Newsvendor Model with Random Supply and Financial Hedging: Utility-Based Approach

by

Filiz SAYIN

A Thesis Submitted to the
Graduate School of Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Master of Science

in

Industrial Engineering

Koç University

July, 2011

Koç University
Graduate School of Sciences and Engineering

This is to certify that I have examined this copy of a master's thesis by

Filiz SAYIN

and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
examining committee have been made.

Committee Members:

Prof. Süleyman Özekici (Advisor)

Prof. Fikri Karaesmen (Advisor)

Prof. Barış Tan

Asst. Prof. Onur Kaya

Asst. Prof. Uğur Çelikyurt

Date:

ABSTRACT

Inventory models have received significant attention in the operations research and management literature. Because of the randomness in demand, these models include uncertainty and this uncertainty generates risks to the managers. In inventory literature, it is mostly assumed that the decision maker is risk-neutral and aims to maximize the expected cash flow. However, today it is known that most managers are risk-sensitive. In this thesis, we consider the single-period, single-item stochastic inventory model where the decision-maker (newsvendor) is risk-averse. Our newsvendor aims to maximize the expected utility of the cash flow, unlike the classical model where the newsvendor is risk-neutral. Moreover, we suppose that there are risks associated with the uncertainty in demand as well as supply. We consider supply randomness based on random yield, random capacity, and both random yield and capacity. Furthermore, we assume that the randomness in demand and supply are correlated with the financial markets. The inventory manager exploits this correlation and manages his risks by investing in a portfolio of financial instruments. The decision problem therefore includes not only the determination of the optimal ordering policy, but also the selection of the optimal portfolio at the same time. We first use a minimum-variance approach to this problem. After finding the optimal portfolio for a given order quantity that minimizes the variance of the cash flow, we determine the optimal order quantity that maximizes the expected utility of the hedged cash flow with this optimal portfolio. Moreover, we also consider the cases in which the randomness in demand and supply is correlated with the financial markets. We analyze these problems in detail and provide a risk-sensitive approach to inventory management. The analyses result in some interesting and explicit characterizations on the structure of the optimal policy. Finally, we present numerical examples to illustrate the effects of parameters on the optimal order quantity, and the effects of utility theory and financial hedging on variance, or risk reduction.

ÖZETÇE

Envanter modelleri, endüstri mühendisliği ve işletme yönetimi literatüründe en çok ele alınan konulardan biridir. Talebin zamanının ve miktarının bilinmiyor oluşundan dolayı, bu modeller rassallık içermektedir. Bu da doğal olarak, yönetici için bir risk oluşturmaktadır. Envanter literatüründe çoğunlukla riske karşı duyarsız olan insanlar ele alınmakta ve beklenen son nakit akışını eniyileme amaçlanmaktadır. Ama günümüzde çoğu yöneticinin karar verirken riske karşı duyarlı olduğu bilinmektedir.

Bu tezde, tek dönem ve tek ürün içeren rassal envanter modelleri, karar vericinin riske karşı duyarlı olduğu durumda tartışılacaktır. Karar vericinin amacı, klasik modellerde olduğu gibi beklenen nakit akışını eniyileme değil, nakit akışının fayda (utility) fonksiyonunu eniyilemek olacaktır. Ayrıca, rassallığın sadece müşteri talebi ile sınırlı olmadığı arzın da rassal olduğu, verilen siparişin hepsinin teslim alınamadığı, modeler dikkate alınacaktır. Tezin ilerleyen kısımlarında, rassallığı meydana getiren müşteri talebinin ve arzın, finansal bazı endeksler ya da varlıklar ile korelasyonu olduğu durumlar tartışılacaktır. Karar verici, bu varlıkların vadeli işlemler ve türev piyasalarında pozisyon alarak, bu korelasyondan yararlanacak ve dönem sonu nakit akışının riskini azaltacaktır. Böylece, karar problemi sadece sipariş miktarını belirlemek değil, aynı zamanda riski azaltacak en iyi portföyü de oluşturmak olacaktır. Bu çok kararlı problemin çözümünde en az-varyans yaklaşımı benimsenecektir. Önce sabit bir sipariş miktarı için dönem sonu nakit akışının varyansını en aza indiren portföy bulunacaktır. Ardından, hesaplanan bu portföy kullanılarak nakit akışından elde edilecek faydanın beklenen değerini ençoklayan sipariş miktarı belirlenecektir. Tüm bu modeler ayrıntısıyla incelenilecek ve envanter yönetimine riske duyarlı bir yaklaşım sergilenecektir. Tüm analizlerin sonucunda eniyi sipariş miktarı hakkında ilginç ve açık sonuçlar elde edilecektir. Son olarak da, simülasyondan faydalanarak, sayısal örneklerle yapılan analizlerin sonuçları değerlendirilecektir.

ACKNOWLEDGMENTS

First, I would like to thank my advisors Prof. Süleyman Özekici and Prof. Fikri Karaesmen for their great supervision and continuous support. They shared their knowledge and experience with me, paid attention to all the steps of my thesis and gave me the guidance that I required. I am proud of being their student.

I would like to thank Prof. Barış Tan, Asst. Prof. Onur Kaya and Asst. Prof. Uğur Çelikyurt for taking part in my thesis committee, for critical reading of this thesis and for their valuable suggestions and comments.

I also thank TÜBITAK (The Scientific and Technological Research Council of Turkey) for their generous financial support.

I am greatful to Müge Güçlü, Buşra Ürek and Umut Arıtürk not only for being wonderful friends, but also for being a family to me. I also would like to thank all my friends at Koç University for their valuable friendship, and for all the fun and good times we shared together.

Last but not least, I thank my parents, Ali and Emel, and my sister, Jülide for always believing in me and for supporting me at every step I have taken so far. Their morale support and encouragements were the driving forces behind all my success. I am very lucky to be part of such a wonderful family. I cannot do anything without their patience, encouragement and love.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

NOMENCLATURE

K	:	Random capacity
U	:	Random yield
D	:	Random demand
y	:	Order quantity
$Q(y)$	:	Random supply if order quantity is y
z_0	:	Initial wealth
c	:	Purchasing cost of one unit of inventory
v	:	Holding cost for one unit of inventory
p	:	Shortage cost for one unit of inventory
s	:	Selling price of one unit of inventory
$\hat{p}$	:	Critical ratio or the probability of satisfying the demand on time
S_0	:	Price of a tradable asset in the market at time 0
S	:	Price of a tradable asset in the market at time T
u	:	Utility function
β	:	Risk tolerance
G_X	:	Cumulative distribution function of X
g_X	:	Probability density function corresponding to G_X

Chapter 1

INTRODUCTION

Managing inventory is crucial for any company dealing with physical products, including manufacturers, wholesalers and retailers. Since inventory management plays a key role for a successful business, many companies all around the world have been overhauling their inventory strategies. Operations research literature contributes by providing powerful tools for inventory management. The single-period, single-item stochastic inventory model is one of the most important problems in the inventory management area. This model deals with the inventory problem in which there is considerable uncertainty about future demands and the product is perishable. The product can be carried in inventory for only a limited period of time before it can be sold, such as periodicals, flowers, fresh fruits and vegetables, seasonal clothing, fashion goods and reservations for a particular flight. The decision maker needs to choose an appropriate order quantity that balances the cost of ordering too many against the cost of ordering too few. Since a newsvendor's decision of choosing the daily newspaper quantity exactly describes the problem, the model has been called the newsvendor model. In the newsvendor literature, it is mostly assumed that the decision maker is risk-neutral and is only concerned about the expected return. So, the expected profit maximization is the common solution approach in this area. However, today most managers and decision makers are risk-sensitive, so they are also interested in decision support tools that involve risk perceptions.

Since the problem includes uncertainty, risk exposure is an inevitable outcome. Often, decision makers take a conservative attitude toward risks. In other words, they are risk-averse and want to avoid risk as much as possible. Risk-averse persons are harmed by a dollar of loss more than they are benefited from a dollar of gain. To get rid of the risk of an undesirable outcome, they may choose an order quantity that results in lower expected profits. The literature reports some methods to control the riskiness of the problem. Within this research stream, expected utility theory, satisfaction probability function and Value-at-Risk (VaR) are the most commonly used methods for modelling risk-sensitivity. In the expected utility theory framework, the aim of the risk-averse decision maker is to maximize the expected utility of the cash flow. In this framework, the utility function corresponds to the satisfaction of the decision maker from the cash flow. Alternatively, the satisfaction probability maximization refers to the maximization of the probability of achieving a "target" level of profit. Finally, VaR measures the potential loss in value of a random function over a defined period for a given confidence interval. The expected utility maximization approach is the focus of this study. As we are dealing with the risk-averse decision makers in this thesis, concave utility functions are used.

In the literature of the newsvendor model, it is mostly assumed that the demand is the only source of randomness. Although the major source of randomness is the demand, supply may also be random. So, supply uncertainty should not be neglected. In recent years, interest in supply uncertainty has increased. The randomness in the supply process may result from several reasons. Two main components of a supply process are production and transportation. Because of the unforeseen events during the production and transportation processes, the quantity received may not be equal to the quantity ordered. Instead, a random amount depending on the order quantity is received. Among many others, long machine downtimes due to unplanned maintenance, strikes, seconds and scraps in a production run, lack of raw material and rework are some reasons which lead to uncertainty during the production stage. Moreover, accidents, deficiencies in the quality of transportation and various environmental factors cause uncertainty during the transportation stage. Due to these problems, the quantity received may be equal to some proportion of the ordered quantity or the capacity of the supplier may be limited by a random number. The existence of the supply randomness increases the level of uncertainty, so does the risk of the decision maker.

Inventory management, like other businesses, is inherently risky. Assuming the costs are fixed, the firm's profit depends on demand and supply. Therefore, the randomness of demand and supply increases the uncertainty of the problem. It seems obvious that risk-averse persons benefit from reducing uncertainty, so managers would want to reduce risk. It is mostly assumed in the literature that the demand and supply are independent from the financial market. However, in most business practices it is possible to find a correlation between the demand and supply with the financial market. Many financial instruments, such as forwards, calls and puts, are available that permit the decision maker to hedge the inventory risk. The impact of financial hedging on decision making is a critical point to analyze. The risk-sensitive decision maker not only tries to maximize the expected utility of the cash flow at the end of the period, but also needs to consider decreasing the risk or the variance of the cash flow by investing in a portfolio of market instruments that are correlated with his random demand and supply.

The motivation of this thesis is to analyze the risk-sensitive approach to the newsvendor problem under random demand and supply. This thesis is divided into two main parts based on the existence of the financial market. In the first part, the newsvendor problem is considered using the expected utility framework without financial hedging. We first analyze the problem when the demand is the only source of uncertainty. Then, we consider the problem when both supply and demand are random, and they are not necessarily independent. Random yield, random capacity and the combination of random yield and capacity is analyzed respectively. Then, the second part of the thesis considers the risk-sensitive newsvendor model with random demand and supply which are correlated with the financial market. After analyzing the model with no supply randomness, random yield, random capacity and random yield and capacity models are discussed. In each case, first a single asset model and then models with multiple assets are analyzed.

The organization of this thesis is as follows. The next chapter contains a review of relevant models in the literature. Chapter 3 focuses on inventory problems with random supply without considering financial hedging. In Chapter 4, we characterize the optimal policy structure for inventory problems with random supply and financial hedging opportunity. In Chapter 5, we discuss the models with illustrative examples. The results and the comments on the results are provided. Finally in Chapter 6, we give a general summary of the thesis and provide some directions for future research.

Chapter 2

LITERATURE REVIEW

Inventory management models, especially newsvendor models, have received significant attention in the literature. Within this literature, much has been written about the newsvendor who aims to maximize expected profit or minimize expected cost. Today, it is known that decision makers are risk-sensitive and risk management in inventory models is quite important. Therefore, the interest in risk-sensitive approaches has been increasing recently. The most common risk measurement techniques used in the literature are utility maximization, satisficing probability maximization and Value at Risk (VaR). We discuss the related literature with these techniques in Section 2.1. Financial hedging is another important issue that considers risk-sensitivity, the literature of which is discussed in Section 2.2. Apart from these, although the literature mostly ignores supply uncertainty, little has been written about random supply. Lastly, in Section 2.3, we discuss the literature about supply uncertainty.

2.1 Risk Sensitive Approaches

Although the majority of the inventory models in the literature are focused on the risk-neutral decision makers whose objective is expected profit maximization, some literature addresses risk-sensitive decision makers. The most common criteria used to model risk-sensitivity are utility and satisficing probability maximization in the literature. Further in this section, VaR is explained in relation to risk-sensitive inventory management.

The discussion of utility maximization began with Lau [37]. Then, Bouakiz and Sobel [7] examined the impact of exponential utility functions on optimal policies for both finite-horizon and infinite-horizon problem. Bouakiz and Sobel [7] show that a base-stock policy is optimal for a multi-period newsvendor problem when the objective is to maximize expected exponential utility of the present value of the net profit. Eeckhoudt et al. [18] study a risk-averse newsvendor who is allowed to obtain additional orders if demand is higher than his initial order. The newsvendor aims to maximize the expected utility of cash flow. They use an argument by Pratt [46] to show how risk-aversion affects optimal order quantity. This argument states that an increase in risk aversion equals the concave transformation of the utility function. They conclude that a risk-averse newsvendor will order less than a risk-neutral newsvendor and a more risk-averse newsvendor orders less. Moreover, they determine comparative statics of cost and price changes. They also analyze two types of changes in the degree of risk: adding background wealth risk and increasing demand risk. In our work, while considering the effects of risk-aversion and parameters we utilize their work and compare our results with theirs. Agrawal and Seshadri [1] also consider a risk-averse newsvendor with the objective of maximizing the expected utility. Their work differs from others because the decision maker does not only decide on an order quantity, but also decides on a selling price and the demand distribution is a function of this selling price. They consider two different cases in which price affects the demand. First, price only affects the scale of the demand distribution. Then, price affects the location of the demand distribution. They find that a risk-averse newsvendor will charge a higher price and order less than the risk-neutral newsvendor if price affects the scale of the demand distribution. And the risk-averse newsvendor will charge a lower price if price affects the location of the distribution, but the effect on the quantity ordered depends on the demand sensitivity to selling price. Agrawal and Seshadri [2] consider the importance of intermediaries in supply chains to reduce the financial risk faced by risk-averse retailers. They show that a risk-neutral distributor can offer a menu of mutually beneficial contracts to retailers so that the supply chain inefficiency can be avoided. Schweitzer and Cachon [8] analyze managers' newsvendor decisions. First, they summarize under which preferences the managerial decisions deviate from the expected profit maximization order. Then they analyze two experiments across different profit-margin conditions. In the first case, the demand distribution is known, and in the second case the demand range is increased. They conclude that for high-profit products the optimal order quantity is less than the order quantity maximizing the expected profit, and for low-profit products the optimal order quantity is more. Chen et al. [10] examine risk-aversion in a multi-period inventory model. Two problems, one where demand does not depend on price and another where demand depends on price, are considered. For different risk-averse utility functions, such as additive utility functions, they obtain the optimal policy. They also extend their work to models in which the decision maker has access to a complete or partially complete financial market.

Keren and Piliskin [33] study an expected utility maximizer newsvendor who is faced with a uniformly distributed demand. They also present a simple example when the newsvendor has an exponential utility function. The model in Ahmed et al. [3] is an extension of a risk-averse, singe-item, multi-period inventory model when the objective function is a coherent risk measure. A coherent risk measure is a quantifier for the risk of financial position which satisfies stochastic dominance conditions. Wang et al. [49] analyze how selling price affects the order quantity decision of a risk-averse newsvendor. They conclude that for bounded decreasing absolute risk aversion utility functions, a risk-averse decision maker orders less as selling price increases if the price is higher than a threshold value. Wang and Webster [51] consider the loss-averse newsvendor model by using a kinked piecewise-linear utility function. A loss-averse newsvendor may order more than a risk-neutral newsvendor if the penalty cost is negligible. Wu et al. [11] study the risk-averse newsvendor model with a mean-variance objective function. They show the importance of the stockout cost on the optimal order quantity. They state that the existence of the stock-out cost may change the conclusion that the risk-averse newsvendor orders less than the risk-neutral newsvendor.

Satisficing probability maximization refers to the probability of achieving a certain level of profit. Lau [37] discusses risk-aversion by considering the satisficing probability maximization method for a single-product model with shortage and salvage costs. This paper considers the problem under two objectives: maximizing expected utility and maximizing the probability of achieving a budgeted profit. Sankarasubramanian and Kumaraswamy [35] also consider the one-period inventory model where it is required to determine the order quantity which maximizes the probability of realizing a predetermined level of profit. Then, Lau and Lau [36] solve the newsvendor problem with two products by using the technique of maximizing the probability of achieving a target profit level. Li et al. [40] and [38] extend Lau and Lau [36] by considering uniformly and exponentially distributed demands. A more recent paper, Parlar and Weng [45], considers the objective of maximizing the expected profit and the probability of exceeding a prespecified and fixed target profit level together. Instead of a fixed target profit level, they introduce a more flexible satisficing objective like the probability of exceeding the expected profit.

VaR is another widely used risk measure of a possible losses. Simons [48], Jorion [27], Dowd [17] and Tapiero [50] contribute to the literature of VaR with their reviews. Gan et al. [21] examine the inventory coordination problem between retailer and supplier by using VaR concept to newsvendor model with a downside risk constraint. Özler et al. [29] study a single-period, multi-product problem by utilizing VaR as the risk measure. They derive the exact distribution function for the two-product newsvendor problem and develop an approximation method for the profit distribution of the multi-product case.

2.2 Financial Hedging

Today, financial markets have a wide range of products and it may be possible to find a financial asset which is correlated with the demand of a product. Therefore, there is an opportunity to use financial instruments to hedge the risk of inventory systems. The earlier paper related to this subject is by Anvari [5] which analyzes a single-period newsvendor model with no set-up costs by using a well-known framework, capital asset pricing model (CAPM). The decision is how much to invest on inventory and on financial assets. The paper clarifies the results by using a numerical example in which the demand is normally distributed. The resulting optimal policy is characterized and compared with the classical expected utility maximization structure. A more recent work is by Caldantey and Hough [9] that considers a related problem. They present results on a non-financial corporation which simultaneously chooses an optimal operating policy and an optimal trading strategy in the financial markets. The risk-averse corporation, with a mean-variance objective, dynamically hedges its profits when the profits are correlated with returns in the financial markets. They discuss how different informational assumptions, regarding whether or not the operational state variables are observable, result in different solution techniques. Chu et al. [13] provide a continuously reviewed model to mitigate inventory risks when uncertain demand is correlated with the financial market. A mean-variance criterion is used to develop an effective financial hedging policy for inventory managers. Gaur and Seshadri [22] present the problem of hedging inventory risk for the newsvendor model when the demand is correlated with the price of a financial asset. Their discussion is motivated by statistical evidence that an inventory index (Redbook), that represents average sales, is highly correlated with a financial index (SP500), that represents average asset prices. They use the SP500 index to construct static hedging strategies using both mean-variance and utility-maximization frameworks. They claim that the SP500 index has a high correlation with the demand as long as the products have discretionary demand. For a single-period problem, where there is a linear dependence between demand and the market index, they derive the hedged-cash flow for a perfectly-correlated arbitrage-free complete market showing that it is possible to make riskless profits from non-financial operations of a firm by using financial instruments. But since perfect correlation is not completely realistic in practice, they extend their framework to fit partially-correlated markets using expected utility maximization. An important aspect they pointed out is that the risk of inventory carrying can be replicated as a financial portfolio by using simple instruments like bonds, futures and options. According to their research, a risk averse decision maker orders more inventory when hedging is applied. They illustrate their work by an example which is also discussed in our work. Ding et al. [16] propose a framework to combine operational and financial hedging. They consider a global firm that sells to both home and foreign markets, and so faces demand and currency exchange uncertainties. Therefore, they study integrated operational and financial hedging by using a mean-variance utility function to model the firm's risk aversion in decision making when there are multiple products and suppliers. From an operational hedging perspective, they suggest that the firm exploit the capacity allocation by delaying the commitment until the demand and exchange rate uncertainties are realized. From a financial hedging perspective, they suggest that the firm use call and put option contracts for currency exchange rates. The firm can improve its profit by using operational hedging and decrease the profit variance by using financial hedging.

2.3 Random Supply

In many supply chain systems, demand is not the only source of uncertainty. Supply may also be random and this randomness contributes to the uncertainty of an inventory system. During production or transportation, the supply process may be disrupted because of some limitations or unforeseen events. Therefore, the amount received will not necessarily be equal to the amount ordered. Supply failures may be caused by natural disasters, labour disputes, machine failures, economic conditions, accidents, wars, terrorism, supplier equipment malfunctions and other causes (Chopra and Sodhi [12]). The work of Norrman and Jansson [26] is motivated by such an example. In 2001, a fire accident occurred at an Ericsson sub-supplier's plant where radio-frequency chips were produced. That accident interrupted the production and resulted in a loss of almost $400 million. That was the primary reason why Ericsson decided to withdraw from the mobile phone business. As an another example, due to financial problems faced by the UK chasis manufacturer UPF Thompson in 2001, the production was disrupted at Land Rover (Juttner [28]). Moreover, as Kharif [34] states, Motorola failed to ship the camera phones during the holiday season in 2003 because of component shortages. Businesses face such tragic examples everyday, so the importance of including random supply into inventory models increases.

Karlin [30] is the earliest paper modeling the fact that the quantity received is not necessarily equal to the quantity ordered. He argued that there is a critical level of initial inventory below which an order should be placed if the holding and shortage cost functions are convex increasing. Shih [47] considers the problem when some percentage of received products are defective. It is assumed that the percentage of defectives is a random variable with a known probability distribution. This work derives the necessary optimality condition for any distribution of the defective percentage. Noori and Keller [32] extend Shih's study by providing closed form solutions for the optimal order quantity for various distributions of the quantity received. They obtain analytical results for the amount received for uniformly and exponentially distributed demand. Gerchak et al. [24] also deal with random yield. They assume that there is an initial stock. They conclude that under random yield the optimal policy is not a order-up-to type anymore. The work of Henig and Gerchak [23] has more general assumptions about the random replenishment distribution and the cost structure. They show that a non-order-up-to policy is optimal in this case. Parlar and Wang [44] analyze a situation in which the newsvendor uses two suppliers, each having random yield. They conclude that diversification can provide a reduction in overall yield variability. Yano and Lee [39] provide a detailed discussion about random supply models. Ciarallo et al. [4] deal with the problem when randomness results from random capacity with a known distribution. They show that a base-stock policy is optimal. Jain and Silver [25] also consider the random capacity model when the newsvendor can assure the availability of a given level of capacity by paying a premium ahead of time. Özekici and Parlar [43] consider the random availability model so that the order is either fully satisfied or remains unfulfilled. Erdem and Özekici [19] follow the work of Özekici and Parlar [43]. They also consider that random capacity is the source of random supply. Gallego and Hu [20] assume that the capacity of the supplier is finite and the retailer receives a random proportion of amount produced. Arifoğlu and Özekici [6] extend Gallego and Hu's work [20] by considering that there is imperfect information and the environment is partially observed. Kazaz [31] analyzes a production plan when yield is also random and it effects the sales and purchasing prices. He illustrates the problem by using olive oil example. In a recent paper, Rekik et al. [15] consider the random yield models when both demand and supply error are uniformly or normally distributed. They also conclude that the random yield causes significant losses. Dada et al. [14] study the procurement problem of the newsvendor when it is supposed that there are multiple suppliers with different qualifications like less expensive or more reliable. Yang et al. [52] also consider a similar problem. They analyze how newsvendor makes a choice between a set of suppliers with different yields and prices.

Okyay et al. [41] summarize the random supply models in the literature under three groups: Random yield, random capacity and random yield and capacity. Let y be the amount ordered and $Q(y)$ be the amount received.

- Random Yield: Only a fraction of amount ordered can enter the stockpile and
$$Q(y) = yU$$
where U represents the proportion of nondefective items received.

- Random Capacity: The supplier has some random replenishment capacity K so that
$$Q(y) = \min\{K, y\}.$$
When an order is placed for y units, the suppliers will ship y if the total amount K of on hand inventory that they posses is greater than y. Or else, they will send all the inventory they posses, which is K.

- Random Yield and Capacity: This is another model that combines the previous two so that
$$Q(y) = U\min\{K, y\}.$$
Once y units are ordered, the supplier can ship at most K and only a proportion U is received in good shape.

Our work is closely related to Okyay et al. [41] and [42]. In Chapter 3, we consider the newsvendor model where there are risks associated with the uncertainty in demand as well as supply, like Okyay et al. [41]. Then in Chapter 4, we consider the model when the randomness in demand and supply is correlated with the financial markets, like Okyay et al. [42]. However, this thesis differs from their works by using the expected utility maximization framework instead of expected cash flow maximization.

Chapter 3

UTILITY-BASED MODELS WITH RANDOM DEMAND AND SUPPLY

In this chapter, we consider the newsvendor problem using a utility based approach. In Section (3.1), we first discuss the results for the standard newsvendor model when the objective is expected utility maximization. Then, we consider models with random supply. In Section (3.2), we discuss random yield models when the amount received is a fraction of the amount ordered. In Section (3.3), we discuss random capacity models when the capacity of supplier is random. Last, the combination of random yield and capacity models is analyzed in Section (3.4).

3.1 Standard Newsvendor Model

The problem of controlling the inventory of a single-item with a stochastic demand over a single-period, called newsvendor problem, is a well-known inventory management problem in the literature. Consider a decision maker (newsvendor) who decides how many items to order in the beginning of the period for sale during that period. If he orders too few, he will lose potential sales. If he orders too much, he must salvage all unsold items at a lower price. The decision-maker aims to find an optimal order quantity which balances these overage and underage costs. It is mostly assumed that the decision maker is risk-neutral and his aim is to maximize the expected profit. However, in practice it is observed that most decision makers are risk-sensitive and expected profit maximization is not adequate for them. They are conservative and prefer to avoid risk as much as possible. Therefore, in order to choose a risk-sensitive order quantity, expected utility theory framework can be used. The objective is to maximize the expected utility of the cash flow rather than the expected cash flow. As risk-averse decision makers are under consideration, concave utility functions are used in this problem.

Eeckhoudt et al. [18] examine the newsvendor problem with a risk-averse decision maker when the aim is to maximize the expected utility of the cash flow. They assume that it is allowed to buy additional newspapers during the selling period. They examine the effects of risk aversion and other parameters on the optimal order quantity. This section of our thesis is such a review of what they show in their paper. In the following sections, we add supply uncertainty to their model.

