Decay Estimates For the Solutions of Linear and Nonlinear Damped Wave Equation

by

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Abstract

In the thesis, we investigate the asymptotic behavior of solutions of linear and nonlinear wave equations. In the first chapter , we give some definitions, lemmas and theorems that will be used. In the second chapter, we establish the decay estimates for solutions of a linear wave equation with mixed boundary conditions . In the third chapter, we study the asymptotic behavior of solutions of initial boundary value problem for the wave equation with nonlinear damping term. In the fourth chapter we derive the decay estimate for solution of the initial boundary value problem for the equation of nonlinear vibrations of an elastic string. What is shown by energy decay for all of these equations is the stability of these equations. As a result, we show the stability of those three different wave equations.

Contents

Chapter 1

Introduction

This thesis is devoted to the study of asymptotic behavior of solutions to initial boundary value problems for damped linear and nonlinear wave equations.

The first problem we consider is the following:

$$u_{tt} - \Delta u + qu = 0, \quad x \in \Omega, t \in \mathbb{R}^+$$

$$u(x,t) = 0, \ x \in \Gamma_0, \ t \in \mathbb{R}^+$$

$$\partial_\nu u + au + lu_t = 0, \ x \in \Gamma_1, t \in \mathbb{R}^+$$

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), x \in \Omega.$$

In the second chapter , following Komornik [3] decay estimate for classical solution of this problem is derived.

In the third chapter by using the so called Nakao inequality we investigate asymptotic behavior of solutions to the following problem

$$\begin{cases} u_{tt} - \sum\limits_{i,j=1}^{n} \left(a_{ij}(x)u_{x_j}\right)_{x_i} + a_0(x)u + \beta\left(x, u_t\right) = f(x,t), \ x \in \Omega, t > 0, \\ u(x,t) = 0, \ x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega, \end{cases}$$

where Ω is a bounded domain with a smooth boundary $\partial\Omega$,

1. $a_{ij}(x)$ and $a_0(x)$ are measurable and bounded functions on Ω :
$a_{ij}(x) = a_{ji}(x)$, $a_0(x) \geq 0 \quad \forall x \in \Omega$
and
$\sum\limits_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \nu^{-1}|\xi|^2 \ , \ \forall \xi \in \mathbb{R}^n$
where $\nu > 0$

2.$\beta(x,z)$ is measurable on $\Omega \times \mathbb{R}$

$$\beta(x,0) = 0,\ |\beta(x,z_1) - \beta(x,z_2)| \le k_0(1 + |z_1|^\gamma + |z_2|^\gamma)|z_1 - z_2|,$$

$$(\beta(x,z_1) - \beta(x,z_2))\,(z_1 - z_2) \ge k_1|z_1 - z_2|^{\gamma+2}$$

3. $0 \le \gamma \le \frac{4}{n-2}$ if $n \ge 3$ $\ 0 \le \gamma \le \infty$ *if* $\ n = 1,2.$

Following Nakao [5], the difference between bounded solution and any solution of this problem tend to zero with a polynomial rate as $t \to \infty$ is shown.

The bounded and the almost periodic solutions for this problem was investigated by Amerio and Prouse[15]. They proved that for any solution $v(t)$ the bounded solution or almost periodic solution $u(t)$ satisfies

$$\lim_{t\to\infty} \|u(t) - v(t)\|_E = 0.$$

Here $\|.\|_E$ denotes the energy norm. They also proved that for each initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$,the problem has a unique weak solution and if $f(x,t)$ is uniformly continuous then the problem has unique bounded weak solution.

In the fourth chaper, following papers of K. Nishihara we study the asymptotic behavior of solutions of the problem:

$$\begin{cases} u_{tt} + (1 + \|A^{\frac{1}{2}}u(t)\|^2)Au(t) + 2\gamma u_t = f(x,t),\ x \in \Omega, t > 0, \\ u(x,t) = 0,\ x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \end{cases}$$

The assumption on A is self adjoint, positive definite operator with discrete spectrum and A^{-1} is compact.

To show the asymptotic behavior of that equation, in Nishihara [7]

$$\|F(A^{\frac{1}{2}})u_t\|, \|F(A^{\frac{1}{2}})A^{\frac{1}{2}}u\| \le C\exp(-\theta t),\ where\ \theta > 0$$

is proven.

Here we also presented the result about continuous dependence of solution to this problem on the data.

The Cauchy and mixed problem for the equation with $\gamma = 0$ was investigated by a number of authors: Pohozaev[12], Perla[11], Nishida[10], Dickey[9], Rivera[13], Greenberg and Hu[16].

When $\gamma > 0$ the global existance and asymptotic behavior of the solutions was shown by Yamada[14] if the data are small in some sense. In [7] no smallness condition is assumed on the data, but sufficient regularity is assumed.

1.1 Prelimineries

1.1.1 Function Spaces

Definition 1.1.1. A Banach space is complete linear normed space.

Definition 1.1.2. A Hilbert space H is a complete inner product space.

Definition 1.1.3. The function v is said to be of class $C^k(G)$ if the derivatives $v', v'', ..., v^{(k)}$ exist and are continuous in G.

Definition 1.1.4. The function f is said to be of class $C^\infty(G)$, if it has derivatives of all orders.

Definition 1.1.5. Let $\Omega \in \mathbb{R}^n$ and f is defined on Ω then $f \in C_0^\infty(\Omega)$ if f is infinitely differentiable function with compact support.

Definition 1.1.6. For $1 \leq p < \infty$, $L^p(\Omega)$ is the set of all measurable functions $u(x)$ in Ω such that the norm

$$\|u\|_{L^p(\Omega)} \leq \left(\int\limits_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}$$

is finite.

Definition 1.1.7. $L^\infty(\Omega)$ is the set of all bounded measurable functions in Ω; the norm is defined by

$$\|u\|_{L^\infty(\Omega)} = ess\ sup_{x\in\Omega}|u(x)|.$$

All $L^p(\Omega)$ spaces are Banach spaces but $L^2(\Omega)$ is a Hilbert space and the inner product on $L^2(\Omega)$ is defined as:

$$(u, v) = \left(\int\limits_\Omega u(x)v(x)dx \right),\ here\ u, v \in L^2(\Omega)$$

Assume $\Omega \subset \mathbb{R}^n$ is bounded, $u \in C^1(\Omega)$ and $\Psi \in C_0^\infty(\Omega)$, by integration by parts we get,

$$\int\limits_\Omega u\Psi_{x_i} dx = -\int\limits_\Omega u_{x_i}\Psi dx. \tag{1.1}$$ wd

Here the term on $\partial\Omega$ vanishes since ψ has compact support.

For the left hand side of the equation (1.1) if we assume that u is locally integrable on Ω, then (1.1) still makes sense. So we can define weak derivative notion as below.

Definition 1.1.8. Let α be multiindex such that $\alpha = (\alpha_1, ..., \alpha_n)$, $|\alpha| = \alpha_1 + ... + \alpha_n$. Suppose that u, v be locally integrable function defined on Ω and

$$\int\limits_\Omega u D^\alpha \Psi dx = (-1)^{|\alpha|} \int\limits_\Omega v \Psi dx, \forall \Psi \in C_0^\infty(\Omega).$$

Then v is called the weak derivative of u in Ω and denoted by $D^\alpha u$.

Now we can define the Sobolev space $H^1(\Omega)$, $H^2(\Omega)$, $H_0^1(\Omega)$ and $H_{\Gamma_0}^1(\Omega)$.

Definition 1.1.9. The Sobolev space $H^1(\Omega)$ is the set of all functions from $L^2(\Omega)$ such that all the first order weak derivatives are also from $L^2(\Omega)$. i.e

$$H^1(\Omega) := \{u \in L^2(\Omega) : u_{x_i} \in L^2(\Omega), 1 \leq i \leq n\}.$$

This space is equipped with the norm:

$$\|u\|_{H^1(\Omega)} = \left(\int\limits_\Omega \sum_{|\alpha| \leq 1} |D^\alpha u|^2 dx \right)^{\frac{1}{2}}.$$

Definition 1.1.10. The Sobolev space $H^2(\Omega)$ is the set of all functions from $L^2(\Omega)$ such that all the first order weak derivatives and second order weak derivatives are also from $L^2(\Omega)$. i.e

$$H^2(\Omega) := \{u \in L^2(\Omega) : u_{x_i}, u_{x_i x_j} \in L^2(\Omega), 1 \leq i, j \leq n\}.$$

This space is equipped with the norm:

$$\|u\|_{H^2(\Omega)} = \left(\int\limits_\Omega \sum_{|\alpha| \leq 2} |D^\alpha u|^2 dx \right)^{\frac{1}{2}}.$$

Definition 1.1.11. The Sobolev space $H_0^1(\Omega)$ is the closure of the space $C_0^\infty(\Omega)$ with respect to the norm on $H^1(\Omega)$. i.e.

$$H_0^1(\Omega) := \{u \in H^1(\Omega) : \ u_m \to u \ where \ (u_m)_{m \in \mathbb{N}} \subset C_0^\infty(\Omega)\}.$$

This space is equipped with the norm:

$$\|u\|_{H^1(\Omega)} = \left(\int\limits_\Omega |\nabla u|^2 dx \right)^{\frac{1}{2}}.$$

Definition 1.1.12. Let Γ_0 ,Γ_1 be the partition of $\partial\Omega$. Then

$$H_{\Gamma_0}^1(\Omega) := \{u \in H^1(\Omega) : u = 0 \ on \ \Gamma_0 \subset \partial\Omega\}.$$

nak3L **Lemma 1.1.13.** ***(Sobolev Lemma) :*** *Let γ be ; $0 \leq \gamma \leq \frac{4}{n-2}$ if $n \geq 3$ $0 \leq \gamma < \infty$ if $n = 1, 2.$ Then*

$$||u||_{L^{\gamma+2}} \leq S_{\gamma+2} ||u||_{H_0^1(\Omega)} \tag{1.2}$$ nak3

for $u \in H_0^1(\Omega)$ where $S_{\gamma+2}$ is a constant depending on Ω and γ.

1.1.2 Some Useful Inequalities

PF **Lemma 1.1.14.** *(Poincare-Fiedrichs Inequality) There exists a constant* $c_0 > 0$ *such that*

$$\|u\|_{L^2(\Omega)} \leq c_0 \|\nabla u\|_{L^2(\Omega)} \ \forall u \in H_0^1(\Omega) \text{ and } \forall u \in H_{\Gamma_0}^1(\Omega).$$

Lemma 1.1.15. ***Cauchy-Schwartz Inequality*** $\forall f(x), g(x) \in L^2(\Omega)$ *satisfies*

$$\int_\Omega |f(x)g(x)|dx \leq \left(\int_\Omega |f(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_\Omega |g(x)|^2 dx \right)^{\frac{1}{2}}$$

Lemma 1.1.16. ***Hölder Inequality*** *Let* $1 \leq p, q \leq \infty$. *Then for all measurable real valued functions* f *and* g *satisfies the inequality*

$$\int_\Omega |f(x)g(x)|dx \leq \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q dx \right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

nak1L **Lemma 1.1.17.** *For any* $a, b \in \mathbb{R}$

$$||a|^\gamma a - |b|^\gamma b| \leq k_0(1 + |a|^\gamma + |b|^\gamma)|a - b|, \quad k_0, \gamma \geq 0 \qquad (1.3)$$ nak1

Proof. Let us consider the function

$$f(x) = |x|^\gamma x.$$

It is clear that

$$f'(x) = (\gamma + 1)|x|^\gamma,$$

and using the mean value theorem we have

$$\begin{aligned} |f(a) - f(b)| &= |(\gamma + 1)||\theta a - (1 - \theta)b|^\gamma |(a - b)| \\ &\leq |(\gamma + 1)|(|a|^\gamma + |b|^\gamma)|(a - b)| \\ &\leq |(\gamma + 1)|(1 + |a|^\gamma + |b|^\gamma)|(a - b)|, \\ &\qquad \text{where } \theta \in (0, 1). \end{aligned}$$

Hence,

$$||a|^\gamma a - |b|^\gamma b| \leq k_0(1 + |a|^\gamma + |b|^\gamma)|a - b|.$$

□

nak2L **Lemma 1.1.18.** *For any $x, y \in \mathbb{R}$*

$$(|x|^\gamma x - |y|^\gamma y)(x-y) \geq k_1|x-y|^{\gamma+2}, \quad k_1, \gamma \geq 0 \tag{1.4}$$ nak2

Proof.

$$J(\gamma+2) := (|x|^\gamma x - |y|^\gamma y)(x-y)$$

$$= \int_0^1 \frac{d}{ds}|sx+(1-s)y|^\gamma (sx+(1-s)yds(x-y)$$

$$= \int_0^1 |sx+(1-s)y|^\gamma |x-y|^2 ds$$

$$+\gamma \int_0^1 |sx+(1-s)y|^{\gamma-2}((sx+(1-s)y)(x-y))^2 ds,$$

$$where\ 0 \leq s \leq 1.$$

Since $\gamma \geq 0$ we get;

$$J(\gamma+2) \geq \int_0^1 |sx+(1-s)y|^\gamma |x-y|^2 ds.$$

If $|x| \geq |y-x|$,then

$$|sx+(1-s)y| = |x-(1-s)(x-y)| \geq |x| - (1-s)|x-y| \geq s|x-y|.$$

So we have,

$$J(\gamma+2) \geq |x-y|^2 \int_0^1 s^\gamma |x-y|^\gamma ds = d_0|x-y|^{\gamma+2},\ where\ d_0 = \frac{1}{\gamma+1}$$

We showed that the inequality (1.4) is true when $|x| \geq |y-x|$.
If $|x| \leq |y-x|$, then

$$|sx+(1-s)y| = |x-(1-s)(x-y)| \leq |x| + (1-s)|x-y| \leq (2-s)|x-y|.$$

Thus we have,

$$|sx+(1-s)y|^\gamma \geq \frac{[|sx+(1-s)y|^2]^{\frac{\gamma+2}{2}}}{(2-s)|x-y|^2}.$$

Integrating the last inequality we obtain

$$
\begin{aligned}
J(\gamma+2) &\geq |x-y|^2 \int\limits_0^1 \frac{[|sx+(1-s)y|^2]^{\frac{\gamma+2}{2}}}{(2-s)|x-y|^2} ds \geq |x|^{\gamma+2} + \frac{1}{2}|y|^{\gamma+2} \\
&\geq \frac{1}{2}(|x|^{\gamma+2} + |y|^{\gamma+2}) \geq d_0|x-y|^{\gamma+2} \geq d_0|x-y|^{\gamma+2}.
\end{aligned}
$$

So we also showed that the inequality (nak2 1.4) is true when $|x| \leq |y-x|$. Hence we are done. □

Lemma 1.1.19. ***Young Inequality*** *Assume that $x, y > 0$ and $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$xy \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}.$$

int5L **Lemma 1.1.20.** ***(Young Inequality with epsilon):***

For any $x, y > 0$ and $p, q > 1$

$$xy \leq \epsilon|x|^p + c(\epsilon)|y|^q,$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $c(\epsilon) = (\epsilon p)^{-\frac{q}{p}} q^{-1}$

GenGronwall **Lemma 1.1.21.** ***Bihari Inequality*** *(BB [1])*

Let K be a nonnegative constant, Φ and Ψ be non-negative continuous functions defined on $[0, \infty)$, and let g be a continuous non-decreasing function defined on $[0, \infty)$ and $g(u) > 0$ on $(0, \infty)$. If Φ satisfies the following integral inequality,

$$\Phi(t) \leq K + \int\limits_0^t \Psi(\tau) g(\Phi(\tau)) d\tau$$

then

$$\Phi(t) \leq G^{-1}\left(\int\limits_0^t \Psi(\tau) d\tau\right),$$

where $G(\eta) = \int\limits_K^\eta \frac{ds}{g(s)}, \eta > K > 0$

Proof. Let us consider the function $\Pi(t) : [0, \infty) \to \mathbb{R}^+$.

$$\Pi(t) = K + \int\limits_0^t \Psi(\tau) g(\Phi(\tau)) d\tau$$

where $K > 0$, Φ and Ψ are non-negative continuous functions defined on $[0, \infty)$.

