Intersection Problems of Steiner Triple Systems

by

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To my parents ...

ABSTRACT

A Steiner triple system of order n ($STS(n)$) is a pair $(S, \mathcal{T})$ where S is a set of symbols of size n and $\mathcal{T}$ is a collection of 3 element subsets of S (*triples*) such that each pair of distinct elements of S belongs to exactly one triple of $\mathcal{T}$. It is known that a Steiner triple system exists if and only if $n \equiv 1, 3 \pmod 6$. Given a Steiner triple system $(S, \mathcal{T})$, the *flower* at an element x of S is defined to be the set of all triples containing the element x. Two Steiner triple systems $(S, \mathcal{T}_1)$ and $(S, \mathcal{T}_2)$ are said to *intersect in k triples* if $|\mathcal{T}_1 \cap \mathcal{T}_2| = k$. For all orders $n \equiv 1, 3 \pmod 6$ let $J(n)$ and $J_f(n)$ be defined as

$J(n) = \{k \mid \exists\ (S, \mathcal{T}_1)$ and $(S, \mathcal{T}_2)$ such that $|\mathcal{T}_1 \cap \mathcal{T}_2| = k\}$ and

$J_f(n) = \{k \mid \exists\ (S, \mathcal{T}_1)$ and $(S, \mathcal{T}_2)$ such that $|\mathcal{T}_1 \cap \mathcal{T}_2| = k + (n-1)/2$ where $(n-1)/2$ of these common triples constitute a common flower$\}$.

This thesis is a complete survey on determining $J(n)$ and $J_f(n)$, i.e. on intersection and flower intersection problems of Steiner triple systems.

ÖZETÇE

n'lik bir Steiner üçlü sistemi (*SÜS(n)*), $(S, \mathcal{T})$ şeklinde ifade edilen bir ikilidir öyle ki S, n elemanlı bir semboller kümesini, $\mathcal{T}$ ise S'nin 3 elemanlı bazı altkümelerinden (üçlü) oluşan bir topluluğu temsil eder ve S'den seçilecek her eleman ikilisi $\mathcal{T}$'nin tam olarak bir üçlüsünde birlikte bulunur. Bilindiği üzere her $n \equiv 1, 3 \pmod 6$ için bir *SÜS(n)* vardır. Bir Steiner üçlü sistemi $(S, \mathcal{T})$ için S'nin x isimli elemanının etrafındaki *çiçek* x elemanını içeren tüm üçlülerin oluşturduğu küme olarak tanımlanır. Eğer $|\mathcal{T}_1 \cap \mathcal{T}_2| = k$ ise $(S, \mathcal{T}_1)$ *ve* $(S, \mathcal{T}_2)$ k *tane üçlüde kesişiyor* denir. $J(n)$ ve $J_f(n)$ kümelerini şu şekilde tanımlayalım:

$J(n) = \{k \mid \exists\ (S, \mathcal{T}_1)$ ve $(S, \mathcal{T}_2)$ öyle ki $|\mathcal{T}_1 \cap \mathcal{T}_2| = k\}$,

$J_f(n) = \{k \mid \exists\ (S, \mathcal{T}_1)$ ve $(S, \mathcal{T}_2)$ öyle ki $|\mathcal{T}_1 \cap \mathcal{T}_2| = k + (n-1)/2$ ve bu üçlülerin $(n-1)/2$ tanesi ortak bir çiçek oluşturur$\}$.

Bu tezde $n \equiv 1, 3 \pmod 6$ şeklindeki tüm n'ler için $J(n)$ ve $J_f(n)$ kümelerini belirliyoruz, başka bir ifadeyle Steiner üçlü sistemlerinin kesişimi ve çiçek kesişimi problemlerini çözüyoruz.

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LIST OF FIGURES

TABLE OF CONTENTS

NOMENCLATURE

PBD	Pairwise balanced design
STS	Steiner triple system
K_n	Complete graph on n vertices
BIBD	Balanced incomplete block design

Chapter 1

PRELIMINARIES

1.1 Definitions

Definition: A *Steiner triple system of order* n ($STS(n)$) is a pair $(S, \mathcal{T})$ where S is a finite set of symbols, $|S| = n$, and $\mathcal{T}$ is a collection of 3 element subsets of S (*triples*) such that each pair of distinct elements of S belongs to exactly one triple of $\mathcal{T}$. A number n is called *admissible* if there exists an $STS(n)$ and the *spectrum* for Steiner triple systems is defined to be the set of integers n for which there exists an $STS(n)$. Given a Steiner triple system $(S, \mathcal{T})$, the *flower* at an element x of S is defined to be the set of all triples containing the element x.

Definition: A *partial triple system* is a pair (S, P) where S is a set of points and P is a collection of triples with the property that every pair of distinct elements of S belongs to at most one triple of P. Two partial triple systems (S, P_1) and (S, P_2) are said to be *balanced* provided P_1 and P_2 cover the same pair of distinct elements of S.

Definition: A *pairwise balanced design* (or PBD) is a pair (S, B) where S is a finite set of symbols, and B is a collection of subsets of S called *blocks* such that each pair of distinct elements of S occurs together in exactly one block of B.

Definition: A *design* is a pair (S, B) such that S is a set of elements called *points* (*vertices*) and B is a collection of non-empty subsets of S called *blocks*.

Definition: A *permutation* on a set S of size n is a bijection from S to S. A system $(S, \mathcal{T}_2)$, which is constructed by performing a permutation on the triples of a system $(S, \mathcal{T}_1)$ is called an *isomorphic disjoint mate* of the system $(S, \mathcal{T}_1)$ whenever these systems do not possess any triples in common. A *transposition* is a permutation which exchanges two elements and keeps the others fixed. A permutation on symbols $i, j, ..., q$ which maps each of i, j to the next, q to i and leaves all other symbols, if any, unchanged is called a *cycle* and is denoted by $(ij...q)$.

Definition: A *1-factorization* of a graph of order n is the decomposition of the edges of the graph into 1-factors where a *1-factor* is a set of $n/2$ disjoint edges, which together cover all vertices of the graph.

Definition: A *latin square* of order n is an $n \times n$ array, each cell of which contains exactly one of the symbols in $\{1, 2, ..., n\}$ such that each row and each column of the array contains each of the symbols in $\{1, 2, ..., n\}$ exactly once. A latin square is said to be *commutative* if cells (i, j) and (j, i) contain the same symbol for all $1 \leq i, j \leq n$. A latin square is said to be *idempotent* if cell (i, i) contains the symbol i for $1 \leq i \leq n$ and a latin square of order $2n$ is said to be *half-idempotent* if for $1 \leq i \leq n$, the cells (i, i) and $(n + i, n + i)$ contain the symbol i.

Definition: A *quasigroup* of order n is a pair $(Q, \circ)$ where Q is a set of size n and $\circ$ is a binary operation on Q such that for every pair of elements $a, b \in Q$, the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions. In our work, a quasigroup is just a latin square with a headline and sideline.

Definition: Let $Q = \{1, 2, ..., 2n\}$ and let $H = \{\{1, 2\}, \{3, 4\}, ..., \{2n-1, 2n\}\}$ where the two element subsets $\{2i - 1, 2i\} \in H$ are called *holes*. A *quasigroup with holes H* is a quasigroup $(Q, \circ)$ of order $2n$ in which for each $h \in H$, $(h, \circ)$ is a subquasigroup of $(Q, \circ)$.

Definition: A *group divisible design* is a triple (X, G, B) where X is a set of points, G is a partition of X into subsets called *groups* and B is a collection of subsets of X called *blocks* such that any pair of points from X appears together either in a group or in a block, but not in both.

1.2 Some Fundamental Constructions

The following constructions will be useful in building some of the other constructions used in the text.

The Quasigroup with Holes Construction for Steiner Triple Systems: Let $(\{1,2,...,2n\},\circ)$ be a commutative quasigroup of order $2n$ with holes $\{2i-1,2i\}$ for all $1 \leq i \leq n$. Then,

(a) $(\{\infty\} \cup \{\{1,2,...,2n\} \times \{1,2,3\}\}, B)$ is an $STS(6n+1)$, where B is defined as follows.

(1) For $1 \leq i \leq n$, let B_i contain the triples in an $STS(7)$ on the symbols $\{\infty\} \cup (\{2i-1,2i\} \times \{1,2,3\})$ and let $B_i \subseteq B$.

(2) For $1 \leq i \neq j \leq 2n$, $\{i,j\} \notin H$ place the triples $\{(i,1),(j,1),(i \circ j,2)\}$, $\{(i,2),(j,2),(i \circ j,3)\}$ and $\{(i,3),(j,3),(i \circ j,1)\}$ in B.

(b) $(\{\infty_1,\infty_2,\infty_3\} \cup (\{1,2,...,2n\} \times \{1,2,3\}), B')$ is an $STS(6n+3)$, where B' is defined by replacing (1) in (a) with:

(1′) For $1 \leq i \leq n$, let B'_i contain the triples in an $STS(9)$ on the symbols $\{\infty_1,\infty_2,\infty_3\} \cup (\{2i-1,2i\} \times \{1,2,3\})$ in which $\{\infty_1,\infty_2,\infty_3\}$ is a triple and let $B'_i \subseteq B'$. □

The systems constructed above have $6n^2+n$ and $6n^2+5n+1$ triples respectively, which are the correct number of triples. We see that all pairs with ∞ (∞_i, $i=1,2,3$ in (b)) being one of the elements and all pairs from the same hole appear in a triple of type (1) ((1′) in (b)). All the other pairs appear in the triples of type (2).

Note that the quasigroup with holes construction for Steiner triple systems allows us to build STS's from quasigroups and using the following construction, one can build bigger quasigroups from STS's.

Construction of Commutative Quasigroups with Holes: For all $n \geq 3$, a commutative quasigroup of order $2n$ with holes $H = \{\{1,2\},\{3,4\},...\{2n-1,2n\}\}$ is constructed as follows.

Let $S = \{1,2,...,2n+1\}$. If $2n+1 \equiv 1$ or $3 \pmod 6$, then we let (S,B) be a Steiner triple system of order $2n+1$, and if $2n+1 \equiv 5 \pmod 6$, then we let (S,B) be a PBD of order $2n+1$ with exactly one block b, of size 5, and the rest of size 3. By renaming the symbols in the triples (blocks) if necessary, we can assume that the only triples containing the symbol $2n+1$ are $\{1,2,2n+1\}$, $\{3,4,2n+1\}$, ..., $\{2n-1,2n,2n+1\}$ and $2n+1 \notin b$ (when forming the quasigroups, these triples will be ignored).

We define a quasigroup $(Q, \circ) = (\{1, 2, ..., 2n\}, \circ)$ as follows:

(1) For each $h \in H = \{\{1, 2\}, \{3, 4\}, ...\{2n - 1, 2n\}\}$, let $(h, \circ)$ be a subquasigroup of $(Q, \circ)$.

(2) For $1 \leq i \neq j \leq 2n$, $\{i, j\} \notin H$ and $\{i, j\} \not\subseteq b$, let $\{i, j, k\}$ be the triple in B containing the symbols i and j and define $i \circ j = k = j \circ i$.

(3) If $2n + 1 \equiv 5 \pmod 6$, then let $(b, \otimes)$ be an idempotent commutative quasigroup of order 5 and for each $\{i, j\} \subseteq b$ define $i \circ j = i \otimes j = j \circ i$. $\square$

Chapter 2

INTRODUCTION

2.1 Overview

Combinatorial design theory is the study of *combinatorial designs* which are collections of subsets of a finite set, having some special intersection properties. Among the oldest and most interesting notions in combinatorial design theory are the *balanced incomplete block designs* ($BIBD$s). A $BIBD$ with parameters (v, b, r, k, λ) is a pair $(X, \mathcal{A})$ where (i): X is a set of v elements (called *points*), (ii): $\mathcal{A}$ is a family of b subsets of X, each of cardinality k (called *blocks*), (iii): every point occurs in exactly r blocks, and (iv): every pair of distinct points occurs in exactly λ blocks. Usually, $k < v$ is also required, which explains the use of the word *incomplete*. A balanced incomplete block design with parameters (v, b, r, k, λ) is denoted by (v, b, r, k, λ)-$BIBD$. From b blocks, each of size k, we have in total bk elements; counting the same set in another way, we have v points, each of which appears r times in a block. Hence, $bk = vr$. A point, say x, appears in r blocks, each time with some other $k-1$ elements, so $\sum_{t \supseteq \{x\}} |t \setminus \{x\}| = r(k-1)$, but the sum on the left hand side can also be counted as $\lambda(v-1)$ since x appears exactly λ times with each of the other $v-1$ points. So, $r(k-1) = \lambda(v-1)$. Each one of these identities allows us to shorten our notation to (v, k, λ)-$BIBD$. A specially interesting case is when $k = 3$ and $\lambda = 1$. We call a $(v, 3, 1)$-$BIBD$ a *Steiner triple system* after Jakob Steiner (1796-1863), an important geometer of his time [15]. The name Steiner triple system is misleading, since W. S. B. Woolhouse [11] is known to be the first one to define this term. In 1844, Woolhouse asked: For which positive integers n does a Steiner triple system of order n exist (denoted by $STS(n)$)? In 1847, this question was solved by T. P. Kirkman

(1806-1895) [15], who is also famous with his 15 *schoolgirls problem*. The problem is to arrange 15 schoolgirls in groups of three for seven days' walks such that each pair of them walk exactly once. This problem is equivalent to constructing an $STS(15)$ where the set of three element blocks (triples) can be partitioned into seven subsets with five triples each, so that each element of the point set appears exactly once in a triple of each such subset. Such Steiner triple systems are also called Kirkman triple systems. Now going back to the existence question of Steiner triple systems, we give constructions different than the ones of Kirkman. The following constructions are due to Raj Chandra Bose (1901-1987) and Thoralf Skolem (1887-1963), respectively, which together guarantee the existence of Steiner triple systems of order n whenever $n \equiv 1$ or $3 \pmod 6$. These constructions make use of the existence of idempotent commutative quasigroups of order $2n+1$ and half-idempotent commutative quasigroups of order $2n$, both for $n \geq 1$. These can be obtained by renaming the tables for the additive group of integers modulo $2n+1$ and $2n$, respectively.

The Bose Construction: Let $v = 6n+3$ and let $(Q, \circ)$ be an idempotent commutative quasigroup of order $2n+1$ where $Q = \{1, 2, ..., 2n+1\}$. Let $S = Q \times \{1, 2, 3\}$, we define T to contain the following types of triples.

(1) For $1 \leq i \leq 2n+1$, $\{(i,1), (i,2), (i,3)\} \in T$.

(2) For $1 \leq i < j \leq 2n+1$, $\{(i,1), (j,1), (i \circ j, 2)\}$, $\{(i,2), (j,2), (i \circ j, 3)\}$, $\{(i,3), (j,3), (i \circ j, 1)\} \in T$.

Then (S, T) is a Steiner triple system of order $6n+3$. Please see Figure 2.1. $\square$

The Skolem Construction: Let $v = 6n+1$ and let $(Q, \circ)$ be a half-idempotent commutative quasigroup of order $2n$ where $Q = \{1, 2, ..., 2n\}$. Let $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$, we define T to contain the following types of triples.

(1) For $1 \leq i \leq n$, $\{(i,1), (i,2), (i,3)\} \in T$.

(2) For $1 \leq i \leq n$, $\{\infty, (n+i, 1), (i, 2)\}$, $\{\infty, (n+i, 2), (i, 3)\}$, $\{\infty, (n+i, 3), (i, 1)\} \in T$.

(3) For $1 \leq i < j \leq 2n$, $\{(i,1), (j,1), (i \circ j, 2)\}$, $\{(i,2), (j,2), (i \circ j, 3)\}$, $\{(i,3), (j,3), (i \circ j, 1)\} \in T$.

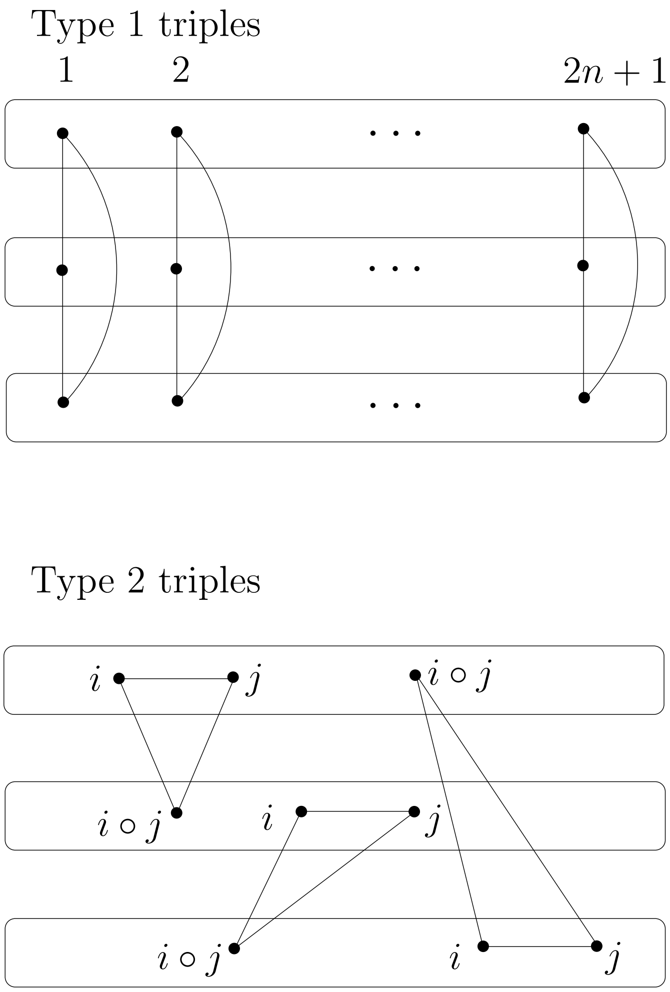


Figure 2.1: The Bose Construction

Then (S, T) is a Steiner triple system of order $6n + 1$. Please see Figure 2.2. $\square$

Theorem 2.1.1 *There exists a Steiner triple system of order n if and only if $n \equiv 1$ or 3 (mod 6).*

Proof: We only need to show that there is no $STS(n)$ for $n \equiv 0, 2, 4$ or 5 (mod 6). Let (S, T) be an $STS(n)$. Any triple in T contains three 2-element subsets. Since each pair of distinct elements of S appears together in exactly one triple of T, we have $3\,|T| = \binom{n}{2}$. So, $|T| = n(n-1)/6$. For any x, let $T(x) = \{t \setminus \{x\}$ such that $x \in t \in T\}$. Then $T(x)$ partitions $S \setminus \{x\}$ into subsets of size two and considering the cardinality of $S \setminus \{x\}$ we see that $n-1$ is even, n is odd. Computing modulo 6, we have $n \equiv 1, 3$ or 5. When $n = 5$, $|T| = n(n-1)/6$ is not an integer. So, the only available cases are $n \equiv 1$ or 3 (mod 6). □

Knowing the set of numbers n for which there exists an $STS(n)$, our next concern is determining the set of numbers k (for fixed n) for which there exists a pair of $STS(n)$'s intersecting in k triples. Throughout this thesis we consider this problem and a variety of it, called *flower intersection problem*, and give complete solutions to both.

