constants $c_1 > 0$ and $A_3 > 0$ such that $\zeta(s)$ has no zeros in the region $\sigma > 1 - c_1, t \leq 2$. In this region

$$\left|\frac{1}{\zeta(s)}\right| \leq A_3.$$

(iii) There exist constants $c_2 > 0$ and $A_4 > 0$ such that $\zeta(s)$ has no zeros in the region $\sigma > 1 - \frac{c_2}{(\log|t|)^9}, t \leq 2$. In this region

$$\left|\frac{1}{\zeta(s)}\right| \leq A_4 \log|t|)^7.$$

The proof of this theorem is based on the so-called 3-4-1 inequality

$$2(1+\cos\theta)^2 = 3 + 4\cos\theta + \cos 2\theta \geq 0$$

applied to the $\zeta(s)$ function. We state this as the following lemma:

Lemma 3.7 (The 3-4-1 inequality for $\zeta(s)$)**.** *We have for $\sigma \geq 1$*

$$|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)| \geq 1.$$

Proof. For $\sigma > 1$,

$$\begin{aligned}
\log|\zeta(s)| = \log\left|\prod_p \left(1-\frac{1}{p^s}\right)^{-1}\right| &= -\Re\sum_p \log\left(1-\frac{1}{p^s}\right) \\
&= \Re\sum_p\sum_{m\geq 1}\frac{1}{mp^{ms}} \\
&= \sum_p\sum_{m\geq 1}\frac{\cos(t\log p^m)}{mp^{m\sigma}}.
\end{aligned}$$

We apply this to σ, $\sigma+it$, $\sigma+2it$ to obtain

$$\begin{aligned}
\log|\zeta(\sigma)| &= \sum_p\sum_{m\geq 1}\frac{1}{mp^{m\sigma}}, \\
\log|\zeta(\sigma+it)| &= \sum_p\sum_{m\geq 1}\frac{\cos(t\log p^m)}{mp^{m\sigma}}, \\
\log|\zeta(\sigma+2it)| &= \sum_p\sum_{m\geq 1}\frac{\cos(2t\log p^m)}{mp^{m\sigma}},
\end{aligned}$$

and thus

$$\begin{aligned}\log|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)| &= 3\log|\zeta(\sigma)|+4\log|\zeta(\sigma+it)|+\log|\zeta(\sigma+2it)|\\ &= \sum_p\sum_{m\geq 1}\frac{3+4\cos(t\log p^m)+\cos(2t\log p^m)}{mp^{m\sigma}}\\ &\geq 0.\end{aligned}$$

The proof is finished once we take exponential of the above inequality. □

Proof of theorem 3.6.

(i) By the Euler product formula (1.17), $\zeta(s)$ has no zeros in the half-plane $\sigma > 1$. So it remains to show that $\zeta(s)$ has no zeros on the line $\sigma = 1$.

For a contradiction assume $\zeta(1+it_0) = 0$ for some $t_0 \in \mathbb{R}$. $\zeta(s)$ is analytic for $\sigma > 0$ with a simple pole at $\sigma = 1$. So clearly $t_0 \neq 0$.

We consider the behaviors of the functions $\zeta(\sigma)$, $\zeta(\sigma+it_0)$ and $\zeta(\sigma+2it_0)$ as $\sigma \to 1^+$.

- $\zeta(\sigma)(\sigma-1)$ is bounded as $\sigma \to 1^+$ since $\zeta(s)$ has a simple pole at $\sigma = 1$.
- The assumption $\zeta(1+it_0) = 0$ and the analyticity of $\zeta(s)$ implies

$$\frac{\zeta(\sigma+it_0)}{\sigma-1} = \frac{\zeta(\sigma+it_0)-\zeta(1+it_0)}{(\sigma+it_0)-(1+it_0)}$$

  stays bounded as $\sigma \to 1^+$.
- the analyticity of $\zeta(s)$ at $1+2-t_0$ guarantees $\zeta(\sigma+2it_0)$ stays bounded as $\sigma \to 1^+$.

Hence the expression

$$|\zeta(\sigma)^3\zeta(\sigma+it_0)^4\zeta(\sigma+2it_0)| = (\sigma-1)|\zeta(\sigma)(\sigma-1)|^3|\zeta(\sigma+2it_0)|\left|\frac{\zeta(\sigma+it_0)}{\sigma-1}\right|^4$$

is of order $O(\sigma-1)$ as $\sigma \to 1^+$. But by Lemma 3.7, it is bounded by 1, hence we get a contradiction.

(ii) In the half-plane $\sigma \geq 2$ the bound on $1/\zeta(s)$ holds trivially. In fact

$$|\zeta(s)| = \left|1+\sum_{n\geq 2}\frac{1}{n^s}\right| \geq 1-\left|\sum_{n\geq 2}\frac{1}{n^s}\right| \geq \sum_{n\geq 2}\frac{1}{n^2} = 2-\frac{\pi^2}{6} > 0$$

so that

$$\left|\frac{1}{\zeta(s)}\right| < \frac{6}{12-\pi^2} \qquad (\sigma \geq 2). \tag{3.5}$$

Therefore it remains to show that $1/\zeta(s)$ is uniformly bounded in the compact rectangle $1-c_1 \leq \sigma \leq 2, |t| \leq 2$. By *(i)*, $\zeta(s)$ has no zeros on the half-plane $\sigma \geq 1$, then $1/\zeta(s)$ is analytic for $\sigma \geq 1$ and bounded in any compact region contained in this region. In particular is bounded in the rectangle $1 \leq \sigma \leq 2, |t| \leq 2$. It follows by compactness that $1/\zeta(s)$ remains bounded in any sufficiently small neighborhood of this rectangle. i.e. in $1-c_1 \leq \sigma \leq 2, |t| \leq 2$ for some constant c_1.

(iii) For $\sigma \geq 2$ the bound holds by (3.5). So we may assume $\sigma \leq 2$, $|t| \geq 2$. We fix a constant A (to be determined later) and consider the range

$$1 + \frac{A}{(\log|t|)^9} \leq \sigma \leq 2. \tag{3.6}$$

By Lemma 3.7 we have $\sigma \geq 1$

$$|\zeta(\sigma+it)| \geq \frac{1}{|\zeta(\sigma)|^3/4}\frac{1}{|\zeta(\sigma+2it)|^1/4}. \tag{3.7}$$

Since $\zeta(s)$ has a simple pole at $s=1$, there exists an absolute constant c_3 such that

$$\zeta(\sigma) \leq \frac{c_3}{\sigma-1}. \qquad (1 \leq \sigma \leq 2)$$

Moreover by theorem 3.2 (ii) we have

$$|\zeta(\sigma+2it)| \leq A_1 \log|2t| \leq 2\log|t|. \qquad (|t| \geq 2)$$

Using these bounds in (3.7) and the bounds on σ in the region (3.6) we obtain

$$\begin{aligned}|\zeta(\sigma+it)| &\geq {c_3}^{-1/4}(\sigma-1)^{3/4}(2A_1)^{-1/4}(\log|t|)^{-1/4}\\ &\geq c_4(A)^{3/4}(\log|t|)^{-7},\end{aligned}$$

where $c_4 = {c_3}^{-1/4}(2A_1)^{-1/4}$ is an absolute constant.

We have proved the asserted bound in the range (3.6) for a constant A. To complete the proof we show that if A is chosen sufficiently small, then the bound of the above type holds also in the range

$$1 - \frac{A}{(\log|t|)^9} \leq \sigma \leq 1 + \frac{A}{(\log|t|)^9}.$$

Write

$$\sigma_1 = 1 - \frac{A}{(\log|t|)^9} \quad \text{and} \quad \sigma_2 = 1 + \frac{A}{(\log|t|)^9}.$$

For $\sigma_1 \le \sigma \le \sigma_2$ we have

$$\begin{aligned}|\zeta(\sigma+it)| &= \left|\zeta(\sigma_2+it) - \int_\sigma^{\sigma_2} \zeta'(u+it)du\right| \\ &\ge |\zeta(\sigma_2+it) - (\sigma_2-\sigma_1)\max_{\sigma_1\le u\le\sigma_2}|\zeta'(u+it)||.\end{aligned}$$

For $|\zeta(\sigma_2+it)|$, we have the bound

$$|\zeta(\sigma_2+it)| \ge c_4(A)^{3/4}(\log|t|)^{-7}$$

since σ_2+it falls into the range (3.6). Moreover by Theorem 3.2 (iii) we have

$$|\zeta'(s)| \le A_2(\log|t|)^2. \qquad (\sigma \ge 1 - \tfrac{1}{12\log|t|})$$

If the constant A is chosen small enough, then the range $\sigma \ge 1 - \frac{A}{(\log|t|)^9}$ is contained in the range $\sigma \ge 1 - \frac{1}{12\log|t|}$ and the bound for $\zeta'(s)$ is valid in this range. Hence we obtain

$$\begin{aligned}|\zeta(\sigma+it)| &\ge c_4(A)^{3/4}(\log|t|)^{-7} - 2A(\log|t|)^{-9}A_2(\log|t|)^2 \\ &= (A)^{3/4}(c_4 - 2(A)^{1/4}A_2)(\log|t|)^{-7}.\end{aligned}$$

Now choosing A so that $c_4 - 2(A)^{1/4}A_2 > 0$, we obtain

$$|\zeta(\sigma+it)| \ge c_5(\log|t|)^{-7}.$$

Setting $c_2 = A$ and $A_4 = 1/c_5$ provides us the asserted estimate.

□

Theorem 3.8 (Upper bounds for $\zeta'(s)/\zeta(s)$)**.** *There exist absolute positive constants $0 < c_6 < 1/2$ and A_5 such that for all s satisfying*

$$\sigma \ge 1 - \frac{c_6}{(\log|t|)^9}, \quad |t| \ge 2 \tag{3.8}$$

we have

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \le A_5(\log|t|)^9;$$

and for all s satisfying

$$\sigma \ge 1 - c_6, \quad |t| \le 2, \quad s \ne 1$$

we have

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \le A_5 \max\left(1, \frac{1}{|\sigma-1|}\right).$$

Proof. Combining the bounds for $1/\zeta(s)$ and $\zeta'(s)$ found in Theorem 3.2 and Theorem 3.6 and choosing the constant c_6 sufficiently small we obtain the asserted bound for $\frac{\zeta'(s)}{\zeta(s)}$ in the range (3.8).

Since $1/\zeta(s)$ is analytic in $\sigma \ge 1$ and $\zeta(s)$ is analytic for $\sigma > 0$ except at $s = 1$, the logarithmic derivative $\frac{\zeta'(s)}{\zeta(s)}$ is analytic in $\sigma \ge 1$ except for a simple pole $s = 1$. Therefore $\frac{\zeta'(s)}{\zeta(s)}(s-1)$ is analytic for $\sigma \ge 1$. By compactness, the analyticity extends to a region $\sigma \ge 1 - c_6$, $|t| \le 2$ provided c_6 is small enough. So this function is bounded in the compact region and

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| = O\left(\frac{1}{|s-1|}\right) = O\left(\frac{1}{|\sigma-1|}\right)$$

in this region.

Moreover, for $\sigma \ge 2$

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| = \left|-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\right| \le \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^2} < \infty,$$

so that we obtain the second estimate. □

Now, as we establish the necessary background, we may prove the Prime Number Theorem given by Theorem 3.1.

Proof of theorem 3.1.
We define the function ψ_1 by

$$\psi_1(x) = \int_0^x \psi(y)dy = \sum_{n\le x}\Lambda(n)(x-n).$$

We apply Perron's formula (1.20) of Theorem 1.6 with $f(n) = \Lambda(n)$ where the corresponding Dirichlet series is $F(s) = \frac{\zeta'(s)}{\zeta(s)}$ which converges absolutely in $\sigma > 1$. Hence Perron's formula gives

$$\psi_1(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)\frac{x^{s+1}}{(s)(s+1)}ds, \tag{3.9}$$

for any $a > 1$ and $x \ge 2$.

We fix $x \geq e$ and let $e \leq T \leq x$ be a parameter which will be determined later as a function of x. Set

$$a = 1 + \frac{1}{\log x} \quad \text{and} \quad b = 1 - \frac{c_6}{(\log T)^9},$$

where c_6 is the constant in Theorem 3.8. Note that since $c_6 < 1/2$ and $e \leq T \leq x$ we have

$$1 < a \leq 2, \quad 1/2 < 1 - c_6 < b < 1.$$

Now we replace the path of integration in (3.9) which is a vertical line by $L = \bigcup_{i=1}^{5} L_i$ where

$$\begin{aligned}
L_1 &= (a - i\infty, a - iT],\\
L_2 &= [a - iT, b - iT],\\
L_3 &= [b - iT, b + iT],\\
L_4 &= [b - iT, a + iT],\\
L_5 &= [a + iT, a + i\infty).
\end{aligned}$$

So we have

$$\psi_1(x) = M + \frac{1}{2\pi i} \sum_{j=1}^{5} I_j,$$

where M is the contribution of the residues in the region enclosed by L_2, L_3, L_4 and $[a - iT, a + iT]$ and I_j denotes the integral over the path L_j.

First we will calculate the main term M. The region is the rectangle with vertices

$$a \mp iT = 1 + \frac{1}{\log x} \mp iT, \quad b \mp iT = 1 - \frac{c_6}{(\log T)^9} \mp iT,$$

which falls within the zero-free region of $\zeta(s)$ given by Theorem 3.6. Thus the integrand $-\frac{\zeta'(s)}{\zeta(s)} \frac{x^{s+1}}{(s)(s+1)}$ has only one singularity generated by the pole of $-\frac{\zeta'(s)}{\zeta(s)}$ at $s = 1$ with residue 1. So residue of the integrand function is

$$Res\left(-\frac{\zeta'(s)}{\zeta(s)} \frac{x^{s+1}}{(s)(s+1)}, 1\right) = \frac{1}{2}x^2.$$

Hence

$$M = \frac{1}{2}x^2. \tag{3.10}$$

This M will be the main term of $\psi_1(x)$. It remains to estimate the contribution of the integrals I_j.

Estimates of I_1 and I_5.

The integrals I_1 and I_5 are along the vertical segments $(a - i\infty, a - iT]$ and $[a + iT, a + i\infty)$. On these segments we have

$$\sigma = a = 1 + \frac{1}{\log x}, \quad |t| \geq T.$$

Therefore

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^a} = -\frac{\zeta'(a)}{\zeta(a)} \ll \frac{1}{a-1} = \log x$$

and

$$\left|\frac{x^{s+1}}{(s)(s+1)}\right| \leq \frac{x^{a+1}}{t^2} = \frac{ex^2}{t^2}.$$

Hence we obtain

$$I_1, I_5 \ll \int_T^{\infty} \log x \frac{x^2}{t^2} dt \ll \frac{x^2 \log x}{T}. \tag{3.11}$$

Estimates of I_2 and I_4.

