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Blow-up theorems for nonlinear evolutionary PDE's

by

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Declaration of Authorship

I, Bilgesu Arif Bilgin, declare that this thesis titled, 'Blow-up theorems for nonlinear evolutionary equations' and the work presented in it are my own. I confirm that:

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"If the facts don't fit the theory, change the facts!"

Albert Einstein

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Abstract

College of Sciences
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Master Thesis

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In this thesis we will study problems of blow-up of solutions of the Cauchy problem and initial-boundary value problems for nonlinear parabolic, nonlinear hyperbolic, nonlinear Schrödinger equations and systems of thermoelasticity type. First we study results on blow-up of solutions for nonlinear heat equation, nonlinear wave equation and nonlinear Schrödinger equation obtained in last years by using the energy methods. Then, we investigate the problem of blow-up and global non-existence of the solution to the initial boundary value problem for a nonlinear thermoelastic system.

Özet

Bu tezde dogrusal olmayan parabolik, hiperbolik, Schrdinger denklemleri ile termoelastik denklem sistemleri icin Cauchy problemi ve baslangc-snr degeri problemlerinin cozumlerinin sonlu zamanda patlamas problemini inceliyoruz.

Acknowledgements

I would like to thank my thesis advisor, Varga Kalantarov, for his guidance on the subject. I also thank my family for believing in me and supporting me with their love during my wirting of this thesis…

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I dedicate this thesis to my dear love, Gizem Gülen...

Chapter 1

Introduction

The simple example of an initial value problem for a nonlinear ordinary differential equation

$$y'(t) = y^2(t), \qquad t > 0,$$

$$y(0) = 1,$$

has the solution

$$y(t) = \frac{1}{1-t}.$$

This example displays an important feature that is common to a large class of nonlinear equations, that is the solution becomes unbounded in a finite time (in this example as $t \to 1^-$), or in other words, the solution blows up in a finite time.

This thesis is devoted to the problem of blow-up of solutions to the Cauchy problem and the initial boundary value problems for nonlinear evolutionary partial differential equations. The question of blow-up of solutions to nonlinear PDE's was under the concern of those who studied the problem of global existence of solutions to nonlinear PDE's. On the other side the blow-up problems appear in the study of a number of physical processes such as wave collapse in nonlinear optics, brake down of waves in nonlinear wave mechanics, and the blow-up of solutions of equations modelling kinetics of chemical reactors. In the last 40 years, there has been an essential amount of activity dealing with the question of blow-up in a finite time. This activity was inspired mainly by the papers of H. Levine [9], [10]. In these papers H. Levine gave a simple but elegant method of finding sufficient conditions for the blow-up of solutions to the Cauchy problem for nonlinear differential-operator equations of the form

$$Pu_t + Au = F(u),$$

and

$$Pu_{tt} + Au = F(u),$$

where P and A are symmetric, positively defined operators.

In this thesis we demonstrate the techniques used to show blowing up of solutions to the Cauchy problem and to the initial boundary value problems for the nonlinear heat equation, wave equation, Schrödinger equation and thermoelastic system.

In this chapter we briefly give the required background information. This includes facts from functional analysis, some well-known inequalities, and auxiliary lemmas which will be our main tools in proving blow-up theorems.

In the second chapter we consider several problems of parabolic type. In the first part we give a result due to M. Jazar and R. Kiwan [6], which makes use of the method of H. Levine. In Section 2.2 we consider the backwards heat equation, which is obtained by reversing the time axis in the heat equation, where the blow-up follows from simple energetic methods. The third part is a demonstration of the method of eigenfunctions, and the fourth part makes use of the Green's function method, which was first introduced by H. Fujita [8]. In Section 2.5 we give an example of blowing up of the derivative for a nonlinear heat equation, where only simple energetic methods are used. In the last two sections of this chapter we illustrate the comperison technique on a nonlinear heat equation with Neumann boundary conditions and with Dirichlet boundary conditions [7], respectively.

Chapter 3 is devoted to the nonlinear wave equations. First, we employ the method of eigenfunctions for an initial boundary value problem. Next, in the second part we use direct energetic methods to obtain a blow-up result. As the tihrd example of this chapter we give a simple application of Levine's lemma to a wave equation [16]. Finally, in the fourth section we consider an initial boundary value problem for a Boussinesq type equation [17].

In Chapter 4 we consider the Cauchy problems for the nonlinear, undamped and damped Schrödinger equations. The global nonexistence results for these problems are due to R. T. Glassey [11], and M. Tsutsumi [12], respectively.

In the final chapter we present the nonexistence results for the one dimensional and multidimensional thermoelastic systems. The work of M. Kirane [13] derives the blowing up of the solution to the Cauchy problem for a one dimensional, nonlinear thermoelastic system by using the generalization of H. Levine's idea, which is due to [4]. The last result presented in the thesis is on the blow-up of the solution to the initial boundary value problem for a multidimensional thermoelastic system. This result is the most complicated out of all that are considered in this thesis, and it is due to S. A. Messaoudi [14].

As a final remark, note that to talk about the blow-up of a solution to a problem one first needs to know that a solution exists for small time. The scope of this thesis is restricted to nonexistence of solutions for large time only. Therefore, local existence results for the problems considered are omitted, and it is always assumed that the data and the domain are smooth enough to guarantee the existence of local solutions.

1.1 Preliminaries

In order to follow the discussions carried out in this thesis one needs to be familiar with some basic concepts in functional analysis. These include the theory of function spaces, and a number of inequalities. In addition to these, we also give the definitions of some widely used terminologies in the PDE theory. First, for the sake of completeness, we define the types of vector spaces that are important in analysis.

Definition 1.1. *A Banach space* is a normed vector space that is complete with respect to its norm.

Definition 1.2. *A Hilbert space* is a vector space that is endowed with an inner product and that is complete with respect to the norm induced by the inner product.

1.1.1 Function spaces

Function spaces are vector spaces which have functions as their elements. Throughout our discussion we will restrict the solutions of the PDE's investigated to belong to some specified function space, and then use some properties of the space in the calculations. Therefore, we present here a list of the function spaces that we will use later.

1. *The Schwarz Space*

   The Schwarz space, $\mathcal{S}$, is the set of all infinitely differentiable funtions defined on $\mathbb{R}^n$ such that for all multiindices α and β,

$$\sup_{x\in\mathbb{R}^n} \left| x^\alpha \left(D^\beta f\right)(x)\right| < +\infty \,.$$

2. L^p *Spaces* $(1 \le p \le +\infty)$

   Let $\Omega \subset \mathbb{R}^n$ be a domain and f be an arbitrary function defined on Ω. If $1 \le p \le +\infty$, we say $f \in L^p(\Omega)$ provided that $\|f\|_{L^p(\Omega)}$ is finite, where for $1 \le p < +\infty$

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f^p(x)|dx\right)^{\frac{1}{p}} ,$$

and for $p = \infty$

$$\|f\|_{L^\infty(\Omega)} := \text{ess sup}_{x\in\Omega}|f(x)|.$$

All L^p spaces just defined are Banach spaces, but only L^2 is a Hilbert space. If $f, g \in L^2(\Omega)$, then their inner product is defined as

$$\langle f, g\rangle = \int_\Omega f(x)g(x)dx.$$

Note that this inner product induces the same norm as that we have defined previously.

3. *Sobolev Spaces*

In order to define the Sobolev spaces we first need the concept of weak derivatives. Let $\Omega \subset \mathbb{R}^n$ and $C_0^\infty(\Omega)$ denote the set of all infinitely differentiable functions with compact support. Suppose $u \in C^1(\Omega)$. Then for any test function $\phi \in C_0^\infty(\Omega)$ the integration by parts formula gives

$$\int_\Omega u\phi_{x_i}dx = -\int_\Omega u_{x_i}\phi dx.$$

Here the boundary term vanishes since ϕ has compact support and therefore vanishes near $\partial\Omega$. More generally, for any positive integer k, if $u \in C^k(\Omega)$ and $\alpha = (\alpha_1, ..., \alpha_n)$ is a multiindex of order $|\alpha| = \alpha_1 + \cdots + \alpha_n = k$, then

$$\int_\Omega uD^\alpha\phi dx = (-1)^{|\alpha|}\int_\Omega D^\alpha u\phi dx,$$

where

$$D^\alpha\phi = \frac{\partial^{|\alpha|}\phi}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}.$$

This equality follows from the divergence theorem; it is just the application of the first one $|\alpha| = k$ times. This observation provides the motivation for the definition of weak derivatives. If $u, v \in L^1_{loc}(\Omega)$, we say that v is the α^{th}-weak partial derivative of u, written $D^\alpha u = v$, provided that

$$\int_\Omega uD^\alpha\phi dx = (-1)^{|\alpha|}\int_\Omega v\phi dx$$

holds for all test funcitons $\phi \in C_c^\infty(\Omega)$. The weak derivative is unique in the sense of almost everywhere equivalence and coincides with the normal derivative for differentiable functions. Now we give the definition of the Sobolev spaces. Let k be a positive integer and $1 \le p \le \infty$. Then the Sobolev space, $W^{k,p}(\Omega)$, is the set of all locally summable functions, for which the α^{th}-weak partial derivative exists and belongs to $L^p(\Omega)$ for all multiindices α of order $\alpha \le k$. If $u \in W^{k,p}(\Omega)$, we define its norm as

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^p dx\right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty \\ \sum_{|\alpha|\le k} \text{ess sup}_{x\in\Omega} |D^\alpha u| & \text{if } p = \infty. \end{cases}$$

With respect to this norm all Sobolev spaces are Banach spaces, and for $p = 2$ they are Hilbert spaces.

Definition 1.3. We denote by

$$W_0^{k,p}(\Omega)$$

the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

For the detailed discussion we refer the reader to [1] or [2].

1.1.2 Some useful inequalities

Inequalities are the fundamental tools used in the analysis of differential equations. Here we present some that will be useful in the main discussion.

1. *Schwarz inequality*

   Let H be a Hilbert space. Then, for any $x, y \in H$ we have

$$|\langle x, y\rangle| \le \|x\|\|y\|. \tag{1.1}$$

2. *Hölder's inequality*

   Let $\Omega \subset \mathbb{R}^n$ be a domain in $\mathbb{R}^n$, and suppose that $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ and $+\infty > p, q \ge 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, $fg \in L^1(\Omega)$, and

$$\int_\Omega |fg|dx \le \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}} \left(\int_\Omega |g|^q dx\right)^{\frac{1}{q}}. \tag{1.2}$$

3. *Jensen inequality*

   Assume that f is a differentiable, convex function defined on $\mathbb{R}$ and Ψ is a continuous, nonnegative, nonzero function defined on the domain $\Omega \subset \mathbb{R}^n$. Then for any contiuous function u defined on Ω we have

$$\frac{\int_\Omega f(u(x))\Psi(x)dx}{\int_\Omega \Psi(x)dx} \ge f\left(\frac{\int_\Omega u(x)\Psi(x)dx}{\int_\Omega \Psi(x)dx}\right). \tag{1.3}$$

   Proof. Let $K := \frac{\int_\Omega u(x)\Psi(x)dx}{\int_\Omega \Psi(x)dx}$. By convexity of f for any $x \in [a, b]$ we have

$$f(u(x)) - f(K) \ge f'(K)(u(x) - K).$$

Multiplying the last inequality by $\Psi(x)$ and integrating over Ω we obtain

$$\int_\Omega f(u(x))\Psi(x)dx - f(K)\int_\Omega \Psi(x)dx \geq$$
$$f'(K)\left[\int_\Omega u(x)\Psi(x)dx - K\int_\Omega \Psi(x)dx\right] = 0,$$

where the last equality is obvious from the definition of K. Since Ψ is continuous, nonnegative and nonzero we have $\int_\Omega \Psi(x)dx > 0$. Therefore dividing the above inequality by $\int_\Omega \Psi(x)dx$ we remain with

$$\frac{\int_\Omega f(u(x))\Psi(x)dx}{\int_\Omega \Psi(x)dx} \geq f(K) = f\left(\frac{\int_\Omega u(x)\Psi(x)dx}{\int_\Omega \Psi(x)dx}\right).$$

□

4. *Young's inequality*

Suppose that $a, b > 0$ and $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.4}$$

5. *L^p embedding theorem*

Suppose that Ω is a bounded domain in $\mathbb{R}^n$,and $1 \leq p \leq q$. If $u \in L^q$, then $u \in L^p$, and there exists a constant C (depending only on Ω, p, q and n) such that

$$\|u\|_{L^p} \leq C\|u\|_{L^q}. \tag{1.5}$$

6. *Poincaré inequality*

Let Ω be a bounded domain in $\mathbb{R}^n$, and assume that $p \geq 1$ and $u \in W_0^{1,p}$. Then, there exists C depending only on Ω and p such that

$$\int_\Omega |u|^p dx \leq C\int_\Omega |\nabla u|^p dx. \tag{1.6}$$

7. *Gagliardo-Nirenberg inequality*

If $u \in H_0^1$, $\Omega \subset \mathbb{R}^n$, the following inequality holds ture:

$$\|u\|_{L^q(\Omega)} \leq \beta\|u\|_{L^2(\Omega)}^{1-\alpha}\|\nabla u\|_{L^2(\Omega)}^{\alpha}, \tag{1.7}$$

where

$$\frac{1}{q} = \alpha\left(\frac{1}{2} - \frac{1}{n}\right) + \frac{1-\alpha}{n},$$

and

$$q \leq \frac{2n}{n-2} \text{ if } n \geq 3,\ q \geq 1 \text{ arbitrary for } n = 1, 2.$$

8. *Sobolev embedding theorem*

   Suppose that $1 \le p < n$, $p^* = \frac{np}{n-p}$, and $u \in W^{1,p}(\mathbb{R}^n)$. Then, $u \in L^{p^*}(\mathbb{R}^n)$, and there exists $C \ge 0$ such that

$$\|u\|_{L^{p^*}} \le C\|\nabla u\|_{L^p}. \tag{1.8}$$

1.2 Auxiliary Material

In this section we will outline the methods used to obtain blow-up results for nonlinear, evolutionary PDE's. The methods used to prove blow-up theorems are based on comparison theorems, Green's function method, eigenfunction method, and the concavity arguments.

1.2.1 Comparison theorems

One way of showing the blow-up of a function is to estimate it from below by a blowing up function. The comparison theorems employ this idea, and in showing that the blowing up function is a lower bound they benefit from the maximum principle. For parabolic equations we have the following maximum principle, which we present without a proof (for the proof we refer the reader to [15]).

Theorem 1.4 (Strong maximum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and let $\Omega_T := \Omega \times (0,T)$. Suppose $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$ satisfies the inequality*

$$u_t + Lu \le 0$$

in Ω_T, where

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x,t)u_{x_i} + c(x,t)u,$$

with

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \ge \lambda|\xi|^2, \;\; \forall \xi \in \mathbb{R}^{\ltimes}, \; \lambda > 0.$$

Define

$$M := \sup_{(x,t)\in\overline{\Omega}_T} u(x,t).$$

Then, for each of the cases

$$c(x,t) \equiv 0, \;\; \text{and } M \text{ is arbitrary},$$

$$c(x,t) \ge 0, \;\; \text{and } M \ge 0,$$

and

$$c(x,t) \text{ is arbitrary, and } M = 0,$$

the following principle holds:

1. *If $u(x_0,t_0) = M$ for some $(x_0,t_0) \in \Omega_T$, then $u(x,t) \equiv M$ for all $(x,t) \in \overline{\Omega_{t_0}}$.*

2. *If $u(x_0,t_0) = M$ for some $x_0 \in \partial\Omega$, but u is not constant in $\overline{\Omega_{t_0}}$, then*

$$\frac{\partial u}{\partial \nu}(x_0,t_0) > 0.$$

Using this principle one can derive the following comparison theorem.

Theorem 1.5 (A comparison theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and let $\Omega_T := \Omega \times (0,T)$. Suppose $u, v \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$ satisfy*

$$v_t - \Delta v - f(v) \leq u_t - \Delta u - f(u), \; (x,t) \in \Omega_T,$$

$$v(x,0) < u(x,0), \; x \in \Omega,$$
$$\frac{\partial v}{\partial \nu}(x,t) \leq \frac{\partial u}{\partial \nu}(x,t), \; x \in \partial\Omega, \; t \in (0,t),$$

where $f \in C^1(\mathbb{R})$. Then,

$$v(x,t) < u(x,t), \text{ for all } (x,t) \in \overline{\Omega}_T.$$

Proof. Let $w = v - u$. First, since f is differentiable, by the mean value theorem for derivatives we have

$$f(v) - f(u) = f'(d)(v - u)$$

for some $d(x,t)$ between v and u. Then, it is easy to see that w satisfies

$$w_t - \Delta w - f'(d)w \leq 0, \; (x,t) \in \Omega_T,$$

$$w(x,0) < 0, \; x \in \Omega,$$
$$\frac{\partial w}{\partial \nu}(x,t) \leq 0, \; x \in \partial\Omega, \; t \in (0,t).$$

Suppose for contradiction that there exists $(x_0,t_0) \in \overline{\Omega}_T$ such that $w(x_0,t_0) \geq 0$. Since $w(x,0) < 0$ for $x \in \Omega$, and w is continuous, there exists a smallest time $t_0 \geq 0$ ($t_0 > 0$ if $x_0 \in \Omega$) such that $w(x_0,t_0) = 0$. Then,

$$M = \sup_{(x,t) \in \overline{\Omega}_{t_0}} w(x,t) = 0.$$

Hence, the strong maximum principle holds for w with $c(x,t) = -f'(d(x,t))$ and $M = 0$. Note that, beacuse of the initial condition for w and our assumption that $w(x_0,t_0) = 0$, w can not be constant in $\overline{\Omega}_{t_0}$. But, this together with the first part of the maximum principle implies $x_0 \notin \Omega$. Also, if $x_0 \in \partial\Omega$, then the boundary condition for w contradicts with the second part of the maximum principle. Hence, we conclude that there exists no $(x_0,t_0) \in \overline{\Omega}_T$ such that $w(x_0,t_0) \geq 0$, that is

$$v(x,t) < u(x,t), \text{ for all } (x,t) \in \overline{\Omega}_T.$$

□

1.2.2 Green's function method

In the Green's function method one choses as the function with which the PDE is multiplied as the Green function of corresponding linear evolutionary problem. Here one exploits the fact that the Green function is positive. This technique is used in section 2.3.

1.2.3 Eigenfunction method

In the eigenfunction method one choses as the function with which the PDE is multiplied as the first eigenfunction of the stationary linear part of the PDE. Again, as in the Green's function method, one benefits from the fact that the first eigenfunction of the corresponding stationary linear problem is positive. The method is illustrated in sections 2.2 and 3.1 for problems of parabolic type, and hyperbolic type, respectively. Using this method one arrives at an ordinary differential equation with convex nonliearity term. We will use the followinf two lemmas to prove blow-up of solutions to nonlinear, parabolic and hyperbolic type equations employing the eigenfunction method.

Lemma 1.6. *Let $\Phi(t)$ be a differentiable function which satisfies*

$$\Phi'(t) \geq h(\Phi(t)) \qquad \text{for all } t \geq 0$$

with $\Phi(0) = \alpha \in \mathbb{R}$, and $h(s) > 0$ for all $s \geq \alpha$. Then

a) $\Phi'(t) > 0$ whenever $\Phi(t)$ exists, and

b) the inequality

$$t \leq \int_{\alpha}^{\Phi(t)} \frac{1}{h(s)} ds$$

holds.