Then, Π is class of $C^1[0, \infty)$ and

$$\Pi_t(t) = \Psi(t)g(\Phi(t)) \leq \Psi(t)g(\Pi(t)), \forall t \in [0, \infty).$$

We can apply the fundemental theorem of calculus to get

$$\frac{\Pi_t(t)}{g(\Pi(t))} = \Psi(t).$$

Integrating both sides of the inequality over the interval $[0, t]$,

$$G(\Pi(t)) \leq \int_0^t \Psi(\tau)d\tau$$

Hence,

$$\Phi(t) \leq \Pi(t) \leq G^{-1}(\int_0^t \Psi(\tau)d\tau),$$

□

ClasGron **Lemma 1.1.22.** ***(Gronwall Inequality):*** Nis2 [7] *If $g(s) = Cs$ then*

$$\Phi(t) \leq K + C \int_0^t \Psi(\tau)\Phi(\tau)d\tau$$

implies

$$\Phi(t) \leq K \exp\left(C \int_0^t \Psi(\tau)d\tau\right).$$

Proof. Taking $g(s) = Cs$ in Lemma GenGronwall 1.1.21, then $G^{-1}(\eta) = K exp(\eta)$ and we obtain

$$\Phi(t) \leq K \exp\left(C \int_0^t \Psi(\tau)d\tau\right).$$

□

Int **Lemma 1.1.23.** *For each $v \in H^2(\Omega) \cap H_0^1(\Omega)$ the following inequality holds true*

$$\|\nabla u\|^2 \leq \|u\|\|\Delta u\| \tag{1.5}$$ inter1

Proof. Since $u = 0$ on the boundary $\partial\Omega$ we have

$$0 = \int_\Omega \nabla \cdot (u\nabla u)dx = \|\nabla u\|^2 + (u, \Delta u).$$

Hence due to the Cauchy-Schwarz inequality we get

$$\|\nabla u\|^2 = -(u, \Delta u) \le \|u\|\|\Delta u\|.$$

□

1.1.3 Auxilary material

kom1T **Theorem 1.1.24.** *(Kom1 [3]) Let $E := \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function, and assume that there exists a constant $T > 0$ such that*

$$\int_t^\infty E(s)ds \le TE(t)\ \forall t \in \mathbb{R}^+ \tag{1.6}$$ kom2

then

$$E(t) \le E(0)e^{\frac{1-t}{T}}\ \ \forall t \ge T. \tag{1.7}$$ kom3

Proof. Let us consider the following function

$$f(x) := e^{\frac{x}{T}} \int_x^\infty E(s)ds,\ \ x \in \mathbb{R}^+.$$

The function f is locally absolutely continuous and nonincreasing on $\mathbb{R}^+$. In fact,by using the fundamental theorem of calculus and the condition (kom2 1.6) we get

$$f'(x) = \frac{1}{T}e^{\frac{x}{T}} \int_x^\infty E(s)ds + e^{\frac{x}{T}}(-E(x)) = \frac{1}{T}e^{\frac{x}{T}}(\int_x^\infty E(s)ds - TE(x)) \le 0, \forall x \in \mathbb{R}^+$$

So f is nonincreasing. Therefore $f(x) \le f(0)$, $\forall x \in \mathbb{R}^+$, and due to (kom2 1.6) we get,

$$\int_x^\infty E(s)ds = f(x) \le f(0) = \int_0^\infty E(s)ds \le TE(0),\ \ \forall x \in \mathbb{R}^+.$$

So we have

$$\int_x^\infty E(s)ds \le TE(0)e^{\frac{-x}{T}},\ \ \forall x \in \mathbb{R}^+. \tag{1.8}$$ kom4

Since E is nonnegative and nonincreasing, we have

$$\int\limits_{x}^{\infty} E(s)ds \geq \int\limits_{x}^{x+T} E(s)ds \geq (x+T-x)E(x+T) = TE(x+T). \qquad (1.9)$$ kom5

By (kom4 1.8) and (kom5 1.9)

$$E(x+T) \leq E(0)e^{\frac{-x}{T}} \quad \forall x \in \mathbb{R}^+.$$

Thus we have

$$E(t) \leq E(0)e^{\frac{1-t}{T}},$$

where $t := x+T$. □

nak4L **Lemma 1.1.25.** Nak [5] *Let $\Phi(t)$ be a positive function on $\mathbb{R}^+$ satisfying :*

$$k\Phi(t)^{\alpha+1} \leq \Phi(t) - \Phi(t+1) \quad \forall t \geq 0, \qquad (1.10)$$ n

for some constants k and $\alpha \geq 0$. Then we have

$$\Phi(t) \leq (\alpha k(t-1) + M^{-\alpha})^{-\frac{1}{\alpha}}, \quad \forall t \geq 0$$

where $M = \max_{t\in[0,1]} \Phi(t)$

Proof. Let us denote $\Phi(t)^{-\alpha} = y(t)$.

$$y(t+1) - y(t) = \int\limits_{0}^{1} \frac{d}{d\theta}(\theta\Phi(t+1) + (1-\theta)\Phi(t))^{-\alpha} d\theta$$

$$= \int\limits_{0}^{1} -\alpha(\theta\Phi(t+1) + (1-\theta)\Phi(t))^{-\alpha-1}(\Phi(t+1) - \Phi(t))d\theta$$

$$\geq \alpha k \Phi(t)^{\alpha+1} \int\limits_{0}^{1} \Phi(t)^{-\alpha-1} d\theta = \alpha k.$$

For $\forall t \geq 1$, choose the integer n as $n \leq t \leq n+1$ and by (n 1.10) we get,

$$y(t) - y(t-n) \geq n\alpha k.$$

Since

$$nk\Phi(t)^{\alpha+1} \leq \Phi(t) - \Phi(t-n),$$

we have

$$y(t) \geq y(t-n) + n\alpha k \geq y(t-n) + (t-1)\alpha k$$

$$\Phi(t)^{-\alpha} \geq (t-1)\alpha k + \Phi(t-n)^{-\alpha}.$$

Then

$$\Phi(t) \leq ((t-1)\alpha k + \Phi(t-n)^{-\alpha})^{\frac{-1}{\alpha}} \leq ((t-1)\alpha k + \max_{t\in[0,1]} \Phi(t)^{-\alpha})^{\frac{-1}{\alpha}} = ((t-1)\alpha k + M^{-\alpha})^{\frac{-1}{\alpha}}.$$

□

nak5L **Lemma 1.1.26.** Nak [5] *Let $\Phi(t)$ be a positive function that satisfies the inequality:*

$$k\Phi(t) \leq \Phi(t) - \Phi(t+1) \quad \textit{for } \forall t \geq 0. \tag{1.11}$$ nak4

with some constant $k < 1$.

Then the following estimate holds true:

$$\Phi(t) \leq Me^{-k't}, \quad t \geq 1. \tag{1.12}$$ nak5

Here $k' = -\log(1-k) > 0$

Proof. By (nak4 1.11)

$$\Phi(t+1) \leq (1-k)\Phi(t).$$

Therefore, if $t \geq 1$, for n such that $n \leq t \leq n+1$, we have

$$\Phi(t) \leq (1-k)\Phi(t-1).$$

Since $0 < k < 1$, $\frac{1}{1-k} \geq 1-k$ and $\Phi(t) \leq \frac{1}{1-k}\Phi(t-1)$.

Thus,

$$\Phi(t) \leq (1-k)^2\Phi(t-2) \cdots \leq (1-k)^n\Phi(t-n) \leq \frac{1}{(1-k)^n}\Phi(t-n)$$

$$\Phi(t) \leq (\frac{1}{1-k})^n\Phi(t-n)$$

$$\Phi(t) \leq (\frac{1}{1-k})^n\Phi(t-n) \leq M(1-k)^{-t} = Me^{-t\log(1-k)} = Me^{-k't}.$$

□

nis4L **Lemma 1.1.27.** *Let $\phi(t)$ be a bounded nonnegative function on $\mathbb{R}^+$, satisfying*

$$\max_{t\leq\tau\leq t+1}\phi(\tau)\leq K_0(\phi(t)-\phi(t+1))+g(t) \tag{1.13}$$ int1

where K_0 is a positive constant and $g(t)$ is a nonnegative function such that

$$g(t)\leq C_0\exp(-\theta t) \text{ with } \theta>0, C_0>0 \tag{1.14}$$ int2

then

$$\phi(t)\leq C'\exp(-\theta' t) \tag{1.15}$$ int3

where $\theta' = \min(\ -\theta, \log\frac{K_0}{K_0+1})$ and C' is a positive constant depending on K_0, C_0, θ.

Proof. By (int1 1.13),

$$\phi(t+1)\leq K_0(\phi(t)-\phi(t+1))+g(t)$$

then

$$\phi(t+1)\leq\frac{K_0}{K_0+1}\phi(t)+\frac{K_0}{K_0+1}g(t)$$

If $t\geq 1$, there exists n such that $n\leq t\leq n+1$. Then we have

$$\phi(t)\leq\left(\frac{K_0}{K_0+1}\right)^n\phi(t-n)+\sum_{i=1}^{n}\left(\frac{K_0}{K_0+1}\right)^i g(t)$$

Since $\frac{K_0}{K_0+1}<1$, then

$$\begin{aligned}\phi(t)&\leq\left(\frac{K_0}{K_0+1}\right)^n\phi(t-n)+\sum_{i=1}^{n}\left(\frac{K_0}{K_0+1}\right)^i g(t)\\&\leq M\left(\frac{K_0}{K_0+1}\right)^t+\sum_{i=1}^{n}\left(\frac{K_0}{K_0+1}\right)^i g(t)\\&\leq M\exp\left(log\left(\frac{K_0}{K_0+1}\right)\right)+C_1\exp(-\theta t)\leq C'\exp(-\theta' t)\\&\quad \textit{where } \theta'=\min(-\theta,\log\frac{K_0}{K_0+1}) \textit{ and } M=\max_{t\in[0,1]}\Phi(t).\end{aligned}$$

Hence we are done.

□

Chapter 2

Estimates of Solutions of Linear Wave Equation

The aim of this chapter is to show the exponential energy decay of the solutions of the wave equation under suitable linear boundary conditions. We use the Theorem 1.1.24 to show that energy norm of the solution of the problem satisfies this theorem. Hence, we conclude that the problem has energy decay. In other words, it is asymptotically stable.

2.1 Linear wave equation

In this section we consider the following problem:

$$u_{tt} - \Delta u + qu = 0, \quad x \in \Omega, t \in \mathbb{R}^+ \tag{2.1}$$

$$u(x,t) = 0, \ x \in \Gamma_0, \ t \in \mathbb{R}^+ \tag{2.2}$$

$$\partial_\nu u + au + lu_t = 0, \ x \in \Gamma_1, t \in \mathbb{R}^+ \tag{2.3}$$

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), x \in \Omega, \tag{2.4}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a bounded domain with a $\mathbf{C}^2$ boundary Γ and $\{\Gamma_0, \Gamma_1\}$ is a partition of boundary $\Gamma, q : \Omega \to \mathbb{R}, a, l := \ \Gamma_1 \to \mathbb{R}^+$ are

continuous and non negative functions. We assume that the initial data u_0 and u_1 are sufficiently smooth functions so that the problem (2.1)-(2.4) has a classical solution.

The following lemma shows that the energy is a non-increasing function of time.

kom1L **Lemma 2.1.1.** *([3]) Suppose that the problem (2.1)-(2.4) has a classical solution. Then the solution of the problem (2.1)-(2.4) satisfies the energy equality:*

$$E(S) - E(T) = \int_S^T \int_{\Gamma_1} l(u_t(t))^2 d\Gamma, \quad \text{such that } 0 \le S < T < \infty. \tag{2.5}$$ kom1

Proof. We multiply the equation (2.1) by u_t and integrate over the region $\Omega \times (S,T)$:

$$0 = \int_S^T \int_\Omega \frac{1}{2}((u_t)^2)_t dxdt - \int_S^T \int_\Omega u_t \Delta u dxdt + \int_S^T \int_\Omega \frac{1}{2} q(u^2)_t dxdt.$$

By using the equality $\nabla u_t \nabla u = \frac{1}{2}(|\nabla u|^2)_t$ and Green's identity we get

$$0 = \frac{1}{2}\left(\int_\Omega (u_t)^2 + qu^2 + |\nabla u|^2 dx\right)\Big|_S^T - \int_S^T \int_\Gamma u_t d_\nu u dxdt.$$

Since $\partial_\nu u + au + lu_t = 0$, we have

$$0 = \frac{1}{2}\left(\int_\Omega (u_t)^2 + qu^2 + |\nabla u|^2 dx + \int_{\Gamma_1} au^2 d\Gamma\right)\Big|_S^T + \int_S^T \int_{\Gamma_1} l(u_t)^2 dxdt$$

or

$$\int_S^T \int_{\Gamma_1} l(u_t)^2 dxdt = E(S) - E(T).$$

Hence $E(S) \ge E(T)$. □

It follows from Lemma 2.1.1 that $lu_t \in L^2(\mathbb{R}^+, L^2(\Gamma_1))$.

In this section our purpose is to show that the energy function tends to zero with an exponential rate as $t \to \infty$ for particular choice of functions l and a.

Assume that there is a point $x_0 \in \mathbb{R}^n$ such that $m \cdot \nu \le 0$ on Γ_0 and $m \cdot \nu \ge 0$ on Γ_1, where $m(x) = x - x_0$ and λ_1 is the biggest constant for which the following inequality holds:

$$\int_\Omega |\nabla u|^2 + qu^2 dx + \int_{\Gamma_1} au^2 d\Gamma \ge \lambda_1 \int_\Omega u^2 dx. \tag{2.6}$$ kom7

By Poincare-Friedrich's inequality and nonnegativity of the functions a and q such a number λ_1 exists.