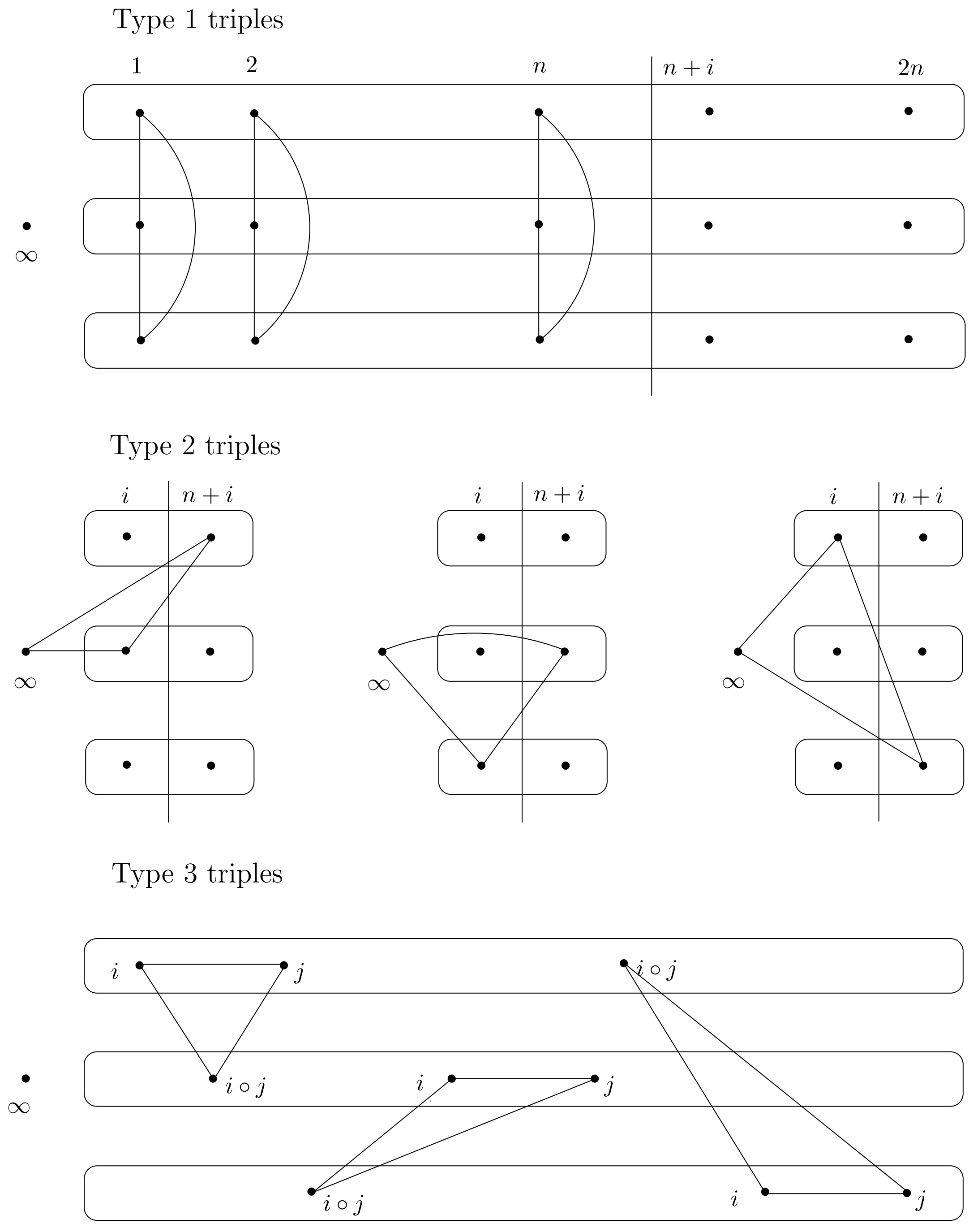


Figure 2.2: The Skolem Construction

Chapter 3

THE BASIC INTERSECTION PROBLEM OF STEINER TRIPLE SYSTEMS

Two Steiner triple systems $(S, \mathcal{T}_1)$ and $(S, \mathcal{T}_2)$ are said to *intersect in k triples* if $| \mathcal{T}_1 \cap \mathcal{T}_2 |= k$. If $k = 0$, $(S, \mathcal{T}_1)$ and $(S, \mathcal{T}_2)$ are said to be *disjoint*, and if $| \mathcal{T}_1 \cap \mathcal{T}_2 |= 1$ they are said to be *almost disjoint*. J. Doyen and C. C. Lindner solved these special cases, respectively, establishing the first results on the intersection problem of Steiner triple systems in [5] and [10]. Then, in 1975, C. C. Lindner and A. Rosa solved completely the problem of determining the set of possible values k of triples in which two Steiner triple systems can intersect [12], [13]. In this chapter, we give a detailed analysis of their proof.

We will denote the number of triples in an $STS(n)$ by t_n; hence $t_n = n(n-1)/6$. We define $I(n) = \{0, 1, ..., t_n - 6,\ t_n - 4,\ t_n\}$. Further, we define $J(n)$ to be the set of all integers k for which there exists a pair of $STS(n)$ intersecting in k triples.

3.1 Elementary Results

Lemma 3.1.1 $J(3) = \{1\}$, $J(7) = \{0, 1, 3, 7\}$.

Proof: When $n = 3$, $t_3 = 1$. Since there is a unique $STS(3)$, $J(3) = \{1\}$. To show that $J(7) = \{0, 1, 3, 7\}$, we explicitly give pairs of $STS(7)$s intersecting in $k = 0, 1, 3, 7$ triples and then show why $2, 4, 5, 6 \notin J(7)$. Let

$T = \{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 1\}, \{6, 7, 2\}, \{7, 1, 3\}\}$.

$T_0 = \{\{6, 5, 3\}, \{5, 4, 2\}, \{4, 3, 1\}, \{3, 2, 7\}, \{2, 1, 6\}, \{1, 7, 5\}, \{7, 6, 4\}\}$.

$T_1 = \{\{2, 3, 7\}, \{3, 5, 6\}, \{1, 2, 5\}, \{\mathbf{4,5,7}\}, \{1, 6, 7\}, \{2, 4, 6\}, \{1, 3, 4\}\}$.

$T_3 = \{\{\mathbf{1,2,4}\}, \{2, 3, 6\}, \{2, 5, 7\}, \{\mathbf{5,6,1}\}, \{3, 4, 5\}, \{4, 6, 7\}, \{\mathbf{7,1,3}\}\}$, where

$|T \cap T_i| = i$. $7 \in J(7)$ is done by taking the same $STS(7)$ twice.

Now, we show $2, 4, 5, 6 \notin J(7)$.

$2 \notin J(7)$: Assume contrarily that $2 \in J(7)$. Let $(S, \mathcal{T}_1)$ and $(S, \mathcal{T}_2)$ be two $STS(7)$ where $\mathcal{T}_1 \cap \mathcal{T}_2 = \{T_1, T_2\}$. We consider two cases $T_1 \cap T_2 = \emptyset$ and $| T_1 \cap T_2 |= 1$.

Case i: Without loss of generality, we assume that $T_1 = \{1, 2, 3\}$ and $T_2 = \{4, 5, 6\}$. Then, also $\{1, 4, 7\} \in \mathcal{T}_1 \cap \mathcal{T}_2$, since the pair $\{1, 4\}$ must appear in a triple in both systems, which gives a contradiction.

Case ii: Without loss of generality, let 1 be the common vertex of the triples T_1 and T_2. 1 appears together with 4 vertices in T_1 and T_2, hence 1 and the remaining two other vertices that have not appeared yet must form a triple, both in $(S, \mathcal{T}_1)$ and $(S, \mathcal{T}_2)$, which gives a contradiction.

$4, 5, 6 \notin J(7)$: Let $\mathcal{T}_1 \cap \mathcal{T}_2 = i$. Assume $i \neq 7$, then there is a triple, say $\{1, 2, 3\} \in \mathcal{T}_1 \setminus \mathcal{T}_2$. Then the pair $\{1, 2\}$ appears in a triple different than $\{1, 2, 3\}$ in $\mathcal{T}_2 \setminus \mathcal{T}_1$, say $\{1, 2, 4\} \in \mathcal{T}_2 \setminus \mathcal{T}_1$. Now the pair $\{1, 4\}$ must appear in a triple of $\mathcal{T}_1 \setminus \mathcal{T}_2$, say $\{1, 4, 5\}$. So, for each a that appears in a triple of $\mathcal{T}_1 \setminus \mathcal{T}_2$, there is at least one other triple in $\mathcal{T}_1 \setminus \mathcal{T}_2$ containing a (in the above case, for 1, these are $\{1, 2, 3\}$ and $\{1, 4, 5\}$). Hence there are at least five elements which appear in a triple of $\mathcal{T}_1 \setminus \mathcal{T}_2$ (in the above case, these are $1, 2, 3, 4, 5$). These five elements appear at least $5 \cdot 2$ times in total in $\mathcal{T}_1 \setminus \mathcal{T}_2$ and hence we must have $\frac{5 \cdot 2}{3} \leq 7 - i$. We have $i < 4$. □

Our work for Lemma 3.1.1 can be generalized to STS's of any order. Now, we will prove that $t_n - 5$, $t_n - 3$, $t_n - 2$ and $t_n - 1$ cannot be in $J(n)$, i.e. there cannot exist a pair of $STS(n)$'s intersecting in $t_n - 5$, $t_n - 3$, $t_n - 2$ or $t_n - 1$ triples, which is the same as saying $J(n) \subseteq I(n)$.

Lemma 3.1.2 *$J(n) \subseteq I(n)$.*

Proof: We have already examined the cases $n = 3$ and $n = 7$. So, assume $n \geq 9$. We remark that our proof for $4, 5, 6 \notin J(7)$ in Lemma 3.1.1 applies for $t_n - 3$, $t_n - 2$, $t_n - 1$ as well (note that in the proof we did not use the information that $n = 7$). So, we only need to show that $t_n - 5 \notin J(n)$. Let $(S, \mathcal{T}_1)$ and $(S, \mathcal{T}_2)$ be two $STS(n)$.

Assume that $\mid \mathcal{T}_1 \setminus \mathcal{T}_2 \mid= 5$. As in the proof of Lemma 3.1.1, each element in a triple of $\mathcal{T}_1 \setminus \mathcal{T}_2$ must appear in at least two triples of $\mathcal{T}_1 \setminus \mathcal{T}_2$. In five triples we have 15 elements in total (where each of the elements is counted at least twice). Then one of the elements must occur in at least three triples of $\mathcal{T}_1 \setminus \mathcal{T}_2$. Denote that element by 1 and let the triples be $\{1,2,3\}, \{1,4,5\}, \{1,6,7\}$. Since each element has to appear in at least two triples of $\mathcal{T}_1 \setminus \mathcal{T}_2$, there must be another triple in $\mathcal{T}_1 \setminus \mathcal{T}_2$ containing the element 2. The same is true for the element 3. Respecting that the elements $4,5,6,7$ also have to appear in at least two triples of $\mathcal{T}_1 \setminus \mathcal{T}_2$, we see that without loss of generality $\mathcal{T}_1 \setminus \mathcal{T}_2 = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{3,5,7\}\}$. The pair of elements that occur together in a triple of $\mathcal{T}_1 \setminus \mathcal{T}_2$ must also occur in a triple of $\mathcal{T}_2 \setminus \mathcal{T}_1$. The pair $\{2,3\}$ should occur in a triple of $\mathcal{T}_2 \setminus \mathcal{T}_1$, but there is no element other than 1 which occurs together with both 2 and 3 in a block of $\mathcal{T}_1 \setminus \mathcal{T}_2$. So, $\{1,2,3\} \in \mathcal{T}_2 \setminus \mathcal{T}_1$. Contradiction. □

Now, we move on to prove the main result which tells that, indeed $J(n) = I(n)$ for all $n \equiv 1,3 \pmod 6$ with the sole exception being $n = 9$. To prove this result, first we will prove some smaller cases and then use induction to handle the general case.

First, note as our third lemma that $J(9) = \{0,1,2,3,4,6,12\}$, whereas $I(9) = \{0,1,2,3,4,5,6,8,12\}$.

Lemma 3.1.3 $J(9) = \{0,1,2,3,4,6,12\}$.

Proof: Consider the $STS(9)$ given by the set of triples $\{\{0,1,2\}$, $\{0,3,6\}$, $\{0,4,8\}$, $\{0,5,7\}$, $\{1,3,8\}$, $\{1,4,7\}$, $\{1,5,6\}$, $\{2,3,7\}$, $\{2,4,6\}$, $\{2,5,8\}$, $\{3,4,5\}$, $\{6,7,8\}\}$.

Applying the following permutations on the above $STS(9)$, we generate new $STS(9)$'s with k triples in common with the above given $STS(9)$, where $k \in \{0,1,2,3,4,6\}$; for $k = 12$, we can take two copies of the above $STS(9)$.

$\mathbf{k = 0}$: $(0,1,6,4,8,7,5)(2,3)$

$\mathbf{k = 1}$: $(0,5,3,4,1,8,6)(2,7)$

$\mathbf{k = 2}$: $(0,7,2,8,1,5,6,4)$

$\mathbf{k=3}$: $(0,3,5,4,6,2,1,7,8)$

$\mathbf{k=4}$: $(0,7,8,6,4,3,2,5,1)$

$\mathbf{k=6}$: $(0,7)(1,3,6,4)$

In [17] and [18] E. Witt proved that any two $STS(9)$'s are isomorphic. In [8] E. S. Kramer and D. M. Mesner used this fact and a computer search to prove that there exists no pair of $STS(9)$'s intersecting in 5 or 8 triples. Moreover, for all $k \in I(9)$, they listed the actual number of pairs of $STS(9)$'s intersecting in k triples. □

3.2 Teirlinck's Algorithm

Before determining the whole set of number of triples in which two Steiner triple systems $(S, \mathcal{T}_1)$ and $(S, \mathcal{T}_2)$ can intersect, we give an algorithm, called Teirlinck's Algorithm, which guarantees that $0 \in J(n)$ for all admissible $n \geq 3$. As we mentioned earlier, such a pair of systems that have no triples in common is called disjoint.

Consider the disjoint systems $(S, \mathcal{T}_1)$ and $(S, \mathcal{T}_2)$, where

$T_1 = \{\{1,2,4\},\{2,3,5\},\{3,4,6\},\{4,5,7\},\{5,6,1\},\{6,7,2\},\{7,1,3\}\}$ and

$T_2 = \{\{1,2,5\},\{2,4,6\},\{4,5,3\},\{5,6,7\},\{6,3,1\},\{3,7,2\},\{7,1,4\}\}$.

We can observe that $T_2 = T_1\alpha$ where α is the following permutation:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 4 & 5 & 6 & 3 & 7 \end{pmatrix}$$

Such a system $(S, \mathcal{T}_2)$, which is constructed by performing a permutation α on a system $(S, \mathcal{T}_1)$ is called an isomorphic disjoint mate of the system $(S, \mathcal{T}_1)$ whenever these systems are disjoint. Given two systems $(S, \mathcal{T}_1)$ and $(S, \mathcal{T}_2)$ of order n, our plan is to construct α as a product of transpositions such that $T_2 = T_1\alpha$. We will denote the transposition that interchanges a and b by (ab). Let $\{c,d,e\} \in \mathcal{T}_1 \cap \mathcal{T}_2$. We define the *spread of* c with respect to $\{c,d,e\}$ as $S(c) = \{c,d,e\} \cup A(c) \cup B(c)$ where $A(c)$ and $B(c)$ are defined as follows:

$A(c) = \{a \mid \{a,x,y\} \in \mathcal{T}_1 \text{ and } \{c,x,y\} \in \mathcal{T}_2 \setminus \{c,d,e\}\}$ and

$B(c) = \{b \mid \{b,z,w\} \in \mathcal{T}_2 \text{ and } \{c,z,w\} \in \mathcal{T}_1 \setminus \{c,d,e\}\}$.

In words, we consider the pair of elements $\{z, w\}$ that forms a triple in $\mathcal{T}_1$ with c. By definition of an STS, this pair of elements also appears together in a triple of $\mathcal{T}_2$. Say the third element of that triple in $\mathcal{T}_2$ is b. All such elements b, except those of $\{c, d, e\}$ constitute $B(c)$. Similarly, considering the pair of elements $\{x, y\}$ that forms a triple in $\mathcal{T}_2$ with c we get elements a such that $\{a, x, y\}$ is a triple of $\mathcal{T}_1$. Such elements a, other than those in $\{c, d, e\}$ constitute $A(c)$.

Given two systems $(S, \mathcal{T}_1)$ and $(S, \mathcal{T}_2)$ of order n, the following algorithm provides a way to manipulate $\mathcal{T}_2$ such that under a certain circumstance the modified STS has fewer triples in common with $\mathcal{T}_1$.