These integrals are along the horizontal segments $[a - iT, b - iT]$ and $[b - iT, a + iT]$. On these segments we have

$$1 - \frac{c_6}{(\log T)^9} = b \leq \sigma \leq a = 1 + \frac{1}{\log x}, \quad t = T.$$

Hence by Theorem 3.8 we have

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \ll (\log T)^9,$$

and

$$\left|\frac{x^{s+1}}{(s)(s+1)}\right| \leq \frac{x^{a+1}}{|s||s+1|} \ll \frac{x^2}{T^2}.$$

Thus we obtain

$$I_2, I_4 \ll \int_a^b (\log T)^9 \frac{x^2}{T^2} d\sigma \ll \frac{x^2 (\log T)^9}{T^2}. \tag{3.12}$$

Estimate of I_3.

This integral is along the vertical segment $[b - iT, b + iT]$ and on this segment we have

$$\sigma = b = 1 - \frac{c_6}{(\log T)^9}, \quad |t| \geq T.$$

Again by Theorem 3.8 we have

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \ll \max\left((\log T)^9, \frac{1}{1-b}\right) \ll (\log T)^9.$$

Also, since

$$\left|\frac{x^{s+1}}{(s)(s+1)}\right| = \frac{x^{b+1}}{|s||s+1|} \ll x^{b+1}\min\left(1, \frac{1}{t^2}\right),$$

we obtain

$$I_3 \ll \int_{-T}^{T} x^{b+1}(\log T)^9 \min\left(1, \frac{1}{t^2}\right) dt \ll x^{b+1}(\log T)^9. \tag{3.13}$$

Now we have estimated all of the integrals and we substitute the estimates (3.10)- (3.13) into (3.9) to get

$$\psi_1(x) = \frac{1}{2}x^2 + R(x,T),$$

where

$$\begin{aligned} R(x,T) &\ll x^2\left(\frac{\log x}{T} + \frac{(\log T)^9}{T^2} + x^{b-1}(\log T)^9\right) \\ &\ll x^2\left(\frac{\log x}{T} + (\log T)^9 \exp\left(-c_6\frac{\log x}{(\log T)^9}\right)\right) \end{aligned}$$

Now we choose T as

$$T = \exp((\log x)^{1/10}).$$

Since $x \geq e$, this choice of T satisfies $e \leq T \leq x$. Substituting T in the above estimate we obtain

$$\begin{aligned} R(x,T) \ll \sum_{j=1}^{5}|I_j| &\ll x^2\left(\log x \exp(-(\log x)^{1/10}) + (\log x)^{9/10}\exp(-c_6(\log x)^{1/10})\right) \\ &\ll x^2\exp(-c_7(\log x)^{1/10}), \end{aligned}$$

with a suitable constant $c_7 > 0$. Hence finally we have for $x \geq e$

$$\psi_1 = \frac{1}{2}x^2 + O(x^2\exp(-c_7(\log x)^{1/10})). \tag{3.14}$$

Transition to $\psi(x)$.

Now we will pass to the estimate for $\psi(x)$ to prove the PNT.

Recall that these two functions are related by

$$\psi_1(x) = \int_0^x \psi(y)dy.$$

We fix $x \geq 6$ and a number $0 < \delta < 1/2$ to be chosen later as a function of x. Note that we have

$$\psi_1(x) - \psi_1(x(1-\delta)) = \int_{x(1-\delta)}^x \psi(y)dy \leq \delta x \psi(x),$$

since $\psi(x)$ is nondecreasing. Now we apply the result (3.14) to $\psi_1(x)$ and $\psi_1(x(1-\delta))$ to get

$$\begin{aligned} \delta\psi(x) &\geq \frac{1}{2}x^2 + O(x^2 A) - \frac{1}{2}x^2(1-\delta)^2 + O(x^2 A') \\ &= \delta x^2 + O(x^2(\delta^2 + A + A')), \end{aligned} \tag{3.15}$$

where

$$A = \exp(-c_7(\log x)^{1/10}) \quad \text{and} \quad A' = \exp(-c_7(\log x(1-\delta))^{1/10}).$$

Since $x(1-\delta) \geq x/2 \geq \sqrt{x}$ we have

$$(\log x(1-\delta))^{1/10} \geq (\log \sqrt{x})^{1/10} \geq \frac{1}{2}(\log x)^{1/10}$$

and hence $A' \leq A^{1/2} \leq 1$. Now (3.15) yields

$$\psi(x) \geq x + O\left(x\left(\delta + \frac{\sqrt{A}}{\delta}\right)\right).$$

Now define δ by

$$\delta = \min(1/2, A^{1/4}).$$

Putting δ in the above equation we obtain

$$\psi(x) \geq x + O(x\delta) = x + O(x\exp(-c_8(\log x)^{1/10})),$$

with $c_8 = \frac{c_7}{4}$.

For the reverse inequality we start from the inequality

$$\psi_1(x(1+\delta)) - \psi_1(x) \geq \delta x \psi(x),$$

and similarly get

$$\psi(x) \leq x + O\left(x\left(\delta + \frac{A}{\delta} + \frac{A'}{\delta}\right)\right),$$

where

$$A = \exp(-c_7(\log x)^{1/10}), \quad A' = \exp(-c_7(\log x(1+\delta))^{1/10}).$$

From this we easily obtain (with the same choice of δ and noting that $A' \leq A$) the reverse side. Hence we obtained the estimate :

$$\psi(x) = x + O(x\delta) = x + O(x\exp(-c_8(\log x)^{1/10}))$$

in the range $x \geq 6$. The estimate holds trivially for $2 \leq x < 6$. So we have completed the proof of PNT in the form given in Theorem 3.1.

Transition to $\pi(x)$.
Now we will derive

$$\pi(x) = li(x) + O(x\exp(-c(\log x)^{1/10})), \tag{3.16}$$

from the equation (3.1) by partial summation.

Consider the function defined as

$$\pi_1(x) = \sum_{n \leq x} \frac{\Lambda(n)}{\log n}.$$

We relate $\pi_1(x)$ to $\pi(x)$ as follows:

$$\begin{aligned}\pi_1(x) &= \sum_{p^m \leq x} \frac{\log p}{m \log p} = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) \\ &= \pi(x) + O(\sqrt{x}), \end{aligned} \tag{3.17}$$

since $\pi(x) = O(x)$ by (1.5).

Now we apply partial summation and have

$$\pi_1(x) = \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(t)}{t\log^2 t}dt$$
$$= \frac{x}{\log x} + O(x\exp(-c_8(\log x)^{1/10} - \log\log x))$$
$$+ \int_2^x \frac{dt}{\log^2 t} + O\left(\int_2^x \frac{\exp(-c_8(\log x)^{1/10}}{\log^2 t}dt\right), \qquad (3.18)$$

where we used (3.1). We have, by integrating by parts,

$$li(x) = \int_2^x \frac{dt}{\log t} = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{dt}{\log^2 t}. \qquad (3.19)$$

Also,

$$\int_2^x \frac{\exp(-c_8(\log t)^{1/10})}{\log^2 t}dt = \int_2^{\sqrt{x}} \frac{\exp(-c_8(\log t)^{1/10})}{\log^2 t}dt + \int_{\sqrt{x}}^x \frac{\exp(-c_8(\log t)^{1/10})}{\log^2 t}dt.$$

The first integral on the right of the last equation is $O(\sqrt{x})$. For the second one we have $(\log t)^{1/10} \geq \frac{1}{2}(\log x)^{1/10}$ since $t \geq \sqrt{x}$ and hence

$$\frac{\exp(-c_8(\log t)^{1/10})}{\log^2 t} \ll \frac{\exp(-c_8/2(\log x)^{1/10})}{\log^2 x}.$$

So we obtained

$$\int_2^x \frac{\exp(-c_8(\log t)^{1/10})}{\log^2 t}dt = O(x\exp(-c_8/2(\log x)^{1/10})). \qquad (3.20)$$

Therefore combining (3.17) - (3.20) we obtain (3.16) with $c = c_8/2$.

3.2 Refined Analytic Techniques, de la Vallée Poussin Error Term

In this part, we will show that the *de la Vallée Poussin-type* error term $x\exp(-c(\log x)^{\alpha})$, which was given in the previous section with $\alpha = 1/10$, can be made better up to $\alpha = 1/2$. We will not fully prove this result, instead we will give references in some places and emphasize the main steps of the proof.

The number α in the error term is closely related to the breadth of the zero-free region of the function $\zeta(s)$. Actually it is determined by the exponent a of the function $\log t$ in the zero-free region

$$\sigma \geq 1 - \frac{c}{(\log t)^a}, \quad |t| \geq 2. \tag{3.21}$$

We have proved in Theorem 3.6 (iii) that a can be 9. But using the functional equation of $\zeta(s)$ and complex function theory we are able to enlarge the region (3.21) to the one with $a = 1$ so that we obtain the error term with $\alpha = 1/2$.

The functional equation of $\zeta(s)$.

Riemann made the greatest contribution to the study of distribution of primes with his memoir in 1859. In his paper he showed that

- The function $\zeta(s)$ satisfies the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{3.22}$$

  where $\Gamma(s)$ is the so-called Gamma function which is analytic in the half plane $\sigma > 0$ with the integal representation $\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx$.

- $\zeta(s)$ can be continued analytically over the whole plane and $\zeta(s)$ is meromorphic with the simple pole at $s = 1$ with residue 1.

The second can be deduced from the functional equation regarding the properties of the $\Gamma(s)$ function.

To use the functional equation effectively we define the function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{3.23}$$

The function $\xi(s)$ is an entire function since it has no pole for $\sigma \geq \frac{1}{2}$ and satisfies $\xi(s - \frac{1}{2}) = \xi(\frac{1}{2} - s)$. Moreover $\xi(s)$ has the product representation

$$\xi(s) = e^{A+Bs}\prod_{\rho}\left(1 - \frac{s}{\rho}\right)e^{s/\rho} \tag{3.24}$$

where A and B are constants and ρ runs through the zeros of $\zeta(s)$ in the *critical strip* $0 < \sigma < 1$. This was proved by Hadamard and lead to improvements in enlarging the zero-free region of $\zeta(s)$. Logaritmic derivative of the equations (3.23) and (3.24) gives

$$\frac{\zeta'(s)}{\zeta(s)} = B - \frac{1}{s-1} + \frac{1}{2}\log\pi - \frac{1}{2}\frac{\Gamma'(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1)} + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \tag{3.25}$$

A zero-free region for $\zeta(s)$.
To establish the zero-free region

$$\sigma \geq 1 - \frac{c}{\log t}, \quad |t| \geq 2, \tag{3.26}$$

we start with employing the 3-4-1 inequality for $\Re\frac{\zeta'(s)}{\zeta(s)}$ and get

$$3\left[-\frac{\zeta'(\sigma)}{\zeta(\sigma)}\right] + 4\left[-\Re\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right] + \left[-\Re\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right] \geq 0. \tag{3.27}$$

This constitute the main inequality leading to a bound for the abscissa of a zero of $\zeta(s)$. We will give bounds for each of the three terms on the left.

The bound for $-\frac{\zeta'(\sigma)}{\zeta(\sigma)}$ in $1 < \sigma \leq 2$ comes from the simple pole of $\zeta(s)$ at $s = 1$. For a positive constant A_1 we have

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} < \frac{1}{\sigma - 1} + A_1.$$

The bounds for the remaining two terms comes from taking the real part of the equation (3.25). The Γ term there is less than $A_2 \log t$ if $t \geq 2$ and $1 < \sigma \leq 2$. Hence in this region

$$-\Re\frac{\zeta'(s)}{\zeta(s)} < A_2 \log t - \sum_{\rho} \Re\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \tag{3.28}$$

Since $\Re\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) \geq 0$ for any ρ, for $s = \sigma + 2it$ we write (3.28) ommitting the whole sum and obtain

$$-\Re\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)} < A_2 \log t.$$

For $s = \sigma + it$, we choose t to coincide with the ordinate γ of a zero $\rho = \beta + i\gamma$ with $\gamma \geq 2$. We take just one term $\frac{1}{\sigma-\beta}$ (the corresponding term to our choise of ρ) and obtain from (3.28)

$$-\Re\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} < A_2 \log t - \frac{1}{\sigma - \beta}.$$

Now we substitute these bouns in the main inequality (3.27) and derive an inequality for the abscissa $\beta$of a zero

$$\frac{4}{\sigma - \beta} < \frac{3}{\sigma - 1} + A_3 \log t$$

where A_3 is a positive constant.

Taking $\sigma = 1 + \frac{\delta}{\log t}$ with $\delta > 0$ and solving for β we get

$$\beta < 1 + \frac{\delta}{\log t} - \frac{4\delta}{(3 + A_3\delta)\log t} = 1 - \frac{c}{\log t} \tag{3.29}$$

choosing δ suitable in relation to A to obtain a constant $c > 0$. Hence the zero-free region (3.26) is established.

The explicit formula for $\psi(x)$.

In the first analytic proof we gave, we worked with the function $\psi_1(x)$ and relate the result to $\psi(x)$. Here, we will work with $\psi_0(x)$ which is $\psi(x)$ when x is not a prime power and $\psi(x) - \Lambda(x)/2$ when it is.

As in the first proof Perron's formula gives

$$\psi_0(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds. \qquad (a > 1)$$

But this time we start with an integral from $a - iT$ to $a + iT$ and regard this as one side of a rectangle extending to the left. We choose T so that the horizontal sides of the rectangle shall avoid the zeros of $\zeta(s)$ in the critical stript. We replace the vertical line of integration $a - iT$ to $a + iT$ by other three sides of the rectangle with vertices

$$a - iT, \quad a + iT, \quad -U - iT, \quad -U + iT. \qquad (U \text{ is an odd integer})$$

The sum of the residues of the integrand at its poles is

$$x - \sum_{|\gamma|<T}\frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \sum_{0<2m<U}\frac{x^{-2m}}{-2m}.$$

The pole at $s = 1$ contributes to x; the pole at $s = 0$ contributes $-\frac{\zeta'(0)}{\zeta(0)}$; each trivial zero $-2m$ contributes $-\frac{x^{-2m}}{-2m}$ and each non-trivial zero ρ contributes $-\frac{x^\rho}{\rho}$.

