THE SIGN CHANGES ON THE AVERAGES OF EULER PHI FUNCTION AND DIVISOR FUNCTION

by

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ABSTRACT

In this study, we analyze the error terms in average orders of Euler phi function and divisor function. Our aim is to obtain a omega type estimation for these error terms and show that they change sign infinitely often. In the first part, we find the average value of the error term in the average order of Euler phi function and obtain a omega type estimation for this error term by employing basic arguments in Number Theory. In the second part, we show that this error term changes sign infinitely often. We accomplish this result by averaging this error term over arithmetic progressions. In the third part we utilize complicated methods to improve the results that we found in the first two parts. Finally, we combine the methods we used up to this section and apply them to the divisor function which is defined to be the sum of divisors of a given number n which is relatively prime to a.

ÖZET

Bu çalışmada, Euler phi fonksiyonu ve bölme fonksiyonunun ortalama büyüme değerinde çıkan hata terimleri analiz edilmiştir. Amacımız bu hata terimleri için omega türü bir değer bulma ve bu hata terimlerinin sonsuz defa işaret değiştirdiğini göstermektir. Birinci bölümde Sayılar Teorisindeki temel argümanları kullanarak, Euler phi fonksiyonunun ortalama büyüme değerinde çıkan hata teriminin, ortalama büyüme değerini hesaplıyoruz ve bu hata terimi için omega türü bir değer buluyoruz. İkinci bölümde, bu hata teriminin sonsuz defa işaret değiştirdiğini gösteriyoruz. Bu sonuç, hata teriminin aritmetik diziler üzerinde toplanması ile elde edilmiştir. Üçüncü bölümde, ilk iki bölümde bulduğumuz sonuçları geliştirmek için, ileri metotlardan faydalanıyoruz. Son olarak, bu bölüme kadar kullandığımız metotları birleştirerek; herhangi bir sayının, verilen bir a sayısı ile aralarında asal olan bölenlerinin toplamı olarak tanımladığımız bölme fonksiyonu üzerinde uyguluyoruz.

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Contents

LIST OF SYMBOLS/ABBREVIATIONS

$E(x)$	$\sum_{n\leq x}\phi(n)-\frac{3}{\pi^2}x^2$.
$H(x)$	$\sum_{n\leq x}\frac{\phi(n)}{n}-\frac{6}{\pi^2}x$.
$\zeta(s)$	The Riemann zeta-function.
$\phi(n)$	The Euler's totient function.
E	The Euler's constant.
$[x]$	The integer part of x.
$\{x\}$	$x-[x]$; the fractional part of x.
$\mu(n)$	The Möbius mu function.
$f(x)=O(g(x))$	$\lvert f(x)\rvert\leq Cg(x)$ where C is an absolute constant.
$f(x)=o(g(x))$	$\lim f(x)/g(x)=0$.
$f(x)\ll g(x)$	$f(x)=O(g(x))$.
$f(x)\sim g(x)$	$\lim f(x)/g(x)=1$.
$f(x)=\Omega_{\pm}(g(x))$	There exist constants c_1 and c_2 such that for infinitely many x, $f(x)\geq c_1\lvert g(x)\rvert$ and for infinitely many x, $f(x)\leq -c_2\lvert g(x)\rvert$
PNT	The Prime Number Theorem.
$v(A)$	Number of distinct prime factors of A.
$\Lambda(n)$	Mangoldt's function.
$\psi(x)$	$\sum_{n\leq x}\Lambda(n)$; the Chebyshev's ψ function.
$\pi(x)$	The number of primes $\leq x$.
$Li(x)$	$\int_2^x\frac{du}{\log u}$; the logarithmic integral.

1 PRELIMINARIES

This chapter includes the basic information needed to understand the text as we frequently will refer in the following chapters. It consists of four main sections and in each of them, we will present the functions and some of their properties that we are going to deal with. We also will introduce some main formulas and tools that are widely used in Analytic Number Theory. All these will be given briefly, without proof, since detailed arguments can be found in [1] or [2].

1.1 Arithmetic Functions $\mu(n)$ and $\phi(n)$

Definition 1. *A real- or complex-valued function defined on the positive integers is called an arithmetic function.*

We give definitions of arithmetic functions $\mu(n)$, $\phi(n)$ and $\Lambda(n)$ which play an important role in the study of divisibility properties of integers and the distribution of primes.

Definition 2. *The Möbius function μ is defined as follows:*

$\mu(1) = 1$;
If $n > 1$, write $n = p_1^{a_1} \cdots p_k^{a_k}$, where $p_1, p_2, \cdots, p_k$ are primes. Then

$$\mu(n) = \begin{cases} (-1)^k & \text{if } a_1 = a_2 = \cdots a_k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

One of the important properties of $\mu(n)$ is,

Theorem 1.1. *If $n \geq 1$, we have*

$$\sum_{d|n} \mu(d) = \left[\frac{1}{n}\right] = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Now we introduce a significant theorem in analytic number theory.

Theorem 1.2 (Möbius Inversion Formula)**.**

$$f(n) = \sum_{d|n} g(d) \Leftrightarrow g(n) = \sum_{d|n} f(d) \mu\left(\frac{n}{d}\right).$$

Where $f(n)$ and $g(n)$ arithmetic functions.

By using Möbius inversion formula we can get many important results. One of them is given in next theorem. First we introduce Mangoldt's function Λ which plays central role in the distribution of primes.

Definition 3.

$$\Lambda(n) = \begin{cases} \log p & \textit{if } n = p^m \textit{ for some prime } p \textit{ and some integer } m \geq 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Theorem 1.3. *If $n \geq 1$, we have*

$$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d} = -\sum_{d|n} \mu(d) \log d.$$

Definition 4. *If $n \geq 1$, the Euler totient $\phi(n)$ is defined to be the number of positive integers not exceeding n which are relatively prime to n; i.e.,*

$$\phi(n) = \sum_{\substack{m=1 \\ (m,n)=1}}^{n} 1.$$

We have some results about function $\phi(n)$ that will be used in next chapters.

Theorem 1.4. *If $n \geq 1$, we have*

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}. \tag{1.1}$$

Theorem 1.5. *For $n \geq 1$, we have*

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right). \tag{1.2}$$

Where p is any prime divisor of n.

Related with this theorem, in a later section we use next result that Mertens proved.

Theorem 1.6. *(Mertens product estimate, strong form) We have*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-E}}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right) \quad \textit{for} \quad x \geq 2.$$

In above equation, the constant E is the Euler's constant which is defined by the equation

$$E = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right).$$

1.2 Dirichlet convolution

Definition 5. *If f and g are two artihmetic functions, we define their Dirichlet product (or Dirichlet convolution) to be the arithmetical function h defined by the equation*

$$h(n) = (f * g)(n) = (g * f)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

An example to this product is that the equation (1.1) can be written as $\phi = \mu * N$. Where N is identity function.

The next theorem relates the product of Dirichlet Series with the Dirichlet convolution of their coefficients.

Theorem 1.7. *Given two functions F(s) and G(s) represented by Dirichlet series,*

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \textit{for} \quad \sigma > a,$$

and

$$G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} \quad \textit{for} \quad \sigma > b.$$

Then in the half-plane where both series converge absolutely we have

$$F(s)G(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}, \tag{1.3}$$

where $h = f * g$, the Dirichlet convolution of f and g:

$$h(n) = \sum_{d|n} f(d) g(\frac{n}{d}).$$

Here σ is real part of s.

Now we apply Theorem 1.6 to the sum

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

We have

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{converges absolutely for} \quad \sigma > 1,$$

and
$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \quad \text{converges absolutely for} \quad \sigma > 1.$$
Taking $f(n) = 1$ and $g(n) = \mu(n)$ in (1.3) we find
$$h(n) = \sum_{d|n} \mu(d).$$
This gives
$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{\sum_{d|n} \mu(d)}{n^s} = 1.$$
In particular, this shows that $\zeta(s) \neq 0$ for $\sigma > 1$ and that
$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \quad \text{for} \quad \sigma > 1. \tag{1.4}$$
In our subject we mainly use this formula for $s = 2$. Euler calculated that $\zeta(2) = \frac{\pi^2}{6}$, therefore we have
$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2}. \tag{1.5}$$

1.3 Multiplicative functions

In this section, we discuss an important subset of arithmetical functions which is called multiplicative functions.

Definition 6. *An arithmetical function f is called multiplicative, if f is not identically zero and if*
$$f(mn) = f(m)f(n) \quad \textit{whenever} \quad (m, n) = 1.$$
A multiplicative function f is called completely multiplicative if we also have
$$f(mn) = f(m)f(n) \quad \textit{for all } m, n.$$

We can give $\mu(n)$, $\phi(n)$ as examples of multiplicative functions. Multiplicativity provides us showing identities for some power of a prime instead of n, which comes from next theorem.

Theorem 1.8. *f is multiplicative if, and only if,*
$$f(p_1^{a_1} \dots p_r^{a_r}) = f(p_1^{a_1}) \dots f(p_r^{a_r})$$
for all primes p_i and all integers $a_i \geq 1$.

The other properties of multiplicative functions are

Theorem 1.9. *If f is multiplicative, we have*

$$\sum_{d|n} \mu(d) f(d) = \prod_{p|n} (1 - f(p)).$$

Theorem 1.10. *Let f be multiplicative arithmetic function such that the series $\sum f(n)$ is absolutely convergent. Then the sum of the series can be expressed as an absolutely convergent infinite product,*

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \left(1 + f(p) + f(p^2) + \cdots\right)$$

extended over all primes. If f is completely multiplicative, the product simplifies to

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \frac{1}{1 - f(p)}.$$

1.4 Elementary Asymptotic Formulas and Partial Summation

To study the average of an arbitrary function $f(n)$ we need a knowledge of its partial sums

$$\sum_{k=1}^{n} f(k).$$

Sometimes it is convenient to replace the upper index n by an arbitrary positive real number x and to consider instead sums of the form

$$\sum_{k \leq x} f(k).$$

This partial sum is also called summatory function of $f(k)$ which is denoted as $F(x)$. Before passing to methods for evaluating $F(x)$, we give a definition for main term in $F(x)$.

Definition 7. *If*

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1,$$

we say that $f(x)$ is asymptotic to $g(x)$ as $x \to \infty$, and we write $f(x) \sim g(x)$ as $x \to \infty$.

For example, in a later section we will prove that

$$\sum_{n \leq x} \phi(n) \sim \frac{3}{\pi^2} x^2 \quad \text{as} \quad x \to \infty.$$

Here the term $\frac{3}{\pi^2}x^2$ is called asymptotic value of the sum.

Sometimes the asymptotic value of a partial sum can be obtained by comparing it with an integral. A summation formula of Euler gives an exact expression for the error made in such an approximation.

Theorem 1.11 (Euler's Summation Formula)**.** *If f has a continuous derivative f' on the interval $[y, x]$, where $0 < y < x$, then*

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)dt + \int_y^x (t-[t])f'(t)dt + f(x)([x]-x) - f(y)([y]-y).$$

The next theorem gives a number of asymptotic formulas which are easy consequences of Euler's summation formula. In equation (1.7), $\zeta(s)$ denotes the Riemann zeta function which is defined by the equation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{if} \quad s > 1,$$

and

$$\zeta(s) = \lim_{n\to\infty} \left(\sum_{n\leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} \right) \quad \text{if} \quad 0 < s < 1.$$

Theorem 1.12.

$$\sum_{n\leq x} \frac{1}{n} = \log x + E + O\left(\frac{1}{x}\right). \tag{1.6}$$

$$\sum_{n\leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s}) \quad \textit{if} \quad s > 0, \quad s \neq 1. \tag{1.7}$$

$$\sum_{n>x} \frac{1}{n^s} = O(x^{1-s}) \quad \textit{if} \quad s > 1. \tag{1.8}$$

$$\sum_{n\leq x} n^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + O(x^{\alpha}) \quad \textit{if} \quad \alpha \geq 0. \tag{1.9}$$

Now we relate the partial sums of arbitarary arithmetical functions f and g with those of their Dirichlet product $f * g$.

Theorem 1.13. *If $h = f * g$, let*

$$H(x) = \sum_{n\leq x} h(n), \quad F(x) = \sum_{n\leq x} f(n), \quad G(x) = \sum_{n\leq x} g(n).$$

Then we have

$$H(x) = \sum_{n\leq x} f(n)G\left(\frac{x}{n}\right) = \sum_{n\leq x} g(n)F\left(\frac{x}{n}\right). \tag{1.10}$$

In Theorem 1.12 if we set $g(n) = 1$ for all n, then $G(x) = [x]$ and (1.10) gives us the following result:

Theorem 1.14. *If* $F(x) = \sum_{n \leq x} f(n)$, *we have*

$$\sum_{n \leq x} \sum_{d|n} f(d) = \sum_{n \leq x} f(n) \left[\frac{x}{n} \right] = \sum_{n \leq x} F\left(\frac{x}{n} \right). \tag{1.11}$$

We end this section by giving the partial summation formula which is one of the most powerful methods for estimating the summatory of arithmetic functions.

Theorem 1.15 (The Partial Summation Formula)**.** *Let x and y be real numbers with $0 < y < x$. Let $f(n)$ be an arithmetic function with summatory function $F(x)$ and $g(t)$ be a function with a continuous derivative on $[y, x]$. Then,*

$$\sum_{y < n \leq x} f(n)g(n) = F(x)g(x) - F(y)g(y) - \int_y^x F(t)g'(t)dt. \tag{1.12}$$

In particular, if $x \geq 2$ and $g(t)$ is continuously differentiable on $[1, x]$, then

$$\sum_{n \leq x} f(n)g(n) = F(x)g(x) - \int_1^x F(t)g'(t)dt. \tag{1.13}$$

2 An Asymptotic Formula for Error the Term Arising From the Summatory function of $\phi(n)$

In Analytic Number Theory, we estimate the averages $\sum_{n\leq x} f(n)$ of arithmetic functions, because they are expected to behave more regularly for large x whereas an arithmetic function itself may behave beyond prediction when n is large. We approach this subject from another perspective that estimates the averages of error terms, and finds that these error terms indeed change sign infinitely often.

Fluctuation of error terms interested mathematicians since beginning of 1900s. After Hadamard and De la Vallée Poussin proved Prime Number Theorem in 1896, E. Schmidt analyzed the error term of $\psi(x)$. In 1903, he proved that $\psi(x) - x$ changes sign infinitely often. After a while, E. Schmidt get the same result for the function $\pi(x) - Li(x)$ under the assumption that Riemann Hypothesis is false. In 1914, Littlewood proved that $\pi(x) - Li(x)$ fluctuates in the case Riemann Hypothesis is true. He showed that $\pi(x) - Li(x) = \Omega_{\pm}(x^{1/2} \log\log\log x)$, where $\Omega_{\pm}$ means that the error term achieves the given order of magnitude infinitely often with both positive and negative signs. For proof of these results, see [2].

In this study, we will mainly focus on Euler's ϕ function which gives remarkable results. The first result on the behaviour of error term in the average of $\phi(n)$ is due to Dirichlet, who proved that $E(x) = O(x^{\delta})$ for some δ, $1 < \delta < 2$, where $E(x)$ is the error term in the average of $\phi(n)$. This was improved by Mertens to $E(x) = O(x \log x)$ [1]. The proof of these estimations are short and elementary. More recently, using the best error term for the Prime Number Theorem, Walfisz improved this to $E(x) = O(x(\log x)^{\frac{2}{3}}(\log\log x)^{\frac{4}{3}})$ [10].

The results above give us an upper bound for the error term. On the other side, in 1930, Pillai and Chowla gave Ω-type estimation for $E(x)$, and showed that $E(x) \neq o(x \log\log\log x)$ [3]. Then, in 1951, Erdös and Shapiro analyzed the error term on the arithmetic progression and proved that $E(x) = \Omega_{\pm}(x \log\log\log\log x)$ [4], which was improved by Montgomery to $E(x) = \Omega_{\pm}(x\sqrt{\log\log x})$ in 1987 [5].

In this chapter, we prove

$$\sum_{n\leq x} E(n) \sim \frac{3}{2\pi^2}x^2$$

and

$$E(x) \neq o(x \log\log\log x).$$

2.1 Average Orders of $\phi(n)$ and $\frac{\phi(n)}{n}$

Let us first find the average order of $\phi(n)$ which is proved by Mertens.

Theorem 2.1. *For $x > 1$, we have*

$$\sum_{n\leq x}\phi(n) = \frac{3}{\pi^2}x^2 + O(x\log x). \tag{2.1}$$

Therefore the average order of $\phi(n)$ is $\frac{3n}{\pi^2}$.