Theorem 3.2.1 *(The Reduction Algorithm) Suppose that* $(S, \mathcal{T}_1)$ *and* $(S, \mathcal{T}_2)$ *are two* $STS(n)$ *such that* $\{1, 2, 3\} \in \mathcal{T}_1 \cap \mathcal{T}_2$ *and* $|S(3)| < n$. *Then there exists a transposition* α *such that* $\mathcal{T}_1 \cap \mathcal{T}_2\alpha \subseteq \mathcal{T}_1 \cap \mathcal{T}_2$ *and* $|\mathcal{T}_1 \cap \mathcal{T}_2\alpha| < |\mathcal{T}_1 \cap \mathcal{T}_2|$.

Proof: Let x be such that $x \notin S(3)$ and let $\alpha = (3x)$. $\mathcal{T}_1 \cap \mathcal{T}_2\alpha$ contains all the triples of $\mathcal{T}_1 \cap \mathcal{T}_2$ which do not include any of x or 3, as well as the triple $\{3, x, a\}$ if $\{3, x, a\} \in \mathcal{T}_1 \cap \mathcal{T}_2$. As triples that do not contain 3 or x do not change under the permutation $(3x)$, if there are some other triples in $\mathcal{T}_1 \cap \mathcal{T}_2\alpha$, these must be either of the form $\{x, a, b\}$ or $\{3, a, b\}$. If $\{x, a, b\} \in \mathcal{T}_1 \cap \mathcal{T}_2\alpha$, then $\{x, a, b\} \in \mathcal{T}_1$ and $\{3, a, b\} \in \mathcal{T}_2$, but this implies $x \in A(3) \subseteq S(3)$. If $\{3, a, b\} \in \mathcal{T}_1 \cap \mathcal{T}_2\alpha$, then $\{3, a, b\} \in \mathcal{T}_1$ and $\{x, a, b\} \in \mathcal{T}_2$ and this implies $x \in B(3) \subseteq S(3)$. Both of these cases contradict with our assumption that $x \notin S(3)$. So, $\mathcal{T}_1 \cap \mathcal{T}_2\alpha \subseteq \mathcal{T}_1 \cap \mathcal{T}_2$. Moreover, $\{1, 2, 3\} \in (\mathcal{T}_1 \cap \mathcal{T}_2) \setminus (\mathcal{T}_1 \cap \mathcal{T}_2\alpha)$. Hence, $|\mathcal{T}_1 \cap \mathcal{T}_2\alpha| < |\mathcal{T}_1 \cap \mathcal{T}_2|$. □

Given two $STS(n)$'s $(S, \mathcal{T}_1)$ and $(S, \mathcal{T}_2)$, as long as the assumption of the theorem is satisfied, we can repeat the above algorithm to find transpositions $\alpha_1, \alpha_2, ..., \alpha_k$ so that $\mathcal{T}_1 \cap \mathcal{T}_2\alpha_1\alpha_2...\alpha_k = \emptyset$. If at some step i the assumption is not satisfied, i.e. if every element in every triple of $\mathcal{T}_1 \cap \mathcal{T}_2\alpha_1\alpha_2...\alpha_i$ has spread (with respect to that triple) equal to S, this algorithm does not help any further. We need another algorithm:

Theorem 3.2.2 *(Teirlinck's Algorithm) Suppose that* $(S, \mathcal{T}_1)$ *and* $(S, \mathcal{T}_2)$ *are two* $STS(n)$ *such that* $\{1, 2, 3\} \in \mathcal{T}_1 \cap \mathcal{T}_2$ *and* $|S(3)| = n$. *Then there exists a transposition*

α such that $\mathcal{T}_1 \cap \mathcal{T}_2\alpha$ contains a triple t with an element e such that $|S(e)| < n$ and $|\mathcal{T}_1 \cap \mathcal{T}_2\alpha| \leq |\mathcal{T}_1 \cap \mathcal{T}_2|$.

Proof: We are going to construct the necessary transposition α by an algorithm, but first we note that given two $STS(n)$'s; $(S, \mathcal{T}_1)$, $(S, \mathcal{T}_2)$ and a triple $\{1, 2, 3\} \in \mathcal{T}_1 \cap \mathcal{T}_2$, for any element of this triple, for example for 1, $S(1) = S$ if and only if $\{1, 2, 3\}$, $A(1)$ and $B(1)$ are all disjoint. This is because $S(1) = \{1, 2, 3\} \cup A(1) \cup B(1)$ and both of $A(1)$ and $B(1)$ should have $(n - 3)/2$ elements by their definition. That said, we describe the algorithm step by step as follows:

1. Let $\{1, 2, 3\} \in \mathcal{T}_1 \cap \mathcal{T}_2$.
2. Suppose $\{3, x, y\}$ is another triple in $\mathcal{T}_2$ containing the symbol 3.
3. Let $\{x, y, c\}$ be the triple containing x and y in $\mathcal{T}_1$. Since we assumed that $S(3) = S$, making use of our observation we conclude that $c \notin \{1, 2, 3\}$.
4. Suppose $\{3, c, d\}$ is the triple containing 3 and c in $\mathcal{T}_2$.
5. As the final step, we find the triple containing c and d in $\mathcal{T}_1$, say $\{c, d, e\}$. Since $S(3) = S$, making use of our observation again, we conclude that $e \notin \{1, 2, 3\}$. Please see Figure 3.1.

We claim that $\alpha = (3e)$ is the required transposition. If so, $\{c, d, e\} \in \mathcal{T}_1 \cap \mathcal{T}_2\alpha$. Denote by $S_\alpha(e)$ the spread of e with respect to $\{c, d, e\}$ after the transposition is applied. So, $|S_\alpha(e)| < n$ because after applying the transposition α, the sets $\{c, d, e\}, A_\alpha(e)$ and $B_\alpha(e)$ are not disjoint. For example, c is in both $\{c, d, e\}$ and $A_\alpha(e)$ (remark that $\{e, x, y\} \in \mathcal{T}_2\alpha$ and $\{c, x, y\} \in \mathcal{T}_1$, hence $c \in A_\alpha(e)$). Finally, we need to show that $|\mathcal{T}_1 \cap \mathcal{T}_2\alpha| \leq |\mathcal{T}_1 \cap \mathcal{T}_2|$. Now, $\{1, 2, 3\}$ is not any more in $\mathcal{T}_1 \cap \mathcal{T}_2\alpha$, but $\{c, d, e\} \in \mathcal{T}_1 \cap \mathcal{T}_2\alpha$. It is enough to show that $\mathcal{T}_1 \cap \mathcal{T}_2\alpha$ does not contain any new triple, other than $\{c, d, e\}$. Assume that there exists a triple in $(\mathcal{T}_1 \cap \mathcal{T}_2\alpha) \setminus (\mathcal{T}_1 \cap \mathcal{T}_2)$. Such a triple must contain at least one of the elements 3 or e. Assume $\{e, a, b\} \in (\mathcal{T}_1 \cap \mathcal{T}_2\alpha) \setminus (\mathcal{T}_1 \cap \mathcal{T}_2)$, where $\{e, a, b\} \neq \{c, d, e\}$ and a and b are arbitrary elements different than e. So, $\{e, a, b\} \in \mathcal{T}_1$ and $\{3, a, b\} \in \mathcal{T}_2$. This means that $3 \in B(e)$. We remember that the spread of each element was assumed to be equal to S. But from $\{c, d, e\} \in \mathcal{T}_1$ and $\{c, d, 3\} \in \mathcal{T}_2$, we also get

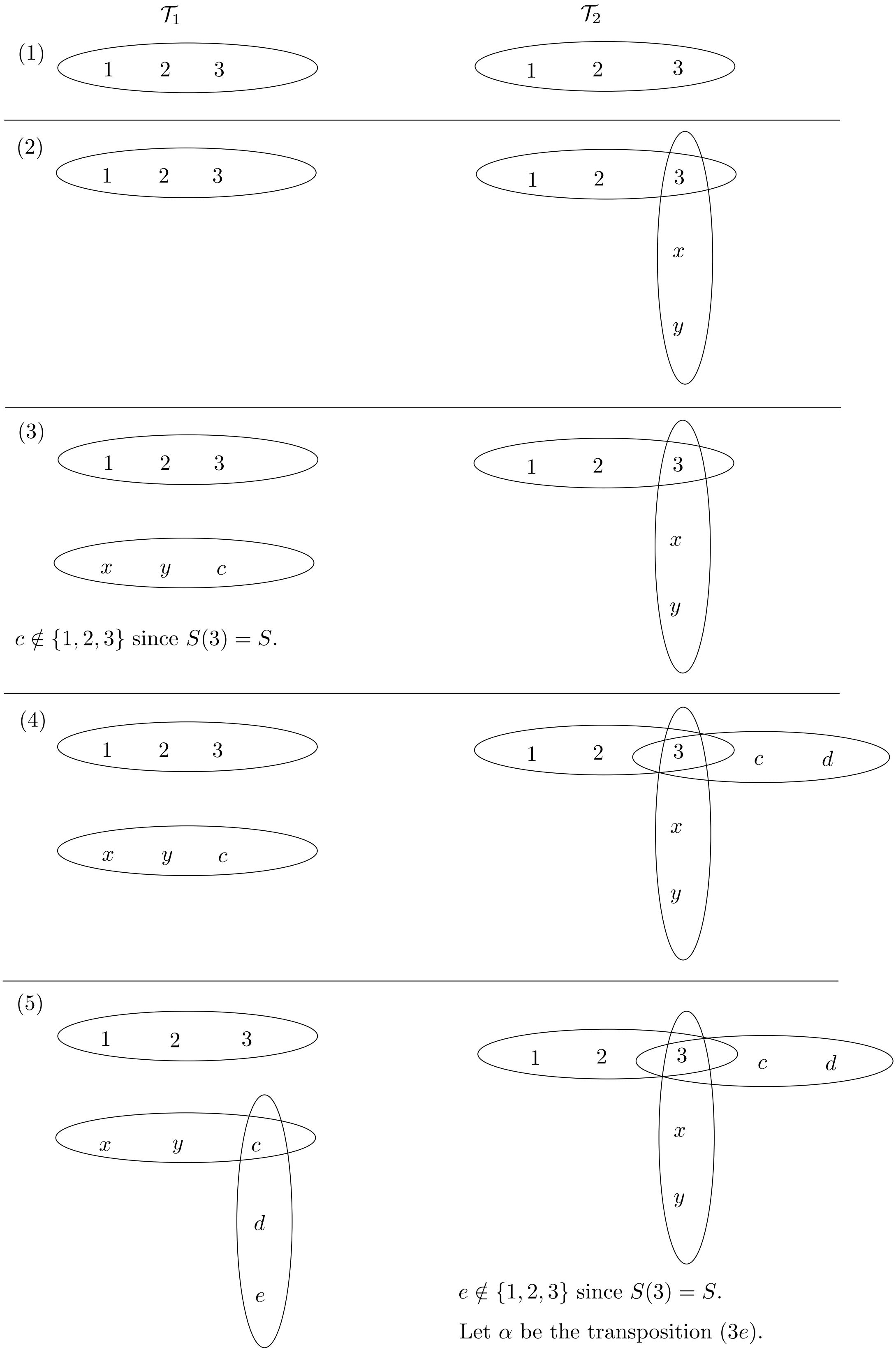


Figure 3.1: Teirlinck's Algorithm

$3 \in B(e)$. The observation we made before describing the algorithm suggests that $\{e, a, b\} = \{c, d, e\}$. Contradiction. This time, assume $\{3, a, b\} \in (\mathcal{T}_1 \cap \mathcal{T}_2\alpha) \setminus (\mathcal{T}_1 \cap \mathcal{T}_2)$, where $\{3, a, b\} \neq \{c, d, e\}$ and a and b are arbitrary elements different than 3. So, $\{3, a, b\} \in \mathcal{T}_1$ and $\{e, a, b\} \in \mathcal{T}_2$. This means again that $3 \in A(e)$, but 3 was already shown to be in $A(e)$ due to a different block. Contradiction. □

Theorem 3.2.3 *Let $(S, \mathcal{T}_1)$ and $(S, \mathcal{T}_2)$ be any two $STS(n)$. Then there exist transpositions $\alpha_1, \alpha_2, ..., \alpha_k$ such that $\mathcal{T}_1 \cap \mathcal{T}_2\alpha_1\alpha_2...\alpha_k = \emptyset$.*

Proof: Repeated applications of the Reduction Algorithm and Teirlinck's Algorithm (needed whenever every element in every triple in the intersection has spread equal to S) produce a pair of disjoint triple systems. □

3.3 Basic Constructions and Lemmas

In this section, we will deal with the main constructions, which we will need to cope both with the smaller cases and the general case.

We start with the well known fact that there exists a 1-factorization of K_{2n} for each n . For the sake of completeness, we give a proof of this fact here. Let us note it as a lemma.

Lemma 3.3.1 *There exists a 1-factorization of K_{2n} for each n.*

Proof: Let $\mathcal{F} = \{F_1, F_2, ..., F_{2n-1}\}$ where $F_i = \{\{i, 2n\}, \{i - j, i + j\}\}$ for $i = 1, 2, ..., 2n - 1,\ j = 1, 2, ..., n - 1$, where $i - j,\ i + j$ are calculated modulo $\mathbb{Z}_{2n-1}$. This gives a 1-factorization of K_{2n}. Please see Figure 3.2. □

2n+1 Construction: We let (S, T) be a Steiner triple system of order n, say $S = \{a_1, a_2, ..., a_n\}$ and $(X, \mathcal{F})$ be a 1-factorization of K_{n+1}, where X is a vertex set such that $X \cap S = \emptyset$ and $\mathcal{F}$ is the union of 1-factors $F_1, F_2, ..., F_n$. We set $S^* = S \cup X$ and define the collection of triples T^* as the union of the following:

(1) $T \subseteq T^*$.

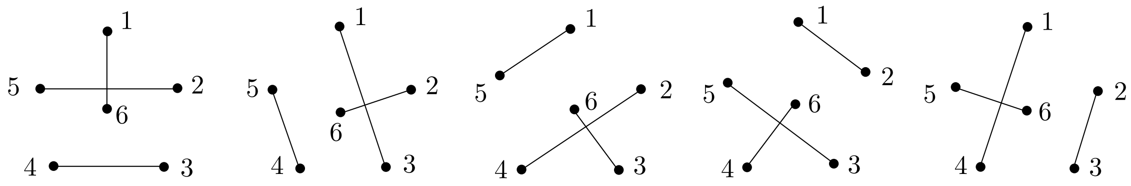


Figure 3.2: A 1-factorization of K_6

(2) For each vertex $a_i \in S$ and for each edge $[x, y]$ of the 1-factor F_i, $i = 1, 2, ..., n$, we place the triple $\{a_i, x, y\}$ in T^*. Please see Figure 3.3.

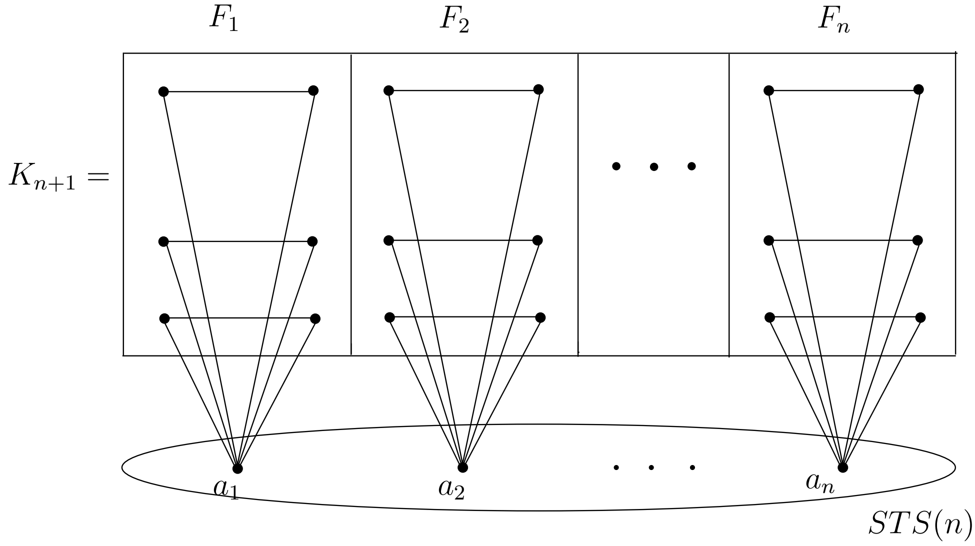


Figure 3.3: $2n + 1$ Construction

(S^*, T^*) is an $STS(2n + 1)$. First we count the number of triples. There are $t_n = n \cdot (n - 1)/6$ triples of type (1) and $n \cdot (n + 1)/2$ triples of type (2). Together, these make $(4n^2 + 2n)/6 = (2n + 1) \cdot 2n/6 = t_{2n+1}$ triples. So, we have the right number of triples. Hence it is enough to show that each pair of vertices of S^* appear together in a triple. All pairs of points, both from S, appear clearly in a triple of type (1), since (S, T) is an STS. All pairs of points, both from X, appear together in a triple of type (2), since to each edge $[x, y]$ corresponds a triple of type (2). And finally, all pairs of vertices such that one is from S, the other one from X appear again in a triple of type (2), because each point a_i of S is joined to both endpoints of all the edges (building a triple of type (2)) of a 1-factor F_i of K_{2n} and a 1-factor covers all vertices of K_{2n}, by definition. □

This construction is an important tool for intersecting Steiner triple systems in certain number of triples because of the following lemma.

Lemma 3.3.2 *If* $k \in J(n)$*, then* $k+s(n+1)/2 \in J(2n+1)$ *for every* $s \in \{0,1,2,...,n\} \setminus \{n-1\}$.

