Anyonic Quantum Computation

by

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This is to certify that I have examined this copy of a master's thesis by

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ABSTRACT

Quantum computation can be done by using non-abelian anyons which are neither bosons nor fermions. Quasi-particles of $\nu = 5/2$ fractional quantum Hall state will provide non-abelian anyons. We discuss a NOT gate operation using these anyons. In the thesis fundamental concepts of quantum computation are reviewed. Non-abelian anyons and topological phases are introduced. Some basic facts about fractional quantum Hall state are given.

ÖZET

Abelyen olmayan anyonlarla kuantum hesaplama yapılabilir. Anyonlar ne fermion ne boson olmayan, farklı parçacıklardır. $\nu = 5/2$ kesirli kuantum Hall durumundaki kuasi-parçacıklar Abelyen olmayan anyonların özelliklerini taşımaktadırlar. Bu tezde $\nu = 5/2$ kesirli kuantum Hall durumu kullanılarak (NOT) operasyonunun yapılabileceği gösterilmektedir. Temel kuantum hesaplama ilkeleri gözden geçirilmiş, Abelyen olmayan anyonlarla topolojik faz tanımları verilmiştir. Kesirli kuantum Hall olayı ile ilgili temel bilgiler de verilmiştir.

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Chapter 1

INTRODUCTION

Difficult problems require ingenious solutions. Quantum computation provides ingenious solutions to very difficult problems such as factorization of very large integers. In 1994 Peter Shor surprised the community of computer scientists with a polynomial time algorithm for quantum computation to solve factoring problem [1, 2].The classical computers can find the prime factors of a k-digit number in a time on the order of $e^{k^{\frac{1}{3}}}$, yet Shor's algorithm demonstrate that a quantum computer can find prime factors in a time of the order of $k^2 \log k \log(\log k)$, which is much faster. This result is important, because powerful encryption techniques make use of the difficulty of factorization and if a quantum computer can be built succesfully, many encryption methods will become useless against quantum computers. This major achivement encouraged many physicsts to pay more attention to quantum computation.

The first idea of quantum computation, possibly, came from Feynman in 1982 [2]. He realized that simulation of many-body Hamiltonians which is hopeless in a classical computer might be solved using properties of quantum mechanical systems. So, the question may comprise the answer. Remarkable advances followed Feynman's idea: other algorithms - for instance searching algorithms- have been developed beside Shor's and the hope for realization of a quantum computer gradually increased in recent years.

Although theoretical achievements manifest that quantum computation is much stronger than classical computation, experimental verification proved tricky. Indeed, few qubit (quantum bit) computers have been built and run, yet they face a big problem: quantum decoherence. Any system prepared in a certain quantum state will interact with its environment, and this interaction destroys completely or changes critically the state of the system, thus eventually canceling the computation. Decoherence along with the other obstacles are referred to as errors. Error correction schemes have also been found [3], yet they require too

many qubits which are already scarce. A more elegant way might be to prevent errors before they occur. An attempt along this line of thought was made by Kitaev [4] in 1997. His idea was to use non-abelian anyons which live on 2-D surfaces and have peculiar properties, for quantum computation.

Classically data is stored in magnetic media, composed of spins. Though each spin is sensitive to environmental factors, a collection of spins is quite intact. The reason is that each spin interacts with its neighbors so that they look at the same direction, hence general alignment of the system does not change. A possible error is avoided at the physical level. So, maybe the same situation can be mimicked in a quantum computer. To do that one should employ non-abelian anyons. A detailed account of their physics will come later, yet it is important to mention their benefit here. Anyons are quasi-particles which live on a 2-D surface. Two or more anyons come together to build a qubit. Qubits formed by non-abelian anyons are topologically protected, meaning that the state of the qubit is unchanged as long as the anyons are kept apart. Local disturbances do not affect the state of the qubit and unitary operations are performed on the qubit just by moving anyons around each other. A question may arise "Why physicists wait to build such a quantum computer?" The answer is that there was no physical system known to experimentalists which resembles the behavior of anyons.

Fortunately, the first sign of the existence of a non-abelian anyon appears in a Fractional Quantum Hall (FQH) state. According to Moore and Read [5], the $\nu = 5/2$ FQH state has indeed non-abelian anyons as excitations or quasiparticles of a 2-D electron system at temperatures close to absolute zero temperature and in a high magnetic field. After many theoretical debates Das Sarma et. al. [6] gave a way to check that assertion, and an application of computation (a NOT gate operation) on the qubits if there are indeed non-abelian anyons. With encouraging experimental advances, a quantum computation by employing non-abelian anyons is no longer a fantasy. Anyonic quantum computation can be done with the non-abelian anyons provided by $\nu = 5/2$ FQH state.

The structure of the thesis is as follows: In Chapter 2, fundamental aspects of quantum computation will be reviewed and definitions of basic concepts will be given. In Chapter 3, anyons and especially non-abelian anyons will be introduced. Theoretical and experimental observations of anyons are given with an emphasis on Quantum Hall Systems in chapter

4. Finally, how different unitary operations can be done by braiding and the experiment proposed by Das Sarma et. al. will be discussed in Chapter 5. We conclude by a short Chapter 6.

Chapter 2

BASICS OF QUANTUM COMPUTATION

A physical system is represented by two objects H and $|\Psi\rangle$. $|\Psi\rangle$ is the state vector of the system, and it is supposed to have all the information about the system. H is the Hamiltonian of that system. In a sense, it governs the system. It decides how the system will change in time by giving the unitary transformation of $|\Psi\rangle$ in time with the equation

$$U\,|\Psi\rangle_{initial} = |\Psi\rangle_{final} \tag{2.1}$$

where U satisfies

$$\frac{dU}{dt} = \frac{-i}{\hbar} HU \tag{2.2}$$

Mathematical realization of $|\Psi\rangle$ or H would be cumbersome, yet a simple framework is enough for our discussion. We adopt the simplest bra-ket notation. So, $|\Psi\rangle$ is a column matrix (a vector) in $\mathbb{C}^d$ with the usual inner product. U or any other operator, unless otherwise stated, is a $d \times d$ complex unitary matrix. Unitary means that its complex transpose is equal to its inverse: $U^\dagger = \left(U^T\right)^*$ and $U^\dagger U = UU^\dagger = I$ where I is the identity matrix. For different dimensions same letter will be used since it is straight forward to see its dimension.

2.1 Definition of Quantum Computation

A quantum computation (QC) is composed of three steps [7]

1. Initialization of a state $|\Psi\rangle$ in a known basis.

2. Unitary evolution of that state in a desired manner.

3. Measuring $|\Psi\rangle$ and extracting the information that we are looking for.

In applications, the most difficult step is the middle one where the essential computation is made. Also most of the errors occur during this step and a state vector might evolve in a fashion that is not desired. So, our focus will be on the second step.

2.2 Quantum Bit (qubit)

Classical information is stored in bits. A bit can take values 0 or 1 corresponding to no signal or signal. Quantum analog of bit is qubit which is short form of quantum bit. A qubit is a unit vector in $\mathbb{C}^2$ where the basis is fixed. Commonly used basis consists of $|0\rangle$ and $|1\rangle$ where $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in vector form. A qubit $|q\rangle$ is a superposition of $|0\rangle$ and $|1\rangle$,

$$q = a\,|0\rangle + b\,|1\rangle \tag{2.3}$$

$a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$. Although qubit has superiority over a bit, the amount of information that we get out of a qubit is the same as a bit. The reason is that a measurement should be done over the qubit to get the information. A measurement is a projection of the qubit along one of the basis vectors. The qubit $|q\rangle$ may assume infinitely many values, but any measurement gives 0 with probability $|a|^2$ and 1 with probability $|b|^2$. So, the situation is even worse than in the classical case, a qubit can store only the same amount of information as a bit and it is not clear what value it will give upon measurement. As we will see, that kind of non-classical properties can be used in favor of computation.

It should also be noted that one can define a "qudit" which is a unit vector in $\mathbb{C}^d$. This vector space has the basis $\{|0\rangle, |1\rangle, |2\rangle, \ldots, |d-1\rangle\}$ and it is possible to do QC in this setting, however, the tradition is to work with qubits and we follow that.

2.3 Initial State

The system is constituted by many qubits, let's say n qubits make up our system than the state vector $|\Psi\rangle$ can be written as a sum of tensor products of one qubit states and

$|\Psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \mathbb{C}^2 \approx \mathbb{C}^{2^n}$

We get an exponentially large (in dimension) Hilbert space. Classically n bits give us $2n$ dimensional space, but in QC we get 2^n dimensional space, hence power of computation may incredibly increase. A basis element of this enormous space is a tensor product of n qubits: $|0\rangle \otimes |1\rangle \otimes \ldots |0\rangle$ and shortly represented as $|01\ldots0\rangle$. For instance a basis of this space for $n = 3$ will be $\{|000\rangle, |001\rangle, |010\rangle, |011\rangle\, |100\rangle, |101\rangle, |111\rangle\}$ just like a representation of numbers 1 to 8 in base-2.

2.4 Quantum Gates

Here we follow circuit model of QC and concepts are adopted from circuit theory [7, 2]. A unitary transformation which is done for QC is called a quantum gate. Quantum gates are different from classical logic gates in that they are reversible, since by definition a unitary matrix has an inverse. They also act on a superposition of basis elements (vectors), in loose sense they do many evaluations in one step. In our simple setting quantum gates are 2×2 or $2n \times 2n$ complex, unitary matrices. It is sufficient to have a specific group of quantum gates to do QC chosen from one-qubit and two-qubit gates which are 2×2 and 4×4 complex, unitary matrices, respectively. Some important examples are given below, both in matrix and bra-ket forms:

$$I = |0\rangle\langle 0| + |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.4}$$

$$X = |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.5}$$

$$P(\varphi) = |0\rangle\langle 0| + e^{i\varphi}|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \tag{2.6}$$

$$\mathbf{H} = \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1|\,|1\rangle + \langle 0| - |1\rangle\langle 1|\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2.7}$$

I is the identity gate, applied to indicate that qubit does not change. X is NOT gate, P is the phase shift, and the last one is the Hadamard gate (not to be confused with the Hamiltonian). When $\mathbf{H}$ is applied to $|0\rangle$ we get an equally weighted qubit

$$\mathbf{H}|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \tag{2.8}$$

Another important gate is controlled-NOT (CNOT) gate. It is a two-qubit gate which changes the second qubit's value only when the first qubit is $|1\rangle$. It can be written as a

tensor product of two one-bit gates

$$C_{NOT} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.9}$$

A unitary transformation U on $|\Psi\rangle$ can be written in terms of one-qubit and two-qubit gates only, as an example for n=4, i. e. $|\Psi\rangle \in \mathbb{C}^{2^4}$, a unitary transformation might be $I \otimes I \otimes X \otimes I$, or $I \otimes C_{NOT} \otimes I$, or $C_{NOT} \otimes C_{NOT}$.