Proof. First note that, $\Phi'(0) \geq h(\Phi(0)) = h(\alpha) > 0$ by the assumption on h. Now, suppose a) is false. Let $t = t_1$ be the first point such that $\Phi'(t_1) = 0$. Since $\Phi'(t) \geq 0$ for all $t \in [0, t_1]$, $\Phi(t_1) \geq \Phi(0) = \alpha$. Hence by the differential inequality and the definition of h, $\Phi'(t_1) \geq h(\Phi(t_1)) > 0$, which is a contradiction. This proves a). Next, a) implies that $\Phi(t) \geq \alpha$ for all $t \geq 0$, and therefore $h(\Phi(t)) > 0$ for all $t \geq 0$. Thus, the differential ineqaulity is seperable for all time, and b) follows by rearrangement, and then integration. □

Lemma 1.7. *Let $\Phi(t)$ be a twice differentiable funciton which satisfies*

$$\Phi''(t) \geq h(\Phi(t)) \qquad \textit{for all } t \geq 0$$

with $\Phi(0) = \alpha \geq 0$, $\Phi'(0) = \beta > 0$. Also suppose $h(s) \geq 0$ for all $s > \alpha$. Then

a) $\Phi'(t) > 0$ whenever $\Phi(t)$ exists, and

b) the inequality

$$t \leq \int_\alpha^{\Phi(t)} \left[\beta^2 + 2\int_\alpha^s h(\xi)d\xi\right]^{-\frac{1}{2}} ds$$

holds.

Proof. Suppose a) is false. Then, let $t = t_1$ be the first point such that $\Phi'(t_1) = 0$. Integrating the differential inequality we obtain

$$\Phi'(t) \geq \Phi'(0) + \int_0^t h(\Phi(s))ds.$$

Thus, at the point $t = t_1$ we have

$$0 = \Phi'(t_1) \geq \Phi'(0) + \int_0^{t_1} h(\Phi(s))ds = \beta + \int_0^{t_1} h(\Phi(s))ds.$$

Since t_1 is the first point where $\Phi'(t) = \beta = 0$, and $\beta > 0$, we have $\Phi'(t) > 0$ for all $t \in [0, t_1)$. Hence $\Phi(t) > \Phi(0) = \alpha$ for all $t \in (0, t_1)$. Therefore, by the assumption on h the integral term on the right hand side of the above inequality is nonnegative. Since $\beta > 0$, we get a contradiction, and this proves a). To prove b), we use a) and multiply the differential inequality by $\Phi'(t)$ to obtain

$$\Phi'(t)\Phi''(t) \geq \Phi'(t)h(\Phi(t)),$$

or

$$\frac{d}{dt}\left[\frac{1}{2}\left(\Phi'(t)\right)^2 - \int_\alpha^{\Phi(t)} h(\xi)d\xi\right] \geq 0.$$

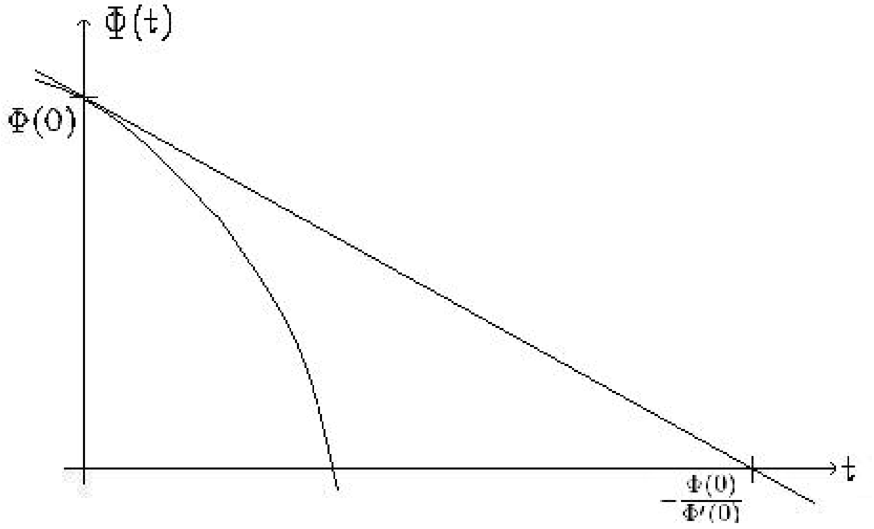

FIGURE 1.1: $\Phi(t)$ crosses zero.

Thus, integrating over $(0, t)$ and remembering that $\Phi(0) = \alpha$, we get

$$\frac{1}{2}\left(\Phi'(t)\right)^2 - \int_{\alpha}^{\Phi(t)} h(\xi)d\xi \geq \frac{1}{2}\left(\Phi'(0)\right)^2 - \int_{\alpha}^{\Phi(0)} h(\xi)d\xi = \frac{\beta^2}{2}.$$

Rearranging the above inequality and then taking the square root, since $\Phi'(t) > 0$, we arrive at

$$\Phi'(t) \geq \left[\beta^2 + 2\int_{\alpha}^{\Phi(t)} h(\xi)d\xi\right]^{\frac{1}{2}}.$$

This equation is seperable, and b) follows directly. □

1.2.4 Concavity methods

Suppose $\Psi(t)$ is a twice differentiable, positive function of time and we want to show that it blows up in finite time. If we define a new function $\Phi(t) := \Psi^{-\alpha}(t)$ with $\alpha > 0$, then it suffices to show that $\Phi(t)$ becomes zero in finite time. This is done by showing that $\Phi(t)$ satisfies a certain differential inequality with certain initial conditions. In both of the concavity methods described below this is the main idea.

Concavity method:

The first idea is due to Levine [9], [10] and it is illustrated in Fig 1.1. What we require is that $\Phi(0) > 0$, $\Phi'(0) < 0$ and $\Phi''(t) \leq 0$ when $t > 0$. If $\Phi(t)$ satisfies these conditions, then its curve will lie under the line with slope equal to $\Phi'(0)$ and hence $\Phi(t)$ will be zero at a time $t \leq -\frac{\Phi(0)}{\Phi'(0)}$. Now, since $\Phi(t) := \Psi^{-\alpha}(t)$, we have

$$\Phi'(t) = -\alpha\Psi^{-(1+\alpha)}(t)\Psi'(t),$$

and

$$\Phi''(t) = \alpha(1+\alpha)\Psi^{-(2+\alpha)}(t)\left(\Psi'(t)\right)^2 - \alpha\Psi^{-(1+\alpha)}(t)\Psi''(t)$$
$$= \alpha\Psi^{-(2+\alpha)}(t)\left[(1+\alpha)\left(\Psi'(t)\right)^2 - \Psi(t)\Psi''(t)\right].$$

Therefore, the conditions stated above in terms of $\Phi(t)$ can be restated in terms of $\Psi(t)$ as $\Psi(0) > 0$, $\Psi'(0) > 0$, and $\Psi(t)\Psi''(t) - (1+\alpha)\left(\Psi'(t)\right)^2 \geq 0$. Also, the blow-up occurs in a time $t \leq \frac{\Psi(0)}{\alpha\Psi'(0)}$. Thus, we have the following lemma.

Lemma 1.8. *Let $\Psi(t)$ be a twice differentiable, positive function, which satisfies, for $t > 0$, the inequality*

$$\Psi(t)\Psi''(t) - (1+\alpha)\left(\Psi'(t)\right)^2 \geq 0$$

with some $\alpha > 0$. If $\Psi(0) > 0$ and $\Psi'(0) > 0$, then there exists a time $t_1 \leq \frac{\Psi(0)}{\alpha\Psi'(0)}$ such that $\Psi(t) \to +\infty$ as $t \to t_1$.

Generelized concavity method:

The second concavity method is a generalization of the first one. This is achieved by generalizing the differential inqeuality that $\Phi(t)$ satisfies, that is one works with a differential inequality of the form

$$\Phi''(t) + C_1\Phi'(t) + C_2\Phi(t) \leq 0,$$

and investigates under which initial conditions this inequality ensures that $\Phi(t)$ becomes zero in finite time. Then, by a similar passage, as done above, one finds the equivalent differential inequality and initial conditions on $\Psi(t)$. The resulting conclusion was first discovered by V. K. Kalantarov and O. A. Ladyzhenskaya [4] and we present it with the following lemma.

Lemma 1.9. *Assume that a twice differentiable, positive function $\Psi(t)$ satisfies for all $t \geq 0$ the inequality*

$$\Psi(t)\Psi''(t) - (1+\gamma)(\Psi')^2(t) \geq -2C_1\Psi(t)\Psi'(t) - C_2\Psi^2(t) \tag{1.9}$$

where $\gamma > 0$ and $C_1, C_2 \geq 0$. Then,

1. *if $\Psi(0) > 0$, $\Psi'(0) + \gamma_2\gamma^{-1}\Psi(0) > 0$, and $C_1 + C_2 > 0$, we have $\Psi(t) \to +\infty$ as*

$$t \to t_1 \geq t_2 = \frac{1}{2\sqrt{C_1^2 + \gamma C_2}} \ln\left(\frac{\gamma_1\Psi(0) + \gamma\Psi'(0)}{\gamma_2\Psi(0) + \gamma\Psi'(0)}\right), \tag{1.10}$$

[6pt] where $\gamma_1 = -C_1 + \sqrt{C_1^2 + \gamma C_2}$, $\gamma_2 = -C_1 - \sqrt{C_1^2 + \gamma C_2}$,

2. *if* $\Psi(0) > 0,\ \Psi'(0) > 0,$ *and* $C_1 = C_2 = 0$,*then* $\Psi(t) \to +\infty$ *as*

$$t \to t_1 \geq t_2 = \frac{\Psi(0)}{\gamma\Psi'(0)}. \tag{1.11}$$

Proof. Set $\Phi(t) := \Psi^{-\gamma}(t)$. Then we have

$$\Phi'(t) = -\gamma\frac{\Psi(t)}{\Psi'(t)}$$

and

$$\Phi''(t) = -\frac{\gamma}{\Psi^{2+\gamma}(t)}\left[\Psi(t)\Psi''(t) - (1+\gamma)(\Psi')^2(t)\right].$$

Using the inequality (1.9) we obtain the differential inequality

$$\Phi''(t) \geq -\frac{\gamma}{\Psi^{2+\gamma}(t)}\left[-2C_1\Psi(t)\Psi'(t) - C_2\Psi^2(t)\right]$$

$$= -2C_1\Phi'(t) + \gamma C_2\Phi(t).$$

Hence we have for all $t \geq 0$

$$\Phi''(t) + 2C_1\Phi'(t) - \gamma C_2\Phi(t) = f(t) \geq 0. \tag{1.12}$$

[6pt] For the first case, where $C_1 + C_2 > 0$, the solution of (1.12) is

$$\begin{aligned}\Phi(t) &= \beta_1 e^{\gamma_1 t} + \beta_2 e^{\gamma_2 t} + (\gamma_1+\gamma_2)^{-1}\int_0^t f(\tau)\left[e^{\gamma_1(t-\tau)} - e^{\gamma_2(t-\tau)}\right]d\tau \\ &\geq \beta_1 e^{\gamma_1 t} + \beta_2 e^{\gamma_2 t} \qquad \forall t \geq 0,\end{aligned} \tag{1.13}$$

where β_1 and β_2 are defined by

$$\begin{cases}\beta_1 + \beta_2 = \Phi(0) \\ \beta_1\gamma_1 + \beta_2\gamma_2 = \Phi'(0).\end{cases}$$

From this we obtain

$$\beta_1 = -(\gamma_1 - \gamma_2)^{-1}\left[\gamma_2\Psi(0) + \gamma\Psi'(0)\right]\Psi^{-1-\gamma}$$

$$\beta_2 = (\gamma_1 - \gamma_2)^{-1}\left[\gamma_1\Psi(0) + \gamma\Psi'(0)\right]\Psi^{-1-\gamma}$$

[6pt] Note that $\gamma_1 > 0$ and $\gamma_2 < 0$,in particular $\gamma_1 > \gamma_2$ In view of this the assumption, $\Psi'(0) + \gamma_2\gamma^{-1}\Psi(0) > 0$, implies that $\beta_1 < 0$ and $\beta_2 > 0$. Combining these with (1.13) we conclude that $\Phi(t)$ vanishes at the time

$$t_2 = \frac{1}{2\sqrt{C_1^2 + \gamma C_2}}\ln\left(\frac{\gamma_1\Psi(0) + \gamma\Psi'(0)}{\gamma_2\Psi(0) + \gamma\Psi'(0)}\right)$$

[6pt] and becomes negative afterwards. This implies that $\Phi(t) \to +\infty$ as $t \to t_1 \leq t_2$.

For the second case, $C_1 = C_2 = 0$ and equation (1.12) imply

$$\Phi''(t) \leq 0.$$

Integrating both sides over $(0,t)$ twice we have

$$\Phi(t) \leq \Phi(0) + \Phi'(0)t$$

and by using the definition of $\Phi(t)$ we obtain

$$\Psi^{-\gamma}(t) \leq \Psi^{-\gamma}(0) - \gamma \frac{\Psi'(0)}{\Psi^{1+\gamma}(0)}$$

and since $\Psi(t) > 0$ for all $t \geq 0$ we have

$$\Psi^{\gamma}(t) \geq \Psi^{\gamma}(0) \left[1 - \gamma t \Psi^{-1}(0)\Psi'(0)\right]^{-1}.$$

Since $\Psi(0) > 0$ and $\Psi'(0) > 0$ we deduce

$$\Psi(t) \to +\infty \text{ as } t \to t_1 \leq t_2 = \frac{\Psi(0)}{\gamma \Psi'(0)}.$$

□

Chapter 2

Nonlinear parabolic equations

nonlinear parabolic equations appear on the study of many processes of natural sciences. They model nonlinear processes in reaction-diffusion theory, kinetics of chemical reactions, combustion theory and a number of other processes.

2.1 A non-local, nonlinear heat equation with Neumann boundary conditions (Concavity method)

In this section we consider the following problem

$$u_t(x,t) - \Delta u(x,t) = |u(x,t)|^{p-1}u(x,t) - \fint_\Omega |u(x,t)|^{p-1}u(x,t)dx, \qquad x \in \Omega,\ t \geq 0 \quad (2.1)$$

$$\frac{\partial u}{\partial \vec{\nu}}(x,t) = 0, \qquad x \in \partial\Omega,\ t \geq 0 \quad (2.2)$$

$$u_0(x) = u(x,0) \qquad \text{with} \qquad \fint_\Omega u_0(x)dx = 0. \quad (2.3)$$

Here, for any function f

$$\fint_\Omega f dx = \frac{1}{|\Omega|}\int_\Omega f dx$$

is the average of the function f over Ω, Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $\vec{\nu}$ is the unit outward normal vector to the boundary of Ω. For simplicity we will assume that $|\Omega| = 1$, so that $\fint_\Omega = \int_\Omega$. First, note that if we integrate equation (2.1) over Ω, since $|\Omega| = 1$, we find that

$$\int_\Omega u_t dx - \int_\Omega \Delta u dx = \int_\Omega |u|^{p-1}u dx - \int_\Omega |u|^{p-1}u dx = 0.$$

In view of the divergence theorem and the condition (2.2) we obtain

$$\frac{d}{dt}\int_{\Omega} u dx = \int_{\Omega} u_t dx = \int_{\Omega} \Delta u dx = \int_{\partial\Omega} \frac{\partial u}{\partial \vec{\nu}} dx = 0. \tag{2.4}$$

From this and the condition (2.3) we also deduce that,

$$\int_{\Omega} u dx = \int_{\Omega} u_0(x) dx = 0. \tag{2.5}$$

Now, if we multiply the equation (2.1) by $u_t(x,t)$ and integrate over Ω, due to (2.4) we see that

$$\int_{\Omega} u_t^2 dx - \int_{\Omega} \Delta u u_t dx = \int_{\Omega} |u|^{p-1} u u_t dx.$$

Noting that

$$\Delta u u_t = \nabla(\nabla u u_t) - \frac{1}{2}\frac{d}{dt}|\nabla u|^2,$$

we obtain from the above equality:

$$\int_{\Omega} u_t^2 dx - \int_{\Omega} \nabla(\nabla u u_t) dx + \frac{1}{2}\frac{d}{dt}\int_{\Omega} |\nabla u|^2 dx = \frac{1}{p+1}\frac{d}{dt}\int_{\Omega} |u|^{p+1} dx.$$

Using the divergence theorem with equation (2.3) we arrive at

$$\frac{d}{dt}\int_{\Omega}\left[\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\right] dx = -\int_{\Omega} u_t^2 dx.$$

If we define

$$E(t) := \int_{\Omega}\left[\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\right] dx, \tag{2.6}$$

we can rewrite the above equality as

$$E'(t) = -\int_{\Omega} u_t^2 dx. \tag{2.7}$$

Let us denote by

$$m(t) := \frac{1}{2}\int_{\Omega} u^2 dx, \tag{2.8}$$

and

$$h(t) := \int_0^t m(s) ds. \tag{2.9}$$

We will show that $h(t)$ blows up in finte time. Before that, with the following lemma, we prepare some results needed for the blow-up theorem.

Lemma 2.1. *If u is a solution of the problem (2.1)-(2.3) and $E(0) \leq 0$ then for all $t \geq 0$. We have*

$$E(t) = E(0) - \int_0^t \int_\Omega u_s^2(x,s)dxds, \tag{2.10}$$

$$m'(t) \geq (p+1) \int_0^t \int_\Omega u_s^2(x,s)dxds, \tag{2.11}$$

$$m'(t) \geq \lambda_1^{-1}(p-1)m(t), \tag{2.12}$$

and

$$\frac{p+1}{2}(h'(t) - h'(0))^2 \leq h(t)h''(t), \tag{2.13}$$

where λ_1 *is the first eigenvalue of the problem*

$$-\Delta\Psi = \lambda\Psi,\ x \in \Omega,$$

$$u = 0,\ \ x \in \partial\Omega.$$

Proof. Integrating (2.7) over $(0,t)$ we arrive at (2.10). Also, note that (2.10) together with the assumption $E(0) \leq 0$ implies $E(t) \leq 0$ for all $t \geq 0$.
Next we calculate

$$\begin{aligned}
m'(t) &= \int_\Omega u u_t dx \\
&= \int_\Omega u \left(\Delta u + |u|^{p-1}u - \int_\Omega |u|^{p-1}u dx \right) dx \\
&= -\int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^{p+1} dx - \int_\Omega |u|^{p-1}u dx \int_\Omega u dx \\
&= -(p+1)E(t) + \frac{(p-1)}{2} \int_\Omega |\nabla u|^2 dx.
\end{aligned}$$

Here we have used the equations (2.1), (2.6) and (2.5). Hence from (2.10) and the assumption that $E(0) \leq 0$ we deduce

$$\begin{aligned}
m'(t) &\geq -(p+1)E(t) \\
&= -(p+1)E(0) + (p+1) \int_0^t \int_\Omega u_s^2(x,s)dxds \\
&\geq (p+1) \int_0^t \int_\Omega u_s^2(x,s)dxds,
\end{aligned}$$

which is (2.11). For (2.12) we use the Poincaré inequality (1.3) and the fact that $E(t) \leq 0$ for all $t \geq 0$.

$$m'(t) = -(p+1)E(t) + \frac{(p-1)}{2}\int_\Omega |\nabla u|^2 dx$$
$$\geq \frac{\lambda_1(p-1)}{2}\int_\Omega |u|^2 dx = \lambda_1(p-1)m(t).$$

Finally using Hölder's inequality (1.2), the definition of $h(t)$ and the equation (2.11) we see that

$$h'(t) - h(0) = \int_0^t m'(s)ds = \int_0^t \int_\Omega u(x,s)u_s(x,s)dxds$$
$$\leq \left(\int_0^t \int_\Omega u^2(x,s)dxds\right)^{\frac{1}{2}} \left(\int_0^t \int_\Omega u_s^2(x,s)dxds\right)^{\frac{1}{2}}$$
$$\leq \left(\frac{2}{p+1}\right)^{\frac{1}{2}} (h(t))^{\frac{1}{2}}(m'(t))^{\frac{1}{2}}$$
$$\leq \left(\frac{2}{p+1}\right)^{\frac{1}{2}} (h(t))^{\frac{1}{2}}(h''(t))^{\frac{1}{2}}.$$

From this the desired inequality (2.13) follows directly, and we are done.