In this standard model, there is a continuous stochastic demand D with a known distribution function $G_D(x) = P\{D \leq x\}$ and a density function g_D. We suppose that the newsvendor has an initial wealth z_0. He buys items at a fixed unit purchase cost c and sells at a fixed unit sale price s. Unsold items can be salvaged at a fixed unit salvage value v. Moreover, if demand exceeds the order quantity, the newsvendor can buy additional items and sell them at the same cost. Therefore, in our problem we use a negative fixed unit shortage penalty p. To avoid trivial situations, $s > c > v \geq 0$ and $(c-s) < p \leq 0$. These parameters also satisfy $0 \leq s+p-c$ and $0 \leq s+p-v$. Our decision maker is risk-averse and his aim is to maximize the expected utility of the cash flow. To avoid trivial situations, we suppose that u is not equal to a constant and it is strictly increasing so that $u' > 0$. Moreover, the utility function is concave with $u'' \leq 0$. Our risk-sensitive decision maker decides on the order quantity y under the random demand D. The aim of the newsvendor is maximizing the expected utility of cash flow by choosing an ordering quantity y, or

$$\max_{y \geq 0} H(y) = E[u(CF(D,y))] \tag{3.1}$$

where $CF(D,y)$ is the random cash flow and it can be written as

$$\begin{aligned} CF(D,y) &= z_0 - cy + s\min\{D,y\} + v\max\{y-D,0\} - p\max\{D-y,0\} \\ &= z_0 - (c-v)y + (s+p-v)\min\{D,y\} - pD. \end{aligned} \tag{3.2}$$

For further analysis, let

$$CF(x,y) = \begin{cases} CF_-(x,y) = z_0 - (c-v)y + (s-v)x & x \leq y \\ CF_+(x,y) = z_0 + (s+p-c)y - px & x \geq y \end{cases}.$$

It clearly follows that $CF(y,y) = CF_-(y,y) = CF_+(y,y) = z_0 + (s-c)y$.

Note that for any random variable X with probability density function g_X, we can write

$$E[u(CF(X,y))] = \int_0^y u(CF_-(x,y))\, g_X(x)\, dx + \int_y^\infty u(CF_+(x,y))\, g_X(x)\, dx$$

and we can easily show that

$$\begin{aligned}\frac{d}{dy}E[u(CF(X,y))] &= -(c-v)\int_0^y u'(CF_-(x,y))\,g_X(x)\,dx \\ &\quad +(s+p-c)\int_y^\infty u'(CF_+(x,y))\,g_X(x)\,dx \\ &= -(c-v)\,E[u'(CF(X,y))1_{\{X\le y\}}] \\ &\quad +(s+p-c)\,E[u'(CF(X,y))1_{\{X>y\}}]. \end{aligned} \tag{3.3}$$

Here, (3.3) follows from

$$E[u'(CF(X,y))1_{\{X\le y\}}] = \int_0^y u'(CF_-(x,y))\,g_X(x)\,dx$$

and

$$E[u'(CF(X,y))1_{\{X>y\}}] = \int_y^\infty u'(CF_+(x,y))\,g_X(x)\,dx.$$

We can also show that

$$\begin{aligned}\frac{d}{dy}E[u'(CF(X,y))1_{\{X\le y\}}] &= u'(CF(y,y))\,g_X(y) \\ &\quad -(c-v)\int_0^y u''(CF_-(x,y))\,g_X(x)\,dx \\ &= u'(CF(y,y))\,g_X(y) \\ &\quad -(c-v)\,E\left[u''(CF_-(X,y))\,1_{X\le y}\right] \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\frac{d}{dy}E[u'(CF(X,y))1_{\{X>y\}}] &= -u'(CF(y,y))\,g_X(y) \\ &\quad +(s+p-c)\int_y^\infty u''(CF_+(x,y))\,g_X(x)\,dx \\ &= -u'(CF(y,y))\,g_X(y) \\ &\quad +(s+p-c)\,E\left[u''(CF_+(X,y))\,1_{\{X>y\}}\right]. \end{aligned} \tag{3.5}$$

In order to solve (3.1), we take the derivative of the objective function with respect to y and set it to zero. By using (3.3) where X is D, the first order condition is

$$\begin{aligned} g(y) &= \frac{d}{dy}E[u(CF(D,y))] \\ &= -(c-v)\,E\left[u'(CF(D,y))\,1_{\{D\le y\}}\right] + (s+p-c)E\left[u'(CF(D,y))\,1_{\{D>y\}}\right] \\ &= -(c-v)\,E\left[u'(CF(D,y))\,1_{\{D\le y\}}\right] \\ &\quad +(s+p-c)\left(E\left[u'(CF(D,y))\right] - E\left[u'(CF(D,y))\,1_{\{D\le y\}}\right]\right) \\ &= 0. \end{aligned} \tag{3.6}$$

Moreover, by using (3.4) and (3.5) where X is D, the second order condition can be obtained as

$$\begin{aligned}\frac{d^2E[u(CF(D,y))]}{dy^2} &= -(c-v)\left(u'(CF(y,y))\,g_D(y)-(c-v)\,E\left[u''(CF_-(D,y))\,1_{D\le y}\right]\right)\\ &\quad +(s+p-c)\begin{pmatrix}(s+p-c)\,E\left[u''(CF_+(D,y))\,1_{\{D>y\}}\right]\\ -u'(CF(y,y))\,g_D(y)\end{pmatrix}\\ &= -(s+p-v)\,u'(CF(y,y))\,g_D(y)\\ &\quad +(c-v)^2\,E\left[u''(CF_-(D,y))\,1_{D\le y}\right]\\ &\quad\quad +(s+p-c)^2\,E\left[u''(CF_+(D,y))\,1_{\{D>y\}}\right]\\ &\le 0. \end{aligned} \tag{3.7}$$

Since the second derivative in (3.7) is negative, the objective function is concave and the second order condition is satisfied. This also implies that $g(y)$ is decreasing in y.

From (3.6), we can conclude that the optimal order quantity y^*satisfies

$$\frac{E\left[u'(CF(D,y^*))\,1_{\{D\le y^*\}}\right]}{E\left[u'(CF(D,y^*))\right]}=\frac{s+p-c}{s+p-v}=\widehat{p} \tag{3.8}$$

where $\widehat{p}$ denotes a critical ratio which clearly satisfies $0\le\widehat{p}\le 1$. We will use the same critical ratio to characterize the optimal order quantity in the rest of this thesis. (3.8) gives the optimality condition provided that $g(0)\ge 0$ and $g(\infty)\le 0$. There may be no solution for (3.8). Since, $g(y)$ is decreasing in y, if $g(0)<0$ or $g(\infty)>0$, there may be no solution for the optimality condition. The optimal solution is $y^*=0$ if $g(0)<0$; that is

$$\begin{aligned} g(0) &= -(s+p-v)E\left[u'(CF(D,0))\,1_{\{D\le 0\}}\right]+(s+p-c)E\left[u'(CF(D,0))\right]\\ &= -(s+p-v)u'(z_0)\,P\{D=0\}+(s+p-c)E\left[u'(CF(D,0))\right]<0. \end{aligned}$$

Equivalently, we can conclude that if

$$P\{D=0\}>\left(\frac{E\left[u'(z_0-pD)\right]}{u'(z_0)}\right)\widehat{p} \tag{3.9}$$

then $y^*=0$. Since we know that $z_0-pD\ge z_0$, and so $u'(z_0-pD)\le u'(z_0)$, the right hand side of (3.9) is clearly between 0 and 1. If $P\{D=0\}=1$, the decision maker clearly orders nothing.

Moreover, the optimal solution is $y^*=\infty$ if $g(\infty)>0$; that is

$$\begin{aligned} g(\infty) &= -(s+p-v)E\left[u'(CF(D,\infty))\,1_{\{D<\infty\}}\right]+(s+p-c)E\left[u'(CF(D,\infty))\right]\\ &= -(c-v)E\left[u'(CF(D,\infty))\right]+(s+p-v)E\left[u'(CF(D,\infty))\right]P\{D=\infty\}>0. \end{aligned}$$

Equivalently, we can conclude that if

$$P\{D=\infty\} > \frac{c-v}{s+p-v} \tag{3.10}$$

then $y^*=\infty$. This argument supposes that u is bounded. Since $s+p-v>c-v$, the ratio in the right hand side of (3.10) is clearly between 0 and 1. If we assume that the demand is finite, or $P\{D=\infty\}=0$, the optimal order quantity y^* is also finite and satisfies (3.8). Moreover, if $P\{D=\infty\}=1$, we have $y^*=\infty$.

As a special case, suppose that the decision maker is risk-neutral, so the utility function is linear, that is $u(x)=a+bx$ so that $u'(x)=b$. Then, the optimality condition in (3.8) reduces to

$$P\{D\leq y^*\}=\widehat{p}$$

which is the same condition as in the standard risk-neutral newsvendor problem.

Up to this point, any concave utility function and any demand distribution is considered. We now solve a special case where the utility function is exponential and the demand is exponentially distributed. The utility function is $u(z)=K-Ce^{-\frac{1}{\beta}z}$ where K is the additive term, C is the multiplicative term and β represents the newsvendor's degree of risk tolerance. The demand D is exponentially distributed and the density function is $g_D(x)=\lambda e^{-\lambda x}$ where λ is the parameter of the distribution. The first order condition in (3.6) can be written as

$$\begin{aligned}\frac{d}{dy}E[u(CF(D,y))] &= -(c-v)\int_0^y \frac{C}{\beta}e^{-\frac{1}{\beta}(z_0-(c-v)y+(s-v)x)}\lambda e^{-\lambda x}dx \\ &\quad +(s+p-c)\int_y^\infty \frac{C}{\beta}e^{-\frac{1}{\beta}(z_0+(s+p-c)y-px)}\lambda e^{-\lambda x}dx \\ &= 0. \end{aligned} \tag{3.11}$$

After some manipulations on (3.11), the optimal order quantity y^* is obtained explicitly as

$$y^*=\frac{\beta}{(s-v)+\lambda\beta}\ln\left(\frac{(s-c+\lambda\beta)(s+p-v)}{(c-v)(-p+\lambda\beta)}\right). \tag{3.12}$$

As β goes zero, the optimal order quantity goes to zero. In other words, if the newsvendor is extremely risk-averse, he will order nothing. Moreover, as β goes infinity, which means that the newsvendor is not risk-averse, the optimal order quantity goes to

$$y^*=\frac{1}{\lambda}\ln\left(\frac{(s+p-v)}{(c-v)}\right).$$

That is the same as the optimal order quantity for risk-neutral newsvendor when the demand is exponentially distributed.

The optimality condition (3.8) states that the optimal order quantity depends on many parameters. In the following subsections, we deal with the effects of these parameters on the optimal order quantity. Each parameter is analyzed one by one.

3.1.1 The Effect of Risk-Aversion

In order to analyze the effect of the risk-aversion on optimal order quantity, Eeckhoudt et al. [18] used an argument by Pratt [46]. Based on this argument, an increase in risk aversion equals to a concave transformation of the utility function. Therefore, in order to show the effect of the risk aversion, we replace $u(x)$ with $k\left(u(x)\right)$, where $k' > 0$ and $k'' < 0$. To analyze the effect of risk-aversion on the order quantity, we will use the Pratt's argument throughout this chapter.

It can be clearly shown that the cash flow is increasing in the demand since

$$CF_{-}\left(x_1, y^*\right) \leq CF\left(y^*, y^*\right) \leq CF_{+}\left(x_2, y^*\right)$$

for all $x_1 \leq y^* \leq x_2$. Then, since the utility function is concave increasing,

$$u\left(CF_{-}\left(x_1, y^*\right)\right) \leq u\left(CF\left(y^*, y^*\right)\right) \leq u\left(CF_{+}\left(x_2, y^*\right)\right)$$

and

$$u'\left(CF_{-}\left(x_1, y^*\right)\right) \geq u'\left(CF\left(y^*, y^*\right)\right) \geq u'\left(CF_{+}\left(x_2, y^*\right)\right).$$

for all $x_1 \leq y^* \leq x_2$. And, since k is another concave function,

$$k'\left(u\left(CF_{-}\left(x_1, y^*\right)\right)\right) \geq k'\left(u\left(CF\left(y^*, y^*\right)\right)\right) \geq k'\left(u\left(CF_{+}\left(x_2, y^*\right)\right)\right) \tag{3.13}$$

for all $x_1 \leq y^* \leq x_2$. We will use (3.13) for further analysis.

We suppose that the objective of the more risk-averse newsvendor is

$$\max_{y \geq 0} \widetilde{H}(y) = E[k\left(u(CF(D, y))\right)]$$

and the first derivative of this objective function is

$$\begin{aligned}
\widetilde{g}\left(y\right) &= -\left(c-v\right)E[k'\left(u(CF(D,y))\right)u'(CF(D,y))1_{\{D\leq y\}}] \\
&\quad +\left(s+p-c\right)E[k'\left(u(CF(D,y))\right)u'(CF(D,y))1_{\{D>y\}}] \\
&= -\left(c-v\right)\int_0^y k'\left(u\left(CF_{-}(x,y)\right)\right)u'\left(CF_{-}(x,y)\right)g_D\left(x\right)dx \\
&\quad +\left(s+p-c\right)\int_y^\infty k'\left(u\left(CF_{+}(x,y)\right)\right)u'\left(CF_{+}(x,y)\right)g_D\left(x\right)dx. \qquad (3.14)
\end{aligned}$$

The optimal order quantity y^* satisfying the optimality condition (3.8) is plugged into equation (3.14). By using the inequality (3.13), it can be shown that

$$\begin{aligned}\widetilde{g}(y^*) &= \Bigg(-(c-v)\int_0^{y^*} k'\left(u\left(CF_-(x,y^*)\right)\right)u'\left(CF_-(x,y^*)\right)g_D(x)\,dx \\ &\qquad + (s+p-c)\int_{y^*}^{\infty} k'\left(u\left(CF_+(x,y^*)\right)\right)u'\left(CF_+(x,y^*)\right)g_D(x)\,dx\Bigg) \\ &\le k'\left(u\left(CF\left(y^*,y^*\right)\right)\right)\Bigg(-(c-v)\int_0^{y^*} u'\left(CF_-(x,y^*)\right)g_D(x)\,dx \\ &\qquad + (s+p-c)\int_{y^*}^{\infty} u'\left(CF_+(x,y^*)\right)g_D(x)\,dx\Bigg) \\ &= 0\end{aligned}$$

so that $\widetilde{g}(y^*) \le 0$. Because $\widetilde{H}$ is concave in y, the inequality implies that the optimal order quantity for the problem with more risk aversion $\widetilde{y}^*$ will be less than y^*. In other words, as risk-averseness increases, the optimal order quantity decreases.

The analysis of the risk-aversion by using Pratt's argument is a review of Eeckhoudt et al. [18]. We further use the same argument by analyzing the effect of risk-aversion for the newsvendor problem with random supply. The following analyses are also done by Eeckhoudt et al. [18], but we follow different methods while doing them.

3.1.2 The Effect of Initial Wealth

In this subsection, we want to see the effect of initial wealth on order quantity. We now redefine the cash flow as

$$CF\left(D,y\left(z_0\right),z_0\right) = z_0 - (c-v)\,y\left(z_0\right) + (s+p-v)\min\left\{D,y\left(z_0\right)\right\} - pD$$

and

$$CF(x,y\left(z_0\right),z_0) = \begin{cases} CF_-(x,y\left(z_0\right),z_0) = z_0 - (c-v)\,y\left(z_0\right) + (s-v)\,x & x \le y\left(z_0\right) \\ CF_+(x,y\left(z_0\right),z_0) = z_0 + (s+p-c)\,y\left(z_0\right) - px & x \ge y\left(z_0\right) \end{cases}.$$

To analyze the effect of initial wealth on the optimal order quantity, we differentiate the optimality condition in (3.8) with respect to z_0. That is

$$\frac{d}{dz_0}\left(\frac{E\left[u'\left(CF\left(D,y\left(z_0\right)^*,z_0\right)\right)1_{\{D\le y(z_0)^*\}}\right]}{E\left[u'\left(CF\left(D,y\left(z_0\right)^*,z_0\right)\right)\right]}\right) = \frac{d}{dz_0}\left(\frac{s+p-c}{s+p-v}\right)$$

or

$$\frac{\left\{\begin{array}{c} E\left[u'\left(CF\left(D,y\left(z_0\right)^*,z_0\right)\right)\right]\frac{d}{dz_0}\left(E\left[u'\left(CF\left(D,y\left(z_0\right)^*,z_0\right)\right)1_{\{D\le y(z_0)^*\}}\right]\right) \\ -E\left[u'\left(CF\left(D,y\left(z_0\right)^*,z_0\right)\right)1_{\{D\le y(z_0)^*\}}\right]\frac{d}{dz_0}\left(E\left[u'\left(CF\left(D,y\left(z_0\right)^*,z_0\right)\right)\right]\right)\end{array}\right\}}{\left(E\left[u'\left(CF\left(D,y\left(z_0\right)^*,z_0\right)\right)\right]\right)^2}=0 \quad (3.15)$$

We know that for any random variable X with a probability density function g_X,

$$E\left[u'\left(\boldsymbol{CF}\right)\right]=E\left[u'\left(\boldsymbol{CF}\right)1_{\{X\le y(z_0)^*\}}\right]+E\left[u'\left(\boldsymbol{CF}\right)1_{\{X>y(z_0)^*\}}\right] \quad (3.16)$$

where $\boldsymbol{CF}=CF\left(X,y\left(z_0\right)^*,z_0\right)$. More precisely,

$$E\left[u'\left(\boldsymbol{CF}\right)1_{\{X\le y(z_0)^*\}}\right]=\int_0^{y(z_0)^*}u'\left(CF_-(x,y\left(z_0\right)^*,z_0)\right)g_X\left(x\right)dx$$

and

$$E\left[u'\left(\boldsymbol{CF}\right)1_{\{X>y(z_0)^*\}}\right]=\int_{y(z_0)^*}^{\infty}u'\left(CF_+(x,y\left(z_0\right)^*,z_0)\right)g_X\left(x\right)dx.$$

Then, we can easily show that

$$\begin{aligned} & \frac{d}{dz_0}\left(E\left[u'\left(\boldsymbol{CF}\right)1_{\{X\le y(z_0)^*\}}\right]\right) \\ = \; & y'\left(z_0\right)^*u'\left(CF(y\left(z_0\right)^*,y^*\left(z_0\right),z_0)\right)g_X\left(y\left(z_0\right)^*\right) \\ & +\int_0^{y(z_0)^*}\left(1-\left(c-v\right)y'\left(z_0\right)^*\right)u''\left(CF_-(x,y\left(z_0\right)^*,z_0)\right)g_X\left(x\right)dx \\ = \; & y'\left(z_0\right)^*u'\left(CF(y\left(z_0\right)^*,y^*\left(z_0\right),z_0)\right)g_X\left(y\left(z_0\right)^*\right) \\ & +\left(1-\left(c-v\right)y'\left(z_0\right)^*\right)E\left[u''\left(\boldsymbol{CF}\right)1_{\{X\le y(z_0)^*\}}\right] \end{aligned} \quad (3.17)$$

and

$$\begin{aligned} & \frac{d}{dz_0}\left(E\left[u'\left(\boldsymbol{CF}\right)1_{\{X>y(z_0)^*\}}\right]\right) \\ = \; & -y'\left(z_0\right)^*u'\left(CF(y\left(z_0\right)^*,y^*\left(z_0\right),z_0)\right)g_X\left(y\left(z_0\right)^*\right) \\ & +\int_{y(z_0)^*}^{\infty}\left(1+\left(s+p-c\right)y'\left(z_0\right)^*\right)u''\left(CF_+(x,y\left(z_0\right)^*,z_0)\right)g_X\left(x\right)dx \\ = \; & -y'\left(z_0\right)^*u'\left(CF(y\left(z_0\right)^*,y^*\left(z_0\right),z_0)\right)g_X\left(y\left(z_0\right)^*\right) \\ & +\left(1+\left(s+p-c\right)y'\left(z_0\right)^*\right)E\left[u''\left(\boldsymbol{CF}\right)1_{\{X>y(z_0)^*\}}\right]. \end{aligned} \quad (3.18)$$

By using (3.16), (3.17) and (3.18), where X is D, (3.15) can be reduced to

$$\frac{\left\{\begin{array}{c} E\left[u'\left(\boldsymbol{CF}\right)1_{\{D>y(z_0)^*\}}\right]\frac{d}{dz_0}\left(E\left[u'\left(\boldsymbol{CF}\right)1_{\{D\le y(z_0)^*\}}\right]\right) \\ -E\left[u'\left(\boldsymbol{CF}\right)1_{\{D\le y(z_0)^*\}}\right]\frac{d}{dz_0}\left(E\left[u'\left(\boldsymbol{CF}\right)1_{\{D>y(z_0)^*\}}\right]\right)\end{array}\right\}}{\left(E\left[u'\left(\boldsymbol{CF}\right)\right]\right)^2}=0$$

or, more precisely

$$\frac{\left\{\begin{array}{c} y'(z_0)^* u'(CF(y(z_0)^*, y^*(z_0), z_0)) f_D(y(z_0)^*) E[u'(\boldsymbol{CF})] \\ +(1-(c-v)y'(z_0)^*) E\left[u'(\boldsymbol{CF}) 1_{\{D>y(z_0)^*\}}\right] E\left[u''(\boldsymbol{CF}) 1_{\{D\leq y(z_0)^*\}}\right] \\ -(1+(s+p-c)y'(z_0)^*) E\left[u'(\boldsymbol{CF}) 1_{\{D\leq y(z_0)^*\}}\right] E\left[u''(\boldsymbol{CF}) 1_{\{D>y(z_0)^*\}}\right] \end{array}\right\}}{(E[u'(\boldsymbol{CF})])^2} = 0. \tag{3.19}$$

From (3.19), the derivative of the optimal order quantity with respect to z_0 can be obtained as

$$y'(z_0)^* = \frac{\left\{\begin{array}{c} -E\left[u'(\boldsymbol{CF}) 1_{\{D>y(z_0)^*\}}\right] E\left[u''(\boldsymbol{CF}) 1_{\{D\leq y(z_0)^*\}}\right] \\ +E\left[u'(\boldsymbol{CF}) 1_{\{D\leq y(z_0)^*\}}\right] E\left[u''(\boldsymbol{CF}) 1_{\{D>y(z_0)^*\}}\right] \end{array}\right\}}{\left\{\begin{array}{c} u'(CF(y(z_0)^*, y^*(z_0), z_0)) f_D(y(z_0)^*) E[u'(\boldsymbol{CF})] \\ -(c-v) E\left[u'(\boldsymbol{CF}) 1_{\{D>y(z_0)^*\}}\right] E\left[u''(\boldsymbol{CF}) 1_{\{D\leq y(z_0)^*\}}\right] \\ -(s+p-c) E\left[u'(\boldsymbol{CF}) 1_{\{D\leq y(z_0)^*\}}\right] E\left[u''(\boldsymbol{CF}) 1_{\{D>y(z_0)^*\}}\right] \end{array}\right\}}. \tag{3.20}$$

In general, the sign of $y'(z_0)^*$ shows how the order quantity is affected by the change in initial wealth. If it is negative, y^* decreases as z_0 increases and if it is positive, y^* increases as z_0 increases. However, the sign of (3.20) cannot be clearly seen because the sign of numerator is indeterminate.

We, therefore, analyze the effect of z_0 for a special case: the exponential utility function where $u(z) = K - Ce^{-\frac{1}{\beta}z}$, $u'(z) = \frac{C}{\beta}e^{-\frac{1}{\beta}z}$ and $u''(z) = -\frac{C}{\beta^2}e^{-\frac{1}{\beta}z}$. Now, the numerator of (3.20) is

$$\left\{\begin{array}{c} -E\left[\frac{C}{\beta}e^{-\frac{1}{\beta}(\boldsymbol{CF})}1_{\{D>y(z_0)^*\}}\right] E\left[-\frac{C}{\beta^2}e^{-\frac{1}{\beta}(\boldsymbol{CF})}1_{\{D\leq y(z_0)^*\}}\right] \\ +E\left[\frac{C}{\beta}e^{-\frac{1}{\beta}(\boldsymbol{CF})}1_{\{D\leq y(z_0)^*\}}\right] E\left[-\frac{C}{\beta^2}e^{-\frac{1}{\beta}(\boldsymbol{CF})}1_{\{D>y(z_0)^*\}}\right] \end{array}\right\}$$

which clearly equals 0. So, it can be concluded that the initial wealth has no effect on order quantity when the utility function is exponential.

Alternatively, we can also show the effect of the initial wealth on the order quantity when the utility function is exponential by writing the first order condition as

$$\begin{aligned} g(y^*) &= -(c-v)\int_0^{y^*} \frac{C}{\beta}e^{-\frac{1}{\beta}(z_0-(c-v)y^*+(s-v)x)} g_D(x)\,dx \\ &\quad +(s+p-c)\int_{y^*}^{\infty} \frac{C}{\beta}e^{-\frac{1}{\beta}(z_0+(s+p-c)y^*-px)} g_D(x)\,dx \\ &= \frac{C}{\beta}e^{-\frac{1}{\beta}(z_0)}\left(\begin{array}{c} (s+p-c)\int_{y^*}^{\infty} e^{-\frac{1}{\beta}((s+p-c)y^*-px)} g_D(x)\,dx \\ -(c-v)\int_0^{y^*} e^{-\frac{1}{\beta}(-(c-v)y^*+(s-v)x)} g_D(x)\,dx \end{array}\right) \\ &= 0 \end{aligned}$$

and the optimal solution that satisfies $g(y^*) = 0$ does not depend on the initial wealth z_0.

3.1.3 The Effect of Salvage Price

This subsection deals with the effect of salvage price on the optimal order quantity. Let's define the cash flow as

$$CF(D, y(v), v) = z_0 - (c - v)\, y(v) + (s + p - v) \min\{D, y(v)\} - pD$$

and

$$CF(x, y(v), v) = \begin{cases} CF_-(x, y(v), v) = z_0 - (c - v)\, y(v) + (s - v)\, x & x \le y(v) \\ CF_+(x, y(v), v) = z_0 + (s + p - c)\, y(v) - px & x \ge y(v) \end{cases}.$$

To show the effect of salvage price, the derivative of the first order condition in (3.6) with respect to v

$$\begin{aligned} \frac{d}{dv} g(y(v)^*) &= -\frac{d}{dv}\left[(c - v)\, E[u'(CF(D, y(v)^*, v)) 1_{\{D \le y(v)^*\}}]\right] \\ &\quad + (s + p - c)\frac{d}{dv} E[u'(CF(D, y(v)^*, v)) 1_{\{D > y(v)^*\}}] \end{aligned}$$

needs to be analyzed.