2.5 Universality of Quantum Gates

A unitary transformation is generally composed of many quantum gates and to do QC it is sufficient to have a finite number of gates. A set of quantum gates is called universal if any unitary transformation may be approximated to arbitrary accuracy only by using these gates. For instance $\{\mathbf{H}, P, C_{NOT}, I\}$ is universal, i.e. any unitary transformation U can be written as $U = U_1 U_2 \ldots U_k$ where each one of $U_1, U_2 \ldots U_k$ is composed only from that set, $U_i = I \otimes P \otimes \ldots \mathbf{H} \otimes C_{NOT}$.

2.6 No-Cloning Theorem

An unknown quantum state cannot be copied. This statement is known as the No-Cloning Theorem. It might look trivial, yet its implications are far reaching. We cannot have a copy of $|\Psi\rangle_{final}$ after a computation, so we do not have backup state vectors to fix some problems if any of them is once realized, or we cannot copy and use that state vector at different places.

There are two operations which can be done to copy a state vector, measurement and unitary evolution. First one selects one of the basis vectors and destroys the state, hence cannot copy a state vector. Second one will not work either. To prove this assertion let's assume there is a U such that it copies an unknown state $|a\rangle$ and pastes it to $|0\rangle$;

$$U|a0\rangle = |aa\rangle \tag{2.10}$$

for all $|a0\rangle$. Also let $|c\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)$ such that $|a\rangle$ and $|b\rangle$ are orthogonal. Then

$$U|c0\rangle = |cc\rangle = \frac{1}{2}(|aa\rangle + |ab\rangle + |ba\rangle + |bb\rangle) \tag{2.11}$$

while

$$U|c0\rangle = \frac{1}{\sqrt{2}}(U|a0\rangle + U|b0\rangle) = \frac{1}{\sqrt{2}}(|aa\rangle + |bb\rangle). \tag{2.12}$$

Two results are not equal to each other, and hence there is no such U.

2.7 Generic Computation Scheme

Suppose a function f will be evaluated by QC where f needs l qubits for input and m qubits for output such that $l + m = n$. Furthermore suppose that there exits a unitary transformation U_f which implements the function f. As a first step, initialize the state vector using Walsh-Hadamard W_l transformation for first l qubits and put $|0\rangle$'s for the rest of m qubits where

$$W_1 = \mathbf{H} \qquad W_l = \mathbf{H} \otimes W_{l-1} \qquad \text{for} \qquad l > 1. \tag{2.13}$$

W_l produces all the basis elements of $\mathbb{C}^{2^l}$ or the binary representation of all the numbers from 0 to $2^l - 1$:

$$W_l|000\ldots0\rangle = \frac{1}{\sqrt{2^l}}(|00\ldots0\rangle + |00\ldots1\rangle + \ldots|11\ldots1\rangle) \tag{2.14}$$

This equation can be written more compactly if elements on the rigth-hand side of the above equation are written as their value in base-10:

$$W_l|0\rangle = \frac{1}{\sqrt{2^l}}\sum_{x=0}^{2^l-1}|x\rangle. \tag{2.15}$$

Initialization of the state will be done as following:

$$|\Psi\rangle_{initial} = W_l \otimes I \otimes I \ldots \otimes I\,|0,0\rangle = \frac{1}{\sqrt{2^l}}\sum_{x=0}^{2^l-1}|x,0\rangle \tag{2.16}$$

In the above equation there are m the identity operators and a ket with comma represents $l + m$ qubits. Then as in the second step, action of U_f on the initial state:

$$U_f|\Psi\rangle_{initial} = \frac{1}{\sqrt{2^l}}\sum_{x=0}^{2^l-1}U_f|x,0\rangle = \frac{1}{\sqrt{2^l}}\sum_{x=0}^{2^l-1}|x,f(x)\rangle \tag{2.17}$$

Here it is crucial to see that U_f acts on each vector in the summation simultaneously and -in a sense- evaluates $f(x)$ for every x. But, it is not possible to extract all the information from this result, and even the desired value of the equation can be got with a probability. The reason behind this difficulty is the last step; measurement.

This is a wide subject by on its own, so we can just touch the problem here. As mentioned above, a measurement projects the state vector onto one of the basis vectors and we do not have control over the selection of any basis vector. Two things can be done to overcome this problem. First is to amplify the coefficient of the desired vector in the sum, so that the probability of finding the desired result gets higher. Second is to use common properties of the function and use them to get the desired result [2].

2.8 Errors

Although the theory of QC is developing very fast, a realization of a quantum computer is lacking. The problem is in the experimental side. The accummulation of errors will force the termination of computation, sometimes at the very beginning. What is meant by error is not usual type of error faced in common experiments like putting a circuit element in a wrong place. Here the cause of the error is quantum mechanical and the main source is decoherence.

Infact all quantum systems are highly fragile and maintaining the state of a quantum system is a difficult matter. Errors faced in experiments are generally the errors caused by interaction of the system with the environment. The state vector of the system will change and data associated with that state vector will no longer be there. Theoretically it is easy to visialize an isolated system from the environment, but in practice it is a great challenge to isolate a physical system from its surroundings. The early physical systems used to construct a quantum computer are composed of certain molecules, ions, or neutral atoms. They are prepared with very special care and it is clear that handling these systems is a very difficult task. It is not possible to put some molecules in free space at specific locations far away from other things and then manipulate these molecules so that a QC is done. Rather than that, molecules stay on a sample inside many other experimental tools, and scientists try to manipulate them by using electric/magnetic fields or by other means. So, the decoherence of the state vector of these molecules is not a possibility, but a reality.

A solution to this problem is quantum error correction that was also proposed by Shor [3]. Quantum error correction is inspired by the classical one. Errors also occur in classical computation, for instance a bit 0 may be read as 1. To overcome such situations bit 0 can be encoded as 000, so if one of the bits changes, by looking at the majority of three bits we conclude that the value of the bit is 0, hence the probability of making a wrong calculation will be decreased. A qubit $|0\rangle$ or $|1\rangle$ may also be encoded as

$$\begin{aligned} \left|0'\right\rangle &= \left(\frac{1}{\sqrt{2}}\left|000\right\rangle + \left|111\right\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}\left|000\right\rangle + \left|111\right\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}\left|000\right\rangle + \left|111\right\rangle\right), \\ \left|1'\right\rangle &= \left(\frac{1}{\sqrt{2}}\left|000\right\rangle - \left|111\right\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}\left|000\right\rangle - \left|111\right\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}\left|000\right\rangle - \left|111\right\rangle\right) \end{aligned} \tag{2.18}$$

Assuming a bit-flip error (undesired X) or a phase-flip error (undesired $P(\pi)$) occur only in a single qubit then it can be fixed by Shor's error correction scheme. What is basically done is that one measures some part of the nine qubit cluster and fixes the error without collapsing whole qubits. In a way, a qubit is encoded nonlocally and by error correction we measure the error rather than the qubit [8].

Other error correction schemes are proposed which use less number of qubits to implement one qubit, but the current situation of quantum computers constructed by few qubits (less than 50) is not promising. To make a safe QC, error rate should be small. The accuracy treshold (above that value error correction is not possible) is between 10^{-4} and 10^{-6}. So 10^4 or 10^6 operations should be carried out without an error at the beginning. Instead of solving this particular problem, one can look for a system that is immune to decoherence. In next chapter we look at exotic particles which might provide such a system.

Chapter 3

ANYONS

In Quantum Mechanics identical particles are divided into two groups: fermions and bosons. They are divided with respect to their behaviour when two identical particles are exchanged. Suppose $|\Psi\rangle$ is the state vector of identical particles and let P_{ij} be the particle exchange operator which exchanges the particles i and j at the positions $\vec{r}_i$ and $\vec{r}_j$ adiabatically. If P_{ij} is applied to $|\Psi\rangle$ twice, by naive logic, $|\Psi\rangle$ should not change, since the particles come back to their initial positions:

$$P_{ij}P_{ij}\,|\Psi\rangle = |\Psi\rangle \qquad \text{and} \qquad P_{ij}^2 = I. \tag{3.1}$$

There are two possibilites for the action of P_{ij};

$$\text{either} \qquad P_{ij}\,|\Psi\rangle = -\,|\Psi\rangle \qquad \text{or} \qquad P_{ij}\,|\Psi\rangle = |\Psi\rangle\,. \tag{3.2}$$

Empirical observations indicate that particles satisfying the first equation (left) obey Fermi-Dirac Statistics, and particles satisfying the second equation (right) obey Bose-Einstein Statistics. They are called fermions and bosons, and common examples for these two groups are electrons and photons, respectively. This straightforward argument is true in 3-dimensional space, but when the identical particles are restricted to move in 2-dimensional space, the above argument fails. A more general description is needed. Wilczek gave a description and named the new type of particles as 'anyon' [9, 10, 11]. Effect of P_{ij} on a state will be:

$$P_{ij}\,|\Psi\rangle = e^{i\theta}\,|\Psi\rangle \qquad 0 \le \theta \le \pi \tag{3.3}$$

where the phase factor θ is the statistics parameter of the system. If $\theta = 0$ particles are bosons, if $\theta = \pi$ particles are fermions. If, on the other hand θ takes values in between then the particles are called 'anyons' since θ can take 'any' value between 0 and π.

A special kind of anyons, non-abelian anyons, can be used for QC which will be discussed at the end of this chapter. But here, in order to understand what is special about 2D surfaces and how different phase factors -other than 0 or π- come into the equation (3.3), we find

it instructive to look at Aharanov-Bohm effect first where topological phase appears. Then we will discuss the Berry phase which is more general and essential.

3.1 Aharonov-Bohm Effect

Classical motion of a particle depends on the fields, not the potentials. Aharonov and Bohm argued that in quantum mechanics even in the absence of fields, potentials have significant effect on the system [12, 13]. They gave two theoretical examples, and one of them was about magnetic vector potential.