□

Now, we are ready to present the blow-up result for $h(t)$.

Theorem 2.2. *Let $p > 1$ and let u be a solution to the problem (2.1)-(2.3) and $u_0 \not\equiv 0$. If $E(0) \leq 0$, then u does not exist for all time.*

Proof. To prove this result, we will use the idea in Levine's lemma 1.6 for the non-negative function $h(t)$. However, since $h(0) = 0$ and the inequality (2.13) found in the preceeding lemma is not quite in the same form as the differential inequality in Levine's lemma, we need some modification. The idea is to choose the initial time $T_i > 0$ so that these assumptions hold. That is, we need to find some initial time T_i such that $h(T_i) > 0$, $h'(T_i) > 0$, and the differential inequality in Levine's lemma should hold for $t \geq T_i$. Then, the result will follow. Now, by the inequality (2.11) $m'(t) \geq 0$ for all $t \geq 0$. Then, since $u_0 \not\equiv 0$, $h'(t) = m(t) \geq m(0) > 0$ for all $t \geq 0$. Therefore, $h(t) > 0$ for any $t > 0$. So, there only remains to show that there exists a time $T_i > 0$ such that the differential inequality in Levine's lemma holds for $h(t)$ whenever $t \geq T_i$. For this, observe that the inequality (2.12) together with $m(0) > 0$ implies that

$$\lim_{t\to\infty} h'(t) = \lim_{t\to\infty} m(t) = +\infty\,.$$

Therefore, for all $0 < B < p+1$ there exists $T_B > 0$, such that for all $t \geq T_B$

$$B\left(h'(t)\right)^2 \leq (p+1)(h'(t) - h'(0))^2.$$

We choose $2 < B < p+1$ ($p > 1$ by assumption), and set $T_i = T_B$. Then, the inequality (2.13) becomes

$$\frac{B}{2}\left(h'(t)\right)^2 \leq h(t)h''(t).$$

Therefore, the differential inequality in Levine's lamma with $\alpha = \frac{B}{2} - 1 > 0$ is satisfied whenever $t \geq T_i$. We are done.

□

Remark 2.3. In [6] the authors tried to prove a global non-existence theorem for the problem:

$$u_t - \Delta u = |u|^p - \fint_\Omega |u|^p dx, \qquad x \in \Omega,\ t > 0,$$

$$\frac{\partial u}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t > 0,$$

$$u(x,0) = u_0(x), \quad \text{with} \quad \int_\Omega u_0 dx = 0.$$

But, the energy equality

$$E(u(t)) = E(u_0) - \int_0^t \int_\Omega u_t^2 dx$$

is derived incorrectly (for the nonlinear term of the form u^p).

Remark 2.4. We can apply the result we obtained in the theorem 2.2 in the study of the following control problem:

$$u_t - \Delta u = |u|^{p-1}u - K(t), \qquad x \in \Omega,\ t > 0,$$

$$\frac{\partial u}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t > 0,$$

$$u(x,t) = u_0(x), \quad \text{with} \quad \int_\Omega u_0 dx = 0,$$

$$\int_\Omega u dx = 0, \qquad t > 0.$$

Here, u_0 is a given function such that

$$\int_\Omega u_0 dx = 0,$$

and $(u(x,t), K(t))$ is the unknown pair of functions.

2.2 A nonlinear backwards heat equation (Direct energetic approach)

In this section we consider the initial boundary value problem for the nonlinear backwards heat equation under the Dirichlet boundary condition. Here, $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$.

$$u_t(x,t) + \Delta u(x,t) = f(u(x,t)), \qquad x \in \Omega,\ t > 0, \tag{2.14}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{2.15}$$

$$u(x,0) = u_0(x) \not\equiv 0, \qquad x \in \Omega, \tag{2.16}$$

where $\frac{\partial u}{\partial \nu}$ is the unit outward normal derivative. We have the following blow-up theorem.

Theorem 2.5. *Suppose that the nonlinearity f satisfies*

$$sf(s) \geq \gamma |s|^p, \qquad \textit{for all } s \in \mathbb{R}, \tag{2.17}$$

where $p > 2$ and $\gamma > 0$ are given numbers. Then, the solution u to the problem (2.14)-(2.16) does not exist for all time.

Proof. Note that if we multiply (2.14) by u and integrate over Ω, using the boundary condition (2.15) we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 = \|\nabla u(t)\|_2^2 + \int_\Omega f(u)u dx.$$

With the assumption (2.17) and the Poincaré inequality (1.6) the above inequality becomes

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 \geq \lambda_1 \|u(t)\|_2^2 + \gamma \int_\Omega |u|^p dx. \tag{2.18}$$

Next, by Hölder's inequality (1.2) with $\frac{1}{p/2} + \frac{1}{p/p-2} = 1$ we have

$$\int_\Omega |u|^2 dx \leq |\Omega|^{\frac{p-2}{p}} \left(\int_\Omega |u|^p dx \right)^{\frac{2}{p}},$$

which implies

$$\int_\Omega |u|^p dx \geq |\Omega|^{\frac{2-p}{2}} \|u(t)\|_2^p.$$

Hence, the inequality (2.18) implies

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 \geq \lambda_1 \|u(t)\|_2^2 + \kappa \|u(t)\|_2^p,$$

where $\kappa = \gamma|\Omega|^{\frac{2-p}{2}}$. If we define $Z(t) := \|u(t)\|_2^2$, the above inequality can be rewritten as

$$Z' \geq 2\lambda_1 Z + \kappa Z^{\frac{p}{2}}.$$

Thus, since $p > 2$, $\|u(t)\|_2^2$ blows up at a finite time

$$T \leq \int_{\|u_0\|_2^2}^{\infty} \frac{ds}{\lambda_1 s + \kappa s^{\frac{p}{2}}} < \infty.$$

□

2.3 A nonlinear heat equation with Dirichlet boundary conditions (Method of eigenfunctions)

Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary. We consider the following initial-boundary value problem

$$u_t(x,t) - \Delta u(x,t) = f(u(x,t)), \qquad x \in \Omega,\ t \geq 0, \tag{2.19}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega, \tag{2.20}$$

$$u(x,0) = u_0(x) \not\equiv 0, \qquad x \in \Omega. \tag{2.21}$$

We will look for a classical solution to this problem. We assume that f is bounded below by a differentiable, convex function g. The conditions on the initial data and g are to be stated later. Suppose $\Psi(x)$ is the first eigenfunction of the Laplace operator under the homogeneous Dirichlet condition, that is $\Psi(x)$ is the solution of the problem

$$\Delta\Psi(x) + \lambda_1\Psi(x) = 0, \qquad x \in \Omega, \tag{2.22}$$

$$\Psi(x) = 0, \qquad x \in \partial\Omega, \tag{2.23}$$

where λ_1 is the first eigenvalue of the Laplace operator. Without loss of generality we take $\Psi(x) > 0$ so that $\int_\Omega \Psi(x)dx = 1$. Now, let us multiply (2.19) by $\Psi(x)$ and then integrate over Ω:

$$\int_\Omega u_t\Psi dx = \int_\Omega \Delta u\Psi dx + \int_\Omega f(u)\Psi dx. \tag{2.24}$$

Note that

$$\begin{aligned} \Delta u\Psi &= \nabla\cdot(\nabla u\Psi) - \nabla u\cdot\nabla\Psi \\ &= \nabla\cdot(\nabla u\Psi) - \nabla\cdot(u\nabla\Psi) + u\Delta\Psi. \end{aligned} \tag{2.25}$$

We observe that in view of equations (2.20) and (2.23), respectively, the divergence theorem implies

$$\int_\Omega \nabla\cdot(u\nabla\Psi)dx = \int_{\partial\Omega} u\nabla\Psi\cdot\vec{\nu}dx = 0,$$

and

$$\int_\Omega \nabla\cdot(\nabla u\Psi)dx = \int_{\partial\Omega} \Psi\nabla u\cdot\vec{\nu}dx = 0,$$

where $\vec{\nu}$ is the outward normal derivative at the boundary of Ω. Therefore, integtating (2.25) we have

$$\int_\Omega \Delta u\Psi dx = \int_\Omega u\Delta\Psi dx.$$

Using this in the equation (2.24) we find

$$\int_\Omega u_t \Psi dx = \int_\Omega u \Delta \Psi dx + \int_\Omega f(u) \Psi dx.$$

Next, because $\Psi(x) > 0$, and f is bounded below by the convex function g, we have

$$\int_\Omega u_t \Psi dx \geq \int_\Omega u \Delta \Psi dx + \int_\Omega g(u) \Psi dx.$$

Since g is convex for the second term on the right hand side of the above inequality we can invoke the Jensen's inequality (1.3). Also, for the first term on the right hand side we use (2.22). Remembering that $\int_\Omega \Psi(x) dx = 1$ we obtain

$$\int_\Omega u_t \Psi dx \geq g\left(\int_\Omega u \Psi dx\right) - \lambda_1 \int_\Omega u \Psi dx. \tag{2.26}$$

Let

$$\Phi(t) := \int_\Omega u(x,t) \Psi(x) dx.$$

Then, since $\Psi(x)$ is independent of time, the above inequality can be rewritten in the following form:

$$\Phi'(t) \geq g(\Phi(t)) - \lambda_1 \Phi(t).$$

Now, in order to invoke lemma 1.6 with $h(s) = g(s) - \lambda_1 s$, we only require that

$$g(s) - \lambda_1 s > 0 \text{ for all } s \geq \Phi(0) = \int_\Omega u_0(x) \Psi(x) dx =: \alpha.$$

Finally, in order to have a blow-up result we need the integral $\int_\alpha^\infty \frac{1}{h(s)} ds$ to converge. Thus, we have proved the following theorem.

Theorem 2.6. *Suppose that u is a classical solution to the problem (2.19)-(2.21). Moreover, assume that f is bounded below by a differentiable, convex function g satisfying*

$$g(s) - \lambda_1 s > 0 \text{ for all } s \geq \int_\Omega u_0(x) \Psi(x) dx,$$

where λ_1 and $\Psi(x)$ are, respectively, the first eigenvalue and eigenfunction to the problem (2.22)-(2.23) with $\int_\Omega \Psi(x) dx = 1$, and $\Psi(x) > 0$ for all $x \in \Omega$. Then, if the integral $\int_\alpha^\infty \frac{1}{g(s) - \lambda_1 s} ds$ converges, u blows up at a finite time less than or equal to

$$T = \int_\alpha^\infty \frac{1}{g(s) - \lambda_1 s} ds.$$

2.4 Cauchy problem for a nonlinear heat equation (Green function method)

Let u be a classical solution to the Cauchy problem for the heat equation

$$u_t(x,t) - \Delta u(x,t) = u^p(x,t), \qquad x \in \mathbb{R}^n,\ t \in (0,T), \tag{2.27}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n, \tag{2.28}$$

where u_0 is a given function and p is a given number.

Theorem 2.7. *If $u_0 \in C_0^2(\mathbb{R}^n)$ is a nonnegative and not identically zero function, and p satisfies the inequality*

$$1 < p < \frac{n+2}{n}, \tag{2.29}$$

then the problem (2.27),(2.28) has not a positive classical solution existing for all $T > 0$.

Proof. We know that the function

$$\Phi(x,s) = \frac{1}{4\pi s}^{n/2} e^{-\frac{|x|^2}{4s}}$$

is a fundamental solution of the heat equation evaluated at a time $s > 0$ (we'll choose it later). It is not difficult to see that

$$\int_{\mathbb{R}^n} \Phi(x,s)dx = 1$$

and

$$\Delta\Phi(x,s) + \frac{n}{2s}\Phi(x,s) \geq 0. \tag{2.30}$$

We consider the function

$$\Psi(t) := \int_{\mathbb{R}^n} u(x,t)\Phi(x,s)dx.$$

By using the equation (2.27) we obtain

$$\begin{aligned}\Psi'(t) &= \int_{\mathbb{R}^n} u_t(x,t)\Phi(x,s)dx = \int_{\mathbb{R}^n} [\Delta u(x,t) + u^p(x,t)]\Phi(x,s)dx \\ &= \int_{\mathbb{R}^n} u(x,t)\Delta\Phi(x,s)dx + \int_{\mathbb{R}^n} u^p(x,t)\Phi(x,s)dx.\end{aligned}$$

By the maximum principle u is positive, so employing the inequality (2.30) we obtain:

$$\Psi'(t) \geq -\frac{n}{2s}\Psi(t) + \int_{\mathbb{R}^n} u^p(x,t)\Phi(x,s)dx. \tag{2.31}$$

Due to the Jensen inequality (1.3) we have

$$\Psi^p(t) \leq \int_{\mathbb{R}^n} u^p(x,t)\Phi(x,s)dx.$$

Thus (2.31) implies:

$$\Psi'(t) \geq -\frac{n}{2s}\Psi(t) + \Psi^p(t). \tag{2.32}$$

Multiplying (2.32) by e^{qt} with $q = \frac{n}{2s}$ we get

$$Z'(t) \geq e^{-q(p-1)t} Z^p(t),$$

where $Z(t) := e^{qt}\Psi(t)$. Integrating the last inequality we obtain

$$Z^{p-1}(t) \geq \frac{Z^{p-1}(0)q}{q - Z^{p-1}(0)(1 - e^{-q(p-1)t})}.$$

This inequality implies that

$$Z(t) \to \infty$$

in a finite time if

$$\Psi(0) = Z(0) > q^{\frac{1}{p-1}},$$

or

$$\frac{1}{4\pi t}^{n/2} \int_{\mathbb{R}^n} u_0(x) e^{-\frac{|x|^2}{4s}} dx > s^{n/2} (\frac{n}{2s})^{\frac{1}{p-1}} = c_0 s^{\frac{n}{2} - \frac{1}{p-1}}, \tag{2.33}$$

where $c_0 > 0$. According to the condition (2.29) $\frac{n}{2} - \frac{1}{p-1} < 0$. Thus for any initial function u_0 we can choose $s > 0$ so large that (2.33) holds. □

2.5 Heat equation with a nonlocal nonlinearity (An example of blow-up of derivative)

Consider the following nonlinear initial boundary value problem for the nonlocal heat equation

$$u_t(x,t) = u_{xx}(x,t) + u(x,t)\left(\int_0^1 u_x^2(x,t)dx\right)^p, \qquad x \in (0,1),\ t>0, \tag{2.34}$$

$$u(x,0) = f(x), \qquad x \in (0,1), \tag{2.35}$$

$$u_x(0,t) = u_x(1,t) = 0, \qquad t>0, \tag{2.36}$$

where $p>0$ is a given number and f is a given function.

Theorem 2.8. *Suppose that u is a classical solution to the problem (2.34)-(2.36), and $f \in C^2[0,1]$ with*

$$\frac{1}{p+1}\left(\int_0^1 (f')^2 dx\right)^{p+1} \geq \int_0^1 (f'')^2 dx. \tag{2.37}$$

Then, u_x blows up in L^2-norm at a time $T \leq \frac{p+1}{2p^2\|f'\|_2^{2p}}$.

Proof. Multiplication of (2.34) by u_{xx} and integration of the resulting equation over $(0,1)$ gives

$$\int_0^1 u_t u_{xx} dx = \int_0^1 u_{xx}^2 dx + \int_0^1 u u_{xx} dx \left(\int_0^1 u_x^2 dx\right)^p. \tag{2.38}$$

Using integration by parts together with (2.36) we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_x(t)\|_2^2 = -\|u_{xx}(t)\|_2^2 + \|u_x(t)\|_2^{2p+2}. \tag{2.39}$$

Next, differentiating (2.34) with respect to x we get

$$u_{xt} = u_{xxx} + u_x\|u_x(t)\|_2^{2p}.$$

Multiplying this equality by u_{xt} and then integrating over $(0,1)$ we arrive at

$$\|u_{xt}(t)\|_2^2 = \int_0^1 u_{xxx}u_{xt}dx + \|u_x(t)\|_2^{2p}\int_0^1 u_x u_{xt} dx.$$

Again, using integration by parts with (2.36) we find

$$\|u_{xt}(t)\|_2^2 = \frac{d}{dt}\left[-\frac{1}{2}\|u_{xx}(t)\|_2^2 + \frac{1}{2(p+1)}\|u_x(t)\|_2^{2(p+1)}\right]. \tag{2.40}$$

Let us define

$$E(t) := -\|u_{xx}(t)\|_2^2 + \frac{1}{(p+1)}\|u_x(t)\|_2^{2(p+1)}.$$

Then, (2.40) implies that

$$E(t) \geq E(0), \qquad \text{for all } t \geq 0.$$

Employing this fact we deduce from (2.39) that

$$\begin{aligned}
\frac{d}{dt}\|u_x(t)\|_2^2 &= 2\left[-\|u_{xx}(t)\|_2^2 + \frac{1}{(p+1)}\|u_x(t)\|_2^{2(p+1)}\right] + \frac{2p}{p+1}\|u_x(t)\|_2^{2(p+1)} \\
&= 2E(t) + \frac{2p}{p+1}\|u_x(t)\|_2^{2(p+1)} \\
&\geq 2E(0) + \frac{2p}{p+1}\|u_x(t)\|_2^{2(p+1)}.
\end{aligned}$$

Note that

$$E(0) = -\|f''\|_2^2 + \frac{1}{(p+1)}\|f'\|_2^{2(p+1)}.$$

Hence, by the assumption (2.37) $E(0) \geq 0$. Therefore, the differential inequality above reduces to

$$\frac{d}{dt}\|u_x(t)\|_2^2 \geq \frac{2p}{p+1}\|u_x(t)\|_2^{2(p+1)}.$$

Integrating we deduce

$$\|u_x(t)\|_2^2 \geq \frac{1}{\left(\|f'\|_2^{-2p} - \frac{2p^2}{p+1}t\right)^{\frac{1}{p}}},$$

which implies u_x blows up in L^2-norm at a time $T \leq \frac{p+1}{2p^2\|f'\|_2^{2p}}$. □

2.6 A nonlinear heat equation with nonhomogeneous Neumann boundary conditions (Comparison technique)

Suppose u is a classical solution to the following nonlinear initial boundary value problem for the heat equation

$$u_t(x,t) - \Delta u(x,t) = u^p(x,t), \qquad x \in \Omega, \quad t > 0, \tag{2.41}$$

$$u(x,0) = u_0(x) \geq d_0, \qquad x \in \Omega, \tag{2.42}$$

$$\frac{\partial u}{\partial \nu}(x,t) = 0, \qquad x \in \partial\Omega, \quad t > 0, \tag{2.43}$$

where $p > 1$, $d_0 > 0$ are numbers and $\vec{\nu}$ is the unit outward normal vector on $\partial\Omega$. Let $v(x,t) = v(t)$ be the solution to the problem

$$v'(t) = v^p(x,t), \qquad t > 0,$$

$$v(0) = \alpha d_0, \qquad 0 < \alpha < 1.$$

Then,

$$v(t) = \alpha d_0 \left(\frac{1}{1 - (p-1)\alpha^{p-1} d_0^{p-1} t} \right)^{\frac{1}{p-1}}.$$

This holds for all $0 < \alpha < 1$. Thereore, we have

$$v(t) \to \infty \text{ as } t \to T \leq \frac{1}{(p-1)d_0^{p-1}}.$$

Moreover, it is easy to see that u and v satisfy the assumptions of Theorem 1.5. Hence, we conclude that

$$u(x,t) \to \infty \text{ as } t \to T \leq \frac{1}{(p-1)d_0^{p-1}}.$$

Thus, we have proved the following theorem.