In what follows we use the notations:

$$k := \frac{1}{R},\ b := \frac{(n-1)}{2R^2},\ Mu := 2m \cdot \nabla u + (n-1)u, \Gamma_m = (m \cdot \nu) d\Gamma$$

$$l := \frac{(m.\nu)}{R},\ a := \frac{(n-1)(m.\nu)}{2R^2}$$

kom2L **Lemma 2.1.2.** *(Kom1 [3]) For any given (u_0, u_1) and $0 \leq S \leq T < \infty$ arbitrarily, the solution of the problem satisfies the following inequality:*

$$2\int_S^T E(t)dt \leq \left[\left|\int_\Omega u_t M u dx\right|\right]_S^T - \int_S^T \int_\Omega (n-2)qu^2 + 2qum \cdot \nabla u dx dt$$
$$+ \int_S^T \int_{\Gamma_0} |\partial_\nu u|^2 d\Gamma_m dt$$
$$+ \int_S^T \int_{\Gamma_1} u_t^2 - |\nabla u|^2 + bu^2 - (ku_t + bu) M u d\Gamma_m dt. \quad (2.7)$$ kom10

Proof. We multiply (lw1 2.1) by $2m \cdot \nabla u$ and integrate over the region $\Omega \times (S, T)$ use the Green's identity and the identities

$$2u_{tt} m \cdot \nabla u = (2u_t m \cdot \nabla u)_t - 2u_t m \cdot \nabla u_t,\ 2u_t \nabla u_t = \nabla u_t^2$$

we get,

$$\int_S^T \int_\Omega -qu2m \cdot \nabla u dx dt = \left(\int_\Omega 2u_t m \cdot \nabla u\right)\Big|_S^T$$
$$+ \int_S^T \int_\Omega 2\nabla u \cdot \nabla (m \cdot \nabla u) - m \cdot \nabla u_t^2 dx dt$$
$$- \int_S^T \int_\Gamma 2\partial_\nu u m \cdot \nabla u d\Gamma dt.$$

Now, applying divergence theorem and using the identities

$$- m \cdot \nabla u_t^2 = -\nabla (m \cdot u_t^2) + \nabla \cdot m(u_t^2), 2\nabla u \cdot \nabla (m \cdot \nabla u)$$
$$= 2\sum_{i,j=1}^n \partial_i u \partial_i (m_j \partial_j u), \partial_j ((\partial_i u)^2) = 2\partial_i \partial_j u \partial_i u$$

we get,

$$\int_S^T \int_\Omega -qu2m \cdot \nabla u dx dt = \left(\int_\Omega 2u_t m \cdot \nabla u dx \right) |_S^T$$
$$- \int_S^T \int_\Gamma 2\partial_\nu u m \cdot \nabla u + (m \cdot \nu) u_t^2 d\Gamma dt$$
$$+ \int_S^T \int_\Omega (\nabla \cdot m) u_t^2 + 2 \sum_{i,j=1}^n \partial_i m_j \partial_i u \partial_j u + \sum_{i,j=1}^n m_j \partial_j ((\partial_i u)^2) dx dt. \quad (2.8)$$

kom43

Again by divergence theorem and the identities

$$m\nabla(|\nabla u|^2) = \nabla(m|\nabla u|^2) - (\nabla \cdot m)|\nabla u|^2,$$
$$\sum_{i,j=1}^n \partial_i m_j \partial_i u \partial_j u = 2|\nabla u|^2, \ \nabla \cdot m = n$$

the equality (kom43 2.8) can be written as,

$$\int_S^T \int_\Gamma 2\partial_\nu u m \cdot \nabla u + (m \cdot \nu)(u_t^2 - |\nabla u|^2) d\Gamma dt =$$
$$\left(\int_\Omega 2u_t m \cdot \nabla u dx\right)|_S^T + \int_S^T \int_\Omega n u_t^2 - (2-n)|\nabla u|^2 + 2qum\nabla u dx dt. \quad (2.9)$$

kom11

Let us multiply the equation (lw1 2.1) by $(n-1)u$, integrate over $\Omega \times (S;T)$ and use the Green's identity:

$$0 = (n-1) \int_\Omega u_t u|_S^T + (n-1) \int_S^T \int_\Omega -u_t^2 + |\nabla u|^2 + qu^2 dx dt$$
$$- (n-1) \int_S^T \int_\Gamma u \partial_\nu u d\Gamma dt. \quad (2.10)$$

kom12

Adding (kom11 2.9) and (kom12 2.10), we get

$$\int_S^T \int_\Gamma \partial_\nu u M u + (m \cdot \nu)(u_t^2 - |\nabla u|^2) d\Gamma dt$$
$$= \left[\int_\Omega u_t M u \right]_S^T + \int_S^T \int_\Omega u_t^2 + |\nabla u|^2 + (n-1)qu^2 + 2qum \cdot \nabla u dx dt. \quad (2.11)$$

kom45

Let us rewrite (kom45 2.11) in the form,

$$\int_S^T \int_\Gamma \partial_\nu u M u + (m\cdot\nu)(u_t^2 - |\nabla u|^2) d\Gamma dt$$
$$= \left[\int_\Omega u_t M u\right]_S^T + \int_S^T \int_\Omega u_t^2 + |\nabla u|^2 + qu^2 + (n-2)qu^2 + 2qum\cdot\nabla u dx dt$$
$$+ \int_S^T \int_{\Gamma_1} \frac{(n-1)(m\cdot\nu)}{2R^2} u^2 d\Gamma dt - \int_S^T \int_{\Gamma_1} \frac{(n-1)(m\cdot\nu)}{2R^2} u^2 d\Gamma dt. \tag{2.12}$$

kom46

i.e.

$$\int_S^T \int_\Gamma \partial_\nu u M u + (m\cdot\nu)(u_t^2 - |\nabla u|^2) d\Gamma dt$$
$$= \left[\int_\Omega u_t M u\right]_S^T + \int_S^T E(t) dt$$
$$+ \int_S^T \int_\Omega (n-2)qu^2 + 2qum\cdot\nabla u dx dt - \int_S^T \int_{\Gamma_1} bu^2 d\Gamma_m dt. \tag{2.13}$$

kom55

Since $u = 0$ on Γ_0 we have $\nabla u = \partial_\nu u \cdot \nu$ on Γ_0. Thus,

$$\int_{\Gamma_0} \partial_\nu u M u d\Gamma = \int_{\Gamma_0} \partial_\nu u (2m\cdot\nabla u) d\Gamma = 2\int_{\Gamma_0} (\partial_\nu u)^2 (m\cdot\nu) d\Gamma, \tag{2.14}$$

kom13

$$\int_{\Gamma_0} -|\nabla u|^2 (m\cdot\nu) d\Gamma = -\int_{\Gamma_0} (\partial_\nu u)^2 |\nu|^2 (m\cdot\nu) d\Gamma = -\int_{\Gamma_0} (\partial_\nu u)^2 (m\cdot\nu) d\Gamma. \tag{2.15}$$

kom14

Therefore by (kom13 2.14) and (kom14 2.15) the left hand side of (kom55 2.13) can be written as :

$$\int_S^T \int_\Gamma \partial_\nu u M u + (m \cdot \nu)(u_t^2 - |\nabla u|^2) d\Gamma dt$$

$$= \int_S^T \int_\Gamma \partial_\nu u M u + \int_S^T \int_{\Gamma_1} (u_t^2 - |\nabla u|^2) d\Gamma_m dt + \int_S^T \int_{\Gamma_0} -|\nabla u|^2 d\Gamma_m dt$$

$$= \int_S^T \int_{\Gamma_1} (-l u_t - a u) M u d\Gamma dt + 2 \int_S^T \int_{\Gamma_0} (\partial_\nu u)^2 d\Gamma_m dt$$

$$+ \int_S^T \int_{\Gamma_1} (u_t^2 - |\nabla u|^2) d\Gamma_m dt - \int_S^T \int_{\Gamma_0} (\partial_\nu u)^2 d\Gamma_m dt$$

$$= \int_S^T \int_{\Gamma_0} (\partial_\nu u)^2 d\Gamma_m dt + \int_S^T \int_{\Gamma_1} u_t^2 - |\nabla u|^2 - (k u_t + b u) M u d\Gamma_m dt.$$

So we get the inequality:

$$2 \int_S^T E(t) dt \leq \left[\left| \int_\Omega u_t M u dx \right| \right]_S^T - \int_S^T \int_\Omega (n-2) q u^2 + 2 q u m \cdot \nabla u dx dt$$

$$+ \int_S^T \int_{\Gamma_0} |\partial_\nu u|^2 d\Gamma_m dt + \int_S^T \int_{\Gamma_1} u_t^2 - |\nabla u|^2 + b u^2 - (k u_t + b u) M u d\Gamma_m dt.$$

Proof follows. □

kom3L **Lemma 2.1.3.** *([4]) Let u be solution of the problem (2.1)-(2.4). Then the following estmate holds:*

$$|\int_\Omega u_t M u dx| \leq 2 R E(t), \quad \forall t \in \mathbb{R}^+. \tag{2.16}$$ kom15

Proof. By Young's inequality with epsilon

$$|\int_\Omega u_t M u dx| \leq \int_\Omega R |u_t|^2 + \frac{|Mu|^2}{4R} dx.$$

$$\int_\Omega |Mu|^2 dx = \int_\Omega |2m\cdot\nabla u|^2 + (n-1)^2u^2 + 4(n-1)um\cdot\nabla u dx$$
$$= \int_\Omega |2m\cdot\nabla u|^2 + (n-1)^2u^2 + (2n-2)m\cdot 2u\nabla u dx.$$

We use divergence theorem and the identities,

$$2u\nabla u = \nabla u^2,$$
$$(2n-2)m\cdot\nabla u^2 = -(2n-2)nu^2 + (2n-2)\nabla\cdot(mu^2)$$

to get

$$\int_\Omega |Mu|^2 dx = \int_\Omega |2m\cdot\nabla u|^2 + ((n-1)^2 - (2n-2)n)u^2 dx$$
$$+ (2n-2)\int_\Gamma (m\cdot\nu)u^2 d\Gamma = \int_\Omega |2m\cdot\nabla u|^2 + (1-n^2)u^2 dx$$
$$+ (2n-2)\int_\Gamma (m\cdot\nu)u^2 d\Gamma. \quad (2.17)$$ kom56

Since $(1-n^2)u^2$ is negative, then we can majorize (kom56 2.17) as:

$$\int_\Omega |Mu|^2 dx \leq 4R^2\int_\Omega |\nabla u|^2 dx + 4R^2\frac{n-1}{2R^2}\int_\Gamma (m\cdot\nu)u^2 d\Gamma$$
$$\leq 4R^2\int_\Omega |\nabla u|^2 dx + 4R^2\int_\Gamma au^2 d\Gamma.$$

As a result:

$$\int_\Omega |u_t Mu| dx \leq R\int_\Omega |u_t|^2 dx + \frac{4R^2}{4R}\Big(\int_\Omega |\nabla u|^2 + qu^2 dx + \int_\Gamma au^2 d\Gamma\Big) = 2RE.$$

□

kom2T **Theorem 2.1.4.** *(Kom1 [3]) Suppose that*

- $q\in C(\overline{\Omega})$,
- $Q_1 = 2R\sup_\Omega q/\sqrt{\lambda_1} < 1$ *if* $n\geq 2$,

Then for every given (u_0, u_1) the solution of the problem satisfies

$$E(t) \leq E(0) \exp(\frac{1-(1-Q_1)t}{2R}), \forall t \in \mathbb{R}^+ \tag{2.18}$$ kom9

Proof. To use the Theorem 2.1.4 we estimate the terms on the right hand side of the inequality (2.7).

By Lemma 2.1.3, the first term of the right hand side of the inequality (2.7) can be estimated as:

$$\left[\left| \int_\Omega u_t M u dx \right| \right]_S^T \leq 2RE(t)|_S^T \leq 2RE(S) + 2RE(T).$$

Since $m \cdot \nu \leq 0$ on Γ_0, we can ignore the third term of the right hand side of the inequality (2.7) . i.e.

$$\int_S^T \int_{\Gamma_0} |\partial_\nu u|^2 d\Gamma_m dt = \int_S^T \int_{\Gamma_0} (m \cdot \nu) |\partial_\nu u|^2 d\Gamma dt \leq 0$$

For the second term we will show that:

$$-\int_S^T \int_\Omega (n-2) q u^2 + 2qum \cdot \nabla u dx dt \leq 2Q_1 \int_S^T E(t) dt.$$

Since $n \geq 3$ and q is a positive function then

$$-\int_S^T \int_\Omega (n-2) q u^2 dx dt \leq 0.$$

By using Young's inequality we get,

$$|u \nabla u| \leq \frac{\lambda_1^{\frac{1}{2}}}{2} u^2 + \frac{1}{2\lambda_1^{\frac{1}{2}}} |\nabla u|^2. \tag{2.19}$$ kom16

Therefore by (2.6) and (2.19), we have,

$$
\begin{aligned}
-\int_S^T\int_\Omega 2qum\cdot\nabla u dxdt &\le 2R\sup_\Omega q\int_S^T\int_\Omega |u\nabla u|dxdt\\
&\le 2R\sup_\Omega q\int_S^T \frac{\lambda_1^{\frac12}}{2}\left(\int_\Omega u^2dx\right)+\frac{1}{2\lambda_1^{\frac12}}\left(\int_\Omega|\nabla u|^2dx\right)dt\\
\le 2R\sup_\Omega q\int_S^T \frac{1}{2\lambda_1^{\frac12}}&\left(\int_\Omega |\nabla u|^2+qu^2dx+\int_{\Gamma_1}au^2d\Gamma\right)+\frac{1}{2\lambda_1^{\frac12}}\left(\int_\Omega|\nabla u|^2dx\right)dt\\
&\le 2R\sup_\Omega q\lambda_1^{-\frac12}\int_S^T\left(\int_\Omega u_t^2+|\nabla u|^2+qu^2dx+\int_{\Gamma_1}au^2d\Gamma\right)dt\\
&\le 2Q_1\int_S^T E(t)dt.
\end{aligned}
$$

Thus, we can deduce that:

$$
\begin{aligned}
2(1-Q_1)\int_S^T E(t)dt &\le 2RE(S)+2RE(T)\\
&+\int_S^T\int_{\Gamma_1} u_t^2-|\nabla u|^2+bu^2-(ku_t+bu)Mud\Gamma_m dt.
\end{aligned}
$$

By Young's inequality, we obtain

$$
2(ku_t+bu)m\cdot\nabla u\le R^2(ku_t+bu)^2+|\nabla u|^2. \tag{2.20}
$$ kom57

We use the particular choice of k, b and (2.20) to make the following estimation on the last term of the right hand side of the inequality (2.7).

$$
\begin{aligned}
&u_t^2-|\nabla u|^2+bu^2-(ku_t+bu)Mu\\
&\quad= u_t^2-|\nabla u|^2+bu^2-2(ku_t+bu)m\cdot\nabla u+(1-n)u(ku_t+bu)\\
&\quad\le u_t^2-|\nabla u|^2+bu^2+R^2(ku_t+bu)^2+|\nabla u|^2+(1-n)u(ku_t+bu)\\
&\quad= u_t^2+b(2-n+R^2b)u^2+R^2k^2(u_t)^2+(1-n+2R^2b)ku_tu\\
&\quad= 2u_t^2+((3-n)/2)bu^2.
\end{aligned}
$$

So we deduce that,

$$2(1-Q_1)\int_S^T E(t)dt \leq 2RE(S) + 2RE(T)$$

$$+\,2\int_S^T\int_{\Gamma_1} u_t^2 d\Gamma_m dt + ((3-n)/2)\int_S^T\int_{\Gamma_1} bu^2 d\Gamma_m dt. \tag{2.21}$$

kom50

According to the particular choice of l,

$$E(S)-E(T) = \int_S^T\int_{\Gamma_1} l(u_t)^2 d\Gamma dt = \frac{1}{R}\int_S^T\int_{\Gamma_1} (u_t)^2 d\Gamma_m dt.$$

Then, the third term of (kom50 2.21) will be:

$$2RE(S) - 2RE(T) = 2\int_S^T\int_{\Gamma_1} (u_t)^2 d\Gamma_m dt.$$

Thus, (kom50 2.21) can be majorized as:

$$2(1-Q_1)\int_S^T E(t)dt \leq 4RE(S) + ((3-n)/2)\int_S^T\int_{\Gamma_1} bu^2 d\Gamma_m dt. \tag{2.22}$$

kom51

Since $n \geq 3$ last term of the (kom51 2.22) is less than 0, then we get

$$2(1-Q_1)\int_S^T E(t)dt \leq 4RE(S).$$

Let T tend to infinity and $S \in \mathbb{R}^+$is fixed, then we obtain,

$$\int_S^\infty E(t)dt \leq \frac{2R}{(1-Q_1)}E(S).$$

Finally, we apply theorem (kom1T 1.1.24) and conclude that

$$E(t) \leq E(0)\exp(1-\frac{(1-Q_1)t}{2R}), \forall t \in \mathbb{R}^+.$$

□

Hence, by Theorem kom2T 2.1.4, we show the asypmtotic stability of the problem.