Consider a constant magnetic field $\vec{B}$ directed in $\hat{z}$ direction in a long solenoid. Send two identical electron beams so that they surround the solenoid and interfere after that. Now, when one looks at the interence pattern, a phase shift will be observed which depends on the flux of the solenoid. Observe that even though magnetic field is zero on the paths of the two beams, vector potential $\vec{A}$ which is given by $\vec{B} = \nabla \times \vec{A}$, is not zero outside the solenoid. Because of that momentum (operator) is not just $\vec{p}$, but $\left(\vec{p} - \frac{q}{c}\vec{A}\right)$ where q is charge of electron and c is the speed of light. Then the 1-electron Hamiltonian will be,

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2. \tag{3.4}$$

Let $\vec{A} = \nabla f$ or $f = \int \vec{A} \cdot d\vec{r'}$ along a path. If $|\Psi_0\rangle$ is the wave function of the electron when $\vec{A} = 0$ or $H_0 = \frac{p^2}{2m}$, then the solution of the Schrödinger equation for H is related to $|\Psi_0\rangle$ as:

$$|\Psi\rangle = e^{\frac{iqf}{\hbar c}} |\Psi_0\rangle \tag{3.5}$$

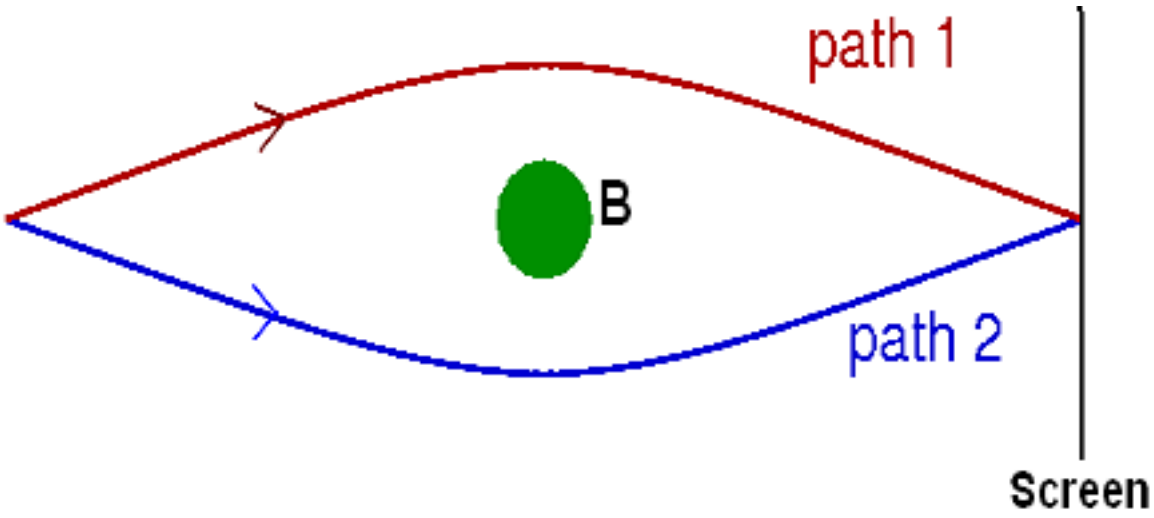

Figure 3.1: Aharonov & Bohm effect

depending on the path taken. When two electrons interfere as shown in the Figure:3.1, we have

$$e^{\frac{iqf_1}{\hbar c}}\left|\Psi_{01}\right\rangle + e^{\frac{iqf_2}{\hbar c}}\left|\Psi_{02}\right\rangle = e^{\frac{iqf_2}{\hbar c}}\left(e^{\frac{iq(f_1-f_2)}{\hbar c}}\left|\Psi_{01}\right\rangle + \left|\Psi_{02}\right\rangle\right). \tag{3.6}$$

The overall phase in front of the paranthesis is not important, and we observe a phase difference of

$$\frac{q(f_1-f_2)}{\hbar c} = \frac{q}{\hbar c}\oint \vec{A}\cdot d\vec{r'} = \frac{q}{\hbar c}\int \nabla\times\vec{A}\cdot d\vec{a'} = \frac{q}{\hbar c}\Phi \tag{3.7}$$

where Φ is the magnetic flux enclosed by the loop. Significance of this result to our interest is that it gives an example to anyonic phase since this situation is reminisence of particle exchange operator P_{ij}.

It should be emphasized that the path taken by the electrons is insignificant, since by the Stoke's theorem what matters is the flux Φ, and not the specific loop that it is enclosed. The phase appeared here is an example of topological phase. Topological phase is a Berry phase in which the specific geometry of the path is unessential, only the topology is important. So, next we look at Berry phase to grasp the idea of topological phase.

3.2 Berry Phase

Berry demonstrated [14] that a quantum system in an eigenstate of a Hamiltonian $H(\mathbf{R})$ will get a phase factor $\gamma_n(C)$, apart from the usual dynamical one, if it is adiabatically transported around a loop C in the paramater space of $\mathbf{R} = (X, Y, \ldots)$. This phase factor

usually is referred to as geometric phase since it will be calculated as a surface integral in the parameter space.

Let the system evolve adiabatically from $t = 0$ to $t = T$ by H where $\mathbf{R}(0) = \mathbf{R}(T)$ and H depends smoothly on $\mathbf{R}$, so that the system comes back to the initial state except with a phase. Assume that the system is initalized at one of the eigenstates $|\psi_n(\mathbf{R})\rangle$ where it satisfies the customary energy eigenvalue equation at time $t = 0$

$$H(\mathbf{R})\,|\psi_n\rangle = E_n(\mathbf{R})\,|\psi_n\rangle\,. \tag{3.8}$$

We assume here $E_n(\mathbf{R})$'s are non-degenerate; degenerate case will come in next section. At a later time $t > 0$ the state vector will be

$$|\Psi(t)\rangle = e^{i\gamma_n(t)} exp\left[\frac{-i}{\hbar}\int_0^t E_n(\mathbf{R}(t'))dt'\right]|\psi_n(\mathbf{R}(t))\rangle\,. \tag{3.9}$$

Then after one cycle, $t = T$, the state vector becomes

$$|\Psi(T)\rangle = e^{i\gamma_n(C)} exp\left[\frac{-i}{\hbar}\int_0^T E_n(\mathbf{R}(t'))dt'\right]|\Psi(0)\rangle \tag{3.10}$$

and the geometric phase factor $\gamma_n(C)$ is found as

$$\gamma_n(C) = i\oint \langle\psi_n(\mathbf{R})|\nabla_R\psi_n(\mathbf{R})\rangle\cdot d\mathbf{R}. \tag{3.11}$$

This line integral can be converted into a surface integral which has the boundary C by Stokes' theorem. (For dimensions higher than three the same argument applies see [15].)

$$\gamma_n(C) = -\int \mathbf{V}_n(\mathbf{R})\cdot d\mathbf{a} \tag{3.12}$$

where $\mathbf{V}_n$, the vector which is analogous to magnetic field, is given by:

$$\mathbf{V}_n = Im\sum_{m\neq n}\frac{\langle\psi_n|\,\nabla_R H\,|\psi_m\rangle\times\langle\psi_m|\,\nabla_R H\,|\psi_n\rangle}{(E_m - E_n)^2}. \tag{3.13}$$

The analogy between the geometric phase and AB phase is apparent. Yet AB phase is a special case, namely a topological phase. The boundary of integral in AB is valid if it includes the solenoid and the shape of it is irrelevant, as long as the integral can be evaluated. Topological phase is a geometric phase which stays constant while the path C is smoothly deformed with fixed initial/end point [16]. This is to say, mathematically, the phase is the same for different paths in the same homotopy class. As an example consider

the circles on a plane with a hole. The circles encircling the hole are in a different homotopy class than the circles not encircling the hole. In AB effect, the hole is the solenoid and circle is the closed loop that the electrons follow.

The requirement of two dimensional space for anyons can be understood by a similar illustration. Consider the paths followed by two particles when they are exchanged twice. First particle goes to the position of second particle then comes back to its original position, while the second particle goes to the position of the first then comes back without colliding. This is topologically equivalent to first particle going around the second particle while the second particle is stationary. Since the first path can be smoothly deformed to the second path and particles do not collide.

In three dimensions, a loop going around a fixed point is topologically the same (homotopic) as not moving the particle ever . This is the reason why in three or higher dimensions there are either fermions or bosons. Yet in two dimensions, a loop around a fixed point cannot be contracted to initial point of the loop (position of first particle) without crossing the fixed point (position of second particle). So in two dimensions, physical system do not necessarily come back exactly to the same initial state when such 'braiding' of anyons take place.

Topological phases are immune to smooth deformations on their paths. This property would be a great power in QC. Anyons with the topological phases are resistant to "small and local" changes, i.e. smooth deformations. So, qubits build up with such anyons would not decohere and the state vector survives. This is the main reason why researchers are turning towards anyons for QC. Yet, this property is not sufficient by itself to achieve QC, anyons should be also non-abelian. In next section we will look at these.

3.3 Non-Abelian Anyons

In the previous section we looked at non-degenerate energy eigenvalues. Now we look at the degenerate case. Let the n'th energy eigenvalue $E_n(\mathbf{R})$ be g-fold degenerate where g is an integer. Also let there be no level crossings, that is, if $E_n(R_0) \neq E_m(R_0)$ for some R_0 in the parameter space then $E_n \neq E_m$ for all other points. In other words, if E_n and E_m are different at $t = 0$, they will be different at all other times. New energy eigenvalue equation

will be

$$H(\mathbf{R})\,|\psi_n, a; \mathbf{R}\rangle = E_n(\mathbf{R})\,|\psi_n, a; \mathbf{R}\rangle \tag{3.14}$$

where the degeneracy parameter $a \in \{1, 2, \ldots, g\}$. Consider again an adiabatic, cyclic evolution in the parameter space $\mathbf{R}(0) = \mathbf{R}(T)$. Hence the energy of the system will not change. Yet, in this case system would not come simply to the same state because there are degenerate energy eigenstates -all of them corresponding to the same energy- and the resulting state will be a linear combination of the degenerate states. If the system started at $|\psi_n, a; \mathbf{R}\rangle$, leaving the details of the calculation to [15], the resulting eigenvector is

$$|\psi_n, a; \mathbf{R}(T)\rangle = e^{-i\int_0^T E_n(R(t))dt} \sum_{b=1}^{g} \left[Pexp(i \oint A_g^n dR) \right]^{ab} |\psi_n, b; \mathbf{R}(0)\rangle\,. \tag{3.15}$$

The term before the summation is the dynamical phase factor which can be neglected in our discussion. P is the path-ordering operator for the exponential integral that arranges the products of operators (matrices).

$$Pexp\left(i\int_{C(0)}^{C(t)} A(x)dx\right) = Texp\left(i\int_0^t A(C(t'))\dot{x}dt'\right) \tag{3.16}$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_0^t dt_1 \int_0^t dt_2 \ldots \int_0^t dt_n A(C(t_1))\dot{x}(t_1) \ldots A(C(t_n))\dot{x}(t_n) \tag{3.17}$$

where T is the time-ordering operator which rearranges the product of operators so that $t_1 > t_2 > t_3 ... > t_n$ [15]. $\mathbf{A}_g^n$ resembles the vector potential of the magnetic field in the AB effect, yet it is a $g \times g$ Hermitian matrix due to the degeneracy of the problem. The term $Pexp(i \oint A_g^n dR)$ is a $g \times g$ unitary matrix which is essential for QC. Basically that term gives the required unitary evolution of the state vector, corresponding to the second part of the QC scheme. The name non-abelian derives from the fact that matrices, especially unitary matrices for computation, are not abelian (commutative) so the name non-abelian fits such matrices/phases.