Theorem 2.9. *Suppose u is a classical solution to the problem (2.41)-(2.43). Then,*

$$u(x,t) \to \infty \text{ as } t \to T \leq \frac{1}{(p-1)d_0^{p-1}}.$$

2.7 A nonlinear heat equation with nonhomogeneous Dirichlet boundary conditions (Comparison technique)

Suppose that the problem

$$u_t(x,t) - u_{xx}(x,t) = u^2(x,t), \tag{2.44}$$

$$u(0,t) = \Psi_0(t), \quad u(1,t) = \Psi_1(t), \tag{2.45}$$

$$u(x,0) = u_0(x), \tag{2.46}$$

has a classical solution in

$$\overline{\Omega}_T = \{0 \le x \le 1,\ 0 \le t \le T\},$$

where $\Psi_1(t)$, $\Psi_2(t)$ are bounded from below by a positive constant $c = \frac{c_1}{c_2}$.It is easy to check that the function

$$v(x,t) = \frac{c_1}{c_2 - tx(1-x)}$$

satisfies the inequality $v_t - v_{xx} - v^2 \le 0$ for $t < 4c_2$ if $c_1 \ge \frac{1}{4} + 8c_2$ and v satisfies the boundary conditions

$$v(0,t) = v(1,t) = \frac{c_1}{c_2}.$$

The function

$$w = (v-u)e^{-\lambda t}$$

is negative on the boundary, and satisfies

$$w_t - w_{xx} + (\lambda - (v+u)w) \le 0.$$

Thus, for $\lambda > 0$ big enough w can not attain a positive maximum in $\Omega_{4c_2}/\Gamma_{4c_2}$. Hence, the function w is nonpositive in Ω_{4c_2}, so

$$u \ge v \qquad \text{in } \Omega_{4c_2}.$$

But, $v \to \infty$ at $x = \frac{1}{2}$ as $t \to 4c_2$. So, the problem has no classical solution in Ω_T for $T \ge 4c_2$. Note, that we may chose c_2 as small as we want and still satisfy $c = \frac{c_1}{c_2}$ for any $c > 0$. Thus, we have proved the following theorem.

Theorem 2.10. *If $\Psi(t)$, and $\Psi_1(t)$ are bounded below by a positive constant, then the problem (2.44)-(2.45) has no classical solution in Ω_T for any $T \ge 4c_2$.*

Chapter 3

Nonlinear hyperbolic equations

This chapter is devoted to the study of solutions to the Cauchy problem and initial boundary value problems for the nonlinear wave equation of the form

$$u_{tt} - \Delta u = f(u), \tag{3.1}$$

and to a one dimensional initial boundary value problem for a nonlinear Boussinesq type equation. The linear case of the wave equation with

$$f(u) = -m^2 u, \quad m \in \mathbb{R}$$

corresponds to the Klein-Gordon equation in relativistic particle physics. In the 1950's equations of this type with the nonlinear term like

$$f(u) = -mu + bu^3$$

were proposed as models in relativistic quantum mechanics. Equations of the form (3.1) appear also in modelling various processes of elasticity.

3.1 Initial boundary value problem for a nonlinear wave equation (Method of eigenfunctions)

Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary. In this chapter we consider the problem

$$u_{tt}(x,t) - \Delta u(x,t) = f(u(x,t)), \qquad x \in \Omega,\ t \geq 0, \tag{3.2}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega,\ t \geq 0, \tag{3.3}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad\qquad x \in \Omega. \tag{3.4}$$

Here f is assumed to be bounded from below by a differentiable, convex function g, and we look for a classical solution to this problem. The conditions on the initial data, the funciton g, and the nonlinearity f are to be stated later. As done in Section 2.2, we denote the first eigenfuntion of the Laplace operator under the homogeneous Dirichlet condition by $\Psi(x)$, and the corresponding eigenvalue by λ_1. Again without loss of generality we take $\int_\Omega \Psi(x)dx = 1$, and $\Psi(x) > 0$ for all $x \in \Omega$. If we multiply equation (3.2) by $\Psi(x)$, by a very similar argumentation to the one done in Section 2.2, we find

$$\int_\Omega u_{tt}\Psi dx + \lambda_1 \int_\Omega u\Psi dx \geq g\left(\int_\Omega u\Psi dx\right).$$

If we let $\Phi(t) := \int_\Omega u(x,t)\Psi(x)dx$, the above ineqaulity becomes

$$\Phi''(t) + \lambda_1\Phi(t) \geq g(\Phi(t)).$$

Now, in order to invoke lemma 1.7 for this defined $\Phi(t)$, we require that

$$\alpha = \Phi(0) = \int_\Omega u_0\Psi dx \geq 0,$$

and

$$\beta = \int_\Omega u_1\Psi dx > 0.$$

Since $\Psi(x) > 0$ when $x \in \Omega$, this is achieved if we assume $u_0(x) \geq 0$, $u_1(x) \geq 0$ for all $x \in \Omega$, and u_1 is not everywhere zero. We also assume that $h(s) := g(s) - \lambda_1 s$ is nonnegative for $s > \alpha$. Finally, the lemma provides a blow-up result if the integral

$$\int_\alpha^\infty \left[\gamma\alpha^2 + \beta^2 - \gamma s^2 + 2\int_\alpha^s g(\xi)d\xi\right]^{-\frac{1}{2}} ds$$

converges. We observe that this is achieved if $g(s)$ grows fast enough as $s \to \infty$. Thus, with this final assumption $\Phi(t)$ blows up in a time less than or equal to $T = \int_\alpha^\infty \left[\gamma\alpha^2 + \beta^2 - \gamma s^2 + 2\int_\alpha^s g(\xi)d\xi\right]^{-\frac{1}{2}} ds$. Therefore, we have established the following theorem.

Theorem 3.1. *Suppose that u is a classical solution to the problem (3.2)-(3.4). As for the initial conditions assume that $u_0(x) \geq 0$, $u_1(x) \geq 0$ for all $x \in \Omega$, and u_1 is not everywhere zero. Moreover, assume that f is bounded below by a differentiable, convex function g satisfying $g(s) - \lambda_1 s \geq 0$ for all $s > \int_\Omega u_0(x)\Psi(x)dx$, where λ_1 and $\Psi(x)$ are, respectively, the first eigenvalue and eigenfunction to the problem (2.22)-(2.23) with*

$\int_\Omega \Psi(x)dx = 1$, *and* $\Psi(x) > 0$ *for all* $x \in \Omega$. *Then, if the integral*

$$\int_\alpha^\infty \left[\gamma\alpha^2 + \beta^2 - \gamma s^2 + 2\int_\alpha^s g(\xi)d\xi\right]^{-\frac{1}{2}} ds$$

converges, u *blows up at a finite time*

$$T^* \leq T = \int_\alpha^\infty \left[\gamma\alpha^2 + \beta^2 - \gamma s^2 + 2\int_\alpha^s g(\xi)d\xi\right]^{-\frac{1}{2}} ds.$$

3.2 An initial boundary value problem for a nonlinear wave equation (direct energetic method)

Let $\Omega \subset \mathbb{R}^n$ be a domain, and consider the following initial boundary value problem:

$$u_{tt}(x,t) - \Delta u(x,t) = F(u(x,t)), \qquad x \in \Omega,\ t \geq 0, \tag{3.5}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega,\ t \geq 0, \tag{3.6}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega. \tag{3.7}$$

Here, F is a nonlinear potential operator, that is there exists another functional G such that

$$\frac{d}{dt} G(u) = (F(u), u_t), \tag{3.8}$$

where for two functions v, ω

$$(v, \omega) = \int_\Omega v\omega dx$$

is their inner product. Moreover, we assume that $F(0) = 0$, and for some $p > 0$

$$(F(u), u) - 2G(u) \geq (u,u)^{1+p}, \tag{3.9}$$

is satisfied. We want to show that for some class of initial data u_0 and u_1 the solution of the problem (3.5), (3.7) blows up in finite time. For this purpose let us take the inner product of the equation (3.5) with u_t. Since Δ is a symmetric operator

$$\frac{d}{dt}(\Delta u, u) = (\Delta u_t, u) + (\Delta u, u_t) = 2(\Delta u, u_t),$$

and therefore we obtain the following equality:

$$\frac{d}{dt}\left[\frac{1}{2}(u_t, u_t) - \frac{1}{2}(\Delta u, u) - G(u)\right] = 0.$$

If we define

$$E(t) := \frac{1}{2}(u_t, u_t) + \frac{1}{2}(\Delta u, u) - G(u)$$

the above equality implies

$$E(t) = E(0), \text{ for all } t \geq 0.$$

Now, the function $\Psi(t) = (u,u)$ satisfies the relation

$$\Psi''(t) = 2(u_t, u_t) + 2(u, u_{tt}) = 2(u_t, u_t) + 2(\Delta u, u) + 2(F(u), u).$$

Using the definition of $E(t)$ we rewrite the above equality in the following form:

$$\Psi''(t) = 4(u_t, u_t) + 2(F(u), u) - 4G(u) - 4E(0).$$

Whenever $E(0) \leq 0$, by the assumption (3.9) this equality implies that

$$\Psi''(t) \geq 2(F(u), u) - 4G(u) \geq 2(u, u)^{1+p} = 2[\Psi(t)]^{1+p}. \tag{3.10}$$

Since $\Psi(t)$ is nonnegative, $\Psi''(t)$ is nonnegative. Therefore, if we assume that $\Psi'(0) = 2(u_0, u_1) \geq 0$, then $\Psi'(t) \geq 0$ for all $t \geq 0$, and hence the inequality (3.10) implies

$$\Psi''(t)\Psi'(t) \geq 2\Psi'(t)[\Psi(t)]^{1+p},$$

or

$$\frac{d}{dt}\left[\frac{1}{2}(\Psi'(t))^2 - 2\int_\alpha^{\Psi(t)} \tau^{1+p} d\tau\right] \geq 0,$$

with $\alpha = (u_0, u_0)$. Integrating this inequality, since $\Psi'(t)$ is nonnegative, we arrive at

$$\Psi'(t) \geq \left[(u_0, u_1)^2 + 4\int_\alpha^{\Psi(t)} \tau^{1+p} d\tau\right]^{\frac{1}{2}}.$$

Finally, from the last inequality we deduce that the solution of the problem (3.5), (3.7) blows up in finite time

$$t \leq t_1 = \int_\alpha^{+\infty}\left[(u_0, u_1)^2 + 4\int_\alpha^{s} \tau^{1+p} d\tau\right]^{-\frac{1}{2}} ds.$$

We summarize what we have proved in the following theorem.

Theorem 3.2. *Suppose u is a solution of the problem (3.5), (3.7), and the assumptions (3.8), (3.9) hold. If $(u_0, u_1) \geq 0$, and*

$$E(0) = \frac{1}{2}(u_1, u_1) + \frac{1}{2}(\Delta u_0, u_0) - G(u_0) \leq 0,$$

then u blows up at a finite time

$$t \leq t_1 = \int_\alpha^{+\infty}\left[(u_0, u_1)^2 + 4\int_\alpha^{s} \tau^{1+p} d\tau\right]^{-\frac{1}{2}} ds.$$

3.3 Cauchy problem for a nonlinear wave equation (Concavity method)

In this section we will apply the the concavity method to the following nonlinear Cauchy problem for the wave equation

$$u_{tt}(x,t) - \Delta u(x,t) = u^p(x,t), \qquad x \in \mathbb{R}^n, \ t \geq 0, \tag{3.11}$$

$$u(x,0) = u_0(x) \not\equiv 0, \qquad u_t(x,0) = u_1(x), \qquad x \in \mathbb{R}^n, \tag{3.12}$$

where $p \geq 2$ is an integer. On this problem we demonstrate the concavity method. We have the following theorem.

Theorem 3.3. *Let u be a solution to the problem (3.11), (3.12). Suppose that the initial functions u_0, u_1 satisfy the following condition*

$$E(0) = \frac{1}{2}\int_{\mathbb{R}^n} (u_1^2(x) + |\nabla u_0(x)|^2) dx - \frac{1}{p+1}\int_{\mathbb{R}^n} u_0^{p+1}(x) dx \leq 0, \tag{3.13}$$

$$\int_{\mathbb{R}^n} u_0(x) u_1(x) dx > 0. \tag{3.14}$$

Then, there exists $T \leq \frac{\|u_0\|}{\alpha(u_0,u_1)}$, where $\alpha = \frac{p-1}{4}$, such that

$$\lim_{t\to T}\int_{\mathbb{R}^n} u^2(x,t)dx \to \infty.$$

Proof. First, let us multiply (3.11) by u_t and integrate over $\mathbb{R}^n$. We find $E'(t) = 0$, where

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} (u_t^2(x,t) + |\nabla u(x,t)|^2) dx - \frac{1}{p+1}\int_{\mathbb{R}^n} u^{p+1}(x,t) dx.$$

Hence, by assumption

$$E(t) = E(0) \leq 0 \text{ for all } t \geq 0. \tag{3.15}$$

Next, to use Levine's Lemma 1.8 we define the function

$$\Psi(t) := \int_{\mathbb{R}^n} u^2(x,t)dx.$$

Now, we are going to show that

$$G(t) := \Psi(t)\Psi''(t) - (1+\alpha)\left(\Psi'(t)\right)^2 \geq 0, \text{ for each } t > 0,$$

$$\Psi(0) > 0, \text{ and } \Psi'(0) > 0.$$

It is easy to see that the last two conditions are satisfied by the assumptions of the theorem. Next, using the definition of $E(t)$ with (3.15) we see that

$$
\begin{aligned}
\Psi''(t) &= 2\int_{\mathbb{R}^n} u_t^2(x,t)dx + 2\int_{\mathbb{R}^n} u(x,t)u_{tt}(x,t)dx \\
&= 2\int_{\mathbb{R}^n} u_t^2(x,t)dx + 2\int_{\mathbb{R}^n} u(x,t)(\Delta u(x,t) + u^p(x,t))dx \\
&= 2\int_{\mathbb{R}^n} u_t^2(x,t)dx - 2\int_{\mathbb{R}^n} |\nabla u(x,t)|^2\, dx + 2\int_{\mathbb{R}^n} u^{p+1}(x,t)dx \\
&= 2(p+1)\left[-\frac{1}{2}\int_{\mathbb{R}^n} u_t^2(x,t)dx - \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x,t)|^2\, dx + \frac{1}{p+1}\int_{\mathbb{R}^n} u^{p+1}(x,t)dx\right] \\
&+ (p+3)\int_{\mathbb{R}^n} u_t^2(x,t)dx + (p-1)\int_{\mathbb{R}^n} |\nabla u(x,t)|^2\, dx \\
&= -2(p+1)E(0) + (p+3)\int_{\mathbb{R}^n} u_t^2(x,t)dx + (p-1)\int_{\mathbb{R}^n} |\nabla u(x,t)|^2\, dx.
\end{aligned}
$$

Since $E(0) \le 0$, we find

$$
\Psi''(t) \ge (p+3)\int_{\mathbb{R}^n} u_t^2(x,t)dx. \tag{3.16}
$$

Taking into account (3.16) we obtain

$$
\begin{aligned}
&\Psi(t)\Psi''(t) - \frac{p+3}{4}\left(\Psi'(t)\right)^2 \\
&\ge (p+3)\int_{\mathbb{R}^n} u_t^2(x,t)dx \int_{\mathbb{R}^n} u^2(x,t)dx - \frac{p+3}{4}\left[2\int_{\mathbb{R}^n} u(x,t)u_t(x,t)dx\right]^2 \\
&= (p+3)\left[\left(\int_{\mathbb{R}^n} u^2(x,t)dx\right)\left(\int_{\mathbb{R}^n} u_t^2(x,t)dx\right) - \left(\int_{\mathbb{R}^n} u(x,t)u_t(x,t)dx\right)^2\right].
\end{aligned}
$$

By the Schwarz inequality (1.1) the right hand side of the last inequality is nonnegative. Hence, we have

$$
\Psi(t)\Psi''(t) - (1 + \frac{p-1}{4})\left(\Psi'(t)\right)^2 \ge 0.
$$

So, the function $\Psi(t)$ satisfies the main inequality of Levine's lemma with $\alpha = \frac{p-1}{4}$. Thus, the statement of the theorem follows. □

Remark 3.4. Similar result is true for the initial boundary value problem

$$
u_{tt}(x,t) - \Delta u(x,t) = |u(x,t)|^{p-1}u(x,t), \qquad x \in \Omega,\ t \in [0,T),
$$

$$
u(x,t) = 0, \qquad x \in \partial\Omega,\ t \in [0,T),
$$

$$
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \Omega,
$$

where $\Omega \subset \mathbb{R}^n$ is a (not necessarily bounded) domain with sufficiently smooth boundary $\partial\Omega$.

Remark 3.5. Following the way of the Theorem 3.3 we can prove blow-up of solutions to the initial boundary value problem for the nonlinear plate equation

$$u_t(x,t) + \Delta^2 u(x,t) - a\Delta u(x,t) = f(u),\ x \in \Omega, t \in [0,T), \tag{3.17}$$

$$u(x,0) = u_0(x),\quad u_t(x,0) = u_1(x),\ x \in \Omega, \tag{3.18}$$

$$u(x,t) = \Delta u(x,t) = 0,\ x \in \Omega, t \in [0,). \tag{3.19}$$

Here $a \geq 0$ is a given number , $\Omega \subset \mathbb{R}^n$ is a domain with smooth boundary $\partial\Omega$, $f(\cdot) \in C^1(\mathbb{R}^n)$ is a given function which satisfies:

$$f(0) = 0,\ f(s)s - 2(2\alpha+1)F(s) \geq 0,\quad F(s) = \int_0^s f(\tau)d\tau,$$

where α is some positive number. We have the following theorem.

Theorem 3.6. *Suppose u is a classical solution to the problem (3.17)-(3.19). If the initial functions* u_0,u_1 *satisfy the conditions*

$$\int_\Omega \left[\frac{1}{2}|\nabla u_0(x)|^2 + \frac{1}{2}|u_1(x)|^2 - F(u_0(x))\right] dx \leq 0,$$

and

$$\int_\Omega u_0(x)u_1(x)dx > 0,$$

then there exists a number $t_1 \leq \frac{\|u_0\|}{\alpha(u_0,u_1)}$ *such that*

$$\|u(t)\| \to +\infty \quad as\ t \to t_1.$$

3.4 A nonlinear Boussinesq type equation

In this section we will deal with the following problem:

$$u_{tt}(x,t) = 3u_{xxxx}(x,t) + u_{xx}(x,t) - K(u^p(x,t))_{xx}, \qquad x \in (0,1),\ t > 0, \tag{3.20}$$

$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \qquad t > 0, \tag{3.21}$$

$$u(x,0) = u_0(x) \not\equiv 0, \quad u_t(x,0) = u_1(x), \qquad x \in [0,1], \tag{3.22}$$

where u_0, u_1 are given functions and $K > 0$, $p > 1$ are given constants.