Note that for any random variable X with a probability density function g_X, we can write

$$\begin{aligned} & \frac{d}{dv} E[u'(CF(X, y(v)^*, v)) 1_{\{X \le y(v)^*\}}] \\ = & \frac{d}{dv}\left(\int_0^{y(v)^*} u'\left(CF_-(x, y(v)^*, v)\right) g_X(x)\, dx\right) \\ = & y'(v)^*\, u'\left(CF(y(v)^*, y(v)^*, v)\right) g_X(y(v)^*) \\ & \quad + \int_0^{y(v)^*} \big(-(c - v)\, y'(v)^* \\ & \qquad\qquad + y(v)^* - x\big)\, u''\left(CF_-(x, y(v)^*, v)\right) g_X(x)\, dx \\ = & y'(v)^*\, u'\left(CF(y(v)^*, y(v)^*, v)\right) g_X(y(v)^*) \\ & \quad - (c - v)\, y'(v)^*\, E\left[u''\left(CF(X, y(v)^*, v)\right) 1_{\{X \le y(v)^*\}}\right] \\ & \qquad + E\left[(y(v)^* - X)\, u''\left(CF(X, y(v)^*, v)\right) 1_{\{X \le y(v)^*\}}\right] \end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
& \frac{d}{dv} E[u'(CF(X, y(v)^*, v)) 1_{\{X > y(v)^*\}}] \\
= & \frac{d}{dv} \left(\int_{y(v)^*}^{\infty} u' \left(CF_+(x, y(v)^*, v) \right) g_X(x) \, dx \right) \\
= & -y'(v)^* u' \left(CF(y(v)^*, y(v)^*, v) \right) g_X(y(v)^*) \\
& \quad + \int_{y(v)^*}^{\infty} (s+p-c) \, y'(v)^* u'' \left(CF_+(x, y(v)^*, v) \right) g_X(x) \, dx \\
= & -y'(v)^* u' \left(CF(y(v)^*, y(v)^*, v) \right) g_X(y(v)^*) \\
& \quad + (s+p-c) \, y'(v)^* E \left[u'' \left(CF(X, y(v)^*, v) \right) 1_{\{X > y(v)^*\}} \right]. \qquad (3.22)
\end{aligned}$$

By using (3.21) and (3.22) where X is D, the derivative of $g(y(v)^*)$ with respect to v can be written as

$$\begin{aligned}
\frac{d}{dv} g(y(v)^*) = & E[u' \left(CF(D, y(v)^*, v) \right) 1_{\{D \le y\}}] \\
& - (s+p-v) \, y'(v)^* u' \left(CF(y(v)^*, y(v)^*, v) \right) g_D(y(v)^*) \\
& + (c-v)^2 y'(v)^* E \left[u'' \left(CF(D, y(v)^*, v) \right) 1_{\{D \le y(v)^*\}} \right] \\
& - (c-v) \, E \left[(y(v)^* - D) \, u'' \left(CF(D, y(v)^*, v) \right) 1_{\{D \le y(v)^*\}} \right] \\
& + (s+p-c)^2 y'(v)^* E \left[u'' \left(CF(D, y(v)^*, v) \right) 1_{\{D > y(v)^*\}} \right] \\
= & \; 0.
\end{aligned}$$

From this equation, we extract the derivative of optimal order quantity with respect to v as

$$y'(v)^* = \frac{\left\{ \begin{array}{c} E[u' \left(CF(D, y(v)^*, v) \right) 1_{\{D \le y\}}] \\ - (c-v) \, E \left[(y(v)^* - D) \, u'' \left(CF(D, y(v)^*, v) \right) 1_{\{D \le y(v)^*\}} \right] \end{array} \right\}}{\left\{ \begin{array}{c} (s+p-v) \, u'[CF(y(v)^*, y(v)^*, v)] f_D(y(v)^*) \\ - (c-v)^2 E \left[u'' \left(CF(D, y(v)^*, v) \right) 1_{\{D \le y(v)^*\}} \right] \\ - (s+p-c)^2 E \left[u'' \left(CF(D, y(v)^*, v) \right) 1_{\{D > y(v)^*\}} \right] \end{array} \right\}}.$$

As $u' > 0$ and $u'' < 0$, both the numerator and the denominator are nonnegative. So, we can conclude that $y'(v)^* \ge 0$ and the optimal order quantity increases as the salvage value increases.

3.1.4 The Effect of Order Cost

In this subsection, we analyze the effect of order cost c on the optimal order quantity. The cash flow can now be written as

$$CF\left(D, y\left(c\right), c\right) = z_0 - \left(c - v\right) y\left(c\right) + \left(s + p - v\right) \min\left\{D, y\left(c\right)\right\} - pD$$

and

$$CF(x, y\left(c\right), c) = \begin{cases} CF_-(x, y\left(c\right), c) = z_0 - \left(c - v\right) y\left(c\right) + \left(s - v\right) x & x \leq y\left(c\right) \\ CF_+(x, y\left(c\right), c) = z_0 + \left(s + p - c\right) y\left(c\right) - px & x \geq y\left(c\right) \end{cases}.$$

To show the effect of the order cost, we investigate

$$\frac{d}{dc}\left(\frac{E\left[u'\left(CF\left(D, y\left(c\right)^*, c\right)\right) 1_{\{D \leq y(c)^*\}}\right]}{E\left[u'\left(CF\left(D, y\left(c\right)^*, c\right)\right)\right]}\right) = \frac{d}{dc}\left(\frac{s + p - c}{s + p - v}\right).$$

As

$$\begin{aligned} E\left[u'\left(CF\left(D, y\left(c\right)^*, c\right)\right)\right] &= E\left[u'\left(CF\left(D, y\left(c\right)^*, c\right)\right) 1_{\{D \leq y(z_0)^*\}}\right] \\ &\quad + E\left[u'\left(CF\left(D, y\left(c\right)^*, c\right)\right) 1_{\{D > y(z_0)^*\}}\right], \end{aligned}$$

it equals

$$\frac{\left\{\begin{array}{l} E\left[u'\left(\mathbf{CF}\right) 1_{\{D > y(c)^*\}}\right] \frac{d}{dc}\left(E\left[u'\left(\mathbf{CF}\right) 1_{\{D \leq y(c)^*\}}\right]\right) \\ -E\left[u'\left(\mathbf{CF}\right) 1_{\{D \leq y(c)^*\}}\right] \frac{d}{dc}\left(E\left[u'\left(\mathbf{CF}\right) 1_{\{D > y(c)^*\}}\right]\right) \end{array}\right\}}{\left(E\left[u'\left(\mathbf{CF}\right)\right]\right)^2} = -\frac{1}{s + p - v} \tag{3.23}$$

where $\mathbf{CF} = CF\left(D, y\left(c\right)^*, c\right).$

For any random variable X with a probability density function g_X,

$$\begin{aligned} &\frac{d}{dc} E\left[u'\left(CF\left(X, y\left(c\right)^*, c\right)\right) 1_{\{X \leq y(c)^*\}}\right] \\ = \; &\frac{d}{dc}\left(\int_0^{y(c)^*} u'\left(CF_-(x, y\left(c\right)^*, c)\right) g_X\left(x\right) dx\right) \\ = \; &y'\left(c\right)^* u'\left(CF(y\left(c\right)^*, y\left(c\right)^*, c)\right) g_X\left(y\left(c\right)^*\right) \\ &\quad - \int_0^{y(c)^*} \left(y\left(c\right)^* + \left(c - v\right) y'\left(c\right)^*\right) u''\left(CF_-(x, y\left(c\right)^*, c)\right) g_X\left(x\right) dx \\ = \; &y'\left(c\right)^* u'\left(CF(y\left(c\right)^*, y\left(c\right)^*, c)\right) g_X\left(y\left(c\right)^*\right) \\ &\quad - \left(y\left(c\right)^* + \left(c - v\right) y'\left(c\right)^*\right) E\left[u''\left(CF\left(X, y\left(c\right)^*, c\right)\right) 1_{\{X \leq y(c)^*\}}\right] \end{aligned} \tag{3.24}$$

and

$$
\begin{aligned}
& \frac{d}{dc} E\left[u'\left(CF\left(X, y\left(c\right)^*, c\right)\right) 1_{\{X>y(c)^*\}}\right] \\
= & \frac{d}{dc}\left(\int_{y(c)^*}^{\infty} u'\left(CF_+(x, y\left(c\right)^*, c)\right) g_X\left(x\right) dx\right) \\
= & -y'\left(c\right)^* u'\left(CF(y\left(c\right)^*, y\left(c\right)^*, c)\right) g_X\left(y\left(c\right)^*\right) \\
& \quad + \int_{y(c)^*}^{\infty}\left(\begin{array}{c} \left(s+p-c\right) y'\left(c\right)^* \\ -y\left(c\right)^* \end{array}\right) u''\left(CF_+(x, y\left(c\right)^*, c)\right) g_X\left(x\right) dx \\
= & -y'\left(c\right)^* u'\left(CF(y\left(c\right)^*, y\left(c\right)^*, c)\right) g_X\left(y\left(c\right)^*\right) \\
& \quad + \left(\begin{array}{c} \left(s+p-c\right) y'\left(c\right)^* \\ -y\left(c\right)^* \end{array}\right) E\left[u''\left(CF\left(X, y\left(c\right)^*, c\right)\right) 1_{\{X>y(c)^*\}}\right]. \qquad (3.25)
\end{aligned}
$$

Hence, by using (3.24) and (3.25), (3.23) can be simplified as

$$
\frac{\left\{\begin{array}{c} y'\left(c\right)^* u'\left(CF(y\left(c\right)^*, y\left(c\right)^*, c)\right) g_D\left(y\left(c\right)^*\right) E\left[u'\left(\mathbf{CF}\right)\right] \\ -\left(\begin{array}{c} y\left(c\right)^* \\ +\left(c-v\right) y'\left(c\right)^* \end{array}\right) E\left[u'\left(\mathbf{CF}\right) 1_{\{D>y(c)^*\}}\right] E\left[u''\left(\mathbf{CF}\right) 1_{\{D\leq y(c)^*\}}\right] \\ -\left(\begin{array}{c} \left(s+p-c\right) y'\left(c\right)^* \\ -y\left(c\right)^* \end{array}\right) E\left[u'\left(\mathbf{CF}\right) 1_{\{D\leq y(c)^*\}}\right] E\left[u''\left(\mathbf{CF}\right) 1_{\{D>y(c)^*\}}\right] \end{array}\right\}}{\left(E\left[u'\left(\mathbf{CF}\right)\right]\right)^2} = -\frac{1}{s+p-v} \qquad (3.26)
$$

where $\mathbf{CF} = CF\left(D, y\left(c\right)^*, c\right)$. From (3.26), the derivative of the optimal order quantity with respect to c can be obtained as

$$
y'\left(c\right)^* = \frac{\left\{\begin{array}{c} -\frac{1}{(s+p-v)}\left(E\left[u'\left(\mathbf{CF}\right)\right]\right)^2 \\ +y\left(c\right)^* E\left[u'\left(\mathbf{CF}\right) 1_{\{D>y(c)^*\}}\right] E\left[u''\left(\mathbf{CF}\right) 1_{\{D\leq y(c)^*\}}\right] \\ -y\left(c\right)^* E\left[u'\left(\mathbf{CF}\right) 1_{\{D\leq y(c)^*\}}\right] E\left[u''\left(\mathbf{CF}\right) 1_{\{D>y(c)^*\}}\right] \end{array}\right\}}{\left\{\begin{array}{c} u'[CF(y\left(c\right)^*, y\left(c\right)^*, c)] g_D\left(y\left(c\right)^*\right) E\left[u'\left(\mathbf{CF}\right)\right] \\ -\left(c-v\right) E\left[u'\left(\mathbf{CF}\right) 1_{\{D>y(c)^*\}}\right] E\left[u''\left(\mathbf{CF}\right) 1_{\{D\leq y(c)^*\}}\right] \\ -\left(s+p-c\right) E\left[u'\left(\mathbf{CF}\right) 1_{\{D\leq y(c)^*\}}\right] E\left[u''\left(\mathbf{CF}\right) 1_{\{D>y(c)^*\}}\right] \end{array}\right\}}. \qquad (3.27)
$$

The sign of $y'\left(c\right)^*$ in (3.27) is indeterminate because despite the nonnegativity of the denominator, the numerator cannot be determined. So, we cannot say anything about how the order quantity is effected.

To analyze the above for special cases, the exponential utility function $u\left(z\right) = K - Ce^{-\frac{1}{\beta}z}$

with $u'(z) = \frac{C}{\beta}e^{-\frac{1}{\beta}z}$ and $u''(z) = -\frac{C}{\beta^2}e^{-\frac{1}{\beta}z}$ is considered. Then, the numerator of (3.27) is

$$\left\{ \begin{array}{c} -\frac{1}{s+p-v}\left(E\left[\frac{C}{\beta}e^{-\frac{1}{\beta}(\boldsymbol{CF})}\right]\right)^2 \\ +y(c)^* E\left[\frac{C}{\beta}e^{-\frac{1}{\beta}(\boldsymbol{CF})}1_{\{D>y(c)^*\}}\right] E\left[-\frac{C}{\beta^2}e^{-\frac{1}{\beta}(\boldsymbol{CF})}1_{\{D\leq y(c)^*\}}\right] \\ -y(c)^* E\left[\frac{C}{\beta}e^{-\frac{1}{\beta}(\boldsymbol{CF})}1_{\{D\leq y(c)^*\}}\right] E\left[-\frac{C}{\beta^2}e^{-\frac{1}{\beta}(\boldsymbol{CF})}1_{\{D>y(c)^*\}}\right] \end{array} \right\}$$

or more explicitly

$$-\frac{1}{s+p-v}\frac{C}{\beta}\left(E\left[e^{-\frac{1}{\beta}\boldsymbol{CF}}\right]\right)^2$$

and so $y'(c)^* \leq 0$ for the exponential utility function. Hence, it can be concluded that as the order cost increases, optimal order quantity decreases in this special case.

3.1.5 The Effect of Penalty Price

The effect of the penalty price p on order quantity is analyzed in this subsection. We can redefine the cash flow as

$$CF(D, y(p), p) = z_0 - (c-v)\,y(p) + (s+p-v)\min\{D, y(p)\} - pD$$

and

$$CF(x, y(p), p) = \left\{ \begin{array}{ll} CF_-(x, y(p), p) = z_0 - (c-v)\,y(p) + (s-v)\,x & x \leq y(p) \\ CF_+(x, y(p), p) = z_0 + (s+p-c)\,y(p) - px & x \geq y(p) \end{array} \right. .$$

To show the effect of the penalty price, we need to investigate the derivative of the first order condition in (3.6) with respect to p, or

$$\begin{aligned} \frac{d}{dp}g(y(p)^*) \quad &= \quad -(c-v)\frac{d}{dp}E[u'(CF(D, y(p)^*, p))1_{\{D\leq y(p)^*\}}] \\ &\quad + \frac{d}{dp}\left((s+p-c)\,E[u'(CF(D, y(p)^*, p))1_{\{D>y(p)^*\}}]\right). \end{aligned} \quad (3.28)$$

Note that for any random variable X with probability density function g_X, we can write that

$$\begin{aligned} & \frac{d}{dp}E[u'(CF(X, y(p)^*, p))1_{\{X\leq y(p)^*\}}] \\ = \; & \frac{d}{dp}\left(\int_0^{y(p)^*} u'(CF_-(x, y(p)^*, p))\,g_X(x)\,dx\right) \\ = \; & y'(p)^*\,u'(CF(y(p)^*, y(p)^*, p))\,g_X(y(p)^*) \\ & \quad + \int_0^{y(p)^*} \left(-(c-v)\,y'(p)^*\right) u''(CF_-(x, y(p)^*, p))\,g_X(x)\,dx \end{aligned}$$

$$\begin{aligned} &= y'(p)^* \, u'\left(CF(y(p)^*, y(p)^*, p)\right) g_X\left(y(p)^*\right) \\ &\qquad - (c-v)\, y'(p)^* \, E\left[u''(CF(X, y(p)^*, p)) 1_{\{X \le y(p)^*\}}\right] \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} &\frac{d}{dp} E[u'(CF(X, y(p)^*, p)) 1_{\{X > y(p)^*\}}] \\ = \;& \frac{d}{dp}\left(\int_{y(p)^*}^{\infty} u'\left(CF_+(x, y(p)^*, p)\right) g_X(x)\, dx\right) \\ = \;& -y'(p)^* \, u'\left(CF(y(p)^*, y(p)^*, p)\right) g_X\left(y(p)^*\right) \\ & + \int_{y(p)^*}^{\infty} \left(y(p)^* + (s+p-c)\, y'(p)^* - x\right) u''\left(CF_+(x, y(p)^*, p)\right) g_X(x)\, dx \\ = \;& -y'(p)^* \, u'\left(CF(y(p)^*, y(p)^*, p)\right) g_X\left(y(p)^*\right) \\ & - E\left[\left(X - y(p)^*\right) u''(CF(X, y(p)^*, p)) 1_{\{X > y(p)^*\}}\right] \\ & \quad + (s+p-c)\, y'(p)^* \, E\left[u''(CF(X, y(p)^*, p)) 1_{\{X > y(p)^*\}}\right]. \end{aligned} \tag{3.30}$$

By using (3.29) and (3.30) where X is D, the derivative of $g\left(y(p)^*\right)$ with respect to p in (3.28) can be written as

$$\begin{aligned} \frac{d}{dp} g\left(y(p)^*\right) &= E[u'(\mathbf{CF}) 1_{\{D > y(p)^*\}}] \\ &\quad - (c-v)\left(y'(p)^* \, u'\left(CF(y(p)^*, y(p)^*, p)\right) g_D\left(y(p)^*\right)\right. \\ &\qquad \left. - (c-v)\, y'(p)^* \, E\left[u''(\mathbf{CF})\, 1_{\{D \le y(p)^*\}}\right]\right) \\ &\quad + (s+p-c)\left(-y'(p)^* \, u'\left(CF(y(p)^*, y(p)^*, p)\right) g_D\left(y(p)^*\right)\right. \\ &\qquad - E\left[\left(D - y(p)^*\right) u''(\mathbf{CF})\, 1_{\{D > y(p)^*\}}\right] \\ &\qquad \left. + (s+p-c)\, y'(p)^* \, E\left[u''(\mathbf{CF})\, 1_{\{D > y(p)^*\}}\right]\right) \\ &= 0 \end{aligned}$$

where $\mathbf{CF} = CF(D, y(p)^*, p)$. From this equation, we extract the derivative of optimal order quantity with respect to p, or

$$y'(p)^* = \frac{\left\{\begin{array}{c} E[u'\left(CF(D, y(p)^*, p)\right) 1_{\{D > y(p)^*\}}] \\ - (s+p-c)\, E\left[\left(D - y(p)^*\right) u''\left(CF(D, y(p)^*, p)\right) 1_{\{D > y(p)^*\}}\right] \end{array}\right\}}{\left\{\begin{array}{c} + (s+p-v)\, u'\left(CF(y(p)^*, y(p)^*, p)\right) g_D\left(y(p)^*\right) \\ - (c-v)^2 \, E\left[u''\left(CF(D, y(p)^*, p)\right) 1_{\{D \le y(p)^*\}}\right] \\ - (s+p-c)^2 \, E\left[u''\left(CF(D, y(p)^*, p)\right) 1_{\{D > y(p)^*\}}\right] \end{array}\right\}}$$

and we can establish that $y'(p)^* \geq 0$ since $u' \geq 0$ and $u'' \leq 0$. Hence, we can conclude that as the penalty cost increases, the optimal order quantity increases.

All analyses done in this section review the work of Eeckhoudt et al. [18]. We use the same argument as Eeckhoudt et al. [18] while analyzing the effect of risk-aversion on the optimal order quantity. However, while analyzing other parameters we use different methods than Eeckhoudt et al. [18]. Analyzing the effect of sale price is much more complicated. Eeckhoudt et al. [18] conclude that as selling price increases, the optimal order quantity increases if the utility function is decreasing partial risk aversion class and the quantity decreases if the class of utility function is constant absolute risk aversion. Moreover, Wang et al. [49] analyze the effect of sale price and conclude that a risk-averse newsvendor orders less than an arbitrarily small quantity as sale price increases if sale price is higher than a threshold value.

In this section, we analyzed the optimal order quantity for expected utility maximization problem when demand is the only source of uncertainty. As it is stated before, demand is not necessarily the only source of uncertainty. In reality, supply uncertainty is another significant form of randomness for inventory management. Due to some unforeseen reasons, supplier could not meet the asked order quantity. Therefore, the amount received can be different than the amount demanded. Starting from the following section, we examine three types of supply randomness; random yield, random capacity and random yield and capacity, respectively. Note that, in literature, there is no such an example that discusses the utility theory and the random supply at the same time. The analysis in the remainder of this chapter, therefore is new.

3.2 Newsvendor Model with Random Yield

This section deals with the newsvendor problems where the supply is subject to yield randomness. In these problems, it is assumed that a random proportion of the asked quantity is received due to production, transportation or other problems on the side of the supplier. Let the amount received from ordering y units be Uy where $0 \leq U \leq 1$ represents the proportion of non-defective items received. We suppose that U has the density function g_U. Moreover, we assume that U and D are not necessarily independent and the conditional density function of demand given $U = w$ is $g_{D|w}$. The cash flow in (3.2) can be updated for the random yield model as

$$CF\left(D,U,y\right)=z_0+\left(s+p-v\right)\min\left\{D,Uy\right\}-(c-v)Uy-pD. \tag{3.31}$$

The aim of the risk-averse decision maker is now

$$\max_{y\geq 0} H(y)=E[u(CF(D,U,y))]$$

where the objective function is

$$E[u(CF(D,U,y))]=E[u(CF(D,U,y))1_{\{D\leq Uy\}}]+E[u(CF(D,U,y))1_{\{D>Uy\}}]. \tag{3.32}$$

For further analysis, let

$$CF(x,w,y)=\begin{cases} CF_-(x,wy)=z_0+(s-v)x-(c-v)wy & x\leq wy \\ CF_+(x,wy)=z_0+(s+p-c)wy-px & x\geq wy \end{cases}$$

for any $x\geq 0$, $y\geq 0$ and $1\geq w\geq 0$, and note that $CF(wy,w,y)=CF_-(wy,wy)=CF_+(wy,wy)=z_0+(s-c)wy$.

Theorem 1 *The optimal order quantity* y^* *satisfies*

$$\frac{E\left[Uu'\left(CF(D,U,y^*)\right)1_{\{D\leq Uy^*\}}\right]}{E\left[Uu'\left(CF(D,U,y^*)\right)\right]}=\widehat{p}. \tag{3.33}$$

Proof. *Note that for any random variables X and V with probability density functions g_X and g_V, we can write*

$$\begin{aligned} E[u(CF(X,V,y))] &= \int_0^\infty\left(\int_0^{wy}u\left(CF_-(x,wy)\right)g_{X|w}\left(x\right)dx\right)g_V(w)dw \\ &\quad+\int_0^\infty\left(\int_{wy}^\infty u\left(CF_+(x,wy)\right)g_{X|w}\left(x\right)dx\right)g_V(w)dw \end{aligned}$$

where $g_{X|w}$ is the conditional probability density function of X given $V=w$. One can show that

$$\begin{aligned} &\frac{d}{dy}E[u(CF(X,V,y))] \\ &= -(c-v)\int_0^\infty w\left(\int_0^{wy}u'\left(CF_-(x,wy)\right)g_{X|w}\left(x\right)dx\right)g_V(w)dw \\ &\quad+(s+p-c)\int_0^\infty w\left(\int_{wy}^\infty u'\left(CF_+(x,wy)\right)g_{X|w}\left(x\right)dx\right)g_V(w)dw \\ &= -(c-v)E\left[Vu'\left(CF(X,V,y)\right)1_{\{X\leq Vy\}}\right] \\ &\quad+(s+p-c)E\left[Vu'\left(CF(X,V,y)\right)1_{\{X>Vy\}}\right]. \end{aligned} \tag{3.34}$$

Moreover, we can also show that

$$
\begin{aligned}
& \frac{d}{dy} E\left[V u'\left(CF(X,V,y)\right) 1_{\{X \le Vy\}}\right] \\
= & \int_0^\infty w^2 u'\left(CF(wy,w,y)\right) g_{X|w}\left(wy\right) g_V(w) dw \\
& \quad -(c-v)\int_0^\infty w^2 \left(\int_0^{wy} u''\left(CF_-(x,wy)\right) g_{X|w}\left(x\right) dx\right) g_V(w) dw \\
= & E\left[V^2 u'\left(CF(Vy,V,y)\right) g_{X|w}\left(Vy\right)\right] \\
& \quad -(c-v) E\left[V^2 u''\left(CF(X,V,y)\right) 1_{\{X \le Vy\}}\right] \qquad (3.35)
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{d}{dy} E\left[V u'\left(CF(X,V,y)\right) 1_{\{X > Vy\}}\right] \\
= & \int_0^\infty w \left(-w u'\left(CF(wy,w,y)\right) g_{X|w}\left(wy\right)\right. \\
& \qquad \left. + (s+p-c)\, w \int_{wy}^\infty u''\left(CF_+(x,wy)\right) g_{X|w}\left(x\right) dx\right) g_V(w) dw \\
= & -E\left[V^2 u'\left(CF(Vy,V,y)\right) g_{X|w}\left(Vy\right)\right] \\
& \quad + (s+p-c)\, E\left[V^2 u''\left(CF(X,V,y)\right) 1_{\{X > Vy\}}\right]. \qquad (3.36)
\end{aligned}
$$

By using (3.34), (3.35) and (3.36) where X is D and V is D, we differentiate the objective function and set it equal to zero, so the first order condition is

$$
\begin{aligned}
g\left(y\right) &= \frac{d}{dy} E[u(CF(D,U,y))] \\
&= -(c-v) E\left[U u'\left(CF(D,U,y)\right) 1_{\{D \le Uy\}}\right] \\
&\qquad + (s+p-c) E\left[U u'\left(CF(D,U,y)\right) 1_{\{D > Uy\}}\right] \\
&= -(s+p-v) E\left[U u'\left(CF(D,U,y)\right) 1_{\{D \le Uy\}}\right] \\
&\qquad + (s+p-c) E\left[U u'\left(CF(D,U,y)\right)\right] \\
&= 0. \qquad (3.37)
\end{aligned}
$$

Moreover, the second derivative of the objective function is

$$\begin{aligned}
\frac{d}{dy}g(y) &= \frac{d^2}{dy^2}E[u(CF(D,U,y))] \\
&= -(s+p-v)E\left[U^2u'(CF(Uy,U,y))\,g_{D|w}(Uy)\right] \\
&\qquad +(c-v)^2E\left[U^2u''(CF(D,U,y))\,1_{\{D\le Uy\}}\right] \\
&\qquad\quad +(s+p-c)^2E\left[U^2u''(CF(D,U,y))\,1_{\{D>Uy\}}\right] \\
&\le 0
\end{aligned}$$

and the second order condition is satisfied. This implies that $g(y)$ is decreasing in y. The objective function is therefore concave and the first order condition in (3.37) is the optimality condition. This yields (3.33). ■

Provided that $g(0) \ge 0$ and $g(\infty) \le 0$, (3.33) is the optimality condition and there exists an optimal order quantity y^* that satisfies (3.33). However, there may be no solution for (3.33).