A crucial point here is that this non-abelian phase should be topological; otherwise it is not suitable for computation. If there are anyons which have topological, non-abelian phases then it is possible, in principle, to apply a desired quantum gate (unitary evolution) by adiabatically changing the parameters $\mathbf{R}(t)$ in a cycle, i.e. $\mathbf{R}(0) = \mathbf{R}(T)$. The parameters $\mathbf{R}(t)$ are generically just the coordinates of anyons, so by moving the anyons on a plane an

essential part of the QC can be done. Of course, there should exist such anyons first and we should know how to manipulate them.

Fortunately, the theoretical work on non-abelian anyons was started by Kitaev's toric code model [4]. Although it involves an abelian, topological phase it is the pioneering work for other theoretical studies. Another model of Kitaev, honey-comb model gives a non-abelian phase [17]. Therefore, it is more appropriate for QC. Yet these models and other similar ones are not practical. They use spin to construct anyons, but any physical realization of them is missing. There are very few physical systems that might give rise to non-abelian anyons. Among these systems the strongest candidate for QC is the $\nu = 5/2$ quantum Hall state. Since the next chapter is devoted to quantum Hall effect, here we discuss other systems.

Transition metal oxides is another possible system[1] that will support non-abelian anyons. It is a benefit that this type of systems are well studied, yet experiments are not done to find a topological phase and finding such a phase is not an easy task. The reason is that there is not a single phenomena to explore their phase diagram and it seems there are different phases all contributing to the phase diagram. However, there are signs for Sr_2RuO_4 to support non-abelian anyonic QC. At low temperatures around 1.5 K, superconductors made out of Sr_2RuO_4 thin films may give the required two dimensional platform for anyonic QC. Another candidate system concerns ultracold atoms[1]. Fast-rotating dilute Bose gases would give a desired physical system, yet determining the non-abelian phase is still a challenging problem. Hence we focus below on the most likely physical system.

Chapter 4

QUANTUM HALL EFFECT

Edwin Hall discovered in 1879 that when a magnetic field is turned on perpendicular to the plane of a current carrying conductor, a voltage difference occurrs which is both transverse to the current and the magnetic field. Consider the situation where current or electric field $\mathbf{E}$ is in the $+\hat{i}$ direction and magnetic field $\mathbf{B}$ is in the $+\hat{k}$ direction, then a voltage difference V_H occurrs across the y-axis and an electric field $\mathbf{E}_H$ is established when the system reaches the steady state. Direction of $\mathbf{E}_H$ depends on the sign of the electric charge of the charge carriers because positive or negative charged particles accumulate both along the positive y-axis. If the charge of a particle is q and its velocity is $\mathbf{v}$ then, in CGS units,

$$\frac{q}{c}\mathbf{v}\times\mathbf{B} = q\mathbf{E}_H \qquad \text{and} \qquad E_H = \frac{v_x B}{c}. \tag{4.1}$$

An important parameter is the Hall resistance which is defined as $\rho_H = \frac{V_H}{I} = \frac{E_H}{j_x}$ where $j_x = nqv_x$ is the surface current density and n is charge density. Hence we get

$$\rho_H = \frac{B}{nqc}. \tag{4.2}$$

In two dimensions, the Hall resistance is equal to Hall resistivity ρ_{xy}. It is the corresponding element of resistivity tensor which is the inverse of conductivity tensor $\rho = \sigma^{-1}$. There is a relation between current density, conductivity tensor and electric field $j_i = \sum_j \sigma_{ij} E_j$ which in matrix form reads

$$\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}. \tag{4.3}$$

The components σ_{xx} and σ_{xy} are [18]

$$\sigma_{xx} = \frac{\sigma_0}{1+\omega^2\tau^2} \qquad \text{and} \qquad \sigma_{xy} = \frac{\sigma_0\omega\tau}{1+\omega^2\tau^2} \tag{4.4}$$

where $\sigma_0 = \frac{nq^2\tau}{m}$ and $\omega = \frac{eB}{mc}$ is the cyclotron frequency. In the limit $\omega\tau > 1$ they become

$$\sigma_{xy} = \frac{nqc}{B} \qquad \text{and} \qquad \sigma_{xx} = 0. \tag{4.5}$$

Since $\sigma_{xx} = \sigma_{yy} = 0$ in an isotropic medium, $\rho_{xy} = \sigma_{xy}^{-1} = \frac{B}{nqc}$ so that the Hall resistivity is equal to Hall resistance and it depends on **B** linearly.

This discovery, named Hall effect, had found many uses in semi-conductor physics. Its quantum analog was even more fruitful. In 1980, Klaus von Klitzing has a similar experiment in the quantum regime; in very intense magnetic fields ($\approx 15T$) and very low temperatures ($\approx 1K$). The resulting Hall resistance is not linear in **B**, but is quantized as a ratio of two fundemental constants, charge of an electron and the Planck's constant: (see Figure:4.1)

$$\rho_H = \frac{1}{\nu}\frac{h}{e^2} \tag{4.6}$$

ν is an integer and it is called the filling factor related to Landau levels. The precision of the ratio is also amazing, with an uncertainity less than 1 part in 10 millions. von Klitzing was awarded the Nobel Prize for Physics for this discovery. Two years later in 1982, Horst L. Störmer and Daniel C. Tsui have performed similar experiments where they find that the filling factor ν can also be a fraction. Robert B. Laughlin has given a theoretical explanation and three of them shared the 1998 Nobel Prize for Physics.

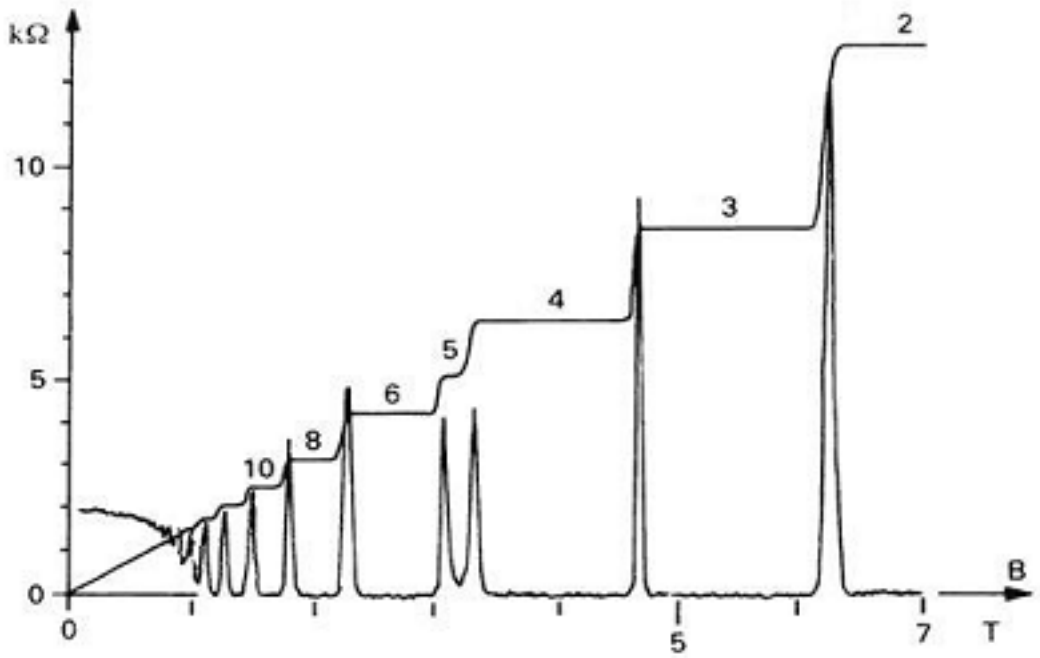

Figure 4.1: Hall resistance plateaus. The numbers indicate the filling factor.

This new phenomena is called either the integer or the fractional Quantum Hall Effect (QHE) depending on the value of ν. The constant ν being a fraction or integer has crucial implications in terms of the underlying physics. So, we will look at them separately, and

later we focus on the $\nu = 5/2$ state and show how non-abelian anyons may exist in this state.

4.1 Integer QHE

In the advanced Hall experiments beside the low temperature and high magnetic fields the thicknes of conducting plate is also extremely small, at the order of 10 Angstroms. MOSFET(metal oxide silicon field effect transistor) devices are used to achieve such thin layers of electrons. With a gate voltage V_g electrons are brought to the interface of Si and SiO_2 layers and in high magnetic field they are confined to two dimensions. Classical reasoning is insufficient to clarify the quantized Hall resistance. The ideal model of electrons in static, uniform magnetic fields (without impurities and neglecting electron-electron interactions) has been solved by Landau [18].

Consider a static, uniform magnetic field along the $\hat{z}$ direction. As in the AB effect, the Hamiltonian changes to

$$\frac{1}{2m}\left[(p_x - qyB)^2 + p_y^2\right]\Psi = E\Psi. \tag{4.7}$$

If we choose the Landau gauge, $A_x = -By$ and $A_y = 0$. Since $[H, p_x] = 0$ we can write the wave function $\Psi = e^{ixk_x}\psi(y)$ where $\psi(y)$ satisfies the simple harmonic oscillator equation shifted from the equilibrium

$$\frac{1}{2m}\left[p_y^2 + \omega^2(y - \frac{\hbar k_x}{m\omega})^2\right]\psi(y) = E_n\psi(y) \tag{4.8}$$

where $E_n = (n + 1/2)\hbar\omega$. Each value of n corresponds to a different Landau level that is degenerate. If periodic boundary conditions are assumed in the x-direction of the plane with sides L_x and L_y, we restrict k_x to be $k_x = \frac{2\pi N}{L_x}$. To be physically consistent shift in the oscillator should be in the range of L_y; So $0 < \frac{\hbar k_x}{m\omega} < L_y$. The maximum number of states is $N = \frac{L_x L_y eB}{hc}$, hence the density of electrons is $n = \frac{eB}{hc}$. If there is also an electric field E_x along the x-direction, electrons would move with the velocity $v_y = \frac{cE_x}{B}$. For a specific Landau level, the current density

$$j_y = nqv_x = q\frac{eB}{ch}\frac{cE_x}{B} = \frac{q^2}{h}E_x. \tag{4.9}$$

$q = e$ for electrons, and if the number of filled Landau levels is ν, then

$$\sigma_{yx} = \nu\frac{e^2}{h} = -\sigma_{xy} \tag{4.10}$$

which gives the Hall resistivity $\rho_{xy} = -\frac{h}{\nu e^2}$. Only the absolute value of ρ_{xy} is significant in experiments, so we get the same result as the value found in experiments.