After the estimates of the integral on horizontal and vertical lines has been done, we let $U \to \infty$ and have the explicit formula (also called the Riemann- Von Mangoldt formula)

$$\psi_0(x) = x - \sum_{|\gamma|<T}\frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - x^{-2}). \tag{3.30}$$

The Prime Number Theorem.

We shall now deduce PNT from (3.30). The main estimate to be done is the sum $\sum_{|\gamma|<T} \frac{x^\rho}{\rho}$. We have by (3.39)

$$|x^\rho| = x^\beta < \exp\left(\frac{-c\log x}{\log T}\right)$$

for large T and $|\gamma| < T$.

For $\gamma > 0$, $|\rho| \geq \gamma$ and it remains to estimate $\sum_{0<\gamma<T} \frac{1}{\gamma}$.

Let $N(t)$ denote the number of zeros in the critical strip with ordinates less than t. Then

$$\sum_{0<\gamma<T} \frac{1}{\gamma} = \int_0^T \frac{dN(t)}{t} = \frac{N(T)}{T} + \int_0^T \frac{dN(t)}{t^2}.$$

$N(t) \ll t\log t$ for large t and hence

$$\sum_{0<\gamma<T} \frac{1}{\gamma} \ll \log^2 T$$

so that

$$\sum_{|\gamma|<T} \left|\frac{x^\rho}{\rho}\right| \ll x\log^2 T \exp\left(\frac{-c\log x}{\log T}\right).$$

All we have to do is now to choose T as a function of x as

$$\log^2 T = \log x.$$

Using this choice of T in the explicit formula, we have for some $c > 0$

$$|\psi(x) - x| \ll x\exp(-c(\log x)^{1/2}).$$

Transition to $\pi(x)$ is done in the same way as in the previous part and gives

$$\pi(x) = li(x) + O(x\exp(-c(\log x)^{1/2})).$$

The Error Term in the PNT

After the PNT had been proved, the main problem has become obtaining the PNT with an error term as good as possible. Riemann, in his paper in 1859 has conjectured the Riemann Hypothesis which states that all non-trivial zeros of the Riemann zeta-function have real part 1/2. As the error term is related to the zero-free region of $\zeta(s)$, the Riemann hypothesis is equivalent to the PNT of the form

$$\psi(x) = x + O_\epsilon(x^{1/2+\epsilon}),$$

for any $\epsilon > 0$. Moreover, this is the best possible error term. Unfortunately this problem is still wide open we are very far from getting what is conjectured. The last progress about the error term has been made by Vinogradov and Korobov in 1958. They have enlarged the zero-free region to $1 - \frac{c_a}{(\log t)^a}$ for any $a > 2/3$ which resulted in the error term

$$\psi(x) = x + O(x \exp(-c(\log x)^{3/5-\epsilon}).$$

4 THE PNT ON ARITHMETIC PROGRESSIONS

The study of primes in arithmetic progressions startes with Dirichlet's theorem. In his theorem, Dirichlet proved the infinitude of primes belonging to a given arithmetic progression, say integers which are $\equiv a \pmod q$ where necessarily $(a, q) = 1$. His was firstly tended to mimick Euclid's proof on the infinitude of primes but the method failed in some cases. Then he attempted to prove his theorem using Euler's mehtod, showing the divergence of the series $\sum_{p \equiv a (\text{mod } q)} \frac{1}{p}$. The tools that Dirichlet introduced and which are now named after him, are the Dirichlet characters and Dirichlet L-functions. Certain properties such as (1.23) of Dirichlet characters are useful when we want to extract terms belonging to a given arithmetic progression. Hence Dirichlet characters and Dirichlet L-functions constitute the key tools for the study of primes in arithmetic progressions.

If we denote the number of primes $\equiv a \pmod q$ where $(a, q) = 1$ by $\pi(x; q, a)$, the PNT for arithmetic progressions gives

$$\pi(x; q, a) = \frac{li(x)}{\phi(q)} - \frac{\chi_1(a) li(x^{\beta_1})}{\phi(q)\beta_1} + O(x \exp(-c(\log x)^{1/2})).$$

The factor $\frac{1}{\phi(q)}$ is not surprising, if we assume that the primes are distributed approximately equally among the residue classes of q. The proof is done following the same procedures in the proof of PNT. But we work with $L(s, \chi)$ instead of $\zeta(s)$. Moreover, the PNT was proved in the form (3.1) which is an estimate of the function $\psi(x)$. Hence as an analogue of $\psi(x)$, we define

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a (\text{mod } q)}} \Lambda(n),$$

and give an equivalent form of the PNT for arithmetic progressions.

Theorem 4.1 (The Prime Number Theroem For Arithmetic Progressions)**.** *There exists an absolute constant $c > 0$ such that if $x \geq 2$ and $(a, q) = 1$, then*

$$\psi(x; q, a) = \frac{x}{\phi(q)} - \frac{\bar{\chi}_1(a) x^{\beta_1}}{\phi(q)\beta_1} + O(x \exp(-c(\log x)^{1/2})). \tag{4.1}$$

where χ_1 denotes the single character modulo q,if it exists, having the exceptional zero β_1; if such χ_1 with β_1 does not exist then the term $-\frac{\bar{\chi}_1(a) x^{\beta_1}}{\phi(q)\beta_1}$ should be omitted.

We will introduce some lemmas and theorems before proving the PNT for arithmetic progressions. All the work done for Riemann zeta function $\zeta(s)$ in the previous section has analogues for the Dirichlet L-function $L(s, \chi)$.

Now we will give these analogues starting from the functional equation.

4.1 The Functional Equation of $L(s,\chi)$

Theorem 4.2 (Functional Equation of $L(s,\chi)$). *Let χ be a primitve character modulo q with $q > 1$.*

If $\chi(-1) = 1$, then $L(s,\chi)$ satisfies

$$\frac{q^{\frac{s}{2}}}{\tau(\chi)}\pi^{-\frac{s}{2}}q^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi) = \pi^{-\frac{1-s}{2}}q^{\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)L(1-s,\bar{\chi}). \tag{4.2}$$

If $\chi(-1) = 1$, then $L(s,\chi)$ satisfies

$$\frac{iq^{\frac{s}{2}}}{\tau(\chi)}\pi^{-\frac{s+1}{2}}q^{\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi) = \pi^{-\frac{2-s}{2}}q^{\frac{2-s}{2}}\Gamma\left(\frac{2-s}{2}\right)L(1-s,\bar{\chi}). \tag{4.3}$$

Here $\Gamma(s)$ is the Gamma function and $\tau(\chi)$ is the Gaussian sum.

Proof. First assume $\chi(-1) = 1$. Beginning with definition of the $\Gamma(s)$

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-t}t^{\frac{s}{2}-1}dt \quad . \qquad (\sigma > 0)$$

Put $t = \frac{n^2\pi x}{q}$,

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-\frac{n^2\pi x}{q}}\left(\frac{n^2\pi x}{q}\right)^{\frac{s}{2}-1}\frac{n\pi}{q}dx = n^s\left(\frac{\pi}{q}\right)^{\frac{s}{2}}\int_0^\infty e^{-\frac{n^2\pi x}{q}}x^{\frac{s}{2}-1}dx.$$

Rearrengement gives

$$\left(\frac{\pi}{q}\right)^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\frac{1}{n^s} = \int_0^\infty e^{-\frac{n^2\pi x}{q}}x^{\frac{s}{2}-1}dx.$$

Multiplying the above equality by $\chi(n)$ and summing over n we get for $\sigma > 1$

$$\left(\frac{\pi}{q}\right)^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi) = \int_0^\infty x^{\frac{s}{2}-1}\left(\sum_{n=1}^{\infty}\chi(n)e^{-\frac{n^2\pi x}{q}}\right)dx. \tag{4.4}$$

Define the function

$$\psi(x,\chi) = \sum_{n=-\infty}^{\infty}\chi(n)e^{-\frac{n^2\pi x}{q}}$$

when $\chi(-1) = 1$. Hence the right handside of the equation (4.4) becomes

$$\frac{1}{2}\int_0^\infty x^{\frac{s}{2}-1}\psi(x,\chi)dx.$$

Now we will derive a functional equation that relates $\psi(x,\chi)$ to $\psi(x^{-1},\bar{\chi})$. Consider

$$\tau(\bar{\chi})\psi(x,\chi) = \sum_{n=-\infty}^{\infty} \chi(n)e^{-\frac{n^2\pi x}{q}}\tau(\bar{\chi}).$$

Replacing $\chi(n)$ by $\frac{1}{\tau(\bar{\chi})}\sum_{m=1}^{q}\bar{\chi}(m)e^{2\pi imn/q}$ by (1.28) in the above equality we obtain

$$\tau(\bar{\chi})\psi(x,\chi) = \sum_{m=1}^{q}\chi(\bar{m})\sum_{n=-\infty}^{\infty} e^{-\frac{n^2\pi x}{q}+\frac{2\pi imn}{q}}. \tag{4.5}$$

Here we will use the identity (8)in [1], §8

$$\sum_{n=-\infty}^{\infty} e^{-(n+\alpha)^2\frac{\pi}{x}+\frac{2\pi imn}{q}} = x^{1/2}\sum_{n=-\infty}^{\infty} e^{-n^2\pi x+2\pi in\alpha}. \tag{4.6}$$

where $\alpha \in \mathbb{R}$ and $x > 0$. Employing the equality (4.6) with x replaced by x/q and $\alpha = m/q$, (4.5) becomes

$$\begin{aligned}
\tau(\bar{\chi})\psi(x,\chi) &= \sum_{m=1}^{q}\bar{\chi}(m)\left(\frac{q}{x}\right)^{1/2}\sum_{n=-\infty}^{\infty} e^{-(n+\frac{m}{q})^2\frac{\pi x}{q}} \qquad (\text{write } l = nq+m)\\
&= \left(\frac{q}{x}\right)^{1/2}\sum_{n=-\infty}^{\infty}\bar{\chi}(l)e^{-\frac{l^2\pi}{xq}}\\
&= \left(\frac{q}{x}\right)^{1/2}\psi(x^{-1},\bar{\chi})
\end{aligned}$$

Now we turn back to (4.4) and split the integral into two parts as

$$\begin{aligned}
\left(\frac{\pi}{q}\right)^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi) &= \frac{1}{2}\int_1^\infty x^{\frac{s}{2}-1}\psi(x,\chi)dx + \frac{1}{2}\int_0^1 y^{\frac{s}{2}-1}\psi(y,\chi)dy\\
&\qquad\qquad (\text{take } y = \tfrac{1}{x})\\
&= \frac{1}{2}\int_1^\infty x^{\frac{s}{2}-1}\psi(x,\chi)dx + \frac{1}{2}\int_1^\infty x^{-\frac{s}{2}-1}\psi(1/x,\chi)dx\\
&= \frac{1}{2}\frac{q^{1/2}}{\tau(\bar{\chi})}\int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\psi(x,\bar{\chi})dx + \frac{1}{2}\int_1^\infty x^{-\frac{s}{2}-1}\psi(1/x,\chi)dx.
\end{aligned} \tag{4.7}$$

Replacing s by $1-s$ and χ by $\bar{\chi}$ in (4.7) we get

$$\begin{aligned}
\left(\frac{\pi}{q}\right)^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) L(1-s,\bar{\chi}) &= \frac{1}{2}\frac{q^{\frac{1}{2}}}{\tau(\chi)} \int_1^\infty x^{\frac{s}{2}-1}\psi(x,\chi)dx \\
&+ \frac{1}{2}\int_1^\infty x^{\frac{-s-1}{2}}\psi(x,\bar{\chi})dx. \\
&= \frac{q^{\frac{1}{2}}}{\tau(\chi)}\frac{1}{2}\int_1^\infty x^{\frac{s}{2}-1}\psi(x,\chi)dx \\
&+ \frac{q^{\frac{1}{2}}}{\tau(\chi)}\frac{1}{2}\frac{q^{\frac{1}{2}}}{\tau(\bar{\chi})}\int_1^\infty x^{\frac{-s-1}{2}}\psi(x,\bar{\chi})dx. \\
&= \frac{q^{\frac{1}{2}}}{\tau(\chi)}\left(\frac{\pi}{q}\right)^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi).
\end{aligned}$$

Above, in the first line we used the functional equation relating $\psi(x,\chi)$ to $\psi(x^{-1},\bar{\chi})$, in the second line we used the fact that $\tau(\chi)\tau(\bar{\chi}) = q$ by (1.30) and finally in the third line we used (4.7) so that we have obtained (4.2).

Now we pass to the case when $\chi(-1) = -1$.

The previous argument fails since $\psi(x,\chi)$ vanishes in this case. Hence, for this case, we define

$$\psi_1(x,\chi) = \sum_{n=-\infty}^{\infty} n\chi(n)e^{-\frac{n^2\pi x}{q}}.$$

We begin with $\Gamma\left(\frac{s+1}{2}\right)$ and similarly get

$$\left(\frac{q}{\pi}\right)^{\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi) = \frac{1}{2}\int_0^\infty x^{\frac{s-1}{2}}\psi_1(x,\chi)dx. \tag{4.8}$$

We appeal to the relation

$$\sum_{n=-\infty}^{\infty} ne^{-\frac{n^2\pi x}{q}+\frac{2\pi imn}{q}} = i\left(\frac{q}{x}\right)^{3/2}\sum_{n=-\infty}^{\infty}\left(n+\frac{m}{q}\right)e^{-\pi\left(n+\frac{m}{q}\right)^2\frac{q}{x}}, \tag{4.9}$$

which can be deduced from (4.6) by differentiating with respect to α and writing x/q in place of x, and m/q in place of α.

As before we use the relation (1.28) and have

$$
\begin{aligned}
\tau(\bar{\chi})\psi_1(x,\chi) &= \sum_{m=1}^{q} \bar{\chi}(m) \sum_{n=-\infty}^{\infty} n e^{-\frac{n^2\pi x}{q}+\frac{2\pi i m n}{q}} \\
&= i\frac{q^{1/2}}{x^{3/2}} \sum_{m=1}^{q} \bar{\chi}(m) \sum_{n=-\infty}^{\infty} (qn+m) e^{-\frac{\pi(qn+m)^2}{xq}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{write } l = nq+m) \\
&= i\frac{q^{1/2}}{x^{3/2}} \sum_{n=-\infty}^{\infty} l\bar{\chi}(l) e^{-\frac{l^2\pi}{xq}} \\
&= i\frac{q^{1/2}}{x^{3/2}} \psi_1(x^{-1},\bar{\chi})
\end{aligned}
\tag{4.10}
$$

Hence we derived the functional equation for $\psi_1(x,\chi)$.