Proof: Let (S,T_1) and (S,T_2) be two $STS(n)$'s intersecting in k triples. Let (S^*,T_1^*) be an $STS(2n+1)$ constructed from (S,T_1) using the $2n+1$ Construction. Let α be a permutation of the elements of S fixing exactly s elements, say $\{a_1,a_2,...,a_s\}$. Clearly, such a permutation exists for $s \in \{0,1,2,...,n\} \setminus \{n-1\}$. Using the same 1-factorization, we define the triples of (S^*,T_2^*) as follows:
(1) $T_2 \subseteq T_2^*$.
(2) For each vertex $a_i \in S$ and for each edge $[x,y]$ of the 1-factor F_i, $i = 1,2,...,n$, we place the triple $\{a_i\alpha, x, y\}$ in T_2^*.

Again by the $2n+1$ Construction, (S^*,T_2^*) is an $STS(2n+1)$. Since each 1-factor contains $(n+1)/2$ edges, (S^*,T_1^*) and (S^*,T_2^*) have $s(n+1)/2$ triples of type (2) in common. In addition, since we chose (S,T_1) and (S,T_2) in such a way that they intersect in k triples, (S^*,T_1^*) and (S^*,T_2^*) have k triples of type (1) in common. So, in total (S^*,T_1^*) and (S^*,T_2^*) have $k+s(n+1)/2$ triples in common. Please see Figure 3.4. □

Lemma 3.3.3 *For* $n \geq 13$*,* $J(n) = I(n)$ *implies* $J(2n+1) = I(2n+1)$.

Proof: We exploit the preceding lemma by putting consecutively $s = 0,1,2,...,n-2$ and $s = n$. Then, the followings are in $J(2n+1)$:

$s = 0$ gives $0,1,2,\ ...\ ,t_n-6,\ t_n-4,\ t_n$.

$s = 1$ gives $0+(n+1)/2,\ 1+(n+1)/2,\ 2+(n+1)/2,\ ...\ ,\ t_n-6+(n+1)/2,$ $t_n-4+(n+1)/2,\ t_n+(n+1)/2$.

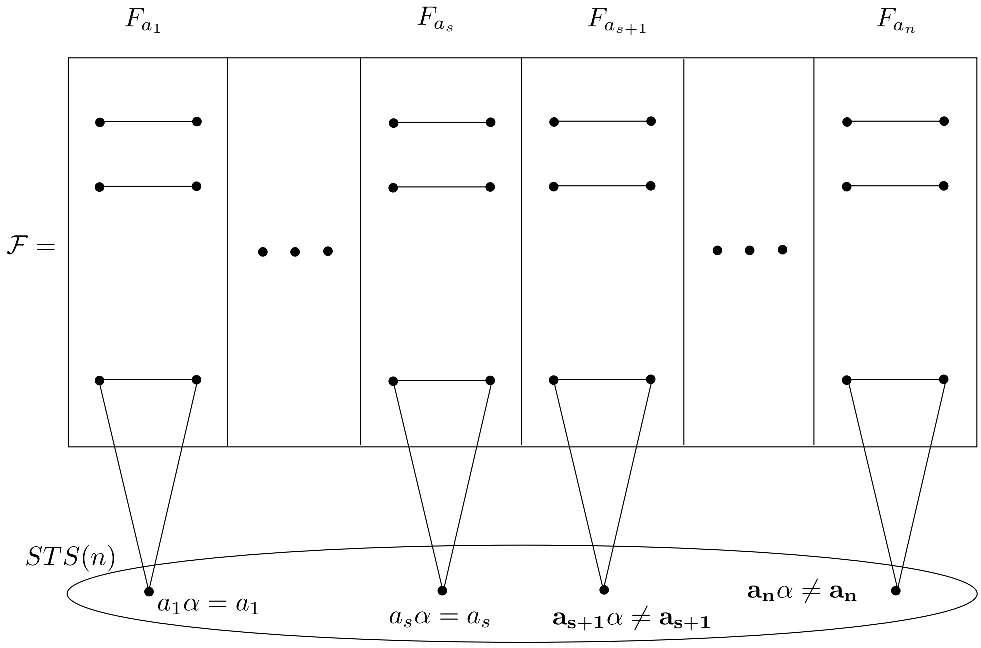


Figure 3.4: $k \in J(n) \Rightarrow k + s(n+1)/2 \in J(2n+1)$

$s = 2$ gives $0 + 2(n+1)/2$, $1 + 2(n+1)/2$, $2 + 2(n+1)/2$, ... , $t_n - 6 + 2(n+1)/2$, $t_n - 4 + 2(n+1)/2$, $t_n + 2(n+1)/2$.

$$\vdots$$

$s = n - 2$ gives $0 + (n-2)(n+1)/2$, $1 + (n-2)(n+1)/2$, $2 + (n-2)(n+1)/2$, ... , $t_n - 6 + (n-2)(n+1)/2$, $t_n - 4 + (n-2)(n+1)/2$, $t_n + (n-2)(n+1)/2$.

$s = n$ gives $0 + n(n+1)/2$, $1 + n(n+1)/2$, $2 + n(n+1)/2$, ... , $t_n - 6 + n(n+1)/2$, $t_n - 4 + n(n+1)/2$, $t_n + n(n+1)/2$.

Put in a closed form, we have $J(n) = I(n)$ for $n \geq 13$ by our assumption. Since $n \geq 13$ implies $t_n - 6 \geq 2(n+1)/2$ (and hence also $t_n - 6 \geq (n+1)/2$); for $t \in I(2n+1)$ such that $0 \leq t \leq n(n+1)/2$, we can write $t = k + s(n+1)/2$, where $k \in I(n)$ and $s \in \{0, 1, ..., n\} \setminus \{n-1\}$. If $t \geq n(n+1)/2$, we can write $t = k + n(n+1)/2$, where $k \in I(n)$. So, assuming $J(n) = I(n)$, we have covered all the numbers in $I(2n+1)$. Hence, $J(2n+1) = I(2n+1)$. □

In what follows, we will need this lemma:

Lemma 3.3.4 [16] *(Stern and Lenz) For any subset* $D \subseteq \{1, 2, ..., \lfloor g/2 \rfloor\}$*, define* $G(D, g)$ *to be the graph with vertex set* $\{0, 1, ..., g-1\}$ *and edge set consisting of all edges having a difference in* D*. If* D *contains an element* d *where* $g/gcd(\{d, g\})$ *is even, then* $G(D, g)$ *has a* 1*-factorization.*

Proof: A proof of this fact can be found in [11] and [16]. □

2n+7 Construction: Let (S, T) be a Steiner triple system of order n (remark that n must be odd), say $S = \{a_1, a_2, ..., a_n\}$ and $(X, \mathcal{F})$ be a factorization of K_{n+7} into n 1-factors $F_1, F_2, ..., F_n$ and $n + 7$ triples generated by the base block $\{0, 1, 3\}$, where X is a vertex set such that $X \cap S = \emptyset$. We set $S^* = S \cup X$ and define the collection of triples T^* as the union of the following:

(1) $T \subseteq T^*$.

(2) For each vertex $a_i \in S$ and for each edge $[x, y]$ of the 1-factor F_i, $i = 1, 2, ..., n$, we place the triple $\{a_i, x, y\}$ in T^*.

(3) Put the $n + 7$ triples generated by the base block $\{0, 1, 3\}$ in T^*. Please see Figure 3.5.

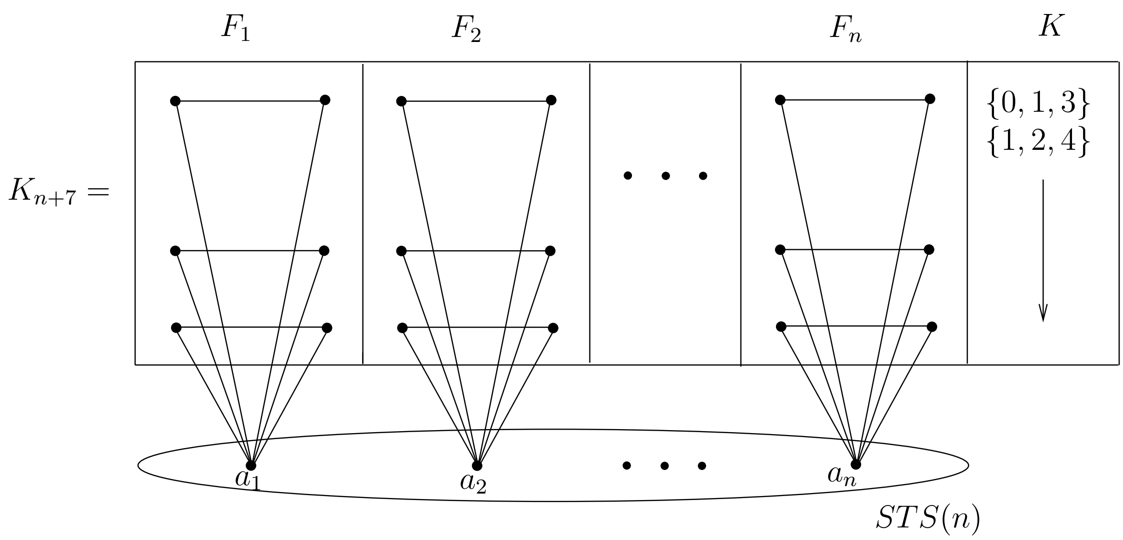


Figure 3.5: $2n + 7$ Construction

(S^*, T^*) is an $STS(2n+7)$. First we need to make clear that such a factorization of K_{n+7} into n 1-factors $F_1, F_2, ..., F_n$ and $n+7$ triples with base block $\{0, 1, 3\}$ exists. This is the point where we need the Stern-Lenz Lemma. Consider K_{n+7} after removing the $n+7$ triples with base block $\{0, 1, 3\}$. The edge set of the remaining graph consists of all the edges with difference in the $\{4,\ 5,\ ...\ ,\ (n+7)/2\}$. In the notation of the Stern-Lenz Lemma, this remaining graph is $G(\{4,\ 5,\ ...\ ,\ (n+7)/2\},\ n+7)$. Taking $d = (n+7)/2$ ($(n+7)/2 \geq 4$), we have $gcd(\{(n+7)/2, n+7\}) = (n+7)/2$. So, in this case $g/gcd(\{d, g\}) = (n+7)/((n+7)/2) = 2$, hence is even. So, there exists a factorization of K_{n+7} into n 1-factors $F_1, F_2, ..., F_n$ and $n+7$ triples with base block $\{0, 1, 3\}$. Now, we count the number of triples. There are $t_n = n \cdot (n-1)/6$ triples of type (1), $n \cdot (n+7)/2$ triples of type (2) and $n+7$ triples of type (3). Together, these make $(4n^2 + 26n + 42)/6 = (2n+7) \cdot (2n+6)/6 = t_{2n+7}$ triples. Hence it is enough to show that each pair of points of S^* appear together in a triple. All pairs of points, both from S, appear in a triple of type (1), since (S, T) is an STS. All pairs of points, both from X, appear together in a triple of type (3) if their difference is 1, 2 or 3 and in a triple of type (2) otherwise. Finally, all pairs of points such that one point is from S, the other one is from X appear again in a triple of type (2), because each point a_i of S is joined to both endpoints of all the edges of a 1-factor F_i of K_{n+7} and a 1-factor covers all points of K_{n+7}, by definition. □

We will make use of the following lemma repeatedly:

Lemma 3.3.5 *If $k \in J(n)$, then $k + s(n+7)/2 + z \in J(2n+7)$ for every $s \in \{0, 1, 2, ..., n\} \setminus \{n-1\}$ where $z \in \{0, n+7\}$.*

Proof: Let (S, T_1) and (S, T_2) be two $STS(n)$'s intersecting in k triples. Let (S^*, T_1^*) be an $STS(2n+7)$ constructed from (S, T_1) using the $2n+7$ Construction. Let α be a permutation of the elements of S fixing exactly s elements, say $\{a_1, a_2, ..., a_s\}$. Such a permutation exists for $s \in \{0, 1, 2, ..., n\} \setminus \{n-1\}$. Using the same 1-factors $F_1, F_2, ..., F_n$ of K_{n+7}, we define the triples of (S^*, T_2^*) as follows:

(1) $T_2 \subseteq T_2^*$.

(2) For each vertex $a_i \in S$ and for each edge $[x, y]$ of the 1-factor F_i, $i = 1, 2, ..., n$,

we place the triple $\{a_i\alpha, x, y\}$ in T_2^*.

(3) Put the $n+7$ triples with base block $\{0,1,3\}$ or $\{0,2,3\}$ in T^*.

Again by the $2n+7$ Construction, (S^*, T_2^*) is an $STS(2n+7)$. Since each 1-factor contains $(n+7)/2$ edges, (S^*, T_1^*) and (S^*, T_2^*) have $s(n+7)/2$ triples of type (2) in common. Since we chose (S, T_1) and (S, T_2) in such a way that they intersect in k triples, (S^*, T_1^*) and (S^*, T_2^*) have k triples of type (1) in common. In addition to these, depending on whether we choose the same base block or not in these two systems, (S^*, T_1^*) and (S^*, T_2^*) have 0 or $n+7$ triples of type (3) in common. So, in total (S^*, T_1^*) and (S^*, T_2^*) have $k + s(n+7)/2 + z$ triples in common. Please see Figure 3.6. □

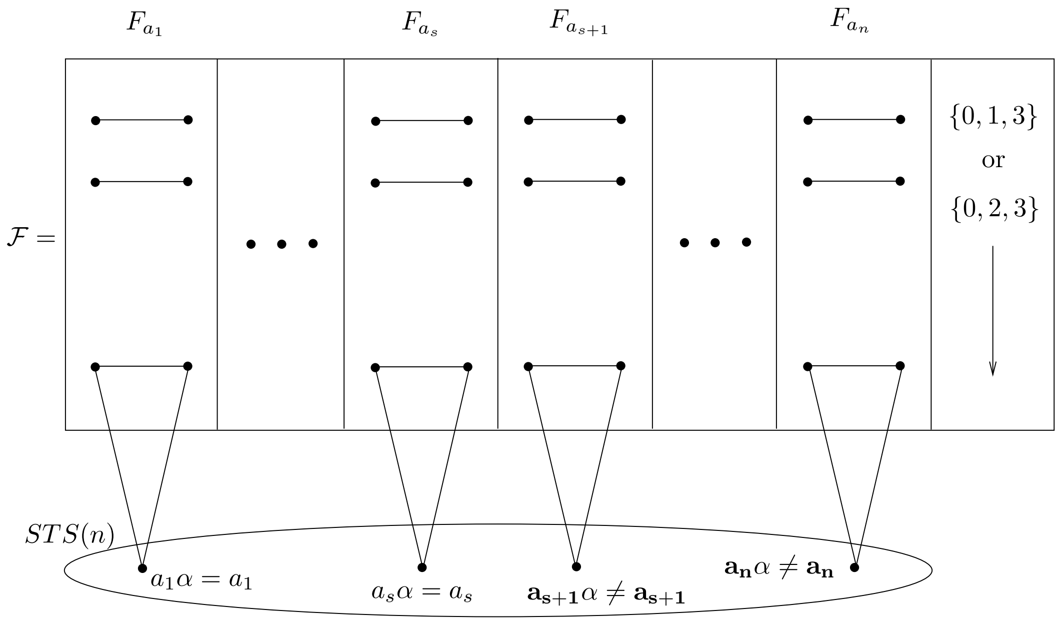


Figure 3.6: $k \in J(n) \Rightarrow k + s(n+7)/2 + z \in J(2n+7)$

Lemma 3.3.6 *For* $n \geq 13$, $J(n) = I(n)$ *implies* $J(2n+7) = I(2n+7)$.

Proof: We go on in a similar way as in the proof of Lemma 3.3.3. First we note that assuming $n \geq 13$ and $J(n) = I(n)$ implies that all $t \leq n+7$ are in $I(n)$. For $t \in I(2n+7)$ such that $0 \leq t \leq n(n+7)/2$, we can write $t = k + s(n+7)/2 + 0$, where

$k \in I(n)$ and $s \in \{0, 1, ..., n\} \setminus \{n-1\}$. If $n(n+7)/2 \leq t \leq n(n+7)/2 + (n+7)$, we can write $t = k + n(n+7)/2 + 0$, where $k \in I(n)$ (we noted that $(n+7) \in I(n)$). If $t \geq n(n+7)/2 + (n+7)$, we can write $t = k + n(n+7)/2 + (n+7)$, where $k \in I(n)$. So, assuming $J(n) = I(n)$, we have covered all the numbers in $I(2n+7)$. Hence, $J(2n+7) = I(2n+7)$. □

3.4 Small Cases

Now, we start dealing with the cases where $n < 27$.

Lemma 3.4.1 $J(13) = I(13)$.

Proof: Consider the $STS(13)$ given by the set of triples $\{\{0,1,2\}, \{0,3,4\}, \{0,5,6\}, \{0,7,8\}, \{0,9,10\}, \{0,11,12\}, \{1,3,5\}, \{1,4,7\}, \{1,6,8\}, \{1,9,11\}, \{1,10,12\}, \{2,3,9\}, \{2,4,5\}, \{2,6,10\}, \{2,7,12\}, \{2,8,11\}, \{3,6,11\}, \{3,7,10\}, \{3,8,12\}, \{4,6,12\}, \{4,8,9\}, \{4,10,11\}, \{5,7,11\}, \{5,8,10\}, \{5,9,12\}, \{6,7,9\}\}$.

Applying the following permutations on the above $STS(13)$, we generate new $STS(13)$'s with k triples in common with the $STS(13)$ given above, where $k \in \{0, 1, ..., 20, 22\}$; for $k = 26$, we can take two copies of the above $STS(13)$.