Landau's theory arrives at the correct value for the Hall resistivity, but it does not answer other questions. According to Landau, the density of states versus the energy curve should be sharp, yet observationally it is broadened like a band. Laughlin proposed a *gedanken* experiment to explain such a broadening[19]. Assume that the conducting plane is wrapped on so that the magnetic field pierces the cylinderic surface at right angle and the current I encloses a loop. Current is related to the energy of the system and the magnetic flux Φ passing through that loop is given by

$$I = c\frac{\Delta U}{\Delta \Phi}, \tag{4.11}$$

when Φ is changed by a flux quantum hc/e, this corresponds to a gauge transformation which extends the -ideally- sharp lines of density of states. The energy will change by NeV_H where N is the number of extended states giving a current

$$I = \frac{Ne^2 V_H}{h} \tag{4.12}$$

and that gives the same Hall resistance $\rho_H = \frac{V_H}{I} = \frac{h}{Ne^2}$.

4.2 Fractional QHE

Tsui et. al. had done experiments in $GaAs$ - $Ga_xAl_{x-l}As$ heterostructures measuring the Hall resistance and found that $\nu = 1/3$ for 2-dimensional electron gas. The fractional filling factor is due to Coulomb interactions among electrons which was neglected in the IQH effect, for the ideal system. Laughlin argued that new states of matter, quasi-holes and quasi-electrons, are responsible for that fraction because they have a charge which is 1/3 of that of an electron [20]. The effect can be seen for the lowest Landau level, $\nu < 1$, because Coulomb interaction is small compared to the energy difference between Landau levels.

Consider a two-dimensional electron gas in the x-y plane, and a perpendicular magnetic field described in symmetric gauge by $\mathbf{A} = B/2(x\hat{j} - y\hat{i})$, and let the magnetic length $l = \sqrt{\frac{c\hbar}{eB}}$ be unity. Then the ground state wave function of n particles reads

$$|\Psi_{z_1,z_2,\ldots,z_n}\rangle = \prod_{j<k}(z_j - z_k)^m e^{-\sum_i \frac{|z_i|^2}{4}} \tag{4.13}$$

where $z = x + iy$ gives the position of the particle in the x-y plane and m is an odd integer. To write that equality Jastrow ansatz [15] is used

$$|\Psi_{z_1,z_2,\ldots,z_n}\rangle = \prod_{j<k} f_1(z_j - z_k) \prod_i f_2(z_i) \tag{4.14}$$

where f_1 stands for pair interactions and f_2 for the response of the system to an external field. f_2 is found from the solution of the Shrödinger equation for the non-interacting system. f_1 is found by the conditions it has to satisfy. It should be an analytic, odd function, because of the antisymmetry condition. Furthermore, conservation of angular momentum fixes the function to be $f_1(z) = z^m$ where m is odd.

An excited state of the system can be generated by inserting in a adiabatic way an infinitely thin solenoid at the position z_0 which has a flux quantum $\Phi = hc/e$. The resulting state vector will be

$$|\Psi_{z_0}\rangle = \prod_{i=1}^{n}(z_i - z_0) \prod_{j<k}(z_j - z_k)^m e^{\sum_i \frac{|z_i|^2}{4}}. \tag{4.15}$$

This describes a vortex since a phase of 2π is gained if one electron makes a loop around z_0. It is called quasi-hole because if an electron happens to be in the position z_0 wave function vanishes. Also note that the charge of it is $e/3$.

Let's look at two quasi-holes state at positions z_a and z_b

$$|\Psi_{z_a,z_b}\rangle = \prod_{i=1}^{n}(z_i - z_a)(z_i - z_b) \prod_{j<k}(z_j - z_k)^m e^{\sum_i \frac{|z_i|^2}{4}} \tag{4.16}$$

If we fix one quasi-hole and move the other around the loop C, the geometric phase will be

$$\gamma(C) = -2\pi \int\int_c d^2 z \rho(z) \tag{4.17}$$

where $\rho(z)$ is the density of particles at z [15]. The phase acquired is $\Delta\gamma(C) = 2\pi/m$ that is the difference between the quasi-hole is in the loop and not. Since m is an odd integer greater than one, $2\pi/m < \pi$ we conclude that the quasi-holes are abelian anyons.

Based on these observations a more elegant theory is developed, namely composite fermions theory. Composite fermions are bound states of electrons and quantized vortices. Electrons, in FQH states, capture even number of vortices and constitute the composite fermions which see a much reduced magnetic field than the electrons. Strong correlations of electrons is substituted by the weak interactions of composite fermions. In a sense FQH

of electrons is equivalent to the IQH effect of composite fermions. This theory identifies the FQH states of electrons for the fractions

$$\nu = \frac{n}{2pn \pm 1} \tag{4.18}$$

where $2p$ is the vorticity of composite fermions and n is the integer filling factor of them. These fractions match with the most of the experimental values, but with exceptions. In the next section we look at such an exception.

4.3 $\nu = 5/2$ *State*

Although Laughlin's theory and composite fermion theory explain the odd denominator filling factor in FQHE, they cannot resolve the even denominator filling factor or the filling factors greater than one. A specific example is $\nu = 5/2$ FQH state. Among the attempts to answer how this state exists, theory of Moore and Read is a widely accepted one[5]. What is interesting to us is that the non-abelian anyons, which Moore and Read called nonabelions, appear as quasi-particles of some FQH states and the $\nu = 5/2$ state is an example. Quasi-particles in this state are regarded as p-wave paired fully spin polarized composite fermions which have charge of $e/4$, that is a quarter of that of an electron. Only the lowest Landau level is full which contributes two filling factors because of two spin directions, and a half of one spin band of the second Landau level is occupied, which all together add up to $2 + 1/2 = 5/2$. The state vector of n particles of this type will be

$$|\Psi_{z_1,z_2,\ldots,z_n}\rangle = pf(\frac{1}{z_i - z_j}) \prod_{z_i<z_j} (z_i - z_j)^2 \prod_i e^{\frac{|z_i|^2}{4}}. \tag{4.19}$$

Here pf denotes the Pfaffian that gives the square root of a skew-symmetric matrix and $\frac{1}{z_i - z_j}$ are the non-diagonal elements of an $n \times n$ matrix[5].

$$pf(A_{ij}) = \frac{1}{2^k k!} \sum_{\sigma \in S_{2k}} sgn(\sigma) \prod_i^k A_{\sigma(2i-1)\sigma(2i)} \tag{4.20}$$

where $k = n/2$, S_{2k} is the symmetric group and $sgn(\sigma)$ is the signature of σ.

Moore and Read have shown that [5] a particle exchange in this setup has non-trivial effects and quasi-particles of the $\nu = 5/2$ state are indeed non-abelian anyons. However, experimental verification is not conclusive. There are many attempts to prove or disprove Moore and Read's argument. First thing to check is the spin polarization of quasi-particles.

There are different approaches [21] to measure the spin polarization for instance: photoluminescense spectroscopy, inelastic light scattering, varying Zeeman energy by changing the magnetic field. First two of these experiments indicate that there is no spin polarization, yet the last one supports the full spin polarization. In higher magnetic fields implication of spin polarization is dominant. This implies a phase transition between spin polarized and unpolarized states. A totally unpolarized state is not plausible because numerics indicate that unpolarized state gives higher energy than the polarized one, however partial polarization is possible. There are also several numerical works [1] which supports the idea of different phases of $\nu = 5/2$ state, the non-abelian phase and a compressible phase (for non-abelian anyons gap between the ground state and excited state should be incompressible).

Another issue is the charge of quasi-particles. If Moore and Read's theory is right then the charge of composite fermions should be one quarter of the electron charge. Experiments to measure that charge has also varying results[22], some of them supports the $e/4$ charge. The discrepancy of result is rooted in the difficulty of measurements and different techniques used. Nevertheless, neither these experiments nor the numerical works eliminate the possibility of having non-abelian anyons in the $\nu = 5/2$ state. On the contrary they gradually began to support that possibility. Of course it is not yet hundred percent proven that the non-abelian anyons exist in $\nu = 5/2$ FQH state. A better way is to try to make a quantum computation and see whether there exist non-abelian anyons or not. In next chapter we look at a proposed experiment which exactly does that.

Chapter 5

QUANTUM COMPUTATION BY USING NON-ABELIAN ANYONS

Having seen non-abelian anyons in 2-dimensions, it is now to answer how QC can be done by using them. As in the circuit model, anyonic QC is (or will be) done in three steps [8]

1. Initialization of the state $|\Psi\rangle$ by creating anyon pairs.

2. Unitary evolution of the state by braiding the anyons adiabatically and keeping the distance between them so that the topology of them is not destroyed .

3. Measuring $|\Psi\rangle$ by combining anyon pairs together and recording whether they are destroyed or not.

We should illustrate what are the corresponding mathematical and physical steps, and point the equivalency of anyonic QC scheme to other proposed schemes.

5.1 Theoretical Model

In order to construct a general theory of anyonic computation, we first assume that there are different kinds of anyons and each of them is labeled by a letter from the finite set $\{a, b, c, ...\}$ (we closely follow [8]). These labels specifys the physical properties of an anyon, a localized object, like the conserved charge of electrons. No label will change if anyons are kept apart or they are adiabatically moved. Local interactions also does not change it which is the most important property of topological phases discussed before. If there is no anyon, the label is set to 1 (some authors use 0). For each label there is an inverse label $\bar{a}$ (also called conjugate label) and fusing a label and its inverse gives 1, that is anyons annihilate.

Two anyons can be combined by bringing them close together and they may give another anyon. This process is called fusion and it is denoted by a cross product $a \times b = c$. But there

may be more than one way of fusing anyons and it is expressed symbolically as

$$a \times b = \sum_c N_{ab}^c c \tag{5.1}$$

where the coefficients N_{ab}^c give the number of different ways that a and b merge to give c. Therefore N_{ab}^c is zero if they do not fuse, otherwise it is a positive integer. N_{ab}^c can be regarded as the Clebsch-Gordan coefficients and labels spins as well as charges. Summation is over all labels, this gives fusion rules, how a and b are related to c. For a label and its inverse, there is a unique way to fuse $a \times \bar{a} = 1$. As there is fusion, there is also splitting. So c can split to its parts a and b. An important splitting is $1 = a \times \bar{a}$ creating anyon and its inverse from vacuum.