Chapter 3

Estimates of Solutions of Wave Equation with Nonlinear Damping Term

In this chapter we consider the wave equation with the nonlinear damping term. We obtain a polynomial decay estimate for the difference of two solutions of the equation under consideration.

3.1 Wave Equation with Nonlinear Damping Term

The problem that we consider in this chapter is:

$$u_{tt}(x,t) - \Delta u(x,t) + |u_t(x,t)|^{\gamma} u_t(x,t) = f(x,t),\ x \in \Omega, t > 0 \tag{3.1}$$ nk1

$$u(x,t) = 0,\ x \in \partial\Omega, t > 0 \tag{3.2}$$ nk2

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \tag{3.3}$$ nk3

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$ and $u_0(x)$, $u_1(x)$ and $f(x,t)$ are sufficiently smooth functions then the problem (nk1 3.1)-(nk3 3.3) has classical solutions.

We suppose that:

$$0 \leq \gamma \leq \frac{4}{n-2}\ if\ n \geq 3,\ 0 \leq \gamma < \infty\ if\ n = 1,2.$$

$$\|u(t)\|_E^2 = \|u_t(t)\|_{L^2(\Omega)}^2 + \|u(t)\|_{H_0^1(\Omega)}^2$$

$$\|f(t)\|_S = \left(\int_t^{t+1} \int_\Omega f(x,s)^{\frac{\gamma+2}{\gamma+1}} dx ds \right)^{\frac{\gamma+1}{\gamma+2}}$$

nk3T **Theorem 3.1.1.** [5] *Let $u(t), v(t)$ be any two solution of the problem* (3.1)-(3.3) *then u, v satisfies*

$$\|u(t)-v(t)\|_E \le \left(\frac{\gamma}{2}K(t-1)+M^{-\gamma}\right)^{-\frac{1}{\gamma}} \text{ for } t \ge 1 \quad \text{if } \gamma > 0$$

and

$$\|u(t)-v(t)\|_E \le Me^{-K't} \quad \text{for } t \ge 1 \ \text{if } \ \gamma = 0$$

where $K' = -\frac{1}{2}\log(1-K) > 0$ and $M = \max_{t\in[0,1]} \|u(t)-v(t)\|_E$ and K is positive constant.

Proof. We multiply the equation (3.1) by $u_t(x,t)$, integrate over $\Omega \times [t, t+1]$ and get

$$\frac{1}{2}\frac{d}{dt}\int_t^{t+1}\int_\Omega u_t^2(x,s)dxds + \frac{1}{2}\frac{d}{dt}\int_t^{t+1}\int_\Omega |\nabla u(x,s)|^2 dxds + \int_t^{t+1}\int_\Omega |u_t(x,s)|^{\gamma+2}dxds = \int_t^{t+1}\int_\Omega u_t(x,s)f(x,s)dxds.$$

Taking $k_2 \le 1$, we have

$$k_2\int_t^{t+1}\int_\Omega |u_t(x,s)|^{\gamma+2}dxds \le \|u(t)\|_E^2 - \|u(t+1)\|_E^2 + \int_t^{t+1}\int_\Omega u_t(x,s)f(x,s)dxds. \quad (3.4)$$ nak60

If $p = \gamma+2$, $q = \frac{\gamma+2}{\gamma+1}$, $\epsilon = \frac{k_2}{2}$, by Young's inequality with epsilon we get

$$u_t(x,t)f(x,t) \le \frac{k_2}{2}|u_t(x,t)|^{\gamma+2} + \frac{\gamma+2}{\gamma+1}\left(\frac{2}{k_2\gamma+2}\right)|f(x,t)|^{\frac{\gamma+2}{\gamma+1}}.$$

Then (3.4) can be written in the form

$$k_2\int_t^{t+1}\int_\Omega |u_t(x,s)|^{\gamma+2}dxds \le \|u(t)\|_E^2 + \int_t^{t+1}\int_\Omega \frac{k_2}{2}|u_t(x,s)|^{\gamma+2}dxds + \int_t^{t+1}\int_\Omega \frac{\gamma+2}{\gamma+1}\left(\frac{2}{k_2\gamma+2}\right)|f(x,s)|^{\frac{\gamma+2}{\gamma+1}}dxds,$$

$$\int_t^{t+1} \|u_t(s)\|^{\gamma+2}_{L^{\gamma+2}(\Omega)} ds \le \frac{2}{k_2} \sup_{t\in\mathbb{R}} (\|u(t)\|^2_E + \frac{\gamma+2}{\gamma+1} \left(\frac{2}{k_2\gamma+2} \right) \|f\|_S^{\frac{\gamma+2}{\gamma+1}}).$$

Take $M_1^{\gamma+2} = \frac{2}{k_2} \sup_{t\in\mathbb{R}}(\|u(t)\|^2_E + \frac{\gamma+2}{\gamma+1}(\frac{2}{k_2\gamma+2})\|f\|_S^{\frac{\gamma+2}{\gamma+1}})$, then

$$\int_t^{t+1} \|u_t(s)\|^{\gamma+2}_{L^{\gamma+2}(\Omega)} ds \le M_1^{\gamma+2}. \tag{3.5}$$

estU

Suppose that u and v are two arbitrary solutions of the problem (3.1)-(3.3). Then the function w is a solution of the equation

$$w_{tt}(x,t) - \Delta w(x,t) + (|u_t(x,t)|^\gamma u_t(x,t) - |v_t(x,t)|^\gamma v_t(x,t)) = 0. \tag{3.6}$$

nak6

Multiplying (3.6) by w_t, integrating over the region $\Omega \times [t, t+1]$ and using Green's identity we get

$$\frac{d}{dt} \int_t^{t+1} \left(\frac{1}{2} \|w_t(s)\|^2_{L^2(\Omega)} + \frac{1}{2} \|\nabla w(s)\|^2_{L^2(\Omega)} \right) ds$$
$$+ \int_t^{t+1} \int_\Omega w_t(x,s)(|u_t(x,s)|^\gamma u_t(x,s) - |v_t(x,s)|^\gamma v_t(x,s)) dx ds = 0,$$

$$\frac{d}{dt} \int_t^{t+1} \|w(s)\|^2_E ds + \int_t^{t+1} \int_\Omega w_t(x,s)(|u_t(x,s)|^\gamma u_t(x,s) - |v_t(x,s)|^\gamma v_t(x,s)) dx ds = 0.$$

By Lemma 1.1.18, we have

$$\frac{d}{dt} \int_t^{t+1} \|w(s)\|^2_E ds + k_1 \int_t^{t+1} \int_\Omega |w_t(x,s)|^{\gamma+2} dx ds \le 0.$$

Integrating the last inequality over the interval $[t, t+1]$, we obtain

$$k_1 \int_t^{t+1} \int_\Omega |w_t(x,s)|^{\gamma+2} dx ds \le \|w(t)\|^2_E - \|w(t+1)\|^2_E := k_1 (A(t))^{\gamma+2}$$

where

$$A(t) = (\frac{1}{k_1}\|w(t)\|_E^2 - \frac{1}{k_1}\|w(t+1)\|_E^2)^{\frac{1}{\gamma+2}}. \qquad (3.7) \quad \boxed{\texttt{nak61}}$$

Using the mean value theorem for integration, we can find two points $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|w_t(t_i)\|_{L^{\gamma+2}(\Omega)} \leq 2A(t) \quad , \ i = 1, 2. \qquad (3.8) \quad \boxed{\texttt{nak8}}$$

Now we will multiply the equation (nak6 3.6) with w and integrate over $\Omega \times [t_1, t_2]$

$$\frac{d}{dt}\int_{t_1}^{t_2}\int_{\Omega} w(x,s)w_t(x,s)dxds - \int_{t_1}^{t_2}\int_{\Omega} w_t^2(x,s)dxds + \int_{t_1}^{t_2}\int_{\Omega} w(x,s)\Delta w(x,s)dxds$$

$$+\int_{t_1}^{t_2}\int_{\Omega} w(x,s)(|u_t(x,s)|^\gamma u_t(x,s) - |v_t(x,s)|^\gamma v_t(x,s))dxds = 0,$$

$$\int_{t_1}^{t_2} \|w(s)\|^2_{H_0^1(\Omega)}ds \leq |(w(t_2), w_t(t_2))| + |(w(t_1), w_t(t_1))|$$

$$+\left|\int_{t_1}^{t_2}(w(s), (|u_t(s)|^\gamma u_t(s) - |v_t(s)|^\gamma v_t(s)))ds\right| + \int_{t_1}^{t_2} \|w_t(s)\|^2_{L^2(\Omega)}ds.$$

By Sobolev lemma and (nak8 3.8)

$$|(w(t_1), w_t(t_1))| \leq \int_{\Omega} |w(t_1)||w_t(t_1)|dx \leq \left(\int_{\Omega} |w(t_1)|^{\frac{\gamma+2}{\gamma+1}}dx\right)^{\frac{\gamma+1}{\gamma+2}} \left(\int_{\Omega} |w_t(t_1)|^{\gamma+2}dx\right)^{\frac{1}{\gamma+2}}$$

$$\leq mes(\Omega)^{\frac{\gamma}{\gamma+2}}\|w(t_1)\|_{L^{\gamma+2}(\Omega)}\|w_t(t_1)\|_{L^{\gamma+2}(\Omega)} \leq mes(\Omega)^{\frac{\gamma}{\gamma+2}}2A(t)\|w(t_1)\|_{L^{\gamma+2}(\Omega)}$$

$$\leq 2mes(\Omega)^{\frac{\gamma}{\gamma+2}}A(t)S_2\|w(t_1)\|_{H_0^1(\Omega)} \leq 2mes(\Omega)^{\frac{\gamma}{\gamma+2}}A(t)S_2\|w(t_1)\|_E$$

$$\leq 2mes(\Omega)^{\frac{\gamma}{\gamma+2}}S_2A(t)\max_{s\in[t,t+1]}\|w(s)\|_E.$$

Similarly,

$$|(w(t_2), w_t(t_2))| \leq 2mes(\Omega)^{\frac{\gamma}{\gamma+2}}S_2A(t)\max_{s\in[t,t+1]}\|w(s)\|_E.$$

By (nak8 3.8) we have,

$$\int_{t_1}^{t_2} \|w_t(s)\|^2_{L^2(\Omega)} ds = \int_{t_1}^{t_2} \int_{\Omega} w_t^2(x,s)\,dx\,ds$$

$$\leq \int_{t_1}^{t_2} mes(\Omega)^{\frac{\gamma}{\gamma+2}} \left(\int_{\Omega} w_t^{2.\frac{\gamma+2}{2}}(x,s)\,dx \right)^{\frac{2}{\gamma+2}} ds \leq mes(\Omega)^{\frac{\gamma}{\gamma+2}} A(t)^2.$$

Employing the Lemma nak1L 1.1.17 we get

$$\left| \int_{t_1}^{t_2} \int_{\Omega} (|u_t(x,s)|^{\gamma} u_t(x,s) - |v_t(x,s)|^{\gamma} v_t(x,s)) w(x,s)\,dx\,ds \right|$$

$$\leq k_0 \int_{t_1}^{t_2} \int_{\Omega} (1 + |u_t(x,s)|^{\gamma} + |v_t(x,s)|^{\gamma}) |w_t(x,s)| |w(x,s)|\,dx\,ds$$

$$\leq k_0 \int_{t_1}^{t_2} mes(\Omega)^{\frac{\gamma}{\gamma+2}} \left(\int_{\Omega} |w_t(x,s)|^{\frac{\gamma+2}{2}.2} dx \right)^{\frac{2}{\gamma+2}\cdot\frac{1}{2}} \left(\int_{\Omega} |w(x,s)|^{\frac{\gamma+2}{2}.2} dx \right)^{\frac{2}{\gamma+2}\cdot\frac{1}{2}} ds$$

$$+ k_0 \int_{t_1}^{t_2} \|u_t(x,s)\|^{\gamma}_{L^{\gamma+2}} \left(\int_{\Omega} |w_t(x,s)|^{\frac{\gamma+2}{2}.2} dx \right)^{\frac{2}{\gamma+2}\cdot\frac{1}{2}} \left(\int_{\Omega} |w(x,s)|^{\frac{\gamma+2}{2}.2} dx \right)^{\frac{2}{\gamma+2}\cdot\frac{1}{2}} ds$$

$$+ k_0 \int_{t_1}^{t_2} \|v_t(x,s)\|^{\gamma}_{L^{\gamma+2}} \left(\int_{\Omega} |w_t(x,s)|^{\frac{\gamma+2}{2}.2} dx \right)^{\frac{2}{\gamma+2}\cdot\frac{1}{2}} \left(\int_{\Omega} |w(x,s)|^{\frac{\gamma+2}{2}.2} dx \right)^{\frac{2}{\gamma+2}\cdot\frac{1}{2}} ds$$

$$= k_0 \int_{t_1}^{t_2} (mes(\Omega)^{\frac{\gamma}{\gamma+2}} + \|u_t(x,s)\|^{\gamma}_{L^{\gamma+2}} + \|v_t(x,s)\|^{\gamma}_{L^{\gamma+2}}) \|w_t(x,s)\|_{L^{\gamma+2}} \|w(x,s)\|_{L^{\gamma+2}} ds.$$