Theorem 3.7. *Let u be a nonnegative solution to the problem (3.20)-(3.22) which satisfies*

$$F'(0) \geq \left[2\pi^2(3\pi^2 - 1)F^2(0) + \frac{32Kp\pi^2}{(p+1)^2(p+3)}F^{\frac{p+3}{2}}(0)\right] > 0, \tag{3.23}$$

where

$$F(t) := \|u(t)\|_2^2.$$

Then, u does not exist for all time.

Proof. Multiply (3.20) by u and integrate by parts to obtain

$$F''(t) = 2\|u_t(t)\|_2^2 + 6\|u_{xx}(t)\|_2^2 - 2\|u_x(t)\|_2^2 + 2\int_0^1 u^{p-1}u_x^2 dx. \tag{3.24}$$

Also, using the Schwarz (1.1) and Poincaré (1.6) inequalities we see that

$$\|u_x(t)\|_2^2 = -\int_0^1 uu_{xx}dx \leq \|u(t)\|_2\|u_{xx}(t)\|_2 \leq \frac{1}{\pi}\|u_x(t)\|_2\|u_{xx}(t)\|_2.$$

Thus,

$$\|u_x(t)\|_2 \leq \frac{1}{\pi}\|u_{xx}(t)\|_2. \tag{3.25}$$

Next, recalling $u \geq 0$, we apply Poincaré's inequality (1.6) to the function $v = u^{\frac{p+1}{2}}$ to find

$$\int_0^1 u^{p-1}u_x^2 dx \geq \frac{4\pi^2}{(p+1)^2}\int_0^1 u^{p+1}dx,$$

and by Hölder's inequality (1.2)

$$\|u(t)\|_2^2 \leq \left[\int_0^1 u^{p+1}dx\right]^{\frac{2}{p+1}}.$$

So, we have

$$\int_0^1 u^{p-1}u_x^2 dx \geq \frac{4\pi^2}{(p+1)^2}F^{\frac{p+1}{2}}(t). \tag{3.26}$$

Now, using (3.25) and (3.26) in (3.24) together with Poincaré's inequality (1.6) we get

$$F''(t) \geq 2LF(t) + 2MF^{\frac{p+1}{2}}(t), \tag{3.27}$$

with

$$L = \pi^2(3\pi^2 - 1) \text{ and } M = \frac{4Kp\pi^2}{(p+1)^2}.$$

This together with the assumption (3.23) implies $F'(t) > 0$ for all $t \geq 0$. Hence, multiplying (3.27) by $F'(t)$ we deduce

$$\frac{d}{dt}\left[\frac{1}{2}\left(F'(t)\right)^2 - LF^2(t) - \frac{4M}{p+3}F^{\frac{p+3}{2}}(t)\right] \geq 0.$$

Integrating and using assumption (3.23) we arrive at

$$F'(t) \geq \left[2LF^2(t) + \frac{8M}{p+3}F^{\frac{p+3}{2}(t)}\right]^{\frac{1}{2}} := Q(F).$$

Thus, since $p > 1$, we have

$$t \leq \int_{F(0)}^{F(t)} \frac{ds}{Q(s)} \leq \int_{F(0)}^{\infty} \frac{ds}{Q(s)} < \infty,$$

that is, u can not exist for all time. □

Chapter 4

Nonlinear Schrödinger equations

The nonlinear Schrödinger equation is a nonlinear evolutionary partial differential equation that describes processes in hydrodynamics, nonlinear optics, nonlinear acoustics and various other nonlinear phenomena.

4.1 Self-focusing nonlinear Schrödinger equation

Consider the Cauchy problem

$$iu_t(x,t) = \Delta u(x,t) + F(|u(x,t)|^2)u(x,t), \qquad x \in \mathbb{R}^n, t > 0, \tag{4.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n \tag{4.2}$$

We shall assume that $u \in \mathcal{S}$, where $\mathcal{S}$ is the Schwarz class. Moreover we assume that F is real-valued and is smooth enough so that a unique local classical solution to (4.1), (4.2) exists. We define

$$G(u) := \int_0^u F(s)ds, \tag{4.3}$$

$$I(t) := \int_{\mathbb{R}^n} |x|^2|u|^2 dx, \tag{4.4}$$

and

$$y(t) := Im \int_{\mathbb{R}^n} x \cdot (\bar{u}\nabla u)dx. \tag{4.5}$$

Lemma 4.1. *If u is a local solution of (4.1) then*

$$\int_{\mathbb{R}^n} |u|^2 dx = \int_{\mathbb{R}^n} |u_0(x)|^2 dx, \tag{4.6}$$

$$E(t) := \int_{\mathbb{R}^n} \left[|\nabla u|^2 - G(|u|^2)\right] dx = \int_{\mathbb{R}^n} \left[|\nabla u_0(x)|^2 - G(|u_0(x)|^2)\right] dx = E(0), \tag{4.7}$$

$$I'(t) = -4y(t), \tag{4.8}$$

and

$$y'(t) = -2\int_{\mathbb{R}^n} |\nabla u|^2 dx + n\int_{\mathbb{R}^n} \left[|u|^2 F(|u|^2) - G(|u|^2)\right] dx. \tag{4.9}$$

Proof. Let us multiply (4.1) by $\bar{u}$ and integrate over $\mathbb{R}^n$:

$$\begin{aligned} i\int_{\mathbb{R}^n} \bar{u}u_t dx &= \int_{\mathbb{R}^n} \Delta u\bar{u} dx + \int_{\mathbb{R}^n} F(|u|^2)u\bar{u} dx \\ &= -\int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} F(|u|^2)|u|^2 dx. \end{aligned}$$

Taking the imaginary part of the above equation we find

$$\frac{d}{dt}\int_{\mathbb{R}^n} |u(x,t)|^2 dx = 0,$$

which implies (4.6), the 1^{st} conservation law.

Next we multiply (4.1) by $\bar{u}_t$ and integrate over $\mathbb{R}^n$:

$$i\int_{\mathbb{R}^n} |u_t|^2 dx = -\int_{\mathbb{R}^n} \nabla u\cdot\nabla\bar{u}_t dx + \int_{\mathbb{R}^n} F(|u|^2)u\bar{u}_t dx.$$

Taking the real part of the above equality we find

$$\begin{aligned} 0 &= -\frac{d}{dt}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} F(|u|^2)\left(|u|^2\right)_t dx \\ &= -\frac{d}{dt}\int_{\mathbb{R}^n} \left[|\nabla u|^2 - G(|u|^2)\right] dx, \end{aligned}$$

which implies (4.7), the 2^{nd} conservation law.

For the 3^{rd} identity we calculate

$$\begin{aligned} I'(t) &= \int_{\mathbb{R}^n} |x|^2\left(\bar{u}u_t + u\bar{u}_t\right) dx \\ &= \int_{\mathbb{R}^n} |x|^2\left\{\bar{u}\left[-i\Delta u - iF(|u|^2)u\right] + u\left[i\Delta\bar{u} + iF(|u|^2)u\right]\right\} dx \\ &= i\int_{\mathbb{R}^n} |x|^2(u\Delta\bar{u} - \bar{u}\Delta u) dx, \end{aligned} \tag{4.10}$$

where we have used (4.1). Note that

$$u\Delta\bar{u} - \bar{u}\Delta u = \nabla\cdot(u\nabla\bar{u} - \bar{u}\nabla u).$$

Combining this with (4.10) we find

$$\begin{aligned} I'(t) &= i \int_{\mathbb{R}^n} |x|^2 \nabla \cdot (u\nabla \bar{u} - \bar{u}\nabla u) dx \\ &= -2i \int_{\mathbb{R}^n} x \cdot (u\nabla \bar{u} - \bar{u}\nabla u) dx \\ &= -2i \int_{\mathbb{R}^n} x \cdot [-2iIm(\bar{u}\nabla u)]\, dx, \end{aligned}$$

which gives (4.8).

For the final identity we have

$$\begin{aligned} y'(t) &= Im \int_{\mathbb{R}^n} x \cdot \nabla u_t \bar{u} dx + Im \int_{\mathbb{R}^n} x \cdot \nabla u \bar{u}_t dx \\ &= -nIm \int_{\mathbb{R}^n} u_t \bar{u} dx - Im \int_{\mathbb{R}^n} x \cdot \nabla \bar{u} u_t dx + Im \int_{\mathbb{R}^n} x \cdot \nabla u \bar{u}_t dx \\ &= -nIm \int_{\mathbb{R}^n} u_t \bar{u} dx + 2Im \int_{\mathbb{R}^n} x \cdot \nabla u \bar{u}_t dx. \end{aligned}$$

Using the equation (4.1) we get

$$\begin{aligned} y'(t) &= -nIm \int_{\mathbb{R}^n} \bar{u}[-i\Delta u - iF(|u|^2)u]dx + 2Im \int_{\mathbb{R}^n} x \cdot \nabla u[i\Delta \bar{u} + iF(|u|^2)\bar{u}]dx \\ &= n \int_{\mathbb{R}^n} F(|u|^2)|u|^2 dx + nRe \int_{\mathbb{R}^n} \Delta u \bar{u} dx + 2Re \int_{\mathbb{R}^n} x \cdot \nabla u[\Delta \bar{u} + F(|u|^2)\bar{u}]dx \\ &= n \int_{\mathbb{R}^n} F(|u|^2)|u|^2 dx - n \int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} x \cdot F(|u|^2)\nabla(|u|^2) dx \\ &+ 2Re \int_{\mathbb{R}^n} x \cdot \nabla u \Delta \bar{u} dx \\ &= n \int_{\mathbb{R}^n} [F(|u|^2)|u|^2 - G(|u|^2)]dx - n \int_{\mathbb{R}^n} |\nabla u|^2 dx + 2Re \int_{\mathbb{R}^n} x \cdot \nabla u \Delta \bar{u} dx. \end{aligned}$$

We claim that

$$\int_{\mathbb{R}^n} x \cdot \nabla u \Delta \bar{u} dx = \frac{n-2}{2} \int_{\mathbb{R}^n} |\nabla u|^2 dx,$$

which together with the above equality implies (4.9). To see the claim we use integration by parts consecutively to find

$$\int_{\mathbb{R}^n} x\cdot\nabla u\Delta\bar{u}dx = \sum_{k=1}^{n}\int_{\mathbb{R}^n} x_k u_{x_k}\sum_{j=1}^{n}\bar{u}_{x_jx_j}dx$$

$$= -\sum_{k=1}^{n}\int_{\mathbb{R}^n} u\Delta\bar{u}dx - \sum_{k=1}^{n}\int_{\mathbb{R}^n} x_k u\sum_{j=1}^{n}\bar{u}_{x_kx_jx_j}dx$$

$$= -n\int_{\mathbb{R}^n} u\Delta\bar{u}dx - \sum_{k=1}^{n}\int_{\mathbb{R}^n} x_k u\bar{u}_{x_kx_kx_k}dx$$

$$-\sum_{k=1}^{n}\sum_{\substack{j=1,\\ j\neq k}}^{n}\int_{\mathbb{R}^n} x_k u\bar{u}_{x_kx_jx_j}dx$$

$$= n\int_{\mathbb{R}^n}|\nabla u|^2dx - \sum_{k=1}^{n}\int_{\mathbb{R}^n} x_k\left[(u\bar{u}_{x_kx_k})_{x_k} - u_{x_k}\bar{u}_{x_kx_k}\right]dx$$

$$+\sum_{k=1}^{n}\sum_{\substack{j=1,\\ j\neq k}}^{n}\int_{\mathbb{R}^n} x_k u_{x_j}\bar{u}_{x_kx_j}dx$$

$$= n\int_{\mathbb{R}^n}|\nabla u|^2dx + \sum_{k=1}^{n}\int_{\mathbb{R}^n} u\bar{u}_{x_kx_k} - \frac{1}{2}\sum_{k=1}^{n}\int_{\mathbb{R}^n}|u_{x_k}|^2dx$$

$$-\frac{1}{2}\sum_{k=1}^{n}\sum_{\substack{j=1,\\ j\neq k}}^{n}\int_{\mathbb{R}^n}|u_{x_j}|^2dx$$

$$= n\int_{\mathbb{R}^n}|\nabla u|^2dx - \int_{\mathbb{R}^n}|\nabla u|^2dx - \frac{1}{2}\int_{\mathbb{R}^n}|\nabla u|^2dx$$

$$-\frac{n-1}{2}\int_{\mathbb{R}^n}|\nabla u|^2dx$$

$$= \frac{n-2}{2}\int_{\mathbb{R}^n}|\nabla u|^2dx.$$

Thus, we are done.

□

Now we state the main result.

Theorem 4.2. *Let u be a classical solution to the Cauchy problem (4.1) with $u_0(x)\in\mathcal{S}$. Also assume that $E(0)\leq 0$, $y(0)>0$, and*

$$sF(s)\geq(1+2/n+\alpha)G(s),\ \textit{for all } s\geq 0 \tag{4.11}$$

with some $\alpha>0$. Then, there exists a finite time T such that

$$\lim_{t\to T^-}\|\nabla u(t)\|_2=\infty.$$

Proof. With the assumption (4.11) and using the definition of $E(t)$ (4.7) the equation (4.9) becomes

$$y'(t) \geq -2\int_{\mathbb{R}^n} |\nabla u|^2 dx + n(2/n+\alpha)\int_{\mathbb{R}^n} G(|u|^2)dx$$
$$= -(2+n\alpha)E(0) + n\alpha\int_{\mathbb{R}^n} |\nabla u|^2 dx.$$

The assumption $E(0) \leq 0$ implies

$$y'(t) \geq n\alpha \int_{\mathbb{R}^n} |\nabla u|^2 dx \geq 0 \tag{4.12}$$

for all $t > 0$. Combining this with the assumption $y(0) > 0$ we deduce

$$y(t) \geq y(0) > 0 \qquad , \forall t \geq 0. \tag{4.13}$$

Then the identity (4.8) implies that

$$0 \leq I(t) \leq I(0) =: I_0 \qquad , \forall t \geq 0. \tag{4.14}$$

Here, note that $I_0 < \infty$ by the assumption $u_0(x) \in \mathcal{S}$. Next, by the definition of $y(t)$ (4.5) and the Hölder inequality (1.2) we have

$$y(t) = |y(t)| = |Im\int_{\mathbb{R}^n} x\cdot(\bar{u}\nabla u)dx| \leq |\int_{\mathbb{R}^n} x\cdot(\bar{u}\nabla u)dx|$$
$$\leq \int_{\mathbb{R}^n} |x||u||\nabla u|dx \leq \left\{\int_{\mathbb{R}^n} |x|^2|u|^2 dx\right\}^{\frac{1}{2}}\left\{\int_{\mathbb{R}^n} |\nabla u|^2 dx\right\}^{\frac{1}{2}},$$

which together with the definition of $I(t)$ (4.4) and the inequality (4.14) implies that

$$y^2(t) \leq I(t)\int_{\mathbb{R}^n} |\nabla u|^2 dx \leq I_0 \int_{\mathbb{R}^n} |\nabla u|^2 dx. \tag{4.15}$$

Now combining (4.12) and (4.15) we get

$$y'(t) \geq \frac{n\alpha}{I_0} y^2(t),$$

or

$$\frac{dy}{y^2(t)} \geq \frac{n\alpha}{I_0} dt.$$

Integrating over $(0,t)$ we obtain

$$-\frac{1}{y(t)} + \frac{1}{y(0)} \geq \frac{n\alpha}{I_0} t,$$

which makes sense since by (4.13) $y(t) > 0$ for all $t \geq 0$. Therefore, we have

$$y(t) \geq \frac{y(0)I_0}{I_0 - n\alpha y(0)t}.$$

Finally, using (4.15) we conclude that

$$\|\nabla u(t)\|_2 = \left\{ \int_{\mathbb{R}^n} |\nabla u|^2 dx \right\}^{\frac{1}{2}} \geq \frac{y(0)I_0^{\frac{1}{2}}}{I_0 - n\alpha y(0)t}.$$

Thus, the solution blows up as $t \to T^-$ for some $T \leq T_0 = \frac{I_0}{n\alpha y(0)}$. This completes the proof.

□

Remark 4.3. In applications the function $F(s)$ has the form

$$F(s) = \kappa s^{\frac{p-1}{2}}.$$

When $p = 3$ we are getting the famous cubic nonlinear Schrödinger equation describing the effect of self-focusing in nonlinear optics. In this case the energy integral is

$$E(t) = \int_{\mathbb{R}^n} \left[|\nabla u(x,t)|^2 - \frac{2\kappa}{p+1} |u(x,t)|^{p+1} \right] dx \equiv E(0). \tag{4.16}$$

Since $\|u(t)\|_2$ is uniformly bounded we can use the Gagliardo-Nirenberg inequality (1.7), and get the estimate

$$\int_{\mathbb{R}^n} |u(x,t)|^{p+1} dx \leq C \|\nabla u(t)\|_2^{\frac{n(p-1)}{2}}.$$

Thus, (4.16) implies

$$\|\nabla u(t)\|_2^2 \leq |E(0)| + \kappa C \|\nabla u(t)\|_2^{\frac{n(p-1)}{2}}.$$

From this inequality we can obtain uniform estimate for

$$\|\nabla u(t)\|_2$$

provided

$$p < 1 + \frac{4}{n}.$$

4.2 A damped nonlinear Schrödinger equation

Consider the Cauchy problem

$$iu_t(x,t) = \Delta u(x,t) + F(|u^2(x,t)|)u(x,t) - i\frac{a}{2}u(x,t), \qquad x \in \mathbb{R}^n,\ t > 0, \tag{4.17}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n \tag{4.18}$$

with $a > 0$.