Corollary 2 *The optimal order quantity is $y^* = 0$ if*

$$P\{D=0\} > \left(\frac{E\left[Uu'(z_0-pD)\right]}{u'(z_0)\,E[U]}\right)\widehat{p}. \tag{3.38}$$

Proof. *As $g(y)$ is decreasing in y, we have $y^* = 0$ if $g(0) < 0$; that is*

$$\begin{aligned}
g(0) &= -(s+p-v)E\left[Uu'(CF(D,U,0))\,1_{\{D\le 0\}}\right] + (s+p-c)E\left[Uu'(CF(D,U,0))\right] \\
&= -(s+p-v)E\left[Uu'(CF(0,U,0))\,1_{\{D=0\}}\right] + (s+p-c)E\left[Uu'(CF(D,U,0))\right] \\
&= -(s+p-v)u'(z_0)\,E[U]\,P\{D=0\} + (s+p-c)E\left[Uu'(z_0-pD)\right] < 0.
\end{aligned}$$

■

Corollary 3 *The optimal order quantity is $y^* = \infty$ if*

$$P\{D=\infty\} > \frac{c-v}{s+p-v}. \tag{3.39}$$

If we assume that demand is finite or $P\{D=\infty\} = 0$, the optimal order quantity is finite and satisfies (3.33). And, if the probability $P\{D=\infty\} = 1$, we have $y^* = \infty$.

Proof. *Again, as $g(y)$ is decreasing in y, we have $y^* = \infty$ if $g(\infty) > 0$; that is*

$$\begin{aligned}
g(\infty) &= -(s+p-v)E\left[Uu'\left(CF(D,U,\infty)\right)1_{\{D<\infty\}}\right] + (s+p-c)E\left[Uu'\left(CF(D,U,\infty)\right)\right] \\
&= -(s+p-v)E\left[Uu'\left(CF(D,U,\infty)\right)\left(1-1_{\{D=\infty\}}\right)\right] \\
&\qquad +(s+p-c)E\left[Uu'\left(CF(D,U,\infty)\right)\right] \\
&= -(c-v)E\left[Uu'\left(CF(D,U,\infty)\right)\right] \\
&\qquad +(s+p-v)E\left[Uu'\left(CF(D,U,\infty)\right)\right]P\{D=\infty\} \\
&> 0.
\end{aligned}$$

This argument supposes that u is bounded. ■

As a special case, let the utility function be linear that is $u(x) = a+bx$, so that $u'(x) = b$. The optimality condition in (3.33) becomes

$$\frac{E\left[U1_{\{D\leq Uy^*\}}\right]}{E[U]} = \widehat{p}$$

which is the same condition as Okyay et al. [41] for the newsvendor problem with random yield. Moreover, when $U = 1$ that is there is no randomness in yield, the optimality condition is

$$\frac{E\left[u'\left(CF(D,y^*)\right)1_{\{D\leq y^*\}}\right]}{E\left[u'\left(CF(D,y^*)\right)\right]} = \widehat{p}$$

which satisfies the optimality condition in (3.8).

The optimality condition in (3.33) depends on many parameters. We analyze the effects of these parameters on the order quantity in following subsections.

3.2.1 The Effect of Risk-Aversion

First, we explore the effect of risk aversion on the optimal order quantity. In order to analyze the effect of the risk-aversion on order quantity, we again use Pratt's argument [46]. So, we replace $u(x)$ with $k(u(x))$ where $k' > 0$ and $k'' < 0$.

The cash flow is increasing in x, since

$$CF_-(x_1, wy^*) \leq CF(wy^*, w, y^*) \leq CF_+(x_2, wy^*)$$

for all $x_1 \leq wy^* \leq x_2$. Then, since u is concave increasing,

$$u\left(CF_-(x_1, wy^*)\right) \leq u\left(CF(wy^*, w, y^*)\right) \leq u\left(CF_+(x_2, wy^*)\right)$$

and

$$u'\left(CF_{-}\left(x_1, wy^*\right)\right) \geq u'\left(CF\left(wy^*, w, y^*\right)\right) \geq u'\left(CF_{+}\left(x_2, wy^*\right)\right)$$

for all $x_1 \leq wy^* \leq x_2$. Moreover, since k is another concave increasing function,

$$k'\left(u\left(CF_{-}\left(x_1, wy^*\right)\right)\right) \geq k'\left(u\left(CF\left(wy^*, w, y^*\right)\right)\right) \geq k'\left(u\left(CF_{+}\left(x_2, wy^*\right)\right)\right). \tag{3.40}$$

Let the objective of the more risk-averse decision maker be

$$\max_{y \geq 0} \widetilde{H}\left(y\right) = E[k\left(u(CF(D, U, y))\right)]$$

and the first derivative of the objective function is

$$\begin{aligned} \widetilde{g}\left(y\right) &= -(c-v)\int_0^\infty w\left(\int_0^{wy} k'\left(u\left(\mathbf{CF}_{-}\right)\right)u'\left(\mathbf{CF}_{-}\right)g_{D|w}\left(x\right)dx\right)g_U(w)dw \\ &\quad +(s+p-c)\int_0^\infty w\left(\int_{wy}^\infty k'\left(u\left(\mathbf{CF}_{+}\right)\right)u'\left(\mathbf{CF}_{+}\right)g_{D|w}\left(x\right)dx\right)g_U(w)dw \end{aligned} \tag{3.41}$$

where $\mathbf{CF}_{-} = CF_{-}(x, wy)$ and $\mathbf{CF}_{+} = CF_{+}(x, wy)$

We substitute the optimal order quantity y^* that satisfies (3.8) in (3.41). By using inequality (3.40), it can be shown that

$$\begin{aligned} \widetilde{g}\left(y^*\right) &= \left(-(c-v)\int_0^\infty w\left(\int_0^{wy^*} k'\left(u\left(\mathbf{CF}_{-}\right)\right)u'\left(\mathbf{CF}_{-}\right)g_{D|w}\left(x\right)dx\right)g_U(w)dw\right. \\ &\quad \left. +\,(s+p-c)\int_0^\infty w\left(\int_{wy^*}^\infty k'\left(u\left(\mathbf{CF}_{+}\right)\right)u'\left(\mathbf{CF}_{+}\right)g_{D|w}\left(x\right)dx\right)g_U(w)dw\right) \\ &< k'\left(u\left(\mathbf{CF}\right)\right)\left(-(c-v)\int_0^\infty w\left(\int_0^{wy^*} u'\left(\mathbf{CF}_{-}\right)f_{D|w}\left(x\right)dx\right)f_U(w)dw\right. \\ &\quad \left. +\,(s+p-c)\int_0^\infty w\left(\int_{wy^*}^\infty u'\left(\mathbf{CF}_{+}\right)f_{D|w}\left(x\right)dx\right)f_U(w)dw\right) \\ &= 0 \end{aligned}$$

where $\mathbf{CF}_{-} = CF_{-}(x, wy^*)$, $\mathbf{CF}_{+} = CF_{+}(x, wy^*)$ and $\mathbf{CF} = CF\left(wy^*, w, y^*\right)$. Therefore, $\widetilde{g}\left(y^*\right) < 0$. Since $\widetilde{H}$ is concave in y, the more risk-averse utility function orders less than y^*. We can conclude that as risk aversion rises the optimal order quantity diminishes.

3.2.2 The Effect of Initial Wealth

In this subsection, we analyze the effect of initial wealth z_0 on the optimal order quantity. We now redefine the cash flow as

$$CF\left(D,U,y\left(z_0\right),z_0\right)=z_0+(s+p-v)\min\left\{D,Uy\left(z_0\right)\right\}-(c-v)Uy\left(z_0\right)-pD$$

and

$$CF(x,w,y\left(z_0\right),z_0)=\begin{cases} CF_-(x,wy\left(z_0\right),z_0)=z_0+(s-v)x-(c-v)wy\left(z_0\right) & x\le wy \\ CF_+(x,wy\left(z_0\right),z_0)=z_0+(s+p-c)wy\left(z_0\right)-px & x\ge wy \end{cases}.$$

To analyze the effect of the initial wealth, the derivative of the optimal order condition in (3.33) with respect z_0 is needed. So, we need calculate

$$\frac{d}{dz_0}\left(\frac{E\left[Uu'\left(CF(D,U,y\left(z_0\right)^*,z_0)\right)1_{\{D\le Uy(z_0)^*\}}\right]}{E\left[Uu'\left(CF(D,U,y\left(z_0\right)^*,z_0)\right)\right]}\right)=\frac{d}{dz_0}\left(\frac{s+p-c}{s+p-v}\right),$$

or more explicitly

$$\frac{\left\{\begin{array}{c} E\left[Uu'\left(\mathbf{CF}\right)1_{\{D>Uy(z_0)^*\}}\right]\frac{d}{dz_0}\left(E\left[Uu'\left(\mathbf{CF}\right)1_{\{D\le Uy(z_0)^*\}}\right]\right) \\ -E\left[Uu'\left(\mathbf{CF}\right)1_{\{D\le Uy(z_0)^*\}}\right]\frac{d}{dz_0}\left(E\left[Uu'\left(\mathbf{CF}\right)1_{\{D>Uy(z_0)^*\}}\right]\right)\end{array}\right\}}{\left(E\left[Uu'\left(\mathbf{CF}\right)\right]\right)^2}=0 \tag{3.42}$$

where $\mathbf{CF}=CF(D,U,y\left(z_0\right)^*,z_0)$ and

$$\begin{aligned} E\left[Uu'\left(CF(D,U,y\left(z_0\right)^*)\right)\right] &= E\left[Uu'\left(CF(D,U,y\left(z_0\right)^*)\right)1_{\{D\le Uy(z_0)^*\}}\right] \\ &\quad +E\left[Uu'\left(CF(D,U,y\left(z_0\right)^*)\right)1_{\{D>Uy(z_0)^*\}}\right].\end{aligned}$$

For any random variables X and V with probability density functions g_X and g_V, we can show that

$$\begin{aligned} & \frac{d}{dz_0}\left(E\left[Vu'\left(\boldsymbol{CF}\right)1_{\{X\le Vy(z_0)^*\}}\right]\right) \\ = & \frac{d}{dz_0}\left(\int_0^\infty w\left(\int_0^{wy(z_0)^*}u'\left(\mathbf{CF}_-\right)g_{X|w}\left(x\right)dx\right)g_V(w)dw\right) \\ = & \int_0^\infty w\left(wy'\left(z_0\right)^*u'\left(CF(wy\left(z_0\right)^*,w,y\left(z_0\right)^*,z_0)\right)g_{X|w}\left(wy\left(z_0\right)^*\right)\right. \\ & \quad \left.+\left(\int_0^{wy(z_0)^*}\left(1-(c-v)\,wy'\left(z_0\right)^*\right)u''\left(\mathbf{CF}_-\right)g_{X|w}\left(x\right)dx\right)\right)g_V(w)dw \\ = & \; y'\left(z_0\right)^*E\left[V^2u'\left(CF(Vy\left(z_0\right)^*,V,y\left(z_0\right)^*,z_0)\right)g_{X|w}\left(Vy\left(z_0\right)^*\right)\right] \\ & \quad +E\left[Vu''\left(\boldsymbol{CF}\right)1_{\{X\le Vy(z_0)^*\}}\right] \\ & \qquad -(c-v)\,y'\left(z_0\right)^*E\left[V^2u''\left(\boldsymbol{CF}\right)1_{\{X\le Vy(z_0)^*\}}\right] \end{aligned} \tag{3.43}$$

where $\mathbf{CF}_{-} = CF_{-}(x, wy(z_0)^*, z_0)$ and

$$\begin{aligned}
&\frac{d}{dz_0}\left(E\left[Vu'(\boldsymbol{CF})\, 1_{\{X>Vy(z_0)^*\}}\right]\right)\\
= \;& \frac{d}{dz_0}\left(\int_0^\infty w\left(\int_{wy(z_0)^*}^\infty u'(\mathbf{CF}_+)\, g_{D|w}(x)\,dx\right) g_U(w)dw\right)\\
= \;& \int_0^\infty w\left(-wy'(z_0)^*\, u'(CF(wy(z_0)^*, w, y(z_0)^*, z_0))\, g_{D|w}(wy(z_0)^*)\right.\\
&\left.\quad + \int_{wy(z_0)^*}^\infty \left(1+(s+p-c)\, wy'(z_0)^*\right) u''(\mathbf{CF}_+)\, g_{D|w}(x)\,dx\right) g_U(w)dw\\
= \;& -y'(z_0)^*\, E\left[V^2 u'(CF(Vy(z_0)^*, V, y(z_0)^*, z_0))\, g_{D|w}(Vy(z_0)^*)\right]\\
&\quad + E\left[Vu''(\boldsymbol{CF})\, 1_{\{X>Vy(z_0)^*\}}\right]\\
&\qquad + (s+p-c)\, y'(z_0)^*\, E\left[V^2 u''(\boldsymbol{CF})\, 1_{\{X>Vy(z_0)^*\}}\right] \qquad (3.44)
\end{aligned}$$

where $\mathbf{CF}_{+} = CF_{+}(x, wy(z_0)^*, z_0)$ and $\boldsymbol{CF} = CF(X, V, y(z_0)^*, z_0)$.

By using (3.43) and (3.44) where X is D and V is U, by letting $\boldsymbol{CF} = CF(D, U, y(z_0)^*, z_0)$ the equation (3.42) can be written as

$$\left\{\begin{array}{c}
y'(z_0)^*\, E[Uu'(\boldsymbol{CF})]\, E\left[U^2 u'(CF(Uy(z_0)^*, U, y(z_0)^*, z_0))\, g_{D|w}(Uy(z_0)^*)\right]\\
+E\left[Uu'(\boldsymbol{CF})\, 1_{\{D>Uy(z_0)^*\}}\right] E\left[Uu''(\boldsymbol{CF})\, 1_{\{D\le Uy(z_0)^*\}}\right]\\
-(c-v)\, y'(z_0)^*\, E\left[Uu'(\boldsymbol{CF})\, 1_{\{D>Uy(z_0)^*\}}\right] E\left[U^2 u''(\boldsymbol{CF})\, 1_{\{D\le Uy(z_0)^*\}}\right]\\
-E\left[Uu'(\boldsymbol{CF})\, 1_{\{D\le Uy(z_0)^*\}}\right] E\left[Uu''(\boldsymbol{CF})\, 1_{\{D>Vy(z_0)^*\}}\right]\\
-(s+p-c)\, y'(z_0)^*\, E\left[Uu'(\boldsymbol{CF})\, 1_{\{D\le Uy(z_0)^*\}}\right] E\left[U^2 u''(\boldsymbol{CF})\, 1_{\{D>Uy(z_0)^*\}}\right]
\end{array}\right\} = 0.$$

We can then obtain the derivative of the optimal order quantity with respect to z_0 as

$$y'(z_0)^* = \frac{\left\{\begin{array}{c}
E\left[Uu'(\boldsymbol{CF})\, 1_{\{D>Uy(z_0)^*\}}\right] E\left[Uu''(\boldsymbol{CF})\, 1_{\{D\le Uy(z_0)^*\}}\right]\\
-E\left[Uu'(\boldsymbol{CF})\, 1_{\{D\le Uy(z_0)^*\}}\right] E\left[Uu''(\boldsymbol{CF})\, 1_{\{D>Vy(z_0)^*\}}\right]
\end{array}\right\}}{\left\{\begin{array}{c}
E[Uu'(\boldsymbol{CF})]\, E\left[U^2 u'(CF(Uy(z_0)^*, U, y(z_0)^*, z_0))\, g_{D|w}(Uy(z_0)^*)\right]\\
-(c-v)\, E\left[Uu'(\boldsymbol{CF})\, 1_{\{D>Uy(z_0)^*\}}\right] E\left[U^2 u''(\boldsymbol{CF})\, 1_{\{D\le Uy(z_0)^*\}}\right]\\
-(s+p-c)\, E\left[Uu'(\boldsymbol{CF})\, 1_{\{D\le Uy(z_0)^*\}}\right] E\left[U^2 u''(\boldsymbol{CF})\, 1_{\{D>Uy(z_0)^*\}}\right]
\end{array}\right\}}. \qquad (3.45)$$

The sign of the denominator is nonnegative, but the sign of the numerator is indeterminate. Therefore, the effect of initial wealth is unclear.

We then explore the numerator for a special utility function, the exponential utility function, that is $u(z) = K - Ce^{-\frac{1}{\beta}z}$, $u'(z) = \frac{C}{\beta}e^{-\frac{1}{\beta}z}$ and $u''(z) = -\frac{C}{\beta^2}e^{-\frac{1}{\beta}z}$. The numerator for the exponential utility function is

$$\left\{\begin{array}{c} E\left[U\frac{C}{\beta}e^{-\frac{1}{\beta}[\boldsymbol{CF}]}1_{\{D>Uy(z_0)^*\}}\right]E\left[-U\frac{C}{\beta^2}e^{-\frac{1}{\beta}[\boldsymbol{CF}]}1_{\{D\leq Uy(z_0)^*\}}\right] \\ -E\left[U\frac{C}{\beta}e^{-\frac{1}{\beta}[\boldsymbol{CF}]}1_{\{D\leq Uy(z_0)^*\}}\right]E\left[-U\frac{C}{\beta^2}e^{-\frac{1}{\beta}[\boldsymbol{CF}]}1_{\{D>Vy(z_0)^*\}}\right] \end{array}\right\}$$

which equals zero. It can therefore be concluded that the optimal order quantity for the exponential utility function is not affected by the initial wealth.

3.2.3 The Effect of Salvage Price

This subsection deals with the effect of the salvage price on order quantity. We redefine the cash flow as

$$CF\left(D,U,y\left(v\right),v\right)=z_0+\left(s+p-v\right)\min\left\{D,Uy\left(v\right)\right\}-(c-v)Uy\left(v\right)-pD$$

and

$$CF(x,w,y\left(v\right),v)=\left\{\begin{array}{ll} CF_{-}(x,wy\left(v\right),v)=z_0+(s-v)x-(c-v)wy\left(v\right) & x\leq wy \\ CF_{+}(x,wy\left(v\right),v)=z_0+(s+p-c)wy\left(v\right)-px & x\geq wy \end{array}\right. .$$

To show the effect of the salvage price, we take the derivative of the first order condition in (3.37) with respect to v; that is

$$\begin{aligned} \frac{d}{dv}g\left(y\left(v\right)^*\right) &= -\frac{d}{dv}\left[(c-v)E\left[Uu'\left(CF(D,U,y\left(v\right)^*,v)\right)1_{\{D\leq Uy(v)^*\}}\right]\right] \\ &\quad +(s+p-c)\frac{d}{dv}\left[E\left[Uu'\left(CF(D,U,y\left(v\right)^*,v)\right)1_{\{D>Uy(v)^*\}}\right]\right] \\ &= 0. \end{aligned} \tag{3.46}$$

Note that for any random variables X and V with probability density functions g_X and g_V, we can write

$$\begin{aligned} &\frac{d}{dv}E\left[Vu'\left(\boldsymbol{CF}\right)1_{\{X\leq Vy(v)^*\}}\right] \\ =\ & \frac{d}{dv}\left(\int_0^\infty w\left(\int_0^{wy(v)^*}u'\left(\mathbf{CF}_{-}\right)g_{X|w}\left(x\right)dx\right)g_V(w)dw\right) \\ =\ & \int_0^\infty w\left(wy'\left(v\right)^*u'\left(\mathbf{CF}_{-}\right)g_{X|w}\left(wy\left(v\right)^*\right)\right. \\ &\quad \left.+\int_0^{wy(v)^*}\left(-x+wy\left(v\right)^*-(c-v)wy'\left(v\right)^*\right)u''\left(\mathbf{CF}_{-}\right)g_{X|w}\left(x\right)dx\right)g_V(w)dw \end{aligned}$$

$$
\begin{aligned}
&= y'(v)^* E\left[V^2 u'\left(CF(Vy(v)^*, V, y(v)^*, v)\right) g_{X|w}\left(Vy(v)^*\right)\right] \\
&\qquad +E\left[V\left(Vy(v)^* - X\right) u''\left(\boldsymbol{CF}\right) 1_{\{X \le Vy(v)^*\}}\right] \\
&\qquad\qquad -(c-v)y'(v)^* E\left[V^2 u''\left(\boldsymbol{CF}\right) 1_{\{X \le Vy(v)^*\}}\right]
\end{aligned} \tag{3.47}
$$

and

$$
\begin{aligned}
& \frac{d}{dv} E\left[V u'\left(\boldsymbol{CF}\right) 1_{\{X > Vy(v)^*\}}\right] \\
= \; & \frac{d}{dv}\left(\int_0^\infty w \left(\int_{wy(v)^*}^\infty u'\left(\mathbf{CF}_+\right) g_{X|w}(x)\, dx\right) g_V(w) dw\right) \\
= \; & \int_0^\infty w \left(-wy'(v)^* u'\left(CF(wy(v)^*, w, y(v)^*, v)\right) g_{X|w}\left(wy(v)^*\right)\right. \\
& \quad \left. + \int_{wy(v)^*}^\infty \left((s+p-c)wy'(v)^*\right) u''\left(\mathbf{CF}_+\right) g_{X|w}(x)\, dx\right) g_V(w) dw \\
= \; & -y'(v)^* E\left[V^2 u'\left(CF(Vy(v)^*, V, y(v)^*, v)\right) g_{X|w}\left(Vy(v)^*\right)\right] \\
& \quad +(s+p-c)y'(v)^* E\left[V^2 u''\left(\boldsymbol{CF}\right) 1_{\{X > Vy(v)^*\}}\right].
\end{aligned} \tag{3.48}
$$

where $\mathbf{CF}_- = CF_-(x, wy(v)^*, v)$, $\mathbf{CF}_+ = CF_+(x, wy(v)^*, v)$ and $\boldsymbol{CF} = CF(X, V, y(v)^*, v)$. By using (3.47) and (3.48) where X is D and V is U, the equation (3.46) can be written as

$$
\begin{aligned}
\frac{d}{dv} g\left(y(v)^*\right) &= E\left[U u'\left(\widehat{\mathbf{CF}}\right) 1_{\{D \le Uy(v)^*\}}\right] \\
& \quad -(c-v)\left(y'(v)^* E\left[U^2 u'\left(\widehat{\mathbf{CF}}\right) g_{D|w}\left(Uy(v)^*\right)\right]\right. \\
& \qquad + E\left[U\left(Uy(v)^* - X\right) u''\left(\boldsymbol{CF}\right) 1_{\{D \le Uy(v)^*\}}\right] \\
& \qquad \left. - (c-v)y'(v)^* E\left[U^2 u''\left(\boldsymbol{CF}\right) 1_{\{D \le Vy(v)^*\}}\right]\right) \\
& \quad +(s+p-c)\left(-y'(v)^* E\left[U^2 u'\left(\widehat{\mathbf{CF}}\right) g_{D|w}\left(Uy(v)^*\right)\right]\right. \\
& \qquad \left. + (s+p-c)y'(v)^* E\left[U^2 u''\left(\boldsymbol{CF}\right) 1_{\{D > Uy(v)^*\}}\right]\right) \\
&= 0
\end{aligned}
$$

where $\widehat{\mathbf{CF}} = CF(Uy(v)^*, U, y(v)^*, v)$. Then, the derivative of the optimal order quantity with respect to v is

$$y'(v)^* = \frac{\left\{ \begin{array}{c} E\left[Uu'\left(CF(Uy\,(v)^*\,, U, y\,(v)^*\,, v)\right) 1_{\{D\le Uy(v)^*\}}\right] \\ -(c-v)E\left[U\left(Uy\,(v)^* - X\right) u''\left(\boldsymbol{CF}\right) 1_{\{D\le Uy(v)^*\}}\right] \end{array} \right\}}{\left\{ \begin{array}{c} +(s+p-v)E\left[U^2 u'\left(CF(Uy\,(v)^*\,, U, y\,(v)^*\,, v)\right) f_{D|w}\left(Uy\,(v)^*\right)\right] \\ -(c-v)^2 E\left[U^2 u''\left(\boldsymbol{CF}\right) 1_{\{D\le Vy(v)^*\}}\right] \\ -(s+p-c)^2 E\left[U^2 u''\left(\boldsymbol{CF}\right) 1_{\{D> Uy(v)^*\}}\right] \end{array} \right\}} \ge 0$$

since $u' \ge 0$ and $u'' \le 0$. So, as the salvage value increases, the optimal order quantity increases.

3.2.4 The Effect of Order Cost

This subsection deals with the effect of the order cost on the optimal order quantity. The cash flow can now be written as

$$CF\left(D, U, y\left(c\right), c\right) = z_0 + (s+p-v)\min\left\{D, Uy\left(c\right)\right\} - (c-v)Uy\left(c\right) - pD$$

and

$$CF(x, w, y\left(c\right), c) = \left\{ \begin{array}{ll} CF_-(x, wy\left(c\right), c) = z_0 + (s-v)x - (c-v)wy\left(c\right) & x \le wy \\ CF_+(x, wy\left(c\right), c) = z_0 + (s+p-c)wy\left(c\right) - px & x \ge wy \end{array} \right. .$$

We then investigate the derivative of the optimal order condition in (3.33) with respect c, that is

$$\frac{d}{dc}\left(\frac{E\left[Uu'\left(CF(D, U, y\left(c\right)^*, c)\right) 1_{\{D\le Uy(c)^*\}}\right]}{E\left[Uu'\left(CF(D, U, y\left(c\right)^*, c)\right)\right]} \right) = \frac{d}{dc}\left(\frac{s+p-c}{s+p-v}\right),$$

or more explicitly

$$\frac{\left\{ \begin{array}{c} E\left[Uu'\left(\mathbf{CF}\right) 1_{\{D> Uy(c)^*\}}\right] \frac{d}{dc}\left(E\left[Uu'\left(\mathbf{CF}\right) 1_{\{D\le Uy(c)^*\}}\right]\right) \\ -E\left[Uu'\left(\mathbf{CF}\right) 1_{\{D\le Uy(c)^*\}}\right] \frac{d}{dc}\left(E\left[Uu'\left(\mathbf{CF}\right) 1_{\{D> Uy(c)^*\}}\right]\right) \end{array} \right\}}{\left(E\left[Uu'\left(\mathbf{CF}\right)\right]\right)^2} = -\frac{1}{s+p-v} \qquad (3.49)$$

as

$$E\left[Uu'\left(\mathbf{CF}\right)\right] = E\left[Uu'\left(\mathbf{CF}\right) 1_{\{D\le Uy(c)^*\}}\right] + E\left[Uu'\left(\mathbf{CF}\right) 1_{\{D> Uy(c)^*\}}\right]$$

where $\mathbf{CF} = CF(D, U, y\left(c\right)^*, c)$.