Due to the estimate (estU 3.5) and the Sobolev lemma we obtain

$$\le k_0(mes(\Omega)^{\frac{\gamma}{\gamma+2}} + M_1^\gamma + M_2^\gamma)\int_{t_1}^{t_2} \|w_t(s)\|_{L^{\gamma+2}}\|w(s)\|_{L^{\gamma+2}}ds$$

$$\le k_0(mes(\Omega)^{\frac{\gamma}{\gamma+2}} + M_1^\gamma + M_2^\gamma)\int_{t_1}^{t_2} \|w_t(s)\|_{L^{\gamma+2}}S_{\gamma+2}\|w(s)\|_{H_0^1(\Omega)}ds$$

$$\le k_0(mes(\Omega)^{\frac{\gamma}{\gamma+2}} + M_1^\gamma + M_2^\gamma)S_{\gamma+2}\int_{t_1}^{t_2} \|w_t(s)\|_{L^{\gamma+2}}\|w(s)\|_E ds$$

$$\le k_0(mes(\Omega)^{\frac{\gamma}{\gamma+2}} + M_1^\gamma + M_2^\gamma)S_{\gamma+2}\max_{s\in[t,t+1]}\|w(s)\|_E\int_{t_1}^{t_2} \|w_t(s)\|_{L^{\gamma+2}}ds$$

$$\le k_0(mes(\Omega)^{\frac{\gamma}{\gamma+2}} + M_1^\gamma + M_2^\gamma)S_{\gamma+2}\max_{s\in[t,t+1]}\|w(s)\|_E A(t).$$

Therefore,

$$\int_{t_1}^{t_2} \|w(s)\|_{H_0^1(\Omega)} \le C_0 A(t)^2 + C_1 A(t)\max_{s\in[t,t+1]}\|w(s)\|_E \qquad (3.9)$$ nak33

$$C_0 = mes(\Omega)^{\frac{\gamma}{\gamma+2}}$$

$$C_1 = 4S_2 mes(\Omega)^{\frac{\gamma}{\gamma+2}} + k_0 S_{\gamma+2}(mes(\Omega)^{\frac{\gamma}{\gamma+2}} + M_1^\gamma + M_2^\gamma).$$

We use (nak61 3.7) and get

$$\int_{t_1}^{t_2} \|w(s)\|_E dt = \int_{t_1}^{t_2} \|w(s)\|^2_{H_0^1(\Omega)}ds + \int_{t_1}^{t_2}\int_\Omega |w_t(s)|^2 ds$$

$$\le \int_{t_1}^{t_2} \|w(s)\|^2_{H_0^1(\Omega)}ds + mes(\Omega)^{\frac{\gamma}{\gamma+2}}\int_{t_1}^{t_2}\left(\int_\Omega |w_t(s)|^{\gamma+2}\right)^{\frac{2}{\gamma+2}}$$

$$\le \int_{t_1}^{t_2} \|w(s)\|^2_{H_0^1(\Omega)}ds + mes(\Omega)^{\frac{\gamma}{\gamma+2}}A(t)^2. \qquad (3.10)$$ nak34

By (nak33 3.9) and (nak34 3.10), we get

$$\int_{t_1}^{t_2} \|w(s)\|_E \le mes(\Omega)^{\frac{\gamma}{\gamma+2}}A(t)^2 + C_0A(t)^2 + C_1A(t)\max_{s\in[t,t+1]}\|w(s)\|_E$$

$$= 2C_0A(t)^2 + C_1A(t)\max_{s\in[t,t+1]}\|w(s)\|_E.$$

Using the mean value theorem for integration there exists $t^* \in [t_1, t_2]$ such that

$$\|w(t^*)\|_E \le 4C_0 A(t)^2 + 2C_1 A(t) \max_{s\in[t,t+1]} \|w(s)\|_E$$

When we multiply (nak6 3.6) by w_t and integrate $\Omega \times [t^*, s]$ (or $\Omega \times [t^*, s]$), we have

$$\pm\|w(s)\|_E \mp \|w(t^*)\|_E + \int_{t_1}^{t_2} (|u_t(s)|^\gamma u_t(s) - |v_t(s)|^\gamma v_t(s), w_t(s))ds = 0.$$

Then $\forall s \in [t, t+1]$ we obtain

$$\|w(s)\|_E^2 \le \|w(t^*)\|_E^2 + \int_t^{t+1} |(|u_t(s)|^\gamma u_t(s) - |v_t(s)|^\gamma v_t(s), w_t(s))|ds.$$

Thus,

$$\max_{s\in[t,t+1]} \|w(s)\|_E^2 \le \|w(t^*)\|_E^2 + \int_t^{t+1} |(|u_t(s)|^\gamma u_t(s) - |v_t(s)|^\gamma v_t(s), w_t(s))|ds. \quad (3.11)$$ nak36

Similar to what we did above we obtain

$$\int_t^{t+1} |(|u_t(s)|^\gamma u_t(s) - |v_t(s)|^\gamma v_t(s), w_t(s))|ds \le k_0 \int_t^{t+1} |1 + |u_t(s)|^\gamma + |v_t(s)|^\gamma||w_t(s)||w_t(s)|ds$$
$$\le k_0(mes(\Omega)^{\frac{\gamma}{\gamma+2}} + M_1^\gamma + M_2^\gamma)A(t)^2.$$

Then, (nak36 3.11) can be written in the form

$$\max_{s\in[t,t+1]} \|w(s)\|_E^2 \le \|w(t^*)\|_E^2 + C_2 A(t)^2$$
$$\le 4C_0 A(t)^2 + 2C_1 A(t) \max_{s\in[t,t+1]} \|w(s)\|_E + C_2 A(t)^2.$$

such that $C_2 = k_0(mes(\Omega)^{\frac{\gamma}{\gamma+2}} + M_1^\gamma + M_2^\gamma)$.
By Young's inequality, we have

$$2C_1 A(t) \max_{s\in[t,t+1]} \|w(s)\|_E \le 2C_1^2 A(t)^2 + \frac{1}{2} \max_{s\in[t,t+1]} \|w(s)\|_E^2 \quad (3.12)$$ nak37

We use (nak37 3.12) to obtain

$$\max_{s\in[t,t+1]} \|w(s)\|_E^2 \le (8C_0 + 2C_2 + 4C_1^2)A(t)^2.$$

Then,

$$\max_{s\in[t,t+1]}(\|w(s)\|_E^2)^{\frac{\gamma+2}{2}} \leq (8C_0+2C_2+4C_1^2)^{\frac{\gamma+2}{2}}\frac{1}{k_1}(\|w(t)\|_E^2-\|w(t+1)\|_E^2).$$

Let's take $K = \frac{k_1}{(8C_0+2C_2+4C_1^2)^{\frac{\gamma+2}{2}}}$ then,

$$K(\|w(s)\|_E^2)^{\frac{\gamma+2}{2}} \leq (\|w(t)\|_E^2 - \|w(t+1)\|_E^2) \quad ,\forall t.$$

If we take $\Phi = \|w(t)\|_E^2$ and $0 < \frac{\gamma+2}{2}$ by Nakao lemma 1, we get

$$\|w(t)\|_E \leq (\frac{\gamma}{2}K(t-1) + M^{-\gamma})^{-\frac{1}{\gamma}} for\ t \geq 1.$$

And if $0 = \frac{\gamma+2}{2}$ then by Nakao lemma 2 we get the result

$$\|w(t)\|_E \leq Me^{-K't} \quad for\ t \geq 1.$$

□

Chapter 4

An Equation of Nonlinear Vibration of Elastic String

In this chapter we study the asymptotic behavior of solutions to the initial boundary value problem for the equation of the nonlinear vibration of elastic string. We will show that the energy norm of the solution of this problem decays with an exponential rate as $t \to +\infty$. We also prove that the solution of the problem continuously depends on data.

The problem we consider in this chapter is:

$$u_{tt}(x,t) - (1 + \|\nabla u(t)\|^2)\Delta u(t)(x,t) + 2\gamma u_t(x,t) = f(x,t),\ x \in \Omega, t > 0, \quad (4.1) \quad \boxed{\texttt{ns1}}$$

$$u(x,t) = 0,\ x \in \partial\Omega, t > 0, \quad (4.2) \quad \boxed{\texttt{ns2}}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega. \quad (4.3) \quad \boxed{\texttt{ns3}}$$

We assume that the data u_0, u_1 and f are so smooth ,then the problem has a classical solution. We also assume that:

$$\int_0^\infty \|f(\tau)\|^2 d\tau < \infty, \quad (4.4) \quad \boxed{\texttt{ns9}}$$

$$\delta_0(t) = \left(\int_t^{t+1} \|f(\tau)\|^2 d\tau \right)^{\frac{1}{2}} \le C_0 e^{-\theta_0 t} \quad (4.5) \quad \boxed{\texttt{ns10}}$$

for some constants $C_0, \theta_0 > 0$,

$$\left(\int_0^\infty \|\Delta f(\tau)\|^2 d\tau \right)^{\frac{1}{2}} < \infty, \quad (4.6) \quad \boxed{\texttt{ns11}}$$

$$\delta_{00}(t) = \left(\int_{t}^{t+1} \|\Delta f(\tau)\|^2 d\tau \right)^{\frac{1}{2}} \leq C_{00} e^{-\theta_{00} t} \qquad (4.7)$$ ns12

for some constants $C_{00}, \theta_{00} > 0$.

4.1 Asymptotic behavior of Solutions of the Equation Nonlinear Vibration of an Elastic String

nis5L **Lemma 4.1.1.** *([7]) Suppose that the assumption (4.4) is satisfied. Then for solution of the problem (4.1) - (4.3) the following estimates hold true*

$$\|u_t(t)\|, \|\nabla u(t)\|, \int_0^{\infty} \|u_t(\tau)\|^2 d\tau \leq C_1, \ \forall t \geq 0,$$

where C_1 is constant independent of t.

Proof. Let us multiply the equation (4.1) by $2u_t$ in $L^2(\Omega)$

$$2(u_{tt}(t), u_t(t)) - 2(1 + \|\nabla u(t)\|^2)(u_t(t), \Delta u(t)) + 4\gamma \|u_t(t)\|^2 = 2(f(t), u_t(t)).$$

Using Young's inequality we get

$$\frac{d}{dt}\left(\|u_t(t)\|^2 + \|\nabla u(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^4 \right) + 4\gamma \|u_t(t)\|^2 \leq \frac{1}{2\gamma}\|f(t)\|^2 + 2\gamma \|u_t(t)\|^2 \qquad (4.8)$$ nis14

Denoting $E(t) = \|u_t(t)\|^2 + \|\nabla u(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^4$, and integrating (4.8) over the interval $[0, t]$ we obtain

$$E(t) + 2\gamma \int_0^t \|u_t(\tau)\|^2 d\tau \leq \frac{1}{2\gamma} \int_0^t \|f(\tau)\|^2 d\tau + E(0).$$

Thus the inequality implies

$$\|u_t(t)\|, \|\nabla u(t)\| \ and \int_0^{\infty} ||u_t(\tau)|^2 d\tau \leq C_1, \forall t \geq 0.$$

□

nis6L **Lemma 4.1.2.** *[7] Suppose that the assumptions (4.4) and (4.5) are satisfied. Then the solution of problem (4.1) - (4.3) satisfies*

$$\|u_t(t)\|, \|\nabla u(t)\| \leq C_2 \exp(-\theta_0' t) \ for\ some\ \theta_0' > 0, \ \forall t \geq 0. \qquad (4.9)$$ nis85

Proof. Integrating (4.8) over the interval $(t, t+1)$ we get

$$2\gamma \int_{t}^{t+1} \|u_t(\tau)\|^2 d\tau \leq \frac{1}{2\gamma} \int_{t}^{t+1} \|f(\tau)\|^2 d\tau + E(t) - E(t+1) \equiv B(t)^2 \qquad (4.10)$$

nis15

There exist two points $t_1 \in \left[t, t+\frac{1}{4}\right]$ and $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$2\gamma \|u_t(t_i)\|^2 \leq 4B(t)^2 \qquad i = 1, 2 \qquad (4.11)$$

nis16

Multiplying (4.1) by u in $L^2(\Omega)$, integrating the obtained inequality over $[t_1, t_2]$ and using Cauchy-Schwartz inequality , we get

$$\begin{aligned}\int_{t_1}^{t_2} \left(1 + \|\nabla u(\tau)\|^2\right) \|\nabla u(\tau)\|^2 d\tau &= \int_{t_1}^{t_2} (-(u_{tt}(\tau), u(\tau)) - 2\gamma(u_t(\tau), u(\tau)) + (f(\tau), u(\tau)) d\tau \\ &\leq |(u_t(t_1), u(t_1))| + |(u_t(t_2), u(t_2))| \\ &\quad + \int_{t_1}^{t_2} \|u_t(\tau)\|^2 d\tau + \int_{t_1}^{t_2} (2\gamma \|u_t(\tau)\| + \|f(\tau)\|) \|u(\tau)\| d\tau\end{aligned}$$

Due to inequalities (4.10) and (4.11), we obtain

$$\begin{aligned}&\leq 2\left(\frac{2}{\gamma}\right)^{\frac{1}{2}} B(t) \max_{\tau\in[t,t+1]} \|u(\tau)\| + \frac{B(t)^2}{2\gamma} + (2\,(2\gamma)^{\frac{1}{2}}\, B(t) + \delta_0(t)) \max_{\tau\in[t,t+1]} \|u(\tau)\| \\ &\leq (2(\left(\frac{2}{\gamma}\right)^{\frac{1}{2}} + (2\gamma)^{\frac{1}{2}}) B(t) + \delta_0(t)) \max_{\tau\in[t,t+1]} \|u(\tau)\| + \frac{B(t)^2}{2\gamma} \qquad (4.12)\end{aligned}$$

nis17

Adding (4.10) and (4.12) we obtain,

$$\int_{t_1}^{t_2} E(\tau) d\tau \leq C\left\{B(t) + \delta_0(t)\right\} \max_{\tau\in[t,t+1]} E(\tau)^{\frac{1}{2}} + B(t)^2.$$

So there exists $t^* \in [t_{1,} t_2]$ such that

$$E(t^*) \leq 2C\left\{B(t) + \delta_0(t)\right\} \max_{\tau\in[t,t+1]} E(\tau)^{\frac{1}{2}} + B(t)^2.$$

If we integrate (4.8) from t^* to s with $s \in [t, t+1]$, we get

$$\begin{aligned}E(s) &\leq E(t^*) - 2\gamma \int_{t^*}^{s} \|u_t(\tau)\|^2 d\tau + \frac{1}{2\gamma} \int_{t^*}^{s} \|f(\tau)\|^2 d\tau \\ &\leq C\left\{ (B(t) + \delta_0(t)) \max_{\tau\in[t,t+1]} E(\tau)^{\frac{1}{2}} + B(t)^2 + \delta_0(t)^2 \right\}.\end{aligned}$$

Using Young's inequality, we obtain

$$\max_{\tau\in[t,t+1]} E(\tau) \le C_\epsilon B(t)^2 + \epsilon \max_{\tau\in[t,t+1]} E(\tau) + C_\epsilon \delta_0(t)^2 + \epsilon \max_{\tau\in[t,t+1]} E(\tau) + B(t)^2 + \delta_0(t)^2. \quad (4.13)$$

Then (4.13) implies

$$\max_{\tau\in[t,t+1]} E(\tau) \le C(B(t)^2 + \delta_0(t)^2) \le C(E(t) - E(t+1)) + c\delta_0(t)^2.$$

By Lemma (1.1.27)

$$E(\tau) \le C_2 \exp(-\theta_0' t) \ for\ some\ \theta_0', C_2 > 0.$$

Hence we get

$$\|u_t(t)\|, \|\nabla u(t)\| \le C_2 \exp(-\theta_0' t) \ for\ some\ \theta_0', C_2 > 0.$$