We shall assume that $u \in \mathcal{S}$, where $\mathcal{S}$ is the Schwarz class. Moreover we assume that F is real-valued, positive and is smooth enough so that a unique local classical solution to (4.17), (4.18) exists. We define

$$G(u) := \int_0^u F(s)ds, \tag{4.19}$$

$$E(t) := \int_{\mathbb{R}^n} \left[|\nabla u|^2 - G(|u|^2)\right] dx, \tag{4.20}$$

$$I(t) := e^{bt} \int_{\mathbb{R}^n} |x|^2|u|^2 dx, \tag{4.21}$$

$$y(t) := -4Im \int_{\mathbb{R}^n} x \cdot (\bar{u}\nabla u)dx, \tag{4.22}$$

and

$$\omega(t) := 8 \int_{\mathbb{R}^n} |\nabla u|^2 dx - 4n \int_{\mathbb{R}^n} \left[F(|u|^2)|u|^2 - G(|u|^2)\right] dx. \tag{4.23}$$

Lemma 4.4. *If u is a local solution of (4.17) and $b \in \mathbb{R}$ then*

$$\int_{\mathbb{R}^n} |u|^2 dx = e^{-at} \int_{\mathbb{R}^n} |u_0(x)|^2 dx, \tag{4.24}$$

$$\begin{aligned} e^{bt}E(t) =& E(0) + b \int_0^t e^{b\tau} E(\tau)d\tau \\ &- a \int_0^t e^{b\tau} \left[\int_{\mathbb{R}^n} |\nabla u|^2 dx - \int_{\mathbb{R}^n} F(|u|^2)|u|^2 dx\right] d\tau, \end{aligned} \tag{4.25}$$

$$I(t) + (a-b) \int_0^t I(\tau)d\tau = I(0) + \int_0^t e^{b\tau} y(\tau)d\tau, \tag{4.26}$$

and

$$e^{bt}y(t) + (a-b) \int_0^t e^{b\tau} y(\tau)d\tau = y(0) + \int_0^t e^{b\tau}\omega(\tau)d\tau. \tag{4.27}$$

Proof. Multiplying (4.17) by $2\bar{u}$, taking the imaginary part of the result and integrating over $\mathbb{R}^n$ we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^n} |u|^2 dx + a \int_{\mathbb{R}^n} |u|^2 dx = 0$$

which implies (4.24). Next we multiply (4.17) by $2\bar{u}_t$, then take the real part of the result and integrate over $\mathbb{R}^n$. This gives

$$0 = -\frac{d}{dt}\left\{ \int_{\mathbb{R}^n} \left[|\nabla u|^2 - G(|u|^2)\right] dx \right\} - i\frac{a}{2}\int_{\mathbb{R}^n} (u\bar{u}_t - \bar{u}u_t)dx.$$

Using (4.17) again we get

$$\begin{aligned} E'(t) &= -i\frac{a}{2}\int_{\mathbb{R}^n}\left[u\left(i\Delta\bar{u} + iF(|u|^2)\bar{u} - \frac{a}{2}\bar{u}\right) - \bar{u}\left(-i\Delta u - iF(|u|^2)u - \frac{a}{2}u\right)\right]dx \\ &= -a\int_{\mathbb{R}^n}|\nabla u|^2 dx + a\int_{\mathbb{R}^n} F(|u|^2)|u|^2 dx. \end{aligned}$$

From this (4.25) follows. For the next identity observe that

$$\begin{aligned} I'(t) &= bI(t) + e^{bt}\int_{\mathbb{R}^n}|x|^2(u\bar{u}_t + \bar{u}u_t)dx \\ &= bI(t) + e^{bt}\int_{\mathbb{R}^n}|x|^2\left[u\left(i\Delta\bar{u} + iF(|u|^2)\bar{u} - \frac{a}{2}\bar{u}\right) + \bar{u}\left(-i\Delta u - iF(|u|^2)u - \frac{a}{2}u\right)\right]dx \\ &= (b-a)I(t) + ie^{bt}\int_{\mathbb{R}^n}|x|^2(u\Delta\bar{u} - \bar{u}\Delta u)dx \\ &= (b-a)I(t) + e^{bt}y(t), \end{aligned}$$

where the last equality is detailed in the calculations of the previous section. Integrating this result over $(0,t)$ gives (4.26). Finally, noting back to the calculations of the final identity in the lemma of the previous chapter we see

$$\begin{aligned} \frac{d}{dt}\left\{e^{bt}y(t)\right\} &= be^{bt}y(t) + e^{bt}\left[4nIm\int_{\mathbb{R}^n}u_t\bar{u}dx - 8Im\int_{\mathbb{R}^n}x\cdot\nabla u\bar{u}_t dx\right] \\ &= be^{bt}y(t) + e^{bt}\omega(t) + e^{bt}\left[-2anIm\int_{\mathbb{R}^n}|u|^2dx + 4a\int_{\mathbb{R}^n}x\cdot\nabla u\bar{u}dx\right] \\ &= (b-a)e^{bt}y(t) + e^{bt}\omega(t). \end{aligned}$$

Integrating this result over $(0,t)$ we find (4.27).

□

Next we show the global nonexistence of the solution to the Cauchy problem (4.17).

Theorem 4.5. *Suppose u is a solution to the Cauchy problem (4.17). Assume that*

$$E(0) \leq 0, \tag{4.28}$$

there exists constants $C_n > 1 + 2/n$, $C'_n > 1$ such that

$$C_nG(s) \leq sF(s) \leq C'_nG(s), \qquad \text{for all } s > 0, \tag{4.29}$$

and

$$\frac{a(C_n'-1)}{d_n-1}I(0)+y(0)\leq 0, \tag{4.30}$$

where $d_n=\frac{n}{2}(C_n-1)>1$. *Then* u *blows up in finite time.*

Proof. From (4.25) we have

$$e^{bt}E(t)=E(0)+b\int_0^t e^{b\tau}g(\tau)d\tau,$$

where

$$g(\tau)=-(a-b)\int_{\mathbb{R}^n}|\nabla u|^2dx-b\int_{\mathbb{R}^n}G(|u|^2)dx+a\int_{\mathbb{R}^n}F(|u|^2)|u|^2dx.$$

Using the hypothesis (4.29) and the fact that $G(s)\geq 0$ for all $s\geq 0$ we get

$$\begin{aligned}
g(\tau)&\leq -(a-b)\int_{\mathbb{R}^n}|\nabla u|^2dx+(aC_n'-b)\int_{\mathbb{R}^n}G(|u|^2)dx\\
&\leq -(a-b)\left[\int_{\mathbb{R}^n}|\nabla u|^2dx-\frac{n}{2}(C_n-1)\int_{\mathbb{R}^n}G(|u|^2)dx\right]\\
&= -(a-b)\left[\int_{\mathbb{R}^n}|\nabla u|^2dx-d_n\int_{\mathbb{R}^n}G(|u|^2)dx\right]\\
&:= -(a-b)E_1(t),
\end{aligned}$$

provided that

$$aC_n'-b\leq d_n(a-b),$$

or

$$b\leq\frac{d_n-C_n'}{d_n-1}a<a, \tag{4.31}$$

where the last inequality above is true because $C_n'>1$ and we are free to choose b so that the first inequality is also satisfied. Then we have

$$e^{bt}E(t)=E(0)-(a-b)\int_0^t e^{b\tau}E_1(\tau)d\tau.$$

Now the hypothesis (4.29) yields $d_n>1$. This together with the positivity of $G(s)$ implies $E_1(t)\leq E(t)$. Hence we get

$$e^{bt}E_1(t)\leq E(0)-(a-b)\int_0^t e^{b\tau}E_1(\tau)d\tau,$$

from which we obtain

$$\frac{d}{dt}\left[e^{(a-b)t}\int_0^t e^{b\tau}E_1(\tau)d\tau\right] \leq E(0)e^{(a-b)t}.$$

By hypothesis (4.28) $E(0) \leq 0$. Therefore we obtain

$$\int_0^t e^{b\tau}E_1(\tau)d\tau \leq 0. \tag{4.32}$$

We observe that by the definition of $\omega(t)$ (4.23) and the hypothesis (4.29) we have

$$\begin{aligned}\omega(t) &\leq 8\left\{\int_{\mathbb{R}^n}|\nabla u|^2dx - \frac{n}{2}(C_n-1)\int_{\mathbb{R}^n}G(|u|^2)dx\right\}\\ &= 8E_1(t).\end{aligned}$$

Combining this with (4.32) and (4.27) we arrive at

$$e^{bt}y(t) + (a-b)\int_0^t e^{b\tau}y(\tau)d\tau \leq y(0),$$

from which it follows that

$$\frac{d}{dt}\left[e^{(a-b)t}\int_0^t e^{b\tau}y(\tau)d\tau\right] \leq y(0)e^{(a-b)t}.$$

Hence we have

$$\int_0^t e^{b\tau}y(\tau)d\tau \leq \frac{1}{a-b}\left(1-e^{-(a-b)t}\right)y(0).$$

Above inequality together with (4.26) implies

$$I(t) \leq I(0) + \frac{1}{a-b}\left(1-e^{-(a-b)t}\right)y(0). \tag{4.33}$$

Now define

$$\delta := \frac{d_n - C_n'}{d_n-1}a - b,$$

which implies

$$a-b = \frac{C_n'-1}{d_n-1}a + \delta. \tag{4.34}$$

By (4.31) $\delta \geq 0$. Next set

$$T = -\frac{1}{a-b}\log\frac{(a-b)I(0)+y(0)}{y(0)}.$$

Observe that replacing t with T in (4.33) gives $I(t) \leq 0$. Using (4.34) we obtain

$$T = -\left(\frac{C_n'-1}{d_n-1}a+\delta\right)^{-1}\log P,$$

where

$$P = \frac{\left(\frac{C'_n-1}{d_n-1}a + \delta\right) I(0) + y(0)}{y(0)}$$
$$= \frac{\frac{a(C'_n-1)}{d_n-1} I(0) + y(0)}{y(0)} + \frac{\delta I(0)}{y(0)}.$$

The hypothesis (4.30) together with the definition of $I(t)$ (4.21) gives

$$y(0) < \frac{a(C'_n-1)}{d_n-1} I(0) + y(0) \leq 0,$$

which implies $P > 0$, and we can choose b satisfying (4.31) large enough so that δ is small enough to make $P < 1$. Hence $T > 0$ and therefore by (4.33) there exists T_1 such that $0 < T_1 \leq T$ and

$$\lim_{t \to T_1} I(t) = 0. \tag{4.35}$$

Since $u \in \mathcal{S}$ by integration by parts and the Schwarz inequality (1.1) we have

$$\int_{\mathbb{R}^n} |u|^2 dx = -\int_{\mathbb{R}^n} x_i \left(u\bar{u}_{x_i} + \bar{u}u_{x_i}\right) dx$$
$$\leq 2\left(\int_{\mathbb{R}^n} x_i^2 |u|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |u_{x_i}|^2 dx\right)^{\frac{1}{2}}$$
$$\leq 2\| |x| u(t)\|_{L^2(\mathbb{R}^n)} \|\nabla u(t)\|_{L^2(\mathbb{R}^n)}.$$

Using (4.24), the above inequality and the definition of $I(t)$ (4.21) we get

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^n)} \geq \frac{e^{-at}\|u_0\|^2_{L^2(\mathbb{R}^n)}}{2\||x|u(t)\|_{L^2(\mathbb{R}^n)}}$$
$$= \frac{e^{-(a-b)t}\|u_0\|^2_{L^2(\mathbb{R}^n)}}{2I(t)}.$$

Finally (4.35) implies

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^n)} \to +\infty \qquad as \quad t \to T_1.$$

This completes the proof. □

We conclude the study of the problem (4.17) by a theorem which states that the blow-up of the solution occurs at the origin only.

Theorem 4.6. *Suppose that the assumptions of the preceding theorem hold. Note that by the preceding theorem there exists a maximal T such that*

$$\lim_{t \to T} \int_{\mathbb{R}^n} |x|^2 |u|^2 dx = 0. \tag{4.36}$$

Then if $\max\left(\frac{2n}{n+2}, 1\right) < p < 2$

$$\lim_{t \to T} \|u(t)\|_{L^p(\mathbb{R}^n)} = 0; \tag{4.37}$$

and if $2 < p \leq \infty$ *for any* $\varepsilon > 0$

$$\lim_{t \to T} \|u(t)\|_{L^2(|x|>\varepsilon)} = 0, \tag{4.38}$$

and

$$\lim_{t \to T} \|u(t)\|_{L^p(|x|<\varepsilon)} = +\infty. \tag{4.39}$$

Proof. For the first part suppose $\max\left(\frac{2n}{n+2}, 1\right) < p < 2$. By the Hölder inequality (1.2) with $\frac{1}{\frac{2}{2-p}} + \frac{1}{\frac{2}{p}} = 1$ we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |u|^p dx &= \int_{|x|>R} |u|^p dx + \int_{|x|<R} |u|^p dx \\
&\leq \left(\int_{|x|>R} |x|^{-\frac{2p}{2-p}} dx \right)^{\frac{2-p}{2}} \left(\int_{|x|>R} |x|^2 |u|^2 \right)^{\frac{p}{2}} \\
&\quad + \left(\int_{|x|<R} 1 dx \right)^{\frac{2-p}{2}} \left(\int_{|x|<R} |u|^2 dx \right)^{\frac{p}{2}}.
\end{aligned}$$

The integral $\int_{|x|>R} |x|^{-\frac{2p}{2-p}} dx$ converges if and only if $\frac{2p}{2-p} > n$ if and only if $p > \frac{2n}{n+2}$. This is satisfied by the assumption on p. Therefore we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |u|^p dx &\leq C_1 R^{\frac{n(2-p)}{2} - p} \left(\int_{|x|>R} |x|^2 |u|^2 \right)^{\frac{p}{2}} + C_2 R^{\frac{n(2-p)}{2}} \left(\int_{|x|<R} |u|^2 dx \right)^{\frac{p}{2}} \\
&\leq C_1 R^{\frac{n(2-p)}{2} - p} \left(\int_{|x|>R} |x|^2 |u|^2 \right)^{\frac{p}{2}} + C_2 R^{\frac{n(2-p)}{2}} \left(\int_{|x|<R} |u_0(x)|^2 dx \right)^{\frac{p}{2}},
\end{aligned}$$

where $C_1, C_2 > 0$ are constants and in the last inequality we have used the equality (4.24). Now since $p < 2$, for any $\delta > 0$ we can pick R small enough so that

$$C_2 R^{\frac{n(2-p)}{2}} \left(\int_{|x|<R} |u_0(x)|^2 dx \right)^{\frac{p}{2}} < \frac{\delta}{2}.$$

With this fixed value of R by (4.36) there exists $t_0 < T$ such that whenever $t \geq t_0$ and $t < T$ we have

$$C_1 R^{\frac{n(2-p)}{2} - p} \left(\int_{|x|>R} |x|^2 |u|^2 \right)^{\frac{p}{2}} < \frac{\delta}{2}.$$

Therefore for arbitrary $\delta > 0$ there exists t with $t_0 \leq t < T$ such that

$$\int_{\mathbb{R}^n} |u(x,t)|^p dx < \delta.$$

From this (4.37) follows. For the second part we observe

$$\int_{|x|>\varepsilon} |u|^2 dx \leq \frac{1}{\varepsilon^2} \int_{\mathbb{R}^n} |x|^2 |u|^2 dx,$$

which tends to zero as $t \to T$ and this gives (4.38). Next using the Hölder inequality (1.2) in the equation (4.24) we obtain

$$\begin{aligned} e^{-at}\|u_0\|^2_{L^2(\mathbb{R}^n)} &= \|u(t)\|^2_{L^2(\mathbb{R}^n)} \\ &= \|u(t)\|^2_{L^2(|x|<\varepsilon)} + \|u(t)\|^2_{L^2(|x|>\varepsilon)} \\ &\geq \|u(t)\|^2_{L^q(|x|<\varepsilon)} \|u(t)\|^2_{L^p(|x|<\varepsilon)} + \|u(t)\|^2_{L^2(|x|>\varepsilon)}, \end{aligned} \tag{4.40}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $p, q \geq 1$. In case $n = 1, 2$ (4.37) states that $\lim_{t\to T} \|u(t)\|_{L^p(\mathbb{R}^n)} = 0$ for $1 < p < 2$, hence since $\lim_{t\to T} e^{-at}\|u_0\|^2_{L^2(\mathbb{R}^n)} > 0$ ($E(0) < 0$ implies this), (4.38) implies that for $2 < q < +\infty$

$$\lim_{t\to T} \|u(t)\|^2_{L^q(|x|<\varepsilon)} = +\infty\,. \tag{4.41}$$

If $n \geq 3$ the same argument applies for $\frac{2n}{n+2} < p < 2$, and hence (4.41) holds for $2 < q < \frac{2n}{n-2}$. But when $q > 2$ Hölder's inequality (1.2) gives

$$\|u(t)\|^q_{L^q(|x|<\varepsilon)} \leq \|u(t)\|^2_{L^2(|x|<\varepsilon)} \|u(t)\|^{q-2}_{L^\infty(|x|<\varepsilon)},$$

and $\|u(t)\|_{L^2(|x|<\varepsilon)} \leq \|u(t)\|^2_{L^2(\mathbb{R}^n)}$ is finite by (4.24). It follows that

$$\lim_{t\to T} \|u(t)\|_{L^\infty(|x|<\varepsilon)} = +\infty\,.$$

This implies (4.39) and the proof is complete.

□

Chapter 5

Nonlinear thermoelasticity equations

5.1 A nonlinear thermoelastic system in one dimension (Generalized concavity method)

Consider the Cauchy problem for the one-dimensional, nonlinear thermoelasticity equation

$$u_{tt}(x,t) = au_{xx}(x,t) + b\theta_x(x,t) + du_x(x,t) - mu_t(x,t) + f\left(t, u(x,t)\right) \tag{5.1}$$

$$c\theta_t(x,t) = K\theta_{xx}(x,t) + bu_{xt}(x,t) + pu_x(x,t) + q\theta_x(x,t) \tag{5.2}$$

with $x \in \mathbb{R}$, $t > 0$, and

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x), \quad x \in \mathbb{R}. \tag{5.3}$$

The coefficients are such that

$$a, b, c, K > 0$$

and

$$d, m, p, q \geq 0.$$

In order to show the blow-up of the solution of the problem (5.1)-(5.3), we need to make some restrictions on the function $f(t,u)$ and the initial data u_0, u_1, and θ_0. We assume that there exists a functional $F(t,u)$, such that

$$F_u(t,u) = f(t,u). \tag{5.4}$$

Also, we assume that

$$uf(t,u) \geq 2(1+2\gamma)F(t,u) \tag{5.5}$$

for some $\gamma \geq (1/4)((1+b^2/ac)^{\frac{1}{2}} - 1)$.

In addition to (5.4) and (5.5) we further require either

$$\begin{cases} F_t(t,u) \geq 2(\varepsilon_2 - m)F(t,u) & \text{if } m+p+d > 0 \\ F_t(t,u) \geq 0 & \text{if } m = p = d = 0 \end{cases} \tag{5.6}$$

with $\varepsilon_2 - m$ greater than or equal to the positive root of

$$(4amc)\xi^2 - (cd^2)\xi - p^2 m = 0,$$

or

$$\begin{cases} F_t(t,u) \geq 2(\tilde{\varepsilon}_2 - m)F(t,u) + \frac{p^2}{4K}u^2 & \text{if } m+d > 0 \\ F_t(t,u) \geq \frac{p^2}{4K}u^2 & \text{if } m = d = 0 \end{cases} \tag{5.7}$$

with $\tilde{\varepsilon}_2 \geq m + \frac{d}{2a}$.

As for the initial data we require

$$u_0 \in H^2(\mathbb{R}), \quad u_1 \in H^1(\mathbb{R}), \quad \theta_0 \in H^1(\mathbb{R}), \tag{5.8}$$

and

$$\int_{-\infty}^{+\infty} u_0^2 dx \leq \int_{-\infty}^{+\infty} u_0 u_1 dx \tag{5.9}$$

Before stating the blow-up theorem let us multiply the first equation of (5.1) by u_t and the second equation by θ, then add them and integrate over $\mathbb{R}$. We have

$$\int_{-\infty}^{+\infty} \left[u_t u_{tt} - a u_t u_{xx} - d u_t u_x + m u_t^2 - u_t f(t,u)\right] dx$$
$$+ \int_{-\infty}^{+\infty} \left[c\theta\theta_t - K\theta\theta_{xx} - p\theta u_x - q\theta\theta_x\right] dx = 0.$$

In view of (5.4) and using integration by parts we obtain

$$\int_{-\infty}^{+\infty}\left[mu_t^2+K\theta_x^2+F_t(t,u)-du_tu_x-p\theta u_x\right]dx$$
$$+\frac{d}{dt}\int_{-\infty}^{+\infty}\left[\frac{1}{2}u_t^2+\frac{a}{2}u_x^2-F(t,u)+\frac{c}{2}\theta^2\right]dx=0.$$

If we define

$$E(t):=\int_{-\infty}^{+\infty}\left[\frac{1}{2}u_t^2+\frac{a}{2}u_x^2-F(t,u)+\frac{c}{2}\theta^2\right]dx, \tag{5.10}$$

then the above equality can be rewritten as

$$E'(t)=\int_{-\infty}^{+\infty}\left[-mu_t^2-K\theta_x^2-F_t(t,u)+du_tu_x+p\theta u_x\right]dx \tag{5.11}$$

Now we state the blow-up theorem and give its proof.