For any random variables X and V with probability density functions g_X and g_V, we can show that

$$\begin{aligned}
&\frac{d}{dc}E\left[Vu'\left(\boldsymbol{CF}\right)1_{\{X\leq Vy(c)^*\}}\right] \\
=\;& \frac{d}{dc}\left(\int_0^\infty w\left(\int_0^{wy(c)^*} u'\left(\mathbf{CF}_-\right)g_{X|w}\left(x\right)dx\right)g_V(w)dw\right) \\
=\;& \int_0^\infty w\left(wy'\left(c\right)^* u'\left(CF(wy\left(c\right)^*, w, y\left(c\right)^*, c)\right)g_{X|w}\left(wy\left(c\right)^*\right)\right. \\
&\qquad \left. + \int_0^{wy(c)^*}\left(-wy\left(c\right)^* - \left(c-v\right)wy'\left(c\right)^*\right)u''\left(\mathbf{CF}_-\right)g_{X|w}\left(x\right)dx\right)g_V(w)dw \\
=\;& y'\left(c\right)^* E\left[V^2u'\left(CF(Vy\left(c\right)^*, V, y\left(c\right)^*, c)\right)g_{X|w}\left(Vy\left(c\right)^*\right)\right] \\
&\quad - y\left(c\right)^* E\left[V^2u''\left(\boldsymbol{CF}\right)1_{\{X\leq Vy(c)^*\}}\right] \\
&\qquad - \left(c-v\right)y'\left(c\right)^* E\left[V^2u''\left(\boldsymbol{CF}\right)1_{\{X\leq Vy(c)^*\}}\right] \qquad (3.50)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dc}E\left[Vu'\left(\boldsymbol{CF}\right)1_{\{X>Vy(c)^*\}}\right] \\
=\;& \frac{d}{dc}\left(\int_0^\infty w\left(\int_{wy(c)^*}^{\infty} u'\left(\mathbf{CF}_+\right)g_{X|w}\left(x\right)dx\right)g_V(w)dw\right) \\
=\;& \int_0^\infty w\left(-wy'\left(c\right)^* u'\left(CF(wy\left(c\right)^*, w, y\left(c\right)^*, c)\right)g_{X|w}\left(wy\left(c\right)^*\right)\right. \\
&\qquad \left. + \int_{wy(c)^*}^{\infty}\left(-wy\left(c\right)^* + \left(s+p-c\right)wy'\left(c\right)^*\right)u''\left(\mathbf{CF}_+\right)g_{X|w}\left(x\right)dx\right)g_V(w)dw \\
=\;& -y'\left(c\right)^* E\left[V^2u'\left(CF(Vy\left(c\right)^*, V, y\left(c\right)^*, c)\right)g_{X|w}\left(Vy\left(c\right)^*\right)\right] \\
&\quad - y\left(c\right)^* E\left[V^2u''\left(\boldsymbol{CF}\right)1_{\{X>Vy(c)^*\}}\right] \\
&\qquad + \left(s+p-c\right)y'\left(c\right)^* E\left[V^2u''\left(\boldsymbol{CF}\right)1_{\{X>Vy(c)^*\}}\right] \qquad (3.51)
\end{aligned}$$

where $\mathbf{CF}_- = CF_-(x, wy\left(c\right)^*, c)$, $\mathbf{CF}_+ = CF_+(x, wy\left(c\right)^*, c)$ and $\boldsymbol{CF} = CF(X, V, y\left(c\right)^*, c)$.

Then, by using (3.50) and (3.51) where X is D and V is U, the derivative of the optimality condition in (3.49) equals

$$\frac{\left\{\begin{array}{c} y'(c)^{*} E\left[Uu'(\boldsymbol{CF})\right] E\left[U^{2}u'\left(CF(Vy(c)^{*},V,y(c)^{*},c)\right) g_{D|w}\left(Vy(c)^{*}\right)\right] \\ -y(c)^{*} E\left[Uu'(\boldsymbol{CF}) 1_{\{D>Uy(c)^{*}\}}\right] E\left[U^{2}u''(\boldsymbol{CF}) 1_{\{D\le Uy(c)^{*}\}}\right] \\ -(c-v)\, y'(c)^{*} E\left[Uu'(\boldsymbol{CF}) 1_{\{D>Uy(c)^{*}\}}\right] E\left[U^{2}u''(\boldsymbol{CF}) 1_{\{D\le Uy(c)^{*}\}}\right] \\ +y(c)^{*} E\left[Uu'(\boldsymbol{CF}) 1_{\{D\le Uy(c)^{*}\}}\right] E\left[U^{2}u''(\boldsymbol{CF}) 1_{\{D>Uy(c)^{*}\}}\right] \\ -(s+p-c)\, y'(c)^{*} E\left[Uu'(\boldsymbol{CF}) 1_{\{D\le Uy(c)^{*}\}}\right] E\left[U^{2}u''(\boldsymbol{CF}) 1_{\{D>Uy(c)^{*}\}}\right] \end{array}\right\}}{\left(E\left[Uu'\left(CF(D,U,y(c)^{*},c)\right)\right]\right)^{2}}$$

$$= \quad -\frac{1}{s+p-v}.$$

Then, the derivative of the optimal order quantity with respect to c is

$$y'(c)^{*} = \frac{\left\{\begin{array}{c} -\frac{1}{s+p-v}\left(E\left[Uu'(\boldsymbol{CF})\right]\right)^{2} \\ +y(c)^{*} E\left[Uu'(\boldsymbol{CF}) 1_{\{D>Uy(c)^{*}\}}\right] E\left[U^{2}u''(\boldsymbol{CF}) 1_{\{D\le Uy(c)^{*}\}}\right] \\ -y(c)^{*} E\left[Uu'(\boldsymbol{CF}) 1_{\{D\le Uy(c)^{*}\}}\right] E\left[U^{2}u''(\boldsymbol{CF}) 1_{\{D>Uy(c)^{*}\}}\right] \end{array}\right\}}{\left\{\begin{array}{c} E\left[Uu'(\boldsymbol{CF})\right] E\left[U^{2}u'\left(CF(Vy(c)^{*},V,y(c)^{*},c)\right) f_{D|w}\left(Vy(c)^{*}\right)\right] \\ -(c-v) E\left[Uu'(\boldsymbol{CF}) 1_{\{D>Uy(c)^{*}\}}\right] E\left[U^{2}u''(\boldsymbol{CF}) 1_{\{D\le Uy(c)^{*}\}}\right] \\ -(s+p-c) E\left[Uu'(\boldsymbol{CF}) 1_{\{D\le Uy(c)^{*}\}}\right] E\left[U^{2}u''(\boldsymbol{CF}) 1_{\{D>Uy(c)^{*}\}}\right] \end{array}\right\}}$$

The sign of the denominator is positive, but the sign of the numerator is indeterminate. So, the sign of $y'(c)^{*}$ is indeterminate and the effect of initial wealth is unclear.

Again, we analyze the numerator for the exponential utility function, that is $u(z) = K - Ce^{-\frac{1}{\beta}z}$, $u'(z) = \frac{C}{\beta}e^{-\frac{1}{\beta}z}$ and $u''(z) = -\frac{C}{\beta^{2}}e^{-\frac{1}{\beta}z}$. The numerator for the exponential utility function is

$$\left\{\begin{array}{c} -\frac{1}{s+p-v}\left(E\left[U\frac{C}{\beta}e^{-\frac{1}{\beta}(\boldsymbol{CF})}\right]\right)^{2} \\ +y(c)^{*} E\left[U\frac{C}{\beta}e^{-\frac{1}{\beta}(\boldsymbol{CF})}1_{\{D>Uy(c)^{*}\}}\right] E\left[-U^{2}\frac{C}{\beta^{2}}e^{-\frac{1}{\beta}(\boldsymbol{CF})}1_{\{D\le Uy(c)^{*}\}}\right] \\ -y(c)^{*} E\left[U\frac{C}{\beta}e^{-\frac{1}{\beta}(\boldsymbol{CF})}1_{\{D\le Uy(c)^{*}\}}\right] E\left[-U^{2}\frac{C}{\beta^{2}}e^{-\frac{1}{\beta}(\boldsymbol{CF})}1_{\{D>Uy(c)^{*}\}}\right] \end{array}\right\} < 0$$

and so $y'(c)^{*} \le 0$. So, it can be concluded that the optimal order quantity for the exponential utility function decreases as c increases.

3.2.5 The Effect of Penalty Price

Now, we want to analyze the effect of the penalty price on order quantity. We can redefine the cash flow as

$$CF\left(D,U,y(p),p\right) = z_{0} + (s+p-v)\min\left\{D,Uy(p)\right\} - (c-v)Uy(p) - pD$$

and

$$CF(x,w,y(p),c) = \begin{cases} CF_-(x,wy(p),p) = z_0 + (s-v)x - (c-v)wy(p) & x \le wy \\ CF_+(x,wy(p),p) = z_0 + (s+p-c)wy(p) - px & x \ge wy \end{cases}.$$

To analyze the effect of penalty price p on the optimal order quantity, the derivative of the first order condition in (3.37) with respect to p is determined. That is

$$\begin{aligned} \frac{d}{dv} g\left(y(p)^*\right) &= -(c-v)\frac{d}{dp}\left[E\left[Uu'\left(CF(D,U,y(p)^*,p)\right)1_{\{D\le Uy(p)^*\}}\right]\right] \\ &\quad + \frac{d}{dp}\left[(s+p-c)E\left[Uu'\left(CF(D,U,y(p)^*,p)\right)1_{\{D>Uy(p)^*\}}\right]\right] \\ &= 0. \end{aligned} \tag{3.52}$$

Note that for any random variables X and V with probability density functions g_X and g_V, we can write that

$$\begin{aligned} &\frac{d}{dp}E\left[Vu'\left(\boldsymbol{CF}\right)1_{\{X\le Vy(p)^*\}}\right] \\ =\ & \frac{d}{dp}\left(\int_0^\infty w\left(\int_0^{wy(p)^*} u'\left(CF_-(x,w,y(p)^*,p)\right)g_{X|w}(x)\,dx\right)g_V(w)dw\right) \\ =\ & \int_0^\infty w\left(wy'(p)^*u'\left(CF(wy'(p)^*,w,y(p)^*,p)\right)g_{X|w}\left(wy'(p)^*\right)\right. \\ &\quad \left. + \int_0^{wy(p)^*}\left(-(c-v)\,wy'(p)^*\right)u''\left(CF_-(x,wy(p)^*,p)\right)g_{X|w}(x)\,dx\right)g_V(w)dw \\ =\ & y'(p)^*E\left[V^2u'\left(CF(Vy'(p)^*,V,y(p)^*,p)\right)g_{X|w}\left(Vy'(p)^*\right)\right] \\ &\quad -(c-v)\,y'(p)^*E\left[V^2u''\left(\boldsymbol{CF}\right)1_{\{X\le Vy(p)^*\}}\right] \end{aligned} \tag{3.53}$$

and

$$\begin{aligned} &\frac{d}{dp}E\left[Vu'\left(\boldsymbol{CF}\right)1_{\{X>Vy(p)^*\}}\right] \\ =\ & \frac{d}{dp}\left(\int_0^\infty w\left(\int_{wy(p)^*}^\infty u'\left(\mathbf{CF}_+\right)g_{X|w}(x)\,dx\right)g_V(w)dw\right) \\ =\ & \int_0^\infty w\left(-wy'(p)^*u'\left(CF(wy(p)^*,w,y(p)^*,p)\right)g_{X|w}\left(wy(p)^*\right)\right. \\ &\quad \left. + \int_{wy(p)^*}^\infty\left(-\left(x-wy(p)^*\right)+(s+p-c)\,wy'(p)^*\right)u''\left(\mathbf{CF}_+\right)g_{X|w}(x)\,dx\right)g_V(w)dw \\ =\ & -y'(p)^*E\left[V^2u'\left(CF(Vy(p)^*,V,y(p)^*,p)\right)g_{X|w}\left(Vy(p)^*\right)\right] \\ &\quad -E\left[V\left(X-Vy(p)^*\right)u''\left(\boldsymbol{CF}\right)1_{\{X>Vy(p)^*\}}\right] \\ &\quad +(s+p-c)\,y'(p)^*E\left[V^2u''\left(\boldsymbol{CF}\right)1_{\{X>Vy(p)^*\}}\right] \end{aligned} \tag{3.54}$$

where $\mathbf{CF}_+ = CF_+(x, wy(p)^*, p)$ and $\boldsymbol{CF} = CF(X, V, y(p)^*, p)$. By using (3.53) and (3.54), where X is D and V is U, the equation (3.52) can be written as

$$\begin{aligned}\frac{d}{dv}g(y(p)^*) &= E\left[Uu'(\boldsymbol{CF})1_{\{D>Uy(p)^*\}}\right] \\ &\quad -(c-v)\Big(y'(p)^* E\left[U^2u'\left(\widehat{CF}\right)g_{D|w}\left(Uy'(p)^*\right)\right] \\ &\qquad -(c-v)y'(p)^* E\left[U^2u''(\boldsymbol{CF})1_{\{D\le Uy(p)^*\}}\right]\Big) \\ &\quad +(s+p-c)\Big(-y'(p)^* E\left[U^2u'\left(\widehat{CF}\right)g_{D|w}\left(Uy(p)^*\right)\right] \\ &\qquad - E\left[U\left(D-Uy(p)^*\right)u''(\boldsymbol{CF})1_{\{D>Uy(p)^*\}}\right] \\ &\qquad + (s+p-c)y'(p)^* E\left[U^2u''(\boldsymbol{CF})1_{\{D>Uy(p)^*\}}\right]\Big) \\ &= 0\end{aligned}$$

where $\widehat{CF} = CF(Uy'(p)^*, U, y(p)^*, p)$. Then, the derivative of the optimal order quantity with respect to p is

$$y'(p)^* = \frac{\left\{\begin{array}{c} E\left[Uu'(\boldsymbol{CF})1_{\{D>Uy(p)^*\}}\right] \\ -(s+p-c)E\left[U\left(D-Uy(p)^*\right)u''(\boldsymbol{CF})1_{\{D>Uy(p)^*\}}\right]\end{array}\right\}}{\left\{\begin{array}{c} +(s+p-v)E\left[U^2u'\left(CF(Uy(p)^*, U, y(p)^*, p)\right)g_{D|w}\left(Uy'(p)^*\right)\right] \\ -(c-v)^2E\left[U^2u''(\boldsymbol{CF})1_{\{D\le Uy(p)^*\}}\right] \\ -(s+p-c)^2E\left[U^2u''(\boldsymbol{CF})1_{\{D>Uy(p)^*\}}\right]\end{array}\right\}} \ge 0$$

and so we can conclude that as the penalty price increases, the optimal order quantity increases.

Throughout this section, we analyzed the newsvendor problem when demand is not only source of uncertainty but yield is also uncertain. In the following section, we consider the case where supply is uncertain because the supplier has random capacity.

3.3 Newsvendor Model with Random Capacity

This section considers the newsvendor problems where the randomness results from both random demand and supplier's random capacity. In this problem, supplier may not fulfill all asked quantity because of limited capacity. Suppose that the amount received from ordering y units is $\min\{K, y\}$ where the random variable $K \ge 0$ represents the maximum number of units that the supplier can ship. We suppose that K has the distribution function $P\{K \le z\} = G_K(z) > 0$ and density function g_K. Moreover, we assume that D and K

may be dependent and then the conditional density function of demand given $K = z$ is $g_{D|z}$. The cash flow in (3.2) can be updated for the newsvendor problem with random capacity as

$$CF(D, K, y) = z_0 + (s + p - v)\min\{D, K, y\} - (c - v)\min\{K, y\} - pD. \tag{3.55}$$

And, the aim of the newsvendor is

$$\max_{y \geq 0} H(y) = E[u(CF(D, K, y))].$$

For further analysis, we can define the cash flow as

$$CF(x, z \wedge y) = \begin{cases} CF_-(x, z \wedge y) = z_0 + (s - v)\,x - (c - v)\,z \wedge y & x \leq z \wedge y \\ CF_+(x, z \wedge y) = z_0 + (s + p - c)\,z \wedge y - px & x \geq z \wedge y \end{cases}$$

where $z \wedge y = \min\{z, y\}$. Note that $CF(y, y) = CF_-(y, y) = CF_+(y, y) = z_0 + (s - c)\,y$.

Theorem 4 *The optimal order quantity* y^* *satisfies*

$$\frac{E\left[u'\left(CF(D, K, y^*)\right) 1_{\{D \leq y^*, K > y^*\}}\right]}{E\left[u'\left(CF(D, K, y^*)\right) 1_{\{K > y^*\}}\right]} = \widehat{p}. \tag{3.56}$$

Proof. *For any random variables* X *and* Z *with probability density functions* g_X *and* g_Z*, we can write that*

$$\begin{aligned} E[u(CF(X, Z, y))] &= \int_0^\infty \left(\int_0^{z \wedge y} u\left(CF_-(x, z \wedge y)\right) g_{X|z}(x)\,dx\right) g_Z(z)\,dz \\ &\quad + \int_0^\infty \left(\int_{z \wedge y}^\infty u\left(CF_+(x, z \wedge y)\right) g_{X|z}(x)\,dx\right) g_Z(z)\,dz \\ &= \int_0^y \left(\int_0^z u\left(CF_-(x, z)\right) g_{X|z}(x)\,dx\right. \\ &\quad \left. + \int_z^\infty u\left(CF_+(x, z)\right) g_{X|z}(x)\,dx\right) g_Z(z)\,dz \\ &\quad + \int_y^\infty \left(\int_0^y u\left(CF_-(x, y)\right) g_{X|z}(x)\,dx\right. \\ &\quad \left. + \int_y^\infty u\left(CF_+(x, y)\right) g_{X|z}(x)\,dx\right) g_Z(z)\,dz \end{aligned} \tag{3.57}$$

and the derivative of (3.57) is

$$\begin{aligned}\frac{d}{dy}E[u(CF(X,Z,y))] &= -(c-v)\int_y^\infty\left(\int_0^y u'\left(CF_-(x,y)\right)g_{X|z}(x)\,dx\right)g_Z(z)\,dz \\ &\quad +(s+p-c)\int_y^\infty\left(\int_y^\infty u'\left(CF_+(x,y)\right)g_{X|z}(x)\,dx\right)g_Z(z)\,dz \\ &= -(c-v)\,E\left[u'\left(\boldsymbol{CF}\right)1_{\{X\le y,Z>y\}}\right] \\ &\quad +(s+p-c)\,E\left[u'\left(\boldsymbol{CF}\right)1_{\{X>y,Z>y\}}\right] \qquad (3.58)\end{aligned}$$

where $\boldsymbol{CF} = CF(X,Z,y)$. *By using (3.58) where* X *is* D *and* Z *is* K, *the first order condition is*

$$\begin{aligned}g(y) &= \frac{d}{dy}E[u(CF(D,K,y))] \\ &= -(c-v)\,E\left[u'\left(\boldsymbol{CF}\right)1_{\{D\le y,K>y\}}\right]+(s+p-c)\,E\left[u'\left(\boldsymbol{CF}\right)1_{\{D>y,K>y\}}\right] \\ &= -(s+p-v)\,E\left[u'\left(\boldsymbol{CF}\right)1_{\{D\le y,K>y\}}\right]+(s+p-c)\,E\left[u'\left(\boldsymbol{CF}\right)1_{\{K>y\}}\right] \\ &= 0. \qquad (3.59)\end{aligned}$$

This can be also written as

$$g(y) = E\left[u'\left(\boldsymbol{CF}\right)1_{\{K>y\}}\right]\left(-(s+p-v)\frac{E\left[u'\left(\boldsymbol{CF}\right)1_{\{D\le y,K>y\}}\right]}{E\left[u'\left(\boldsymbol{CF}\right)1_{\{K>y\}}\right]}+(s+p-c)\right)=0. \qquad (3.60)$$

Noting that $P\{K>y\}>0$ *and* $u'>0$ *by our assumption, we observe that*

$$E\left[u'\left(\boldsymbol{CF}\right)1_{\{K>y\}}\right]>0.$$

The equation (3.60) can be rewritten as

$$-(s+p-v)h(y)+(s+p-c)=0.$$

where

$$h(y)=\frac{E\left[u'\left(CF(D,K,y)\right)1_{\{D\le y,K>y\}}\right]}{E\left[u'\left(CF(D,K,y)\right)1_{\{K>y\}}\right]}. \qquad (3.61)$$

In the newsvendor model with random capacity, the concavity of the objective function is not necessarily satisfied because $g(y)$ *in (3.60) is not necessarily decreasing. So, the existence and uniqueness of* y^* *satisfying (3.56) depends on the structure of* $h(y)$ *and we need to put some restrictions on it. Suppose that* $h(y)$ *is strictly increasing in* y. *If* $h(0)\le\widehat{p}\le h(\infty)$, *then there exists a unique* $0\le y^*\le\infty$ *that satisfies the optimality condition in (3.56).* ■

Corollary 5 *The optimal order quantity is $y^* = 0$ when*

$$P\{D = 0|K > 0\} > \left(\frac{E[u'(z_0 - pD)]}{u'(z_0)}\right)\widehat{p}. \tag{3.62}$$

Proof. *We can declare that we have $y^* = 0$ if $h(0) > \widehat{p}$; that is*

$$\begin{aligned} h(0) &= \frac{E\left[u'(CF(D,K,0))\,1_{\{D\le 0,K>0\}}\right]}{E\left[u'(CF(D,K,0))\,1_{\{K>0\}}\right]} \\ &= \frac{E\left[u'(CF(D,K,0))\,1_{\{D=0,K>0\}}\right]}{E\left[u'(CF(D,K,0))\,1_{\{K>0\}}\right]} \\ &= \frac{E\left[u'(z_0)\,1_{\{D=0,K>0\}}\right]}{E\left[u'(z_0 - pD)\,1_{\{K>0\}}\right]} \\ &= \frac{u'(z_0)}{E[u'(z_0 - pD)]}P(D = 0|K > 0) > \widehat{p} \end{aligned}$$

which yields to (3.62). ■

Corollary 6 *The optimal order quantity is $y^* = \infty$ if*

$$P\{D = \infty|K = \infty\} > 1 - \widehat{p}. \tag{3.63}$$

We can conclude that if the demand is finite, $P\{D = \infty\} = 0$, the optimal order quantity is clearly finite.

Proof. *We can also argue that $y^* = \infty$ if $h(\infty) < \widehat{p}$; that is*

$$\begin{aligned} h(\infty) &= \frac{E\left[u'(CF(D,K,\infty))\,1_{\{D<\infty,K=\infty\}}\right]}{E\left[u'(CF(D,K,\infty))\,1_{\{K=\infty\}}\right]} \\ &= \frac{E\left[u'(CF(D,K,\infty))\,1_{\{K=\infty\}}\right] - E\left[u'(CF(D,K,\infty))\,1_{\{D=\infty,K=\infty\}}\right]}{E\left[u'(CF(D,K,\infty))\,1_{\{K=\infty\}}\right]} \\ &= 1 - \frac{E\left[u'(CF(\infty,\infty,\infty))\,1_{\{D=\infty,K=\infty\}}\right]}{E\left[u'(CF(D,\infty,\infty))\,1_{\{K=\infty\}}\right]} \\ &= 1 - P(D = \infty|K = \infty) < \widehat{p}. \end{aligned}$$

This argument supposes that the utility function u is bounded. ■

Moreover, as $h(y)$ is increasing, it follows from (3.60) that the derivative $g(y)$ is nonnegative on $[0, y^*)$ and nonpositive on $[y^*, \infty)$. So, we can argue that the objective function is in increasing on $[0, y^*)$ and decreasing on $[y^*, \infty)$. Therefore, we have a quasi-concave objective function and y^* satisfying (3.56) is indeed the optimal solution.

As a special case, suppose that the newsvendor is risk-neutral, that is the utility function is linear. The utility function is $u(x) = a + bx$, so that $u'(x) = b$. The optimality function can be rewritten as

$$\frac{P\{D \leq y^*, K > y^*\}}{P\{K > y^*\}} = P\{D \leq y^* | K > y^*\} = \widehat{p}$$

which is the same optimality condition as Okyay et al. [41] for the newsvendor model with random capacity. Moreover, suppose that there is no capacity restriction, $K = \infty$, the optimality condition (3.56) yields to

$$\frac{E\left[u'[CF(D,K,y^*)]1_{\{D \leq y^*\}}\right]}{E\left[u'[CF(D,K,y^*)]\right]} = \widehat{p}$$

which satisfies the optimality condition of standard model (3.8).

3.3.1 The Effect of Risk-Aversion

This subsection clarifies the effect of risk-aversion on the optimal order quantity. In order to analyze this effect, we again apply Pratt's argument which demonstrates that the utility function can be replaced with its concave transformation to show the increase in risk-aversion. Suppose that we have two different decision makers with the same cash flow but different utility functions, the first one with utility function $u(x)$ and more risk-averse one with utility function $k(u(x))$, where $k' > 0$ and $k'' < 0$.

The cash flow is increasing in x, or

$$CF_-(x_1, z \wedge y^*) \leq CF(z \wedge y^*, z \wedge y^*) \leq CF_+(x_2, z \wedge y^*)$$

for all $x_1 \leq \min\{z, y^*\} \leq x_2$, and then

$$u'(CF_-(x_1, z \wedge y^*)) \geq u'(CF(z \wedge y^*, z \wedge y^*)) \geq u'(CF_+(x_2, z \wedge y^*)).$$

And, for concave increasing function k

$$k'(u(\mathbf{CF}_-)) \geq k'(u(\mathbf{CF})) \geq k'(u(\mathbf{CF}_+)) \tag{3.64}$$

where $\mathbf{CF}_- = CF_-(x_1, z \wedge y^*)$, $\mathbf{CF} = CF(z \wedge y^*, z \wedge y^*)$ and $\mathbf{CF}_+ = CF_+(x_2, z \wedge y^*)$.