$\mathbf{k=0}$: $(0,10,12,8,11,9)(1,4,3,5)(2,7)$

$\mathbf{k=1}$: $(0,2,3,6)(1,7,11)(4,12,9,10,8,5)$

$\mathbf{k=2}$: $(0,8,9,12,10,7,5,6,4,11,3,1,2)$

$\mathbf{k=3}$: $(0,3,8,12,1,6,9)(2,10,7,5)$

$\mathbf{k=4}$: $(0,10,6,1,5,9,4,7,12,11,2)$

$\mathbf{k=5}$: $(0,2,1,10,8,5,7,4,12)(3,6,9,11)$

$\mathbf{k=6}$: $(0,3,12,4,8,5,2,9,11)(1,10,6,7)$

$\mathbf{k=7}$: $(0,11,2)(1,12,8,3)(4,6,9)$

$\mathbf{k=8}$: $(1,5,4,2,6)(3,7)(8,11,12)$

$\mathbf{k=9}$: $(0,12,9)(1,7,5,10,11,8,4,2,3)$

$\mathbf{k=10}$: $(0,4,9,12,6,10,2,3,8,1)(5,7,11)$

$\mathbf{k=11}$: $(1,12,10,2,7,3,9,6,11)(4,8)$

$\mathbf{k = 12}:$ $(0, 6, 12, 5, 4, 11, 9, 3, 7, 8)(1, 2, 10)$

$\mathbf{k = 13}:$ $(0, 7, 6)(1, 3, 12, 2, 10, 9, 5, 8, 4)$

$\mathbf{k = 14}:$ $(0, 3, 8, 11, 5)(1, 7, 6, 4, 9)(2, 10, 12)$

$\mathbf{k = 15}:$ $(1, 3, 6, 10, 11, 8, 9, 12, 7)(2, 4, 5)$

$\mathbf{k = 16}:$ $(0, 11, 6, 5, 7, 3)(1, 8)(4, 12, 9, 10)$

$\mathbf{k = 17}:$ $(1, 5, 2, 6)(3, 9, 7, 8)(4, 1, 0)$

$\mathbf{k = 18}:$ $(0, 3)(1, 10, 8, 6, 2, 7, 11)(5, 9, 12)$

$\mathbf{k = 19}:$ $(1, 7, 6)(2, 8, 5)(3, 12, 9)(4, 11, 10)$

$\mathbf{k = 20}:$ $(0, 6, 4, 8, 11, 7, 2, 3, 1, 10)(5, 12, 9)$

$\mathbf{k = 22}:$ $(0, 6, 3, 10, 8)(1, 4, 2, 12, 9)(5, 11, 7)$ □

Lemma 3.4.2 $J(15) = I(15)$.

Proof: First, let us prepare the ingredients for the proof. Below is a quasigroup, $(Q, \circ_1)$ of order 4. Applying the permutation $(\sigma_1\sigma_2)(\sigma_3\sigma_4)$ to the rows of $(Q, \circ_1)$ gives a quasigroup intersecting in no cells, and the permutation $(\sigma_1\sigma_2)$ to the rows of $(Q, \circ_1)$ gives a quasigroup intersecting in 8 cells with $(Q, \circ_1)$. Taking two copies the same quasigroup $(Q, \circ_1)$, we get 16 cells in the intersection.

$\circ_1$				
σ_1	1	2	3	4
σ_2	2	3	4	1
σ_3	3	4	1	2
σ_4	4	1	2	3

And below are two idempotent commutative quasigroups of order 7 intersecting in 12 cells above the main diagonal:

$\circ_1$							
	1	**6**	**5**	**7**	2	**3**	4
	6	2	7	**5**	4	1	**3**
	5	7	3	**6**	1	4	2
	7	5	6	4	**3**	**2**	**1**
	2	4	1	3	5	7	**6**
	3	1	4	2	7	6	**5**
	4	3	2	1	6	5	7

and

$\circ_2$							
	1	**6**	**5**	**7**	4	**3**	2
	6	2	1	**5**	7	4	**3**
	5	1	3	**6**	2	7	4
	7	5	6	4	**3**	**2**	**1**
	4	7	2	3	5	1	**6**
	3	4	7	2	1	6	**5**
	2	3	4	1	6	5	7

Let $(Q, \circ)$ be a quasigroup of order 4 and let $S = \{\infty_1, \infty_2, \infty_3\} \cup (Q \times \{1, 2, 3\})$. Further, we let $S(i) = \{\infty_1, \infty_2, \infty_3\} \cup (Q \times \{i\})$, $i = 1, 2, 3$ and $(S(i), T(i))$ be an $STS(7)$ where $\{\infty_1, \infty_2, \infty_3\}$ is a triple in both of $T(2)$ and $T(3)$. We set $T(2)^* = T(2) \setminus \{\infty_1, \infty_2, \infty_3\}$ and $T(3)^* = T(3) \setminus \{\infty_1, \infty_2, \infty_3\}$. Now, we are ready to define a collection of triples T on S as follows:

$T = \{T(1) \cup T(2)^* \cup T(3)^*\} \cup \{(a, 1), (b, 2), (a \circ b, 3) | a, b \in Q\}$.

(S, T) is an $STS(15)$. Clearly, $| \; S \; | = 15$. Pairs with both elements from $\{\infty_1, \infty_2, \infty_3\}$ appear in a triple of $T(1)$, pairs with one element from $\{\infty_1, \infty_2, \infty_3\}$ and the other element from $(Q \times \{i\})$ and also pairs with both elements from $(Q \times \{i\})$ appear in a triple of $T(1)$, if $i = 1$; in a triple of $T(i)^*$ if $i = 2, 3$. And finally, pairs with one element from $(Q \times \{i\})$ and the other from $(Q \times \{j\})$, where $i \neq j$, $i, j = 1, 2, 3$ appear in a triple of the form $\{(a, 1), (b, 2), (a \circ b, 3)\}$. In our construction, we have $(| \; STS(7) \; |) + (| \; STS(7) \; | -1) + (| \; STS(7) \; | -1) + 4 \cdot 4 = 35$ triples, which is equal to the number of triples in an $STS(15)$. Hence (S, T) is an $STS(15)$.

We consider two $STS(15)$'s (S, T_1) and (S, T_2) constructed in this way. By Lemma 3.1.1, $J(7) = \{0, 1, 3, 7\}$, so we can find a pair of $STS(7)$'s $(S(1), T_1(1))$ and $(S(1), T_2(1))$ intersecting in $0, 1, 3$ or 7 triples. Renaming the labels if necessary so that $\{\infty_1, \infty_2, \infty_3\}$ is a triple, we can also find a pair of $STS(7)$'s $(S(2), T_1(2))$, $(S(2), T_2(2))$ and $(S(3), T_1(3))$, $(S(3), T_2(3))$ which intersect in $0, 1, 3$ or 7 triples. So, $x = | \; T_1(1) \cap T_2(1) \; | \in \{0, 1, 3, 7\}$, $y = | \; T_1(2)^* \cap T_2(2)^* \; | \in \{0, 2, 6\}$ and $z = | \; T_1(3)^* \cap T_2(3)^* \; | \in \{0, 2, 6\}$. And using the quasigroups of order 4 which we constructed before, we can have w other common triples in T_1 and T_2, where $w \in \{0, 8, 16\}$. In total, $| T_1 \cap T_2 |= x + y + z + w$, where $x \in \{0, 1, 3, 7\}$, $y, z \in \{0, 2, 6\}$, $w \in \{0, 8, 16\}$. This way, we can write the numbers in the set $\{0, 1, 2, ..., 35\} \setminus \{26, 30, 32, 33, 34\}$ as a sum of the form $x + y + z + w$. We remind that $I(15) = \{0, 1, 2, ..., 35\} \setminus \{30, 32, 33, 34\}$. So, only 26 is missing. We need to construct a pair of $STS(15)$'s intersecting in 26 triples to complete the proof.

Now, let $S = \{\infty\} \cup (X \times \{1, 2\})$ where $| X |= 7$ and define collection of triples T_1, T_2 as the union of the following:

(1) $\{\infty, (x, 1), (x, 2)\} \in T_1$ and T_2 for all $x \in X$.

(2) Put the same $STS(7)$ on $X \times \{2\}$ in each of T_1 and T_2.

(3) Let $(X, \circ_1)$ and $(X, \circ_2)$ be a pair of idempotent and commutative quasigroups on X which intersect in exactly 12 cells above the main diagonal. For each $x \neq y \in X$, we put $\{(x, 1), (y, 1), (x \circ_1 y, 2)\}$ in T_1 and $\{(x, 1), (y, 1), (x \circ_2 y, 2)\}$ in T_2.

Clearly, in each system, we have 7 triples of type (1), again 7 triples of type (2) and 21 triples of type (3). So, altogether there are 35 triples. Note that pairs with one element being ∞ appear in type (1) triples. Pairs with both elements from the second level occur in type (2) triples. Pairs with both elements from the first level occur in type (3) triples. The pair $\{(a, 1), (b, 2)\}$ is in a triple of type (1) if $a = b$, in a triple of type (3) otherwise. Remark that the quasigroup used to build the triples of the type (3) is idempotent. Hence (S, T_1) and (S, T_2) are $STS(15)$'s having 7 triples of the type (1), 7 triples of the type (2) and 12 triples of the type (3) in common. So, we have $7 + 7 + 12 = 26 \in J(15)$. □

Lemma 3.4.3 $J(19) = I(19)$.

Proof: Again, we prepare the necessary ingredients first. Below is a commutative quasigroup $(Q, \circ_1)$ of order 6 with holes of size 2.

$\circ_1$						
	1	2	5	6	3	4
	2	1	6	5	4	3
	5	6	3	4	1	2
	6	5	4	3	2	1
	3	4	1	2	5	6
	4	3	2	1	6	5

Changing i (where $i = 0, 1, 2$ or 3) of the 2×2 subsquares other than those constituting the holes and the corresponding subsquares below the diagonal independently from the form

a	b
b	a

to

b	a
a	b

, we get a new commutative quasigroup of order 6 with holes of size 2 intersecting in $0, 4, 8$ or 12 cells with $(Q, \circ_1)$ above the diagonal of the holes.

Now let $S = \{\infty\} \cup (\{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3\})$. We define a collection of triples T on S using the quasigroup with holes construction for Steiner triple systems of section 1.2. to get an $STS(19)$.

Now we consider two $STS(19)$'s (S, T_1) and (S, T_2) constructed in the above way (not necessarily from the same $STS(7)$'s or from the same quasigroups on each level). Since $J(7) = \{0, 1, 3, 7\}$, the $STS(7)$'s we define on each hole $\{2i-1, 2i\}$, $i = 1, 2, 3$ in T_1 and T_2 can intersect in $0, 1, 3$ or 7 triples. Using the pairs of commutative quasigroups of order 6 with holes of size 2 intersecting in $0, 4, 8, 12$ triples, the collection of triples of type $\{(a, 1), (b, 1), (a \circ_1 b, 2)\}$ in T_1 and T_2 intersect in $0, 4, 8$ or 12 triples. The same applies to the collection of triples of type $\{(a, 2), (b, 2), (a \circ_1 b, 3)\}$ and $\{(a, 3), (b, 3), (a \circ_1 b, 1)\}$ as well. So, using this construction, we can intersect (S, T_1) and (S, T_2) in $x_1 + x_2 + x_3 + y_1 + y_2 + y_3$ triples where $x_1, x_2, x_3 \in \{0, 1, 3, 7\}$ and $y_1, y_2, y_3 \in \{0, 4, 8, 12\}$. We immediately observe that all the elements of the set $\{0, 1, 2, ..., 57\} \setminus \{48, 52, 54, 55, 56\}$ can be written as a sum $x_1+x_2+x_3+y_1+y_2+y_3$. We note that $I(19) = \{0, 1, 2, ..., 57\} \setminus \{52, 54, 55, 56\}$, so constructing a pair of $STS(19)$'s intersecting in 48 triples will complete the proof. Taking $n = 9$, $k = 3$ and $s = 9$ in the $2n+1$ Construction we get $48 \in J(19)$ by lemma 3.3.2. $\square$

Lemma 3.4.4 $J(21) = I(21)$.

Proof: The proof is similar to the proof of the Lemma 3.4.2. We take a quasigroup of order 6, say $(Q, \circ)$. We construct new quasigroups of order 6 intersecting in $0, 4, 8, 12, 16, 20, 24, 28, 32, 36$ cells with $(Q, \circ)$ by changing respectively $9, 8, 7, 6, 5, 4, 3, 2, 1, 0$ of the nine 2×2 subsquares of $(Q, \circ)$ from the form $\begin{array}{|c|c|}\hline \mathrm{a} & \mathrm{b} \\ \hline \mathrm{b} & \mathrm{a} \\ \hline \end{array}$ to $\begin{array}{|c|c|}\hline \mathrm{b} & \mathrm{a} \\ \hline \mathrm{a} & \mathrm{b} \\ \hline \end{array}$. Now let $(Q, \circ)$ be a quasigroup of order 6 and let $S = \{\infty_1, \infty_2, \infty_3\} \cup (Q \times \{1, 2, 3\})$. Further, we let $S(i) = \{\infty_1, \infty_2, \infty_3\} \cup (Q \times \{i\})$, $i = 1, 2, 3$ and $(S(i), T(i))$ be an $STS(9)$ where $\{\infty_1, \infty_2, \infty_3\}$ is a triple in both of $T(2)$ and $T(3)$. We set $T(2)^* = T(2) \setminus \{\infty_1, \infty_2, \infty_3\}$ and $T(3)^* = T(3) \setminus \{\infty_1, \infty_2, \infty_3\}$. Now, we are ready to define a collection of triples T on S as follows:

$T = \{T(1) \cup T(2)^* \cup T(3)^*\} \cup \{(a, 1), (b, 2), (a \circ b, 3)\}$ where $a, b \in Q$.

Pairs with both elements from $\{\infty_1, \infty_2, \infty_3\}$ appear in a triple of $T(1)$, pairs with one element from $\{\infty_1, \infty_2, \infty_3\}$ and the other element from $(Q \times \{i\})$ and pairs with both elements from $(Q \times \{i\})$ appear in a triple of $T(1)$, if $i = 1$, in a triple of $T(i)^*$ if $i = 2, 3$. And finally, the pairs with one element from $(Q \times \{i\})$ and the other from $(Q \times \{j\})$, where $i \neq j$, $i, j = 1, 2, 3$ appear in a triple of the form $\{(a, 1), (b, 2), (a \circ b, 3)\}$. In our construction, we have $\mid STS(9) \mid + (\mid STS(9) \mid - 1) + (\mid STS(9) \mid - 1) + 6 \cdot 6 = 70$ triples, which is equal to the number of triples in an $STS(21)$. Hence (S, T) is an $STS(21)$.

We consider two $STS(21)$'s (S, T_1) and (S, T_2) constructed in this way. Since $J(9) = \{0, 1, 2, 3, 4, 6, 12\}$, we can find a pair of $STS(9)$'s $(S(1), T_1(1))$ and $(S(1), T_2(1))$ intersecting in $0, 1, 2, 3, 4, 6, 12$ triples. So, we have $x = \mid T_1(1) \cap T_2(1) \mid \in \{0, 1, 2, 3, 4, 6, 12\}$, $y = \mid T_1(2)^* \cap T_2(2)^* \mid \in \{0, 1, 2, 3, 5, 11\}$ and $z = \mid T_1(3)^* \cap T_2(3)^* \mid \in \{0, 1, 2, 3, 5, 11\}$. Using the quasigroups of order 6 which we constructed before, we can have w other common triples in T_1 and T_2, where $w \in \{0, 4, 8, 12, 16, 20, 24, 28, 32, 36\}$. In total, $\mid T_1 \cap T_2 \mid = x + y + z + w$ and we can write the numbers in the set $\{0, 1, 2, ..., 70\} \setminus \{63, 65, 67, 68, 69\}$ as a sum of the form $x + y + z + w$. We remind that $I(21) = \{0, 1, 2, ..., 70\} \setminus \{65, 67, 68, 69\}$. So, only 63 and 66 are missing. Now, using the $2n + 7$ Construction, we cover these cases. Taking $k = 0$, $s = 7$ and $k = 3$, $s = 7$ in the $2n + 7$ Construction we get $63 \in J(21)$ and $66 \in J(21)$, respectively by lemma 3.3.5. □

Lemma 3.4.5 $J(25) = I(25)$.

Proof: Below is a pair of commutative quasigroups of order 8 with holes of size 2, intersecting in 6 cells above the main diagonal.

$\circ_1$								
	1	2	6	7	8	3	4	5
	2	1	8	5	4	7	6	3
	6	8	3	4	**7**	1	**5**	2
	7	5	4	3	2	**8**	1	**6**
	8	4	7	2	5	6	**3**	1
	3	7	1	8	6	5	2	**4**
	4	6	5	1	3	2	7	8
	5	3	2	6	1	4	8	7

and

$\circ_2$								
	1	2	8	5	4	7	6	3
	2	1	6	7	8	3	4	5
	8	6	3	4	**7**	2	**5**	1
	5	7	4	3	1	**8**	2	**6**
	4	8	7	1	5	6	**3**	2
	7	3	2	8	6	5	1	**4**
	6	4	5	2	3	1	7	8
	3	5	1	6	2	4	8	7

Moreover, the following quasigroup does not intersect in any cell above the main diagonal with the first quasigroup above.

$\circ_3$								
	1	2	8	6	7	4	5	3
	2	1	5	7	3	8	4	6
	8	5	3	4	2	7	6	1
	6	7	4	3	8	1	2	5
	7	3	2	8	5	6	1	4
	4	8	7	1	6	5	3	2
	5	4	6	2	1	3	7	8
	3	6	1	5	4	2	8	7

Taking the same quasigroup of order 8 with holes of size 2, we get a pair of quasigroups of order 8 with holes of size 2, intersecting in 24 cells.

Let $S = \{\infty\} \cup (\{1,2,3,4,5,6,7,8\} \times \{1,2,3\})$. Define a collection of triples T on S using the quasigroup with holes construction for Steiner triple systems of Section 1.2. to get an $STS(25)$.