As in the previous case we split the integral in (4.8) into two parts and use (4.10) to have

$$
\left(\frac{q}{\pi}\right)^{\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi) = \frac{1}{2}\int_1^{\infty} x^{\frac{s-1}{2}} \psi_1(x,\chi)dx + \frac{1}{2}\frac{iq^{1/2}}{\tau(\bar{\chi})}\int_1^{\infty} x^{\frac{-s}{2}} \psi_1(x,\bar{\chi})dx.
\tag{4.11}
$$

Replacing s by $1-s$ and χ by $\bar{\chi}$ in (4.11) we obtain

$$
\begin{aligned}
\left(\frac{q}{\pi}\right)^{\frac{2-s}{2}} \Gamma\left(\frac{2-s}{2}\right) L(1-s,\bar{\chi}) &= \frac{1}{2}\int_1^{\infty} x^{\frac{-s}{2}} \psi_1(x,\bar{\chi})dx \\
&+ \frac{1}{2}\frac{iq^{\frac{1}{2}}}{\tau(\chi)}\int_1^{\infty} x^{\frac{s-1}{2}} \psi_1(x,\bar{\chi})dx. \\
&= \frac{iq^{\frac{1}{2}}}{\tau(\chi)}\frac{iq^{\frac{1}{2}}}{\tau(\bar{\chi})}\frac{1}{2}\int_1^{\infty} x^{\frac{-s}{2}} \psi_1(x,\bar{\chi})dx \\
&+ \frac{iq^{\frac{1}{2}}}{\tau(\chi)}\frac{1}{2}\int_1^{\infty} x^{\frac{s-1}{2}} \psi_1(x,\bar{\chi})dx \\
&= \frac{iq^{\frac{1}{2}}}{\tau(\chi)}\left(\frac{q}{\pi}\right)^{\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi).
\end{aligned}
$$

Above we used (4.10) in the first line and in the second line we used $\tau(\chi)\tau(\bar{\chi}) = -q$ by (1.30). Hence we have proved (4.3).

□

It is more convenient, for future reference, to put these two forms of functional equations into a single one in the following lemma. We will do this by introducing a number a, defined as

$$a = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

Lemma 4.3. *Let χ be a primitive character modulo q. If*

$$\xi(s,\chi) = \left(\frac{q}{\pi}\right)^{\frac{s+a}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi), \tag{4.12}$$

then

$$\xi(1-s,\bar{\chi}) = \frac{i^a q^{1/2}}{\tau(\chi)} \xi(s,\chi). \tag{4.13}$$

From the functional equation, regarding the properties of the Γ function, we can deduce the following for $L(s,\chi)$ when χ modulo q is primitive:

- The zeros of $\xi(s,\chi)$ (if any exist) are all situated in the critical strip $0 \leq \sigma \leq 1$, with neither $s = 0$ or $s = 1$ being a zero. These zeros are placed symmetrically about the line $\sigma = \frac{1}{2}$.

- The zeros of $L(s,\chi)$ are identical (in position and order of multiplicity) with those of $\xi(s,\chi)$, except that $L(s,\chi)$ has a simple zero at each point $s = -a, -a-2, -a-4, \ldots$ coming from the simple poles of $\Gamma\left(\frac{s+a}{2}\right)$.

- In conclusion, $L(s,\chi)$ has no zeros in the half-plane $\sigma > 1$ (can be seen from the Euler product formula as well), has possible *critical zeros* which we usually denote by $\rho = \beta + i\gamma$ in the critical strip $0 \leq \sigma \leq 1$, and simple zeros $-a - 2m$ in the halfplane $\sigma \leq 0$. The zeros of $L(s,\chi)$ in $\sigma < 0$ and $s = 0$ in the case when $a = 0$ are called the *trivial zeros* of $L(s,\chi)$.

We defined $\xi(x,\chi)$ in the previous theorem to use the functional equation more effectively. That is because $\xi(x,\chi)$ is entire and the zeros of $\xi(x,\chi)$ coincide exactly with the non-trivial (or critical) zeros of $L(s,\chi)$. Also,

$$|\xi(x,\chi)| < e^{C|s|\log|s|} \quad \text{as } |s| \to \infty, \text{ for some constant } C > 0,$$

but

$$|\xi(x,\chi)| < e^{C'|s|}$$

does not hold for any constant $C' > 0$ when $|s|$ is large. Hence $\xi(x,\chi)$ is of order at most 1. (Remember that an entire function $f(z)$ is said to be of finite order if there is a number $\alpha > 0$ such that $f(z) = O(e^{|z|^\alpha})$ as $|z| \to \infty$.) Since $\xi(x,\chi)$ satisfies these, then by Lemma 10.11 of [2], the next theorem follows.

Theorem 4.4 (Product Formula for $\xi(s,\chi)$)**.** *Let χ be a primitive character modulo q. The function $\xi(s,\chi)$ necessarily has the form*

$$\xi(s,\chi) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \tag{4.14}$$

where A and B are constants and ρ runs over the critical zeros of $L(s,\chi)$.

The product formula is useful when we construct the zero-free region for $L(s,\chi)$. We need estimates on $\frac{L'(s,\chi)}{L(s,\chi)}$ and the product formula provides an expression for $\frac{L'(s,\chi)}{L(s,\chi)}$ via logarithmic differentiation as follows:

We first take the logaritmic derivative of (4.12) to obtain

$$\frac{\xi'(s,\chi)}{\xi(s,\chi)} = \frac{1}{2}\log\left(\frac{q}{\pi}\right) + \frac{\Gamma'\left(\frac{s+a}{2}\right)}{2\Gamma\left(\frac{s+a}{2}\right)} + \frac{L'(s,\chi)}{L(s,\chi)}. \tag{4.15}$$

Then we take the logaritmic derivative of (4.13).

$$\frac{\xi'(s,\chi)}{\xi(s,\chi)} = B(\chi) + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \tag{4.16}$$

Combining these two gives us the following theorem:

Theorem 4.5. *If χ is a primitive character modulo q and $B(\chi)$ is the number in Theorem 4.4, then we have*

$$\frac{L'(s,\chi)}{L(s,\chi)} = -\frac{1}{2}\log\left(\frac{q}{\pi}\right) - \frac{\Gamma'\left(\frac{s+a}{2}\right)}{2\Gamma\left(\frac{s+a}{2}\right)} + B(\chi) + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \tag{4.17}$$

In the next lemma we will try to give an expression for $B(\chi)$ in (4.17).

Lemma 4.6. *Let χ be a primitive character modulo q and $B(\chi)$ be the number in Theorem 4.4. Then*

$$B(\chi) = -\frac{1}{2}\log\left(\frac{q}{\pi}\right) - -\frac{L'(1,\bar{\chi})}{L(1,\bar{\chi})} + \frac{1}{2}E + (1-a)\log 2. \tag{4.18}$$

Moreover,

$$\Re B(\chi) = \frac{1}{2}\sum_{\rho}\left(\frac{1}{1-\rho} + \frac{1}{\rho}\right) = \sum_{\rho}\Re\frac{1}{\rho}. \tag{4.19}$$

Here, E is the Euler's constant.

Proof. Taking $s=0$ in (4.16) we get

$$B(\chi) = \frac{\xi'(0,\chi)}{\xi(0,\chi)} = -\frac{\xi'(1,\bar{\chi})}{\xi(1,\bar{\chi})}$$

by the functional equation (4.13) of $\xi(s,\chi)$. Now taking $s=1$ and $\chi=\bar{\chi}$ in (4.15) we obtain

$$B(\chi) = -\frac{1}{2}\log\left(\frac{q}{\pi}\right) - \frac{L'(1,\bar{\chi})}{L(1,\bar{\chi})} - \frac{\Gamma'\left(\frac{1+a}{2}\right)}{2\Gamma\left(\frac{1+a}{2}\right)}.$$

Since $\frac{\Gamma'\left(\frac{1+a}{2}\right)}{2\Gamma\left(\frac{1+a}{2}\right)} = -\frac{1}{2}E - (1-a)\log 2$ (given by (C.14) in [2]) we have proved (4.18).

From (4.18) it can be seen that $B(\bar{\chi}) = \bar{B(\chi)}$ since $L(1,\bar{\chi}) = L(\bar{1,\chi})$. Taking $s=1$ in (4.16)

$$B(\chi) + \sum_{\rho}\left(\frac{1}{1-\rho} + \frac{1}{\rho}\right) = \frac{\xi'(1,\chi)}{\xi(1,\chi)} = -\frac{\xi'(0,\bar{\chi})}{\xi(0,\bar{\chi})} = -B(\bar{\chi}) = -\bar{B(\chi)},$$

so that

$$\Re B(\chi) = -\frac{1}{2}\sum_{\rho}\left(\frac{1}{1-\rho} + \frac{1}{\rho}\right) = -\frac{1}{2}\sum_{\rho}\left(\Re\frac{1}{1-\rho} + \Re\frac{1}{\rho}\right).$$

Since for a critical zero ρ, we have $0 \le \Re\rho \le 1$ and $\rho \notin \{0,1\}$ we see that

$$\Re\frac{1}{1-\rho} \ge 0, \quad \Re\frac{1}{\rho} \ge 0$$

so that $\sum_{\rho}\Re\frac{1}{1-\rho}$ and $\sum_{\rho}\Re\frac{1}{\rho}$ both converge absolutely.(Since their sum converges and they both consist of nonnegative terms.)

Also, the map $\rho \mapsto 1-\bar{\rho}$ merely permutes the zeros of $L(s,\chi)$ so we write $1-\bar{\rho}$ in place of ρ in the sum and this would not change the sum because of absolute convergence. Hence

$$\begin{aligned}
\Re B(\chi) &= -\frac{1}{2}\sum_{\rho}\Re\frac{1}{1-\rho} - \frac{1}{2}\sum_{\rho}\Re\frac{1}{\rho} \\
&= -\frac{1}{2}\sum_{\rho}\Re\frac{1}{\bar{\rho}} - \frac{1}{2}\sum_{\rho}\Re\frac{1}{\rho} \\
&= \sum_{\rho}\Re\frac{1}{\rho}.
\end{aligned}$$

□

4.2 A Zero-free Region for $L(s,\chi)$

The next lemma can be regarded as the key to establish the zero-free region. As in the proof of zero-free region for $\zeta(s)$, it is based on the 3-4-1 inequality and is an analogue of the eqn (3.27) for $L(s,\chi)$. Throughout this section, c_i is a positive absolute constant for each i.

Lemma 4.7. *For any Dirichlet character χ modulo q and $\sigma > 1$ we have*

$$3\left[-\frac{L'(\sigma,\chi_0)}{L(\sigma,\chi_0)}\right] + 4\left[-\Re\frac{L'(\sigma+it,\chi)}{L(\sigma+it,\chi)}\right] + \left[-\Re\frac{L'(\sigma+2it,\chi^2)}{L(\sigma+2it,\chi^2)}\right] \geq 0. \quad (4.20)$$

Proof. Recall the Euler product formula for $L(s,\chi)$ for $\sigma > 1$

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right).$$

Logarithmic differentiation gives

$$\begin{aligned}
-\frac{L'(s,\chi)}{L(s,\chi)} &= \sum_p \sum_{m=1}^{\infty} \frac{\log p\,\chi(p^m)}{p^{ms}} \\
&= \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^s} \\
&= \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)e^{-it\log n}}{n^\sigma}.
\end{aligned}$$

If we set $\Re\chi(n)e^{-it\log n} = \cos\theta$, then $\Re\chi^2(n)e^{-2it\log n} = \cos 2\theta$ and hence

$$-\Re\frac{L'(\sigma+it,\chi)}{L(\sigma+it,\chi)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)\cos\theta}{n^\sigma}, -\Re\frac{L'(\sigma+2it,\chi^2)}{L(\sigma+2it,\chi^2)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)\cos 2\theta}{n^\sigma}.$$

So the RHS of the equation (4.20) becomes

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^\sigma}(3 + 4\cos\theta + \cos 2\theta). \quad (4.21)$$

Since $3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2 \geq 0$, each term in the sum (4.21) is ≥ 0 and we are done. □

Lemma 4.8. *Let χ be a primitive character modulo q and $1 \leq \sigma \leq 2$. Then*

$$-\Re\frac{L'(s,\chi)}{L(s,\chi)} < -\sum_\rho \Re\frac{1}{s-\rho} + c_1\mathscr{L} \quad (4.22)$$

where $\mathscr{L} = \log q + \log(|t|+2)$ and c_1 is a positive absolute constant.

Proof. Using (4.17) of Theorem 4.5 we have

$$\frac{L'(s,\chi)}{L(s,\chi)} = -\frac{1}{2}\log\left(\frac{q}{\pi}\right) - \Re\frac{\Gamma'\left(\frac{s+a}{2}\right)}{2\Gamma\left(\frac{s+a}{2}\right)} + \Re B(\chi) + \Re\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right).$$

By (4.19) we have $\Re B(\chi) = \sum_{\rho} \Re\frac{1}{\rho}$ where the series is absolutely convergent with nonnegative terms and hence any part of can be omitted. Noting that the Γ term is $O(\log|t|+2)$ by Theorem C.1 in [2] for $1 \le \sigma \le 2$, we obtain (4.22). □

We are ready to give the zero-free region for $L(s,\chi)$. We will consider the cases when χ is complex and when χ is real seperately.

Theorem 4.9. *For any complex character χ modulo q, $L(s,\chi)$ has no zero in the region*

$$\sigma \ge \begin{cases} 1 - \frac{c_5}{\log q|t|} & \text{if } |t| \ge 1, \\ 1 - \frac{c_5}{\log q} & \text{if } |t| \le 1. \end{cases}$$

where c_5 is a positive absolute constant.

Proof. We will WLOG assume $t \ge 0$, for the zeros of $L(s,\chi)$ with $t < 0$ are the complex conjugates of the zeros of $L(s,\bar{\chi})$ since $L(s,\chi) = \overline{L(s,\bar{\chi})}$.