The objective of the newsvendor with more risk averse utility function is

$$\max_{y\geq 0}\widetilde{H}(y) = E[k\left(u(CF(D,K,y))\right)]$$

and the first derivative of objective function is

$$\begin{aligned}\widetilde{g}\left(y\right) &= -(c-v)\int_{y}^{\infty}\left(\int_{0}^{y}k'\left(u\left(CF_{-}(x,y)\right)\right)u'\left(CF_{-}(x,y)\right)g_{D|z}\left(x\right)dx\right)g_{K}\left(z\right)dz \\ &\quad +(s+p-c)\int_{y}^{\infty}\left(\int_{y}^{\infty}k'\left(u\left(CF_{+}(x,y)\right)\right)u'\left(CF_{+}(x,y)\right)g_{D|z}\left(x\right)dx\right)g_{K}\left(z\right)dz \\ &= E\left[k'\left(u\left(\boldsymbol{CF}\right)\right)u'\left(\boldsymbol{CF}\right)1_{\{K>y\}}\right]\left(-(s+p-v)\widetilde{h}\left(y\right)+(s+p-c)\right) \end{aligned}\tag{3.65}$$

where

$$\widetilde{h}\left(y\right)=\frac{E\left[k'\left(u\left(\boldsymbol{CF}\right)\right)u'\left(\boldsymbol{CF}\right)1_{\{D\leq y,K>y\}}\right]}{E\left[k'\left(u\left(\boldsymbol{CF}\right)\right)u'\left(\boldsymbol{CF}\right)1_{\{K>y\}}\right]}.\tag{3.66}$$

To compare the optimal order quantities that solve (3.56) and (3.65), we substitute the optimal order quantity solving y^* (3.56) to equation (3.65). The result is

$$\begin{aligned}\widetilde{g}\left(y^*\right) &= \Bigg(-(c-v)\int_{y^*}^{\infty}\left(\int_{0}^{y^*}k'\left(u\left(\mathbf{CF}_{-}\right)\right)u'\left(\mathbf{CF}_{-}\right)g_{D|z}\left(x\right)dx\right)g_{K}\left(z\right)dz \\ &\quad + (s+p-c)\int_{y^*}^{\infty}\left(\int_{y^*}^{\infty}k'\left(u\left(\mathbf{CF}_{+}\right)\right)u'\left(\mathbf{CF}_{+}\right)g_{D|z}\left(x\right)dx\right)g_{K}\left(z\right)dz\Bigg) \\ &\leq k'\left(u\left(\mathbf{CF}\right)\right)\Bigg(-(c-v)\int_{y^*}^{\infty}\left(\int_{0}^{y^*}u'\left(\mathbf{CF}_{-}\right)g_{D|z}\left(x\right)dx\right)g_{K}\left(z\right)dz \\ &\quad + (s+p-c)\int_{y^*}^{\infty}\left(\int_{y^*}^{\infty}u'\left(\mathbf{CF}_{+}\right)g_{D|z}\left(x\right)dx\right)g_{K}\left(z\right)dz\Bigg)\end{aligned}$$

where $\mathbf{CF}_{-} = CF_{-}(x,y^*)$, $\mathbf{CF}_{+} = CF_{+}(x,y^*)$ and $\mathbf{CF} = CF\left(\min\left\{z,y^*\right\},\min\left\{z,y^*\right\}\right)$. So, we can conclude that $\widetilde{g}\left(y^*\right)\leq 0$. By supposing $\widetilde{h}\left(y\right)$ increasing in y and $\widetilde{h}\left(y\right) > h\left(y\right)$ or

$$\frac{E\left[k'\left(u\left(CF(D,K,y)\right)\right)u'\left(CF(D,K,y)\right)1_{\{D\leq y,K>y\}}\right]}{E\left[k'\left(u\left(CF(D,K,y)\right)\right)u'\left(CF(D,K,y)\right)1_{\{K>y\}}\right]} > \frac{E\left[u'\left(CF(D,K,y)\right)1_{\{D\leq y,K>y\}}\right]}{E\left[u'\left(CF(D,K,y)\right)1_{\{K>y\}}\right]},$$

for any concave k, we can conclude that a more risk-averse newsvendor will order less. In other words, as risk aversion increases, the optimal order quantity decreases.

As the concavity of the objective function cannot be obtained for inventory models with random capacity, the effects of other parameters on the optimal order quantity are not resulted in explicit and nice characterizations. Therefore, we do not discuss them in our thesis.

3.4 Newsvendor Model with Random Yield and Capacity

In Section 3.2 and Section 3.3, the newsvendor model for random yield and random capacity is analyzed separately. This section considers the case where both random capacity and random yield exists. It is accepted that the capacity of the supplier is limited by a random number K and the newsvendor receives a random proportion of the amount produced. That is to say that when y is ordered, $U \min\{K, y\}$ is received where U and K are random variables and represent the proportion of non-defective items received and the maximum number of units that the supplier can ship, respectively. Suppose that D, U and K are not necessarily independent and they have a joint distribution function, $F_{DKU}(x, z, w) = P\{D \leq x, K \leq z, U \leq w\}$. Moreover, we also assume that the conditional density functions $g_{K|U=w}$ and $g_{D|K=z,U=w}$ all exist. Then, the cash flow is

$$CF(D, K, U, y) = z_0 - (c - v) U \min\{K, y\} + (s + p - v) \min\{D, UK, Uy\} - pD \quad (3.67)$$

and the aim of the risk-averse newsvendor is,

$$\max_{y \geq 0} H(y) = E[u(CF(D, K, U, y))].$$

For further analysis, let

$$CF(x, w(z \wedge y)) = \begin{cases} CF_-(x, w(z \wedge y)) = z_0 - (c - v) w(z \wedge y) & x \leq w(z \wedge y) \\ \qquad + (s - v) x & \\ CF_+(x, w(z \wedge y)) = z_0 + (s + p - c) w(z \wedge y) & \\ \qquad - px & x \geq w(z \wedge y) \end{cases}$$

and $CF(wy, wy) = CF_-(wy, wy) = CF_+(wy, wy) = z_0 + (s - c) wy$.

Theorem 7 *The optimal order quantity y^* satisfies*

$$\frac{E\left[Uu'(CF(D, K, U, y^*)) 1_{\{D \leq Uy^*, K > y^*\}}\right]}{E\left[Uu'(CF(D, K, U, y^*)) 1_{\{K > y^*\}}\right]} = \widehat{p}. \quad (3.68)$$

Proof. *Note that for any random variables X, V and Z with the probability density functions g_X, g_V and g_Z, we can write that*

$$\begin{aligned} & E[u(CF(X, Z, V, y))] \\ = & \int_0^\infty g_V(w)\, dw \int_0^\infty g_{Z|w}(z)\, dz \left(\int_0^{w(z \wedge y)} u(CF_-(x, w(z \wedge y)))\, g_{X|zw}(x)\, dx \right. \\ & \left. + \int_{w(z \wedge y)}^\infty u(CF_+(x, w(z \wedge y)))\, g_{X|zw}(x)\, dx \right) \end{aligned}$$

$$
\begin{aligned}
= \quad & \int_0^\infty g_V(w)\,dw \left(\int_0^y g_{Z|w}(z)\,dz \left(\int_0^{wz} u\left(CF_-(x,wz)\right) g_{X|zw}(x)\,dx \right.\right. \\
& \qquad \left. + \int_{wz}^\infty u\left(CF_+(x,wz)\right) g_{X|zw}(x)\,dx \right) \\
& \quad + \int_y^\infty g_{Z|w}(z)\,dz \left(\int_0^{wy} u\left(CF_-(x,wy)\right) g_{X|zw}(x)\,dx \right. \\
& \qquad \left.\left. + \int_{wy}^\infty u\left(CF_+(x,wy)\right) g_{X|zw}(x)\,dx \right)\right).
\end{aligned}
$$

And we can show that

$$
\begin{aligned}
& \frac{d}{dy} E[u(\boldsymbol{CF})] \\
= \quad & \int_0^\infty g_V(w)\,dw \left(\int_y^\infty g_{Z|w}(z)\,dz \left(-(c-v)\,w \int_0^{wy} u'(\mathbf{CF}_-)\, g_{X|zw}(x)\,dx \right.\right. \\
& \qquad \left.\left.\left. + (s+p-c)\,w \left(\int_{wy}^\infty u'(\mathbf{CF}_+)\, g_{X|zw}(x)\,dx \right)\right)\right)\right) \\
= \quad & -(c-v)\,E\left[V u'(\boldsymbol{CF})\, 1_{\{X \le Vy, Z>y\}}\right] + (s+p-c)\,E\left[V u'(\boldsymbol{CF})\, 1_{\{X > Vy, Z>y\}}\right]
\end{aligned}
\tag{3.69}
$$

where $\mathbf{CF}_- = CF_-(x, wy)$, $\mathbf{CF}_+ = CF_+(x, wy)$ *and* $\boldsymbol{CF} = CF(X, Z, V, y)$. *By using (3.69), the first order condition can be written as*

$$
\begin{aligned}
g(y) \quad = \quad & \frac{d}{dy} E[u(CF(D,K,U,y))] \\
= \quad & -(c-v)\,E\left[U u'(CF(D,K,U,y))\, 1_{\{D \le Uy, K>y\}}\right] \\
& \quad + (s+p-c)\,E\left[U u'(CF(D,K,U,y))\, 1_{\{D > Uy, K>y\}}\right] \\
= \quad & -(s+p-v)\,E\left[U u'(CF(D,K,U,y))\, 1_{\{D \le Uy, K>y\}}\right] \\
& \quad + (s+p-c)\,E\left[U u'(CF(D,K,U,y))\, 1_{\{K>y\}}\right] \\
= \quad & 0.
\end{aligned}
\tag{3.70}
$$

The derivative (3.70) can also be written as

$$
g(y) = E\left[U u'(CF(D,K,U,y))\, 1_{\{K>y\}}\right] \left(-(s+p-v)\,h(y) + (s+p-c)\right) = 0 \tag{3.71}
$$

where

$$
h(y) = \frac{E\left[U u'(CF(D,K,U,y))\, 1_{\{D \le Uy, K>y\}}\right]}{E\left[U u'(CF(D,K,U,y))\, 1_{\{K>y\}}\right]}. \tag{3.72}
$$

Noting that $P\{K > y\} > 0$ *for all* y *by our assumption and assuming that* $U > 0$ *and* $u' > 0$, $E\left[Uu'(CF(D,K,U,y))\,1_{\{K>y\}}\right] > 0$, *(3.71) can be also written as*

$$-(s+p-v)\,h(y) + (s+p-c) = 0. \tag{3.73}$$

The existence and uniqueness of the optimal order quantity y^* *depends on the structure of* $h(y)$ *in (3.72), like in the previous section. More precisely, suppose that* $h(y)$ *is increasing in* y. *If* $h(0) \leq \widehat{p} \leq h(\infty)$, *then the first order condition in (3.73) is the optimality condition and there exists a* $0 \leq y^* \leq \infty$ *that satisfies the optimality condition* $h(y^*) = \widehat{p}$, *so as* $g(y^*) = 0$. ■

Corollary 8 *The optimal order quantity is* $y^* = 0$ *if*

$$P\{D = 0|K > 0\} > \widehat{p}\frac{E\left[Uu'(z_0 - pD)\right]}{u'(z_0)\,E\left[U\right]}. \tag{3.74}$$

Proof. *Moreover, we can also assert that we have* $y^* = 0$ *if* $h(0) > \widehat{p}$; *that is*

$$\begin{aligned} h(0) &= \frac{E\left[Uu'(CF(D,K,U,0))\,1_{\{D\leq 0,K>0\}}\right]}{E\left[Uu'(CF(D,K,U,0))\,1_{\{K>0\}}\right]} \\ &= \frac{E\left[Uu'(CF(0,K,U,0))\,1_{\{D=0,K>0\}}\right]}{E\left[Uu'(CF(D,K,U,0))\,1_{\{K>0\}}\right]} \\ &= \frac{E\left[Uu'(z_0)\,1_{\{D=0,K>0\}}\right]}{E\left[Uu'(z_0 - pD)\,1_{\{K>0\}}\right]} \\ &= \frac{u'(z_0)\,E\left[U\right]}{E\left[Uu'(z_0 - pD)\right]}P\{D = 0|K > 0\} > \widehat{p}. \end{aligned}$$

■

Corollary 9 *We have* $y^* = \infty$ *if*

$$P\{D = \infty|K = \infty\} > 1 - \widehat{p}. \tag{3.75}$$

We can argue that if the demand is finite, the optimal order quantity is clearly finite.

Proof. *Moreover, we can also argue that* $y^* = \infty$ *if* $h(\infty) < \widehat{p}$; *that is*

$$\begin{aligned} h(\infty) &= \frac{E\left[Uu'(CF(D,K,U,\infty))\,1_{\{D<\infty,K=\infty\}}\right]}{E\left[Uu'(CF(D,K,U,\infty))\,1_{\{K=\infty\}}\right]} \\ &= \frac{E\left[Uu'(CF(D,K,U,\infty))\,1_{\{K=\infty\}}\right] - E\left[Uu'(CF(D,K,U,\infty))\,1_{\{D=\infty,K=\infty\}}\right]}{E\left[Uu'(CF(D,K,U,\infty))\,1_{\{K=\infty\}}\right]} \\ &= 1 - P\{D = \infty|K = \infty\} < \widehat{p}. \end{aligned}$$

■

Furthermore, as $h(y)$ is increasing, it follows from (3.71) that the derivative $g(y)$ is nonnegative on $[0, y^*)$ and nonpositive on $[y^*, \infty)$. So, it can be argued that the objective function is in increasing on $[0, y^*)$ and decreasing on $[y^*, \infty)$. The objective function is quasi-concave and y^* satisfying (3.68) is surely the optimal solution.

As a special case, suppose that the newsvendor is risk-neutral that is the utility function is linear. The optimality condition then becomes

$$\frac{E\left[U1_{\{D\leq Uy^*, K>y^*\}}\right]}{E\left[U1_{\{K>y^*\}}\right]} = \widehat{p}$$

which is the same condition as Okyay et al. [41] for the newsvendor model with random yield and capacity. Moreover, by supposing $K = \infty$ (random yield model), the optimality condition is

$$\frac{E\left[Uu'[CF(D, U, y^*)]1_{\{D\leq Uy^*\}}\right]}{E\left[Uu'[CF(D, U, y^*)]\right]} = \widehat{p}$$

which satisfies (3.33). By supposing $U = 1$ (random capacity model), the optimality condition is

$$\frac{E\left[u'[CF(D, K, y^*)]1_{\{D\leq y^*, K>y^*\}}\right]}{E\left[u'[CF(D, K, y^*)]1_{\{K>y^*\}}\right]} = \widehat{p}$$

which satisfies (3.56). Lastly, by supposing $U = 1$ and $K = \infty$ (standard model), the optimality condition is

$$\frac{E\left[u'[CF(D, y^*)]1_{\{D\leq y^*\}}\right]}{E\left[u'[CF(D, y^*)]\right]} = \widehat{p}$$

which satisfies (3.8).

3.4.1 The Effect of Risk-Aversion

As before, we use Pratt's argument to analyze the effect of the risk-aversion on the optimal order quantity. Again, let k be a concave function where $k' > 0$ and $k'' < 0$.

The cash flow is increasing in x that is

$$CF_-\left(x_1, w\left(z \wedge y^*\right)\right) \leq CF\left(w\left(z \wedge y^*\right), w\left(z \wedge y^*\right)\right) \leq CF_+\left(x_2, w\left(z \wedge y^*\right)\right)$$

for all $x_1 \leq w\left(z \wedge y^*\right) \leq x_2$. Then,

$$u'\left(\mathbf{CF}_-\right) \geq u'\left(CF\left(w\left(z \wedge y^*\right), w\left(z \wedge y^*\right)\right)\right) \geq u'\left(\mathbf{CF}_+\right)$$

and

$$k'\left(u\left(\mathbf{CF}_-\right)\right) \geq k'\left(CF\left(w\left(z \wedge y^*\right), w\left(z \wedge y^*\right)\right)\right) \geq k'\left(u\left(\mathbf{CF}_+\right)\right) \tag{3.76}$$

where $\mathbf{CF}_{-} = CF_{-}(x_1, w(z \wedge y^*))$ and $\mathbf{CF}_{+} = CF_{+}(x_2, w(z \wedge y^*))$.

The aim of our more risk-averse newsvendor is,

$$\max_{y \geq 0} \widetilde{H}(y) = E[k(u(CF(D,K,U,y)))]$$

and the first derivative of objective function is

$$\begin{aligned}
& \widetilde{g}(y) \\
= \; & -(c-v) \int_0^\infty w g_U(w)\, dw \int_y^\infty g_{K|w}(z)\, dz \int_0^{wy} k'(u(\mathbf{CF}_{-}))\, u'(\mathbf{CF}_{-})\, g_{D|zw}(x)\, dx \\
& + (s+p-c) \int_0^\infty w g_U(w)\, dw \int_y^\infty g_{K|w}(z)\, dz \int_{wy}^\infty k'(u(\mathbf{CF}_{+}))\, u'(\mathbf{CF}_{+})\, g_{D|zw}(x)\, dx \\
= \; & E\left[U k'(u(\boldsymbol{CF}))\, u'(\boldsymbol{CF})\, 1_{\{K>y\}}\right] \Big(-(s+p-v)\, \widetilde{h}(y) + (s+p-c) \Big) \qquad (3.77)
\end{aligned}$$

where $\mathbf{CF}_{-} = CF_{-}(x, wy)$, $\mathbf{CF}_{+} = CF_{+}(x, wy)$ and

$$\widetilde{h}(y) = \frac{E\left[U k'(u(CF(D,K,U,y)))\, u'(CF(D,K,U,y))\, 1_{\{D \leq Uy, K>y\}}\right]}{E\left[U k'(u(CF(D,K,U,y)))\, u'(CF(D,K,U,y))\, 1_{\{K>y\}}\right]}. \qquad (3.78)$$

Moreover, when we substitute the optimal order quantity for the newsvendor problem with utility function u into the equation (3.77), we obtain

$$\begin{aligned}
& \widetilde{g}(y) \\
= \; & -(c-v) \int_0^\infty w g_U(w)\, dw \int_{y^*}^\infty g_{K|w}(z)\, dz \int_0^{wy^*} k'(u(\mathbf{CF}_{-}))\, u'(\mathbf{CF}_{-})\, g_{D|zw}(x)\, dx \\
& + (s+p-c) \int_0^\infty w g_U(w)\, dw \int_{y^*}^\infty g_{K|w}(z)\, dz \int_{wy^*}^\infty k'(u(\mathbf{CF}_{+}))\, u'(\mathbf{CF}_{+})\, g_{D|zw}(x)\, dx \\
< \; & k'(u(CF(w(z \wedge y^*), w(z \wedge y^*))))\, g(y^*)
\end{aligned}$$

where $\mathbf{CF}_{-} = CF_{-}(x, wy^*)$, $\mathbf{CF}_{+} = CF_{+}(x, wy^*)$ and

$$\begin{aligned}
g(y^*) \; = \; & -(c-v) \int_0^\infty w f_U(w)\, dw \int_{y^*}^\infty f_{K|w}(z)\, dz \int_0^{wy^*} u'(\mathbf{CF}_{-})\, f_{D|zw}(x)\, dx \\
& + (s+p-c) \int_0^\infty w f_U(w)\, dw \int_{y^*}^\infty f_{K|w}(z)\, dz \int_{wy^*}^\infty u'(\mathbf{CF}_{+})\, f_{D|zw}(x)\, dx \\
= \; & 0.
\end{aligned}$$

Therefore, we can conclude that $\widetilde{g}(y^*) < 0$. By considering $\widetilde{h}(y)$ increases in y and $\widetilde{h}(y) > h(y)$ that is

$$\frac{E\left[U k'(u(\mathbf{CF}))\, u'(\mathbf{CF})\, 1_{\{D \leq Uy, K>y\}}\right]}{E\left[U k'(u(\mathbf{CF}))\, u'(\mathbf{CF})\, 1_{\{K>y\}}\right]} > \frac{E\left[U u'(\mathbf{CF})\, 1_{\{D \leq Uy, K>y\}}\right]}{E\left[U u'(\mathbf{CF})\, 1_{\{K>y\}}\right]}$$

for any concave k where $\mathbf{CF} = CF(D, K, U, y)$, we can conclude that as risk-aversion increases, the optimal order quantity decreases.

Similar to the random capacity, we do not include the discussions about the effects of other parameters on the optimal order quantity in our thesis.

Up to this point, we considered the newsvendor model where the decision maker is risk-averse. First, we analyzed the standard model in Section 3.1 and then we investigated the models with random supply in Sections 3.2-3.4. In the following chapter, we will consider the case when there exists a financial hedging opportunity to decrease the risk further.

Chapter 4

UTILITY-BASED MODELS WITH HEDGING

In the previous chapter, we consider the newsvendor problem where the cash flow is random due to the stochastic nature of demand and supply. As we assume that the decision maker is risk-averse, we have tried to maximize the expected utility of the cash flow. In this chapter, we further consider the existence of a financial market in which there are financial securities correlated with demand and supply. Therefore, the decision maker needs to decide not only how much to order from supplier, but also how much to invest on a portfolio of financial securities.

Okyay et al. [42] consider the inventory management problem with hedging and provide a risk-sensitive solution approach to this problem by considering both the mean and the variance of cash flow. The first aim is to find an optimal portfolio of financial securities that minimizes the variance of the hedged cash flow for any possible order quantity. Then, the mean of the hedged cash flow with this optimal portfolio is maximized by choosing an optimal order quantity. In our thesis, we use the same risk-sensitive, two-step solution approach. Although the first step remains the same, as a second step we aim to maximize the expected utility of the hedged cash flow. To analyze the second step, we initially repeat the first step of Okyay et al. [42].

We assume that the length of the period is T during which the risk-free interest rate is r. Similar to the previous chapter, the newsvendor buys the items at c, sells them for s, returns the unsold portion at v, and compensates the stock-outs at p which satisfy $s > ce^{rT} > v \geq 0$ and $(ce^{rT} - s) < p \leq 0$ to avoid trivial situations. All cash flows occur at time T except for the cash payment made at time 0 to purchase inventory. Let $\mathbf{X}$ denote the vector of random variables corresponding to demand and supply uncertainties and S denote the price of a primary asset in the market at the end of the period. The random vector $\mathbf{X}$ and the financial variable S are correlated. Suppose that there are $n \geq 1$ derivative securities in the market where $f_i(S)$ is the net payoff of the ith derivative security of the primary asset at the end of the period. In other words, it is the payoff $\hat{f}_i(S)$ received at time T minus its investment cost f_i^T so that $f_i(S) = \hat{f}_i(S) - f_i^T$. Let f_i^0 denote the price of the ith derivative security at the beginning of the period when it is purchased, and we then have $f_i^T = e^{rT} f_i^0$. If the market is complete with some risk-neutral probability measure Q, then it is well-known that $f_i^0 = e^{-rT} E_Q[\hat{f}_i(S)]$ and this will lead to $E_Q[f_i(S)] = E_Q\left[\hat{f}_i(S) - f_i^T\right] = 0$. We do not necessarily suppose that the market is complete. However, the consequences of such a market will be analyzed in our numerical illustrations in the last chapter. Moreover, let α_i denote the amount of security i in the portfolio. The total hedged cash flow at time T is given by

$$CF_{\boldsymbol{\alpha}}(\mathbf{X}, S, y) = CF(\mathbf{X}, y) + \sum_{i=1}^{n} \alpha_i f_i(S) \tag{4.1}$$

where $CF(\mathbf{X}, y)$ denotes the unhedged cash flow. The first step of our solution algorithm is to find the optimal portfolio $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ to minimize the variance of the total cash flow for a given order quantity y. So, the optimization problem is

$$\min_{\boldsymbol{\alpha}} Var\left(CF(\mathbf{X}, y) + \sum_{i=1}^{n} \alpha_i f_i(S)\right) \tag{4.2}$$

Once the optimal solution $\boldsymbol{\alpha}^*(y)$ is determined for any order quantity y, the risk-averse decision maker chooses the optimal order quantity by solving

$$\max_{y \geq 0} E\left[u\left(CF(\mathbf{X}, y) + \sum_{i=1}^{n} \alpha_i^*(y) f_i(S)\right)\right]. \tag{4.3}$$

Note that we do not impose nonnegativity restrictions on the portfolio $\boldsymbol{\alpha}$ implying that shortselling is possible.

First, suppose that there is a single derivative security in the market where $f(S)$ is the payoff of that derivative security of the primary asset and let α denote the amount of that security. Thus, the optimization problem becomes

$$\min_{\alpha} Var(CF(\mathbf{X}, y) + \alpha f(S))$$

where the objective function can be written as

$$\begin{aligned} Var(CF_\alpha(\mathbf{X}, S, y)) &= Var(CF(\mathbf{X}, y) + \alpha f(S)) \\ &= \alpha^2 Var(f(S)) \\ &\quad + 2\alpha Cov(f(S), CF(\mathbf{X}, y)) \\ &\quad + Var(CF(\mathbf{X}, y)). \end{aligned} \tag{4.4}$$

Proposition 10 *When there is a single asset, the optimal asset quantity that minimizes the variance of cash flow is*

$$\alpha^*(y) = -\frac{Cov\left(f(S), CF(\mathbf{X}, y)\right)}{Var\left(f(S)\right)}. \tag{4.5}$$

Proof. *The gradient of the objective function in (4.4) with respect to* α *is*

$$\frac{\partial}{\partial \alpha}\left(Var\left(CF_\alpha(\mathbf{X}, S, y)\right)\right) = 2\alpha Var\left(f(S)\right) + 2Cov\left(f(S), CF(\mathbf{X}, y)\right)$$

and the Hessian is

$$\frac{\partial^2}{\partial \alpha^2}\left(Var\left(CF_\alpha(\mathbf{X}, S, y)\right)\right) = 2Var\left(f(S)\right) \geq 0.$$

As the second order condition is satisfied, we can conclude that the first order condition gives the optimal solution. ■

We will use this optimal portfolio for single asset in the following sections.