Now consider two $STS(25)$'s (S,T_1) and (S,T_2) constructed in the above way (not necessarily from the same $STS(7)$'s or from the same quasigroups on each level). Since $J(7) = \{0,1,3,7\}$, the $STS(7)$'s we define on each hole $\{2i-1, 2i\}$, $i = 1,2,3,4$ in T_1 and T_2 can intersect in $0,1,3$ or 7 triples. Using the pairs of commutative quasigroups of order 8 with holes of size 2 intersecting in $0,6,24$ triples, the collection of triples of type $\{(a,1),(b,1),(a \circ_1 b, 2)\}$ in T_1 and T_2 intersect in $0,6$ or 24 triples. The same applies also to the collection of triples of type $\{(a,2),(b,2),(a \circ_2 b,3)\}$ and $\{(a,3),(b,3),(a \circ_3 b,1)\}$. So, using this construction only, we can intersect (S,T_1) and (S,T_2) in $x_1 + x_2 + x_3 + x_4 + y_1 + y_2 + y_3$ triples where $x_1, x_2, x_3, x_4 \in \{0,1,3,7\}$ and $y_1, y_2, y_3 \in \{0,6,24\}$. We immediately observe that all the elements of the set $\{0,1,2,...,100\} \setminus \{91,95,97,98,99\}$ can be written as a sum $x_1+x_2+x_3+x_4+y_1+y_2+y_3$. Note that $I(25) = \{0,1,2,...,100\} \setminus \{91,95,97,98,99\}$, so only 91 is missing. We cover that case using the $2n+7$ Construction. Take $k = 3$, $s = 9$ and $z = 9+7$ in the $2n+7$ Construction to get $91 \in J(25)$. □

3.5 The Main Result

After proving the validity of some necessary constructions and small cases, it is time to prove the main result.

Theorem 3.5.1 *$J(n) = I(n)$ for all $n \equiv 1$ or $3 \pmod 6$ except for $n = 9$, $J(9) = \{0,1,2,3,4,6,12\}$.*

Proof: We considered all the admissible cases up to $n = 25$. The next admissible case is $n = 27$. Assume that $n \geq 27$ and for all m such that $13 \leq m < n$, $J(m) = I(m)$. Since $n \equiv 1$ or $3 \pmod 6$, it is true that $n \equiv 1,3,7$ or $9 \pmod{12}$. We examine this in two cases $n \equiv 3$ or $7 \pmod{12}$ and $n \equiv 1$ or $9 \pmod{12}$. If $n \equiv 3$ or $7 \pmod{12}$, then $(n-1)/2 \equiv 1$ or $3 \pmod 6$ (i.e. there exists an $STS((n-1)/2))$ and $(n-1)/2 \geq 13$. Then, by our assumption on m, $J((n-1)/2) = I((n-1)/2)$. By Lemma 3.3.3, we have $J(n) = I(n)$. If $n \equiv 1$ or $9 \pmod{12}$, then $(n-7)/2 \equiv 1$ or $3 \pmod 6$ (i.e. there exists $STS((n-7)/2))$ and $(n-7)/2 \geq 13$ (remark that the conditions $n \equiv 1$ or 9 $\pmod{12}$ and $n \geq 27$ together imply that $n \geq 33$). Then, again by our assumption on m, $J((n-7)/2) = I((n-7)/2)$. By Lemma 3.3.6, we have $J(n) = I(n)$. So, in both cases $J(n) = I(n)$. □

Chapter 4

THE FLOWER INTERSECTION PROBLEM OF STEINER TRIPLE SYSTEMS

The flower intersection problem of Steiner triple systems is a special case of the basic intersection problem, which we investigated in the first part of this work. Given a Steiner triple system $(S, \mathcal{T})$, the *flower* at an element x of S is the set of all triples containing the element x. This time, we take two $STS(n)$ with a common flower and we are interested in the problem of finding the set of possible number of triples in common (other than the ones in the common flower). The set of numbers k, for which there exists a pair of $STS(n)$'s intersecting in k triples other than the triples in the common flower will be denoted by $J_f(n)$. This problem is completely solved by D.G. Hoffman and C.C. Lindner [7].

Clearly, a flower in an $STS(n)$ consists of $(n-1)/2$ triples. So, the biggest possible number in $J_f(n)$ is $n(n-1)/6 - (n-1)/2$. We define $I_f(n)$ to be the set $\{0, 1, ..., t_n - 6 - (n-1)/2, t_n - 4 - (n-1)/2, t_n - (n-1)/2\}$, where $t_n = n(n-1)/6$. From Lemma 3.1.2, we conclude that $J_f(n) \subseteq I_f(n)$.

4.1 Small Cases

We start determining $J_f(n)$ when $n \leq 27$. It is trivial that $J_f(1) = I_f(1) = \{0\}$ and $J_f(3) = I_f(3) = \{0\}$. We move on to the next smallest case.

Lemma 4.1.1 $J_f(7) = I_f(7) = \{0, 4\}$.

Proof: From Lemma 3.1.1, we know that $J(7) = I(7) = \{0, 1, 3, 7\}$. So, $J_f(7) \subseteq \{0, 4\}$. Taking the same $STS(7)$ twice, we get $4 \in J_f(7)$. And below are two $STS(7)$'s, $(S, \mathcal{T}_1)$ and $(S, \mathcal{T}_2)$ with a common flower at 1, intersecting in no other triple:

$\mathcal{T}_1 = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,4,7\},\{3,5,6\}\}$.
$\mathcal{T}_2 = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,7\},\{2,5,6\},\{3,4,6\},\{3,5,7\}\}$. □

Lemma 4.1.2 $J_f(9) = \{0,2,8\}$, *where* $I_f(9) = \{0,1,2,4,8\}$.

Proof: Lemmas 3.1.2 and 3.1.3 together imply that $J_f(9) \subseteq \{0,2,8\}$. Taking the same $STS(9)$ twice, we get $8 \in J_f(9)$. Let

$\mathcal{T} = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{1,8,9\},\{2,4,7\},\{2,6,9\},\{2,8,5\},\{3,4,9\},$
$\{3,6,5\},\{3,8,7\},\{\mathbf{4},\mathbf{6},\mathbf{8}\},\{\mathbf{5},\mathbf{7},\mathbf{9}\}\}$,
$\mathcal{T}_0 = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{1,8,9\},\{2,4,9\},\{2,5,7\},\{2,6,8\},\{3,4,6\},$
$\{3,5,8\},\{3,7,9\},\{4,7,8\},\{5,6,9\}\}$,

$\mathcal{T}_2 = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{1,8,9\},\{2,4,9\},\{2,6,5\},\{2,8,7\},\{3,4,7\},$
$\{3,6,9\},\{3,8,5\},\{\mathbf{4},\mathbf{6},\mathbf{8}\},\{\mathbf{5},\mathbf{7},\mathbf{9}\}\}$, where $\mathcal{T}$ and $\mathcal{T}_i$ with a common flower at 1 intersect in i triples for $i = 0,2$. □

Lemma 4.1.3 *Let* $n \geq 13$ *be admissible, i.e.* $n \equiv 1,3 \pmod 6$ *and* $r = (n-1)/2$. *If* $k \in I_f(n)$ *with* $k \geq 2r(r-3)/3$, *then* $k \in J_f(n)$.

Proof: Let (S,T_1) and (S,T_2) be two $STS(n)$'s with $k+r$ triples in common, where the existence of such a pair is given by Theorem 3.5.1. The number of triples not in common is then $(2r+1)2r/6-(k+r)$. Assume that the systems do not have a flower in common. If a point $x \in S$ is contained in a triple of $T_1 \setminus T_2$, then it must be contained in at least one other such triple. Arguing in the same way for all n points, one sees that in order not to have a common triple, (S,T_1) and (S,T_2) must have at least $2(2r+1)/3$ triples not in common. So, if $2(2r+1)/3 > (2r+1)2r/6-(k+r)$, equivalently if $k > (2r^2-6r-2)/3$, then (S,T_1) and (S,T_2) have a common flower. From $n \equiv 1,3$ (mod 6) and $2r+1 = n$, it follows that $r \equiv 0,1$ (mod 3). For $r \equiv 0$ (mod 3), k being an integer and $k > (2r^2-6r-2)/3$ means $k > \left\lceil \frac{2r^2-6r-2}{3} \right\rceil = (2r^2-6r)/3$. For $r \equiv 1$ (mod 3), k being an integer and $k > (2r^2-6r-2)/3$ means $k > \left\lceil \frac{2r^2-6r-2}{3} \right\rceil = (2r^2-6r-2)/3$. Moreover, $k > \left\lceil \frac{2r^2-6r-2}{3} \right\rceil$ implies $k \geq 1+\left\lceil \frac{2r^2-6r-2}{3} \right\rceil = (2r^2-6r+1)/3 > (2r^2-6r)/3$. Combining the results for $r \equiv 0$ (mod 3) and $r \equiv 1$ (mod 3), we can state that $k \geq 2r(r-3)/3$, then (S,T_1) and (S,T_2) have a common flower. □

Lemma 4.1.4 $J_f(13) = I_f(13)$.

Proof: By the preceding lemma, it is enough to show that $k \in J_f(13)$ for $0 \le k \le 11$. Below we explicitly give pairs of $STS(13)$'s, $(S, \mathcal{T}_1)$ and $(S, \mathcal{T}_2)$ on the set of symbols $\{x\} \cup \{1, 2, ..., 12\}$ with a common flower at x, intersecting in exactly $k = 0, 1, ..., 11$ other triples, respectively.

k = 0: $\mathcal{T}_1 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{6,7,10\},$
$\{2,10,11\},\{5,7,8\},\{3,5,6\},\{6,9,11\},\{8,11,12\},\{3,4,8\},\{1,9,12\},\{4,7,12\},$
$\{1,6,8\},\{3,7,11\},\{8,9,10\},\{2,3,12\},\{4,5,9\},\{1,4,11\},\{1,2,5\},\{1,3,10\},$
$\{2,7,9\},\{5,10,12\},\{2,4,6\}\}$.
$\mathcal{T}_2 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{3,2,4\},$
$\{8,4,6\},\{7,9,6\},\{3,11,8\},\{7,10,8\},\{12,5,8\},\{5,1,4\},\{2,10,1\},\{3,10,12\},$
$\{12,9,4\},\{11,7,4\},\{3,7,5\},\{9,1,8\},\{10,5,6\},\{11,12,1\},\{7,2,12\},\{11,2,6\},$
$\{2,9,5\},\{3,1,6\},\{11,10,9\}\}$.

k = 1: $\mathcal{T}_1 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{1,11,12\},$
$\{4,8,11\},\{8,9,12\},\{5,6,9\},\{1,4,6\},\{4,7,12\},\{5,10,12\},\{1,2,5\},\{3,7,10\},$
$\{2,10,11\},\{3,4,5\},\{5,7,8\},\{1,9,10\},\{2,6,7\},\{2,3,12\},\{1,3,8\},\{3,6,11\},$
$\{7,9,11\},\{6,8,10\},\{2,4,9\}\}$.
$\mathcal{T}_2 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{1,11,12\},$
$\{8,4,6\},\{7,9,6\},\{3,11,8\},\{7,10,8\},\{12,5,8\},\{5,1,4\},\{2,10,1\},\{3,10,12\},$
$\{12,9,4\},\{11,7,4\},\{3,7,5\},\{9,1,8\},\{10,5,6\},\{3,2,4\},\{7,2,12\},\{11,2,6\},$
$\{2,9,5\},\{3,1,6\},\{11,10,9\}\}$.

k = 2: $\mathcal{T}_1 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{2,3,4\},$
$\{7,8,10\},\{4,5,12\},\{7,9,11\},\{3,6,11\},\{1,5,9\},\{4,8,9\},\{6,9,10\},\{1,2,6\},$
$\{3,7,12\},\{1,10,12\},\{1,4,11\},\{4,6,7\},\{2,9,12\},\{5,6,8\},\{2,10,11\},\{1,3,8\},$
$\{2,5,7\},\{8,11,12\},\{3,5,10\}\}$.
$\mathcal{T}_2 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{2,3,4\},$
$\{7,8,10\},\{7,9,6\},\{3,11,8\},\{8,4,6\},\{12,5,8\},\{5,1,4\},\{2,10,1\},\{3,10,12\},$
$\{12,9,4\},\{11,7,4\},\{3,7,5\},\{9,1,8\},\{10,5,6\},\{11,12,1\},\{7,2,12\},\{11,2,6\},$

$\{2,9,5\},\{3,1,6\},\{11,10,9\}\}$.

k = 3: $\mathcal{T}_1 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{3,8,11\},$
$\{1,3,6\},\{1,11,12\},\{5,6,9\},\{1,4,8\},\{4,7,12\},\{5,10,12\},\{1,2,5\},\{3,7,10\},$
$\{2,10,11\},\{3,4,5\},\{8,9,12\},\{5,7,8\},\{1,9,10\},\{2,6,7\},\{2,3,12\},\{4,6,11\},$
$\{7,9,11\},\{6,8,10\},\{2,4,9\}\}$.
$\mathcal{T}_2 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{3,8,11\},$
$\{1,3,6\},\{1,11,12\},\{3,2,4\},\{7,10,8\},\{12,5,8\},\{5,1,4\},\{2,10,1\},\{3,10,12\},$
$\{12,9,4\},\{11,7,4\},\{3,7,5\},\{9,1,8\},\{10,5,6\},\{7,9,6\},\{7,2,12\},\{11,2,6\},$
$\{2,9,5\},\{8,4,6\},\{11,10,9\}\}$.

k = 4: $\mathcal{T}_1 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{4,8,11\},$
$\{2,10,11\},\{1,3,8\},\{7,9,11\},\{1,4,6\},\{4,7,12\},\{5,10,12\},\{1,2,5\},\{3,7,10\},$
$\{3,6,11\},\{3,4,5\},\{1,11,12\},\{5,7,8\},\{1,9,10\},\{2,6,7\},\{2,3,12\},\{8,9,12\},$
$\{5,6,9\},\{6,8,10\},\{2,4,9\}\}$.
$\mathcal{T}_2 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{4,8,11\},$
$\{2,10,11\},\{1,3,8\},\{7,9,11\},\{1,4,2\},\{5,1,9\},\{12,7,2\},\{4,7,3\},\{6,5,4\},$
$\{12,11,3\},\{6,3,10\},\{6,9,2\},\{9,4,12\},\{5,7,10\},\{6,8,7\},\{5,3,2\},\{6,1,11\},$
$\{5,8,12\},\{1,12,10\},\{9,8,10\}\}$.

k = 5: $\mathcal{T}_1 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{1,5,9\},$
$\{7,9,11\},\{1,10,12\},\{2,10,11\},\{1,3,8\},\{2,3,4\},\{4,8,9\},\{6,9,10\},\{1,2,6\},$
$\{7,8,10\},\{4,5,12\},\{1,4,11\},\{4,6,7\},\{2,9,12\},\{5,6,8\},\{3,7,12\},\{3,6,11\},$
$\{2,5,7\},\{8,11,12\},\{3,5,10\}\}$.
$\mathcal{T}_2 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{1,5,9\},$
$\{7,9,11\},\{1,10,12\},\{2,10,11\},\{1,3,8\},\{6,7,8\},\{8,9,10\},\{2,7,12\},\{1,2,4\},$
$\{3,4,7\},\{4,5,6\},\{3,11,12\},\{3,6,10\},\{2,6,9\},\{4,9,12\},\{5,7,10\},\{4,8,11\},$
$\{2,3,5\},\{1,6,11\},\{5,8,12\}\}$.

k = 6: $\mathcal{T}_1 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{1,5,9\},$
$\{3,7,12\},\{3,6,11\},\{2,3,4\},\{4,8,9\},\{2,9,12\},\{4,5,12\},\{7,9,11\},\{6,9,10\},$
$\{1,2,6\},\{7,8,10\},\{1,10,12\},\{1,4,11\},\{4,6,7\},\{5,6,8\},\{2,10,11\},\{1,3,8\},$

$\{2,5,7\},\{8,11,12\},\{3,5,10\}\}$.
$\mathcal{T}_2 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{1,5,9\},$
$\{3,7,12\},\{3,6,11\},\{2,3,4\},\{4,8,9\},\{2,9,12\},\{3,5,8\},\{1,8,12\},\{1,3,10\},$
$\{9,10,11\},\{6,7,9\},\{1,4,6\},\{1,2,11\},\{7,8,11\},\{4,5,7\},\{6,8,10\},\{4,11,12\},$
$\{2,7,10\},\{2,5,6\},\{5,10,12\}\}$.

k = 7: $\mathcal{T}_1 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{1,4,11\},$
$\{3,5,10\},\{4,5,9\},\{1,5,6\},\{2,3,11\},\{5,8,12\},\{2,5,7\},\{10,11,12\},\{3,7,12\},$
$\{1,2,10\},\{6,8,10\},\{8,9,11\},\{6,7,11\},\{1,3,8\},\{7,9,10\},\{2,6,9\},\{3,4,6\},$
$\{2,4,12\},\{4,7,8\},\{1,9,12\}\}$.
$\mathcal{T}_2 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{1,4,11\},$
$\{3,5,10\},\{4,5,9\},\{1,5,6\},\{2,3,11\},\{5,8,12\},\{2,5,7\},\{2,4,6\},\{6,8,11\},$
$\{7,8,9\},\{1,2,9\},\{7,10,11\},\{1,3,12\},\{9,11,12\},\{3,6,7\},\{1,8,10\},\{2,10,12\},$
$\{4,7,12\},\{3,4,8\},\{6,9,10\}\}$.

k = 8: $\mathcal{T}_1 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{7,8,10\},$
$\{4,6,8\},\{1,3,6\},\{4,7,11\},\{3,10,12\},\{6,7,9\},\{3,8,11\},\{1,2,10\},\{1,9,11\},$
$\{2,4,9\},\{1,4,12\},\{5,7,12\},\{3,4,5\},\{5,9,10\},\{6,10,11\},\{8,9,12\},\{1,5,8\},$
$\{2,5,6\},\{2,11,12\},\{2,3,7\}\}$.
$\mathcal{T}_2 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{7,8,10\},$
$\{4,6,8\},\{1,3,6\},\{4,7,11\},\{3,10,12\},\{6,7,9\},\{3,8,11\},\{1,2,10\},\{2,3,4\},$
$\{5,8,12\},\{1,4,5\},\{4,9,12\},\{3,5,7\},\{1,8,9\},\{5,6,10\},\{1,11,12\},\{2,7,12\},$
$\{2,6,11\},\{2,5,9\},\{9,10,11\}\}$.

k = 9: $\mathcal{T}_1 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{1,5,9\},$
$\{3,7,12\},\{7,9,11\},\{2,3,4\},\{1,2,6\},\{7,8,10\},\{1,4,11\},\{4,6,7\},\{2,5,7\},$
$\{4,5,12\},\{3,6,11\},\{4,8,9\},\{6,9,10\},\{1,10,12\},\{2,9,12\},\{5,6,8\},\{2,10,11\},$
$\{1,3,8\},\{8,11,12\},\{3,5,10\}\}$.
$\mathcal{T}_2 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{1,5,9\},$
$\{3,7,12\},\{7,9,11\},\{2,3,4\},\{1,2,6\},\{7,8,10\},\{1,4,11\},\{4,6,7\},\{2,5,7\},$
$\{3,5,6\},\{4,9,12\},\{1,8,12\},\{1,3,10\},\{6,10,11\},\{4,5,8\},\{6,8,9\},\{2,9,10\},$
$\{5,10,12\},\{3,8,11\},\{2,11,12\}\}$.