First assume χ is primitive. We will use Lemma 4.7 and find bounds for each of the three terms on LHS of equation (4.20). The bound on the first term follows easily from the bound on $-\frac{\zeta'(\sigma)}{\zeta(\sigma)}$. (Since it has a simple pole at $s=1$).

$$-\frac{L'(\sigma,\chi_0)}{L(\sigma,\chi_0)} = \sum_{n=1}^{\infty}\frac{\chi_0(n)\Lambda(n)}{n^{\sigma}} \le \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma}} = -\frac{\zeta'(\sigma)}{\zeta(\sigma)} < \frac{1}{\sigma-1} + c_2 \tag{4.23}$$

for $1 \le \sigma \le 2$ and an absolute constant $c_2 > 0$. Now we choose t to be the imaginary part of a nontrivial zero $\rho = \beta + i\gamma$ of $L(s,\chi)$. Then by (4.22) in Lemma 4.8, where in the sum we omit the terms other than the one corresponding to the zero ρ we chose, we have for $1 \le \sigma \le 2$

$$-\Re\frac{L'(\sigma+it,\chi)}{L(\sigma+it,\chi)} < -\frac{1}{\sigma-\beta} + c_1\mathscr{L}. \tag{4.24}$$

To estimate the last term we cannot always apply Lemma 4.8 since χ^2 may not be primitive. Instead, if χ_1 is the primitive character modulo q_1 inducing χ^2 (χ is complex so that χ^2 is nonprincipal so is χ_1.) then Lemma 4.8 is valid for χ_1 and omitting the whole sum in (4.22) we have

$$-\Re\frac{L'(\sigma+2it,\chi_1)}{L(\sigma+2it,\chi_1)} < c_1(\log q_1 + \log(t+2)). \tag{4.25}$$

Recall the relation

$$L(s,\chi^2) = L(s,\chi_1)\prod_{p|q}\left(1-\frac{\chi_1(p)}{p^s}\right).$$

Lograithmic differentiation of the above equality gives

$$\begin{aligned}\left|\frac{L'(s,\chi^2)}{L(s,\chi^2)} - \frac{L'(s,\chi_1)}{L(s,\chi_1)}\right| &= \left|\sum_{p|q}\frac{\chi_1(p)p^{-s}\log p}{1-\chi_1(p)p^{-s}}\right| \\ &\le \sum_{p|q}\frac{p^{-\sigma}\log p}{1-p^{-\sigma}} \\ &\le \sum_{p|q}\log p \\ &\le \log q.\end{aligned}$$

Combining this with (4.25) we conclude

$$-\Re\frac{L'(\sigma+2it,\chi^2)}{L(\sigma+2it,\chi^2)} < c_3\mathscr{L}. \tag{4.26}$$

The estimates (4.23), (4.24) and (4.26) when substituted in the main inequality (4.20) gives

$$\frac{4}{\sigma-\beta} < \frac{3}{\sigma-1} + c_4\mathscr{L}. \tag{4.27}$$

Here we take $\sigma = 1 + \frac{\delta}{\mathscr{L}}$ for some constant $\delta > 0$. Then solving for β in (4.27)

$$\beta < 1 + \frac{\delta}{\mathscr{L}} - \frac{4\delta}{(3+c_4\delta)\mathscr{L}},$$

and if δ is chosen suitably in relation to c_4 we obtain

$$\beta < 1 - \frac{c_5}{\mathscr{L}}$$

for a constant $c_5 > 0$.

It remains to consider the case when χ is not primitive. Let χ_1 modulo q_1 be the primitive character inducing χ. Since χ is complex so is χ_1 and the above result applies to χ_1. i.e., $L(s, \chi_1)$ has no zero in the region

$$\sigma \geq \begin{cases} 1 - \frac{c_5}{\log q|t|} & \text{if } |t| \geq 1, \\ 1 - \frac{c_5}{\log q} & \text{if } |t| \leq 1. \end{cases}$$

But the relation $L(s, \chi) = L(s, \chi_1) \prod_{p|q} \left(1 - \frac{\chi_1(p)}{p^s}\right)$ implies that the only zeros of $L(s, \chi)$ other than the zeros of $L(s, \chi_1)$ are on $\sigma = 0$. So $L(s, \chi)$ has no zero in the above region as well. □

Theorem 4.10. *For any real nonprincipal character χ modulo q, $L(s, \chi)$ has at most one zero in the region*

$$\sigma \geq \begin{cases} 1 - \frac{c_{14}}{\log q|t|} & \textit{if } |t| \geq 1, \\ 1 - \frac{c_{14}}{\log q} & \textit{if } |t| \leq 1 \end{cases}$$

where c_{14} is a positive absolute constant.

Proof. Again we may assume $t \geq 0$. the argument in the previous proof works but needs modification for $-\Re \frac{L'(\sigma+2it,\chi^2)}{L(\sigma+2it,\chi^2)}$ since this time χ^2 is the principal character.

So we have (4.23) and (4.24) as well, and for $-\Re \frac{L'(\sigma+2it,\chi^2)}{L(\sigma+2it,\chi^2)}$ we use the expression (3.25) for $\frac{\zeta'(s)}{\zeta(s)}$

$$\frac{\zeta'(s)}{\zeta(s)} = B - \frac{1}{s-1} + \frac{1}{2}\log \pi - \frac{1}{2}\frac{\Gamma'(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1)} + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

The Γ term is $O(\log t)$ for $t \geq 2$ and $1 \leq \sigma \leq 2$ (Theorem C.1 of [2]) and in this region

$$-\Re\frac{\zeta'(s)}{\zeta(s)} < \Re\frac{1}{s-1} - \Re\sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + c_6 \log t.$$

The sum over the critical zeros can be neglected since it is positive. Hence for $t \geq 0$, $1 \leq \sigma \leq 2$ we have

$$-\Re\frac{\zeta'(s)}{\zeta(s)} < \Re\frac{1}{s-1} + c_6 \log(t+2).$$

The relation between $\frac{\zeta'(s)}{\zeta(s)}$ and $-\Re\frac{L'(\sigma+2it,\chi^2)}{L(\sigma+2it,\chi^2)}$ can be set up via the relation $L(s,\chi_0) = \zeta(s)\prod_{p|q}\left(1-\frac{1}{p^s}\right)$. Again, logarithmic differentiation gives

$$\left|\frac{L'(s,\chi_0)}{L(s,\chi_0)} - \frac{\zeta'(s)}{\zeta(s)}\right| = \left|\sum_{p|q}\frac{p^{-s}\log p}{1-p^{-s}}\right| \leq \sum_{p|q}\log p \leq \log q.$$

Hence

$$-\Re\frac{L'(\sigma+2it,\chi^2)}{L(\sigma+2it,\chi^2)} < \Re\frac{1}{\sigma-1+2it} + c_7\mathscr{L}. \tag{4.28}$$

The estimates (4.23), (4.24) and (4.28) when substituted in the main inequality (4.20) gives

$$\frac{4}{\sigma-\beta} < \frac{3}{\sigma-1} + \Re\frac{1}{\sigma-1+2it} + c_8\mathscr{L},$$

where now $t=\gamma\geq 0$, the imaginary part of some critical zero $\rho=\beta+i\gamma$ of $L(s,\chi)$. Here if we take $\sigma = 1+\frac{\delta}{\mathscr{L}}$ for some constant $\delta>0$, we obtain

$$\frac{4}{\sigma-\beta} < \frac{3\mathscr{L}}{\sigma} + \frac{\mathscr{L}\delta}{\delta^2+4t^2\mathscr{L}^2} + c_8\mathscr{L}.$$

Subject to the condition $t=\gamma\geq\frac{\delta}{\mathscr{L}}$ where now $\mathscr{L}=\log q+\log(\gamma+2)$, we obtain

$$\frac{4}{\sigma-\beta} < \frac{3\mathscr{L}}{\sigma} + \frac{\mathscr{L}}{5\delta} + c_8\mathscr{L}.$$

whence

$$\beta < 1 - \frac{4-5c_8\delta}{16+5c_8\delta}\frac{\delta}{\mathscr{L}}.$$

If we choose δ sufficiently small in relation to c_8 and note that $\gamma\geq\frac{\delta}{\log q}$ implies $\gamma\geq\frac{\delta}{\mathscr{L}}$, we have proved the following:

There exists a positive absolute constant c_9 such that if $0<\delta<c_9$ and χ is a real nonprincipal character modulo q, then any zero $\beta+i\gamma$ of $L(s,\chi)$ for which

$$|\gamma| \geq \frac{\delta}{\log q}$$

satisfies

$$\beta < 1-\frac{\delta}{5\mathscr{L}}$$

where $\mathscr{L}=\log q+\log(\gamma+2)$.

We have omitted (and also will omit) the requirement that χ should be primitive since the case when χ is not primitive can be treated similarly as in the previous theorem.

To complete the proof, it remains to consider the zeros with $0 \leq \gamma < \frac{\delta}{\log q}$.We shall show that there is at most one zero with $\beta > 1 - \frac{\delta'}{\log q}$ for a suitable constant δ' and if there is one, it must be real.

We consider $\frac{L'(\sigma,\chi)}{L(\sigma,\chi)}$. We have the bound for $1 < \sigma \leq 2$

$$-\frac{L'(\sigma,\chi)}{L(\sigma,\chi)} = \sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{n^{\sigma}} \geq -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} = \frac{\zeta'(\sigma)}{\zeta(\sigma)} > -\frac{1}{\sigma-1} - c_{10}. \quad (4.29)$$

On the other hand χ is real and the zeros of $L(s,\chi)$ occur in conjugate pairs. Hence Lemma 4.8 gives

$$-\frac{L'(\sigma,\chi)}{L(\sigma,\chi)} < -\Re \sum_{\rho} \frac{1}{\sigma-\rho} + c_{11} \log q.$$

If we fix two complex conjugate zeros ρ and $\bar{\rho}$ and omit the rest of the sum above we get

$$-\frac{L'(\sigma,\chi)}{L(\sigma,\chi)} < -\frac{2(\sigma-\beta)}{(\sigma-\beta)^2+\gamma^2} + c_{11} \log q. \quad (4.30)$$

Combining (4.29) and (4.30) we have

$$\frac{2(\sigma-\beta)}{(\sigma-\beta)^2+\gamma^2} < \frac{1}{\sigma-1} + c_{12} \log q, \quad (4.31)$$

with an appropriate choice of c_{12} for $\sigma > 1$.(This hold trivially for a suitable choice of c_{12} for $\sigma > 2$ too.)

Now we take $\sigma = 1 + \frac{2\delta}{\log q}$. Then our assumption that $|\gamma| < \frac{\delta}{\log q}$ implies

$$|\gamma| < \frac{\delta}{\log q} = \frac{1}{2}(\sigma-1) < \frac{1}{2}(\sigma-\beta).$$

This, with the inequality (4.31) implies that

$$-\frac{1}{\sigma-1} < c_{12} \log q - \frac{8}{5(\sigma-\beta)}$$

and thus solving for β above

$$\beta < 1 - \frac{6 - 20c_{12}\delta}{1 + 2c_{12}\delta} \frac{\delta}{\log q}.$$

If δ is chosen sufficiently small in relation to c_{12} we can have

$$\beta < 1 - \frac{\delta}{\log q}.$$

This argument for two complex conjugate zeros also works when there are two real zeros or a double real zero. Hence we have proved:

There exists a positive absolute constant c_{13} such that if $0 < \delta < c_{13}$ and χ is a real nonprincipal character modulo q, then the only possible zero of $L(s, \chi)$ satisfying

$$|\gamma| < \frac{\delta}{\log q}, \quad \beta > 1 - \frac{\delta}{\log q}$$

is a simple real zero.

Combining the results of two cases, we obtain the statement of the theorem.

□

For future reference we bring together the Theorem 4.9 and Theorem 4.10 and state the result as follows:

Theorem 4.11 (Zero-free Region for $L(s, \chi)$)**.** *There exists a positive absolute constant c_{15} with the following property. If χ is a complex character modulo q, then $L(s, \chi)$ has no zero in the region*

$$\sigma \geq \begin{cases} 1 - \frac{c_{15}}{\log q|t|} & \text{if } |t| \geq 1, \\ 1 - \frac{c_{15}}{\log q} & \text{if } |t| \leq 1. \end{cases} \tag{4.32}$$

If χ is a real nonprincipal character, the only possible zero of $L(s, \chi)$ in this region is a real simple zero.

4.3 On the Zeros of $L(s, \chi)$

Definition 6. *Let $c = c_{15} < \frac{1}{4}$ be the constant in Theorem 4.11 (we may take it to be $< \frac{1}{4}$). For a nonprincipal character χ modulo q we call a zero $\rho = \beta + i\gamma$ exceptional if it satisfies*

$$|\gamma| < 1 \quad \text{and} \quad \beta > 1 - \frac{c}{\log q}.$$

By Theorem 4.11 if it exists, then it is real and unique, so we denote it by β_1.

Actually, if there exist values of q for which $L(s, \chi)$ has an exceptional zero, then such values of q are very rare. This is a result due to Landau and Page has deduced that if q belongs to a positive set of integers which are bounded, then there is at most one real primitive χ modulo q for some q such that $L(s, \chi)$ has a possible exceptional zero. We will prove these results in the next lemma.

Lemma 4.12. *(i) There exists an absolute constant* $c_1 > 0$ *such that for any two distinct real primitive characters to the moduli* q_1, q_2 *respectively; if the corresponding L-functions have real zeros* β_1, β_2, *then*

$$\min(\beta_1, \beta_2) < 1 - \frac{c_1}{\log q_1 q_2}.$$

(Note that the possibility $q_1 = q_2$ *is not excluded.)*

(ii) For a suitable constant c_2, *there is at most one real primitive character modulo* $q \leq z$, $(z \geq)3$ *such that* $L(s, \chi)$ *has an exceptional zero with* $\beta > 1 - \frac{c_2}{\log q}$.

(iii) There is an absolute constant c_3 *such that if* χ *is a real nonprincipal character modulo* q *and* $L(s, \chi)$ *has an exceptional zero* β_1, *then* β_1 *satisfies*

$$\beta_1 \leq 1 - \frac{c_3}{q^{1/2} \log^2 q}.$$

Proof. See [2] §11.2. □

Let us now give some results concerning the zeros of $L(s, \chi)$ and the number of zeros of $L(s, \chi)$. We will use them when we prove the explicit formula and the PNT for arithmetic progressions.

Lemma 4.13. *If* $\rho = \beta + i\gamma$ *runs through all nontrivial zeros of* $L(s, \chi)$, *then for any real number* t

$$\sum_{\rho} \frac{1}{1 + (t - \gamma)^2} = O(\mathscr{L}),$$

where $\mathscr{L} = \log q + \log(|t| + 2)$.