Moreover, suppose that there are n derivative securities in the market. Then, the optimization problem is

$$\min_{\boldsymbol{\alpha}} Var\left(CF(\mathbf{X}, y) + \boldsymbol{\alpha}^{\mathbf{T}}\mathbf{f}(S)\right)$$

where $\boldsymbol{\alpha}$ is a column vector with entries $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\mathbf{f}(S)$ is another column vector with entries $\mathbf{f} = (f_1(S), f_2(S), \cdots, f_n(S))$. The objective function can be written as

$$\begin{aligned} Var\left[CF_{\boldsymbol{\alpha}}(\mathbf{X}, S, y)\right] &= Var\left(CF(\mathbf{X}, y) + \sum_{i=1}^{n} \alpha_i f_i(S)\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j Cov\left(f_i(S), f_j(S)\right) \\ &\quad +2\sum_{i=1}^{n} \alpha_i Cov\left(f_i(S), CF(\mathbf{X}, y)\right) \\ &\quad +Var\left(CF(\mathbf{X}, y)\right). \end{aligned}$$

We can rewrite the objective function in compact matrix notation as

$$Var\left(CF_{\boldsymbol{\alpha}}(\mathbf{X}, S, y)\right) = \boldsymbol{\alpha}^T \mathbf{C} \boldsymbol{\alpha} + 2\boldsymbol{\alpha}^T \boldsymbol{\mu}(y) + Var\left(CF(\mathbf{X}, y)\right) \tag{4.6}$$

where $\boldsymbol{\alpha}^T$ is the transpose of $\boldsymbol{\alpha}$, $\mathbf{C}$ is the covariance matrix of the securities with entries

$$C_{ij} = Cov\left(f_i(S), f_j(S)\right)$$

and $\boldsymbol{\mu}(y)$ is a vector with entries

$$\mu_i(y) = Cov(f_i(S), CF(\mathbf{X}, y)).$$

Proposition 11 *When there are multiple securities in the market, the optimal portfolio is*

$$\boldsymbol{\alpha}^*(y) = -\mathbf{C}^{-1}\boldsymbol{\mu}(y). \tag{4.7}$$

Proof. *By taking the gradient of the objective function (4.6) and setting it equal to zero, the first order condition is obtained as*

$$\frac{\partial}{\partial \boldsymbol{\alpha}}(Var(CF_{\boldsymbol{\alpha}}(\mathbf{X}, S, y))) = 2\mathbf{C}\boldsymbol{\alpha} + 2\boldsymbol{\mu}(y) = 0$$

and the Hessian is

$$\frac{\partial^2}{\partial \boldsymbol{\alpha}^2}(Var(CF_{\boldsymbol{\alpha}}(\mathbf{X}, S, y))) = 2\mathbf{C} \geq 0$$

as the covariance matrix $\mathbf{C}$ *is always positive definite. So, the second order condition is satisfied and the first order condition gives the optimality condition.* ■

We will use this optimal portfolio for multiple securities in the following sections.

In Section (4.1), we first consider the standard model where demand is the only source of randomness. Then, in Sections (4.2)-(4.4), we consider the random supply models, including random yield, random capacity, and random yield and capacity models, respectively.

4.1 Standard Newsvendor Model

Suppose that there is no randomness in the supply. Recall that the random demand D have a cumulative distribution $G_D(x) = P\{D \leq x\}$ and a density function g_D and the random demand D and the financial variable S are correlated. Then, the unhedged cash flow equals to

$$\begin{aligned} CF_\alpha(\mathbf{X}, S, y) &= CF(D, y) + \alpha f(S) \\ &= -\left(ce^{rT} - v\right)y + (s + p - v)\min\{D, y\} - pD + \boldsymbol{\alpha}^{\mathbf{T}}\mathbf{f}(S) \end{aligned} \tag{4.8}$$

where $\mathbf{X} = \{D\}$.

We analyze the problem first for a single security and then for multiple securities.

4.1.1 Hedging with Only One Security

Suppose that there is a single derivative security in the market where $f(S)$ is the payoff of that derivative security of the primary asset and let α denote the amount of that security. Thus, the optimization problem becomes

$$\min_{\alpha} Var\left(CF_{\alpha}\left(D,S,y\right)\right).$$

The optimal asset quantity for a single asset in (4.5) can be updated as

$$\alpha^{*}\left(y\right)=-\frac{Cov\left(f\left(S\right),CF\left(D,y\right)\right)}{Var\left(f\left(S\right)\right)}$$

since $\mathbf{X}=\{D\}$. We can rewrite the optimal asset quantity as

$$\alpha^{*}\left(y\right)=-\left(s+p-v\right)\beta_{D}\left(y\right)+p\beta_{D}\left(\infty\right) \tag{4.9}$$

where

$$\beta_{D}\left(y\right)=\frac{Cov\left(f\left(S\right),\min\left\{D,y\right\}\right)}{Var\left(f\left(S\right)\right)}$$

and

$$\beta_{D}\left(\infty\right)=\frac{Cov\left(f\left(S\right),D\right)}{Var\left(f\left(S\right)\right)}.$$

Once we obtain the optimal portfolio $\alpha^{*}(y)$, we can use it to maximize the expected utility of the hedged cash flow. So, the optimization problem is

$$\max_{y\geq 0} E\left[u\left(CF_{\alpha^{*}(y)}\left(D,S,y\right)\right)\right] \tag{4.10}$$

where the hedged cash flow at time T is given by

$$\begin{aligned}CF_{\alpha^{*}(y)}\left(D,S,y\right) &= CF\left(D,y\right)+\alpha^{*}\left(y\right)f\left(S\right)\\ &= -\left(ce^{rT}-v\right)y+\left(s+p-v\right)\min\left\{D,y\right\}-pD+\alpha^{*}\left(y\right)f\left(S\right).\end{aligned} \tag{4.11}$$

We can also write the hedged cash flow as

$$CF_{\alpha^{*}(y)}\left(x,t,y\right)=\begin{cases} CF_{-}(x,t,y)=-\left(ce^{rT}-v\right)y+\left(s-v\right)x+\alpha^{*}\left(y\right)f\left(t\right) & x\leq y\\ CF_{+}(x,t,y)=\left(s+p-ce^{rT}\right)y-px+\alpha^{*}\left(y\right)f\left(t\right) & x\geq y\end{cases} \tag{4.12}$$

where $CF_-(y,t,y) = CF_+(y,t,y) = \left(s - ce^{rT}\right) y + \alpha^*(y) f(t)$. Then, the objective function becomes

$$E\left[u\left(CF_{\alpha^*(y)}(D,S,y)\right)\right] = \int_0^y E_x\left[u\left(CF_-(x,S,y)\right)\right] g_D(x)\,dx + \int_y^\infty E_x\left[u\left(CF_+(x,S,y)\right)\right] g_D(x)\,dx \qquad (4.13)$$

where $E_x[Y] = E[Y|D=x]$ is a conditional expectation.

We now redefine the critical ratio as

$$\widehat{p} = \frac{s+p-ce^{rT}}{s+p-v}$$

because the cash flow involves the compounded order cost with risk-free rate r.

Theorem 12 *The optimality condition is*

$$\frac{E\left[u'\left(CF(D,S,y^*)\right)1_{\{D\leq y^*\}}\right] + \beta_D'(y^*)\,E\left[f(S)\,u'\left(CF(D,S,y^*)\right)\right]}{E\left[u'\left(CF(D,S,y^*)\right)\right]} = \widehat{p} \qquad (4.14)$$

where $\widehat{p}$ is the same critical ratio. The only difference from $\widehat{p}$ in (3.8) is that the order cost is discounted with the risk-free interest rate r.

Proof. *By defining the derivative of $\alpha^*(y)$ in (4.9) as*

$$\frac{d}{dy}\alpha^*(y) = -(s+p-v)\,\beta_D'(y),$$

one can show that the derivative of (4.13) is

$$\begin{aligned}
&\frac{d}{dy}E\left[u\left(CF_{\alpha^*(y)}(D,S,y)\right)\right] \\
= &-\left(ce^{rT}-v\right)\int_0^y E_x\left[u'\left(CF_-(x,S,y)\right)\right]g_D(x)\,dx \\
&+\left(s+p-ce^{rT}\right)\int_y^\infty E_x\left[u'\left(CF_+(x,S,y)\right)\right]g_D(x)\,dx \\
&-(s+p-v)\,\beta_D'(y)\int_0^\infty E_x\left[f(S)\,u'\left(CF_-(x,S,y)\right)\right]g_D(x)\,dx \\
= &-\left(ce^{rT}-v\right)E\left[u'\left(CF(D,S,y)\right)1_{\{D\leq y\}}\right] \\
&+\left(s+p-ce^{rT}\right)E\left[u'\left(CF(D,S,y)\right)1_{\{D>y\}}\right] \\
&-(s+p-v)\,\beta_D'(y)\,E\left[f(S)\,u'\left(CF(D,S,y)\right)\right] \qquad (4.15)
\end{aligned}$$

where

$$\beta_D'(y) = \frac{Cov\left(f(S),1_{\{D>y\}}\right)}{Var\left(f(S)\right)}.$$

Then, the first order condition can be written as

$$\begin{aligned} g(y) &= E\left[u'(CF(D,S,y))\right]\left(\left(s+p-ce^{rT}\right)-(s+p-v)\,h(y)\right) \\ &= 0 \end{aligned} \tag{4.16}$$

where

$$h(y) = \frac{E\left[u'(CF(D,S,y))\,1_{\{D\le y\}}\right] + \beta_D'(y)\,E\left[f(S)\,u'(CF(D,S,y))\right]}{E\left[u'(CF(D,S,y))\right]}. \tag{4.17}$$

Noting that $u' > 0$ by our assumption, the first order condition (4.16) can be written as

$$\frac{E\left[u'(CF(D,S,y))\,1_{\{D\le y\}}\right] + \beta_D'(y)\,E\left[f(S)\,u'(CF(D,S,y))\right]}{E\left[u'(CF(D,S,y))\right]} = \frac{s+p-ce^{rT}}{s+p-v}. \tag{4.18}$$

The existence and uniqueness of the optimal order quantity depends on the structure of $h(y)$. By supposing that $h(y)$ is increasing in y and letting $h(0) \le \widehat{p} \le h(\infty)$, the first order condition in (4.18) is the optimality condition. ■

Moreover, we can conclude that if $h(0) > \widehat{p}$, we have $y^* = 0$ and if $h(\infty) < \widehat{p}$, we have $y^* = \infty$. Furthermore, as $h(y)$ is increasing, the derivative $g(y)$ is nonnegative on $[0, y^*)$ and nonpositive on $[y^*, \infty)$. So, it can be argued that the objective function is in increasing on $[0, y^*)$ and decreasing on $[y^*, \infty)$. The objective function is quasi-concave and y^* satisfying (4.14) is indeed the optimal solution.

In this chapter we consider two special cases. The first special case supposes that there is no hedging opportunity, or $\alpha = 0$. Therefore, the optimality condition is

$$\frac{E\left[u'(CF(D,y^*))\,1_{\{D\le y^*\}}\right]}{E\left[u'(CF(D,y^*))\right]} = \widehat{p}$$

which satisfies (3.8). Moreover, the second special case supposes that the utility function is linear; that is $u(x) = a + bx$. The optimality condition becomes

$$P\{D \le y^*\} + \beta_D'(y^*)\,E\left[f(S)\right] = \widehat{p}$$

which is the same condition as Okyay et al. [42].

4.1.2 Hedging with Multiple Securities

Now, suppose that there are n derivative securities in the market. Then, the optimization problem is

$$\min_{\boldsymbol{\alpha}} Var\left(CF_{\boldsymbol{\alpha}}(D,S,y)\right).$$

The optimal portfolio in (4.7) can be written as

$$\boldsymbol{\alpha}^{*}(y) = -\mathbf{C}^{-1}\boldsymbol{\mu}(y) \tag{4.19}$$

where

$$C_{ij} = Cov\left(f_i(S), f_j(S)\right)$$

and

$$\mu_i(y) = Cov\left(f_i(S), CF(D, y)\right).$$

Then, the optimal portfolio $\boldsymbol{\alpha}^{*}(y)$ is used to maximize the utility of the expected cash flow. So, the new optimization problem is

$$\max_{y} E\left[u\left(CF_{\boldsymbol{\alpha}^{*}(y)}(D, S, y)\right)\right] \tag{4.20}$$

and the hedged cash flow can also be represented using

$$CF_{\boldsymbol{\alpha}^{*}(y)}(x,t,y) = \begin{cases} CF_{-}(x,t,y) = -\left(ce^{rT} - v\right)y + (s-v)x - \boldsymbol{\mu}(y)^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{f}(t) & x \le y \\ CF_{+}(x,t,y) = \left(s + p - ce^{rT}\right)y - px - \boldsymbol{\mu}(y)^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{f}(t) & x \ge y \end{cases}$$

where $CF_{-}(y,t,y) = CF_{+}(y,t,y) = \left(s - ce^{rT}\right)y - \boldsymbol{\mu}(y)^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{f}(t)$. Then, the objective function can be written as

$$\begin{aligned} E\left[u\left(CF_{\boldsymbol{\alpha}^{*}(y)}(x, S, y)\right)\right] &= \int_0^y E_x\left[u\left(CF_{-}(x,S,y)\right)\right] g_D(x)\,dx \\ &\quad + \int_y^\infty E_x\left[u\left(CF_{+}(x,S,y)\right)\right] g_D(x)\,dx. \end{aligned} \tag{4.21}$$

Theorem 13 *The optimal order quantity y^{*} satisfies*

$$\frac{\left\{\begin{array}{c} E\left[u'\left(CF_{\alpha^{*}(y^{*})}(D,S,y^{*})\right)1_{\{D\le y^{*}\}}\right] \\ +\mathbf{C}^{-1}Cov\left(\mathbf{f}(S), 1_{\{D>y^{*}\}}\right)E\left[\mathbf{f}(S)\,u'\left(CF_{\alpha^{*}(y^{*})}(D,S,y^{*})\right)\right] \end{array}\right\}}{E\left[u'\left(CF_{\alpha^{*}(y^{*})}(D,S,y^{*})\right)\right]} = \widehat{p}. \tag{4.22}$$

Proof. *First, we take the derivative of (4.21) as*

$$\begin{aligned}
&\frac{d}{dy}E\left[u\left(CF_{\boldsymbol{\alpha}^*(y)}\left(x,S,y\right)\right)\right] \\
= \quad & -\left(ce^{rT}-v\right)\int_0^y E_x\left[u'\left(CF_-(x,S,y)\right)\right]g_D\left(x\right)dx \\
& +\left(s+p-ce^{rT}\right)\int_y^\infty E_x\left[u'\left(CF_+(x,S,y)\right)\right]g_D\left(x\right)dx \\
& -\boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}\int_0^\infty E_x\left[\mathbf{f}\left(S\right)u'\left(CF_-(x,S,y)\right)\right]g_D\left(x\right)dx \\
= \quad & -\left(ce^{rT}-v\right)E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(D,S,y\right)\right)1_{D\le y}\right] \\
& +\left(s+p-ce^{rT}\right)E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(D,S,y\right)\right)1_{D>y}\right] \\
& -\boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{f}\left(S\right)E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(D,S,y\right)\right)\right] \qquad (4.23)
\end{aligned}$$

where the derivative of $\boldsymbol{\mu}(y)$ *equals*

$$\boldsymbol{\mu}'\left(y\right)=\frac{d}{dy}Cov\left(\mathbf{f}\left(S\right),CF\left(D,y\right)\right)=\left(s+p-v\right)Cov\left(\mathbf{f}\left(S\right),1_{\{D>y\}}\right). \qquad (4.24)$$

Therefore, by using (4.23) and (4.24), the first order condition can be written as

$$\begin{aligned}
g\left(y\right) &= E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(D,S,y\right)\right)\right]\left(\left(s+p-ce^{rT}\right)-\left(s+p-v\right)h\left(y\right)\right) \\
&= 0 \qquad (4.25)
\end{aligned}$$

where

$$h\left(y\right)=\frac{\left\{\begin{array}{c} E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(D,S,y\right)\right)1_{\{D\le y\}}\right] \\ +\mathbf{C}^{-1}Cov\left(\mathbf{f}\left(S\right),1_{\{D>y\}}\right)E\left[\mathbf{f}\left(S\right)u'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(D,S,y\right)\right)\right]\end{array}\right\}}{E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(D,S,y\right)\right)\right]}.$$

Note that by our assumption $u'\left(CF_{\boldsymbol{\alpha}^*(y)}(D,S,y)\right)>0$ *for all* y*, then the first order condition in (4.25) can be written as*

$$\frac{\left\{\begin{array}{c} E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(D,S,y\right)\right)1_{\{D\le y\}}\right] \\ +\mathbf{C}^{-1}Cov\left(\mathbf{f}\left(S\right),1_{\{D>y\}}\right)E\left[\mathbf{f}\left(S\right)u'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(D,S,y\right)\right)\right]\end{array}\right\}}{E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(D,S,y\right)\right)\right]}=\frac{s+p-ce^{rT}}{s+p-v}. \qquad (4.26)$$

Again, the existence and uniqueness of the optimal order quantity depends on the structure of $h(y)$*. Supposing that* $h(y)$ *is increasing in* y *and* $h(0)\le\widehat{p}\le h(\infty)$*, the first order condition in (4.26) is the optimality condition.* ■

Again, for the first special case, when $\boldsymbol{\alpha}^*(y) = 0$, the optimality condition is

$$\frac{E\left[u'\left(CF(D,y^*)\right)1_{\{D\leq y^*\}}\right]}{E\left[u'\left(CF(D,y^*)\right)\right]} = \widehat{p}.$$

which clearly satisfies (3.8). And, for the second special case, $u(x) = a + bx$, the optimality condition becomes

$$P\{D \leq y^*\} + \mathbf{C}^{-1}Cov\left(\mathbf{f}(S), 1_{\{D>y^*\}}\right)E\left[\mathbf{f}(S)\right] = \widehat{p}$$

which is the same condition as Okyay et al. [42].

In following sections, we add supply uncertainty into our model. Therefore, both demand and supply increase the uncertainty of the problem. Moreover, the randomness in demand and supply is correlated with the financial markets.

4.2 Newsvendor Model with Random Yield

This section deals with the case when the supply is subject to yield randomness and the amount received from ordering y units is Uy where $0 \leq U \leq 1$ has the density function g_U and the conditional density function of D given $U = w$ is $g_{D|w}$. Then, the unhedged cash flow is

$$CF(\mathbf{X}, y) = -\left(ce^{rT} - v\right)Uy + (s + p - v)\min\{D, Uy\} - pD$$

where $\mathbf{X} = \{D, U\}$. Moreover, D and U is correlated with S.

We analyze the model with hedging on a single security or multiple securities, respectively.

4.2.1 Hedging with Only One Security

Suppose that there is only one derivative security in the market where $f(S)$ is the payoff of that derivative security and let α denote the amount of that security. First, we want to solve the following optimization problem

$$\min_{\alpha} Var\left(CF_{\alpha}(\mathbf{X}, S, y)\right)$$

where $\mathbf{X} = \{D, U\}$. The optimal portfolio in (4.5) can be rewritten as

$$\alpha^*(y) = -\frac{Cov\left(f(S), CF(D, U, y)\right)}{Var\left(f(S)\right)}$$

or more explicitly

$$\alpha^*(y) = \left(ce^{rT} - v\right) y\beta_U - (s+p-v)\,\beta_{D,U}(y) + p\beta_D(\infty) \tag{4.27}$$

where

$$\beta_U = \frac{Cov\left(f(S), U\right)}{Var\left(f(S)\right)},$$

$$\beta_{D,U}(y) = \frac{Cov\left(f(S), \min\{D, Uy\}\right)}{Var\left(f(S)\right)}$$

and

$$\beta_D(y) = \frac{Cov\left(f(S), \min\{D, y\}\right)}{Var\left(f(S)\right)}.$$

We then use the optimal asset quantity $\alpha^*(y)$ to find an optimal order quantity that maximizes the expected utility of the hedged cash flow, or

$$\max_{y \geq 0} E\left[u\left(CF_{\alpha^*(y)}(\mathbf{X}, S, y)\right)\right].$$

For further analysis, we can redefine the hedged cash flow as

$$CF_{\alpha^*(y)}(x,w,t,y) = \begin{cases} CF_-(x,w,t,y) = -\left(ce^{rT} - v\right) wy + (s-v)\,x & x \leq wy \\ \qquad +\alpha^*(y)\,f(t) & \\ CF_+(x,w,t,y) = \left(s+p-ce^{rT}\right) wy - px & x \geq wy \\ \qquad +\alpha^*(y)\,f(t) & \end{cases}$$

where $CF_-(wy,w,t,y) = CF_+(wy,w,t,y) = \left(s - ce^{rT}\right) wy + \alpha^*(y)\,f(t)$. Then, the objective function can be written as

$$E\left[u\left(CF_{\alpha^*(y)}(\mathbf{X}, S, y)\right)\right] = \int_0^\infty \left(\int_0^{wy} E_{x,w}\left[u\left(CF_-(x,w,S,y)\right)\right] g_{D|w}(x)\,dx + \int_{wy}^\infty E_{x,w}\left[u\left(CF_+(x,w,S,y)\right)\right] g_{D|w}(x)\,dx\right) g_U(w)dw \tag{4.28}$$

by letting $E_{x,w}[Y] = E[Y|D=x, U=w]$ denote a conditional expectation.

Theorem 14 *The optimal order quantity y^* satisfies*

$$\frac{E\left[Uu'\left(CF(\mathbf{X},S,y^*)\right) 1_{\{D \leq Uy^*\}}\right]}{E\left[Uu'\left(CF(\mathbf{X},S,y^*)\right)\right]} + \left(\beta'_{D,U}(y^*) - \frac{\left(ce^{rT} - v\right)}{(s+p-v)}\beta_U\right) \frac{E\left[f(S)\,u'\left(CF(\mathbf{X},S,y^*)\right)\right]}{E\left[Uu'\left(CF(\mathbf{X},S,y^*)\right)\right]} = \widehat{p}. \tag{4.29}$$

Proof. *We take the derivative of (4.28) as*

$$\begin{aligned}
&\frac{d}{dy}E\left[u\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)\right] \\
= & \int_0^\infty \left(\int_0^{wy} \left(-\left(ce^{rT}-v\right)E_{x,w}\left[wu'\left(CF_-(x,w,S,y)\right)\right]\right.\right. \\
&\left.\left. + \frac{d\alpha^*(y)}{dy}E_{x,w}\left[f(S)\,u'\left(CF_-(x,w,S,y)\right)\right]\right) g_{D|w}(x)\,dx\right) g_U(w)dw \\
&+\int_0^\infty \left(\int_{wy}^\infty \left(\left(s+p-ce^{rT}\right)E_{x,w}\left[wu'\left(CF_+(x,w,S,y)\right)\right]\right.\right. \\
&\left.\left. + \frac{d\alpha^*(y)}{dy}E_{x,w}\left[f(S)\,u'\left(CF_+(x,w,S,y)\right)\right]\right) g_{D|w}(x)\,dx\right) g_U(w)dw
\end{aligned} \tag{4.30}$$

where

$$\frac{d\alpha^*(y)}{dy} = \left(ce^{rT}-v\right)\beta_U - (s+p-v)\,\beta'_{D,U}(y)\,.$$

The derivative (4.30) can be also written as

$$\begin{aligned}
&\frac{d}{dy}E\left[u\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)\right] \\
= & -\left(ce^{rT}-v\right)E\left[Uu'\left(CF\left(\mathbf{X},S,y\right)\right)1_{\{D\le Uy\}}\right] \\
&+\left(s+p-ce^{rT}\right)E\left[Uu'\left(CF\left(\mathbf{X},S,y\right)\right)1_{\{D> Uy\}}\right] \\
&+\left(\left(ce^{rT}-v\right)\beta_U\right. \\
&\left. -(s+p-v)\,\beta'_{D,U}(y)\right)E\left[f(S)\,u'\left(CF\left(\mathbf{X},S,y\right)\right)\right].
\end{aligned}$$

Then, the first order condition becomes

$$\begin{aligned}
g(y) &= (s+p-v)\,E\left[Uu'\left(CF\left(\mathbf{X},S,y\right)\right)\right]\left(\frac{\left(s+p-ce^{rT}\right)}{(s+p-v)} - h(y)\right) \\
&= 0
\end{aligned} \tag{4.31}$$

where

$$\begin{aligned}
h(y) &= \frac{E\left[Uu'\left(CF\left(\mathbf{X},S,y\right)\right)1_{\{D\le Uy\}}\right]}{E\left[Uu'\left(CF\left(\mathbf{X},S,y\right)\right)\right]} \\
&\quad + \left(\beta'_{D,U}(y) - \frac{\left(ce^{rT}-v\right)}{(s+p-v)}\beta_U\right)\frac{E\left[f(S)\,u'\left(CF\left(\mathbf{X},S,y\right)\right)\right]}{E\left[Uu'\left(CF\left(\mathbf{X},S,y\right)\right)\right]}.
\end{aligned}$$

Note that $u' > 0$ by our assumption and

$$E\left[Uu'\left(CF\left(\mathbf{X},S,y\right)\right)\right] > 0.$$

So, the first order condition in (4.31) is updated as

$$\begin{aligned} & \frac{E\left[Uu'\left(CF\left(\mathbf{X},S,y\right)\right)1_{\{D\leq Uy\}}\right]}{E\left[Uu'\left(CF\left(\mathbf{X},S,y\right)\right)\right]} \\ & \quad +\left(\beta'_{D,U}\left(y\right)-\frac{\left(ce^{rT}-v\right)}{\left(s+p-v\right)}\beta_{U}\right)\frac{E\left[f\left(S\right)u'\left(CF\left(\mathbf{X},S,y\right)\right)\right]}{E\left[Uu'\left(CF\left(\mathbf{X},S,y\right)\right)\right]} \\ = & \frac{\left(s+p-ce^{rT}\right)}{\left(s+p-v\right)}. \end{aligned} \tag{4.32}$$

The existence and uniqueness of the optimal order quantity depends on the structure of $h(y)$. Again, by supposing that $h(y)$ is increasing in y and $h(0) \leq \widehat{p} \leq h(\infty)$, the first order condition in (4.32) is the optimality condition. ■

For the first special case, $\alpha = 0$, the first order condition is

$$\frac{E\left[Uu'\left(CF\left(\mathbf{X},S,y^{*}\right)\right)1_{\{D\leq Uy^{*}\}}\right]}{E\left[Uu'\left(CF\left(\mathbf{X},S,y^{*}\right)\right)\right]}=\widehat{p}$$

which satisfies the optimality condition in (3.8). And for the second special case, $u(x) = a + bx$, the optimality condition is

$$\frac{E\left[U1_{\{D\leq Uy^{*}\}}\right]}{E\left[U\right]}+\left(\beta'_{D,U}\left(y^{*}\right)-\frac{\left(ce^{rT}-v\right)}{\left(s+p-v\right)}\beta_{U}\right)\frac{E\left[f\left(S\right)\right]}{E\left[U\right]}=\widehat{p}$$

which equals to optimality condition in Okyay et al. [42].