The order of the fusion is irrelevant, that fusion is a commutative and associative operation

$$a \times b = b \times a \qquad \text{and} \qquad (a \times b) \times c = a \times (b \times c). \tag{5.2}$$

since the total charge would not depend on the order of fusion, it is not like a chemical reaction. Yet in computation, we consider anyons in a sequence from left to right as $(a, b, c, d, ...)$ and this is different from, say $(c, b, d, a, ...)$. Although anyons are in a plane, it is possible to put a curve passing, say over all the anyons and think that curve as a line giving us their sequence. This order is crucial, because the braiding depends on it (see Appendix A:Braiding).

5.1.1 Fusion Space

Let V_{ab}^c be the fusion space of a and b fusing to c. Let each one of fusion ways is denoted as

$$|ab : c, \mu\rangle \qquad \mu = 1, 2, \ldots, N_{ab}^c \tag{5.3}$$

where μ represents the fusion way. It can be interpreted as the state vector of a and b when they are to fuse to c in the μ'th way. These vectors constitute an orthonormal basis of the Hilbert space V_{ab}^c. The dual vectors of them $\{\langle ab : c, \mu|\}$ constitute the Hilbert space of splitting V_c^{ab} which is the dual of V_{ab}^c. Two basis vectors from V_c^{ab} and V_{ab}^c are mutually orthogonal

$$\langle ab : c', \mu' | ab : c, \mu \rangle = \delta_{cc'}\delta_{\mu\mu'}. \tag{5.4}$$

For a pair of anyons, the full Hilbert space will be $\bigoplus_c V_{ab}^c$. Anyons are non-abelian if the dimension of the full Hilbert space is greater than two

$$dim\left(\bigoplus_c V_{ab}^c\right) = \sum_c N_{ab}^c \geq 2 \tag{5.5}$$

When two non-abelian anyons are brought together, different fusion channels are open, on the other hand for abelian anyons there is one way to merge: N_{ab}^c is 1 only for a specific c and zero for the rest. Different ways correspond to the degenerate ground states of the system of anyons. For many anyons if at least one pair is non-abelian, it is sufficient to call all the anyons non-abelian. Remember that this is a collective property of anyons rather than individual property of each anyon. So it is protected from local disturbances to specific anyons, once again emphasizing the topological aspect of anyonic QC.

5.1.2 Braiding R, F, and B matrices

In the computation process there would be n anyons in consecutive order $(a_1, a_2, a_3, ..., a_n)$ to be braided. Braiding means, in this simple setting, rearranging the initial order by exchanging a single neighbouring pair at each step. Then it is sufficient to know how two adjacent anyons will be braided and what is its effect on the fusion space.

Since anyons have specific order, V_{ab}^c and V_{ba}^c are not equal to each other, yet they are isomorphic. Isomorphism is associated with the (counterclockwise) exchange operator $R : V_{ab}^c \to V_{ba}^c$. Its action is

$$R\left|ab : c, \mu\right\rangle = \sum_{\mu} \left|ba : c, \mu'\right\rangle R_{\mu'}^{\mu} \tag{5.6}$$

R is called the "R-matrix" that has non-trivial effects on state vectors. If R is applied twice, it becomes an automorphism of V_{ab}^c and called a monodromy operator. R^2 is equivalent to moving one anyon around other completing a full cycle in the counterclockwise direction.

Fusion space of more than two anyons can be written as a tensor product of fusion spaces of two anyons. For three anyons (a, b, c) all combining to give d, the fusion space V_{abc}^d will be

$$V_{abc}^d \cong \bigoplus_e V_{ab}^e \otimes V_{ec}^d \cong \bigoplus_k V_{ak}^d \otimes V_{bc}^k, \tag{5.7}$$

since there are two ways $(a \times b) \times c$ and $a \times (b \times c)$. The two Hilbert spaces are equivalent, hence there is an isomorphism F which relates the basis vectors of $V^d_{(ab)c}$ and $V^d_{a(bc)}$

$$|(ab)c : d, e\mu\nu\rangle = |ab : e, \mu\rangle \otimes |ec : d, \nu\rangle\,, \qquad \left|a(bc) : d, k\mu'\nu'\right\rangle = \left|ak : d, \mu'\right\rangle \otimes \left|bc : k, \nu'\right\rangle \tag{5.8}$$

F is called the "F-matrix" and it acts on the basis states as

$$F\,|(ab)c : d, e\mu\nu\rangle = \sum_{k\mu'\nu'} \left|a(bc) : d, k\mu'\nu'\right\rangle F^e_k. \tag{5.9}$$

As explained above, there is not a unique way to construct the fusion space of many anyons. For the general case where there are n anyons, however, it is appropriate to fix a construction. Suppose, n anyons $(a_1, a_2, a_3, ..., a_n)$ are fused to give a total charge c in the following way. First a_1 and a_2 are fused to b_1, then b_1 and a_3 to give b_2, and so on.

$$V^c_{a_1a_2\ldots a_n} \cong \bigoplus_{b_1b_2\ldots b_{n-2}} V^{b_1}_{a_1a_2} \otimes V^{b_2}_{b_1a_3} \otimes V^{b_3}_{b_2a_4} \otimes \ldots \otimes V^c_{b_{n-2}a_n}. \tag{5.10}$$

There will be $(n-2)$ tensor products and $(n-1)$ fusion spaces of anyon pairs. A basis for $V^c_{a_1a_2\ldots a_n}$ can be given as product of bases of fusion spaces

$$\{\ |a_1a_2 : b_1, \mu_1\rangle\, |b_1a_3 : b_2, \mu_2\rangle \ldots |b_{n-2}a_n : c, \mu_{n-2}\rangle\ \}\,. \tag{5.11}$$

Hence the dimension of $V^c_{a_1a_2\ldots a_n}$ is exponentially large

$$dim(V^c_{a_1a_2\ldots a_n}) = N^c_{a_1a_2\ldots a_n} = \sum_{b_1b_2\ldots b_{n-2}} N^{b_1}_{a_1a_2} N^{b_2}_{b_1a_3} N^{b_3}_{b_2a_4} \ldots N^c_{b_{n-2}a_n}. \tag{5.12}$$

This decomposition is not compulsory, other choices are possible. Since $N^c_{a_1a_2\ldots a_n}$ does not change upon changing the decomposition and different decompositions are physically the same, all of them will give c at the end.

A plausible question is how will the braiding be represented in this bigger space $V^c_{a_1a_2\ldots a_n}$. The R-matrix exchanges anyon couples, but in $V^{b_j}_{b_{j-1}a_{j+1}}$, the b_{j-1} is not an anyon, it is rather a summation index. R can exchange a_j with a_{j+1} not b_{j-1} with a_{j+1}. To resolve this problem we should use the F-matrix.

Suppose a_j and a_{j+1} are to be braided. First thing is to bring a_j and a_{j+1} in the same fusion space of a pair. Let F-matrix act on $(\ldots \otimes V^{b_{j-1}}_{b_{j-2}a_j} \otimes V^{b_j}_{b_{j-1}a_{j+1}} \otimes \ldots)$ and turn it to $(\ldots \otimes V^{b_j}_{b_{j-2}b_{j-1}} \otimes V^{b_{j-1}}_{a_ja_{j+1}} \otimes \ldots)$. Then the R-matrix exchanges the anyons having

$(\ldots \otimes V^{b_{j-1}}_{a_{j+1}a_j} \otimes \ldots)$. However, the process is not yet finished, because we changed the standard decomposition. So again using the F-matrix, but this time with its inverse F^{-1}, we bring back the space to its early form $(\ldots \otimes V^{b_{j-1}}_{b_{j-2}a_{j+1}} \otimes V^{b_j}_{b_{j-1}a_j} \otimes \ldots)$. Of course, F or R cannot act directly on the vectors of $V^c_{a_1a_2\ldots a_n}$, rather the followingnew operators can act on them:

$$R_j = I \otimes \ldots I \otimes R \otimes I \ldots I, \tag{5.13}$$

where R-matrix acts on the j'th fusion space;

$$F_{j,j+1} = I \otimes \ldots I \otimes F \otimes I \ldots I, \tag{5.14}$$

where F-matrix acts on the j'th and $j+1$'th fusion spaces, with $j > 1$. For $j = 1$, R_1 is enough since the specific fusion space is $V^{b_1}_{a_1a_2}$.

The composition of these consecutive operations is denoted by B_j. It corresponds to the braiding of j'th anyon with $j+1$'th anyon in the standard basis and it is called braid matrix or "B-matrix". We write

$$B_j = F^{-1}_{j,j+1} R_j F_{j,j+1}. \tag{5.15}$$

For example, for three anyons case its action is (symbolically):

$$|(ab)c : d\rangle \to \sum |a(bc) : d\rangle \to \sum |a(cb) : d\rangle \to \sum |(ac)b : d\rangle \,. \tag{5.16}$$

5.1.3 Pentagon and Hexagon equations

One purpose of introducing R and F matrices is to relate different bases of the full Hilbert space of n anyons. They are basically useful isomorphisms. Two different bases can be related using R and F matrices in many ways, but the resulting isomorphism should be the same. This consistency is essential and there is a theorem which guaranties consistency of R and F matrices if pentagon and hexagon equations, which will be explained below, are satisfied.

Consider the following two bases of four anyons, first in the standard basis which fuse anyons from left to right, and second in a basis which fuses the anyons from right to left. These bases can be related to each other by using the F-matrix in two different ways. Symbolically they are shown as below, respectively. If the end bases (points) are fixed, the

shape of a pentagon appears and this justifies the naming.

$$(((a_1a_2)a_3)a_4) \longrightarrow ((a_1a_2)(a_3a_4)) \longrightarrow (a_1(a_2(a_3a_4))) \tag{5.17}$$

$$(((a_1a_2)a_3)a_4) \longrightarrow ((a_1(a_2a_3))a_4) \longrightarrow (a_1((a_2a_3)a_4)) \longrightarrow (a_1(a_2(a_3a_4))). \tag{5.18}$$

In equation form, they read

$$F_{23}F_{34}\left|L,b_1b_2\right\rangle = \sum_{xy}\left|R,xy\right\rangle (F_{34})^{b_2}_{x}(F_{23})^{b_1}_{y}, F_{23}F_{34}F_{23}\left|L,b_1b_2\right\rangle = \sum_{zkl}\left|R,kl\right\rangle (F_{23})^{b_1}_{z}(F_{34})^{z}_{k}(F_{23})^{b_2}_{l}.$$

Solving these two equations, we get the pentagon equation

$$(F_{34})^{b_2}_{x}(F_{23})^{b_1}_{y} = \sum_{z}(F_{23})^{b_1}_{z}(F_{34})^{z}_{x}(F_{23})^{b_2}_{y}. \tag{5.19}$$

Another consistency equation is found by considering the left (standard) basis and a new basis which shuffles three anyons. There are again two ways to relate these bases by using F and R matrices. They are

$$((a_1a_2)a_3) \xrightarrow{F} ((a_1(a_2a_3)) \xrightarrow{R} ((a_2a_3)a_1) \xrightarrow{F} (a_2(a_3a_1)) \tag{5.20}$$

and

$$((a_1a_2)a_3) \xrightarrow{R} ((a_2a_1)a_3)) \xrightarrow{F} (a_2(a_1a_3)) \xrightarrow{R} (a_2(a_3a_1)), \tag{5.21}$$

respectively. Left basis is transformed to new basis in two different ways;

$$F_{23}R_1F_{23}\left|L,b_1\right\rangle = \sum_{xy}\left|N,y\right\rangle (F_{23})^{x}_{y}(R_1)_x(F_{23})^{b_1}_{x}, R_1F_{23}R_1\left|L,b_1\right\rangle = \sum_{y}\left|N,y\right\rangle (R_1)_y(F_{23})^{b_1}_{y}(R_1)_{b_1}.$$

By solving the above equations, the hexagon equation is found

$$(R_1)_y(F_{23})^{b_1}_{y}(R_1)_{b_1} = \sum_{x}(F_{23})^{x}_{y}(R_1)_x(F_{23})^{b_1}_{x}. \tag{5.22}$$

R and F matrices do not have to obey other rules to make the anyon model valid. So, pentagon and hexagon equations, which are called Moore-Seiberg polynomial equations, are tests for the anyon model. Of course, it is another issue that whether there exist such anyons or not. But theoretically it is safe to work with an anyon model where R and F matrices satisfy these two equations.