Proof. We start with the inequality (4.22) in lemma 4.8 for $1 < \sigma \leq 2$

$$-\Re \frac{L'(s, \chi)}{L(s, \chi)} < -\sum_{\rho} \Re \frac{1}{s - \rho} + O(\mathscr{L}).$$

Take $s = 2 + it$ above. Since $\frac{L'(2,\chi)}{L(2,\chi)} \leq \frac{\zeta'(2)}{\zeta(2)} = O(1)$, we can conclude

$$\sum_{\rho} \Re \frac{1}{s-\rho} = O(\mathscr{L}).$$

Moreover,

$$\Re \frac{1}{s-\rho} = \frac{2-\beta}{(2-\beta)^2 + (t-\gamma)^2} \geq \frac{1}{4 + (t-\gamma)^2} \geq \frac{1}{4} \frac{1}{1 + (t-\gamma)^2}.$$

So the result follows. □

Lemma 4.14. *(i) The number of zeros in $t-1 < \gamma < t+1$ is $O(\mathscr{L})$ for all $t \in \mathbb{R}$.*
(ii) For every $t \in \mathbb{R}$,

$$\sum_{\substack{\rho \\ |t-\gamma| \geq 1}} \frac{1}{(t-\gamma)^2} = O(\mathscr{L}).$$

(iii) For all s satisfying $-1 \leq \sigma \leq 2$, $|t| \geq 2$ and $L(s,\chi) \neq 0$, we have

$$\frac{L'(s,\chi)}{L(s,\chi)} = \sum_{\substack{\rho \\ |t-\gamma| < 1}} \frac{1}{(s-\rho)} + O(\mathscr{L}).$$

Here $\mathscr{L} = \log q + \log(|t| + 2)$.

Proof. (i) and (ii) are immediate from the lemma. For (iii), we apply theorem 4.5 to s and $2 + it$ and subtract to have

$$\frac{L'(s,\chi)}{L(s,\chi)} - \frac{L'(2+it,\chi)}{L(2+it,\chi)} = -\frac{\Gamma'\left(\frac{s+a}{2}\right)}{2\Gamma\left(\frac{s+a}{2}\right)} + \frac{\Gamma'\left(\frac{2+a+it}{2}\right)}{2\Gamma\left(\frac{2+a+it}{2}\right)} + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{2+it-\rho} \right).$$

By Stirling's formula (5) in [1], §10,

$$-\frac{\Gamma'\left(\frac{s+a}{2}\right)}{2\Gamma\left(\frac{s+a}{2}\right)} + \frac{\Gamma'\left(\frac{2+a+it}{2}\right)}{2\Gamma\left(\frac{2+a+it}{2}\right)} = O\left(\frac{1}{|t|}\right).$$

Furthermore, $\frac{L'(2+it,\chi)}{L(2+it,\chi)} = O(1)$. Hence

$$\frac{L'(s,\chi)}{L(s,\chi)} = O(1) + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{2+it-\rho} \right).$$

Now for the terms in the sum above with $|\gamma - t| \geq 1$, we have

$$\left| \frac{1}{s-\rho} + \frac{1}{2+it-\rho} \right| = \frac{2-\sigma}{|(s-\rho)(2+it-\rho)|} \leq \frac{3}{|\gamma - t|^2},$$

and by (ii)

$$\sum_{\substack{\rho \\ |t-\gamma|\geq 1}} \left(\frac{1}{s-\rho} + \frac{1}{2+it-\rho} \right) = O\left(\sum_{\rho} \frac{3}{|\gamma - t|^2} \right) = O(\mathscr{L}).$$

For the terms with $|\gamma - t| < 1$ we have $|2+it-\rho| \geq 1$ and by (i), number of such terms is $O(\mathscr{L})$. Hence the result follows. □

Lemma 4.15. *Let H be the half-plane $\sigma \leq -1$, when circles of radius say 1/2 around trivial zeros are excluded. Then*

$$\left| \frac{L'(s,\chi)}{L(s,\chi)} \right| \ll \log(q|s|), \qquad (\forall s \in H)$$

where the constant is absolute.

Proof. We use the relation $\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})} = \pi^{-1/2} 2^{1-s} \cos \frac{s\pi}{2} \Gamma(s)$ from [1],§10 in the functional equations (4.2) and (4.3) and obtain the unsymmmetric form of the functional equation

$$L(1-s,\chi) = \epsilon(\chi) \pi^{-s} 2^{1-s} \cos \frac{(s-a)\pi}{2} \Gamma(s) L(s,\bar{\chi}),$$

where $|\epsilon(\chi)| = \left| \frac{i^a q^{1/2}}{\tau(\bar{\chi})} \right| = 1$. Taking the logarithmic derivative and replacing s by $1-s$ we obtain

$$-\frac{L'(s,\chi)}{L(s,\chi)} = \log q - \log 2\pi - \frac{1}{2}\pi \cot \left(\frac{\pi(s+a)}{2} \right) + \frac{\Gamma'(1-s)}{\Gamma(1-s)} + \frac{L'(1-s,\bar{\chi})}{L(1-s,\bar{\chi})}.$$

If $s \in H$, then $\Re 1 - s \geq 2$ and thus $\frac{L'(1-s,\bar{\chi})}{L(1-s,\bar{\chi})} = O(1)$. Also, $\frac{\Gamma'(1-s)}{\Gamma(1-s)} = O(\log 2|s|)$ and $\cot\left(\frac{\pi(s+a)}{2}\right) = O(1)$. These bounds substituted in the above equality give the result. □

Definition 7. *For a Dirichlet character χ modulo q and $T > 0$, we define $N(T,\chi)$ to be the number of zeros of $L(s,\chi)$ in the rectangle $0 < \sigma < 1, |t| < T$.*

We can give estimates on $N(T,\chi)$ using the argument principal method from complex analysis. Here we are goning the state the result in the next theorem. For details and the proof see [1], §16.

Theorem 4.16 (The Number $N(T,\chi)$)**.** *For any primitive Dirichlet character χ modulo $q \geq 3$ and $T \geq 2$ we have*

$$\frac{1}{2}N(T,\chi) = \frac{T}{2\pi}\log\frac{qT}{2\pi} - \frac{T}{2\pi} + O(\log T + \log q). \tag{4.33}$$

For a nonprimitive character χ modulo q induced by χ_1 modulo q_1, then the above estimate holds for $N(T,\chi)$ with q_1 in place of q.

4.4 The Explicit Formulae

As an analogue of Chebyshev's ψ function we define $\psi(x,\chi)$

$$\psi(x,\chi) = \sum_{n\leq x}\chi(n)\Lambda(n),$$

and set $\psi_0(x,\chi) = \psi(x,\chi)$ when x is not a prime power, and $\psi_0(x,\chi) = \psi(x,\chi) - \frac{\chi(x)\Lambda(x)}{2}$ when x is a prime power.

Also, as we will use in the next proofs, let us denote by $b(\chi)$ the 0th coefficient in the Laurent expansion of $\frac{L'(s,\chi)}{L(s,\chi)}$ at $s=0$, and $< x >$ denotes the distance from x to the nearest prime power.

Theorem 4.17 (The Explicit Formula for $\psi_0(x,\chi)$)**.** *For any primitive Dirichlet character χ modulo $q \geq 3$, $x \geq 2$ and $T \geq 2$ we have*

$$\psi_0(x,\chi) = -\sum_{|\gamma|<T}\frac{x^\rho}{\rho} - (1-a)\log x - b(\chi) + \sum_{m=1}^{\infty}\frac{x^{a-2m}}{2m-a} + R(x,T) \tag{4.34}$$

where $\rho = \beta + i\gamma$ runs through all nontrivial zeros of $L(s,\chi)$ with $|\gamma| < T$ and

$$R(x,T) \ll \frac{x}{T}\log^2(qxT) + \log x \min\left(1, \frac{x}{T < x >}\right) \tag{4.35}$$

where the implied constant is absolute.

In the proof of Theroem 4.15, we make use of the discontinuous integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s}ds = \begin{cases} 0 & \text{if } 0 < y < 1, \\ 1/2 & \text{if } y = 1, \\ 1 & \text{if } y > 1. \end{cases}$$

The following lemma gives a more precise statement about this integral. For the proof see the Lemma in [1], §17.

Lemma 4.18. *If*

$$I(y,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s} ds, \quad \delta(y) = \begin{cases} 0 & \text{if } 0 < y < 1, \\ 1/2 & \text{if } y = 1, \\ 1 & \text{if } y > 1, \end{cases}$$

then for $y > 0$, $c > 0$ and $T > 0$ we have

$$|I(y,T) - \delta(y)| = \begin{cases} y^c min\left(1, \frac{1}{\pi T|\log y|}\right) & \text{if } y \neq 1, \\ \frac{c}{\pi T} & \text{if } y = 1. \end{cases}$$

Now let us define

$$J(y,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \left[-\frac{L'(s,\chi)}{L(s,\chi)}\right] \frac{y^s}{s} ds.$$

Lemma 4.19. *For any $x > 2$ and $T > 0$, if we take $c = 1 + \frac{1}{\log x}$, then*

$$|J(x,T) - \psi_0(x,\chi)| \ll \frac{x \log^2 x}{T} + \log x \min\left(1, \frac{x}{T < x >}\right). \tag{4.36}$$

Proof. For $\sigma > 1$ we have

$$J(x,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{n^s}\frac{x^s}{s} ds = \sum_{n=1}^{\infty} \chi(n)\Lambda(n) \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{(x/n)^s}{s} ds,$$

where the change of order is permitted since the series $\sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{n^s}$ is uniformly convergent in the path of integration. Now applying lemma 4.17 with $y = x/n$ we obtain

$$\begin{aligned} \left|J(x,T) - \sum_{n=1}^{\infty} \chi(n)\Lambda(n)\delta\left(\frac{x}{n}\right)\right| &\leq \sum_{\substack{n=1 \\ n\neq x}}^{\infty} \Lambda(n) \left|\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{(x/n)^s}{s} ds - \delta\left(\frac{x}{n}\right)\right| \\ &< \sum_{\substack{n=1 \\ n\neq x}}^{\infty} \Lambda(n) \left(\frac{x}{n}\right)^c \min\left(1, \frac{x}{T|\log(x/n)|}\right) + \frac{c}{T}\Lambda(x). \end{aligned}$$

Since $\sum_{n=1}^{\infty} \chi(n)\Lambda(n)\delta\left(\frac{x}{n}\right) = \psi_0(x,\chi)$ the above inequality becomes

$$|J(x,T) - \psi_0(x,\chi)| < \sum_{\substack{n=1 \\ n\neq x}}^{\infty} \Lambda(n) \left(\frac{x}{n}\right)^c \min\left(1, \frac{x}{T|\log(x/n)|}\right) + \frac{c}{T}\Lambda(x). \tag{4.37}$$

We will divide the terms of the series in (4.37) into three groups and find their contributions to the sum seperately.

The terms with $n \le \frac{3}{4}x$ or $n \ge \frac{5}{4}x$. For these $|\log(x/n)|$ has positive lower bound and hence their contribution is

$$\ll \sum_{n=1}^{\infty} \Lambda(n) \left(\frac{x}{n}\right)^c = \frac{x^c}{T}\left(-\frac{\zeta'(c)}{\zeta(c)}\right) \ll \frac{x^c}{T(c-1)} = \frac{ex\log x}{T}.$$

The terms with $\frac{3}{4}x < n < x$. If we denote by x_1 the largest prime less than x, then we can take $\frac{3}{4}x < x_1 < x$ since otherwise the terms would vanish. For the term $n = x_1$,

$$\log\left(\frac{x}{n}\right) = -\log\left(1 - \frac{x - x_1}{x}\right) \ge \frac{x - x_1}{x},$$

since $log(1-y) \ge y$ for $0 \le y < 1$. Hence the term $n = x_1$ contributes

$$\ll \Lambda(x_1)\min\left(1, \frac{x}{T(x-x_1)}\right) \ll \log x \min\left(1, \frac{x}{T(x-x_1)}\right).$$

For the remaining terms of this group, we put $n = x_1 - \nu$ where $0 < \nu < \frac{1}{4}x$. Then we have

$$\log\left(\frac{x}{n}\right) > \log\left(\frac{x_1}{n}\right) = -\log\left(1 - \frac{\nu}{x_1}\right) \ge \frac{\nu}{x_1}.$$

Hence these terms contribute

$$\ll \sum_{0<\nu<\frac{1}{4}x} \Lambda(x_1 - \nu)\frac{x_1}{T\nu} \ll \frac{x_1 \log x}{T} \sum_{0<\nu<\frac{1}{4}x} \frac{1}{\nu} \ll \frac{x\log^2 x}{T}.$$

The terms with $x < n < \frac{5}{4}x$ These terms are dealt with similary but instead of x_1 we use x_2, the least prime power greater than x. The contribution of these terms is

$$\ll \log x \min\left(1, \frac{x}{T(x_2 - x)}\right) + \frac{x\log^2 x}{T}.$$

So, collecting all estimates and noting that $\frac{c}{T}\Lambda(x) \ll \frac{x\log^2 x}{T}$ the lemma follows. $\square$

Now we are in a position to prove the explicit formula (4.34).

Proof of Theorem 4.17. The first thing to be done is to replace the vertical line $[c-iT, c+iT]$ of integration in $J(x,T)$ by other three sides of the rectangle with vertices at

$$c-iT, \quad c+iT, \quad -U-iT, \quad -U+iT,$$

where U is a large integer $\equiv 1+a \pmod 2$ and $c = 1+\frac{1}{\log x}$ as in the previous lemma. So the vertical segment $[-U-iT, -U+iT]$ passes halfway between two trivial zeros of $L(s,\chi)$.

Let us compute the residues of the integrand at its poles in the rectangle. $L(s,\chi)$ has no poles so that the integrand $-\frac{L'(s,\chi)}{L(s,\chi)}\frac{x^s}{s}$ has poles at the zeros of $L(s,\chi)$ and at $s=0$. $Res\left(\frac{L'(s,\chi)}{L(s,\chi)}, s=\rho\right) = 1$ for any simple zero of $L(s,\chi)$ whether trivial or not. Hence the sum of the residues in the rectangle is

$$-\sum_{|\gamma|<T}\frac{x^\rho}{\rho} - \sum_{0<2m-a<U}\frac{x^{-(2m-a)}}{-(2m-a)} - Res\left(\frac{L'(s,\chi)}{L(s,\chi)}\frac{x^s}{s}, s=0\right).$$

The situation differs according to the value of $\chi(-1)$. If $\chi(-1) = -1$, then $L(0,\chi) \neq 0$ and $s=0$ is a simple pole. In this case, $Res\left(\frac{L'(s,\chi)}{L(s,\chi)}, s=0\right)$ $= \frac{L'(0,\chi)}{L(0,\chi)} = b(\chi)$ if we recall that $b(\chi)$ was the 0th coefficient of the Laurent expansion of $L(s,\chi)$.