Theorem 5.1. *Suppose that u_0, u_1, and θ_0 satisfy (5.8), (5.9), and are such that $E(0)<0$. Also suppose that $f(t,u)$ satisfies either (5.4), (5.5), (5.6) or (5.4), (5.5), (5.7). Then, there exists a positive number $T<+\infty$ such that*

$$\lim_{t\to T^-}\int_{-\infty}^{+\infty}u^2(x,t)dx=+\infty\,.$$

Proof. Let $\Psi(t)=\int_{-\infty}^{+\infty}u^2(x,t)dx+\beta(t+t_0)^2$, where positive β and t_0 are to be determined later. Then

$$\Psi'(t)=2\left\{\int_{-\infty}^{+\infty}uu_tdx+\beta(t+t_0)\right\}$$

and

$$\Psi''(t)=2\left\{\int_{-\infty}^{+\infty}(u_t^2+uu_{tt})dx+\beta\right\}.$$

Therefore, we have

$$\Psi\Psi''-(1+\gamma)\Psi'^2=2\Psi\left\{\int_{-\infty}^{+\infty}(u_t^2+uu_{tt})dx+\beta\right\}-4(1+\gamma)\left\{\int_{-\infty}^{+\infty}uu_tdx+\beta(t+t_0)\right\}^2.$$

We apply the Schwarz inequality (1.1) to the second term on the right hand side twice (first in L^2, second in $\mathbb{R}^2$) in the following way

$$\left\{\int_{-\infty}^{+\infty} uu_t dx + \beta(t+t_0)\right\}^2 \le \left\{\left(\int_{-\infty}^{+\infty} u^2 dx\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty} u_t^2 dx\right)^{\frac{1}{2}} + \sqrt{\beta}(t+t_0)\sqrt{\beta}\right\}^2$$

$$\le \left\{\int_{-\infty}^{+\infty} u^2(x,t)dx + \beta(t+t_0)^2\right\}\left\{\int_{-\infty}^{+\infty} u_t^2 dx + \beta\right\}$$

$$= \Psi\left\{\int_{-\infty}^{+\infty} u_t^2 dx + \beta\right\}.$$

Hence, we arrive at

$$\Psi\Psi'' - (1+\gamma)\Psi'^2 \ge 2\Psi\left\{-(1+2\gamma)\left(\int_{-\infty}^{+\infty} u_t^2 dx + \beta\right) + \int_{-\infty}^{+\infty} uu_{tt} dx\right\}. \tag{5.12}$$

Next, we multiply the first equation of (5.1) by u, integrate over $\mathbb{R}$ and use integration by parts to get

$$\int_{-\infty}^{+\infty} uu_{tt} dx = -a\int_{-\infty}^{+\infty} u_x^2 dx - b\int_{-\infty}^{+\infty} u_x\theta dx - m\int_{-\infty}^{+\infty} uu_t dx + \int_{-\infty}^{+\infty} uf(t,u)dx$$

$$\ge -a\int_{-\infty}^{+\infty} u_x^2 dx - b\int_{-\infty}^{+\infty} u_x\theta dx - \frac{m}{2}\Psi' + \int_{-\infty}^{+\infty} uf(t,u)dx$$

Combining this with (5.12) we obtain

$$\Psi\Psi'' - (1+\gamma)\Psi'^2 \ge 2\Psi\Bigg\{-(1+2\gamma)\left(\int_{-\infty}^{+\infty} u_t^2 dx + \beta\right) - a\int_{-\infty}^{+\infty} u_x^2 dx$$

$$-b\int_{-\infty}^{+\infty} u_x\theta dx - \frac{m}{2}\Psi' + \int_{-\infty}^{+\infty} uf(t,u)dx\Bigg\}.$$

Next, we apply the ε-inequality to the term $\int_{-\infty}^{+\infty} u_x\theta dx$

$$\Psi\Psi'' - (1+\gamma)\Psi'^2 \ge 2\Psi\Bigg\{-(1+2\gamma)\left(\int_{-\infty}^{+\infty} u_t^2 dx + \beta\right) - \left(a + \frac{b^2}{4\varepsilon_1}\right)\int_{-\infty}^{+\infty} u_x^2 dx$$

$$-\varepsilon_1\int_{-\infty}^{+\infty} \theta^2 dx - \frac{m}{2}\Psi' + \int_{-\infty}^{+\infty} uf(t,u)dx\Bigg\},$$

and then use the definition of $E(t)$, (5.10) to obtain

$$\Psi\Psi'' - (1+\gamma)\Psi'^2 \ge 2\Psi\Bigg\{-2(1+2\gamma)E(t) + \left(2a\gamma - \frac{b^2}{4\varepsilon_1}\right)\int_{-\infty}^{+\infty} u_x^2 dx - \frac{m}{2}\Psi'$$

$$+(c(1+2\gamma)-\varepsilon_1)\int_{-\infty}^{+\infty} \theta^2 dx - \beta(1+2\gamma)$$

$$+\int_{-\infty}^{+\infty}\left[uf(t,u) - 2(1+2\gamma)F(t,u)\right]dx\Bigg\}.$$

Now, we pick $\varepsilon_1 = \frac{b^2}{8a\gamma}$ so that the coefficient of $\int_{-\infty}^{+\infty} u_x^2 dx$ vanishes. Also applying assumption (5.5), and noting that the restriction on γ in (5.5) together with this picked value of ε_1 implies $c(1+2\gamma) - \varepsilon_1 \ge 0$ we remain with

$$\Psi\Psi'' - (1+\gamma)\Psi'^2 \geq 2\Psi\left\{-2(1+2\gamma)E(t) - \frac{m}{2}\Psi' - \beta(1+2\gamma)\right\}.$$

Now, suppose we have shown that $E(t) \leq 0$ for all $t \geq 0$. Then we have

$$\Psi\Psi'' - (1+\gamma)\Psi'^2 \geq 2\Psi\left\{-(1+2\gamma)(2E(0)+\beta) - \frac{m}{2}\Psi'\right\}.$$

We choose $\beta \leq -2E(0) > 0$ and we remain with

$$\Psi\Psi'' - (1+\gamma)\Psi'^2 \geq -m\Psi\Psi'.$$

This is the inequality (2.3) in Lemma 1 with $C_1 = \frac{m}{2}$, $C_2 = 0$, $\gamma_1 = 0$ and $\gamma_2 = -m$. Since β and t_0 are positive $\Psi(0) = \int_{-\infty}^{+\infty} u_0^2 dx + \beta t_0^2 > 0$. If $m = 0$ we choose t_0 so large that $\Psi'(0) = 2\left\{\int_{-\infty}^{+\infty} u_0 u_1 dx + \beta t_0\right\} > 0$ (This is possible by assumption (5.8)), so that the requirements of the second part of Lemma 1 are fullfilled and we have $\lim_{t\to T^-}\int_{-\infty}^{+\infty} u^2(x,t)dx = +\infty$ for $T = \frac{\Psi(0)}{\gamma\Psi'(0)}$. If $m \neq 0$ in order to invoke the first part of the Lemma 1 we require $\Psi'(0) - \frac{m}{\gamma}\Psi(0) > 0$, that is

$$\int_{-\infty}^{+\infty} u_0^2 dx + \beta t_0^2 < \frac{2\gamma}{m}\left(\int_{-\infty}^{+\infty} u_0 u_1 dx + \beta t_0\right),$$

or

$$(\beta)t_0^2 - \left(\frac{2\gamma\beta}{m}\right)t_0 + \left(\int_{-\infty}^{+\infty} u_0^2 dx - \frac{2\gamma}{m}\int_{-\infty}^{+\infty} u_0 u_1 dx\right) < 0.$$

The assumption (5.9) ensures that the parabola in the above inequality has a positive and a non-positive root, so that a positive t_0 satisfying the inequality exists. Thus, by the first part of the Lemma 1 $\lim_{t\to T^-}\int_{-\infty}^{+\infty} u^2(x,t)dx = +\infty$ for $T = \frac{1}{m}\ln\left(\frac{\gamma\Psi'(0)}{\gamma\Psi'(0)-m\Psi(0)}\right)$.

There only remains to show that $E(t) \leq 0$ for all $t \geq 0$. We consider two cases.

When (5.6) is satisfied we use the ε-inequality in the equation (5.11) and then the definition of $E(t)$ (5.10) to obtain

$$
\begin{aligned}
E'(t) = &\int_{-\infty}^{+\infty} \left[-mu_t^2 - K\theta_x^2 - F_t(t,u) + du_t u_x + p\theta u_x\right] dx \\
\leq &(\varepsilon_2 - m) \int_{-\infty}^{+\infty} u_t^2 dx - \int_{-\infty}^{+\infty} F_t(t,u) dx + \varepsilon_3 \int_{-\infty}^{+\infty} \theta^2 dx \\
&+ \frac{1}{4}\left(\frac{d^2}{\varepsilon_2} + \frac{p^2}{\varepsilon_3}\right) \int_{-\infty}^{+\infty} u_x^2 dx \\
= &2(\varepsilon_2 - m)E(t) + \left\{\frac{1}{4}\left(\frac{d^2}{\varepsilon_2} + \frac{p^2}{\varepsilon_3}\right) - a(\varepsilon_2 - m)\right\} \int_{-\infty}^{+\infty} u_x^2 dx \\
&+ \int_{-\infty}^{+\infty} \left[2(\varepsilon_2 - m)F(t,u) - F_t(t,u)\right] dx \\
&+ \left[\varepsilon_3 - c(\varepsilon_2 - m)\right] \int_{-\infty}^{+\infty} \theta^2 dx.
\end{aligned}
\tag{5.13}
$$

1. If $m = p = d = 0$, then it is easy to see that (5.6) together with the equality (5.11) implies that $E'(t) \leq 0$ for all $t \geq 0$. Therefore the assumption $E(0) \leq 0$ implies that $E(t) \leq 0$ for all $t \geq 0$.

2. If $m = 0$ and $p + d > 0$ choose $\varepsilon_3 = c\varepsilon_2$ and pick ε_2 so that $\frac{1}{4}\left(d^2/\varepsilon_2 + p^2/\varepsilon_3\right) - a(\varepsilon_2 - m) \leq 0$. Clearly $\varepsilon_2 \geq \left((cd^2 + p^2)/4ac\right)^{\frac{1}{2}}$ will do. Now (5.13) and (5.6) imply that $E'(t) \leq 2\varepsilon_2 E(t)$ and again since $E(0) \leq 0$ we have $E(t) \leq 0$ for all $t \geq 0$.

3. If $m \neq 0$, then choose $\varepsilon_3 = c(\varepsilon_2 - m)$ and choose ε_2 such that $\varepsilon_2 - m > 0$ and

$$\frac{1}{4}\left(\frac{d^2}{\varepsilon_2} + \frac{p^2}{c(\varepsilon_2 - m)}\right) - a(\varepsilon_2 - m) \leq 0.$$

It is enough to have

$$\frac{1}{4}\left(\frac{d^2}{\varepsilon_2 - m} + \frac{p^2}{c(\varepsilon_2 - m)}\right) - a(\varepsilon_2 - m) \leq 0,$$

which is satisfied if we choose $\varepsilon_2 \geq m + \left((cd^2 + p^2)/4ac\right)^{\frac{1}{2}}$. Then again by (5.6), (5.13) gives $E'(t) \leq 2\varepsilon_2 E(t)$ and again since $E(0) \leq 0$ we have $E(t) \leq 0$ for all $t \geq 0$.

For the second case if (5.7) is satisfied we use the ε-inequality in a different way

$$
\begin{aligned}
E'(t) = &\int_{-\infty}^{+\infty} \left[-mu_t^2 - K\theta_x^2 - F_t(t,u) + du_t u_x + p\theta u_x\right] dx \\
\leq &2(\tilde{\varepsilon}_2 - m)E(t) + \left[\frac{d^2}{4\tilde{\varepsilon}_2} - a(\tilde{\varepsilon}_2 - m)\right] \int_{-\infty}^{+\infty} u_x^2 dx \\
&+ (\varepsilon_3 - K) \int_{-\infty}^{+\infty} \theta_x^2 dx - (\tilde{\varepsilon}_2 - m) \int_{-\infty}^{+\infty} \theta^2 dx \\
&+ \int_{-\infty}^{+\infty} \left\{2(\tilde{\varepsilon}_2 - m)F(t,u) + \frac{p^2}{4\varepsilon_3}u^2 - F_t(t,u)\right\} dx.
\end{aligned}
\tag{5.14}
$$

We choose $\varepsilon_3 = K$ and $\tilde{\varepsilon}_2 - m > 0$ such that $\frac{d^2}{4\tilde{\varepsilon}_2} - a(\tilde{\varepsilon}_2 - m) \leq 0$. For this it is enough to have $\frac{d^2}{4(\tilde{\varepsilon}_2 - m)} - a(\tilde{\varepsilon}_2 - m) \leq 0$ which is satisfied if $\tilde{\varepsilon}_2 \geq m + \frac{d}{2a}$. Then with the assumption (5.7) (5.14) yields $E'(t) \leq (\tilde{\varepsilon}_2 - m)E(t)$. Hence, since $E(0) \leq 0$, we deduce that $E(t) \leq 0$ for all $t \geq 0$. This completes the proof.

□

5.2 A multidimensional nonlinear system thermoelastic type

Consider the problem

$$\begin{aligned} u_{tt}(x,t) = \Delta u(x,t) &+ \vec{b}\cdot\nabla\theta(x,t) + \vec{D}\cdot\nabla u(x,t) \\ &- mu_t + e^{\beta t}u(x,t)|u|^{p-2}(x,t), \quad x\in\Omega,\ t\geq 0, \end{aligned} \tag{5.15}$$

$$c\theta_t(x,t) = div\left(\nabla\theta(x,t) + \vec{b}u_t(x,t)\right) + \vec{R}\cdot\nabla u(x,t), \quad x\in\Omega,\ t\geq 0, \tag{5.16}$$

$$u(x,t) = 0, \qquad \theta(x,t) = 0, \qquad x\in\partial\Omega, \quad t\geq 0, \tag{5.17}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad \theta(x,0) = \theta_0(x), \qquad x\in\Omega. \tag{5.18}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ $(n\geq 1)$; $\vec{b}\neq 0$, $\vec{D}$, $\vec{R}$ are constant vectors in $\mathbb{R}^n$; $c,m>0$, $p>2$, $\beta>0$ are given numbers; and u_0, u_1, θ_0 are given initial functions. We have got the following result about the blow-up of the solutions of the problem (5.15)-(5.18).

Theorem 5.2. *Suppose that*

$$2<p\leq\frac{2n}{(n-2)} \text{ for } n\geq 3 \text{ and } p>2 \text{ when } n\leq 2.$$

Assume also that

$$\beta\geq\sqrt{\frac{(cd^2+r^2)}{c}}.$$

Then, for each $T>0$ there exists $\lambda>0$ such that if the initial conditions are such that

$$E(0) = \frac{1}{2}\left(\int_\Omega u_1^2dx + \int_\Omega|\nabla u_0|^2dx + c\int_\Omega\theta_0^2dx\right) - \frac{1}{p}\int_\Omega|u_0|^pdx < -\lambda,$$

then the solution of (5.15)-(5.18) blows up at a time $T^*\leq T$.

Suppose that (u,θ) is a classical solution to the problem (5.15)-(5.18). Let us multiply (5.15) and (5.16) by u_t and θ, respectively; add them up and integrate the obtained relation over Ω. Using integration by parts we find

$$\begin{aligned} 0 =& \frac{d}{dt}\left[\frac{1}{2}\left(\int_\Omega u_t^2dx + \int_\Omega|\nabla u|^2dx + c\int_\Omega\theta^2dx\right) - \frac{1}{p}e^{\beta t}\int_\Omega|u|^pdx\right] \\ &+ \frac{\beta}{p}e^{\beta t}\int_\Omega|u|^pdx + m\int_\Omega u_t^2dx - \int_\Omega u_t\vec{D}\cdot\nabla udx - \int_\Omega\theta\vec{R}\cdot\nabla udx + \int_\Omega|\nabla\theta|^2dx. \end{aligned} \tag{5.19}$$

The derivative terms in the above equation motivates us in defining the energy as

$$E(t) := \frac{1}{2}\left(\int_\Omega u_t^2dx + \int_\Omega|\nabla u|^2dx + c\int_\Omega\theta^2dx\right) - \frac{1}{p}e^{\beta t}\int_\Omega|u|^pdx. \tag{5.20}$$

In the following lemma, we show that if $E(0) \le 0$, then $E(t) \le 0$ for all $t \ge 0$.

Lemma 5.3. *If* $E(0) \le 0$ *and*

$$\beta \ge \sqrt{\frac{cd^2 + r^2}{c}}. \tag{5.21}$$

Then

$$E'(t) \le -\int_\Omega |\nabla\theta|^2 dx. \tag{5.22}$$

Proof. From (5.19) and the definition of $E(t)$ (5.20) we have

$$E'(t) = -\frac{\beta}{p} e^{\beta t} \int_\Omega |u|^p dx - m \int_\Omega u_t^2 dx + \int_\Omega u_t \vec{D} \cdot \nabla u dx + \int_\Omega \theta \vec{R} \cdot \nabla u dx - \int_\Omega |\nabla\theta|^2 dx.$$

It is easy too see that

$$\begin{aligned}
\left| \int_\Omega u_t \vec{D} \cdot \nabla u dx \right| &\le \left(\int_\Omega (u_t)^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega \left|\vec{D} \cdot \nabla u\right|^2 dx \right)^{\frac{1}{2}} \\
&\le \left(\int_\Omega (u_t)^2 \, dx \right)^{\frac{1}{2}} \left|\vec{D}\right| \left(\int_\Omega |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \\
&\le \varepsilon_1 \int_\Omega (u_t)^2 \, dx + \frac{\left|\vec{D}\right|^2}{4\varepsilon_1} \int_\Omega |\nabla u|^2 \, dx,
\end{aligned}$$

and similarly

$$\left| \int_\Omega \theta \vec{R} \cdot \nabla u dx \right| \le \varepsilon_2 \int_\Omega \theta^2 dx + \frac{\left|\vec{R}\right|^2}{4\varepsilon_2} \int_\Omega |\nabla u|^2 \, dx.$$

Using these in the above equality we obtain

$$\begin{aligned}
E'(t) \le &-\frac{\beta}{p} e^{\beta t} \int_\Omega |u|^p dx + (\varepsilon_1 - m) \int_\Omega u_t^2 dx + \left(\frac{d^2}{4\varepsilon_1} + \frac{r^2}{4\varepsilon_2} \right) \int_\Omega |\nabla u|^2 dx \\
&+ \varepsilon_2 \int_\Omega \theta^2 dx - \int_\Omega |\nabla\theta|^2 dx.
\end{aligned}$$

Here $d = \left|\vec{D}\right|$ and $r = \left|\vec{R}\right|$. By using the definition of $E(t)$ we can rewrite the last inequality in the following form:

$$\begin{aligned}
E'(t) \le &-\int_\Omega |\nabla\theta|^2 dx + 2(\varepsilon_1 - m)E(t) + [\varepsilon_2 - c(\varepsilon_1 - m)] \int_\Omega \theta^2 dx \\
&+ \left[\left(\frac{d^2}{4\varepsilon_1} + \frac{r^2}{4\varepsilon_2} \right) - (\varepsilon_1 - m) \right] \int_\Omega |\nabla u|^2 dx + \frac{1}{p} [2(\varepsilon_1 - m) - \beta] e^{\beta t} \int_\Omega |u|^p dx.
\end{aligned}$$

We pick $\varepsilon_2 = c(\varepsilon_1 - m)$ and ε_1 large enough so that

$$\left(\frac{d^2}{4\varepsilon_1} + \frac{r^2}{4c(\varepsilon_1 - m)} \right) - (\varepsilon_1 - m) \le 0.$$

It is enough to have

$$\left(\frac{d^2}{4(\varepsilon_1 - m)} + \frac{r^2}{4c(\varepsilon_1 - m)}\right) - a(\varepsilon_1 - m) \leq 0,$$

or equivalently

$$\varepsilon_1 \geq m + \sqrt{\frac{(cd^2 + r^2)}{4c}}.$$

Using the assumption (5.21) we remain with

$$E'(t) \leq -\int_\Omega |\nabla\theta|^2 dx + 2(\varepsilon_1 - m)E(t). \tag{5.23}$$

From this it follows that

$$E(t) \leq \left[E(0) - \int_0^t \int_\Omega |\nabla\theta(x,\tau)|^2 e^{-2(\varepsilon_1 - m)\tau} dx d\tau\right] e^{2(\varepsilon_1 - m)\tau}.$$

Since $E(0) \leq 0$, we deduce that $E(t) \leq 0$ for all $t \geq 0$. From this and the differential inequality 5.23 we deduce that

$$E'(t) \leq -\int_\Omega |\nabla\theta|^2 dx,$$

and this completes the proof. □

Lemma 5.4. *Suppose $p \leq \frac{2n}{(n-2)}$. Then there exists a constant $C' \geq 1$, depending only on n, p and the domain, such that*

$$\|u(t)\|_p^s \leq C'\left(\|\nabla u(t)\|_2^2 + \|u(t)\|_p^p\right) \tag{5.24}$$

for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq p$.