Proof. By using (1.1), (1.5) and Theorem 1.11 we have

$$\begin{aligned}
\sum_{n\leq x}\phi(n) &= \sum_{n\leq x}\sum_{d|n}\mu(d)\frac{n}{d} = \sum_{\substack{q,d\\ qd\leq x}}\mu(d)q = \sum_{d\leq x}\mu(d)\sum_{q\leq x/d}q \\
&= \sum_{d\leq x}\mu(d)\left\{\frac{1}{2}\left(\frac{x}{d}\right)^2 + O\left(\frac{x}{d}\right)\right\} \\
&= \frac{1}{2}x^2\sum_{d\leq x}\frac{\mu(d)}{d^2} + O\left(x\sum_{d\leq x}\frac{1}{d}\right) \\
&= \frac{1}{2}x^2\left\{\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2} - \sum_{d>x}\frac{\mu(d)}{d^2}\right\} + O(x\log x) \\
&= \frac{1}{2}x^2\left\{\frac{6}{\pi^2} + O\left(\frac{1}{x}\right)\right\} + O(x\log x) \\
&= \frac{3}{\pi^2}x^2 + O(x\log x).
\end{aligned}$$

□

By using above result and partial summation formula, we get the following formula.

Theorem 2.2. *For $x > 1$, we have*

$$\sum_{n\leq x}\frac{\phi(n)}{n} = \frac{6}{\pi^2}x + O(\log x). \tag{2.2}$$

Therefore the average order of $\frac{\phi(n)}{n}$ is $\frac{6}{\pi^2}$.

Proof. Applying partial summation formula and using

$$\sum_{n\leq x}\phi(n) = \frac{3}{\pi^2}x^2 + O(x\log x),$$

we get

$$\begin{aligned}
\sum_{n\le x}\frac{\phi(n)}{n} &= \frac{3}{\pi^2}x+O(\log x)+\int_1^x\left\{\frac{3}{\pi^2}+O\left(\frac{\log t}{t^2}\right)\right\}dt\\
&= \frac{3}{\pi^2}x+O(\log x)+\int_1^x\frac{3}{\pi^2}dt+O\left(\int_1^x\frac{\log t}{t^2}dt\right)\\
&= \frac{6}{\pi^2}x+O(\log x).
\end{aligned}$$

□

2.2 Average Order of Error term in Partial Sum of $\phi(n)$

First, let us define

$$E(x)=\sum_{n\le x}\phi(n)-\frac{3}{\pi^2}x^2$$

and

$$H(x)=\sum_{n\le x}\frac{\phi(n)}{n}-\frac{6}{\pi^2}x.$$

In this section, we will show that

$$\sum_{x\le R}E(x)\sim\frac{3}{2\pi^2}R^2.$$

Theorem 2.3. *If*

$$\sum_{n\le x}a_n=o(x)$$

and

$$|a_n|<K<\infty,$$

then

$$\sum_{n\le x}a_n\left\{\frac{x}{n}\right\}^2=o(x).$$

Where $\left\{\frac{x}{n}\right\}$ *denotes the fractional part of* $\frac{x}{n}$.

For the proof of this theorem, we refer to [3].

Now we will make some preparation for the next theorem. Let us put

$$M(x)=\sum_{n\le x}\mu(n).$$

Then one of the implications of the PNT is,

$$M(x) = o(x) \tag{2.3}$$

and we can obtain from De la Vallée Poussin type zero free region for the Riemann zeta function

$$M(x) = O(x \exp(-c\sqrt{\log x})).$$

Lemma 2.4. *We have*

$$\sum_{n>x} \frac{\mu(n)}{n^2} = o\left(\frac{1}{x}\right). \tag{2.4}$$

Proof. For proving our result, it is sufficient to take $M(x) = O(\frac{x}{\log x})$. By partial summation formula, we have

$$\begin{aligned}
\sum_{x<n\leq y} \frac{\mu(n)}{n^2} &= \frac{M(y)}{y^2} - \frac{M(x)}{x^2} + 2\int_x^y \frac{M(t)}{t^3} dt \\
&= O\left(\frac{1}{y\log y}\right) + O\left(\frac{1}{x\log x}\right) + O\left(\int_x^y \frac{1}{t^2\log t} dt\right) \\
&= O\left(\frac{1}{y\log y}\right) + O\left(\frac{1}{x\log x}\right) + O\left(\frac{1}{\log x}\int_x^y \frac{1}{t^2} dt\right) \\
&= O\left(\frac{1}{y\log y}\right) + O\left(\frac{1}{x\log x}\right) + O\left(\frac{1}{\log x}\left(\frac{1}{x} - \frac{1}{y}\right)\right).
\end{aligned}$$

By letting $y \to \infty$, we get

$$\sum_{n>x} \frac{\mu(n)}{n^2} = O\left(\frac{1}{x\log x}\right) = o\left(\frac{1}{x}\right).$$

$\square$

Now we use (1.1), (1.5), and Theorem 1.11 with above result to get

$$\begin{aligned}
\sum_{n\leq x} \frac{\phi(n)}{n} &= \sum_{n\leq x}\sum_{d|n} \frac{\mu(d)}{d} = \sum_{n\leq x} \frac{\mu(n)}{n}\left[\frac{x}{n}\right] \\
&= x\sum_{n\leq x} \frac{\mu(n)}{n^2} - \sum_{n\leq x} \frac{\mu(n)}{n}\left\{\frac{x}{n}\right\} \\
&= x\left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} - \sum_{n>x} \frac{\mu(n)}{n^2}\right) - \sum_{n\leq x} \frac{\mu(n)}{n}\left\{\frac{x}{n}\right\} \\
&= \frac{6}{\pi^2}x + o(1) - \left(\sum_{n\leq x} \frac{\mu(n)}{n}\left\{\frac{x}{n}\right\}\right).
\end{aligned} \tag{2.5}$$

Recall that

$$H(x) = \sum_{n \le x} \frac{\phi(n)}{n} - \frac{6}{\pi^2}x.$$

From (2.5), we have

$$H(x) = -\left(\sum_{n \le x} \frac{\mu(n)}{n} \left\{ \frac{x}{n} \right\} \right) + o(1). \tag{2.6}$$

Now we are ready to prove our main result for $H(x)$.

Theorem 2.5. *We have*

$$\sum_{x \le R} H(x) \sim \frac{3}{\pi^2} R \quad \text{as} \quad R \to \infty. \tag{2.7}$$

Proof. First observe that it is sufficient to prove the theorem only when R is an integer. Let R be an integer and consider,

$$\sum_{x \le R} H(x) = \sum_{x \le R} \sum_{n \le x} \frac{\phi(n)}{n} - \sum_{x \le R} \frac{6}{\pi^2} x = \sum_{x \le R} \sum_{n \le x} \frac{\phi(n)}{n} - \frac{3}{\pi^2}(R^2 + R).$$

We write double sum as,

$$\sum_{x \le R} \sum_{n \le x} \frac{\phi(n)}{n} = (0) + \left(\frac{\phi(1)}{1} \right) + \left(\frac{\phi(1)}{1} + \frac{\phi(2)}{2} \right) + \\ \cdots + \left(\frac{\phi(1)}{1} + \frac{\phi(2)}{2} + \cdots + \frac{\phi(R)}{R} \right).$$

In the above equation we complete each parenthesis to $\left(\frac{\phi(1)}{1} + \frac{\phi(2)}{2} + \cdots + \frac{\phi(R)}{R} \right)$, and obtain

$$\begin{aligned} \sum_{x \le R} H(x) &= (R+1) \sum_{n \le R} \frac{\phi(n)}{n} - \sum_{n \le R} \phi(n) - \frac{3}{\pi^2}(R^2 + R) \\ &= \left\{ R \sum_{n \le R} \frac{\phi(n)}{n} - \sum_{n \le R} \phi(n) \right\} + \frac{6}{\pi^2} R + O(\log R) - \frac{3}{\pi^2}(R^2 + R) \\ &= \left\{ R \sum_{n \le R} \frac{\phi(n)}{n} - \sum_{n \le R} \phi(n) \right\} - \frac{3}{\pi^2}(R^2 - R) + O(\log R). \end{aligned} \tag{2.8}$$

Combining with (2.5), we see that

$$\sum_{n\le R}\frac{\phi(n)}{n}=R\sum_{n\le R}\frac{\mu(n)}{n^2}-\sum_{n\le R}\frac{\mu(n)}{n}\left\{\frac{R}{n}\right\}.$$

Moreover, using the formula we show that

$$\sum_{n\le R}\phi(n)=\sum_{d\le R}\mu(d)\sum_{q\le\frac{R}{d}}q=\frac{1}{2}\sum_{d\le R}\mu(d)\left\{\left[\frac{R}{d}\right]^2+\left[\frac{R}{d}\right]\right\}.$$

(2.8) becomes

$$\sum_{x\le R}H(x)=R^2\sum_{n\le R}\frac{\mu(n)}{n^2}-R\sum_{n\le R}\frac{\mu(n)}{n}\left\{\frac{R}{n}\right\}$$
$$\underbrace{-\frac{1}{2}\sum_{n\le R}\mu(n)\left\{\left[\frac{R}{n}\right]^2+\left[\frac{R}{n}\right]\right\}}_{A}-\frac{3}{\pi^2}(R^2-R)+O(\log R).$$

$$\begin{aligned}A&=-\frac{1}{2}\sum_{n\le R}\mu(n)\left[\frac{R}{n}\right]^2-\frac{1}{2}\sum_{n\le R}\mu(n)\left[\frac{R}{n}\right]\\&=-\frac{1}{2}R^2\sum_{n\le R}\frac{\mu(n)}{n^2}+R\sum_{n\le R}\frac{\mu(n)}{n}\left\{\frac{R}{n}\right\}-\underbrace{\frac{1}{2}\sum_{n\le R}\mu(n)\left\{\frac{R}{n}\right\}^2}_{A_1}\end{aligned}$$
$$\underbrace{-\frac{1}{2}\sum_{n\le R}\mu(n)\left[\frac{R}{n}\right]}_{A_2}.$$

Since

$$\sum_{n\le R}\mu(n)=o(R),$$

Theorem 2.3 gives $A_1=o(R)$. And by using Theorem 1.13 and Theorem 1.1, we see that

$$A_2=-\frac{1}{2}\sum_{n\le R}\sum_{d|n}\mu(d)=-\frac{1}{2}.$$

This gives,

$$A=-\frac{1}{2}R^2\sum_{n\le R}\frac{\mu(n)}{n^2}+R\sum_{n\le R}\frac{\mu(n)}{n}\left\{\frac{R}{n}\right\}+o(R).$$

Going back and putting A into our main equation, we have

$$\begin{aligned}\sum_{x\le R} H(x) = R^2 \sum_{n\le R} \frac{\mu(n)}{n^2} - R\sum_{n\le R} \frac{\mu(n)}{n}\left\{\frac{R}{n}\right\} - \frac{1}{2}R^2 \sum_{n\le R}\frac{\mu(n)}{n^2} \\ + R\sum_{n\le R}\frac{\mu(n)}{n}\left\{\frac{R}{n}\right\} + o(R) - \frac{3}{\pi^2}(R^2 - R) + O(\log R) \\ = \frac{1}{2}R^2 \sum_{n\le R}\frac{\mu(n)}{n^2} - \frac{3}{\pi^2}R^2 + \frac{3}{\pi^2}R + o(R) \\ = \frac{1}{2}R^2\left\{\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2} - \sum_{n>R}\frac{\mu(n)}{n^2}\right\} - \frac{3}{\pi^2}R^2 + \frac{3}{\pi^2}R + o(R) \\ = \frac{3}{\pi^2}R + o(R). \qquad (2.9)\end{aligned}$$

Which gives the desired result. □

Now we relate $H(x)$ to $E(x)$ in the next theorem.

Theorem 2.6. *We have*

$$E(R) = RH(R) + o(R) \quad as \quad R \to \infty. \qquad (2.10)$$

Proof.

$$\begin{aligned}E(R) = \sum_{n\le R}\phi(n) - \frac{3}{\pi^2}R^2 = \sum_{d\le R}\mu(d)\sum_{q\le \frac{R}{d}} q - \frac{3}{\pi^2}R^2 \\ = \frac{1}{2}\sum_{n\le R}\mu(n)\left\{\left[\frac{R}{n}\right]^2 + \left[\frac{R}{n}\right]\right\} - \frac{3}{\pi^2}R^2 \\ = \frac{1}{2}\sum_{n\le R}\mu(n)\left[\frac{R}{n}\right]^2 + \frac{1}{2}\sum_{n\le R}\mu(n)\left[\frac{R}{n}\right] - \frac{3}{\pi^2}R^2 \\ = \frac{1}{2}R^2\sum_{n\le R}\frac{\mu(n)}{n^2} - R\sum_{n\le R}\frac{\mu(n)}{n}\left\{\frac{R}{n}\right\} + \frac{1}{2}\sum_{n\le R}\mu(n)\left\{\frac{R}{n}\right\}^2 \\ + \frac{1}{2}\sum_{n\le R}\mu(n)\left[\frac{R}{n}\right] - \frac{3}{\pi^2}R^2.\end{aligned}$$

Again by a similar argument as in the previous theorem, we have

$$E(R) = -R\sum_{n\le R}\frac{\mu(n)}{n}\left\{\frac{R}{n}\right\} + o(R).$$

and by equation (2.6), we have

$$H(R) = -\left(\sum_{n\le R} \frac{\mu(n)}{n}\left\{\frac{R}{n}\right\}\right) + o(1),$$

and

$$E(R) = RH(R) + o(R) \quad \text{follows.}$$

□

Theorem 2.7. *We have*

$$\sum_{x\le R} E(x) \sim \frac{3}{2\pi^2}R^2 \quad as \quad R \to \infty. \tag{2.11}$$

Proof. From Theorem 2.6, we have

$$\sum_{x\le R} E(x) = \sum_{x\le R} xH(x) + o\left(\sum_{x\le R} x\right) = \sum_{x\le R} xH(x) + o(R^2).$$

We apply partial summation formula and (2.7) to deduce that

$$\begin{aligned}\sum_{x\le R} xH(x) &= \frac{3}{\pi^2}R^2 + o(R^2) - \int_1^R \frac{3}{\pi^2}tdt - \int_1^R f(t)dt \\ &= \frac{3}{2\pi^2}R^2 + o(R^2) - \int_1^R f(t)dt.\end{aligned}$$

Where $f(t)$ is a function with $f(t) = o(t)$. To complete the proof, it suffices to show that

$$\lim_{R\to\infty} \frac{1}{R^2}\int_1^R f(t)dt = 0. \tag{2.12}$$

Given $\epsilon > 0$, there exists a n (depending only on ϵ) such that $\left|\frac{f(R)}{R}\right| < \epsilon$ if $R \ge n$.

$$\begin{aligned}\left|\frac{1}{R^2}\int_1^R f(t)dt\right| &\le \left|\frac{1}{R^2}\int_1^n f(t)dt\right| + \left|\frac{1}{R^2}\int_n^R f(t)dt\right| \\ &\le \frac{(n-1)f(n^*)}{R^2} + \frac{1}{R^2}\int_n^R \epsilon tdt \\ &= \frac{(n-1)f(n^*)}{R^2} + \frac{\epsilon}{R^2}\left(\frac{R^2}{2} - \frac{n^2}{2}\right)\end{aligned}$$

$$= \frac{(n-1)f(n^*)}{R^2} + \frac{\epsilon}{2} - \frac{n^2}{2R^2}$$

here

$$f(n^*) = \sup_{t\in[1,n]} |f(t)|.$$

By letting $R \to \infty$ we find

$$\limsup_{R\to\infty} \left| \frac{1}{R^2} \int_1^R f(t)dt \right| \leq \frac{\epsilon}{2}$$

and since ϵ is arbitrary, this proves (2.12).

Therefore, we have

$$\sum_{x\leq R} xH(x) = \frac{3}{2\pi^2} R^2 + o(R^2).$$

And this completes the proof of

$$\sum_{x\leq R} E(x) \sim \frac{3}{2\pi^2} R^2.$$

□

2.3 An Important Result of $E(x)$

In this section we prove that $E(x) \neq o(x \log\log\log x)$ which is a Ω-type estimation for $E(x)$. First we make some preparations.

Let $P(a, b)$ denote the product of the primes between a and b, i.e.

$$\prod_{a<p<b} p$$

and $P(a, b) = 1$, if we have no prime between a and b. Let x_0 be the least positive solution of the following system of congruences which is clearly solvable:

$$\begin{aligned}
x &\equiv 0\ (mod\ 2),\\
x+1 &\equiv 0\ (mod\ P(2,2^3)),\\
x+3 &\equiv 0\ (mod\ P(2^3,2^{3^2})),\\
\cdots\quad &\cdots\\
x+2k-1 &\equiv 0\ (mod\ P(2^{3^{k-1}},2^{3^k})).
\end{aligned}$$

Some observation on these congruences are now in order.

1. x_0 is even, from first congruence.

2. $x_0 > 2^{3^{k-1}} - (2k-1)$, from last congruence.

3. $x_0 < P(1, 2^{3^k})$, by Chinese Remainder theorem.