However, if we assume $\chi(-1) = 1$, then it has a double pole at $s=0$. Now, since $L(s,\chi)$ has asimple pole at $s=0$, its Laurent expansion near $s=0$ is of the form

$$\frac{L'(s,\chi)}{L(s,\chi)} = \frac{1}{s} + b(\chi) + \ldots.$$

Also,

$$\frac{x^s}{s} = \frac{1}{s} + \log x + \ldots.$$

Hence the Laurent expansion of $\frac{L'(s,\chi)}{L(s,\chi)}\frac{x^s}{s}$ is of the form

$$\frac{L'(s,\chi)}{L(s,\chi)}\frac{x^s}{s} = \frac{1}{s^2} + (\log x + b(\chi))\frac{1}{s} + \ldots,$$

which gives us the residue $Res\left(\frac{L'(s,\chi)}{L(s,\chi)}\frac{x^s}{s}, s=0\right) = \log x + b(\chi)$.

Thus, assuming that there is no zero with $\gamma = T$, we obtain

$$J(x,T) = \frac{1}{2\pi i}\int_L \left(-\frac{L'(s,\chi)}{L(s,\chi)}\right)\frac{x^s}{s}ds - \sum_{|\gamma|<T}\frac{x^\rho}{\rho} - (1-a)\log x - b(\chi)$$
$$- \sum_{0<2m-a<U}\frac{x^{-(2m-a)}}{-(2m-a)}. \qquad (4.38)$$

Here the integral is over the path $L = L_1 \cup L_2 \cup L_3$ where

$$L_1 = [-U+iT, c+iT],$$
$$L_2 = [-U-iT, -U+iT],$$
$$L_3 = [c-iT, -U-iT].$$

Now we are going the estimate the integral on these segments seperately:

Estimates along the horizontal lines L_1 and L_3.
In order to have good bounds on the integrand on thes lines, we wish to stay away from the zeros of $L(s,\chi)$. This is possible with an appropriate choice of T. Let $T \geq 2$, then by Lemma 4.13 (iii) there are at most $O(\log(qT))$ zeros with $|\gamma - T| < 1$. Among the ordinates of these zeros, there must be a gap of length$\gg \frac{1}{\log(qT)}$. So if we change T by a bounded amount, we can ensure that

$$|\gamma - T| \gg \frac{1}{\log(qT)}, \qquad (4.39)$$

for all zeros $\rho = \beta + i\gamma$.

We also recall the Lemma 4.14(iii) which implies

$$\frac{L'(s,\chi)}{L(s,\chi)} = \sum_{|\gamma-T|<1}\frac{1}{s-\rho} + O(\log(qT)),$$

for all $s = \sigma + iT$ with $-1 \leq \sigma \leq 2$. With our choice of T satisfying (4.39),we have $\frac{1}{s-\rho} \leq \frac{1}{|\gamma-T|} \ll \log(qT)$ and number of terms is also $\ll \log(qT)$. Hence

$$\frac{L'(\sigma+iT,\chi)}{L(\sigma+iT,\chi)} \ll \log^2(qT), \qquad (4.40)$$

for $-1 \leq \sigma \leq 2$.

Hence the contribution from $L_1 \cap \{-1 \leq \sigma \leq 2\} = [-1+iT, c+iT]$ to

$\int_L \left(-\frac{L'(s,\chi)}{L(s,\chi)}\right) \frac{x^s}{s} ds$ is (remember $c = 1 + (\log x)^{-1}$)

$$\ll \int_{-1}^{c} \log^2(qT)\frac{x^\sigma}{T} d\sigma = \frac{x^c \log^2(qT)}{T\log x} \ll \frac{x\log^2(qT)}{T\log x}. \qquad (4.41)$$

For $L_1 \cap \{\sigma < -1\} = [-U + iT, -1 + iT)$ we use the bound on $L(s,\chi)$ in Lemma 4.15

$$\left|\frac{L'(s,\chi)}{L(s,\chi)}\frac{x^s}{s}\right| \ll \frac{\log(q|s|)x^\sigma}{|s|} \ll \frac{\log(qT)x^\sigma}{T} \qquad (4.42)$$

since $T \geq 2$. Hence the contribution from $L_1 \cap \{\sigma < -1\}$ is

$$\ll \int_{-U}^{-1} \frac{x^\sigma}{T} d\sigma = \frac{\log(qT)}{Tx\log x} - \frac{\log(qT)}{Tx^U \log x} \ll \frac{\log(qT)}{Tx\log x}. \qquad (4.43)$$

Combining (4.41) and (4.43), the total contribution along L_1 is

$$\ll \frac{x\log^2(qT)}{T\log x} + \frac{\log(qT)}{Tx\log x} \ll \frac{x\log^2(qT)}{T\log x}. \qquad (4.44)$$

Along L_3, we have $s = \sigma - iT$ where on L_1 we had $s = \sigma + iT$. Thus, the bounds in (4.40) and (4.42) also hold on L_3 is again (4.44).

Estimates along the line L_2 .

On the vertical part $L_2 = [-U - iT, -U + iT]$, we are in $\{\sigma \leq -1\}$ and the bound in (4.42) holds with a slight modification.

$$\left|\frac{L'(s,\chi)}{L(s,\chi)}\frac{x^s}{s}\right| \ll \frac{\log(qU)x^{-U}}{U}.$$

Thus the contribution from L_3 is

$$\ll \frac{\log(qU)}{Ux^U}\int_{-T}^{T} dt \ll \frac{T\log(qU)}{Ux^U}. \qquad (4.45)$$

We have completed the estimation of the integral over the horizontal and vertical sides of L. Combining the estimates (4.44) and (4.45), we have proved that

$$\int_L \left(-\frac{L'(s,\chi)}{L(s,\chi)}\right) \frac{x^s}{s} ds = O\left(\frac{x\log^2(qT)}{T\log x} + \frac{T\log(qU)}{Ux^U}\right).$$

Now we will use the above estimate in (4.38) and let $U \to \infty$ which makes $\frac{T\log(qU)}{Ux^U} \to 0$. We obtain

$$J(x,T) = -\sum_{|\gamma|<T} \frac{x^\rho}{\rho} - (1-a)\log x - b(\chi) + \sum_{m=1}^{\infty} \frac{x^{-(2m-a)}}{-(2m-a)}. \tag{4.46}$$

We finally use (4.46) in (4.36) of Lemma 4.19 and obtain the statement of the theorem.

Note that we put some restriction on T to obtain (4.39) but this can ber removed now. We have changed T by a bounded amount and that effects the sum $\sum_{|\gamma|<T} \frac{x^\rho}{\rho}$ by $O(\log(qT))$ terms. Since each term in the sum is $O(x/T)$, the total change in the sum is $O(x\log(qT)/T)$ which is covered by (4.35).

$\square$

This form (4.34) of the explicit formula is of little use with the unknown $b(\chi)$. Moreover, for the terms $\frac{x^\rho}{\rho}$, ρ may be too close to 0 or 1. In the next lemma, we will find an expression for $b(\chi)$ and then we will give the explicit formula for $\psi(x,\chi)$ in which the possible problematic zero is visible explicitely and which is also valid for nonprimitive characters.

Lemma 4.20. *The number $b(\chi)$ in (4.34) satisfies*

$$b(\chi) = O(\log q) - \sum_{\substack{\rho \\ |\gamma|<1}} \frac{1}{\rho},$$

where ρ runs through all nontrivial zeros of $L(s,\chi)$.

Proof. We start with recalling equation (4.17)

$$\frac{L'(s,\chi)}{L(s,\chi)} = -\frac{1}{2}\log\left(\frac{q}{\pi}\right) - \frac{\Gamma'\left(\frac{s+a}{2}\right)}{2\Gamma\left(\frac{s+a}{2}\right)} + B(\chi) + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

Replacing s by 2 above we obtain

$$\frac{L'(2,\chi)}{L(2,\chi)} = -\frac{1}{2}\log\left(\frac{q}{\pi}\right) - \frac{\Gamma'\left(\frac{2+a}{2}\right)}{2\Gamma\left(\frac{2+a}{2}\right)} + B(\chi) + \sum_\rho \left(\frac{1}{2-\rho} + \frac{1}{\rho}\right).$$

$\frac{L'(2,\chi)}{L(2,\chi)} = O(1)$ and $\frac{\Gamma'\left(\frac{2+a}{2}\right)}{2\Gamma\left(\frac{2+a}{2}\right)} = O(1)$, so if we subtract the two equations we get

$$\frac{L'(s,\chi)}{L(s,\chi)} = O(1) - \frac{\Gamma'\left(\frac{s+a}{2}\right)}{2\Gamma\left(\frac{s+a}{2}\right)} + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{2-\rho}\right). \tag{4.47}$$

Now, $b(\chi)$ is by definiton the 0th coefficient of the Laurent expansion of $\frac{L'(s,\chi)}{L(s,\chi)}$ at $s = 0$. So we are going to determine $b(\chi)$ from the Laurent expansion of the RHS of (4.47) at $s = 0$. If $\chi(-1) = -1$, so that $a = 1$ and $\frac{\Gamma'\left(\frac{s+a}{2}\right)}{2\Gamma\left(\frac{s+a}{2}\right)}$ is regular at $s = 0$; if $\chi(-1) = 1$, so that $a = 0$ and $\frac{\Gamma'\left(\frac{s+a}{2}\right)}{2\Gamma\left(\frac{s+a}{2}\right)}$ has the Laurent expansion

$$\frac{\Gamma'\left(\frac{s+a}{2}\right)}{2\Gamma\left(\frac{s+a}{2}\right)} = -\frac{2}{s} + constant + \ldots$$

By its definition, in the former case $b(\chi) = \frac{L'(0,\chi)}{L(0,\chi)}$; and it is the constant in the expansion of $\frac{L'(s,\chi)}{L(s,\chi)}$ in the latter case. Hence it satisfies

$$b(\chi) = O(1) - \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{2-\rho}\right). \tag{4.48}$$

We will consider the sum in (4.48). For the terms with $|\gamma| \geq 1$, we have

$$\sum_{\substack{\rho \\ |\gamma|\geq 1}} \left|\frac{1}{s-\rho} + \frac{1}{2-\rho}\right| = 2\sum_{\substack{\rho \\ |\gamma|\geq 1}} \left|\frac{1}{\rho(2-\rho)}\right| \ll \sum_{\substack{\rho \\ |\gamma|\geq 1}} \frac{1}{\gamma^2} = O(\log q),$$

where in the last step we used Lemma 4.13 with $t = 0$. For the terms with $|\gamma| < 1$,

$$\sum_{\substack{\rho \\ |\gamma|<1}} \left|\frac{1}{(2-\rho)}\right| \ll \sum_{\substack{\rho \\ |\gamma|<1}} \frac{1}{1+\gamma^2} = O(\log q),$$

again by using Lemma 4.13 with $t = 0$. Hence these two bounds when substituted in (4.48) gives the result. $\square$

Theorem 4.21 (The Explicit Formula for $\psi(x,\chi)$)**.** *Let χ be a nonprincipal character modulo q and $2 \leq T \leq x$. Then*

$$\psi(x,\chi) = -\left\{\begin{array}{ll} \frac{x^{\beta_1}}{\beta_1} & \text{if } \beta_1 \text{ exists} \\ 0 & \text{otherwise} \end{array}\right\} - \sideset{}{'}\sum_{|\gamma|<T} \frac{x^{\rho}}{\rho} + R(x,T) \tag{4.49}$$

where $\sum'$ *denotes the summation over all the zeros* $\rho = \beta + i\gamma$ *of* $L(s,\chi)$ *with* $0 < \beta < 1$*, excluding the zeros* β_1 *and* $1 - \beta_1$ *if they exists and where*

$$|R(x,T)| \ll \frac{x \log^2(qx)}{T} + x^{1/4} \log x, \tag{4.50}$$

where the implied constant is absolute.

Proof. We will first assume that χ is primitive. We may assume x to be an integer since replacing x by the closest integer effects $\psi(x,\chi)$ by an amount $O(\log x)$ and this is covered by the bound in (4.50). Furthermore, since x is an integer, $< x > \geq 1$ and the bound in (4.35) can be replaced by

$$|R(x,T)| \ll \frac{x \log^2(qxT)}{T}.$$

If we take $2 \leq T \leq x$ then Theorem 4.17 gives

$$\psi_0(x,\chi) = -\sum_{|\gamma|<T} \frac{x^\rho}{\rho} - (1-a)\log x - b(\chi) + \sum_{m=1}^{\infty} \frac{x^{a-2m}}{2m-a} + O(\frac{x \log^2(qxT)}{T}) \tag{4.51}$$

We have the expression for $b(\chi)$ from Lemma 4.20:

$$b(\chi) = O(\log q) - \sum_{\substack{\rho \\ |\gamma|<1}} \frac{1}{\rho}.$$

Also we have

$$\sum_{m=1}^{\infty} \frac{x^{a-2m}}{2m-a} \leq \sum_{k=1}^{\infty} x^{-k} = \frac{1}{x-1} \text{and} \psi(x,\chi) = \psi_0(x,\chi) + O(\log x).$$

Substituting these three in (4.51) above and noting that $T \leq x$we obtain

$$\psi(x,\chi) = -\sum_{|\gamma|<T} \frac{x^\rho}{\rho} + \sum_{\substack{\rho \\ |\gamma|<1}} \frac{1}{\rho} + O(\frac{x \log^2(qx)}{T}). \tag{4.52}$$

There are two cases. If $L(s,\chi)$ does not have any exceptional zero, then $\beta < 1 - \frac{c}{\log q}$ holds for all ρ with $|\gamma| < 1$. As can be deduced from the functional equation, zeros are placed symmetrically about the line $\sigma = \frac{1}{2}$ so that we have $\beta > \frac{c}{\log q}$. Thus, $\frac{1}{\rho} = O(\log q)$ for all ρ. Moreover, the number of zeros with $|\gamma| < 1$ is $O(\log q)$ by Theorem 4.16 with $T = 2$. All of these result in

$$\sum_{\substack{\rho \\ |\gamma|<1}} \frac{1}{\rho} = O(\log^2 q), \tag{4.53}$$

which implies that (4.49) and (4.50) hold in this case.