$\mathbf{k = 10}$: $\mathcal{T}_1 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{3,7,12\},$
$\{7,9,11\},\{2,3,4\},\{4,8,9\},\{1,2,6\},\{7,8,10\},\{4,6,7\},\{2,9,12\},\{2,10,11\},$
$\{2,5,7\},\{1,5,9\},\{4,5,12\},\{3,6,11\},\{6,9,10\},\{1,10,12\},\{1,4,11\},\{5,6,8\},$
$\{1,3,8\},\{8,11,12\},\{3,5,10\}\}$.
$\mathcal{T}_2 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{3,7,12\},$
$\{7,9,11\},\{2,3,4\},\{4,8,9\},\{1,2,6\},\{7,8,10\},\{4,6,7\},\{2,9,12\},\{2,10,11\},$
$\{2,5,7\},\{1,3,11\},\{3,6,10\},\{4,11,12\},\{1,8,12\},\{1,4,5\},\{6,8,11\},\{5,6,9\},$
$\{5,10,12\},\{3,5,8\},\{1,9,10\}\}$.

$\mathbf{k = 11}$: $\mathcal{T}_1 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{3,6,11\},$
$\{5,6,9\},\{1,4,6\},\{4,7,12\},\{3,7,10\},\{3,4,5\},\{5,7,8\},\{1,9,10\},\{2,6,7\},$
$\{7,9,11\},\{6,8,10\},\{4,8,11\},\{8,9,12\},\{5,10,12\},\{1,2,5\},\{2,10,11\},\{1,11,12\},$
$\{2,3,12\},\{1,3,8\},\{2,4,9\}\}$.
$\mathcal{T}_2 = \{\{x,1,7\},\{x,2,8\},\{x,3,9\},\{x,4,10\},\{x,5,11\},\{x,6,12\},\{3,6,11\},$
$\{5,6,9\},\{1,4,6\},\{4,7,12\},\{3,7,10\},\{3,4,5\},\{5,7,8\},\{1,9,10\},\{2,6,7\},$
$\{7,9,11\},\{6,8,10\},\{2,9,12\},\{2,4,11\},\{2,5,10\},\{3,8,12\},\{10,11,12\},\{1,8,11\},$
$\{1,5,12\},\{4,8,9\},\{1,2,3\}\}$. □

Before continuing with the case $n = 15$, first we need a new definition. We call a commutative latin square of order 8 on the symbols $0, 1, \ldots, 7$ a *special latin square*, whenever the 2×2 subsquares on the diagonal are of the form $\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline\end{array}$ so that the latin square looks as follows:

Lemma 4.1.5 *The set of integers k for which there exists a pair of special latin squares* (*of order* 8) *agreeing in exactly k of the* 24 *cells above the 2×2 diagonal boxes, contains* (*among others*) *all non-negative integers $k \leq 14$.*

Proof: Such an integer k can be written in the form $k = x+y+z$, where $x, z \in \{0, 4\}$, $0 \leq y \leq 9$ with $y \neq 5, 7$. For $0 \leq k \leq 4$, take $y = k$, $x = 0$, $z = 0$; for $k = 5$, $y = 1$, $x = 4$, $z = 0$; for $k = 6$, take $y = 6$, $x = 0$, $z = 0$; for $k = 7$, take $y = 3$, $x = 4$, $z = 0$; for $8 \leq k \leq 12$, take $y = k - 8$, $x = 4$, $z = 4$; for $k = 13$, take $y = 9$, $x = 4$, $z = 0$

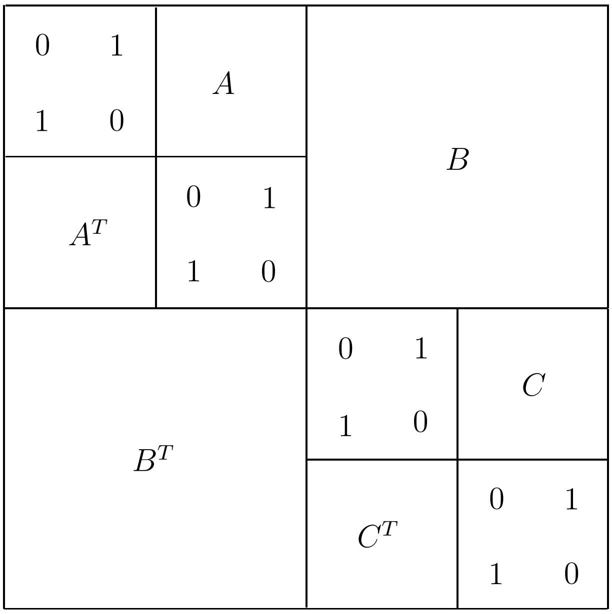


Figure 4.1: A special latin square

and finally for $k = 14$, take $y = 6$, $x = 4$, $z = 4$. Furthermore, let $B_1 = (Q, \circ)$ and $B_2 = (Q, \circ_i)$ as constructed below, where $Q = \{4, 5, 6, 7\}$ and $i = 0, 1, 2, 3, 4, 6, 8, 9$. We remark that for each $i \in \{0, 1, 2, 3, 4, 6, 8, 9\}$, $(Q, \circ)$ and $(Q, \circ_i)$ agree in exactly i cells.

$\circ$				
	4	5	6	7
	5	6	7	4
	6	7	4	5
	7	4	5	6

$\circ_0$				
	7	6	5	4
	6	5	4	7
	5	4	7	6
	4	7	6	5

$\circ_1$				
	4	7	5	6
	7	4	6	5
	5	6	7	4
	6	5	4	7

$\circ_2$				
	4	6	7	5
	7	4	5	6
	5	7	6	4
	6	5	4	7

$\circ_3$				
	4	5	7	6
	5	4	6	7
	7	6	5	4
	6	7	4	5

$\circ_4$				
	4	5	6	7
	7	4	5	6
	5	6	7	4
	6	7	4	5

$\circ_6$				
	4	5	6	7
	5	4	7	6
	7	6	5	4
	6	7	4	5

$\circ_8$				
	4	5	6	7
	5	6	7	4
	7	4	5	6
	6	7	4	5

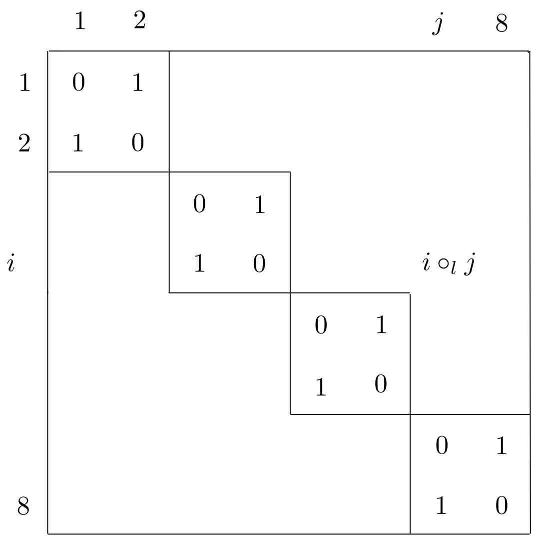
$\circ_9$				
	4	5	6	7
	5	7	4	6
	6	4	7	5
	7	6	5	4

Now form pairs of special latin squares agreeing in exactly $k \in \{0, 1, ..., 14\}$ cells above the 2×2 diagonal boxes as follows. For the first special latin square, let $A_1 = C_1 = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & 2 \\ \hline\end{array}$. For the second special latin square, if $x = 0$, let $A_2 = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 3 \\ \hline\end{array}$ and if $x = 4$, let $A_2 = A_1$. If $z = 0$, let $C_2 = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 3 \\ \hline\end{array}$ and if $z = 4$, let $C_2 = C_1$. The special latin square defined by A_1, B_1, C_1 agrees in exactly $k = x + y + z$ cells above the 2×2 diagonal boxes with the special latin square defined by A_2, B_2, C_2. □

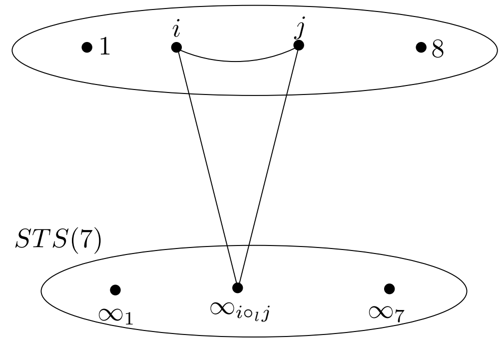
Lemma 4.1.6 $J_f(15) = I_f(15)$.

Proof: By Lemma 4.1.3, it is sufficient to show that the statement holds for $0 \leq k \leq 18$. We write all such numbers k as a sum of the form $k = x + y$ where $x \in \{0, 4\}$ and $0 \leq y \leq 14$. Lemma 4.1.5 guarantees us the existence of a pair of special latin squares $(L_1, \circ_1)$ and $(L_2, \circ_2)$ of order 8 which have exactly y cells in common above the diagonal blocks. Next, we construct two $STS(15)$'s (S, T_1) and (S, T_2) on the set $\{\infty_i : 1 \leq i \leq 7\} \cup \{1, 2, ..., 8\}$. We start by defining two $STS(7)$'s on $\{\infty_i : 1 \leq i \leq 7\}$ with collection of triples T_1^* and T_2^* such that T_1^* and T_2^* have a common flower at ∞_1 and x additional common triples. This can be done since $J_f(7) = I_f(7) = \{0, 4\}$ (see Lemma 4.1.1). In addition, for each pair of numbers i, j such that $1 \leq i < j \leq 8$, put the triple $(\infty_{i \circ_l j}, i, j)$ in T_l for $l = 1, 2$. Please see Figure 4.2.

(S, T_l) forms an $STS(15)$. Observe that each pair of points from the set $\{1, 2, ..., 8\}$ appears together exactly once in a triple of each of the systems (S, T_1) and (S, T_2) since on a latin square, $i \circ_l j$ $(l = 1, 2)$ always has a unique solution for each pair of numbers i and j. Also, each pair of points of the form $\{\infty_i, j\}$, where $(\{\infty_i : 1 \leq i \leq 7\})$



Put $\{\infty_{i \circ_l j}, i, j\}$ for $1 \leq i < j \leq 8$.
$(L_1, \circ_1)$ and $(L_2, \circ_2)$ have y common cells above the diagonal boxes.



(S, T_1^*) and (S, T_2^*) have a common flower at ∞_1 and x other common triples.

Figure 4.2: $J_f(15)$ Construction

and $j \in \{1, 2, ..., 8\}$ appears together exactly once in a triple of each of the (S, T_l)'s $(l = 1, 2)$, since there exists a unique solution x to the equation $i = x \circ_l j$. Finally, pairs of points from the set $\{\infty_i : 1 \leq i \leq 7\}$ appear together exactly once in a triple of the corresponding $STS(7)$.

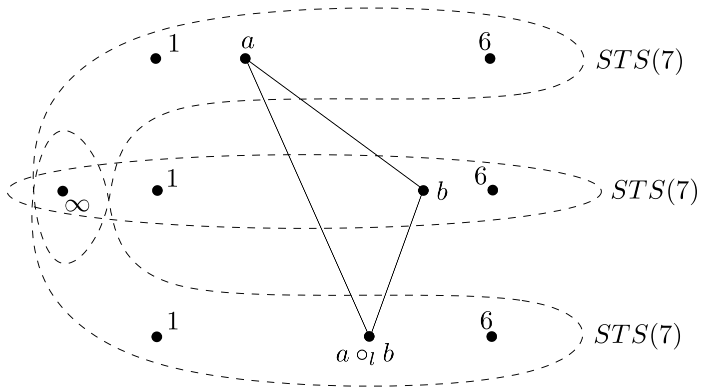
Note that by the definition of special latin squares, the positions of cells with the label 1 are fixed, so the triples $\{\infty_1, 1, 2\}$, $\{\infty_1, 3, 4\}$, $\{\infty_1, 5, 6\}$, $\{\infty_1, 7, 8\}$ are contained in both of the systems (S, T_1) and (S, T_2). These four triples and the three triples of the common flower of $STS(7)$'s at ∞_1 constitute a common flower at ∞_1 for (S, T_l)'s $(l = 1, 2)$. In total, apart from the triples of the common flower, (S, T_l)'s $(l = 1, 2)$ have $k = x + y$ triples in common, x of them coming from the $STS(7)$'s, and y of them coming from the special latin squares $(L_1, \circ_1)$ and $(L_2, \circ_2)$. □

Lemma 4.1.7 $J_f(19) = I_f(19)$.

Proof: For the proof, we will use a result by H.L. Fu [6], which tells that for $n \geq 5$ there exists a pair of latin squares on the same set of n symbols agreeing in exactly k

cells if and only if $k \in A(n^2)$, where $A(n^2) = \{0, 1, ..., n^2\} \backslash \{n^2-5, n^2-3, n^2-2, n^2-1\}$.

Each $k \in I_f(19) = \{0, 1, ..., 42, 44, 48\}$ can be written as a sum $k = x_1+x_2+x_3+y$, where $x_1, x_2, x_3 \in \{0, 4\}$ and $y \in A(36)$. For $0 \leq k \leq 30$, let $y = k$ and $x_i = 0$, for $i = 1, 2, 3$. For $31 \leq k \leq 48$ such that $k \notin \{43, 45, 46, 47\}$, let $y = k-12$ and $x_i = 4$. Next, construct two $STS(19)$'s (S, T_1) and (S, T_2) on the set $\infty \cup (\{1, 2, ..., 6\} \times \{1, 2, 3\})$. For each $i = 1, 2, 3$, we define a pair of $STS(7)$'s with collection of triples $T_1(i)$ and $T_2(i)$ on $\infty \cup (\{1, 2, ..., 6\} \times \{i\})$ such that $T_1(i)$ and $T_2(i)$ have a common flower (consisting of three triples) at ∞ and x_i additional common triples since $J_f(7) = I_f(7) = \{0, 4\}$. Let $(L_1, \circ_1)$ and $(L_2, \circ_2)$ be two latin squares of order 6 on $\{1, 2, ..., 6\}$ which agree in exactly y cells. In addition to the triples of the sets $T_l(1)$, $T_l(2)$, $T_l(3)$, the collection T_l $(l = 1, 2)$ contains also the triples $\{(a, 1), (b, 2), (a \circ_l b, 3)\}$ $(l = 1, 2)$ for each pair $1 \leq a, b \leq 6$. Please see Figure 4.3.



$(S, T_1(i))$ and $(S, T_2(i))$ have a common flower at ∞ and x_i common triples where $i = 1, 2, 3$.

Figure 4.3: $J_f(19)$ Construction

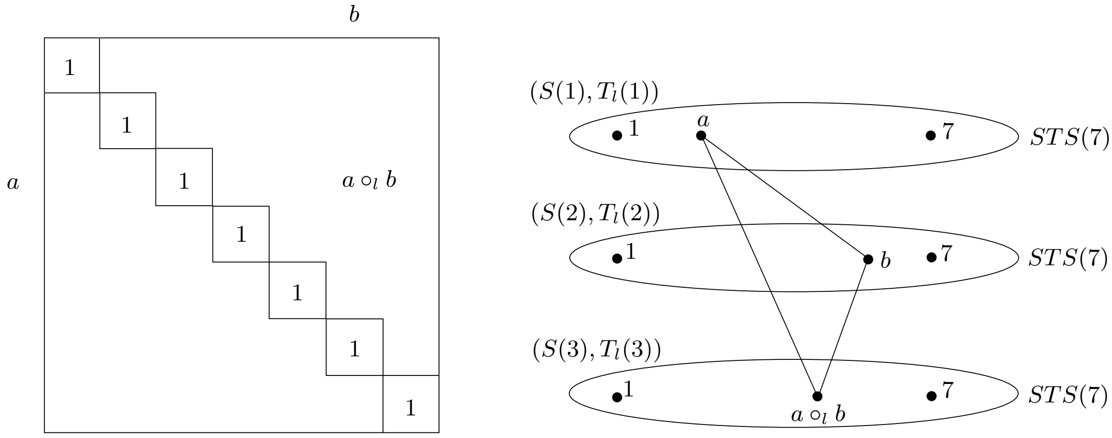
Both (S, T_1) and (S, T_2) are $STS(19)$'s. For each $l = 1, 2$, pairs containing the element ∞ appear together exactly once in the corresponding $STS(7)$, pairs of points with the same second coordinate i, where $i = 1, 2, 3$, appear together exactly once in one of the triples from the set $T_l(i)$ and pairs of points of different second coordinate appear together exactly once in a triple of the form $\{(a,1),(b,2),(a \circ_l b,3)\}$.

Note that we constructed the two $STS(19)$'s so that they contain the same triples involving ∞ in each of the $T_l(i)$. These triples constitute a common flower at ∞ for the $STS(19)$'s. In total, apart from the triples of the common flower, (S,T_1) and (S,T_2) have $k = x_1 + x_2 + x_3 + y$ triples in common, x_i $(i = 1,2,3)$ of them coming from the collections $T_l(i)$ $(l = 1,2)$ and y of them coming from the latin squares $(L_1, \circ_1)$ and $(L_2, \circ_2)$. □

Lemma 4.1.8 $J_f(21) = I_f(21)$.