□

Lemma 4.1.3. [7] *If we assume (4.6) in addition to the assumptions of Lemma 4.1.2 for the problem (4.1) - (4.3), we have*

$$\|\Delta u_t(t)\|, \|\Delta\nabla u(t)\|, \int_0^\infty \|\Delta u_t(\tau)\|^2 d\tau \le C_3$$

Proof. Multiplying (4.1) by $2\Delta^2 u_t$ in $L^2(\Omega)$ we get

$$\frac{d}{dt}\left(\|\Delta u_t(t)\|^2\right) + \frac{d}{dt}\left(\|\Delta\nabla u(t)\|^2\right) + \frac{d}{dt}\left(\|\nabla u(t)\|^2\|\Delta\nabla u(t)\|^2\right) - 2(\nabla u(t), \nabla u_t(t))(\|\Delta\nabla u(t)\|^2) + 4\gamma\|\Delta u_t(t)\|^2 = 2(\Delta f(t), \Delta u_t(t)). \quad (4.14)$$

Due to the inequality (1.5) and Poincare-Friedrich's inequality we have

$$\|\nabla u_t(t)\| \le C\|\Delta u_t(t)\| \quad (4.15)$$

By using Young's inequality, Cauchy- Schwartz inequality and (4.15) the equation (4.14) can be written as

$$\begin{aligned}\frac{d}{dt}E_0(t) + 2\gamma\|\Delta u_t(t)\|^2 &\le \frac{1}{2\gamma}\|\Delta f(t)\|^2 + 2(\nabla u(t), \nabla u_t(t))(\|\Delta\nabla u(t)\|^2)\\ &\le \frac{1}{2\gamma}\|\Delta f(t)\|^2 + C\|\nabla u(t)\|.\|\nabla u_t(t)\|.E_0(t)\\ &\le \frac{1}{2\gamma}\|\Delta f(t)\|^2 + C\|\nabla u(t)\|E_0(t)^{\frac{1}{2}}.E_0(t) \quad (4.16)\end{aligned}$$

where

$$E_0(t) = \|\Delta u_t(t)\|^2 + \|\Delta\nabla u(t)\|^2 + \|\nabla u(t)\|^2\|\Delta\nabla u(t)\|^2.$$

Integrating (4.16) and using (4.15) , (4.9) we get

$$E_0(t) + 2\gamma\int\limits_0^t \|\Delta u_t(\tau)\|^2 d\tau \le E_0(0) + \frac{1}{2\gamma}\int\limits_0^\infty \|\Delta f(\tau)\|^2 d\tau + C\int\limits_0^t \exp(-\theta_1'\tau)\,(E_0(\tau))^{\frac{1}{2}}\,E_0(\tau)d\tau.$$

By the Bihari Lemma 1.1.21 we have

$$E_0(t) \le G^{-1}\left(C\int\limits_0^t \exp(-\theta_1'\tau)d\tau\right) < \infty,$$

where $G(\eta) = \int\limits_k^\eta \frac{ds}{s^{\frac{3}{2}}}$, and $k = E_0(0) + \frac{1}{2\gamma}\int\limits_0^\infty \|\Delta f(\tau)\|^2 d\tau < \infty$.
Thus $E_0(t) \le C_3$. Hence,

$$\|\Delta u_t(t)\|, \|\Delta\nabla u(t)\|, \int\limits_0^\infty \|\Delta u_t(t)\|^2 dt \le C_3.$$

□

Theorem 4.1.4. [7] *Suppose that (4.4),(4.5),(4.6) and (4.7) are satisfied. Then the following estimates for the solutions of (4.1) - (4.3) holds true:*

$$\|\Delta u_t(t)\|, \|\Delta\nabla u(t)\| \le C_{00}e^{-\theta_{00}t} \quad \textit{for some } C_{00}, \theta_{00} > 0. \qquad (4.17)$$

Proof. Integrating (4.16) over the interval $(t, t+1)$ and using (4.15) we get,

$$2\gamma\int\limits_t^{t+1} \|\Delta u_t(\tau)\|^2 d\tau \le C\int\limits_t^{t+1} \exp(-\theta_1'\tau)\,(E_0(\tau))^{\frac{1}{2}}\,E_0(\tau)d\tau + \frac{1}{2\gamma}\int\limits_t^{t+1} \|\Delta f(\tau)\|^2 d\tau + E_0(t) - E_0(t+1). \qquad (4.18)$$

By Lemma 4.1.3 ,relations (4.7) and (4.18) we have

$$2\gamma \int_t^{t+1} \|\Delta u_t(\tau)\|^2 d\tau \le C\delta_{00}^2(t) + C\exp(-\theta_1'\tau) + E_0(t) - E_0(t+1) \equiv B_0(t)^2 \tag{4.19}$$

nis20

Therefore, there exist two points $t_1 \in \left[t, t+\frac{1}{4}\right]$ and $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$2\gamma\|\Delta u_t(t_i)\|^2 \le 4B_0(t)^2, \quad i = 1,2 \tag{4.20}$$

nis21

Multiplying (4.1) by $\Delta^2 u(x,t)$ in $L^2(\Omega)$ we get

$$2\left(1+\|\nabla u(t)\|^2\right)\|\Delta\nabla u(t)\|^2 = -2\frac{d}{dt}\left(\Delta u(t), \Delta u_t(t)\right) + 2\|\Delta u_t(t)\|^2 - 4\gamma\left(\Delta u(t), \Delta u_t(t)\right) + 2\left(\Delta u(t), \Delta f(t)\right) \tag{4.21}$$

nis22

Integrating (4.21) over (t_1, t_2) and using Cauchy- Schwartz inequality, we get

$$\int_{t_1}^{t_2} 2\left(1+\|\nabla u(\tau)\|^2\right)\|\Delta\nabla u(\tau)\|^2 d\tau \le 2|\Delta u(t_1), \Delta u_t(t_1)| + 2|\Delta u(t_2), \Delta u_t(t_2)|$$

$$+2\int_{t_1}^{t_2} \|\Delta u_t(\tau)\|^2 d\tau + 4\gamma\int_{t_1}^{t_2} \|\Delta u(\tau)\|\|\Delta u_t(\tau)\| d\tau + 2\int_{t_1}^{t_2} \|\Delta u(\tau)\|\|\Delta f(\tau)\| d\tau.$$

By (4.19) and (4.20),we obtain

$$\le CB_0(t)\max_{\tau\in[t,t+1]}\|\Delta u(\tau)\| + CB_0(t)^2 + \int_{t_1}^{t_2}\left(4\gamma\|\Delta u_t(\tau)\| + 2\|\Delta f(\tau)\|\right)\|\Delta u(\tau)\| d\tau$$

$$\le CB_0(t)\max_{\tau\in[t,t+1]}\|\Delta u(\tau)\| + CB_0(t)^2 + C(B_0(t)+\delta_{00}(t))\max_{\tau\in[t,t+1]}\|\Delta u(\tau)\|. \tag{4.22}$$

nis23

Adding (4.19) and (4.22) we obtain

$$E_0(t) \le C(B_0(t)+\delta_{00}(t))\max_{\tau\in[t,t+1]}\|\Delta u(\tau)\| + CB_0(t)^2,$$

$$\max_{\tau\in[t,t+1]} E_0(t) \le C(B_0(t)+\delta_{00}(t))\max_{\tau\in[t,t+1]} E_0(t)^{\frac{1}{2}} + CB_0(t)^2.$$

By Young inequality and definition of $B_0(t)^2$, we get

$$\max_{\tau\in[t,t+1]} E_0(t) \leq C\delta_{00}(t)^2 + C\exp(-\theta_1'\tau) + E_0(t) - E_0(t+1)$$

By Nakao Lemma (nis4L 1.1.27), we get

$$E_0(t) \leq C_{00}\exp(-\theta_0 t) \quad \textit{for some } C_{00}, \theta_0 > 0$$

Hence,

$$\|\Delta u_t(t)\|, \|\Delta\nabla u(t)\| \leq C_{00}e^{-\theta_0 t}.$$

□

4.1.1 Continuous Dependence on Data

Let us consider the problem

$$U_{tt}(x,t) - (1+\|\nabla U(t)\|^2)\Delta U(x,t) + 2\gamma U_t(x,t) = h(x,t), \tag{4.23}$$ ns4

$$U(x,t) = 0 \quad on \quad \partial G, \tag{4.24}$$ ns5

$$U(x,0) = U_0, U_t(x,0) = U_1. \tag{4.25}$$ ns6

Assume that $f(x,t) = h(x,t) = 0$ on $\partial\Omega$.

nis2T **Theorem 4.1.5.** *Nis1 [6] Let U be the solution of the problem (ns4 4.23)- (ns6 4.25) and u be the solution of the problem (ns1 4.1)-(ns3 4.3).*

If $(u_0 - U_0, u_1 - U_1, f-h)$ is sufficiently small i.e

$$\|\Delta(u_0-U_0)\|^2 + \|\nabla(u_1-U_1)\|^2 + \int_0^\infty \|\nabla(f-h)(\tau)\|^2 d\tau \leq \epsilon^2 \ \textit{forsome } \epsilon > 0, \tag{4.26}$$ nis25

then the following estimates hold true:

$$\|\nabla(u_t(t)-U_t(t))\|^2 + \|\Delta(u(t)-U(t))\|^2 \leq C\epsilon^2, \forall t \geq 0.$$

$$\int_0^t \|\nabla(u_t(\tau)-U_t(\tau))\|^2 d\tau + \int_0^t \|\Delta(u(\tau)-U(\tau))\|^2 d\tau \leq C\epsilon^2, \forall t \geq 0.$$

Proof. By Lemma 4.1.1 we have

$$\|u_t(t)\|, \|\nabla u(t)\|, \int\limits_0^t \|u_t(\tau)\|^2 d\tau \le C, \forall t \ge 0$$

and

$$\|U_t(t)\|, \|\nabla U(t)\|, \int\limits_0^t \|U_t(\tau)\|^2 d\tau \le C, \forall t \ge 0.$$

Then $\|v(t)\| = \|u(t) - U(t)\| \le \|u(t)\| + \|U(t)\|$. Then,

$$\|v_t(t)\|, \|\nabla v(t)\|, \int\limits_0^t \|v_t(\tau)\|^2 d\tau \le C, \forall t \ge 0,$$

where $v(t) = u(t) - U(t)$.

Let $u(x,t)$ be a solution of (4.1) - (4.3). Then

$$(U_{tt}(x,t)+v_{tt}(x,t))-(1+\|\nabla U(t)+\nabla v(t)\|^2)\Delta(U(x,t)+v(x,t))+2\gamma(U_t x,(t)+v_t(x,t)) = f(x,t).$$

By (4.23), we obtain

$$\begin{aligned} v_{tt}(x,t) - (1 + \|\nabla v(t)\|^2 + \|\nabla U(t)\|^2)\Delta v(x,t) + 2\gamma v_t(x,t) \\ = 2(\nabla U(t), \nabla v(t))\Delta v(x,t) \\ + (2(\nabla U(t), \nabla v(t)) + \|\nabla v(t)\|^2)\Delta U(x,t) + g(x,t), \quad (4.27) \end{aligned}$$

nis27

where $g(x,t) = f(x,t) - h(x,t), \quad v_0 = u_0 - U_0, \quad v_1 = u_1 - U_1.$

Multiplying (4.27) by $2v_t$ and integrate over Ω, we get

$$\begin{aligned} \frac{d}{dt}(\|v_t(t)\|^2 + \|\nabla v(t)\|^2 + \frac{1}{2}\|\nabla v(t)\|^4 + \|\nabla U(t)\|^2\|\nabla v(t)\|^2) + 4\gamma\|v_t(t)\|^2 = \\ 2(\nabla U(t), \nabla U_t(t))\|\nabla v(t)\|^2 - 2\frac{d}{dt}((\nabla U(t), \nabla v(t))\|\nabla v(t)\|^2) \\ + 2((\nabla U_t(t), \nabla v(t)) + (\nabla U(t), \nabla v_t(t)))\|\nabla v(t)\|^2 \\ + 2((\nabla U(t), \nabla v(t)) + \|\nabla v(t)\|^2)(\Delta U(t), v_t(t)) + 2(g(t), v_t(t)). \quad (4.28) \end{aligned}$$

nis28

Integrate both sides of the (4.28) from 0 to t, since $-(\nabla U, \nabla v_t) =$

$(\Delta U, v_t)$ we get,

$$(\|v_t(t)\|^2+\|\nabla v(t)\|^2+\frac{1}{2}\|\nabla v(t)\|^4+\|\nabla U(t)\|^2\|\nabla v(t)\|^2)$$

$$-(\|v_t(0)\|^2+\|\nabla v(0)\|^2+\tfrac{1}{2}\|\nabla v(0)\|^4+\|\nabla U(0)\|^2\|\nabla v(0)\|^2)+4\gamma\int_0^t\|v_t(\tau)\|^2d\tau$$

$$=2\int_0^t(\nabla U(\tau),\nabla U_t(\tau))\|\nabla v(\tau)\|^2d\tau-2(\nabla U(t),\nabla v(t))\|\nabla v(t)\|^2+2(\nabla U(0),\nabla v(0))\|\nabla v(0)\|^2$$

$$+2\int_0^t(\nabla U_t(\tau),\nabla v(\tau))\|\nabla v(\tau)\|^2d\tau+2\int_0^t(\nabla U(\tau),\nabla v(\tau))(\Delta U(\tau),v_t(\tau))d\tau+2\int_0^t(g(\tau),v_t(\tau))d\tau.$$

By Hölder inequality and Young's inequality we can write our equation as:

$$\|\nabla v_t(t)\|^2+\|\nabla v(t)\|^2+\frac{1}{2}\|\nabla v(t)\|^4+\|\nabla U(t)\|^2\|\nabla v(t)\|^2+4\gamma\int_0^t\|\nabla v_t(\tau)\|^2d\tau$$

$$\leq\int_0^t(\|\nabla U(\tau)\|^2+\|\nabla U_t(\tau)\|^2)(\|\nabla v(\tau)\|^2+2\|\nabla v_t(\tau)\|^2)d\tau+(\|\nabla U(t)\|^2+\|\nabla v(t)\|^2)\|\nabla v(t)\|^2$$

$$+2(\|\nabla U(0)\|.\|\nabla v(0)\|)(\|\nabla v(0)\|^2+2\|v_t(0)\|^2)+(\|\nabla v(0)\|^2+2\|v_t(0)\|^2)$$

$$+(\frac{1}{2}\|\nabla v(0)\|^2+\|\nabla U(0)\|^2)(\|\nabla v(0)\|^2+2\|v_t(0)\|^2)$$

$$+2\int_0^t\|\nabla U_t(\tau)\|.\|\nabla v(\tau)\|(\|\nabla v(\tau)\|^2+2\|v_t(\tau)\|^2)d\tau$$

$$+\int_0^t\|\nabla U(\tau)\|.\|\Delta U(\tau)\|(\|\nabla v(\tau)\|^2+2\|v_t(\tau)\|^2)d\tau$$

$$+2\gamma\int_0^t\|v_t(\tau)\|^2d\tau+\frac{1}{2\gamma}\int_0^t\|g(\tau)\|^2d\tau.$$