Now we clearly have $2^{3^{k-2}} < x_0$. Taking logarithm of both sides

$$\begin{aligned}
3^{k-2}\log 2 &< \log x_0\\
(k-2)\log 3 + \log\log 2 &< \log\log x_0\\
k\log 3 - 2\log 3 + \log\log 2 &< \log\log x_0 \quad \text{follows.}
\end{aligned}$$

After arranging constants, we have

$$k < H_1 \log\log x_0, \quad \text{where } H_1 > 0. \tag{2.13}$$

On the other side, we have $x_0 < P(1, 2^{3^k})$, and similarly

$$\begin{aligned}
x_0 &< \prod_{1<p<2^{3^k}} p\\
\log x_0 &< \sum_{p\le 2^{3^k}} \log p = O\left(2^{3^k}\right)\\
\log\log x_0 &< O\left(3^k\right)\\
\log\log\log x_0 &< O\left(k\right).
\end{aligned}$$

After arranging constant, we have

$$k > H_2 \log\log\log x_0 \quad \text{where} \quad H_2 > 0. \tag{2.14}$$

From Theorem 1.6, we have

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)\sim\frac{e^{-E}}{\log x}, \tag{2.15}$$

where E is Euler's contant, and p runs through primes. It follows that

$$\prod_{x<p\le x^3}\left(1-\frac{1}{p}\right)=\frac{1}{3}+o(1). \tag{2.16}$$

Hence there is a positive integer f, independent of k, such that

$$\prod_{x<p\le x^3}\left(1-\frac{1}{p}\right)<\frac{1}{2}, \tag{2.17}$$

when $x\ge 2^{3^f}$.

Observe that $\frac{\phi(n)}{n}<\frac{1}{2}$ when n is even, and since x_0 is even

$$\frac{\phi(x_0+t)}{x_0+t}<\frac{1}{2}, \tag{2.18}$$

when t is even. We will get a similar estimate for odd t as well.

When t is any odd number from the set $2f+1,\ 2f+3,\cdots,2k-1$ with $k\ge f+1$. Since x_0 is the least positive solution of the congurances, we have

$$x_0+t=rP(2^{3^{s-1}},2^{3^s}),$$

for some r positive integer and for $s\in[f+1,k]$, so by (1.2) and (2.17) we have

$$\frac{\phi(x_o+t)}{x_0+t}=\prod_{p|(x_0+t)}\left(1-\frac{1}{p}\right)\le\prod_{2^{3^{s-1}}<p\le 2^{3^s}}\left(1-\frac{1}{p}\right)<\frac{1}{2}, \tag{2.19}$$

since $2^{3^{s-1}}\ge 2^{3^f}$.

For the remaining values of t from $1,\ 2,\cdots,2f$, we have

$$\frac{\phi(x_0+t)}{x_0+t}<1. \tag{2.20}$$

Theorem 2.8. *We have*

$$E(x)\ne o(x\log\log\log x). \tag{2.21}$$

Proof.

$$\begin{aligned}E(x_0+2k)-E(x_0) &= \sum_{x_0+1}^{x_0+2k}\phi(n)-\frac{3}{\pi^2}(x_0+2k)^2+\frac{3}{\pi^2}(x_0)^2\\ &= \sum_{x_0+1}^{x_0+2k}\phi(n)-\frac{12}{\pi^2}k^2-\frac{12}{\pi^2}x_0k\\ &= -\frac{12}{\pi^2}k^2-\frac{12}{\pi^2}x_0k+\sum_{n=x_0+1}^{x_0+2f}\phi(n)+\sum_{n=x_0+2f+1}^{x_0+2k}\phi(n).\end{aligned} \tag{2.22}$$

From (2.18), (2.19), (2.20) and (2.22), we obtain

$$\begin{aligned}E(x_0+2k)-E(x_0) &< -\frac{12}{\pi^2}k^2-\frac{12}{\pi^2}x_0k+\sum_{n=x_0+1}^{x_0+2f}n+\sum_{n=x_0+2f+1}^{x_0+2k}\frac{n}{2}\\ &= -\frac{12}{\pi^2}k^2-\frac{12}{\pi^2}x_0k+f(2x_0+2f+1)\\ &+\frac{1}{2}(k-f)(2x_0+2k+2f+1)\\ &= x_0k\left(1-\frac{12}{\pi^2}\right)+O(x_0)+O(k^2)\\ &= x_0k\left(1-\frac{12}{\pi^2}\right)+O(x_0)+O(\log\log^2 x_0)\end{aligned} \tag{2.23}$$

since $f=O(1)$ and $k=O(\log\log x_0)$.

From (2.14), (2.23) and using $\left(1-\frac{12}{\pi^2}\right)<0$, we have

$$|E(x_o+2k)-E(x_0)| > H_2\left(\frac{12}{\pi^2}-1\right)x_0\log\log\log x_0+O(x_0). \tag{2.24}$$

Now suppose that $E(x)=o(x\log\log\log x)$. Since $k=O(\log\log x_0)$ from (2.13),

$$|E(x_0+2k)-E(x_0)| = o(x_0\log\log\log x_0)$$

which contradicts with equation (2.24), and hence we have

$$E(x)\neq o(x\log\log\log x).$$

□

3 The First Result on the Sign Changes of the Error Term in the Average Order of $\phi(n)$

In the study of analyzing the error term $E(x)$, fluctuation of this error term is the most important part that mathematicians interested in. Related with this subject, the first question that one can ask whether $E(x) < 0$ for some positive x. Sylvester tabulated $\phi(n)$, $\sum_{m \leq n} \phi(m)$, and $(3/\pi^2)n$ for $n = 1, 2, \cdots, 1000$ [9]. He conjectured that $E(x) > 0$ whenever x is a positive integer [8], [9], but he failed to note that $E(820) < 0$. In 1936, Sarma disproved the conjecture and showed that $E(820) = -9.092\ldots < 0$ [7].

In this section we prove that $E(x)$ changes sign infinitely often. We have the result $E(x) = \Omega_{\pm}(x \log\log\log\log x)$. In other words, there exists a positive constant c and infinitely many integers x such that

$$E(x) > cx \log\log\log\log x$$

and infinitely many integers x such that

$$E(x) < -cx \log\log\log\log x.$$

3.1 Evaluation of Certain Sums

Let us start with a lemma which relates $H(x)$ to $E(x)$.

Lemma 3.1. *For integral x,*

$$\sum_{n \leq x} H(n) = \frac{3}{\pi^2}x + (x+1)H(x) - E(x). \tag{3.1}$$

Proof. By using (2.8), we get

$$\begin{aligned}
\sum_{n \leq x} H(n) &= (x+1)\sum_{n \leq x} \frac{\phi(n)}{n} - \sum_{n \leq x} \phi(n) - \frac{3}{\pi^2}(x^2 + x) \\
&= (x+1)\left(\frac{6}{\pi^2}x + H(x)\right) - \sum_{n \leq x} \phi(n) - \frac{3}{\pi^2}(x^2 + x) \\
&= \frac{3}{\pi^2}x + (x+1)H(x) - E(x). \qquad (3.2)
\end{aligned}$$

□

Lemma 3.2. *We have the following estimates.*

1.

$$\sum_{d \leq x} \frac{1}{d} H\left(\frac{x}{d}\right) = O(1). \tag{3.3}$$

2.

$$\sum_{d\le x} H\left(\frac{x}{d}\right) = O(x). \tag{3.4}$$

3.

$$\sum_{d\le x} E\left(\frac{x}{d}\right) = O(x). \tag{3.5}$$

Proof. In the following three proof, we use Theorem 1.11.

1.

$$\begin{aligned}
x + O(1) &= \sum_{n\le x} 1 = \sum_{n\le x} \frac{1}{n} \sum_{d|n} \phi(d) = \sum_{dq\le x} \frac{\phi(d)}{dq} \\
&= \sum_{q\le x} \frac{1}{q} \sum_{d\le x/q} \frac{\phi(d)}{d} = \sum_{q\le x} \frac{1}{q} \left\{ H\left(\frac{x}{q}\right) + \frac{6}{\pi^2}\frac{x}{q} \right\} \\
&= \sum_{q\le x} \frac{1}{q} H\left(\frac{x}{q}\right) + \frac{6}{\pi^2} x \sum_{q\le x} \frac{1}{q^2} \\
&= \sum_{q\le x} \frac{1}{q} H\left(\frac{x}{q}\right) + x + O(1).
\end{aligned}$$

This gives

$$\sum_{q\le x} \frac{1}{q} H\left(\frac{x}{q}\right) = O(1).$$

2. By using $H(x) = O(\log x)$, we have

$$\begin{aligned}
\sum_{d\le x} H\left(\frac{x}{d}\right) &= O\left(\sum_{d\le x} \log \frac{x}{d}\right) = O\left(\log\left(\prod_{d\le x} \frac{x}{d}\right)\right) \\
&= O\left(\log\left(\frac{x^x}{[x]!}\right)\right) = O(x\log x - \log[x]!).
\end{aligned}$$

Making use of the formula

$$\log [x]! = x\log x - x + O(\log x), \ (\textit{see } [1])$$

we have

$$\sum_{d\le x} H\left(\frac{x}{d}\right) = O(x).$$

3. Note that

$$\frac{x^2}{2} + O(x) = \sum_{n \le x} n = \sum_{n \le x} \sum_{d|n} \phi(d) = \sum_{q \le x} \sum_{d \le x/q} \phi(d)$$

$$= \sum_{q \le x} \left\{ \frac{3}{\pi^2} \frac{x^2}{q^2} + E\left(\frac{x}{q}\right) \right\} = \sum_{q \le x} E\left(\frac{x}{q}\right) + \frac{x^2}{2} + O(x).$$

This in turn gives

$$\sum_{d \le x} E\left(\frac{x}{d}\right) = O(x).$$

□

Theorem 3.3. *The formula*

$$\sum_{mn \le x} H(n) = \frac{3}{\pi^2} x \log x + O(x) \tag{3.6}$$

holds.

Proof. From lemma 3.1, we have

$$\sum_{n \le x} H(n) = \frac{3}{\pi^2} x + (x+1) H(x) - E(x). \tag{3.7}$$

Replacing x by x/m in (3.7) and summing over all integral $m \le x$ we have

$$\sum_{m \le x} \sum_{n \le x/m} H(n) = \frac{3}{\pi^2} \sum_{m \le x} \frac{x}{m} + x \sum_{m \le x} \frac{1}{m} H\left(\frac{x}{m}\right) + \sum_{m \le x} H\left(\frac{x}{m}\right) - \sum_{m \le x} E\left(\frac{x}{m}\right).$$

By using lemma 3.2 and (1.6) we have

$$\sum_{mn \le x} H(n) = \frac{3}{\pi^2} x \log x + O(x).$$

□

3.2 The Average of $H(n)$ over Arithmetic Progressions

The main idea of the proof is to evaluate certain averages of $H(n)$ over arithmetic progressions. Let us begin with an important lemma.

Lemma 3.4. *We have*

$$\sum_{\substack{m \le z \\ m \equiv \beta (mod A)}} \frac{\phi(m)}{m} = \frac{C}{A} \sum_{d|(A,\beta)} \frac{\mu(d)}{d} z + O(\log z), \tag{3.8}$$

where

$$C = C(A) = \prod_{p \nmid A} \left(1 - \frac{1}{p^2}\right).$$

Proof. Let us put $m = dq$. Then we have

$$\sum_{\substack{m \leq z \\ m \equiv \beta (mod A)}} \frac{\phi(m)}{m} = \sum_{\substack{m \leq z \\ m \equiv \beta (mod A)}} \sum_{d|m} \frac{\mu(d)}{d} = \sum_{\substack{dq \equiv \beta (mod A) \\ dq \leq z}} \frac{\mu(d)}{d}$$

$$= \sum_{\substack{d \leq z \\ (d,A)|\beta}} \frac{\mu(d)}{d} \sum_{\substack{q \leq z/d \\ q \equiv \frac{\beta}{(d,A)} mod\left(\frac{A}{(d,A)}\right)}} 1$$

$$= \sum_{\substack{d \leq z \\ (d,A)|\beta}} \frac{\mu(d)}{d} \left\{ \frac{(d,A)}{A} \frac{z}{d} + O(1) \right\}.$$

If we take $\tau = (d, A)$, then we have $d = t\tau$ for some positive integer t, so that

$$\sum_{\substack{d \leq z \\ (d,A)|\beta}} \frac{\mu(d)}{d} \left\{ \frac{(d,A)}{A} \frac{z}{d} + O(1) \right\} = \frac{z}{A} \sum_{\tau|(A,\beta)} \tau \sum_{(d,A)=\tau} \frac{\mu(d)}{d^2} + O(\log z)$$

$$= \frac{z}{A} \sum_{\tau|(A,\beta)} \tau \frac{\mu(\tau)}{\tau^2} \sum_{(t,A)=1} \frac{\mu(t)}{t^2} + O(\log z)$$

follows.

From the formula

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} = \prod_{p} \left(1 - \frac{1}{p^2}\right)$$

we see that

$$\sum_{(t,A)=1} \frac{\mu(t)}{t^2} = \prod_{p \nmid A} \left(1 - \frac{1}{p^2}\right) = C.$$

Finally we have,

$$\sum_{\substack{m \leq z \\ m \equiv \beta (mod A)}} \frac{\phi(m)}{m} = C \frac{z}{A} \sum_{\tau|(A,\beta)} \frac{\mu(\tau)}{\tau} + O(\log z)$$

which is the desired result. $\square$

Theorem 3.5. *If A, B are integers with $A > B \geq 0$, then*

$$\sum_{n\leq x} H(An-B) = \frac{1}{A}\sum_{n\leq Ax-B} H(n) + \triangle x + O(\log x), \tag{3.9}$$

where

$$\triangle = \triangle(A,B) = M(A,B) - \frac{3}{\pi^2}$$

and

$$M(A,B) = \begin{cases} \frac{6}{\pi^2}B - \frac{1}{2}\frac{\phi(A)C(A)}{A} - C(A)\sum_{c=1}^{B-1}\frac{\phi(A,c)}{(A,c)} & \textit{for } B \neq 0 \\ \\ \frac{1}{2}\frac{\phi(A)C(A)}{A} & \textit{for } B = 0. \end{cases} \tag{3.10}$$

Proof. It clearly suffices to prove (3.9) for integral x. Thus we may assume that x an integer. We have

$$\begin{aligned} \sum_{n\leq x} H(An-B) &= \sum_{n\leq x}\left\{\sum_{m\leq An-B}\frac{\phi(m)}{m} - \frac{6}{\pi^2}(An-B)\right\} \\ &= \sum_{n\leq x}\sum_{m\leq An-B}\frac{\phi(m)}{m} - \frac{6}{\pi^2}\sum_{n\leq x}(An-B) \\ &= \sum_{m\leq Ax-B}\frac{\phi(m)}{m}\sum_{\frac{m+B}{A}\leq n\leq x} 1 - \frac{3}{\pi^2}(Ax^2+Ax-2Bx) \\ &= \underbrace{\sum_{m\leq Ax-B}\frac{\phi(m)}{m}\left(x - \left[\frac{m+B}{A}\right]\right)}_{K} + \underbrace{\sum_{\substack{m\leq Ax-B \\ m\equiv -B (mod A)}}\frac{\phi(m)}{m}}_{L} \\ &\quad - \frac{3}{\pi^2}(Ax^2+Ax-2Bx). \end{aligned} \tag{3.11}$$

First consider the summation K to get

$$\sum_{m\le Ax-B}\frac{\phi(m)}{m}\left(x-\left[\frac{m+B}{A}\right]\right)=x\sum_{m\le Ax-B}\frac{\phi(m)}{m}-\sum_{m\le Ax-B}\frac{\phi(m)}{m}\left[\frac{m+B}{A}\right]$$

$$=x\sum_{m\le Ax-B}\frac{\phi(m)}{m}$$

$$-\sum_{a=0}^{A-1}\sum_{\substack{m\le Ax-B\\ m+B\equiv a(modA)}}\frac{\phi(m)}{m}\frac{m+B-a}{A}$$

$$=\left\{x\sum_{m\le Ax-B}\frac{\phi(m)}{m}-\frac{1}{A}\sum_{m\le Ax-B}\phi(m)\right\}$$

$$-\sum_{a=0}^{A-1}\frac{B-a}{A}\sum_{\substack{m\le Ax-B\\ m+B\equiv a(modA)}}\frac{\phi(m)}{m}.$$