Proof: For the proof of this lemma, we will use another result by H.L. Fu [6]. It assures us that for $n \geq 5$ there exists a pair of latin squares on the same set of n symbols with the same constant diagonal agreeing in exactly k cells off the main diagonal if and only if $k \in A(n^2 - n)$.

Each $k \in I_f(21) = \{0, 1, ..., 54, 56, 60\}$ can be written as a sum $k = x_1 + x_2 + y + z$, where $x_1, x_2 \in \{0,1,3,7\}$, $y \in \{0,4\}$ and $z \in A(42)$. For $0 \leq k \leq 18$, let $z = k$, $x_i = 0$ for $i = 1,2$ and $y = 0$. For $k \in \{19, 20, ..., 54, 56, 60\}$, let $z = k - 18$, $x_i = 7$ for $i = 1,2$ and $y = 4$. Construct two $STS(21)$'s (S,T_1) and (S,T_2) on the set $\{1,2,...,7\} \times \{1,2,3\}$. For each of $i = 1,2$, we define a pair of $STS(7)$'s $(S(i), T_1(i))$ and $(S(i), T_2(i))$ with collection of triples on $\{1,2,...,7\} \times \{i\}$ such that $T_1(i)$ and $T_2(i)$ have x_i triples in common, which is possible since $J(7) = I(7) = \{0,1,3,7\}$. Further, let $(S(3), T_1(3))$ and $(S(3), T_2(3))$ be a pair of $STS(7)$'s on $\{1,2,...,7\} \times \{3\}$ with a common flower at $(1,3)$ (consisting of three triples) and y additional common triples, which is possible since $J_f(7) = I_f(7) = \{0,4\}$. Also let $(L_1, \circ_1)$ and $(L_2, \circ_2)$ be two latin squares of order 7 on $\{1,2,...,7\}$, both of which have the constant main diagonal of 1's and agree in exactly z additional cells off the main diagonal. In addition to the triples of the sets $T_l(1)$, $T_l(2)$, $T_l(3)$, the collection T_l $(l = 1,2)$ contains also the triples $\{(a,1),(b,2),(a \circ_l b,3)\}$ for each pair of numbers $1 \leq a,b \leq 7$. Please see Figure 4.4.



Put $\{(a, 1), (b, 2), (a \circ_l b, 3)\}$ for $1 \leq a, b \leq 7$. $(L_1, \circ_1)$ and $(L_2, \circ_2)$ have z common cells off the main diagonal.

$(S(i), T_1(i))$ and $(S(i), T_2(i))$ have x_i triples in common, $(S(3), T_1(3))$ and $(S(3), T_2(3))$ have a common flower at $(1, 3)$ and y common triples.

Figure 4.4: $J_f(21)$ Construction

Both (S, T_1) and (S, T_2) constructed this way are $STS(21)$'s. For each $l = 1, 2$, pairs of points with the same second coordinate i, where $i = 1, 2, 3$, appear together exactly once in one of the triples from the set $T_l(i)$ and pairs of points of different second coordinate appear together exactly once in a triple of the form $\{(a, 1), (b, 2), (a \circ_l b, 3)\}$.

Note that by fixing the positions of 1's in $(L_1, \circ_1)$ and $(L_2, \circ_2)$, we forced both (S, T_1) and (S, T_2) to contain the triples $\{(1, 1), (1, 2), (1, 3)\}$, $\{(2, 1), (2, 2), (1, 3)\}$, $\{(3, 1), (3, 2), (1, 3)\}$, $\{(4, 1), (4, 2), (1, 3)\}$, $\{(5, 1), (5, 2), (1, 3)\}$, $\{(6, 1), (6, 2), (1, 3)\}$, $\{(7, 1), (7, 2), (1, 3)\}$. Our construction of $(S(3), T_1(3))$ and $(S(3), T_2(3))$ give the other three triples of the common flower at $(1, 3)$ of the $STS(21)$'s obtained by the $STS(7)$ defined on $\{1, 2, 3, 4, 5, 6, 7\} \times \{3\}$. Totally, apart from the triples of the common flower, (S, T_1) and (S, T_2) have $k = x_1 + x_2 + y + z$ triples in common, x_i $(i = 1, 2)$ of them coming from the collections $T_l(i)$ $(l = 1, 2)$, y of them from $T_l(3)$ $(l = 1, 2)$ and z of them coming from the latin squares $(L_1, \circ_1)$ and $(L_2, \circ_2)$. □

Lemma 4.1.9 $J_f(25) = I_f(25)$.

Proof: The proof of this case resembles very much that of the case where $n = 19$. Here, each $k \in I_f(25) = \{0, 1, ..., 82, 84, 88\}$ can be written as a sum $k = x_1 + x_2 + x_3 +$

y, where $x_1, x_2, x_3 \in \{0, 2, 8\}$ and $y \in A(64)$. For $0 \leq k \leq 24$, let $y = k$ and $x_i = 0$, for $i = 1, 2, 3$. For $25 \leq k \leq 88$ such that $k \notin \{83, 85, 86, 87\}$, let $y = k - 24$ and $x_i = 8$. Define two $STS(25)$'s (S, T_1) and (S, T_2) on the set $\infty \cup (\{1, 2, ..., 8\} \times \{1, 2, 3\})$. For each $i = 1, 2, 3$, we define a pair of $STS(9)$'s with collection of triples $T_1(i)$ and $T_2(i)$ on $\infty \cup (\{1, 2, ..., 8\} \times \{i\})$ such that $T_1(i)$ and $T_2(i)$ have a common flower (consisting of four triples) at ∞ and x_i additional common triples, which is possible since $J_f(9) = \{0, 2, 8\}$ (see Lemma 4.1.2). Let $(L_1, \circ_1)$ and $(L_2, \circ_2)$ be two latin squares of order 8 on $\{1, 2, ..., 8\}$ which agree in exactly y cells. In addition to the triples of the sets $T_l(1)$, $T_l(2)$, $T_l(3)$, the collection T_l $(l = 1, 2)$ contains also the triples $\{(a, 1), (b, 2), (a \circ_l b, 3)\}$ for each pair of numbers $1 \leq a, b \leq 8$.

(S, T_1) and (S, T_2) defined in this way are $STS(25)$'s. For each $l = 1, 2$, pairs containing the element ∞ and pairs of points with the same second coordinate i, where $i = 1, 2, 3$, appear together exactly once in one of the triples from the set $T_l(i)$ and pairs of points of different second coordinate appear together exactly once in a triple of the form $\{(a, 1), (b, 2), (a \circ_l b, 3)\}$.

The way we constructed the two $STS(25)$'s (S, T_1) and (S, T_2) guarantees that they contain the same triples involving ∞ in each of the $T_l(i)$. These triples constitute a common flower at ∞ for the $STS(25)$'s. In total, apart from the common flower, (S, T_1) and (S, T_2) have $k = x_1 + x_2 + x_3 + y$ triples in common, x_i $(i = 1, 2, 3)$ of them coming from the collections $T_l(i)$ $(l = 1, 2)$ and y of them coming from the latin squares $(L_1, \circ_1)$ and $(L_2, \circ_2)$. □

Lemma 4.1.10 $J_f(27) = I_f(27)$.

Proof: The proof of this case is very similar to the case where $n = 21$. Each $k \in I_f(21) = \{0, 1, ..., 98, 100, 104\}$ can be written as a sum $k = x_1 + x_2 + y + z$, where $x_1, x_2 \in \{0, 1, 2, 3, 4, 6, 12\}$, $y \in \{0, 2, 8\}$ and $z \in A(72)$. For $0 \leq k \leq 32$, let $z = k$, $x_i = 0$ for $i = 1, 2$ and $y = 0$. For $33 \leq k \leq 104$ such that $k \notin \{99, 101, 102, 103\}$, let $z = k - 32$, $x_i = 12$ for $i = 1, 2$ and $y = 8$. Construct two $STS(27)$'s (S, T_1) and (S, T_2) on the set $\{1, 2, ..., 9\} \times \{1, 2, 3\}$. For each $i = 1, 2$, define a pair of $STS(9)$'s $(S(i), T_1(i))$ and $(S(i), T_2(i))$ with collection of triples on $\{1, 2, ..., 9\} \times \{i\}$

such that $T_1(i)$ and $T_2(i)$ have x_i triples in common, which is possible since $J(9) = \{0, 1, 2, 3, 4, 6, 12\}$ by Lemma 3.1.3. Further, let $(S(3), T_1(3))$ and $(S(3), T_2(3))$ be a pair of $STS(9)$'s on $\{1, 2, ..., 9\} \times \{3\}$ with a common flower at $(1, 3)$ (consisting of four triples) and y additional common triples, which is possible since $J_f(9) = \{0, 2, 8\}$ by Lemma 4.1.2. Also let $(L_1, \circ_1)$ and $(L_2, \circ_2)$ be two latin squares of order 9 on $\{1, 2, ..., 9\}$, both of which have the constant main diagonal of 1's and agree in exactly z additional cells off the main diagonal. In addition to the triples of the sets $T_l(1)$, $T_l(2)$, $T_l(3)$, the collection T_l $(l = 1, 2)$ contains also the triples $\{(a, 1), (b, 2), (a \circ_l b, 3)\}$ for each pair of numbers $1 \le a, b \le 9$.

Both (S, T_1) and (S, T_2) are $STS(27)$'s. For each $l = 1, 2$, pairs of points with the same second coordinate i where $i = 1, 2, 3$ appear together exactly once in one of the triples from the set $T_l(i)$ and pairs of points of different second coordinate appear together exactly once in a triple of the form $\{(a, 1), (b, 2), (a \circ_l b, 3)\}$.

As a result of fixing the positions of 1's in $(L_1, \circ_1)$ and $(L_2, \circ_2)$, we forced both (S, T_1) and (S, T_2) to contain the triples $\{(1, 1), (1, 2), (1, 3)\}$, $\{(2, 1), (2, 2), (1, 3)\}$, $\{(3, 1), (3, 2), (1, 3)\}$, $\{(4, 1), (4, 2), (1, 3)\}$, $\{(5, 1), (5, 2), (1, 3)\}$, $\{(6, 1), (6, 2), (1, 3)\}$, $\{(7, 1), (7, 2), (1, 3)\}$, $\{(8, 1), (8, 2), (1, 3)\}$, $\{(9, 1), (9, 2), (1, 3)\}$. Furthermore, our construction of $(S(3), T_1(3))$ and $(S(3), T_2(3))$ give the other four triples of the common flower at $(1, 3)$ of the $STS(27)$'s. Totally, apart from the triples of the common flower, (S, T_1) and (S, T_2) have $k = x_1 + x_2 + y + z$ other triples in common, x_i $(i = 1, 2)$ of them coming from the collections $T_l(i)$ for $l = 1, 2$, y of them from $T_l(3)$, for $l = 1, 2$ and z of them coming from the latin squares $(L_1, \circ_1)$ and $(L_2, \circ_2)$. □

4.2 *The Main Result*

Lemma 4.2.1 *There exists a design for each $r \equiv 0, 1$ (mod 3), $r \ge 15$, with all blocks of size $\equiv 0, 1$ (mod 3) where at least one of the blocks is of size t with $6 \le t < r$.*

Proof: It is known that for $t \ne 2, 6$ there exists a design of order $4t$ with 4 disjoint blocks of size t and remaining blocks of size 4. (Equivalently there exists a pair of orthogonal latin squares of order t [11].) For $0 \le s \le t$ we delete $t - s$ points from one of the blocks of size t. What remains is a design with of order $3t+s$ with three blocks of size t, one block of size s and other blocks of size 3 or 4. Denote this design by $D_{s,t}$. When $t=2$ or 6, this construction still works if $s=0$, because in that case only one latin square of order t is needed. Now we restrict s and t to be congruent to 0 or 1 (mod 3) and $t \geq 6$. The designs $D_{s,t}$ constructed with these restrictions cover the case $r=18$ (take $t=6, s=0$) and all the admissible cases where $r \geq 21$ (take s to be the remainder of r when divided by three and t to be equal to $(r-s)/3$).

The only remaining admissible cases are $r=15,16,19$. They are constructed by ad-hoc techniques. For $r=15$ we take a set of size 7, say $\{1,2,...,7\}$ and we define a symmetric latin square $(L,\circ)$ of order 8 on the set $\{\infty_0,\infty_1,...,\infty_7\}$ such that the main diagonal consists of zeroes and for $i \neq j$, the cells are defined by $\infty_i \circ \infty_j = k$ for some $k \in \{1,2,...,7\}$. We define the blocks to be $\{1,2,...,7\}$ and $\{\infty_i,\infty_j,k\}$. Clearly all pairs of points, not both from the set $\{1,2,...,7\}$ appear together in a block of this type. We have a design of order 15 with one block of size 7 and others of size 3. For $r=16$ we consider $D_{0,5}$ and adjoin the point ∞ to the blocks of size 5 so that the vertex set is $\{\infty\} \cup \{1,2,3,4,5\} \times \{1,2,3\}$. For each $i \in \{1,2,3\}$ take the block $\{\infty,(1,i),(2,i),(3,i),(4,i),(5,i)\}$. The remaining blocks are defined by $\{(i,1),(j,2),(i \circ j,3)\}$, where $i \circ j$ is defined in the same way as in $D_{0,5}$. This way we get a design of order 16 with 3 blocks of size 6 and the remaining blocks of size 3. For $r=19$ we consider $D_{0,6}$ and adjoin the point ∞ to the blocks of size 6 so that the vertex set is $\{\infty\} \cup \{1,2,3,4,5,6\} \times \{1,2,3\}$. For each $i \in \{1,2,3\}$ take the block $\{\infty,(1,i),(2,i),(3,i),(4,i),(5,i),(6,i)\}$. The remaining blocks are defined by $\{(i,1),(j,2),(i \circ j,3)\}$, where $i \circ j$ is defined in the same way as in $D_{0,6}$. This way we get a design of order 19 with 3 blocks of size 7 and the remaining blocks of size 3. □

Theorem 4.2.2 *For each $n \equiv 1$ or 3 (mod 6), $J_f(n) = I_f(n)$, except that $J_f(9) = \{0,2,8\}$.*

Proof: Assume that the statement holds for all admissible orders less than n. Let P be a design on the r-set as in the previous lemma, call it R. Suppose that $B_1,B_2,...,B_s$ are the blocks of P where $r_i = |B_i|$, $1 \leq i \leq s$ with $6 \leq r_1 < r$. We construct a pair of Steiner triple systems (S, T_1), (S, T_2) on the set $\{\infty\} \cup R \times \{1, 2\}$. For each $k \in I_f(n)$, we can find a_i's such that $a_i \in I_f(2r_i + 1)$ ($a_i \neq 1$ or 4 whenever $r_i = 4$) with $a_1 + ... + a_s = k$. Since $r_i < r$, by induction hypothesis there exists a pair of systems $(S, T_1(i))$, $(S, T_2(i))$ on $\{\infty\} \cup B_i \times \{1, 2\}$ with exactly $a_i + r_i$ triples in common, r_i of them being the triples of the flower $\{\{\infty, (x, 1), (x, 2)\} : x \in B_i\}$. For $l = 1, 2$, we let T_l consist of the triples in the union of $T_l(i)$ where $1 \leq i \leq s$. Then (S, T_1) and (S, T_2) have exactly $k + r$ triples in common, r of which constitute the flower $\{\{\infty, (x, 1), (x, 2)\} : x \in R\}$ at x. Taking the union of $T_l(i)$'s, for the fixed l, $l = 1, 2$, we guarantee that each pair of vertices occurs together in at least one triple. Moreover, considering any of the (S, T_l)'s, $l = 1, 2$, since $B_i \cap B_j$ for $i \neq j$, $1 \leq i, j \leq s$ cannot contain more than one vertex, we do not see the same pair of vertices more than once in a triple of $(S, T_l(i))$. □

Chapter 5

CONCLUSION

In this thesis we gave complete solutions to the basic intersection problem and flower intersection problem of Steiner triple systems, modifying respectively some parts of the work of C. C. Lindner and A. Rosa published in 1975 [12], [13] and of D. G. Hoffman and C. C. Lindner published in 1987 [7]. In 1982, C. C. Lindner and W. D. Wallis determined for all n, the set of numbers k such that there exists a pair of 1-factorizations on the same set of size n intersecting in k edges [14]. It turned out that their result and the work of J. Doyen on disjoint Steiner triple systems [5] can be combined to show most of the cases of the basic intersection problem of Steiner triple systems [14]. In [7], the authors also pointed out that the flower intersection problem is equivalent to the intersection problem of group divisible designs with group size 2 and block size 3. This observation led to the study of intersection problems of group divisible designs in different settings. In 1992, R. A. R. Butler and D. G. Hoffman solved the case with block size 3 and group size g [1]. A very similar problem to the flower intersection problem of Steiner triple systems is the disjoint intersection problem of Steiner triple systems, where the intersecting triples are required to be pairwise disjoint. This problem was completely solved by Y. M. Chee in 2004 [2]. Since there exists no $STS(6n + 5)$, a pairwise balanced design of order $6n + 5$ with exactly 1 block of size 5 and the rest of size 3, denoted by $PBD(5^*, 3)$, is the closest object with this order to being a Steiner triple system. A natural extension of the basic intersection problem and flower intersection problem is studied for $PBD(5^*, 3)$'s, both of which were solved by S. Küçükçifçi [9]. Another interesting problem arises when we have Steiner triple systems of different orders, say $STS(u)$ with vertex set U and $STS(v)$ with vertex set V such that $U \subseteq V$. P. Danziger, P. Dukes, T. Griggs and

E. Mendelsohn solved this problem for $v - u = 2, 4$ and for $v \geq 2u - 3$ [3]. To sum up, apart from the Steiner triple systems, intersection problem of different designs are studied as well [1], [9] and these intersection problems give rise to other similar problems [2], [3]. There is a variety of this kind of problems which are yet open, hence it is possible to concentrate on these problems and go further in this direction.

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