Denote $E_1(v(t))=(\frac{1}{2}\|\nabla v(t)\|^2+\|\nabla v_t(t)\|^2)$.
By (nis100 4.17)

$$\|v_t(t)\|,\|\nabla v(t)\|,\int_0^t\|v_t\|^2d\tau,\|U_t(t)\|,\|\nabla U(t)\|,\int_0^t\|U_t(\tau)\|^2d\tau\leq C \quad (4.29)$$ nis75

we have

$$E_1(v(t)) + \frac{1}{2}(1 - \|\nabla v(t)\|^2)\|\nabla v(t)\|^2 + 2\gamma \int\limits_0^t \|v_t(\tau)\|^2 d\tau$$

$$\leq C\left(E_1(v(0)) + \int\limits_0^t \|g(\tau)\|^2 d\tau\right) + C\int\limits_0^t c(\tau)E_1(v(\tau))d\tau \quad (4.30)$$ nis30

where $c(\tau) = Ce^{-\theta t},\ \theta > 0$. It follows from (nis30 4.30) that for each $t \leq t^* := \sup\left\{t : \|\nabla v(t)\|^2 \leq 1\right\}$ the following inequality holds true

$$E_1(v(t)) \leq C\left(E_1(v(0)) + \int\limits_0^t \|g(\tau)\|^2 d\tau\right) + C\int\limits_0^t c(\tau)E_1(v(\tau))d\tau.$$

By Lemma ClasGron 1.1.22 Gronwall's inequality and (nis25 4.26) we have

$$E_1(v(t)) \leq C\left(E_1(v(0)) + \int\limits_0^t \|g(\tau)\|^2 d\tau\right) \exp\left(C\int\limits_0^t c(\tau)d\tau\right) \leq C\epsilon^2.$$

If we choose ϵ sufficiently small i.e. $C\epsilon^2 < \frac{1}{4}$, then,

$$E_1(v(t)) = \|v_t(t)\|^2 + \frac{1}{2}\|\nabla v(t)\|^2 < \frac{1}{4} \implies \|\nabla v(t)\|^2 < \frac{1}{2}.$$

Thus for sufficiently small ϵ's $t^* = \infty$.
For sufficiently small ϵ's by (nis30 4.30) and (nis25 4.26) we have

$$\|v_t(t)\|^2 + \|\nabla v(t)\|^2 + 2\gamma\int\limits_0^t \|v_t(t)\|^2 dt \leq C\left(E_1(v(0)) + \int\limits_0^t \|g(\tau)\|^2 d\tau\right)$$

$$\leq C\epsilon^2. \quad (4.31)$$ nis31

Multiply (nis27 4.27) by $2\Delta v_t$ and integrate over Ω we obtain,

$$-\frac{d}{dt}(\|\nabla v_t(t)\|^2 + \|\Delta v(t)\|^2 + \|\nabla v(t)\|^2\|\Delta v(t)\|^2 + \|\nabla U(t)\|^2\|\Delta v(t)\|^2) - 4\gamma\|\nabla v_t(t)\|^2$$
$$+ 2(\nabla v(t), \nabla v_t(t))\|\Delta v(t)\|^2 + 2(\nabla U(t), \nabla U_t(t))\|\Delta v(t)\|^2$$
$$= 2\frac{d}{dt}((\nabla U(t), \nabla v(t))\|\Delta v(t)\|^2) - 2((\nabla U_t(t), \nabla v(t)) + (\nabla U(t), \nabla v_t(t)))\|\Delta v(t)\|^2$$
$$+ 2(2(\nabla U(t), \nabla v(t)) + \|\nabla v(t)\|^2)(\Delta U(t), \Delta v_t(t)) + 2(g(t), \Delta v_t(t)). \quad (4.32)$$ nis70

Since $\|\nabla v(t)\|^2(\Delta U(t), \Delta v_t(t)) = -(\nabla U(t), \nabla v_t(t))\|\Delta v(t)\|^2$, we integrate (nis70 4.32) over the interval $[0,t]$ and using Cauchy -Schwartz inequality ,we get

$$\|\nabla v_t(t)\|^2+\|\Delta v(t)\|^2+\|\nabla v(t)\|^2\|\Delta v(t)\|^2+\|\nabla U(t)\|^2\|\Delta v(t)\|^2+4\gamma\int_0^t \|\nabla v_t(\tau)\|^2 d\tau$$

$$-\left(\|\nabla v_t(0)\|^2+\|\Delta v(0)\|^2+\|\nabla v(0)\|^2\|\Delta v(0)\|^2+\|\nabla U(0)\|^2\|\Delta v(0)\|^2\right)$$

$$\leq 2\int_0^t \|\nabla v(\tau)\|\|\nabla v_t(\tau)\|\|\Delta v(\tau)\|^2 d\tau + 2\int_0^t \|\nabla U(\tau)\|\|\nabla U_t(\tau)\|\|\Delta v(\tau)\|^2 d\tau$$

$$-2(\nabla U(t), \nabla v(t))\|\Delta v(t)\|^2+2(\nabla U(0), \nabla v(0))\|\Delta v(0)\|^2+2\int_0^t \|\nabla U_t(\tau)\|\|\nabla v(\tau)\|\|\Delta v(\tau)\|^2 d\tau$$

$$-4\int_0^t (\nabla U(\tau), \nabla v(\tau))(\Delta U(\tau), \Delta v_t(\tau))d\tau - 2\int_0^t (\nabla g(\tau), \nabla v_t(\tau))d\tau.$$

By Cauchy-Schwartz inequality, Young's inequality and the identity

$$(\nabla U, \nabla v)(\Delta U, \Delta v_t) = -(\Delta U, \Delta v)(\nabla U, \nabla v_t)$$

we have

$$\|\nabla v_t(t)\|^2+\|\Delta v(t)\|^2+\|\nabla v(t)\|^2\|\Delta v(t)\|^2+\|\nabla U(t)\|^2\|\Delta v(t)\|^2+4\gamma\int_0^t \|\nabla v_t(\tau)\|^2 d\tau$$

$$\leq 2\int_0^t \|\nabla v(\tau)\|\|\nabla v_t(\tau)\|\|\Delta v(\tau)\|^2 d\tau+2\int_0^t \|\nabla U(\tau)\|\|\nabla U_t(\tau)\|(\|\nabla v_t(\tau)\|^2+\|\Delta v(\tau)\|^2)d\tau$$

$$+(\|\nabla U(t)\|^2+\|\nabla v(t)\|^2)\|\Delta v(t)\|^2+\|\nabla v_t(0)\|^2+\|\Delta v(0)\|^2+\|\nabla v(0)\|^2\|\Delta v(0)\|^2$$

$$+\|\nabla U(0)\|^2\|\Delta v(0)\|^2+(\|\nabla U(0)\|^2+\|\nabla v(0)\|^2)\|\Delta v(0)\|^2$$

$$+2\int_0^t \|\nabla U_t(\tau)\|\|\nabla v(\tau)\|(\|\nabla v_t(\tau)\|^2+\|\Delta v(\tau)\|^2)d\tau$$

$$+2\int_0^t \|\Delta U(\tau)\|\|\nabla U(\tau)\|(\|\nabla v_t(\tau)\|^2+\|\Delta v(\tau)\|^2)d\tau$$

$$+\frac{1}{2\gamma}\int_0^t \|\nabla g(\tau)\|^2 d\tau+2\gamma\int_0^t \|\nabla v_t(\tau)\|^2 d\tau. \quad (4.33)$$ nis1000

Taking $E_2(v(t)) = \|\nabla v_t(t)\|^2 + \|\Delta v(t)\|^2$ we obtain

$$
\begin{aligned}
&\|\nabla v_t(0)\|^2 + \|\Delta v(0)\|^2 + 2\|\nabla v(0)\|^2\|\Delta v(0)\|^2 + 2\|\nabla U(0)\|^2\|\Delta v(0)\|^2 \\
&\quad \le \|\nabla v_t(0)\|^2 + \|\Delta v(0)\|^2 + 2\|\nabla v(0)\|^2(\|\Delta v(0)\|^2 + \|\Delta v_t(0)\|^2) \\
&\qquad + 2\|\nabla U(0)\|^2(\|\Delta v(0)\|^2 + \|\Delta v_t(0)\|^2) \\
&\qquad \le CE_2(v(0)).
\end{aligned} \tag{4.34}
$$

nis71

By ([nis100]4.17),([nis71]4.34) and ([nis75]4.29) , ([nis1000]4.33)can be written in the form

$$
\begin{aligned}
&E_2(v(t)) + \|\nabla v(t)\|^2\|\Delta v(t)\|^2 + \|\nabla U(t)\|^2\|\Delta v(t)\|^2 + 2\gamma \int_0^t \|\nabla v_t(\tau)\|^2 d\tau \\
&\quad \le C\left(E_2(v(0)) + \int_0^\infty \|\nabla g(\tau)\|^2 d\tau\right) + \int_0^t C\epsilon\|\nabla v_t(\tau)\|\|\Delta v(\tau)\|^2 d\tau \\
&\qquad + (\|\nabla U(t)\|^2 + \|\nabla v(t)\|^2)\|\Delta v(t)\|^2 + \int_0^t Cc(\tau)E_2(v(\tau))d\tau.
\end{aligned}
$$

Then we get,

$$
\begin{aligned}
&E_2(v(t)) + 2\gamma \int_0^t \|\nabla v_t(\tau)\|^2 d\tau - \int_0^t C\epsilon\|\nabla v_t(\tau)\|.\|\Delta v(\tau)\|^2 d\tau \\
&\quad \le C\left(E_2(v(0)) + \int_0^\infty \|\nabla g(\tau)\|^2 d\tau\right) + \int_0^t Cc(\tau)E_2(v(\tau))d\tau.
\end{aligned} \tag{4.35}
$$

nis32

Next, multiplying ([nis27]4.27) by Δv and integrate over Ω we obtain

$$
\begin{aligned}
&-\frac{d}{dt}(v_t(t), \Delta v(t)) - \|\nabla v_t(t)\|^2 + (1 + \|\nabla v(t)\|^2 + \|\nabla U(t)\|^2)\|\Delta v(t)\|^2 + \gamma\frac{d}{dt}\|\nabla v(t)\|^2 \\
&= -2(\nabla U(t), \nabla v(t))\|\Delta v\|^2 - (g(t), \Delta v(t)) - 2(\nabla U(t), \nabla v(t)) + \|\nabla v(t)\|^2)(\Delta U(t), \Delta v(t)) \\
&\le 2\|\nabla U(t)\|\|\nabla v(t)\|(\|\Delta v(t)\|^2 + \|\nabla v_t(t)\|^2) + 2\| U(t)\|.\|\Delta U(t)\|(\|\Delta v(t)\|^2 + \|\nabla v_t(t)\|^2) \\
&\quad + (\|\nabla U(t)\|^2 + \|\nabla v(t)\|^2)\|\Delta v(t)\|^2 + \frac{1}{2}\|g(t)\|^2 + \frac{1}{2}\|\Delta v(t)\|^2.
\end{aligned} \tag{4.36}
$$

nis76

If we integrate (nis76 4.36) from 0 to t, we have

$$\int_0^t \frac{1}{2}\|\Delta v(\tau)\|^2 d\tau \leq -(v_t(0),\Delta v(0))+(v_t(t),\Delta v(t))+\frac{\gamma}{2}\|\nabla v(t)\|^2+\gamma\|\nabla v(0)\|^2$$

$$+\int_0^t Cc(\tau)E_2(v(\tau))d\tau+\frac{1}{2}\int_0^t \|g(\tau)\|^2 d\tau+\int_0^t \|\nabla v_t(\tau)\|^2 d\tau$$

$$\leq \frac{1}{2}\|\nabla v_t(0)\|^2+(\frac{1}{2}+\gamma)\|\nabla v(0)\|^2+\frac{\gamma}{2}\|\nabla v(t)\|^2+\frac{1}{2\gamma}\|\nabla v_t(t)\|^2$$

$$+\int_0^t Cc(\tau)E_2(v(\tau))d\tau+\frac{1}{2}\int_0^t \|g(\tau)\|^2 d\tau+\int_0^t \|\nabla v_t(\tau)\|^2 d\tau.$$

Then we have

$$\int_0^t \frac{1}{2}\|\Delta v(\tau)\|^2 d\tau \leq C\left(E_2v(0)+\int_0^\infty \|g(t)\|^2 dt\right)+\frac{\gamma}{2}\|\nabla v(t)\|^2+\frac{1}{2\gamma}\|\nabla v_t(t)\|^2$$

$$+\int_0^t Cc(\tau)E_2(v(\tau))d\tau \ +\int_0^t \|\nabla v_t(\tau)\|^2 d\tau. \quad (4.37) \quad \text{nis33}$$

If we multiply (nis33 4.37) by γ and add right hand side $\frac{1}{2}\|\Delta v(t)\|^2$ we have

$$\frac{\gamma}{2}\int_0^t \|\Delta v(\tau)\|^2 d\tau \leq C\gamma\left(E_2v(0)+\int_0^\infty \|g(\tau)\|^2 d\tau\right)+\frac{\gamma^2}{2}\|\nabla v(t)\|^2+\gamma\int_0^t Cc(\tau)E_2(v(\tau))d\tau$$

$$+\gamma\int_0^t \|\nabla v_t(\tau)\|^2 d\tau+\frac{1}{2}E_2(v(t)). \quad (4.38) \quad \text{nis34}$$

Adding (nis34 4.38) to (nis32 4.35) , using Poincare inequality to function $g(t)$ and (nis31 4.31), we get

$$\frac{1}{2}E_2(v(t))+\gamma\int_0^t \|\nabla v_t(\tau)\|^2 d\tau+\int_0^t (\frac{\gamma}{2}-C\epsilon\|\nabla v_t(\tau)\|)\|\Delta v(\tau)\|^2 d\tau$$

$$\leq C\left(E_2v(0)+\epsilon^2+\int_0^\infty \|\nabla g(t)\|^2 dt\right)+\int_0^t Cc(\tau)E_2(v(\tau))d\tau. \quad (4.39) \quad \text{nis400}$$

If $t^{**} = \sup\left\{t : \|\nabla v_t(t)\| \leq \frac{\gamma}{2C\epsilon}\right\} > 0$, for $t < t^{**}$ (4.39) is true. So we get

$$E_2(v(t)) \leq C\left(E_2 v(0) + \epsilon^2 + \int\limits_0^\infty \|\nabla g(t)\|^2 dt\right) + \int\limits_0^t Cc(\tau) E_2(v(\tau)) d\tau.$$

By (4.26) and lemma(1.1.22) Gronwall's inequality

$$E_2(v(t)) \leq C\left(E_2 v(0) + \epsilon^2 + \int\limits_0^\infty \|\nabla g(t)\|^2 d\tau\right) \exp \int\limits_0^t Cc(\tau) d\tau \leq C\epsilon^2 \tag{4.40}$$

nis36

If we choose ϵ sufficiently small, then $E_2(v(t)) = \|\nabla v_t(t)\|^2 + \|\Delta v(t)\|^2 \leq \frac{\gamma}{4C\epsilon}$, then $\|\nabla v_t(t)\| < \sqrt{C}\epsilon < \frac{\gamma}{4C\epsilon}$, which means for sufficiently small ϵ , $t^{**} = \infty$.