Next we add and subtract

$$\frac{B-1}{A}\sum_{m\le Ax-B}\frac{\phi(m)}{m}$$

and

$$\frac{3}{\pi^2}[(Ax-B)^2+(Ax-B)]$$

to get

$$K=\frac{1}{A}\left\{(Ax-B+1)\sum_{m\le Ax-B}\frac{\phi(m)}{m}-\sum_{m\le Ax-B}\phi(m)-\frac{3}{\pi^2}[(Ax-B)^2+(Ax-B)]\right\}$$

$$+\frac{B-1}{A}\sum_{m\le Ax-B}\frac{\phi(m)}{m}+\sum_{a=0}^{A-1}\frac{a-B}{A}\sum_{\substack{m\le Ax-B\\ m+B\equiv a(modA)}}\frac{\phi(m)}{m}$$

$$+\frac{3}{\pi^2}\frac{[(Ax-B)^2+(Ax-B)]}{A}.$$

Observe that, as we did in (2.8), the terms in parenthesis are equal to

$$\frac{1}{A}\sum_{m\le Ax-B}H(m).$$

Using this the summation K becomes

$$= \frac{1}{A}\sum_{m\le Ax-B} H(m) + \frac{B-1}{A}\sum_{m\le Ax-B}\frac{\phi(m)}{m} + \sum_{a=0}^{A-1}\frac{a-B}{A}\sum_{\substack{m\le Ax-B\\ m+B\equiv a(modA)}}\frac{\phi(m)}{m}$$

$$+ \frac{3}{\pi^2}\frac{[(Ax-B)^2+(Ax-B)]}{A}.$$

Note that

$$\frac{B-1}{A}\sum_{m\le Ax-B}\frac{\phi(m)}{m} = \frac{B-1}{A}\frac{6}{\pi^2}(Ax-B) + O(\log x)$$

and

$$\frac{3}{\pi^2}\frac{[(Ax-B)^2+(Ax-B)]}{A} = \frac{3}{\pi^2}Ax^2 - \frac{6}{\pi^2}Bx + \frac{3}{A}\frac{B^2}{\pi^2} + \frac{3}{\pi^2} - \frac{3}{A}\frac{B}{\pi^2}.$$

Taking these into account, the summation K further equals

$$= \frac{1}{A}\sum_{m\le Ax-B} H(m) + \frac{3}{\pi^2}Ax^2 - \frac{3}{\pi^2}x + \underbrace{\sum_{a=0}^{A-1}\frac{a-B}{A}\sum_{\substack{m\le Ax-B\\ m+B\equiv a(modA)}}\frac{\phi(m)}{m}}_{K_1} + O(\log x).$$

Now we evaluate the sum K_1 by using lemma (3.4).

$$\begin{aligned} K_1 &= \sum_{a=0}^{A-1}\frac{a-B}{A}\sum_{\substack{m\le Ax-B\\ m+B\equiv a(modA)}}\frac{\phi(m)}{m} \\ &= \sum_{a=0}^{A-1}\frac{(a-B)}{A}\frac{C}{A}(Ax-B)\sum_{d|(A,a-B)}\frac{\mu(d)}{d} + O(\log x) \\ &= \frac{Cx}{A}\sum_{a=0}^{A-1}(a-B)\sum_{d|(A,a-B)}\frac{\mu(d)}{d} + O(\log x). \end{aligned}$$

If we let $c = a - B$, then above equation becomes

$$= \frac{Cx}{A} \sum_{c=-B}^{A-B-1} c \sum_{d|(A,c)} \frac{\mu(d)}{d} + O(\log x)$$

$$= \frac{Cx}{A} \left\{ \sum_{c=0}^{A-B-1} c \sum_{d|(A,c)} \frac{\mu(d)}{d} + \sum_{c=A-B}^{A-1} (c-A) \sum_{d|(A,c)} \frac{\mu(d)}{d} \right\} + O(\log x)$$

$$= \frac{Cx}{A} \left\{ \sum_{c=0}^{A-B-1} c \sum_{d|(A,c)} \frac{\mu(d)}{d} + \sum_{c=A-B}^{A-1} (c) \sum_{d|(A,c)} \frac{\mu(d)}{d} - A \sum_{c=A-B}^{A-1} \sum_{d|(A,c)} \frac{\mu(d)}{d} \right\}$$
$$+ O(\log x)$$

$$= \frac{Cx}{A} \left\{ \sum_{c=0}^{A-1} c \sum_{d|(A,c)} \frac{\mu(d)}{d} - A \sum_{c=A-B}^{A-1} \sum_{d|(A,c)} \frac{\mu(d)}{d} \right\} + O(\log x)$$

$$= \frac{Cx}{A} \left\{ \sum_{c=0}^{A-1} c \sum_{\substack{d|A \\ d|c}} \frac{\mu(d)}{d} - A \sum_{c=1}^{B} \sum_{d|(A,c)} \frac{\mu(d)}{d} \right\} + O(\log x)$$

$$= \frac{Cx}{A} \left\{ \sum_{d|A} \frac{\mu(d)}{d} \sum_{\substack{1 \le c \le A-1 \\ c \equiv 0 (mod\ d)}} c - A \sum_{c=1}^{B} \frac{\phi(A,c)}{(A,c)} \right\} + O(\log x)$$

$$= \frac{Cx}{A} \left\{ \frac{1}{2} \sum_{d|A} \mu(d) \sum_{1 \le c/d \le (A-1)/d} \frac{c}{d} - A \sum_{c=1}^{B} \frac{\phi(A,c)}{(A,c)} \right\} + O(\log x)$$

$$= \frac{Cx}{A} \left\{ \frac{1}{2} \sum_{d|A} \mu(d) \left\{ \left(\frac{A}{d}\right)^2 - \left(\frac{A}{d}\right) \right\} - A \sum_{c=1}^{B} \frac{\phi(A,c)}{(A,c)} \right\} + O(\log x)$$

$$= \frac{Cx}{A} \left\{ \frac{1}{2} A^2 \sum_{d|A} \frac{\mu(d)}{d^2} - \frac{1}{2}\phi(A) - A \sum_{c=1}^{B} \frac{\phi(A,c)}{(A,c)} \right\} + O(\log x)$$

$$= x \left\{ \frac{CA}{2} \sum_{d|A} \frac{\mu(d)}{d^2} - \frac{1}{2} \frac{\phi(A)C}{A} - C \sum_{c=1}^{B} \frac{\phi(A,c)}{(A,c)} \right\} + O(\log x). \tag{3.12}$$

We remark

$$\sum_{c=1}^{B} \frac{\phi(A,c)}{(A,c)} = 0 \quad \text{when} \quad B = 0.$$

Thus, summation K becomes

$$= \frac{1}{A} \sum_{m \leq Ax-B} H(m) + \frac{3}{\pi^2} A x^2 - \frac{3}{\pi^2} x$$
$$+ x \left\{ \frac{CA}{2} \sum_{d|A} \frac{\mu(d)}{d^2} - \frac{1}{2} \frac{\phi(A)C}{A} - C \sum_{c=1}^{B} \frac{\phi(A,c)}{(A,c)} \right\} + O(\log x).$$

Next consider summation L, which is

$$\sum_{\substack{m \leq Ax-B \\ m \equiv -B (modA)}} \frac{\phi(m)}{m}.$$

Using Lemma 3.4, we get

$$L = \sum_{\substack{m \leq Ax-B \\ m \equiv -B (modA)}} \frac{\phi(m)}{m} = \frac{C}{A} \sum_{d|(A,-B)} \frac{\mu(d)}{d} (Ax - B) + O(\log x)$$
$$= Cx \sum_{d|(A,B)} \frac{\mu(d)}{d} - \frac{CB}{A} \sum_{d|(A,-B)} \frac{\mu(d)}{d} + O(\log x)$$
$$= Cx \frac{\phi(A,B)}{(A,B)} + O(\log x).$$

Going back to our main equation (3.11) and inserting summation K and summation L, we obtain

$$\sum_{m \leq x} H(An - B) = \frac{1}{A} \sum_{m \leq Ax-B} H(m) + \frac{3}{\pi^2} A x^2 - \frac{3}{\pi^2} x + Cx \frac{\phi(A,B)}{(A,B)}$$
$$+ x \left\{ \frac{CA}{2} \sum_{d|A} \frac{\mu(d)}{d^2} - \frac{1}{2} \frac{\phi(A)C}{A} - C \sum_{c=1}^{B} \frac{\phi(A,c)}{(A,c)} \right\} + O(\log x).$$

Recall that

$$C = \prod_{p \nmid A} \left(1 - \frac{1}{p^2} \right).$$

Therefore

$$C \sum_{d|A} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} \quad \text{follows.}$$

In this way, one obtains

$$\begin{aligned}\sum_{m\le x} H(An-B) &= \frac{1}{A}\sum_{m\le Ax-B} H(m) + \frac{3}{\pi^2}Ax^2 - \frac{3}{\pi^2}x + \frac{3}{\pi^2}Ax - \frac{x}{2}\frac{\phi(A)C}{A} \\ &- Cx\sum_{c=1}^{B}\frac{\phi(A,c)}{(A,c)} + Cx\frac{\phi(A,B)}{(A,B)} - \frac{3}{\pi^2}[Ax^2 + Ax - 2Bx] \\ &+ O(\log x) \\ &= \frac{1}{A}\sum_{m\le Ax-B} H(m) + \frac{6}{\pi^2}Bx - \frac{x}{2}\frac{\phi(A)C}{A} - Cx\sum_{c=1}^{B}\frac{\phi(A,c)}{(A,c)} \\ &- \frac{3}{\pi^2}x + O(\log x) \\ &= \frac{1}{A}\sum_{m\le Ax-B} H(m) + \triangle x + O(\log x) \qquad (3.13)\end{aligned}$$

which finally gives desired the result. □

Theorem 3.6. *For any integers A and B with $A > B \ge 0$*

$$\sum_{mn\le x} H(An-B) = M(A,B)x\log x + O(x). \qquad (3.14)$$

Proof. From the previous theorem, since $H(x) = O(\log x)$ we have

$$\begin{aligned}\sum_{m\le x} H(An-B) &= \frac{1}{A}\sum_{m\le Ax} H(m) - \frac{1}{A}\sum_{Ax-B<m\le Ax} H(m) + \triangle x + O(\log x) \\ &= \frac{1}{A}\sum_{m\le Ax} H(m) + \triangle x + O(\log x).\end{aligned}$$

If we replace x by x/m in preceding equation, we obtain

$$\sum_{n\le x/m} H(An-B) = \frac{1}{A}\sum_{n\le \frac{Ax}{m}} H(n) + \triangle\frac{x}{m} + O(\log\frac{x}{m}).$$

Summing the above sum over all integers $m \le x$, we have

$$\begin{aligned}\sum_{mn\le x} H(An-B) &= \frac{1}{A}\sum_{m\le x}\sum_{n\le \frac{Ax}{m}} H(n) + \triangle \sum_{m\le x}\frac{x}{m} + O(\sum_{m\le x}\log\frac{x}{m}) \\ &= \frac{1}{A}\sum_{m\le Ax}\sum_{n\le \frac{Ax}{m}} H(n) - \frac{1}{A}\sum_{x<m\le Ax}\sum_{n\le \frac{Ax}{m}} H(n) \\ &+ \triangle x\log x + O(x).\end{aligned}$$

Since

$$\sum_{x<m\le Ax}\sum_{n\le \frac{Ax}{m}} H(n) = \sum_{a=1}^{A-1}\sum_{n\le A-a} H(n) = O(1),$$

we have

$$\sum_{mn\le x} H(An-B) = \frac{1}{A}\sum_{mn\le Ax} H(n) + \triangle x\log x + O(x).$$

Using Theorem 3.3, we have

$$\begin{aligned}\sum_{mn\le x} H(An-B) &= \frac{1}{A}\frac{3}{\pi^2}Ax\log Ax + \triangle x\log x + O(x) \\ &= \frac{3}{\pi^2}x\log x + \triangle x\log x + O(x) \\ &= M(A,B)x\log x + O(x).\end{aligned}$$

□

If we combine Theorem 3.6 with the result

$$\sum_{n\le x} H(n) \sim \frac{3}{\pi^2}x,$$

which we proved in section 2, we get a new result.

Theorem 3.7. *For A, B any integers, $A > B \ge 0$,*

$$\sum_{n\le x} H(An-B) \sim M(A,B)x. \tag{3.15}$$

Proof. From theorem (3.5), we have

$$\sum_{n\le x} H(An-B) = \frac{1}{A}\sum_{n\le Ax} H(n) + \triangle x + O(\log x).$$

If we put

$$\sum_{n\le x} H(n) \sim \frac{3}{\pi^2}x$$

into equation we get

$$\begin{aligned}\sum_{n\le x} H(An-B) &\sim \frac{3}{\pi^2}x + \triangle x \\ &= M(A,B)x.\end{aligned}$$

□

3.3 Sign Changes of $H(x)$

We first note that $A = A_\kappa = \prod_{i=1}^{\kappa} p_i$, where p_i denote the i^{th} prime number and κ is sufficiently large.

Theorem 3.8. *For $B \neq 0$, and fixed, we have*

$$\lim_{\kappa\to\infty}\lim_{x\to\infty}\frac{1}{x\log x}\sum_{mn\le x} H(A_\kappa n - B) = \frac{6}{\pi^2} - H(B-1). \tag{3.16}$$

Proof. First observe that from Theorem 3.6, we have

$$\begin{aligned}\frac{1}{x\log x}\sum_{mn\le x} H(A_\kappa n-B) &= M(A_\kappa,B) + O\left(\frac{1}{\log x}\right) \\ &= \frac{6}{\pi^2}B - \frac{1}{2}\frac{\phi(A_\kappa)C(A_\kappa)}{A_\kappa} - C(A_\kappa)\sum_{c=1}^{B-1}\frac{\phi(A_\kappa,c)}{(A_\kappa,c)} \\ &+ O\left(\frac{1}{\log x}\right).\end{aligned}$$

Now letting $\kappa\to\infty$ gives, $C(A_\kappa)=1$, $\frac{1}{2}\frac{\phi(A_\kappa)C(A_\kappa)}{A_\kappa}=0$ and $(A_\kappa,c)=c$, so

$$\sum_{c=1}^{B-1}\frac{\phi(A_\kappa,c)}{(A_\kappa,c)} = \sum_{c=1}^{B-1}\frac{\phi(c)}{c} = \frac{6}{\pi^2}(B-1) + H(B-1).$$

And after letting $x\to\infty$, we get

$$\lim_{\kappa\to\infty}\lim_{x\to\infty}\frac{1}{x\log x}\sum_{mn\le x} H(A_\kappa n - B) = \frac{6}{\pi^2} - H(B-1).$$

□

From Theorem 3.3 it follows that $H(n)$ is positive for infinitely many n. In the next theorem, we show that one cannot have $H(n) \geq 0$ for all sufficiently large n.

Theorem 3.9. *$H(n) < 0$ for some sufficiently large n.*

Proof. Assume for a contradiction that $H(n) \geq 0$ for all sufficiently large n. Then by using Theorem 3.8, for all sufficiently large B we have

$$\lim_{\kappa\to\infty} \lim_{x\to\infty} \frac{1}{x \log x} \sum_{mn\leq x} H(A_\kappa n - B) = \frac{6}{\pi^2} - H(B-1) \geq 0.$$

By definition of $H(n)$, we write

$$H(B) - H(B-1) = \frac{\phi(B)}{B} - \frac{6}{\pi^2}.$$

This gives

$$\frac{6}{\pi^2} - H(B-1) = \frac{\phi(B)}{B} - H(B).$$

Therefore

$$\frac{\phi(B)}{B} \geq H(B) \geq 0.$$

Choosing $\epsilon > 0$ and a large odd number B such that $\frac{\phi(B)}{B} < \epsilon$, we see that

$$H(B+1) = H(B) - \frac{6}{\pi^2} + \frac{\phi(B+1)}{B+1} \leq \epsilon - \frac{6}{\pi^2} + \frac{1}{2} < 0$$

which gives the required contradiction. □

In the previous section we proved that $H(x) \neq o(\log\log\log x)$. Thus there exist infinitely many integers x such that

$$|H(x)| > c \log\log\log x \quad \text{for some } c > 0. \tag{3.17}$$

Note that using (3.17), we see that given any large number $N \geq 6$, one can find integer B such that $|H(B)| > N$. This brings us to an important result.

Theorem 3.10. *For integral x, we have*

$$\limsup_{x\to\infty} H(x) = \infty \quad \textit{and} \quad \liminf_{x\to\infty} H(x) = -\infty. \tag{3.18}$$

Proof. From implication of (3.17) we have two cases, either $H(B) > N$ or $H(B) < -N$.

Case 1: $H(B) > N$.

From (3.16), we have

$$\lim_{\kappa\to\infty}\lim_{x\to\infty}\frac{1}{x\log x}\sum_{mn\le x}H(A_\kappa n-B)=\frac{\phi(B)}{B}-H(B)<-N+1.$$

Therefore, for all sufficiently large κ, say $\kappa\ge\kappa_0$, we have

$$\lim_{x\to\infty}\frac{1}{x\log x}\sum_{mn\le x}H(A_\kappa n-B)<-N+2.$$

Then for each κ, there exists an $x_0=x_0(\kappa)$ such that for all $x\ge x_0$

$$\sum_{mn\le x}H(A_\kappa n-B)<(-N+3)x\log x. \tag{3.19}$$

We conclude from (3.19) that for each $\kappa\ge\kappa_0$, we have an $n^*=n^*(\kappa)$ such that

$$H(A_\kappa n^*-B)<-N+3\le-\frac{1}{2}N.$$

Since otherwise we would get a contradiction from

$$\sum_{mn\le x}1=\sum_{m\le x}\sum_{n\le x/m}1\sim\sum_{m\le x}\frac{x}{m}\sim x\log x.$$

Case 2: $H(B)<-N$.