Proof. If $\|u(t)\|_p \leq 1$, then $\|u(t)\|_p^s \leq \|u(t)\|_p^2$. Also, since $p \leq \frac{2n}{(n-2)}$ and Ω is bounded, we have $\|u(t)\|_p^2 \leq C_0\|u(t)\|_{\frac{2n}{n-2}}^2$. Finally, the Sobolev embedding theorem (1.8) states $\|u(t)\|_{\frac{2n}{n-2}}^2 \leq C_1\|\nabla u(t)\|_2^2$. Therefore, combining these results we obtain

$$\|u(t)\|_p^s \leq C\|\nabla u(t)\|_2^2 \leq C\left(\|\nabla u(t)\|_2^2 + \|u(t)\|_p^p\right),$$

where $C = C_0C_1$. If $\|u(t)\|_p \geq 1$, then

$$\|u(t)\|_p^s \leq \|u(t)\|_p^p \leq \left(\|\nabla u(t)\|_2^2 + \|u(t)\|_p^p\right).$$

Thus, for $C' \geq \max(1, C)$ (5.24) holds, and we are done.

□

Now, if we introduce for the term $\|\nabla u(t)\|_2^2$ in the inequality found in the above lemma the definition of $E(t)$ (5.20), we find

$$\|u(t)\|_p^s \le C'\left(2E(t) - \|u_t(t)\|_2^2 - 2c\|\theta(t)\|_2^2 + \left(\frac{2}{p}e^{\beta t} + 1\right)\|u(t)\|_p^p\right)$$

If we assume that the assumptions of the Lemma 5.3 hold, then $E(t) \le 0$ for all $t \ge 0$, and the above inequality becomes

$$\|u(t)\|_p^s \le C'\left(\frac{2}{p}e^{\beta t} + 1\right)\|u(t)\|_p^p \tag{5.25}$$

for any $u \in H_0^1(\Omega)$ and $2 \le s \le p$.

Now, we are ready to give the proof of the blow-up theorem.

Proof of the theorem. Set $H(t) = -E(t)$. Then (5.22) becomes

$$H'(t) \ge \int_\Omega |\nabla\theta|^2 dx, \tag{5.26}$$

and (5.20) implies

$$\lambda < -E(0) = H(0) \le H(t) \le \frac{1}{p}e^{\beta t}\|u(t)\|_p^p. \tag{5.27}$$

We define

$$L(t) := H^{1-\alpha}(t) + \varepsilon\int_\Omega uu_t dx + \frac{m\varepsilon}{2}\int_\Omega u^2 dx \tag{5.28}$$

where $\varepsilon > 0$ is arbitrarily small and to be bounded from above later and $\alpha = \frac{p-2}{2p} < 1$. Our aim is to obtain a blowing up ODE of $L(t)$, and from there we will deduce the blow-up of the negative energy, $H(t)$. So let us differentiate $L(t)$ and then use (5.15). We have

$$\begin{aligned} L'(t) =& (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_\Omega uu_{tt}dx + \varepsilon\int_\Omega u_t^2 dx + m\varepsilon\int_\Omega uu_t dx \\ =& (1-\alpha)H^{-\alpha}(t)H'(t) - \varepsilon\int_\Omega |\nabla u|^2 dx + \varepsilon\int_\Omega u_t^2 dx \\ &+ \varepsilon e^{\beta t}\int_\Omega |u|^p dx + \varepsilon\int_\Omega u\vec{b}\cdot\nabla\theta dx. \end{aligned}$$

We replace the term $\varepsilon e^{\beta t}\int_\Omega |u|^p dx$ with its value from the definition of $H(t)$.

$$\begin{aligned} L'(t) =& (1-\alpha)H^{-\alpha}(t)H'(t) - \varepsilon\int_\Omega |\nabla u|^2 dx + \varepsilon\int_\Omega u_t^2 dx \\ &+ p\varepsilon\left[H(t) + \frac{1}{2}\left(\int_\Omega u_t^2 dx + \int_\Omega |\nabla u|^2 dx + c\int_\Omega \theta^2 dx\right)\right] \\ &+ \varepsilon\int_\Omega u\vec{b}\cdot\nabla\theta dx. \end{aligned}$$

Next, we use the ε-inequality to estimate the term $\varepsilon \int_\Omega u\vec{b}\cdot\nabla\theta dx$ to obtain for all $\delta > 0$

$$\begin{aligned}
L'(t) \geq & (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon(\frac{p}{2}-1)\int_\Omega |\nabla u|^2 dx \\
& + \varepsilon(\frac{p}{2}+1)\int_\Omega u_t^2 dx + p\varepsilon H(t) + \frac{cp\varepsilon}{2}\int_\Omega \theta^2 dx \\
& - \varepsilon\delta\int_\Omega u^2 dx - \frac{b\varepsilon}{4\delta}\int_\Omega |\nabla\theta|^2 dx \\
\leq & \left[(1-\alpha)H^{-\alpha}(t) - \frac{b\varepsilon}{4\delta}\right]\|\nabla\theta(t)\|_2^2 + \varepsilon(\frac{p}{2}-1)\|\nabla u(t)\|_2^2 \\
& + \varepsilon(\frac{p}{2}+1)\|u_t(t)\|_2^2 + p\varepsilon H(t) + \frac{cp\varepsilon}{2}\|\theta(t)\|_2^2 - b\varepsilon\delta\|u(t)\|_2^2
\end{aligned}$$

where $b = \|\vec{b}\|$,and in the second step we have used (5.26). Now we pick $\delta = \frac{H^\alpha(t)}{M} > 0$, where M large is to be selected. We get

$$\begin{aligned}
L'(t) \geq & \left[(1-\alpha) - \frac{bM\varepsilon}{4}\right]H^{-\alpha}(t)\|\nabla\theta(t)\|_2^2 + \varepsilon(\frac{p}{2}-1)\|\nabla u(t)\|_2^2 \\
& + \varepsilon(\frac{p}{2}+1)\|u_t(t)\|_2^2 + p\varepsilon H(t) + \frac{cp\varepsilon}{2}\|\theta(t)\|_2^2 - \frac{b\varepsilon}{M}H^\alpha(t)\|u(t)\|_2^2.
\end{aligned} \tag{5.29}$$

By the L^p embedding theorem (1.5), and (5.27), respectively, we have the estimate

$$\begin{aligned}
H^\alpha(t)\|u(t)\|_2^2 \leq & CH^\alpha(t)\|u(t)\|_p^2 \\
\leq & \frac{C}{p^\alpha}e^{\alpha\beta t}\|u(t)\|_p^{2+\alpha p} \\
\leq & K_T\|u(t)\|_p^{2+\alpha p},
\end{aligned}$$

where $K_T = \frac{C}{p^\alpha}e^{\alpha\beta T}$ (T is the given positive number in the statement of the theorem.). Now notice that $s = 2+\alpha p = 2 + \frac{p-2}{2} = \frac{p}{2}+1 < p$, because $p > 2$, which allows us to use the inequality (5.25) to obtain from the above inequality the following one:

$$\begin{aligned}
H^\alpha(t)\|u(t)\|_2^2 \leq & \kappa_T\|u(t)\|_p^p \\
\leq & \kappa_T\left(H(t) + \|u_t(t)\|_2^2 + \|u(t)\|_p^p + \|\theta(t)\|_2^2\right),
\end{aligned}$$

where $\kappa_T \geq K_T C'\left(\frac{2}{p}e^{\beta T}+1\right)$. Inserting this into equation (5.29) we get

$$\begin{aligned}
L'(t) \geq & \left[(1-\alpha) - \frac{bM\varepsilon}{4}\right]H^{-\alpha}(t)\|\nabla\theta(t)\|_2^2 + \varepsilon(\frac{p}{2}-1)\|\nabla u(t)\|_2^2 \\
& + \varepsilon(\frac{p}{2}+1)\|u_t(t)\|_2^2 + p\varepsilon H(t) + \frac{cp\varepsilon}{2}\|\theta(t)\|_2^2 \\
& - \frac{b\varepsilon}{M}\kappa_T\left(H(t) + \|u_t(t)\|_2^2 + \|u(t)\|_p^p + \|\theta(t)\|_2^2\right).
\end{aligned}$$

Next, we split the term $p\varepsilon H(t)$ as $\varepsilon(p-\mu)H(t) + \varepsilon\mu H(t)$ for $\mu > 0$, and then use the definition of $H(t) = -E(t)$ (5.20) to obtain

$$\begin{aligned}
L'(t) \geq & \left[(1-\alpha) - \frac{bM\varepsilon}{4}\right] H^{-\alpha}(t)\|\nabla\theta(t)\|_2^2 + \varepsilon(\frac{p}{2} - 1 - \frac{\mu}{2})\|\nabla u(t)\|_2^2 \\
& + \varepsilon(\frac{p}{2} + 1 - \frac{\mu}{2})\|u_t(t)\|_2^2 + \varepsilon(p-\mu)H(t) + c\varepsilon\left(\frac{p}{2} - \frac{\mu}{2}\right)\|\theta(t)\|_2^2 + \frac{\varepsilon\mu}{p}e^{\beta t}\|u(t)\|_p^p \\
& - \frac{b\varepsilon}{M}\kappa_T\left(H(t) + \|u_t(t)\|_2^2 + \|u(t)\|_p^p + \|\theta(t)\|_2^2\right) \\
\geq & \left[(1-\alpha) - \frac{bM\varepsilon}{4}\right] H^{-\alpha}(t)\|\nabla\theta(t)\|_2^2 + \varepsilon(\frac{p}{2} - 1 - \frac{\mu}{2})\|\nabla u(t)\|_2^2 \\
& + \varepsilon(\frac{p}{2} + 1 - \frac{\mu}{2})\|u_t(t)\|_2^2 + \varepsilon(p-\mu)H(t) + c\varepsilon\left(\frac{p}{2} - \frac{\mu}{2}\right)\|\theta(t)\|_2^2 + \frac{\varepsilon\mu}{p}\|u(t)\|_p^p \\
& - \frac{b\varepsilon}{M}\kappa_T\left(H(t) + \|u_t(t)\|_2^2 + \|u(t)\|_p^p + \|\theta(t)\|_2^2\right).
\end{aligned}$$

We choose $\mu < p-2$ so that the coefficients of all the terms where the parameter μ appears become positive. Next, we can drop the $\|\nabla u(t)\|_2^2$ term, and then we choose M large enough so that the above inequality takes the form

$$L'(t) \geq \left[(1-\alpha) - \frac{bM\varepsilon}{4}\right] H^{-\alpha}(t)\|\nabla\theta(t)\|_2^2 + \varepsilon\Gamma_T\left(H(t) + \|u_t(t)\|_2^2 + \|u(t)\|_p^p + \|\theta(t)\|_2^2\right), \tag{5.30}$$

with $\Gamma_T > 0$. After M is chosen we choose ε so small that $(1-\alpha) - \frac{bM\varepsilon}{4} > 0$ and

$$L(0) = H^{1-\alpha}(0) + \varepsilon\int_\Omega u_0 u_1 dx + \frac{m\varepsilon}{2}\int_\Omega u_0^2 dx > 0. \tag{5.31}$$

Then (5.30) becomes

$$L'(t) \geq \Lambda_T\left(H(t) + \|u_t(t)\|_2^2 + \|u(t)\|_p^p + \|\theta(t)\|_2^2\right) \geq 0, \tag{5.32}$$

where $\Lambda_T = \varepsilon\Gamma_T$. Hence, $L(t) \geq L(0) > 0$ for all $t \geq 0$. Now, by the Schwarz inequality (1.1) and the L^p embedding theorem (1.5) we have

$$|\int_\Omega u u_t dx| \leq \|u(t)\|_2\|u_t(t)\|_2 \leq C\|u(t)\|_p\|u_t(t)\|_2$$

which implies

$$|\int_\Omega u u_t dx|^{\frac{1}{1-\alpha}} \leq C\|u(t)\|_p^{\frac{1}{1-\alpha}}\|u_t(t)\|_2^{\frac{1}{1-\alpha}}.$$

Then, Young's inequality (1.4) with $\frac{1}{p(1-\alpha)} + \frac{1}{2(1-\alpha)} = 1$ implies

$$\begin{aligned}
|\int_\Omega u u_t dx|^{\frac{1}{1-\alpha}} & \leq C\left(\|u(t)\|_p^p + \|u_t(t)\|_2^2\right) \\
& \leq C\left(H(t) + \|u_t(t)\|_2^2 + \|u(t)\|_p^p + \|\theta(t)\|_2^2\right).
\end{aligned} \tag{5.33}$$

Also, by L^p imbedding theorem (1.5) we have the estimate

$$|m\int_\Omega u^2dx|^{\frac{1}{1-\alpha}} = m^{\frac{1}{1-\alpha}}\|u(t)\|_2^{\frac{2}{1-\alpha}} \le C\|u(t)\|_p^{\frac{2}{1-\alpha}}.$$

Now, if $\|u(t)\|_p \ge 1$, since $\frac{2}{1-\alpha} = \frac{4p}{p+2} < p$ (because $2 < p \Rightarrow \frac{4}{p+2} < 1$, and multiply both sides by p), we have

$$|m\int_\Omega u^2dx|^{\frac{1}{1-\alpha}} \le C\|u(t)\|_p^{\frac{2}{1-\alpha}} \le C\|u(t)\|_p^p \le C\left(H(t) + \|u_t(t)\|_2^2 + \|u(t)\|_p^p + \|\theta(t)\|_2^2\right).$$

If $\|u(t)\|_p \le 1$, since for all $t \ge 0$ $H(t) \ge H(0) > \lambda > 0$, we have

$$|m\int_\Omega u^2dx|^{\frac{1}{1-\alpha}} \le C\|u(t)\|_p^{\frac{2}{1-\alpha}} \le C \le \frac{C}{\lambda}\left(H(t) + \|u_t(t)\|_2^2 + \|u(t)\|_p^p + \|\theta(t)\|_2^2\right).$$

Therefore, in any case we have

$$|m\int_\Omega u^2dx|^{\frac{1}{1-\alpha}} \le C\left(H(t) + \|u_t(t)\|_2^2 + \|u(t)\|_p^p + \|\theta(t)\|_2^2\right). \tag{5.34}$$

(5.32) together with (5.33) and (5.34) implies that

$$L'(t) \ge C_T\left(H(t) + |\int_\Omega uu_tdx|^{\frac{1}{1-\alpha}} + |m\int_\Omega u^2dx|^{\frac{1}{1-\alpha}}\right) \tag{5.35}$$

Finally since $\frac{1}{1-\alpha} > 1$, $f(x) = x^{\frac{1}{1-\alpha}}$ is a convex function for $x \ge 0$. Therefore, we have

$$\begin{aligned}
L^{\frac{1}{1-\alpha}}(t) &= \left[H^{1-\alpha}(t) + \varepsilon\int_\Omega uu_tdx + \frac{m\varepsilon}{2}\int_\Omega u^2dx\right]^{\frac{1}{1-\alpha}} \\
&\le \left[|H(t)|^{1-\alpha} + \left|\varepsilon\int_\Omega uu_tdx\right| + \left|\frac{m\varepsilon}{2}\int_\Omega u^2dx\right|\right]^{\frac{1}{1-\alpha}} \\
&\le C\left[H(t) + |\int_\Omega uu_tdx|^{\frac{1}{1-\alpha}} + |m\int_\Omega u^2dx|^{\frac{1}{1-\alpha}}\right].
\end{aligned}$$

This together with (5.35) gives

$$L'(t) \ge \gamma L^{\frac{1}{1-\alpha}}(t), \tag{5.36}$$

or

$$\frac{dL}{L^{\frac{1}{1-\alpha}}} \ge \gamma dt.$$

Integrating both sides over $(0,t)$ we obtain

$$\frac{1}{1-\frac{1}{1-\alpha}}\left[L^{(1-\frac{1}{1-\alpha})}(t) - L^{(1-\frac{1}{1-\alpha})}(0)\right] \ge \gamma t.$$

Noting that $1 - \frac{1}{1-\alpha} = -\frac{p-2}{p+2}$, and remembering $L(t) > 0$ for all $t \geq 0$ we deduce

$$L^{\frac{p-2}{p+2}}(t) \geq \frac{1}{L^{-\frac{p-2}{p+2}}(0) - \gamma t \frac{p-2}{p+2}}.$$

This implies $L(t) \to +\infty$ as $t \to \frac{p+2}{\gamma(p-2)} L^{-\frac{p-2}{p+2}}(0)$. So by choosing λ large enough we have

$$L(t) \to +\infty \text{ as } t \to T^* \leq T.$$

This and the definition of $L(t)$, since ε is arbitrarily small, imply that H(t) blows up, which in turn implies that $\|u(t)\|_p^p \to +\infty$ as $t \to T^* \leq T$. □

Remark 5.5. If we make the change of variables

$$\theta = \xi e^{-\alpha t}, \qquad \text{and} \qquad u = v e^{-\alpha t},$$

with $\alpha = \frac{\beta}{p-2} > 0$, then the equations (5.15), (5.16) become

$$v_{tt} = \Delta v + \vec{b} \cdot \nabla \xi + \vec{D} \cdot \nabla v - (m - 2\alpha) v_t + \alpha(m - \alpha) v + v|v|^{p-2},$$

$$c\xi_t = div\left(\nabla \xi + \vec{b} v_t\right) + \left(\vec{R} + \alpha \vec{b}\right) \cdot \nabla v + c\alpha \xi.$$

Theorem 5.2 requires $\beta \geq \sqrt{\frac{cd^2+r^2}{c}}$. Therefore, if

$$\alpha \geq \frac{1}{p-2} \sqrt{\frac{cd^2 + r^2}{c}},$$

and the other coefficients are as stated before, then the statement of Theorem 5.2 holds also for this system under homogeneous boundary and any initial conditions.

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