If $L(s,\chi)$ has an exceptional zero β_1, we denote by $\sum'$ the summation over all nontrivial zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ excluding the zeros β_1 and $1-\beta_1$. Then from (4.52) we get

$$\psi(x,\chi) = -\sum_{|\gamma|<T}{}' \frac{x^\rho}{\rho} + \sum_{\substack{\rho \\ |\gamma|<1}}{}' \frac{1}{\rho} - \frac{x^{\beta_1}-1}{\beta_1} - \frac{x^{1-\beta_1}-1}{1-\beta_1} + O(\frac{x\log^2(qx)}{T}).$$

The second sum can be absorbed in the error term as in the previous case. We keep $-\frac{x^{\beta_1}}{\beta_1}$ and omit the term $\frac{1}{\beta_1} = O(1)$. For the term $\frac{x^{1-\beta_1}-1}{1-\beta_1}$, by mean value theorem, there exists some σ between $1-\beta_1$ and 0 satisfying

$$\frac{x^{1-\beta_1}-1}{1-\beta_1} = x^\sigma \log x < x^{1/4}\log x,$$

since $\sigma < 1-\beta_1 < 1/4$. Hence (4.49) and (4.50) hold in this case too and we have proved the theorem for primitive characters.

Now we will extend the proof to the case when χ is not primitive. Let χ_1 modulo q_1 be the primitive character inducing χ. Then the theorem holds for χ_1 and

$$\psi(x,\chi_1) = -\left\{ \begin{array}{ll} \frac{x^{\beta_1}}{\beta_1} & \text{if } \beta_1 \text{ exists} \\ 0 & \text{otherwise} \end{array} \right\} - \sum_{|\gamma|<T}{}' \frac{x^\rho}{\rho} + O(\frac{x\log^2(q_1 x)}{T} + x^{1/4}\log x). \tag{4.54}$$

Above, the summation is over the nontrivial zeros of $L(s,\chi_1)$ excluding β_1 and $1-\beta_1$ if they exist. We will relate (4.54) to $\psi(x,\chi)$.

The difference between $\psi(x,\chi)$ and $\psi(x,\chi_1)$ is

$$\begin{aligned} |\psi(x,\chi) - \psi(x,\chi_1)| \le \sum_{\substack{n\le x \\ (n,q)>1}} \Lambda(n) &= \sum_{p|q} \sum_{\substack{m\ge 1 \\ p^m \le x}} \log p \\ &= O(\log x) \sum_{p|q} \log p \\ &= O(\log x \log q), \end{aligned}$$

and it is covered by the error term in (4.54).

Next, we will explore the relation between zeros of $L(s,\chi)$ and the zeros of $L(s,\chi_1)$. By the identity $L(s,\chi) = L(s,\chi_1)\prod_{p|q}\left(1 - \frac{\chi_1(p)}{p^s}\right)$, they have exactly the same zeros in $0 < \sigma < 1$. Moreover, if $L(s,\chi)$ has an exceptional zero β_1, then it is the unique exceptional zero of $L(s,\chi_1)$ too since $\beta_1 > 1 - \frac{c}{\log q} \geq 1 - \frac{c}{\log q_1}$. But not necessarily vice versa. $L(s,\chi_1)$ can have an exceptional zero where $L(s,\chi)$ does not have one. In the former case, we see that (4.49) and (4.50) holds noting that $\frac{x\log^2(q_1 x)}{T} \ll \frac{x\log^2(qx)}{T}$.

In the latter case the term $\frac{x^{1-\beta_1}}{1-\beta_1}$ appears in the sum $\sum'_{|\gamma|<T} \frac{x^\rho}{\rho}$ for $L(s,\chi)$ but not in the sum $\sum'_{|\gamma|<T} \frac{x^\rho}{\rho}$ for $L(s,\chi_1)$. Since β_1 is an exceptional zero for $L(s,\chi_1)$ whereas it is not exceptional for $L(s,\chi)$, we have

$$1 - \frac{c}{\log q_1} < \beta_1 \leq 1 - \frac{c}{\log q},$$

which implies

$$\frac{x^{1-\beta_1}}{1-\beta_1} = O(\log q) + \frac{x^{1-\beta_1} - 1}{1-\beta_1} = O(\log q + x^{1/4}\log x).$$

Hence, (4.49) and (4.50) is satisfied in this case also. □

4.5 The Proof of PNT for Arithmetic Progressions

Finally we are in a position to prove the PNT for arithmetic progressions. Let us restate the Theorem 4.1 and then prove. ,

Theorem 4.22 (Prime Number Theroem For Arithmetic Progressions)**.** *There exists an absolute constant $c > 0$ such that if $x \geq 2$ and $(a,q) = 1$, then*

$$\psi(x;q,a) = \frac{x}{\phi(q)} - \frac{\chi_1(a)x^{\beta_1}}{\phi(q)\beta_1} + O(x\exp(-c(\log x)^{1/2})). \tag{4.55}$$

where χ_1 denotes the single character modulo q,if it exists, having the exceptional zero β_1; if such χ_1 with β_1 does not exist then the term $-\frac{\chi_1(a)x^{\beta_1}}{\phi(q)\beta_1}$ should be omitted.

Proof. Recall that we hace defined the function $\psi(x;q,a)$ in the beginning of this chapter as

$$\psi(x;q,a) = \sum_{\substack{n\leq x \\ n\equiv a (\mathrm{mod}\ q)}} \Lambda(n).$$

It is immediate from (1.23) that

$$\psi(x;q,a) = \frac{1}{\phi(q)} \sum_{\chi} \bar{\chi}(a)\psi(x,\chi). \tag{4.56}$$

where the sum is over the characters modulo q.

The contribution of the principal character to the sum on the right of (4.56) provides the main term. We have

$$|\psi(x,\chi_0)-\psi(x)| = \sum_{\substack{n\le x\\(n,q)>1}} \Lambda(n) = \sum_{p|q}\sum_{\substack{m\ge 1\\p^m\le x}} \log p \ll \log x \log q.$$

By de la Vallée Poussin's form of the PNT, namely

$$\psi(x) = x + O(x\exp(-c_1(\log x)^{1/2})),$$

we obtain

$$\begin{aligned}\psi(x;q,a) &= \frac{x}{\phi(q)} + \frac{1}{\phi(q)}\sum_{\chi\ne\chi_0}\chi(a)\psi(x,\chi)\\ &\quad + O\left(\frac{1}{\phi(q)}[x\exp(-c_1(\log x)^{1/2}) + \log x\log q]\right). \qquad (4.57)\end{aligned}$$

For $\chi \ne \chi_0$ we have by Theorem 4.21

$$\psi(x,\chi) = -\left\{\begin{array}{ll}\frac{x^{\beta_1}}{\beta_1} & \text{if } \beta_1 \text{ exists}\\ 0 & \text{otherwise}\end{array}\right\} - {\sum_{|\gamma|<T}}' \frac{x^\rho}{\rho} + R_1(x,T) \qquad (4.58)$$

where $\sum'$ denotes the summation over all the zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ with $0<\beta<1$, excluding the zeros β_1 and $1-\beta_1$ if they exists and where

$$|R_1(x,T)| \ll \frac{x\log^2(qx)}{T} + x^{1/4}\log x,$$

provided $2 \le T \le x$. The term $\frac{x^{\beta_1}}{\beta_1}$ occurs for at most one real nonprincipal character, say χ_1, modulo q by Lemma 4.12. Since the sum excludes the possible exceptional zero, by Theorem 4.11, all the zeros in the sum on the right of (4.58) satisfies

$$\beta < 1 - \frac{c_2}{\log(qT)}$$

for a positive absolute constant c_2. Hence

$$|x^\rho| = x^\beta < x\exp\left(-\frac{c_2(\log x)}{\log(qT)}\right),$$

and thus

$${\sum_{|\gamma|<T}}' \frac{x^\rho}{\rho} < x\exp\left(-\frac{c_2(\log x)}{\log(qT)}\right){\sum_{|\gamma|<T}}' \frac{1}{|\rho|}. \qquad (4.59)$$

The sum $\sum'_{|\gamma|<T} \frac{1}{|\rho|}$ in the right of (4.58) with terms $|\gamma| \geq 1$ can be evaluated using the result on $N(T,\chi)$. By Theorem 4.16, $N(t,\chi) \ll t\log(qt)$ and hence

$$\begin{aligned}\sum_{1\leq|\gamma|<T}' \frac{1}{|\rho|} = \int_1^T t^{-1} dN(T,\chi) &\leq \frac{N(T,\chi)}{T} + \int_1^T t^{-2} N(T,\chi)dt \\ &\ll \log(qT) + \int_1^T t^{-1}\log(qt)dt \\ &\ll \log T \log(qT). \end{aligned} \tag{4.60}$$

For the terms with $|\gamma| < 1$ of the sum, we use the argument in (4.53) and obtain

$$\sum_{|\gamma|<1}' \frac{1}{|\rho|} \ll \log^2 q.$$

Combining (4.60) and (4.61) we finally get

$$\sum_{|\gamma|<t}' \frac{1}{|\rho|} \ll \log T \log(qT) + \log^2 q \leq \log^2(qT). \tag{4.61}$$

Now we substitute (4.59) and (4.62) in (4.58) to obtain

$$\psi(x,\chi) = -\left\{ \begin{array}{ll} \frac{x^{\beta_1}}{\beta_1} & \text{if } \beta_1 \text{ exists} \\ 0 & \text{otherwise} \end{array} \right\} + R_2(x,T), \tag{4.62}$$

where

$$|R_2(x,T)| \ll x\log^2(qT)\exp\left(-\frac{c_2\log x}{\log(qT)}\right) + \frac{x\log^2(qx)}{T} + x^{1/4}\log x.$$

Now let us assume $q \leq \exp(C(\log x)^{1/2})$ for any constant $C > 0$. We choose also $T = \exp(C(\log x)^{1/2})$ to find that the error term $R_2(x,T)$ in (4.63) is

$$\begin{aligned}|R_2(x,T)| &\ll x[2C(\log x)^{1/2}]^2 \exp\left(-\frac{c_2}{2C}\frac{\log x}{(\log x)^{1/2}}\right) \\ &\quad + x\log^2(x)\exp(-C(\log x)^{1/2}) + x^{1/4}\log x \\ &\ll x\exp(-c_3(\log x)^{1/2}), \end{aligned} \tag{4.63}$$

provided $c_3 > 0$ is a constant satisfying $c_3 < \min(c_2/2C, C)$.

Now we put (4.62) with (4.63) in the equation (4.57) to obtain the statement of the theorem with $c = \min(c_1, c_3)$ under the assumption $q \leq$

$\exp(C(\log x)^{1/2})$. (We also used that the possible χ_1 is real so that $\bar{\chi}_1 = \chi_1$.) Here, c is only depending on C.

On the other hand if we assume $q \geq \exp(C(\log x)^{1/2})$, the theorem is still valid, but it is worse than trivial. Because the largest term in $\psi(x; q, a)$ is $\leq \log x$, and the number of terms is $\leq x/q + 1$ so that

$$\psi(x; q, a) \leq \log x(x/q + 1) \ll x \exp(-C/2(\log x)^{1/2}).$$

□

Theorem 4.23. *For $q \leq /(\log x)^{1-\delta}$ for some $\delta > 0$, we have*

$$\psi(x; q, a) = \frac{x}{\phi(q)} + O(x \exp(-c(\log x)^{1/2})). \tag{4.64}$$

One of the main difficulties encountered in the theory of distribution of primes in arithmetic progressions is the possible term $-\frac{\chi_1(a)x^{\beta_1}}{\phi(q)\beta_1}$. The only universal bound we have for β_1 is the one given in Lemma 4.12, that is $\beta_1 \leq 1 - \frac{c_4}{q^{1/2}\log^2 q}$ where $c_4 > 0$ is some absolute constant. With this bound on β_1 we have

$$-\frac{\chi_1(a)x^{\beta_1}}{\phi(q)\beta_1} \ll \frac{x}{\phi(q)} \exp\left(-c_4 \frac{\log x}{q^{1/2}\log^2 q}\right). \tag{4.65}$$

This will be of the same order with the error term in (4.55) if we impose a very severe restriction on q such as

$$\leq /(\log x)^{1-\delta} \text{for some} \delta > 0.$$

This limitation on q results in the following theorem:

Theorem 4.24. *If $q \leq /(\log x)^{1-\delta}$ for some $1 > \delta > 0$ and $(q, a) = 1$, we have*

$$\psi(x; q, a) = \frac{x}{\phi(q)} + O(x \exp(-c(\log x)^{1/2})).$$

where $c > 0$ and the constant implied by the O term are absolute.

These results, though seem weak, are the only results which are effective in the sense that the constants c, c_i and the ones implied by the symbol O are computable. There are also results which are stronger, but ineffective. The best result is due to Walfisz who applied Siegel's theorem to primes in arithmetic progressions and obtained [2], Corollary 11.19,

Theorem 4.25 (The Siegel-Walfisz Theorem)**.** *Let N be any positive constant. If $q \leq (\log x)^N$ and $(q, a) = 1$, then*

$$\psi(x; q, a) = \frac{x}{\phi(q)} + O_N(x \exp(-c(\log x)^{1/2})).$$

The various results for $\psi(x;q,a)$ found through this section have analogues for the prime counting function $\pi(x;q,a)$, the number of primes up to that are congruent to $a \pmod q$. The transition is made by partial summation as in the transition in the PNT. We end this section by stating the result we have for $\pi(x;q,a)$:

Theorem 4.26. *There exists an absolute constant $c > 0$ such that if $x \geq 2$ and $(a,q) = 1$, then*

$$\pi(x;q,a) = \frac{li(x)}{\phi(q)} - \frac{\chi_1(a)li(x^{\beta_1})}{\phi(q)\beta_1} + O(x\exp(-c(\log x)^{1/2})). \tag{4.66}$$

where χ_1 denotes the single character modulo q,if it exists, having the exceptional zero β_1; if such χ_1 with β_1 does not exist then the term $-\frac{\chi_1(a)li(x^{\beta_1})}{\phi(q)\beta_1}$ should be omitted.

The Riemann Hypothesis, as in the situation with the PNT, gives the best possible result on the PNT on arithmetic progressions.It can be extended to Dirichlet L-functions named the Generalized Riemann Hypothesis which states that all non-trivial zeros of $L(s,\chi)$ have real part 1/2. Assuming the Generalized Riemann Hypothesis, we have for $q \leq x$,

$$\psi(x;q,a) = \frac{x}{\phi(q)} + O_\epsilon(x^{1/2+\epsilon}), \tag{4.67}$$

for any $\epsilon > 0$.

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