We have

$$\lim_{\kappa\to\infty}\lim_{x\to\infty}\frac{1}{x\log x}\sum_{mn\le x}H(A_\kappa n-B)=\frac{\phi(B)}{B}-H(B)>N$$

Therefore, for all sufficiently large κ, say $\kappa\ge\kappa_0$, we have

$$\lim_{x\to\infty}\frac{1}{x\log x}\sum_{mn\le x}H(A_\kappa n-B)>N-1.$$

Then for each κ there exists an $x_0=x_0(\kappa)$ such that for all $x\ge x_0$

$$\sum_{mn\le x}H(A_\kappa n-B)>(N-3)x\log x. \tag{3.20}$$

We conclude from (3.20) that for each $\kappa\ge\kappa_0$, we have an $n^*=n^*(\kappa)$ such that

$$H(A_\kappa n^*-B)>N-3\ge\frac{1}{2}N.$$

□

Theorem 3.11. *For integral x, we have*

$$\limsup_{x\to\infty} \frac{E(x)}{x} = \infty \quad \textit{and} \quad \liminf_{x\to\infty} \frac{E(x)}{x} = -\infty. \tag{3.21}$$

Proof. From Theorem (2.6), we have

$$\frac{E(x)}{x} = H(x) + O(1).$$

Hence, (3.21) follows immediately from the previous theorem. □

3.4 More Precise Results

In this section, we prove our main theorem

Theorem 3.12. *For $c > 0$, there exist infinitely many x such that*

$$H(x) > c \log\log\log\log x \tag{3.22}$$

and infinitely many x such that

$$H(x) < -c \log\log\log\log x. \tag{3.23}$$

Proof. Recall that from Lemma 3.4, we have

$$\sum_{\substack{m\le z \\ m\equiv\beta(modA)}} \frac{\phi(m)}{m} = \frac{C}{A} \sum_{d|(A,\beta)} \frac{\mu(d)}{d} z + O(\log z).$$

To analyze the error term more carefully in the proof of Lemma 3.4, let $\tau = (d, A)$ and $d = k\tau$, so that

$$\begin{aligned}
O\left(\sum_{\substack{d\le z \\ (d,A)|\beta}} \frac{|\mu(d)|}{d} \right) &= O\left(\sum_{\substack{d\le z \\ \tau|\beta\ \tau|A}} \frac{\mu^2(d)}{d} \right) \\
&= O\left(\sum_{\tau|\beta\ \tau|A} \frac{\mu^2(\tau)}{\tau} \sum_{k\le z/\tau} \frac{\mu^2(k)}{k} \right) \\
&= O\left(\sum_{\tau|(A,\beta)} \mu^2(\tau) \log \frac{z}{\tau} \right).
\end{aligned}$$

Using above result, we have from Lemma 3.4 that

$$\sum_{\substack{m \leq z \\ m \equiv \beta (mod A)}} \frac{\phi(m)}{m} = \frac{C}{A} \sum_{d|(A,\beta)} \frac{\mu(d)}{d} z + O\left(\sum_{\tau|(A,\beta)} \mu^2(\tau) \log \frac{z}{\tau} \right). \tag{3.24}$$

Using (3.24) instead of (3.8) in the proof of Theorem 3.5, we obtain for integral x,

$$\begin{aligned} \sum_{\substack{m \leq Ax-B \\ m \equiv -\beta (mod A)}} \frac{\phi(m)}{m} &= \frac{C}{A} \sum_{d|(A,\beta)} \frac{\mu(d)}{d}(Ax-B) + O\left(\sum_{\tau|(A,\beta)} \mu^2(\tau) \log \frac{Ax-B}{\tau} \right) \\ &= \frac{C}{A} \sum_{d|(A,\beta)} \frac{\mu(d)}{d}(Ax-B) + O\left(\sum_{\tau|(A,\beta)} \mu^2(\tau) \log Ax \right) \\ &= \frac{C}{A} \sum_{d|(A,\beta)} \frac{\mu(d)}{d}(Ax-B) + O(2^{v(A)} \log Ax). \end{aligned}$$

This gives

$$\sum_{n \leq x} H(An-B) = \frac{1}{A} \sum_{n \leq Ax} H(n) + \triangle x + O(2^{v(A)} \log Ax). \tag{3.25}$$

Combining (3.25) with

$$\sum_{n \leq x} H(n) \sim \frac{3}{\pi^2} x,$$

we get

$$\frac{1}{A} \sum_{n \leq Ax} H(n) + \triangle x \sim \frac{3}{\pi^2} x + \triangle x = M(A,B)x.$$

Using this in (3.25), we get

$$\sum_{n \leq x} H(An-B) = M(A,B)x + O(2^{v(A)} \log Ax) + o(x) \tag{3.26}$$

where both the O and o are uniform in A. Now, let us take

$$x = A = \prod_{p \leq B} p.$$

Then

$$C(A) = \prod_{p > B} \left(1 - \frac{1}{p^2}\right).$$

Here $C(A)$ depends on B, and we know that $\frac{6}{\pi^2} \leq C(A) \leq 1$. Therefore, we can find constants $c_1 > 0$ and $c_2 > 0$ such that $1 - \frac{c_1}{B} < C(A) < 1 - \frac{c_2}{B}$.

In this way, (3.26) becomes

$$\begin{aligned}\frac{1}{A}\sum_{n\leq x} H(An-B) &= M(A,B) + O\left(\frac{2^{v(A)+1}\log A}{A}\right) + o(1)\\ &= M(A,B) + O(1)\\ &= \frac{6}{\pi^2}B - \frac{1}{2}\frac{\phi(A)}{A}C(A) - C(A)\sum_{c=1}^{B-1}\frac{\phi(A,c)}{(A,c)} + O(1).\end{aligned}$$

Here we have

$$\sum_{c=1}^{B-1}\frac{\phi(A,c)}{(A,c)} = \frac{6}{\pi^2}B - \frac{6}{\pi^2} + H(B-1) = \frac{6}{\pi^2}B + H(B) - \frac{\phi(B)}{B}.$$

This gives

$$\begin{aligned}\frac{1}{A}\sum_{n\leq x} H(An-B) - \frac{6}{\pi^2}B + \frac{1}{2}\frac{\phi(A)}{A}C(A) + C(A)\frac{6}{\pi^2}B + C(A)H(B)&\\ -C(A)\frac{\phi(B)}{B} = O(1).&\end{aligned} \tag{3.27}$$

Since we have $1 - \frac{c_1}{B} < C(A) < 1 - \frac{c_2}{B}$, for some c_1, c_2 positive

$$\begin{aligned}\frac{1}{A}\sum_{n\leq x} H(An-B) - \frac{6}{\pi^2}B + \frac{1}{2}\frac{\phi(A)}{A}C(A) + C(A)\frac{6}{\pi^2}B + C(A)H(B)&\\ -C(A)\frac{\phi(B)}{B} \geq&\\ \frac{1}{A}\sum_{n\leq x} H(An-B) - \frac{6}{\pi^2}B + \left(1-\frac{c_1}{B}\right)\frac{6}{\pi^2}B + \left(1-\frac{c_2}{B}\right)\left(H(B) - \frac{\phi(B)}{B}\right)&\end{aligned}$$

follows. From (3.27), we have

$$\begin{aligned}\frac{1}{A}\sum_{n\leq x} H(An-B) - \frac{6c_1}{\pi^2} + H(B) - \frac{\phi(B)}{B} - \frac{c_2H(B)}{B}&\\ + \frac{c_2\phi(B)}{B^2} = O(1).&\end{aligned} \tag{3.28}$$

To simplify (3.28), note that $\frac{6c_1}{\pi^2} = O(1)$, $\frac{\phi(B)}{B} \leq 1$, $\frac{c_2H(B)}{B} = O(1)$ and

$\frac{c_2\phi(B)}{B^2} < 1$. Consequently,

$$\frac{1}{A}\sum_{n\leq x} H(An-B) + H(B) = O(1) \quad \text{holds.} \tag{3.29}$$

This means there exist l independent of A,B such that

$$\left|\frac{1}{A}\sum_{n\leq x} H(An-B) + H(B)\right| \leq l \tag{3.30}$$

for sufficiently large B.

We know that for infinitely many B

$$|H(B)| > c\log\log\log B \quad \text{holds.}$$

There are two cases:

Case 1: $H(B) > c\log\log\log B$.

First observe that, since we take

$$A = \prod_{p\leq B} p,$$

for large B, we have

$$\log A = \sum_{p\leq B} \log p \sim B.$$

(3.30) implies that there exist $n^* \leq A$, such that

$$\begin{aligned} H(An^*-B) &\leq l - c\log\log\log B \\ &\leq -\frac{1}{2}c\log\log\log B \\ &\leq -c_1\log\log\log\log(An^*-B). \end{aligned}$$

Case 2: $H(B) < -c\log\log\log B$.

Again (3.30) implies that there exist an $n^* \leq A$ such that

$$\begin{aligned} H(An^*-B) &\geq c\log\log\log B - l \\ &\geq \frac{1}{2}c\log\log\log B \\ &\geq c_1\log\log\log\log(An^*-B). \end{aligned}$$

Which concludes the proof. $\square$

Theorem 3.13. *For $c > 0$, there exist infinitely many x such that*

$$E(x) > cx \log \log \log \log x$$

and infinitely many x such that

$$E(x) < -cx \log \log \log \log x.$$

Proof. From theorem 2.6 we have

$$\frac{E(x)}{x} = H(x) + O(1).$$

Thus, combining this with theorem 3.12, we have the desired result. $\square$

4 Improved Result of Sign Changes of Error Term in Mean of $\phi(n)$

In this secton we show that $E(x) = \Omega_{\pm}(x\sqrt{\log\log x})$, which is a further improvement result of the previous section. In other words, there exist a positive constant c and infinitely many integers x such that

$$E(x) > cx\sqrt{\log\log x}$$

and infinitely many integer x such that

$$E(x) < -cx\sqrt{\log\log x}.$$

Here we introduce a new approach to the problem such as complex integration with employing Perron's formula. Now let us begin with some preliminaries.

4.1 Basic lemmas

First we define $s(x)$ to be the 'saw tooth' function. Which is, $s(x)$ has period 1, $s(0) = 0$, and $s(x) = \frac{1}{2} - \{x\}$ for $0 < \{x\} < 1$.

Lemma 4.1. *We have*

$$H(x) = \frac{1}{2}\frac{\phi(x)}{x} + \sum_{d\le y}\mu(d)\frac{s(x/d)}{d} + O\left(e^{-c_1\sqrt{\log x}}\right) \tag{4.1}$$

uniformly for $x \ge 2$, $y \ge xe^{-c_1\sqrt{\log x}}$.

Here we set $\phi(x) = 0$, if x is not an integer. The constants c_i are positive and absolute.

Proof. Observe that we have three cases for y; namely $y = x$, $y > x$ and $xe^{-c_1\sqrt{\log x}} \le y < x$. First consider the case $y = x$. We recall that

$$\frac{\phi(n)}{n} = \sum_{d|n}\frac{\mu(d)}{d}.$$

It follows that

$$\begin{aligned}\sum_{n\le x}\frac{\phi(n)}{n} &= \sum_{n\le x}\sum_{d|n}\frac{\mu(d)}{d}\\ &= \sum_{d\le x}\frac{\mu(d)}{d}\sum_{q\le x/d}1\end{aligned}$$

$$= x\sum_{d\le x}\frac{\mu(d)}{d^2} - \sum_{d\le x}\frac{\mu(d)}{d}\left\{\frac{x}{d}\right\}$$
$$= \frac{6}{\pi^2}x - x\sum_{d\ge x}\frac{\mu(d)}{d^2} - \sum_{d\le x}\frac{\mu(d)}{d}\left\{\frac{x}{d}\right\}.$$

This gives

$$H(x) = -x\sum_{d\ge x}\frac{\mu(d)}{d^2} - \sum_{d\le x}\frac{\mu(d)}{d}\left\{\frac{x}{d}\right\}. \tag{4.2}$$

Here we put

$$L(x) = \sum_{d\le x}\frac{\mu(d)}{d}.$$

From the PNT with De la Vallée Poussin error term, we know that $L(x) \ll e^{-c_0\sqrt{\log x}}$ for $x \ge 2$. Hence, by using partial summation in (4.2)

$$\sum_{d\ge x}\frac{\mu(d)}{d^2}$$

becomes

$$\sum_{d\ge x}\frac{\mu(d)}{d^2} = \sum_{x\le d<y}\frac{\mu(d)}{d}\frac{1}{d} = \frac{L(y)}{y} - \frac{L(x)}{x} + \int_x^y \frac{L(t)}{t^2}dt$$
$$\ll \frac{e^{-c_0\sqrt{\log y}}}{y} - \frac{e^{-c_0\sqrt{\log x}}}{x} + \int_x^y \frac{e^{-c_0\sqrt{\log t}}}{t^2}dt.$$

Letting $y \to \infty$, we have

$$\sum_{d\ge x}\frac{\mu(d)}{d^2} \ll \frac{1}{x}e^{-c_0\sqrt{\log x}}.$$

Using this, and adding $L(x)/2$ to (4.2) we see that

$$H(x) = \sum_{d\le x}\frac{\mu(d)}{d}\left(\frac{1}{2} - \left\{\frac{x}{d}\right\}\right) + O(e^{-c_0\sqrt{\log x}}). \tag{4.3}$$

In the case when x is not an integer, we have $s(x) = 1/2 - \{x\}$ and $\phi(x) = 0$ so that

$$H(x) = \frac{1}{2}\frac{\phi(x)}{x} + \sum_{d\le y}\mu(d)\frac{s(x/d)}{d} + O\left(e^{-c_1\sqrt{\log x}}\right)$$

by setting $c_1 = c_0$. On the other hand, if x is an integer, we have $s(x/d) = 0$

while $d|x$. If we calculate the sum in this case we get

$$\frac{1}{2}\sum_{d\leq x}\frac{\mu(d)}{d}=\frac{1}{2}\frac{\phi(x)}{x}.$$

Putting this into (4.3), we see that

$$H(x)=\frac{1}{2}\frac{\phi(x)}{x}+\sum_{d\leq y}\mu(d)\frac{s(x/d)}{d}+O\left(e^{-c_1\sqrt{\log x}}\right)$$

again by setting $c_1=c_0$.

When $y>x$, using the case $y=x$ it suffices to show that

$$\sum_{x<d\leq y}\mu(d)\frac{s(x/d)}{d}=O(e^{-c_0\sqrt{\log x}}).$$

Applying partial summation, we get

$$\sum_{x<d\leq y}\mu(d)\frac{s(x/d)}{d}=\left(\frac{1}{2}-\frac{x}{y}\right)L(y)-\frac{1}{2}L(x)-x\int_x^y\frac{L(t)}{t^2}dt.$$

Since $y>x$, $L(y)<L(x)\ll e^{-c_0\sqrt{\log x}}$ so that

$$\sum_{x<d\leq y}\mu(d)\frac{s(x/d)}{d}\ll e^{-c_0\sqrt{\log x}}\quad\text{follows}$$

by taking $c_0=c_1$ when $y>x$.

When $xe^{-c_1\sqrt{\log x}}\leq y\leq x$, again it suffices to show that

$$\sum_{y<d\leq x}\mu(d)\frac{s(x/d)}{d}\ll e^{-c_1\sqrt{\log x}}.$$

Suppose k is a positive integer such that $1\leq k\leq\frac{x}{y}$. Since $s(x/d)$ is monotonic for $\frac{x}{k+1}<d<\frac{x}{k}$, we have

$$\begin{aligned}\sum_{\frac{x}{k+1}<d<\frac{x}{k}}\mu(d)\frac{s(x/d)}{d}&=\frac{1}{2}L\left(\frac{x}{k}\right)-\frac{1}{2}L\left(\frac{x}{k+1}\right)+x\int_{x/(k+1)}^{x/k}\frac{L(t)}{t^2}dt\\&\ll e^{-c_0\sqrt{\log\frac{x}{k+1}}}\\&\ll e^{-\frac{1}{2}c_0\sqrt{\log\frac{x}{k}}}.\end{aligned}$$

Summing over both sides by $1 \le k \le \frac{x}{y}$, we find that

$$\begin{aligned}
\sum_{1\le k\le x/y} \sum_{\frac{x}{k+1}<d<\frac{x}{k}} \mu(d)\frac{s(x/d)}{d} &= \sum_{y<d\le x} \mu(d)\frac{s(x/d)}{d} \\
&\ll \sum_{1\le k\le x/y} e^{-\frac{1}{2}c_0\sqrt{\log \frac{x}{k}}} \\
&\ll \left(\frac{x}{y}-1\right) e^{-\frac{1}{2}c_0\sqrt{\log y}} \\
&\ll \frac{x}{y} e^{-\frac{1}{2}c_0\sqrt{\log y}} \\
&\ll \frac{e^{-\frac{1}{2}c_0\sqrt{\log xe^{-c_1\sqrt{\log x}}}}}{e^{-c_1\sqrt{\log x}}} \\
&\ll \frac{e^{-\frac{1}{2}c_0\sqrt{\log x - c_1\sqrt{\log x}}}}{e^{-c_1\sqrt{\log x}}} \\
&\ll e^{-\frac{1}{2}c_0\sqrt{\log x}+c_1\sqrt{\log x}}.
\end{aligned}$$

Taking $4c_1 = c_0$ the above expression becomes

$$\sum_{y<d\le x} \mu(d)\frac{s(x/d)}{d} \ll e^{-2c_1\sqrt{\log x}+c_1\sqrt{\log x}} = e^{-c_1\sqrt{\log x}}.$$

This completes the proof. □

Lemma 4.2. *Let b and $r > 0$ be relatively prime integers, and let β be a real number. Then*

$$\sum_{n=1}^{r} s(bn/r+\beta) = s(r\beta). \tag{4.4}$$

Proof. First observe that from definition of $s(x)$ both sides are periodic with period $\frac{1}{r}$, so we may assume that $0 \le \beta < \frac{1}{r}$. The numbers bn run through a complete residue system ($mod\ r$), so we may assume that $b = 1$. Then we have two cases; $\beta = 0$ or $0 < \beta < 1/r$.