5.1.4 Fibonacci Anyons

It is instructive to give a simple example, so we will look at Fibonacci anyons. In this model, there are two types of anyons; trivial anyon -or vacua- denoted by 1 and non-trivial one a (one may also use 0 and 1, respectively). Fusion rules are trivial

$$a \times 1 = 1 \times a = a \qquad \text{and} \qquad 1 \times 1 = 1, \tag{5.23}$$

except for the pair aa which fuses to either 1 or a

$$a \times a = 1 \qquad \text{or} \qquad a \times a = a \tag{5.24}$$

which means $\bar{a} = a$, from the left equation. Furthermore observe that $\sum_{c=1}^{a} N_{aa}^{c} = 2$. This implies that the Fibonacci model is non-abelian.

A meaningful creation of a pair will be $1 = a \times a$. We consider the n non-trivial anyons fusing to 1 giving the Hilbert space $V^1_{aa\ldots a}$, or more compactly $V^1_{a^n}$:

$$V^1_{a^n} \cong \bigoplus_{b_1 b_2 \ldots b_{n-3}} V^{b_1}_{aa} \otimes V^{b_2}_{b_1 a} \otimes V^{b_3}_{b_2 a} \otimes \ldots \otimes V^{a}_{b_{n-3} a} \tag{5.25}$$

It has the basis

$$\{\ |aa : b_1, \mu_1\rangle\, |b_1 a : b_2, \mu_2\rangle\, |b_2 a : b_3, \mu_3\rangle \ldots |b_{n-3} a : a, \mu_{n-3}\rangle\ \}\,. \tag{5.26}$$

This basis might be written more compactly only in terms of b_j's omitting μ_j's as $\{|b_1 b_2 b_3 \ldots b_{n-3}\rangle\}$, since b_j depends on b_{j-1} and there are only two possibilities. Notice that two consecutive labels cannot be both 1, for instance if b_2 is 1 than b_3 has to be a since $1 \times a = a$.

For two anyons aa there is clearly one way to give 1, and three anyons also fuse to 1 in one way because b_1 in $\bigoplus_{b_1} V^{b_1}_{aa} V^1_{b_1 a}$ cannot be a, since the pair $b_1 a$ might fuse to a rather than 1. For more than three anyons, $n > 3$, we can use the above notation and write

$$\begin{aligned} \{|1\rangle\,, |a\rangle\} \\ \{|1a\rangle\,, |aa\rangle\,, |a1\rangle\} \\ \{|1a1\rangle\,, |aa1\rangle\,, |a1a\rangle\, |1aa\rangle\,, |aaa\rangle\} \\ \vdots \end{aligned} \tag{5.27}$$

By looking at this pattern, if the dimension of $V^1_{a^n}$ is denoted by N^1_n, we infer the recursion formula

$$N^1_n = N^1_{n-1} + N^1_{n-2}. \tag{5.28}$$

This is the formula that generates the Fibonacci numbers, and hence the name of the model follows.

5.2 Equivalence of Anyonic QC

There are several approaches to QC; trough the use of quantum Turing machines, or quantum circuits, or modular functors [23]. The first two is well studied, but the last one, which we called anyonic QC, is a new model. So it should be shown that it is equivalent to other two models. Here we illustrate how it can be simulated with quantum circuits. Reminding the steps, we first create m anyon pairs a and $\bar{a}$ from vacuum so that there are $n = 2m$ anyons. Next we rearrange them by braiding, thatis applying the B-matrix. Finally we combine the anyon pairs and record which pair annihilates and which does not.

A first thing to note is that the full Hilbert space of anyons is not written as a direct tensor product, but a sum of tensor products of fusion spaces. It is not clear how qubits can be implemented in this Hilbert space, which is written as

$$V^1_{a_1 a_2 \ldots a_n} \cong \bigoplus_{b_1 b_2 \ldots b_{n-2}} V^{b_1}_{a_1 a_2} \otimes V^{b_2}_{b_1 a_3} \otimes V^{b_3}_{b_2 a_4} \otimes \ldots \otimes V^1_{b_{n-2} a_n} \tag{5.29}$$

when all n anyons are created from vacuum, having zero charge. Observe that b_{n-2} has to be $\bar{a}_n$ since there is one way to fuse to 1, hence there is not a sum over b_{n-2}. Then the full Hilbert space can be written as

$$V^1_{a_1 a_2 \ldots a_n} \cong \bigoplus_{b_1 b_2 \ldots b_{n-3}} V^{b_1}_{a_1 a_2} \otimes V^{b_2}_{b_1 a_3} \otimes V^{b_3}_{b_2 a_4} \otimes \ldots \otimes V^{\bar{a}_n}_{b_{n-3} a_{n-1}} \tag{5.30}$$

It is composed of tensor products of $(n-2)$ fusion spaces, yet it is not a single product. To overcome this problem consider the Hilbert space of all combinations of all pairs

$$\mathtt{H}_d = \bigoplus_{a,b,c} V^c_{ab} \tag{5.31}$$

where d stands for the dimension of $\mathtt{H}_d$ and is given by

$$d = \sum_{a,b,c} N^c_{ab}. \tag{5.32}$$

d does not depend on n, it is determined by the type of anyons that are used not with the number of them. $V^1_{a_1 a_2 \ldots a_n}$ is contained in a bigger space $\mathbf{H}$ which is a tensor product of $(n-2)$ of $\mathtt{H}_d$'s:

$$V^1_{a_1 a_2 \ldots a_n} \subset \mathbf{H} = \mathtt{H}_d \otimes \mathtt{H}_d \otimes \ldots \otimes \mathtt{H}_d. \tag{5.33}$$

H might be considered as a Hilbert space of $(n-2)$ qudits and its dimension is d^{n-2}. Hence the state of anyons can be simulated by $(n-2)$ qudits, instead of qubits.

Having introduced the qudits, the second thing is to show the equality of the B-matrix thus braiding with the quantum gates. It should be shown that acting on the state of anyons with B-matrix is equivalent to a unitary transformation on **H**. This is trivially true, since B_j is composed of R_j and $F_{j,j+1}$which are both isomorphisms. That is to say B_j is also an isomorphism. An isomorphism of Hilbert spaces is an orthogonal transformation. B_j is defined for $V^1_{a_1 a_2 \ldots a_n}$ which is a subspace of **H**, so B_j is a unitary transformation in **H** and corresponds to a two-qudit quantum gate. It is a two-qudit gate because the B-matrix acts on two fusion spaces and each fusion space is regarded as a qudit. Also note that when the state of anyons is written in the space **H** using another basis rather than the basis of $V^1_{a_1 a_2 \ldots a_n}$, the properties of R, F, and B matrices are protected, and our conclusions hold in both cases.

The final thing to check for the consistency of anyonic QC is fusion of anyons. A measurement consists of looking at the result of fusion of a pair of anyons. To do this in the standard basis, we should bring the anyon pair together in the same fusion space with the help of F-matrix to have $V^{b_{j-1}}_{a_j a_{j+1}}$. This fusion space is implemented by a qudit in **H**. When a_j and a_{j+1} fuse together their fusion space vanishes. Hence the qudit associated with it collapses as it should when a measurement is done. So fusion of anyon pair really means a measurement of a qudit. This concludes the equivalency of anyonic QC with the quantum circuit model.

5.3 NOT gate in $\nu = 5/2$ state

As in the early works on QC, theory precedes the experiment in anyonic QC. Although there is a good theory of computation by using anyons which minimizes errors, the actual anyons are missing and for that reason physicists do not pay much attention to it. However, this attitude has been changing with the possibility of existence of non-abelian anyons in $\nu = 5/2$ FQH state as discussed previously. Quasi-particles in that state are promising candidates enabling the anyonic QC. An essential step will be to make a simple gate operation by using them. Das Sarma et. al. [6] have proposed an experiment in which a NOT operation is performed by the qubits that are constructed by the quasi-particles in the $\nu = 5/2$ state.