By (4.40)and (4.26) the inequality (4.35) can be written as:

$$\begin{aligned} &E_2(v(t)) + 2\gamma \int\limits_0^t \|\nabla v_t(\tau)\|^2 d\tau \\ &\quad \leq C\left(E_2(v(0)) + C\epsilon^2 + C\int\limits_0^\infty \|\nabla g(t)\|^2 dt + \int\limits_0^t Cc(\tau) E_2(v(\tau))\right) d\tau \\ &\quad \leq C\epsilon^2 \end{aligned} \tag{4.41}$$

nis37

By (4.38) and (4.41) we have

$$\int\limits_0^t \|\Delta v(\tau)\|^2 d\tau \leq C\epsilon^2 \quad . \tag{4.42}$$

nis38

Hence we get the results:

$$\|\nabla v_t(t)\|^2 + \|\Delta v(t)\|^2 \leq C\epsilon^2$$

$$\int\limits_0^t \|\nabla v_t(\tau)\|^2 d\tau + \int\limits_0^t \|\Delta v(\tau)\|^2 d\tau \leq C\epsilon^2$$

□

[6] *Let U be the solution of the problem (4.23)- (4.25) and u be the solution of the problem (4.1)-(4.3). If $(u_0 - U_0, u_1 - U_1, f - h)$ is sufficiently*

small in the sense (nis25 4.26) *and*

$$\left(\int_t^{t+1} \|\nabla(f-h)(\tau)\|^2 d\tau\right)^{\frac{1}{2}} \leq C_2 e^{-\theta' t} \text{ for } \theta' > 0 \qquad (4.43)$$ nis39

then we have:

$$\|\nabla(u(t)-U(t))\|^2 + \|u_t(t)-U_t(t)\|^2 \leq C_3 e^{-\theta'' t}.$$

nis3T **Theorem 4.1.6.** *Proof.* Integrating (nis28 4.28) over the interval $[t, t+1]$ we get

$$\begin{aligned}
&\|v_t(t+1)\|^2 + \|\nabla v(t+1)\|^2 + \frac{1}{2}\|\nabla v(t+1)\|^4 + \|\nabla U(t+1)\|^2 \|\nabla v(t+1)\|^2 \\
&\quad - (\|v_t(t)\|^2 + \|\nabla v(t)\|^2 + \frac{1}{2}\|\nabla v(t)\|^4 + \|\nabla U(t)\|^2\|\nabla v(t)\|^2) \\
&\quad + 4\gamma \int_t^{t+1} \|v_t(\tau)\|^2 d\tau = 2\int_t^{t+1} (\nabla U(\tau), \nabla U_t(\tau))\|\nabla v(\tau)\|^2 d\tau \\
&\quad - 2(\nabla U(t+1), \nabla v(t+1))\|\nabla v(t+1)\|^2 + 2(\nabla U(t), \nabla v(t))\|\nabla v(t)\|^2 \\
&+2\int_t^{t+1} (\nabla U_t(\tau), \nabla v(\tau))\|\nabla v(\tau)\|^2 d\tau + 2\int_t^{t+1} (\nabla U(\tau), \nabla v(\tau))(\Delta U(\tau), v_t(\tau)) d\tau + 2\int_t^{t+1} (g(\tau), v_t(\tau)) d\tau.
\end{aligned}$$

By the Hölder inequality and Young's inequality

$$\|v_t(t+1)\|^2 + \|\nabla v(t+1)\|^2 + \frac{1}{2}\|\nabla v(t+1)\|^4$$
$$- (\|v_t(t)\|^2 + \|\nabla v(t)\|^2 + \frac{1}{2}\|\nabla v(t)\|^4) + 4\gamma \int\limits_t^{t+1} \|v_t(\tau)\|^2 d\tau$$
$$\leq \|\nabla U(t)(\tau)\|^2 \|\nabla v(t)(\tau)\|^2 + \int\limits_t^{t+1} (\|\nabla U(\tau)\|^2 + \|\nabla U_t(\tau)\|^2)\|\nabla v(\tau)\|^2 d\tau$$
$$+ 2\|\nabla U(t+1)\|\|\nabla v(t+1)\|\|\nabla v(t+1)\|^2$$
$$+ 2\|\nabla U(t)\|\|\nabla v(t)\|\|\nabla v(t)\|^2 + 2\int\limits_t^{t+1} (\|\nabla U_t(\tau)\|\|\nabla v(\tau)\|)\|\nabla v(\tau)\|^2 d\tau$$
$$+2\int\limits_t^{t+1} \|\nabla U(\tau)\|\|\Delta U(\tau)\|\|\nabla v(\tau)\|\|v_t(\tau)\| d\tau \frac{1}{2\gamma}\int\limits_t^{t+1} \|g(\tau)\|^2 d\tau + 2\gamma \int\limits_t^{t+1} \|v_t(\tau)\|^2 d\tau$$
$$\leq \|\nabla U(t)\|^2 \|\nabla v(t)\|^2 + \int\limits_t^{t+1} (\|\nabla U(\tau)\|^2 + \|\nabla U_t(\tau)\|^2)\|\nabla v(\tau)\|^2 d\tau$$
$$+ 2\|\nabla U(t+1)\|\|\nabla v(t+1)\|\|\nabla v(t+1)\|^2$$
$$+ 2\|\nabla U(t)\|\|\nabla v(t)\|\|\nabla v(t)\|^2 + 2\int\limits_t^{t+1} (\|\nabla U_t(\tau)\|\|\nabla v(\tau)\|)\|\nabla v(\tau)\|^2 d\tau$$
$$+\int\limits_t^{t+1} \|\nabla U(\tau)\|\|\Delta U(\tau)\|(\|\nabla v(\tau)\|^2 + \|v_t(\tau)\|^2) d\tau + \frac{1}{2\gamma}\int\limits_t^{t+1} \|g(\tau)\|^2 d\tau + 2\gamma \int\limits_t^{t+1} \|v_t(\tau)\|^2 d\tau.$$

By (nis31 4.31) and (nis36 4.40)

$$\|v_t(t+1)\|^2 + \|\nabla v(t+1)\|^2 + \frac{1}{2}\|\nabla v(t+1)\|^4 - (\|v_t(t)\|^2 + \|\nabla v(t)\|^2 + \frac{1}{2}\|\nabla v(t)\|^4)$$
$$+ 2\gamma \int\limits_t^{t+1} \|v_t(\tau)\|^2 d\tau \leq Ce^{-\theta_1 t}.$$

Let's denote $E_3(v(t)) = \|v_t(t)\|^2 + \|\nabla v(t)\|^2 + \frac{1}{2}\|\nabla v(t)\|^4$,then

$$2\gamma \int\limits_t^{t+1} \|v_t(\tau)\|^2 d\tau \leq Ce^{-\theta_1 t} + E_3(t) - E_3(t+1) \ = B(t)^2 \qquad (4.44) \quad \text{nis40}$$

where $\theta_1 = \min(\theta', \theta)$.

By (nis39 4.43) and Poincare-Friedrich's inequality,we obtain

$$\left(\int_t^{t+1} \|g(\tau)\|^2 d\tau\right)^{\frac{1}{2}} \leq \frac{1}{\lambda_1}\left(\int_t^{t+1} \|\nabla g(\tau)\|^2 d\tau\right)^{\frac{1}{2}} \leq Ce^{-\theta' t}.$$

By (nis40 4.44) there exist two points $t_1 \in [t, t+\frac{1}{4}], t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$2\gamma\|v_t(t_i)\|^2 \leq 4B(t)^2 \qquad i = 1, 2. \tag{4.45}$$ nis41

By Teorem nis8L 4.1.4

$$\int_t^{t+1} \|\Delta U(\tau)\| d\tau \leq Ce^{-\theta t}. \tag{4.46}$$ nis42

Multiply (nis27 4.27) by v, then we get

$$\begin{aligned}&\frac{d}{dt}(v(t), v_t(t)) - \|v_t(t)\|^2 + \|\nabla v(t)\|^2 + \|\nabla v(t)\|^4 + \|\nabla v(t)\|^2\|\nabla U(t)\|^2 + 2\gamma(v(t), v_t(t)) \\ &= -2(\nabla U(t), \nabla v(t))\|\nabla v(t)\|^2 + 2((\nabla U(t), \nabla v(t)) + \|\nabla v(t)\|^2)(\Delta U(t), v(t)) + (g(t), v(t)).\end{aligned}$$

Then,

$$\begin{aligned}&\|\nabla v(t)\|^2 + \|\nabla v(t)\|^4 + \|\nabla v(t)\|^2\|\nabla U(t)\|^2 \\ &\quad \leq -\frac{d}{dt}(v(t), v_t(t)) + \|v_t(t)\|^2 + 4\|\nabla U(t)\|.\|\nabla v(t)\|.\|\nabla v(t)\|^2 \\ &\qquad + (g(t), v(t)) - 2\gamma(v(t), v_t(t)) + 2\|\nabla v(t)\|^2\|\nabla U(t)\|^2.\end{aligned}$$

By (nis31 4.31) and $\|\nabla v_t(t)\|^2$, $\|\Delta v(t)\|^2 \leq C\epsilon^2$

$$\begin{aligned}\|\nabla v(t)\|^2 + \|\nabla v(t)\|^4 &\leq -\frac{d}{dt}(v(t), v_t(t)) + \|v_t(t)\|^2 \\ &\quad + Ce^{-\theta t} + 2\gamma\|v(t)\|\|v_t(t)\| + \|v(t)\|\|g(t)\|. \end{aligned}\tag{4.47}$$ nis43

If we integrate (nis43 4.47) over $[t_1, t_2]$,we have

$$\begin{aligned}\int_{t_1}^{t_2} (1+\|\nabla v(\tau)\|^2)\|\nabla v(\tau)\|^2 d\tau &\leq |v(t_1), v_t(t_1)| + |v(t_2), v_t(t_2)| + \int_{t_1}^{t_2} \|v_t(\tau)\|^2 d\tau \\ &\quad + \int_{t_1}^{t_2} (2\gamma\|v_t(\tau)\| + \|g(\tau)\|) d\tau + Ce^{-\theta t}.\end{aligned}$$

Since $Ce^{-\theta t} < CB(t)^2$ we have

$$\int_{t_1}^{t_2} (1 + \|\nabla v(\tau)\|^2)\|\nabla v(\tau)\|^2 d\tau$$

$$\leq \max_{t\leq\tau\leq t+1} \|v(\tau)\|(|v_t(t_1)| + |v_t(t_2)|) + CB(t)^2$$

$$+ \max_{t\leq\tau\leq t+1} \|v(\tau)\| \int_{t_1}^{t_2} (2\gamma\|v_t(\tau)\| + \|g(\tau)\|)d\tau + CB(t)^2$$

$$\leq CB(t) \max_{t\leq\tau\leq t+1} \|v(\tau)\| + CB(t)^2. \quad (4.48) \quad \boxed{\texttt{nis44}}$$

Adding (4.44) and (4.47),we get

$$\int_{t_1}^{t_2} E_3(\tau)d\tau \leq CB(t) \max_{t\leq\tau\leq t+1} E_3(\tau)^{\frac{1}{2}} + CB(t)^2.$$

Thus there exists a point $t^* \in [t_1, t_2]$ such that

$$E_3(t^*) \leq 2C \left\{ B(t) \max_{t\leq\tau\leq t+1} E_3(\tau)^{\frac{1}{2}} + B(t)^2 \right\}. \quad (4.49) \quad \boxed{\texttt{nis45}}$$

We integrate (4.28) from t^* to $s \in [t, t+1]$:

$$\|v_t(s)\|^2 + \|\nabla v(s)\|^2 + \frac{1}{2}\|\nabla v(s)\|^4$$

$$= \|v_t(t^*)\|^2 + \|\nabla v(t^*)\|^2 + \frac{1}{2}\|\nabla v(t^*)\|^4 + \int_{t^*}^{s} -\frac{d}{dt}(\|\nabla U(\tau)\|^2\|\nabla v(\tau)\|^2)d\tau$$

$$-\int_{t^*}^{s} 4\gamma\|v_t(\tau)\|d\tau + 2\int_{t^*}^{s} (\nabla U(\tau), \nabla U_t(\tau))\|\nabla v(\tau)\|^2 d\tau - 2\int_{t^*}^{s} \frac{d}{dt}(\nabla U(\tau), \nabla v(\tau))\|\nabla v(\tau)\|^2 d\tau$$

$$+2\int_{t^*}^{s} (\nabla U_t(\tau), \nabla v(\tau))\|\nabla v(\tau)\|^2 d\tau - 2\int_{t^*}^{s} (\nabla U(\tau), \nabla v(\tau))(\Delta U(\tau), v_t(\tau))d\tau + 2\int_{t^*}^{s} (g(\tau), v_t(\tau))d\tau.$$

Then

$$E_3(s) \leq E_3(t^*) + \|\nabla U(t^*)\|^2 + \|\nabla v(t^*)\|^2 + \|\nabla U(t^*)\|^2 + \|\nabla v(t^*)\|^2$$

$$+ \int_{t^*}^{s} (\|\nabla U(\tau)\|^2 + \|\nabla U_t(\tau)\|^2)\|\nabla v(\tau)\|^2 d\tau + 2\|\nabla U(s)\|\|\nabla v(s)\|\|\nabla v(s)\|^2$$

$$+ 2\|\nabla U(t^*)\|\|\nabla v(t^*)\|\|\nabla v(t^*)\|^2 + 2\int_{t^*}^{s} \|\nabla U_t(\tau)\|\|\nabla v(\tau)\|\|\nabla v(\tau)\|^2 d\tau$$

$$+ 2\int_{t^*}^{s} \|\nabla U(\tau)\|\|\Delta U(\tau)\|\|\nabla v(\tau)\|\|v_t(\tau)\|d\tau + 2\int_{t^*}^{s} \|g(\tau)\|\|v_t(\tau)\|d\tau.$$

By (nis45 4.49), Theorem nis8L 4.1.4 , nis2T 4.1.5 and

$$\|v_t(t)\|, \|\nabla v(t)\|, \int_0^t \|v_t(\tau)\|^2 d\tau, \|U_t(t)\|, \|\nabla U(t)\|, \int_0^t \|U_t(\tau)\|^2 d\tau \leq C,$$

we get

$$E_3(s) \leq E_3(t^*) + Ce^{-\theta t} + Ce^{-\theta' t}.$$

$$E_3(s) \leq C\left\{B(t) \max_{t\leq\tau\leq t+1} E_3(\tau)^{\frac{1}{2}} + B(t)^2\right\} \quad \forall s \in [t, t+1],$$

$$\max_{t\leq\tau\leq t+1} E_3(\tau) \leq C\left\{B(t) \max_{t\leq\tau\leq t+1} E_3(\tau)^{\frac{1}{2}} + B(t)^2\right\},$$

$$\max_{t\leq\tau\leq t+1} E_3(\tau) \leq \frac{1}{2} \max_{t\leq\tau\leq t+1} E_3(\tau) + CB(t)^2.$$

Then,

$$\max_{t\leq\tau\leq t+1} E_3(\tau) \leq 2CB(t)^2 \leq C(E_3(t) - E_3(t+1) + Ce^{-\theta_1 t}.$$

By Lemma nis4L 1.1.27 we get,

$$\|\nabla v(t)\|^2 + \|v_t(t)\|^2 \leq C_3 e^{-\theta'' t} \ for\ some \theta'' > 0.$$

□

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