If $\beta = 0$ then left hand side is

$$\begin{aligned}
\sum_{n=1}^{r} s(bn/r+\beta) = \sum_{n=1}^{r-1} s(n/r) &= \sum_{n=1}^{r-1}(\frac{1}{2}-\frac{n}{r}) \\
&= \frac{1}{2}\sum_{n=1}^{r-1} 1 - \frac{1}{r}\sum_{n=1}^{r-1} n \\
&= \frac{r-1}{2} - \frac{1}{r} r(r-1)/2 = 0
\end{aligned}$$

which gives $s(r\beta) = 0$.

If $0 < \beta < 1$, then sum is

$$\begin{aligned}
\sum_{n=1}^{r} s(n/r + \beta) &= \sum_{n=0}^{r-1} s(n/r + \beta) \\
&= \sum_{n=0}^{r-1} (\frac{1}{2} - n/r - \beta) \\
&= \frac{1}{2} - \beta + \sum_{n=1}^{r-1} (\frac{1}{2} - \frac{n}{r} - \beta) \\
&= \frac{1}{2} - \beta + \frac{r-1}{2} - \frac{1}{2r} r(r-1) - \beta(r-1) \\
&= \frac{1}{2} - r\beta = s(r\beta).
\end{aligned}$$

□

Now in next lemma we extend the sum in (4.4) over an arbitrary interval $1 \le n \le N$.

Lemma 4.3. *Let b and $r > 0$ be relatively prime integers, and let β be a real number. Then for any positive N,*

$$\sum_{n=1}^{N} s(nb/r + \beta) = \frac{N}{r} s(r\beta) + O(r). \tag{4.5}$$

Proof. Let $N = Qr + R$, $0 \le R < r$. By Lemma 4.2

$$\begin{aligned}
\sum_{n=1}^{N} s(nb/r + \beta) &= \sum_{n=1}^{Qr} s(nb/r + \beta) + \sum_{n=1}^{R} s(nb/r + \beta) \\
&= Qs(r\beta) + \sum_{n=1}^{R} s(nb/r + \beta) \\
&= \frac{N}{r} s(r\beta) + O(1) + O(R) \\
&= \frac{N}{r} s(r\beta) + O(r).
\end{aligned}$$

□

Lemma 4.4. *If q is a positive integer, $q \le e^{c_2\sqrt{\log N}}$, α is a non-integral* *real number, $0 < \alpha < q$, then*

$$\sum_{n=1}^{N} H(nq+\alpha) = C(q,\alpha)N + O(Ne^{-c_2\sqrt{\log N}}). \tag{4.6}$$

Where

$$C(q,\alpha) = \frac{6}{\pi^2}\left(\prod_{p|q}(1-p^{-2})\right)^{-1} \sum_{d|q} \mu(d)\frac{s(\alpha/d)}{d}. \tag{4.7}$$

Proof. We apply Lemma 4.1 by taking $y = (N+1)qe^{-c_1\sqrt{\log N}}$. Put $x = nq+\alpha$, where n is an integer and $n \leq N$. Since $nq+\alpha$ is non-integral we have $\phi(nq+\alpha) = 0$, so that

$$H(nq+\alpha) = \sum_{d\leq y} \mu(d)\frac{s((nq+\alpha)/d)}{d} + O(e^{-c_1\sqrt{\log(nq+\alpha)}}). \tag{4.8}$$

If we sum over both sides of (4.8) up to N, we get

$$\sum_{n=1}^{N} H(nq+\alpha) = \sum_{n=1}^{N}\sum_{d\leq y} \mu(d)\frac{(s(nq+\alpha)/d)}{d} + O\left(\sum_{n=1}^{N} e^{-c_1\sqrt{\log(nq+\alpha)}}\right). \tag{4.9}$$

In (4.9), the error term is estimated to be

$$\begin{aligned}\sum_{n=1}^{N} e^{-c_1\sqrt{\log(nq+\alpha)}} &\ll \sum_{n\leq N^{1/4}} e^{-c_1\sqrt{\log nq}} + \sum_{N^{1/4}<n\leq N} e^{-c_1\sqrt{\log nq}} \\ &\ll N^{1/4} + (N-N^{1/4})e^{\frac{-c_1}{2}\sqrt{\log N}}.\end{aligned}$$

Here observe that $N^{1/4} < Ne^{\frac{-c_1}{2}\sqrt{\log N}}$. Taking $c_2 = \frac{c_1}{2}$ we see that the error term is

$$\ll Ne^{-c_2\sqrt{\log N}}. \tag{4.10}$$

Now, we apply Lemma 4.3 to the double sum in (4.9) by taking $r = \frac{d}{(d,q)}$ and $\beta = \frac{\alpha}{d}$. In this way, one obtains

$$\begin{aligned}\sum_{d\leq y}\frac{\mu(d)}{d}\sum_{n=1}^{N} s\left(\frac{nq+\alpha}{d}\right) &= \sum_{d\leq y}\frac{\mu(d)}{d}\left(\frac{N}{d}(d,q)s\left(\frac{\alpha}{(d,q)}\right) + O(d)\right) \\ &= N\sum_{d\leq y}\mu(d)\frac{(d,q)}{d^2}s\left(\frac{\alpha}{(d,q)}\right) + O\left(\sum_{d\leq y}1\right).\end{aligned} \tag{4.11}$$

Here error term is

$$\sum_{d\le y} 1 \ll (N+1)qe^{-c_1\sqrt{\log N}} = Nqe^{-c_1\sqrt{\log N}} + qe^{-c_1\sqrt{\log N}}$$
$$\ll Nqe^{-c_1\sqrt{\log N}}$$
$$\ll Ne^{c_2-c_1\sqrt{\log N}}.$$

Again taking $c_2 = \frac{c_1}{2}$, we see that

$$O\left(\sum_{d\le y} 1\right) \ll Ne^{-c_2\sqrt{\log N}}. \tag{4.12}$$

To estimate the main term in (4.11), we take d square free, since otherwise $\mu(d) = 0$. Let us write $d = ef$ with $e|q$, $(f,q) = 1$. Clearly, $e = (d,q)$. Thus the main term in equation (4.11) becomes

$$N\sum_{d\le y}\mu(d)\frac{(d,q)}{d^2}s\left(\frac{\alpha}{(d,q)}\right) = N\sum_{\substack{ef\le y\\(f,q)=1}}\frac{\mu(ef)}{e^2f^2}es\left(\frac{\alpha}{e}\right)$$
$$= N\sum_{e\le y}\frac{\mu(e)}{e}s\left(\frac{\alpha}{e}\right)\sum_{\substack{f\le y/e\\(f,q)=1}}\frac{\mu(f)}{f^2}. \tag{4.13}$$

Here observe that

$$\sum_{\substack{f\le y/e\\(f,q)=1}}\frac{\mu(f)}{f^2} = \sum_{\substack{f=1\\(f,q)=1}}^{\infty}\frac{\mu(f)}{f^2} - \sum_{\substack{f>y/e\\(f,q)=1}}\frac{\mu(f)}{f^2}$$
$$= \frac{6}{\pi^2}\left(\prod_{p|q}(1-p^{-2})\right)^{-1} + O\left(\sum_{f>y/e}\frac{1}{f^2}\right)$$
$$= \frac{6}{\pi^2}\left(\prod_{p|q}(1-p^{-2})\right)^{-1} + O(e/y).$$

Therefore, equation (4.13) becomes

$$= N\frac{6}{\pi^2}\left(\prod_{p|q}(1-p^{-2})\right)^{-1}\sum_{e|q}\frac{\mu(e)}{e}s\left(\frac{\alpha}{e}\right) + O\left(N\sum_{e|q}\frac{1}{y}\right). \tag{4.14}$$

The proof will be complete, if we can show that $O\left(N\sum_{e|q}\frac{1}{y}\right) \ll Ne^{-c_2\sqrt{\log N}}$. Since e is square free,

$$\begin{aligned}\frac{N}{y}\sum_{e|q}1 = \frac{N}{y}2^{w(q)} &\ll \frac{2^{w(q)}}{q}e^{c_1\sqrt{\log N}}\\ &\ll e^{c_1\sqrt{\log N}}\\ &\ll Ne^{-c_1/2\sqrt{\log N}} = Ne^{-c_2\sqrt{\log N}} \quad \text{follows.}\end{aligned}$$

Finally combining (4.9),(4.10),(4.12) and (4.14), we have

$$\sum_{n=1}^{N} H(nq+\alpha) = C(q,\alpha)N + O(Ne^{-c_2\sqrt{\log N}}) \quad \text{as desired.}$$

$\square$

4.2 Sign Changes in the Mean of $\phi(n)$

Before proving our main theorems we recall the following result which is a consequence of the Mellin transform.

Let $\alpha(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$ be Dirichlet series of a_n with $\sigma_0 > max(0,\sigma_c)$ where σ_c is abscissa of convergence, and for a positive integer k put

$$C_k(x) = \frac{1}{k!}\sum_{n\le x} a_n(x-n)^k.$$

Then we have

$$C_k(x) = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\alpha(s)\frac{x^{s+k}}{s(s+1)\cdots(s+k)} \tag{4.15}$$

for $x > 0$. We refer the proof of this formula to [2].

Theorem 4.5. *For* $x \ge 2$

$$H(x) = \frac{E(x)}{x} + O(e^{-c\sqrt{\log x}}). \tag{4.16}$$

Where c *is a positive absolute constant.*

Proof. From Theorem 2.1 and 2.2, we have

$$H(x) = \sum_{n\le x}\frac{\phi(n)}{n} - \frac{6}{\pi^2}x$$

and

$$\frac{E(x)}{x} = \sum_{n \le x} \frac{\phi(n)}{x} - \frac{3}{\pi^2}x.$$

By using these

$$\sum_{n \le x} \left(1 - \frac{n}{x}\right) \frac{\phi(n)}{n} = \sum_{n \le x} \frac{\phi(n)}{n} - \sum_{n \le x} \frac{\phi(n)}{x} = H(x) + \frac{3}{\pi^2}x - \frac{E(x)}{x}. \quad (4.17)$$

It suffices to show that

$$\sum_{n \le x} \left(1 - \frac{n}{x}\right) \frac{\phi(n)}{n} = \frac{3}{\pi^2}x + O(e^{-c\sqrt{\log x}}).$$

First recall that

$$\sum_{n \le x} \frac{\phi(n)}{n^{s+1}} = \frac{\zeta(s)}{\zeta(s+1)} \quad \text{for} \quad \sigma > 1.$$

By (4.15), we have

$$\sum_{n \le x} \left(1 - \frac{n}{x}\right) \frac{\phi(n)}{n} = \frac{1}{x} \sum_{n \le x} (x - n) \frac{\phi(n)}{n} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\zeta(s) x^s}{\zeta(s+1)s(s+1)} \quad (4.18)$$

for $a > 1$.

Let $\wp$ denote the contour $\sigma = \frac{-c}{\log \tau}$, $-\infty < t < +\infty$, where $\tau = |t| + 4$. On this contour we have $\frac{1}{\zeta(s+1)} \ll \log \tau$ and $\zeta(s) \ll \tau^{1/2} \log \tau$.

Before estimating the integral we need some preparation. While the path is $\gamma(t) = -\frac{c}{\log(t+4)} + it$, and we have $s(s+1)$ in the integral, it follows that

$$\begin{aligned}(-\frac{c}{\log(t+4)} + it)(1 - \frac{c}{\log(t+4)} + it) = &-\frac{c}{\log(t+4)} + it + \frac{c^2}{\log^2(t+4)} \\ &-i\frac{ct}{\log(t+4)} - i\frac{ct}{\log(t+4)} - t^2.\end{aligned}$$

This gives

$$\frac{1}{s(s+1)} \ll \frac{1}{(t+4)^2}.$$

First we estimate the integral on $\gamma(t) = -\frac{c}{\log(t+4)} + it$ where $0 \le t < \infty$.

$$\begin{aligned}\left|\int_\gamma\right| &= \left|\int_0^T \frac{\zeta(\gamma(t))x^{\gamma(t)}}{\zeta(1+\gamma(t))\gamma(t)(\gamma(t)+1)}\gamma'(t)dt\right| \\ &\ll \int_0^T \frac{|\zeta(\gamma(t))|\left|x^{\gamma(t)}\right|}{|\zeta(1+\gamma(t))||\gamma(t)||(\gamma(t)+1)|}|\gamma'(t)|dt \\ &\ll \int_0^T \frac{|\zeta(\gamma(t))|\left|x^{\gamma(t)}\right|}{|\zeta(1+\gamma(t))|(t+4)^2}|\gamma'(t)|dt.\end{aligned}$$

On $\gamma(t)$ we have $\frac{1}{\zeta(s+1)} \ll \log\tau$ and $\zeta(s) \ll \tau^{1/2}\log\tau$. Therefore,

$$\begin{aligned}\left|\int_\wp\right| &\ll \int_0^T \frac{\tau^{1/2}\log^2\tau\left|x^{\gamma(t)}\right|}{(t+4)^2}|\gamma'(t)|dt \\ &= \int_0^T \frac{(t+4)^{1/2}\log^2(t+4)\left|x^{\gamma(t)}\right|}{(t+4)^2}|\gamma'(t)|dt \\ &\ll \int_0^T \frac{\log(t+4)e^{-c\frac{\log x}{\log(t+4)}}}{(t+4)^{3/2}}dt \\ &= \int_0^{e^{\sqrt{\log x}}} \frac{\log(t+4)e^{-c\frac{\log x}{\log(t+4)}}}{(t+4)^{3/2}}dt + \int_{e^{\sqrt{\log x}}}^T \frac{\log(t+4)e^{-c\frac{\log x}{\log(t+4)}}}{(t+4)^{3/2}}dt \\ &\ll e^{-c\sqrt{\log x}}\int_0^{e^{\sqrt{\log x}}} \frac{1}{(t+4)^{5/4}}dt + e^{-c\frac{\log x}{\log T}}\int_{e^{\sqrt{\log x}}}^T \frac{1}{(t+4)^{5/4}} \\ &\ll e^{-c\sqrt{\log x}} + e^{-c\frac{\log x}{\log(T+4)}}\left(-\frac{4}{(T+4)^{1/4}} + \frac{4}{(e^{\sqrt{\log x}}+4)^{1/4}}\right) \quad \text{follows.}\end{aligned}$$

By letting $T \to \infty$ we obtain

$$\int_\gamma \ll e^{-c\sqrt{\log x}} \quad \text{when} \quad 0 \le t < \infty. \tag{4.19}$$

Second, to estimate the integral on $\gamma(t) = -\frac{c}{\log(t+4)} + it$, where $-\infty < t \le 0$, note that

$$\begin{aligned}\left|\int_\gamma\right| &= \left|\int_{-T}^0 \frac{\zeta(\gamma(t))x^{\gamma(t)}}{\zeta(1+\gamma(t))\gamma(t)(\gamma(t)+1)}\gamma'(t)dt\right| \\ &\ll \int_{-T}^0 \frac{|\zeta(\gamma(t))|\left|x^{\gamma(t)}\right|}{|\zeta(1+\gamma(t))||\gamma(t)||(\gamma(t)+1)|}|\gamma'(t)|dt \\ &\ll \int_{-T}^0 \frac{|\zeta(\gamma(t))|\left|x^{\gamma(t)}\right|}{|\zeta(1+\gamma(t))|(|t|+4)^2}|\gamma'(t)|dt.\end{aligned}$$