Consider the Hall bar with three pairs of gate electrodes placed on the opposite sides as in the Figure:5.1 . When a current of quasiparticles is sent (from the left) it may go straight or reflected from the electrode pairs and turn back. These reflections may occur if appropriate voltages are applied to the gate pairs, and they are associated with tunneling amplitudes between the edges which are t_1 for M and N, t_s for A and B, and t_2 for the gates P and Q. If there are quasiparticles (anyons) in the region between the gate pairs M-N and P-Q, and the current (also anyons) encircle them then this process is a unitary transformation U on the state of the anyons(braiding). Let the state vector $|\xi\rangle$ stand for the case that the current turns back from the first gate pair, and $U|\xi\rangle$ denotes the case if it is reflected from the last gate pair. Then the four-terminal longitudinal conductivity will be

$$\sigma_{xx} \propto |t_1|^2 + |t_2|^2 + 2Re\left\{t_1^* t_2 e^{i\phi} \langle\xi| U |\xi\rangle\right\}. \tag{5.34}$$

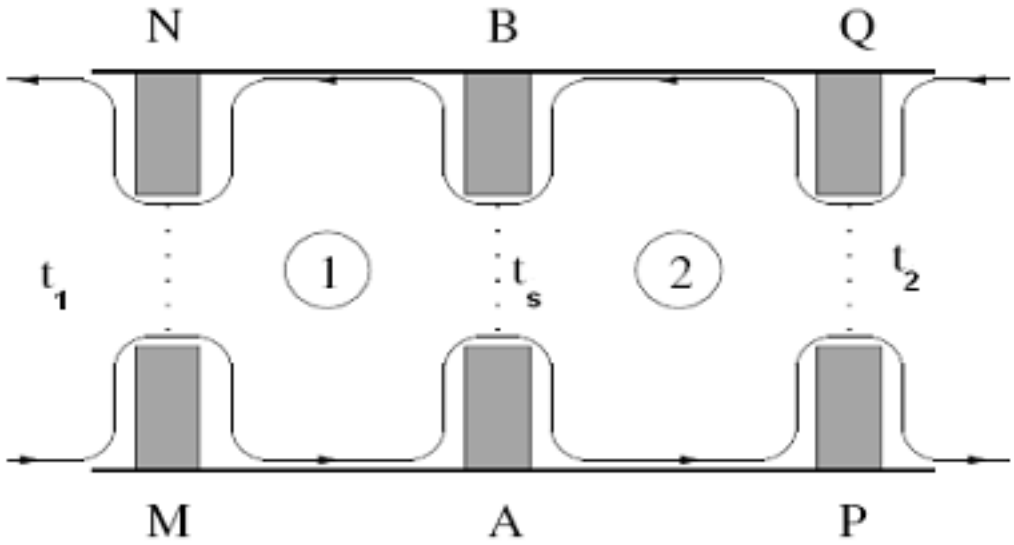

Figure 5.1: Hall bar and gate electrodes. Adopted from [6]

If there are an odd number of quasiparticles the term $\langle\xi| U |\xi\rangle$ vanishes. If there are an even number of them the expectation value would get plus or minus sign depending on the quaisparticles inside the region [1].

Two individually gated antidots (the opposite of a quantum dot) are placed in the two *cells* formed by the electrodes to hold quasiparticles. Electrons are excluded from these region by applying an electrostatic potential. If there is one (or an odd number of) $e/4$

charged quasiparticle on each antidot, they form a qubit. This qubit can be initialized by a measurement of σ_{xx}. The value of the qubit is $|0\rangle$ when $\sigma_{xx} \propto |t_1|^2 + |t_2|^2 + 2Re\left\{t_1^* t_2 e^{i\phi}\right\}$ and it is $|1\rangle$ when $\sigma_{xx} \propto |t_1|^2 + |t_2|^2 - 2Re\left\{t_1^* t_2 e^{i\phi}\right\}$.

Now in the second step, a quantum NOT gate is operated on the qubit by using the middle gate electrodes. If a quasiparticle (or an odd number of them) is tunneled from A to B so that the braiding is completed around the quasiparticle in the antidot 1, the value of the qubit is changed from $|0\rangle$ to $|1\rangle$ or from $|1\rangle$ to $|0\rangle$, and hence the NOT gate is applied to the qubit. But if an even number of quasiparticles tunnel from A to B, the qubit does not change. To ensure that only one quasiparticle travels from A to B, another antidot will be inserted between them. By tuning the voltage properly at gates A, B and at the new antidot, a single quasiparticle would encircle the antidot 1.

In the last step, again using the first and last gate pairs and measuring the longitudinal conductivity, the state of the qubit will be determined, and the computation ends. Notice that this process also proves that the quasiparticles in the $\nu = 5/2$ state are indeed non-abelian anyons because the value of σ_{xx} has changed although the gate voltages, magnetic field, and other properties of the Hall bar remains the same. If they were abelian anyons σ_{xx} would not have altered.

An error takes place only by unintended tunnelling across the gates because only way to change the state vector is by braiding. If a quasiparticle tunnels from A to B a bit flip error occurs while if it tunnels from P to Q a phase flip error occur. Both processes are related to the longitudinal resistivity which is very small since the density of quasiparticles decay exponentially at low temperatures. Thermally activated form of the error rate is [6]

$$\frac{\Gamma}{\Delta} \propto \frac{T}{\Delta} e^{-\Delta/T} < 10^{-30} \tag{5.35}$$

where Δ is the energy gap. For plausible systems this gives error rate of 10^{-15} which is much better then the other proposed QC systems.

Anyon model of the $\nu = 5/2$ state has three labels $\{1, \psi, \sigma\}$. The nontrivial fusion rules are

$$\sigma \times \sigma = 1 \quad \text{or} \quad \sigma \times \sigma = \psi, \qquad \sigma \times \psi = \sigma \qquad \text{and} \qquad \psi \times \psi = 1 \tag{5.36}$$

In the above proposed computation two anyons that are kept in the antidots are of type σ and their fusion will either give 1 or ψ, corresponding to the qubit values $|0\rangle$ and $|1\rangle$,

respectively. When another σ encircles one of them, the state of the qubit flips. Other gates might also be constructed by braiding, yet this setting is not enough for universal QC. This problem can be solved by two non-topological operations; a single-qubit $\pi/8$ phase gate and a two-qubit measurement[1].

Chapter 6

CONCLUSIONS

The idea of quantum computation is one of the most brilliant ideas of current physics community. Thus, scientists are working seriously to make a quantum computer because it will dethrone the classical computation and information processing. On the way to realize this idea, maybe the biggest obstacle is decoherence of the physical systems that are used in the construction of quantum computers.

A smart way to avoid decoherence and minimize errors is to use non-abelian anyons which are particles living only on 2-dimensional surfaces. Anyons are strange particles which do not obey Fermi or Bose statistics, rather they have fractional statistics. The interchange of two anyons results in a geometric phase factor $e^{i\phi}$ with $0 < \phi < \pi$. Furthermore, if anyons are non-abelian then this phase factor is topological, i.e. it is a unitary matrix which switchs the eigenstate of the system with a different one in the same degenerate eigenstate space. So, an interchange of two non-abelian anyons results in a quantum gate operator. In this sense, a QC can be carried out by braiding non-abelian anyons and measuring the system at the end. Here, the problem is to find such physical objects.

In this work, we look at a possible system, $\nu = 5/2$ FQH state, that provides non-abelian anyons and the machinery to implement a quantum computation (NOT operator) by using these non-abelian anyons. Only the theoretical aspects of computation is adressed. The actual experiment has not yet been done because of the difficulties faced in dealing with QHE systems. Such systems exist under conditions near absolute zero temperature and at intense magnetic fields, so to find a way to manipulate the quasi-particles is a difficult task. Even if the experiments verify the theory there is still a major problem: the anyons of the $\nu = 5/2$ state do not match the sufficient conditions to give a universal set of quantum gates. But this situation can be handled by methods that are less safe. Also notice that there are other possible anyonic system which may provide universal quantum gates. The $\nu = 12/5$ state is one example. It is speculated that this state might have Fibonacci anyons

which enables universal QC[1].

Currently, other proposed quantum computer schemes have better applications, yet they are still far from QC that will prove Shor's factoring algorithm. Quantum computers built so far have very few qubits and even if there are sufficient number of qubits, the error rate is too high for reliable computation. Anyonic quantum computation is in its beginning period, so it is coming from behind, but in few years time it may catch on. The biggest advantage of anyonic QC is that a state of the system does not decohere. Although locally it can be disturbed, globally the state of the system is intact because topological phase of the anyons do not change unless anyons are brought closer. If qubits are build in these systems they are going to be robust. The $\nu = 5/2$ FQH state is the most likely case where such qubits are about to be built. It will not be a surprise, if quantum computation is realized by non-abelian anyons in the near future.

Appendix 1

BRAIDING

The braid group B_n can be defined in several ways [24]. Here we look at three formations since they all give good insights. Let's start with the usual one. Consider the n disjoint strings in the $2+1$ dimensional space (x, y, t) with their end points are fixed on the two parallel planes, for instance $(k_i, 0, 0)$ and $(k_f, 0, 1)$ where $k_i, k_f \in \{1, 2, ..., n\}$. k_i and k_f represent the initial and final positions of k'th string, respectively. Each interweaving is an element of B_n unless they are homotopic, i.e. they can be mapped to each other continuously or more mathematically

Definition 1 *Let X and Y be topological spaces, and $f, g : X \to Y$ are continuous functions. f and g are homotopic if there exists a continuous family of maps $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$.*

Being homotopic defines an equivalence relation on the set of continuous maps from X to Y. So instead of the specific braiding, elements are taken from the equivalence class of interweavings. For example the identity element is n straight lines as well as n curly strings where none of them is interweaved with another. The group operation is connecting the top of the first element to the bottom of the second element, with an appropriate scaling to fix the length in the t direction.

The second picture provides elements of B_n as the history of n noncolliding particles in the complex plane $\mathbb{C}$ with the initial and final positions are fixed. Elements of the group have the form

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), ..., \alpha_n(t)) \tag{A.1}$$

Every argument is a continuous function $\alpha_i : [0, 1] \to \mathbb{C}$ and $\alpha_i(t) \neq \alpha_j(t)$ for all t unless $i = j$. Again we consider $[\alpha]$ instead of α but the latter is handy so we use it. The group

multiplication is

$$\alpha\beta = \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2 \\ \beta(2t-1) & 1/2 \leq t \leq 1 \end{cases} \tag{A.2}$$

That picture is the closest to the physical system. There are n anyons which are moved on the plane without thouching each other than come back to the initial position, only with a different combination for instance first particle comes to the position of third one and so on. Identity element has constant arguments $\alpha_i(t) = i$ for $i \in 1, 2, ..., n$. The inverse of an element is constituted by $\alpha_i^{-1} = \alpha_i(1-t)$. These properties are sufficient to make this set a group, namely closure, associativity (which is straightforward), identity, and inverse element. Yet, B_n is not commutative for $n > 2$ (B_1 is trivial group and $B_2 \cong \mathbb{Z}$) as the name of the *nonabelian* anyons suggests. Also note that B_n is infinite group for $n > 1$.

By projection of the braids in the first picture to (x, t) plane we get a third picture where braids are vertical (in time direction) drawings. B_n can be constructed algebraically by generators σ_i for $1 \leq i \leq n$. Consider the n vertical lines in a row (the identity element), σ_i interweaves the i'th strand with the $i+1$'th strand so that the left one (i'th one) crosses over the right one and leaves all other strands as they are. This operation is equivalent to the counterclockwise interchange of i'th and $i+1$'th anyons. The inverse of generators σ_i^{-1} interweaves the same strands but this time $i+1$'th strand crosses over the i'th one. Elements of the braid group can be obtained by the generators $\{\sigma_1, \sigma_2, ..., \sigma_{n-1}\}$. For example an element of B_8 is represented by $\sigma_7\sigma_2\sigma_3$ in the order left to right and it is not equal to $\sigma_3\sigma_2\sigma_7$. Generators satisfy two defining relations

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i & |i-j| &\geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} & 1 &\leq i \leq 1 \end{aligned} \tag{A.3}$$

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