On $\gamma(t)$ we have $\frac{1}{\zeta(s+1)} \ll \log\tau$ and $\zeta(s) \ll \tau^{1/2}\log\tau$. Therefore

$$\begin{aligned}
\left|\int_{\wp}\right| &\ll \int_{-T}^{0} \frac{\tau^{1/2}\log^2\tau\,\left|x^{\gamma(t)}\right|}{(|t|+4)^2}|\gamma'(t)|dt \\
&= \int_{-T}^{0} \frac{(|t|+4)^{1/2}\log^2(|t|+4)\,\left|x^{\gamma(t)}\right|}{(|t|+4)^2}|\gamma'(t)|dt \\
&\ll \int_{-T}^{0} \frac{\log(|t|+4)e^{-c\frac{\log x}{\log(|t|+4)}}}{(|t|+4)^{3/2}}dt \\
&= \int_{-e^{\sqrt{\log x}}}^{0} \frac{\log(|t|+4)e^{-c\frac{\log x}{\log(|t|+4)}}}{(|t|+4)^{3/2}}dt + \int_{-T}^{-e^{\sqrt{\log x}}} \frac{\log(|t|+4)e^{-c\frac{\log x}{\log(|t|+4)}}}{(|t|+4)^{3/2}}dt \\
&\ll e^{-c\sqrt{\log x}} \int_{-e^{\sqrt{\log x}}}^{0} \frac{1}{(|t|+4)^{5/4}}dt + e^{-c\frac{\log x}{\log T}} \int_{-T}^{-e^{\sqrt{\log x}}} \frac{1}{(|t|+4)^{5/4}} \\
&\ll e^{-c\sqrt{\log x}} + e^{-c\frac{\log x}{\log(T+4)}}\left(-\frac{4}{(e^{\sqrt{\log x}}+4)^{1/4}} + \frac{4}{(T+4)^{1/4}}\right) \quad \text{follows.}
\end{aligned}$$

Again, by letting $T \to \infty$, we obtain

$$\int_{\gamma} \ll e^{-c\sqrt{\log x}} \quad \text{when} \quad -\infty < t \leq 0. \tag{4.20}$$

Now after these estimations we evaluate the residue. Observe that in the region between $\wp$ and the orginal line of integration, we have the only pole at $s = 1$. From the contour $\wp$, $\zeta(s+1)$ has no critical zero since $\wp$ in the zero free region for $\zeta(s+1)$. And also we take in account the pole $s = 0$ for $\zeta(s+1)$, but note that $\frac{1}{\zeta(s+1)s}$ has a removable singularity at $s = 0$. Hence we need to consider only the simple pole at $s = 1$. So residue calculation gives

$$2\pi i \lim_{s\to 1}(s-1)\frac{\zeta(s)x^s}{\zeta(s+1)s(s+1)} = 2\pi i\frac{3}{\pi^2}x. \tag{4.21}$$

Thus from (4.19),(4.20),(4.21), our integral is

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\zeta(s)x^s}{\zeta(s+1)s(s+1)} = \frac{3}{\pi^2}x + O(e^{-c\sqrt{\log x}}).$$

This gives

$$H(x) = \frac{E(x)}{x} + O(e^{-c\sqrt{\log x}}).$$

□

This result brings us to our main theorem.

Theorem 4.6. *We have*

$$E(x) = \Omega_{\pm}(x\sqrt{\log\log x}). \tag{4.22}$$

Proof. From Theorem 4.5, it suffices to show that $H(x) = \Omega_{\pm}(\sqrt{\log\log x})$.

Now we apply lemma 4.4 with choice

$$q = \prod_{\substack{p\leq z\\ p\equiv 3(mod 4)}} p,$$

where we choose $z \approx \sqrt{\log N}$ and $\omega(q)$ is even, so that $q \equiv 1(mod4)$ and $q \leq e^{c_2\sqrt{\log N}}$.

Observe that if $d|q$, then $\frac{q}{d} \equiv 1(mod4)$ or $\frac{q}{d} \equiv 3(mod4)$ when $\omega(d)$ is even or odd. If $\omega(d)$ is even then $\frac{q}{d} \equiv 1(mod4)$ and if $\omega(d)$ is odd then $\frac{q}{d} \equiv 3(mod4)$. By using this we show that

$$\mu(d)s\left(\frac{q}{4d}\right) = \frac{1}{4} \quad \text{for all} \quad d|q.$$

In the case $\omega(d)$ is even, we have $\mu(d) = 1$ and $\frac{q}{d} \equiv 1(mod4)$ which give $\mu(d)s\left(\frac{q}{4d}\right) = 1\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{4}$. And on the other side, if $\omega(d)$ is odd, then we have $\mu(d) = -1$ and $\frac{q}{d} \equiv 3(mod4)$ which gives

$$\mu(d)s\left(\frac{q}{4d}\right) = (-1)\left(\frac{1}{2} - \frac{3}{4}\right) = \frac{1}{4}.$$

Now we choose α in Lemma 4.4 for proving the claims $H(x)$ is positive for infinitely many x and $H(x)$ is negative for infinitely many x. In the first case we choose $\alpha = \frac{q}{4}$, so that

$$\begin{aligned} C(q, q/4) &= \frac{6}{\pi^2}\left(\prod_{p|q}(1-p^{-2})\right)^{-1}\sum_{d|q}\mu(d)\frac{s(q/4d)}{d} \\ &= \frac{6}{\pi^2}\left(\prod_{p|q}(1-p^{-2})\right)^{-1}\sum_{d|q}\frac{1}{4d} \\ &\approx \sum_{d|q}\frac{1}{d} \\ &= \prod_{\substack{p\le z \\ p\equiv 3(mod4)}}\left(1+\frac{1}{p}\right). \end{aligned}$$

Taking logarithm of both sides, we get

$$\begin{aligned} \log(C(q, q/4)) &\approx \sum_{\substack{p\le z \\ p\equiv 3(mod4)}} \log\left(1+\frac{1}{p}\right) \\ &= \sum_{\substack{p\le z \\ p\equiv 3(mod4)}} \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{p^n n} \\ &= \sum_{\substack{p\le z \\ p\equiv 3(mod4)}} \frac{1}{p} + \sum_{\substack{p\le z \\ p\equiv 3(mod4)}} \sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{p^n n}. \end{aligned}$$

By Mertens estimation, we have

$$\sum_{\substack{p\le z \\ p\equiv k(modq)}} \frac{1}{p} = \frac{1}{\phi(q)}\log\log z + O(1).\ (see\ [2])$$

Here p is prime and $(k, q) = 1$. This gives

$$\sum_{\substack{p\le z \\ p\equiv 3(mod4)}} \frac{1}{p} = \frac{1}{2}\log\log z + O(1).$$

And, note that

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{p^n n} &\le \sum_{n=2}^{\infty}\frac{1}{p^n n}\\ &\le \frac{1}{2p^2}\sum_{n=0}^{\infty}\frac{1}{p^n}\\ &= \frac{1}{2(p^2-p)} \le \frac{1}{p^2}.\end{aligned}$$

This gives

$$\sum_{\substack{p\le z\\ p\equiv 3(mod4)}}\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{p^n n} = O(1).$$

Hence, we have

$$C(q,q/4) \approx \sqrt{\log z} \approx \sqrt{\log\log N}. \tag{4.23}$$

From equation (4.6), we see that there are x such that $H(x)$ is positive and furthermore $H(x) = \Omega(\sqrt{\log\log x})$.

Since the average value of $H(nq+\alpha)$ for $1 \le n \le N$ is asymptotic to $C(q, \frac{q}{4}) \approx \sqrt{\log\log N}$, we have $H(x) \ne o(\sqrt{\log\log x})$.

In the second case we choose $\alpha = \frac{3q}{4}$. In this case we show

$$\mu(d)s\left(\frac{\alpha}{d}\right) = -\frac{1}{4}.$$

If $\omega(d)$ is even, then we have $\mu(d) = 1$ and $\frac{q}{d} \equiv 1(mod4)$ which gives

$$\mu(d)s\left(\frac{3q}{4d}\right) = 1\left(\frac{1}{2}-\frac{3}{4}\right) = -\frac{1}{4}.$$

And on the other side, if $\omega(d)$ is odd, we have $\mu(d) = -1$ and $\frac{q}{d} \equiv 3(mod4)$ which gives

$$\mu(d)s\left(\frac{3q}{4d}\right) = (-1)\left(\frac{1}{2}-\frac{1}{4}\right) = -\frac{1}{4}.$$

This gives us,

$$C(q,3q/4) = -C(q,q/4) \approx -\sqrt{\log\log N} \tag{4.24}$$

Thus, again from equation (4.6), we obtain there are x such that $H(x)$ is negative and does not exceed $-\sqrt{\log\log x}$.

From both cases $\alpha = \frac{q}{4}$ and $\alpha = \frac{3q}{4}$ we obtain that $H(x) = \Omega_{\pm}(\sqrt{\log \log x})$, and hence from Theorem 4.5, we have

$$E(x) = \Omega_{\pm}(x\sqrt{\log \log x}).$$

$\square$

5 Fluctuation on the Averages of the Sum of Divisor Function

In this section, we apply the methods that we used in preceding sections and show the existence of fluctuations on the error term for the average order of divisors of n which are coprime to any given integer a. We define this a function as,

$$D_a(n) = \sum_{\substack{d|n \\ (d,a)=1}} d. \tag{5.1}$$

Where $a > 1$ is an integer.

5.1 Average order of $D_a(n)$ and $\frac{D_a(n)}{n}$

Let us first find the average of $D_a(n)$.

Theorem 5.1. *For all $x \geq 2$, we have*

$$\sum_{n \leq x} D_a(n) = \frac{\pi^2 x^2 \phi(a)}{12a} + O(x \log x). \tag{5.2}$$

Proof.

$$\begin{aligned}
\sum_{n \leq x} D_a(n) &= \sum_{n \leq x} \sum_{\substack{d|n \\ (d,a)=1}} d = \sum_{n \leq x} \sum_{\substack{d|n \\ (n/d,a)=1}} \frac{n}{d} \\
&= \sum_{d \leq x} \sum_{\substack{q \leq x/d \\ (q,a)=1}} q = \sum_{d \leq x} \sum_{q \leq x/d} q \sum_{\substack{r|q \\ r|a}} \mu(r) \\
&= \sum_{d \leq x} \sum_{\substack{r \leq x/d \\ r|a}} r\mu(r) \sum_{s \leq x/dr} s \\
&= \frac{x^2}{2} \sum_{d \leq x} \frac{1}{d^2} \sum_{r|a} \frac{\mu(r)}{r} + O\left(x \sum_{d \leq x} \frac{1}{d} \right) \\
&= \frac{\phi(a)}{a} \frac{x^2}{2} \sum_{d \leq x} \frac{1}{d^2} + O(x \log x) \\
&= \frac{\pi^2 x^2 \phi(a)}{12a} + O(x \log x).
\end{aligned}$$

□

By using above theorem and partial summation formula we get following result.

Theorem 5.2. *For all $x \geq 2$, we have*

$$\sum_{n \leq x} \frac{D_a(n)}{n} = \frac{\pi^2 x \phi(a)}{6a} + O(\log x). \tag{5.3}$$

Proof.

$$\begin{aligned}\sum_{n \leq x} \frac{D_a(n)}{n} &= \frac{\pi^2 x^2 \phi(a)}{x 12a} + O(\log x) + \int_1^x \frac{\pi^2 \phi(a)}{12a} dt \\ &= \frac{\pi^2 x \phi(a)}{6a} + O(\log x).\end{aligned}$$

□

5.2 Fluctuation on the averages of $D_a(n)$

Before we start the section, we introduce some definitions. We define

$$K_a(x) := \sum_{n \leq x} D_a(n) - \frac{\pi^2 x^2 \phi(a)}{12a} \tag{5.4}$$

$$F_a(x) := \sum_{n \leq x} \frac{D_a(n)}{n} - \frac{\pi^2 x \phi(a)}{6a} \tag{5.5}$$

for $a = p_1^{\beta_1} p_2^{\beta_2} \dots p_s^{\beta_s}$ where $\beta_i > 0$ and $1 \leq i \leq s$. And we use p, q to denote primes.

In this section, we prove that $K_a(x) = \Omega_{\pm}(x \log \log x)$. Let us begin with some lemmas for proving the theorem.

Lemma 5.3. *For each natural number n, we have*

$$\frac{D_a(n)}{n} = \sum_{d|n} \frac{1}{d} \prod_{p|(a,d)} (1-p). \tag{5.6}$$

Proof. Observe that both sides in the equation are multiplicative function of n, it sufficies to check the equality for $n = q^b$, where q is a prime and b is a nonnegative integer. We have two cases here, $q|a$ or $q \nmid a$.

If $q|a$, then

$$\frac{D_a(q^b)}{q^b} = \frac{1}{q^b} \sum_{\substack{d|q^b \\ (d,a)=1}} d = \frac{1}{q^b}$$

and

$$\sum_{d|q^b}\frac{1}{d}\prod_{p|(a,d)}(1-p)=1+(1-q)\left(\frac{1}{q}+\frac{1}{q^2}+\ldots+\frac{1}{q^b}\right)=\frac{1}{q^b}.$$

Which gives the identity.

If $q\nmid a$

$$\frac{D_a(q^b)}{q^b}=\frac{1}{q^b}\sum_{\substack{d|q^b\\(d,a)=1}}d=\frac{(1+q+q^2+\ldots+q^b)}{q^b}$$

and

$$\sum_{d|q^b}\frac{1}{d}\prod_{p|(a,d)}(1-p)=\sum_{d|q^b}\frac{1}{d}=\left(1+\frac{1}{q}+\ldots+\frac{1}{q^b}\right)=\frac{(1+q+q^2+\ldots+q^b)}{q^b}.$$

Which completes the formula. $\square$

Lemma 5.4. *We have*

$$\sum_{n\le x}\frac{1}{n}\prod_{p|(a,n)}(1-p)=\begin{cases}\log p_1+O\left(\frac{1}{x}\right) & \textit{if } s=1,\\ O\left(\frac{1}{x}\right) & \textit{if } s>1.\end{cases}\tag{5.7}$$

Proof. From Theorem 1.8, we have

$$\begin{aligned}\sum_{n\le x}\frac{1}{n}\prod_{p|(a,n)}(1-p)&=\sum_{n\le x}\frac{1}{n}\sum_{d|(a,n)}\mu(d)d=\sum_{d|a}\mu(d)\sum_{n\le x}\frac{d}{n}\sum_{d|n}1\\&=\sum_{d|a}\mu(d)\sum_{q\le x/d}\frac{1}{q}=\sum_{d|a}\mu(d)\left(\log\frac{x}{d}+E+O\left(\frac{d}{n}\right)\right)\\&=(\log x+E)\sum_{d|a}\mu(d)-\sum_{d|a}\mu(d)\log d+O\left(\frac{1}{x}\right).\end{aligned}$$

From Theorem 1.3, we have $-\sum_{d|a}\mu(d)\log d=\Lambda(a)$, so that

$$\sum_{n\le x}\frac{1}{n}\prod_{p|(a,n)}(1-p)=(\log x+E)\sum_{d|p_1\cdots p_s}\mu(d)+\Lambda(a)+O\left(\frac{1}{x}\right)\quad\text{follows.}$$

Consider the cases that $s=1$ or $s>1$. If $s=1$, $\Lambda(a)=\log p_1$ and

$\sum_{d|p_1} \mu(d) = 0$, so

$$\sum_{n \le x} \frac{1}{n} \prod_{p|(a,n)} (1-p) = \log p_1 + O\left(\frac{1}{x}\right).$$

If $s > 1$, then $\Lambda(a) = 0$ and $\sum_{d|p_1 \cdots p_s} \mu(d) = 0$ so that

$$\sum_{n \le x} \frac{1}{n} \prod_{p|(a,n)} (1-p) = O\left(\frac{1}{x}\right).$$

□

Lemma 5.5. *We have*

$$F_a(x) = -\sum_{k=1}^{\infty} \frac{1}{k} \left\{\frac{x}{k}\right\} \prod_{p|(a,k)} (1-p). \tag{5.8}$$

Proof. First observe that, from Theorem 1.9 and using the fact that

$$\frac{1}{k^2} \prod_{p|(a,k)} (1-p)$$

is completely multiplicative, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{k^2} \prod_{p|(a,k)} (1-p) &= \prod_q \left(\frac{1}{1 - \left\{ \frac{1}{q^2} \prod_{p|(a,q)} (1-p) \right\}} \right) \\
&= \prod_q \left(\sum_{i=0}^{\infty} \frac{1}{q^{2i}} \prod_{p|(a,q^i)} (1-p) \right) \\
&= \prod_{q|a} \left(1 + \sum_{i=0}^{\infty} \frac{1-q}{q^{2i}} \right) \prod_{q\nmid a} \sum_{i=0}^{\infty} \frac{1}{q^{2i}} \\
&= \prod_{q|a} \left(1 + (1-q) \left(\frac{1}{1-\frac{1}{q^2}} - 1 \right) \right) \prod_{q\nmid a} \left(1 - \frac{1}{q^2}\right)^{-1} \\
&= \prod_{q|a} \left(1 + \frac{1}{q}\right)^{-1} \prod_{q\nmid a} \left(1 - \frac{1}{q^2}\right)^{-1} \\
&= \prod_{q|a} \left(1 - \frac{1}{q}\right) \prod_{q} \left(1 - \frac{1}{q^2}\right)^{-1} = \frac{\pi^2 \phi(a)}{6a}.
\end{aligned} \tag{5.9}$$

By using equation (5.5), Lemma 5.3 and above result, we get

$$\begin{aligned}
F_a(x) &= \sum_{n\le x}\sum_{k|n}\frac{1}{k}\prod_{p|(a,k)}(1-p) - x\sum_{k=1}^{\infty}\frac{1}{k^2}\prod_{p|(a,k)}(1-p)\\
&= \sum_{k\le x}\frac{1}{k}\sum_{q\le x/k}\prod_{p|(a,k)}(1-p) - x\sum_{k=1}^{\infty}\frac{1}{k^2}\prod_{p|(a,k)}(1-p)\\
&= \sum_{k\le x}\frac{1}{k}\left[\frac{x}{k}\right]\prod_{p|(a,k)}(1-p) - x\sum_{k=1}^{\infty}\frac{1}{k^2}\prod_{p|(a,k)}(1-p).
\end{aligned}$$

Since $\left[\frac{x}{k}\right] = 0$ for $x > k$

$$\begin{aligned}
F_a(x) &= \sum_{k=1}^{\infty}\frac{1}{k}\left[\frac{x}{k}\right]\prod_{p|(a,k)}(1-p) - x\sum_{k=1}^{\infty}\frac{1}{k^2}\prod_{p|(a,k)}(1-p)\\
&= \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{x}{k}-\left\{\frac{x}{k}\right\}\right)\prod_{p|(a,k)}(1-p) - x\sum_{k=1}^{\infty}\frac{1}{k^2}\prod_{p|(a,k)}(1-p)\\
&= -\sum_{k=1}^{\infty}\frac{1}{k}\left\{\frac{x}{k}\right\}\prod_{p|(a,k)}(1-p) \quad \text{follows.}
\end{aligned}$$

□

Lemma 5.6. *We have*

$$F_a(x) = -\sum_{k\le y}\frac{1}{k}\left\{\frac{x}{k}\right\}\prod_{p|(a,k)}(1-p) + O(1) \tag{5.10}$$

uniformly for $x \ge 2$, $y \ge \frac{1}{2}\sqrt{x}$.

Proof. By Lemma 5.5, it is sufficient to prove that

$$\sum_{k\ge y}\frac{1}{k}\left\{\frac{x}{k}\right\}\prod_{p|(a,k)}(1-p) = O(1).$$

Here if $y \ge \frac{x}{2}$, then

$$\sum_{k\ge y}\frac{1}{k}\left\{\frac{x}{k}\right\}\prod_{p|(a,k)}(1-p) \ll \sum_{k\ge y}\frac{1}{k}\left\{\frac{x}{k}\right\}\prod_{p|a}(1-p) \ll \sum_{k\ge y}\frac{x}{k^2} \ll \frac{x}{y} \ll 1. \tag{5.11}$$

Now we assume that $\frac{\sqrt{x}}{2} \le y < \frac{x}{2}$. By (5.11) we have

$$\sum_{k \ge x} \frac{1}{k} \left\{ \frac{x}{k} \right\} \prod_{p|(a,k)} (1-p) = O(1).$$

Therefore, it suffices to prove that

$$\sum_{y<k\le x} \frac{1}{k} \left\{ \frac{x}{k} \right\} \prod_{p|(a,k)} (1-p) = O(1).$$

Let M be an integer with $M \le x/y < M+1$. Then, for each integer t with $2 \le t \le M+1$, $\left\{\frac{x}{k}\right\}$ is monotone in the range $x/t < k \le x/(t-1)$. Hence, by Lemma 5.4 and partial summation, we have

$$\sum_{x/t<k\le x/(t-1)} \frac{1}{k} \left\{ \frac{x}{k} \right\} \prod_{p|(a,k)} (1-p) = O\left(\frac{t}{x}\right)$$

and

$$\sum_{y<k\le x/M} \frac{1}{k} \left\{ \frac{x}{k} \right\} \prod_{p|(a,k)} (1-p) = O\left(\frac{M}{x}\right).$$

Thus, for $y \ge \frac{\sqrt{x}}{2}$, we have

$$\begin{aligned}\sum_{y<k\le x} \frac{1}{k} \left\{ \frac{x}{k} \right\} \prod_{p|(a,k)} (1-p) &= \sum_{2\le t\le M} \sum_{x/t<k\le x/(t-1)} \frac{1}{k} \left\{ \frac{x}{k} \right\} \prod_{p|(a,k)} (1-p) \\ &\quad + \sum_{y<k\le x/M} \frac{1}{k} \left\{ \frac{x}{k} \right\} \prod_{p|(a,k)} (1-p) \\ &\ll \frac{1}{x} \sum_{1\le t\le x/y} t \ll \frac{1}{x}\frac{x^2}{y^2} \ll 1.\end{aligned}$$

which gives desired result. □

Lemma 5.7. *We have*

$$\frac{K_a(x)}{x} - F_a(x) = O(1). \tag{5.12}$$

Proof. By (5.4), we have

$$\frac{K_a(x)}{x} = \frac{1}{x} \sum_{n\le x} D_a(n) - \frac{\pi^2 x \phi(a)}{12a}. \tag{5.13}$$

From Lemma 5.3

$$\begin{aligned}\sum_{n\le x} D_a(n) &= \sum_{n\le x} n \sum_{d|n} \frac{1}{d} \prod_{p|(a,d)} (1-p) = \sum_{d\le x}\sum_{q\le x/d} q \prod_{p|(a,d)} (1-p)\\ &= \sum_{d\le x}\left(\frac{1}{2}\left[\frac{x}{d}\right]^2 + \frac{1}{2}\left[\frac{x}{d}\right]\right) \prod_{p|(a,d)} (1-p)\\ &= \sum_{d\le x}\left(\frac{x^2}{2d^2} + \frac{x}{2d} - \frac{x}{d}\left\{\frac{x}{d}\right\}\right) \prod_{p|(a,d)} (1-p) + O(x).\end{aligned}$$

From Lemma 5.4, we have

$$\sum_{d\le x} \frac{1}{d} \prod_{p|(a,d)} (1-p) = O(1/x).$$

Therefore,

$$\begin{aligned}\sum_{n\le x} D_a(n) &= \sum_{d\le x}\left(\frac{x^2}{2d^2} - \frac{x}{d}\left\{\frac{x}{d}\right\}\right) \prod_{p|(a,d)} (1-p) + O(x)\\ &= \frac{x^2}{2}\sum_{d\le x} \frac{1}{d^2} \prod_{p|(a,d)} (1-p) - x\sum_{d\le x} \frac{1}{d}\left\{\frac{x}{d}\right\} \prod_{p|(a,d)} (1-p) + O(x).\end{aligned}$$

Using Lemma 5.6, we get

$$\sum_{n\le x} D_a(n) = \frac{x^2}{2}\sum_{d\le x} \frac{1}{d^2} \prod_{p|(a,d)} (1-p) + xF_a(x) + O(x). \tag{5.14}$$

If we put (5.14) into (5.13), then

$$\begin{aligned}\frac{K_a(x)}{x} &= \frac{x}{2}\sum_{d\le x} \frac{1}{d^2} \prod_{p|(a,d)} (1-p) + F_a(x) - \frac{\pi^2 x\phi(a)}{12a} + O(1)\\ &= \frac{x}{2}\sum_{d=1}^{\infty} \frac{1}{d^2} \prod_{p|(a,d)} (1-p) - \frac{\pi^2 x\phi(a)}{12a} - \frac{x}{2}\sum_{d>x} \frac{1}{d^2} \prod_{p|(a,d)} (1-p)\\ &+ F_a(x) + O(1) \quad \text{follows.}\end{aligned}$$

Form (5.9), we have

$$\frac{K_a(x)}{x} - F_a(x) = -\frac{x}{2}\sum_{d>x}\frac{1}{d^2}\prod_{p|(a,d)}(1-p) + O(1)$$
$$= O\left(x\sum_{d>x}\frac{1}{d^2}\right) + O(1) = O(1).$$

This completes the proof. □

Lemma 5.8. *If b, r are positive integers such that $(b,r)=1$ and β is real number, then for any positive integer N, we have*

$$\sum_{n=1}^{N}\left\{\frac{nb}{r}+\beta\right\} = \frac{N}{r}\{r\beta\} + \frac{N}{r}\left(\frac{r-1}{2}\right) + O(r). \tag{5.15}$$

Proof. By Lemma 4.3, we have

$$\sum_{n=1}^{N}\left(\frac{1}{2} - \left\{\frac{nb}{r}+\beta\right\}\right) = \frac{N}{r}\left(\frac{1}{2} - \{r\beta\}\right) + O(r).$$

Note that

$$-\sum_{n=1}^{N}\left\{\frac{nb}{r}+\beta\right\} = -\frac{N}{2} + \frac{N}{2r} - \frac{N}{r}\{r\beta\} + O(r).$$

This gives

$$\sum_{n=1}^{N}\left\{\frac{nb}{r}+\beta\right\} = \frac{N}{r}\{r\beta\} + \frac{N}{r}\left(\frac{r-1}{2}\right) + O(r).$$

□

Lemma 5.9. *Let $A = \frac{m!}{p_1^{\beta_1}\cdots p_s^{\beta_s}}$ be an integer with $(A,a)=1$ and $A \geq ma$. Then*

$$\sum_{k\leq A}\frac{(k,A)}{k^2}\prod_{p|(a,k)}(1-p) \geq \frac{\phi(a)}{a}\log m + O(1). \tag{5.16}$$

Proof. By using Theorem 1.8, we have

$$
\begin{aligned}
\sum_{k\le A}\frac{(k,A)}{k^2}\prod_{p\mid(a,k)}(1-p) &= \sum_{k\le A}\frac{(k,A)}{k^2}\sum_{\substack{d\mid a\\ d\mid k}}\mu(d)d \\
&= \sum_{d\mid a}\sum_{k\le A}\frac{(k,A)}{k^2}\sum_{d\mid k}\mu(d)d \\
&= \sum_{d\mid a}\frac{\mu(d)}{d}\sum_{q\le A/d}\frac{(q,A)}{q^2} \\
&= \sum_{d\mid a}\frac{\mu(d)}{d}\sum_{q\le A/a}\frac{(q,A)}{q^2} + \sum_{d\mid a}\frac{\mu(d)}{d}\sum_{A/a<q\le A/d}\frac{(q,A)}{q^2} \\
&= \frac{\phi(a)}{a}\underbrace{\sum_{q\le A/a}\frac{(q,A)}{q^2}}_{A_1} + \underbrace{\sum_{d\mid a}\frac{\mu(d)}{d}\sum_{A/a<q\le A/d}\frac{(q,A)}{q^2}}_{A_2}.
\end{aligned}
$$

Since $\frac{A}{a}\ge m$, we have

$$
\begin{aligned}
A_1 = \sum_{q\le A/a}\frac{(q,A)}{q^2} \ge \sum_{\substack{q\le m\\ (q,a)=1}}\frac{(q,A)}{q^2} &= \sum_{\substack{q\le m\\ (q,a)=1}}\frac{1}{q} \\
&= \sum_{q\le m}\frac{1}{q}\sum_{\substack{d\mid q\\ d\mid a}}\mu(d) = \sum_{d\mid a}\sum_{q\le m}\frac{1}{q}\sum_{d\mid q}\mu(d) \\
&= \sum_{d\mid a}\frac{\mu(d)}{d}\sum_{r\le m/d}\frac{1}{r} = \sum_{d\mid a}\frac{\mu(d)}{d}\Big(\log\frac{m}{d}+O(1)\Big) \\
&= \frac{\phi(a)}{a}\log m + O(1).
\end{aligned}
$$

And

$$
A_2 = \sum_{d\mid a}\frac{\mu(d)}{d}\sum_{A/a<q\le A/d}\frac{(q,A)}{q^2} \ll \sum_{A/a<k\le A}\frac{(k,A)}{k^2} \ll \sum_{A/a<k\le A}\frac{1}{k} \ll 1.
$$

Using A_1 and A_2 in the above equation, we obtain

$$
\sum_{k\le A}\frac{(k,A)}{k^2}\prod_{p\mid(a,k)}(1-p) \ge \frac{\phi(a)}{a}\log m + O(1).
$$

□

Theorem 5.10. *For any integer $a > 1$, we have*

$$K_a(x) = \Omega_{\pm}(x \log\log x). \tag{5.17}$$

Proof. Let A be as in Lemma 5.9 and let B be an integer with $0 \le B < A$. By Lemma 5.6, we have

$$\begin{aligned}\frac{1}{A}\sum_{n=1}^{A} F_a(nA+B) &= -\frac{1}{A}\sum_{n=1}^{A}\sum_{k\le A}\frac{1}{k}\left\{\frac{nA+B}{k}\right\}\prod_{p\mid(a,k)}(1-p) + O(1)\\ &= -\frac{1}{A}\sum_{k\le A}\frac{1}{k}\left(\prod_{p\mid(a,k)}(1-p)\right)\sum_{n=1}^{A}\left\{\frac{nA+B}{k}\right\} + O(1).\end{aligned} \tag{5.18}$$

By using Lemma 5.8 by putting $r = \frac{k}{(k,A)}$ and $\beta = \frac{B}{k}$, the equation (5.18) becomes

$$= -\sum_{k\le A}\frac{(k,A)}{k^2}\left(\prod_{p\mid(a,k)}(1-p)\right)\left(\left\{\frac{B}{(k,A)}\right\} + \frac{k}{2(k,A)} - \frac{1}{2}\right) + O(1).$$

Using Lemma 5.4, this further equals

$$= -\sum_{k\le A}\frac{(k,A)}{k^2}\left(\prod_{p\mid(a,k)}(1-p)\right)\left(\left\{\frac{B}{(k,A)}\right\} - \frac{1}{2}\right) + O(1). \tag{5.19}$$

In equation (5.19), we take $B = 0$ to obtain

$$\frac{1}{A}\sum_{n=1}^{A} F_a(nA) = \frac{1}{2}\sum_{k\le A}\frac{(k,A)}{k^2}\prod_{p\mid(a,k)}(1-p) + O(1). \tag{5.20}$$

Taking $B = A - 1$, we have

$$\begin{aligned}\frac{1}{A}\sum_{n=1}^{A} F_a(nA+A-1) &= -\sum_{k\le A}\frac{(k,A)}{k^2}\left(\prod_{p\mid(a,k)}(1-p)\right)\left(\frac{1}{2} - \frac{1}{(k,A)}\right) + O(1)\\ &= -\frac{1}{2}\sum_{k\le A}\frac{(k,A)}{k^2}\prod_{p\mid(a,k)}(1-p) + O(1).\end{aligned} \tag{5.21}$$

Note that

$$\log\log(A^2+B) \ll \log\log A \ll \log(m\log m) \ll \log m. \tag{5.22}$$

Using Lemma 5.9 in equation (5.20), we have

$$\frac{1}{A}\sum_{n=1}^{A} F_a(nA) \geq \frac{\phi(a)}{2a}\log m + O(1).$$

From (5.22), we get

$$F_a(x) > \log\log x. \tag{5.23}$$

Using Lemma 5.9 in equation (5.21), we have

$$\frac{1}{A}\sum_{n=1}^{A} F_a(nA + A - 1) \leq -\frac{\phi(a)}{2a}\log m + O(1).$$

Again from (5.22), we get

$$F_a(x) \leq -\log\log x. \tag{5.24}$$

Finally from (5.23) and (5.24) we get

$$F_a(x) = \Omega_{\pm}(\log\log x).$$

Therefore, by Lemma 5.7

$$K_a(x) = \Omega_{\pm}(x\log\log x) \quad \text{follows.}$$

□

6 Conclusion

In this thesis, we investigate the error terms in the averages of the arithmetic functions $\phi(n)$ and $D_a(n)$. By using basic concepts of the Number Theory, we gave Ω-type estimations for these error terms and by evaluating the averages on the arihmetic progressions, we showed that these error terms change sign infinitely many times. Both results provided us to get better prediction for the artihmetic functions $\phi(n)$ and $D_a(n)$, when n is a large number. However, an answer can be given to Montgomery's conjecture $E(x) = O(x \log\log x)$ [5], if one improves the results of this thesis.

Moreover, one can obtain similar results for the Jordan totient function which is defined as

$$J_k(n) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right)^k.$$

It seems with some minor modifications, the current methods would work for the Jordan totient function to obtain similar results. On the other hand, in each section we mainly used the different properties of Möbius function. Therefore, any function which is related to the Möbius function can be studied for similar results.

Consequently, in the major part of Analytic Number Theory we face with error terms, to understand the behaviour of arithmetic functions, further we should analyze these error terms by obtaining big-O and Ω-type estimations.

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