4.2.2 Hedging with Multiple Securities

Now, we assume that there are n financial securities in the market. The first step of our solution technique is to solve the optimization problem,

$$\min_{\boldsymbol{\alpha}} Var\left(CF_{\boldsymbol{\alpha}}\left(\mathbf{X},S,y\right)\right)$$

where $\mathbf{X} = \{D, U\}$. The optimal portfolio in (4.7) can be updated as

$$\boldsymbol{\alpha}^{*}\left(y\right)=-\mathbf{C}^{-1}\boldsymbol{\mu}\left(y\right). \tag{4.33}$$

where

$$C_{ij}=Cov\left(f_{i}\left(S\right),f_{j}\left(S\right)\right)$$

and

$$\mu_i(y) = Cov\left(f_i(S_i), CF(D,U,y)\right).$$

In the second step of our algorithm, the optimal $\boldsymbol{\alpha}^*(y)$ is used to maximize the expected utility of the hedged cash flow. Then, the optimization problem is

$$\max_y E\left[u\left(CF_{\boldsymbol{\alpha}^*(y)}(D,U,S,y)\right)\right]$$

where the hedged cash flow can be defined as

$$CF_{\boldsymbol{\alpha}^*(y)}(x,w,t,y) = \begin{cases} CF_-(x,w,t,y) = -\left(ce^{rT}-v\right)wy + (s-v)x & x \le wy \\ \qquad -\boldsymbol{\mu}(y)^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{f}(t) & \\ CF_+(x,w,t,y) = \left(s+p-ce^{rT}\right)wy - px & x > wy \\ \qquad -\boldsymbol{\mu}(y)^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{f}(t) & \end{cases}$$

where $CF_-(wy,w,t,y) = CF_+(wy,w,t,y) = \left(s-ce^{rT}\right)wy - \boldsymbol{\mu}(y)^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{f}(t)$. Then, the objective function is

$$\begin{aligned} E\left[u\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)\right] &= \int_0^\infty\left(\int_0^{wy} E_{x,w}\left[u\left(\mathbf{CF}_-\right)\right]g_{D|w}(x)\,dx\right)g_U(w)dw \\ &\quad+\int_0^\infty\left(\int_{wy}^\infty E_{x,w}\left[u\left(\mathbf{CF}_+\right)\right]g_{D|w}(x)\,dx\right)g_U(w)dw \end{aligned} \tag{4.34}$$

where $\mathbf{CF}_- = CF_-(x,w,S,y)$ and $\mathbf{CF}_+ = CF_+(x,w,S,y)$.

Theorem 15 *The optimal order quantity y^* satisfies*

$$\frac{E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y^*)}(\mathbf{X},S,y^*)\right)1_{\{D\le wy^*\}}\right]}{E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y^*)}(\mathbf{X},S,y^*)\right)\right]}+\frac{\boldsymbol{\mu}(y^*)^{\mathbf{T}}\mathbf{C}^{-1}}{(s+p-v)}\frac{E\left[\mathbf{f}(S)\,u'\left(CF_{\boldsymbol{\alpha}^*(y^*)}(\mathbf{X},S,y^*)\right)\right]}{E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y^*)}(\mathbf{X},S,y^*)\right)\right]} = \widehat{p}. \tag{4.35}$$

Proof. *The derivative of (4.34) is*

$$\begin{aligned}
&\frac{d}{dy}E\left[u\left(CF_{\boldsymbol{\alpha}^*(y)}\left(\mathbf{X},S,y\right)\right)\right] \\
=\ & \int_0^\infty \left(-\left(ce^{rT}-v\right)\left(\int_0^{wy} E_{x,w}\left[wu'\left(\mathbf{CF}_-\right)\right]g_{D|w}\left(x\right)dx\right)\right. \\
&\quad +\left(s+p-ce^{rT}\right)\left(\int_{wy}^\infty E_{x,w}\left[wu'\left(\mathbf{CF}_+\right)\right]g_{D|w}\left(x\right)dx\right) \\
&\quad \left.-\boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}\left(\int_0^\infty E_{x,w}\left[\mathbf{f}\left(S\right)u'\left(\mathbf{CF}_{\boldsymbol{\alpha}^*(y)}\right)\right]g_{D|w}\left(x\right)dx\right)\right)g_U(w)dw \\
=\ & -\left(ce^{rT}-v\right)E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(\mathbf{X},S,y\right)\right)1_{\{D\le wy\}}\right] \\
&\quad +\left(s+p-ce^{rT}\right)E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(\mathbf{X},S,y\right)\right)1_{\{D>wy\}}\right] \\
&\quad -\boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}E\left[\mathbf{f}\left(S\right)u'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(\mathbf{X},S,y\right)\right)\right]
\end{aligned} \tag{4.36}$$

where $\mathbf{CF}_{\boldsymbol{\alpha}^*(y)} = CF_{\boldsymbol{\alpha}^*(y)}(x,w,S,y)$ *and*

$$\boldsymbol{\mu}\left(y\right) = -\left(ce^{rT}-v\right)yCov\left(\mathbf{f}\left(S\right),U\right)+\left(s+p-v\right)Cov\left(\mathbf{f}\left(S\right),\min\left\{D,Uy\right\}\right)-pCov\left(\mathbf{f}\left(S\right),D\right),$$

the derivative of which is

$$\boldsymbol{\mu}'\left(y\right) = -\left(ce^{rT}-v\right)Cov\left(\mathbf{f}\left(S\right),U\right)+\left(s+p-v\right)Cov\left(\mathbf{f}\left(S\right),1_{\{D>Uy\}}\right).$$

Hence, the first order condition can be written as

$$\begin{aligned}
g\left(y\right) &= -\left(s+p-v\right)E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(\mathbf{X},S,y\right)\right)1_{\{D\le wy\}}\right] \\
&\quad +\left(s+p-ce^{rT}\right)E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(\mathbf{X},S,y\right)\right)\right] \\
&\quad -\boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}E\left[\mathbf{f}\left(S\right)u'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(\mathbf{X},S,y\right)\right)\right] \\
&= \left(s+p-v\right)E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(\mathbf{X},S,y\right)\right)\right]\left(\frac{s+p-ce^{rT}}{s+p-v}-h\left(y\right)\right) \\
&= 0
\end{aligned} \tag{4.37}$$

where

$$h\left(y\right) = \frac{E\left[Uu'\left(CF\left(\mathbf{X},S,y\right)\right)1_{\{D\le Uy\}}\right]}{E\left[Uu'\left(CF\left(\mathbf{X},S,y\right)\right)\right]} + \frac{\boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}E\left[\mathbf{f}\left(S\right)u'\left(CF\left(\mathbf{X},S,y\right)\right)\right]}{\left(s+p-v\right)E\left[Uu'\left(CF\left(\mathbf{X},S,y\right)\right)\right]}.$$

Note that $u' > 0$ *by our assumption and*

$$E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(\mathbf{X},S,y\right)\right)\right] > 0.$$

Then, (4.37) can be written as

$$\begin{aligned} & \frac{E\left[Uu'\left(CF\left(\mathbf{X},S,y\right)\right)1_{\{D\leq Uy\}}\right]}{E\left[Uu'\left(CF\left(\mathbf{X},S,y\right)\right)\right]} \\ & \quad + \frac{\boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}E\left[\mathbf{f}\left(S\right)u'\left(CF\left(\mathbf{X},S,y\right)\right)\right]}{\left(s+p-v\right)E\left[Uu'\left(CF\left(\mathbf{X},S,y\right)\right)\right]} \\ = & \frac{s+p-ce^{rT}}{s+p-v}. \end{aligned} \tag{4.38}$$

The existence and uniqueness of y^* again depends on the structure of $h\left(y\right)$. We suppose that $h\left(y\right)$ is increasing in y and if $h\left(0\right) \leq \widehat{p} \leq h\left(\infty\right)$, the first order condition in (4.38) is the optimality condition. ■

If we consider the first special case, the optimality condition is

$$\frac{E\left[Uu'\left(CF\left(\mathbf{X},S,y^*\right)\right)1_{\{D\leq Uy^*\}}\right]}{E\left[Uu'\left(CF\left(\mathbf{X},S,y^*\right)\right)\right]} = \widehat{p}$$

which satisfies the optimality condition in (3.8). Else if we consider the second special case, $u(x) = a + bx$, the optimality function becomes as

$$\frac{E\left[U1_{\{D\leq Uy^*\}}\right]}{E\left[U\right]} + \frac{\boldsymbol{\mu}\left(y^*\right)^{\mathbf{T}}\mathbf{C}^{-1}E\left[\mathbf{f}\left(S\right)\right]}{\left(s+p-v\right)E\left[U\right]} = \widehat{p}$$

which is the same condition as Okyay et al. [42].

4.3 Newsvendor Model with Random Capacity

In this section we analyze the newsvendor problem when the capacity of supplier is random, so the amount received from ordering y units is $\min\{K,y\}$ where K is a random variable with the density function be g_K and the conditional density function of D given $K=z$ is $g_{D|z}$. Both D and K are correlated with S. The unhedged cash flow equals

$$CF\left(\mathbf{X},y\right) = -\left(ce^{rT}-v\right)\min\{K,y\} + \left(s+p-v\right)\min\{D,K,y\} - pD \tag{4.39}$$

where $\mathbf{X} = \{D,K\}$.

We again analyze the single security and multiple securities cases in turn.

4.3.1 Hedging with Only One Security

Again, we suppose that there is only one derivative security with payoff $f(S)$ and amount α. The first problem is to find the optimal α for a given order quantity y by solving

$$\min_{\alpha} Var\left(CF_{\alpha}\left(\mathbf{X}, S, y\right)\right)$$

where $\mathbf{X} = \{D, K\}$. The optimal portfolio in (4.5) can be updated as

$$\alpha^*(y) = -\frac{Cov\left(f(S), CF(D, K, y)\right)}{Var\left(f(S)\right)}$$

or

$$\alpha^*(y) = \left(ce^{rT} - v\right)\beta_K(y) - (s + p - v)\,\beta_{D,K}(y) + p\beta_D(\infty) \tag{4.40}$$

where

$$\beta_K(y) = \frac{Cov\left(f(S), \min\{K, y\}\right)}{Var\left(f(S)\right)},$$

$$\beta_{D,K}(y) = \frac{Cov\left(f(S), \min\{D, K, y\}\right)}{Var\left(f(S)\right)}$$

and

$$\beta_D(y) = \frac{Cov\left(f(S), \min\{D, y\}\right)}{Var\left(f(S)\right)}.$$

Then, the optimal $\alpha^*(y)$ can be used to maximize the expected utility of hedged cash flow. Then, the optimization problem is

$$\max_{y} E\left[u\left(CF_{\alpha^*(y)}\left(\mathbf{X}, S, y\right)\right)\right].$$

We can define the hedged cash flow as

$$CF_{\alpha^*(y)}(x, z, t, y) = \begin{cases} CF_-(x, z\wedge y, t, y) = -\left(ce^{rT} - v\right) z\wedge y + (s - v)\,x + \alpha^*(y)\,f(t) & x \le z\wedge y \\ CF_+(x, (z\wedge y), t, y) = \left(s + p - ce^{rT}\right) z\wedge y - px + \alpha^*(y)\,f(t) & x \ge z\wedge y \end{cases}$$

where

$$\begin{aligned} CF_-(z\wedge y, z\wedge y, t, y) &= CF_+(z\wedge y, z\wedge y, t, y) \\ &= \left(s - ce^{rT}\right) z\wedge y + \alpha^*(y)\,f(t). \end{aligned}$$

By using this definition, we write the objective function as

$$\begin{aligned}
& E\left[u\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)\right] \\
= & \int_0^y g_K(z)dz\left(\int_0^z E_{x,z}\left[u\left(CF_-(x,z,S,y)\right)\right]g_{D|z}\left(x\right)dx\right. \\
& \left.+\int_z^\infty E_{x,z}\left[u\left(CF_+(x,z,S,y)\right)\right]g_{D|z}\left(x\right)dx\right) \\
& +\int_y^\infty g_K(z)dz\left(\int_0^y E_{x,z}\left[u\left(CF_-(x,y,S,y)\right)\right]g_{D|z}\left(x\right)dx\right. \\
& \left.+\int_y^\infty E_{x,z}\left[u\left(CF_+(x,y,S,y)\right)\right]g_{D|z}\left(x\right)dx\right)
\end{aligned} \tag{4.41}$$

where the conditional expectation is $E_{x,z}\left[Y\right]=E\left[Y|D=x,K=z\right].$

Theorem 16 *The optimal order quantity y^*satisfies*

$$\left(\beta'_{D,K}\left(y^*\right)-\frac{\left(ce^{rT}-v\right)}{\left(s+p-v\right)}\beta'_K\left(y^*\right)\right)\frac{E\left[f\left(S\right)u'\left(\boldsymbol{CF}\right)\right]}{E\left[u'\left(\boldsymbol{CF}\right)1_{\{K>y^*\}}\right]}+\frac{E\left[u'\left(\boldsymbol{CF}\right)1_{\{D\leq y^*,K>y^*\}}\right]}{E\left[u'\left(\boldsymbol{CF}\right)1_{\{K>y^*\}}\right]}=\widehat{p} \tag{4.42}$$

where $\boldsymbol{CF}=CF_{\alpha^*(y^*)}\left(\mathbf{X},S,y^*\right)$.

Proof. *The first derivative of (4.41) is*

$$\begin{aligned}
& \frac{d}{dy}E\left[u\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)\right] \\
= & \frac{d\alpha^*\left(y\right)}{dy}\int_0^\infty g_K(z)dz\left(\int_0^\infty E_{x,z}\left[f\left(S\right)u'\left(CF_{\alpha^*(y)}\left(x,z,S,y\right)\right)\right]g_{D|z}\left(x\right)dx\right) \\
& -\left(ce^{rT}-v\right)\int_y^\infty g_K(z)dz\left(\int_0^y E_{x,z}\left[u'\left(CF_-(x,y,S,y)\right)\right]g_{D|z}\left(x\right)dx\right) \\
& +\left(s+p-ce^{rT}\right)\int_y^\infty g_K(z)dz\left(\int_y^\infty E_{x,z}\left[u'\left(CF_+(x,y,S,y)\right)\right]g_{D|z}\left(x\right)dx\right) \\
= & \left(\left(ce^{rT}-v\right)\beta'_K\left(y\right)-\left(s+p-v\right)\beta'_{D,K}\left(y\right)\right)E\left[f\left(S\right)u'\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)\right] \\
& -\left(ce^{rT}-v\right)E\left[u'\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)1_{\{D\leq y,K>y\}}\right] \\
& +\left(s+p-ce^{rT}\right)E\left[u'\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)1_{\{D>y,K>y\}}\right].
\end{aligned}$$

Then, the first order condition is

$$\begin{aligned}
g\left(y\right) &= \left(s+p-v\right)E\left[u'\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)1_{\{K>y\}}\right]\left(\frac{s+p-ce^{rT}}{s+p-v}-h\left(y\right)\right) \\
&= 0.
\end{aligned} \tag{4.43}$$

where

$$\begin{aligned} h(y) &= \left(\beta'_{D,K}(y) - \frac{(ce^{rT}-v)}{(s+p-v)}\beta'_K(y)\right)\frac{E\left[f(S)\,u'\left(CF_{\alpha^*(y)}(\mathbf{X},S,y)\right)\right]}{E\left[u'\left(CF_{\alpha^*(y)}(\mathbf{X},S,y)\right)1_{\{K>y\}}\right]} \\ &\quad + \frac{E\left[u'\left(CF_{\alpha^*(y)}(\mathbf{X},S,y)\right)1_{\{D\leq y,K>y\}}\right]}{E\left[u'\left(CF_{\alpha^*(y)}(\mathbf{X},S,y)\right)1_{\{K>y\}}\right]}. \end{aligned}$$

Noting that $u' > 0$ *and* $P\{K > y\} > 0$ *by our assumption, we observe that*

$$E\left[u'\left(CF_{\alpha^*(y)}(\mathbf{X},S,y)\right)1_{\{K>y\}}\right] > 0$$

and the first order condition (4.43) can be written as

$$\begin{aligned} &\left(\beta'_{D,K}(y) - \frac{(ce^{rT}-v)}{(s+p-v)}\beta'_K(y)\right)\frac{E\left[f(S)\,u'\left(CF_{\alpha^*(y)}(\mathbf{X},S,y)\right)\right]}{E\left[u'\left(CF_{\alpha^*(y)}(\mathbf{X},S,y)\right)1_{\{K>y\}}\right]} \\ &\quad + \frac{E\left[u'\left(CF_{\alpha^*(y)}(\mathbf{X},S,y)\right)1_{\{D\leq y,K>y\}}\right]}{E\left[u'\left(CF_{\alpha^*(y)}(\mathbf{X},S,y)\right)1_{\{K>y\}}\right]} \\ = &\ \frac{s+p-ce^{rT}}{s+p-v}. \end{aligned} \tag{4.44}$$

Again, the existence and uniqueness of the optimal order quantity y^* *depends on the structure of* $h(y)$. *By supposing that* $h(y)$ *is increasing in* y, *if* $h(0) \leq \widehat{p} \leq h(\infty)$, *the first order condition in (4.44) is the optimality condition.* ■

By considering the first special case, the optimality condition becomes

$$\frac{E\left[u'(\boldsymbol{CF})\,1_{\{D\leq y^*,K>y^*\}}\right]}{E\left[u'(\boldsymbol{CF})\,1_{\{K>y^*\}}\right]} = \widehat{p}$$

which satisfies the optimality condition in (3.8). Moreover, by considering the second special case, the optimality condition is

$$P\{D \leq y^*|K > y^*\} + \left(\beta'_{D,K}(y^*) - \frac{(ce^{rT}-v)}{(s+p-v)}\beta'_K(y^*)\right)\frac{E[f(S)]}{P\{K>y^*\}} = \widehat{p}$$

which is the same condition as Okyay et al. [42].

4.3.2 Hedging with Multiple Securities

We further suppose that there are n derivative securities in the market. The first optimization problem is to find the optimal $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ to minimize the variance of the total hedged cash flow for a given order quantity y, or

$$\min_{\alpha} Var\left(CF_{\boldsymbol{\alpha}}\left(\mathbf{X}, S, y\right)\right)$$

where $\mathbf{X} = \{D, K\}$. The optimal portfolio in (4.7) can be written as

$$\boldsymbol{\alpha}^*\left(y\right) = -\mathbf{C}^{-1}\boldsymbol{\mu}\left(y\right) \tag{4.45}$$

where

$$C_{ij} = Cov\left(f_i\left(S\right), f_j\left(S\right)\right)$$

and

$$\mu_i\left(y\right) = Cov\left(f_i\left(S_i\right), CF\left(D, K, y\right)\right)$$

that denotes the covariance between the financial securities and the cash flow.

Then, as a second step, by using the optimal portfolio $\boldsymbol{\alpha}^*\left(y\right)$ the optimization problem is

$$\max_{y} E\left[u\left(CF_{\boldsymbol{\alpha}^*(y)}\left(\mathbf{X}, S, y\right)\right)\right]$$

where the hedged cash flow can be written as

$$CF_{\boldsymbol{\alpha}^*(y)}\left(x, z\wedge y, t, y\right) = \begin{cases} CF_-(x, z\wedge y, S, y) = -\left(ce^{rT} - v\right) z\wedge y + (s-v)\,x & x \leq z\wedge y \\ \qquad -\boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{f}\left(t\right) & \\ CF_+(x, z\wedge y, S, y) = \left(s + p - ce^{rT}\right) z\wedge y - px & x \geq z\wedge y \\ \qquad -\boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{f}\left(t\right) & \end{cases}$$

where $z\wedge y = \min\{z, y\}$ and it follows that $CF_-(z\wedge y, z\wedge y, S, y) = CF_+(z\wedge y, z\wedge y, S, y) = \left(s - ce^{rT}\right) z\wedge y - \boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{f}\left(S\right)$. Then, the objective function can be written as

$$\begin{aligned} & E\left[u\left(CF_{\boldsymbol{\alpha}^*(y)}\left(\mathbf{X}, S, y\right)\right)\right] \\ = \; & \int_0^y g_K(z)dz\left(\int_0^z E_{x,z}\left[u\left(CF_-(x, z, S, y)\right)\right] g_{D|z}\left(x\right)dx \right. \\ & \left. \qquad + \int_z^\infty E_{x,z}\left[u\left(CF_+(x, z, S, y)\right)\right] g_{D|z}\left(x\right)dx\right) \\ & + \int_y^\infty g_K(z)dz\left(\int_0^y E_{x,z}\left[u\left(CF_-(x, y, S, y)\right)\right] g_{D|z}\left(x\right)dx \right. \\ & \left. \qquad + \int_y^\infty E_{x,z}\left[u\left(CF_+(x, y, S, y)\right)\right] g_{D|z}\left(x\right)dx\right). \end{aligned} \tag{4.46}$$

Theorem 17 *The optimal order quantity y^* satisfies*

$$\frac{\boldsymbol{\mu}(y^*)^{\mathbf{T}}\mathbf{C}^{-1}}{(s+p-v)}\frac{E\left[\mathbf{f}(S)\,u'\left(CF_{\boldsymbol{\alpha}^*(y^*)}(\mathbf{X},S,y^*)\right)\right]}{E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y^*)}(\mathbf{X},S,y^*)\right)1_{\{K>y^*\}}\right]} + \frac{E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y^*)}(\mathbf{X},S,y^*)\right)1_{\{D\le y^*,K>y^*\}}\right]}{E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y^*)}(\mathbf{X},S,y^*)\right)1_{\{K>y^*\}}\right]} = \widehat{p}. \tag{4.47}$$

Proof. *The first derivative of (4.46) is*

$$\begin{aligned}
&\frac{d}{dy}E\left[u\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)\right]\\
=\;& -\boldsymbol{\mu}(y)^{\mathbf{T}}\mathbf{C}^{-1}\int_0^\infty g_K(z)dz\left(\int_0^\infty E_{x,z}\left[\mathbf{f}(S)\,u'\left(CF_{\boldsymbol{\alpha}^*(y)}(x,z\wedge y,S,y)\right)\right]g_{D|z}(x)\,dx\right)\\
&+\int_y^\infty g_K(z)dz\left(-\left(ce^{rT}-v\right)\int_0^y E_{x,z}\left[u'\left(CF_-(x,y,S,y)\right)\right]g_{D|z}(x)\,dx\right.\\
&\left.\qquad+\left(s+p-ce^{rT}\right)\int_y^\infty E_{x,z}\left[u'\left(CF_+(x,y,S,y)\right)\right]g_{D|z}(x)\,dx\right)\\
=\;& -\boldsymbol{\mu}(y)^{\mathbf{T}}\mathbf{C}^{-1}E\left[\mathbf{f}(S)\,u'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)\right]\\
&-\left(ce^{rT}-v\right)E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)1_{\{D\le y,K>y\}}\right]\\
&+\left(s+p-ce^{rT}\right)E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)1_{\{D>y,K>y\}}\right].
\end{aligned}$$

Then, the first order condition is obtained as

$$\begin{aligned}
g(y) &= (s+p-v)\,E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)1_{\{K>y\}}\right]\left(\frac{s+p-ce^{rT}}{s+p-v}-h(y)\right)\\
&= 0
\end{aligned} \tag{4.48}$$

where

$$h(y) = \frac{\boldsymbol{\mu}(y)^{\mathbf{T}}\mathbf{C}^{-1}}{s+p-v}\frac{E\left[\mathbf{f}(S)\,u'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)\right]}{E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)1_{\{K>y\}}\right]}+\frac{E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)1_{\{D\le y,K>y\}}\right]}{E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)1_{\{K>y\}}\right]}.$$

Noting that $u'>0$ and $P\{K>y\}>0$ by our assumption, we observe that

$$E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)1_{\{K>y\}}\right]>0.$$

Then, (4.48) can be written as

$$\begin{aligned}
&\frac{\boldsymbol{\mu}(y)^{\mathbf{T}}\mathbf{C}^{-1}}{s+p-v}\frac{E\left[\mathbf{f}(S)\,u'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)\right]}{E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)1_{\{K>y\}}\right]}\\
&\quad+\frac{E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)1_{\{D\le y,K>y\}}\right]}{E\left[u'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)1_{\{K>y\}}\right]}\\
=\;&\frac{s+p-ce^{rT}}{s+p-v}.
\end{aligned} \tag{4.49}$$

Again, the existence and uniqueness of the optimal order quantity y^* depends on the structure of $h(y)$. By supposing that $h(y)$ is increasing in y and $h(0) \le \widehat{p} \le h(\infty)$, (4.49) gives the optimality condition. ■

For the first special case, the optimality condition is

$$\frac{E\left[u'\left(CF\left(\mathbf{X}, S, y^*\right)\right) 1_{\{D \le y^*, K > y^*\}}\right]}{E\left[u'\left(CF\left(\mathbf{X}, S, y^*\right)\right) 1_{\{K > y^*\}}\right]} = \widehat{p}$$

which satisfies the optimality condition (3.8). Moreover, for the second special case, the optimality condition is

$$P\{D \le y^* | K > y^*\} + \frac{\mathbf{C}^{-1} \boldsymbol{\mu}'(y^*) E[\mathbf{f}(S)]}{(s + p - v) P\{K > y^*\}} = \widehat{p}$$

which satisfies the condition in Okyay et al. [42].

4.4 Newsvendor Model with Random Yield and Capacity

This section deals with the newsvendor problem when supply randomness is resulted from both yield and capacity. So, the amount received from ordering y unit is $U \min\{K, y\}$ where U and K are random variables with density functions $g_U(w)$ and $g_K(z)$. Moreover, we suppose that D, U and K are not necessarily independent and have a joint distribution function, $G_{DKU}(x, z, w) = P\{D \le x, K \le z, U \le w\}$. The conditional distribution function of D for given $K = z$ and $U = w$ is $g_{D|zw}$ and the conditional distribution function of K for a given $U = w$ is $g_{K|w}$. We also suppose that D, U and K are correlated with S. The unhedged cash flow equals to

$$CF(\mathbf{X}, y) = -\left(ce^{rT} - v\right) U \min\{K, y\} + (s + p - v) \min\{D, UK, Uy\} - pD \tag{4.50}$$

where $\mathbf{X} = (D, U, K)$.

4.4.1 Hedging with Only One Security

Suppose that there is only one derivative security in the market , so the first optimization problem is

$$\min_{\alpha} Var\left(CF_\alpha\left(\mathbf{X}, S, y\right)\right)$$

where $\mathbf{X} = (D, U, K)$. The optimal asset quantity in (4.5) can be written as

$$\alpha^*(y) = -\frac{Cov\left(f(S), CF(D,U,K,y)\right)}{Var\left(f(S)\right)}$$

or more explicitly

$$\alpha^*(y) = \left(ce^{rT} - v\right)\beta_{K,U}(y) - (s+p-v)\,\beta_{D,K,U}(y) + p\beta_D(\infty) \qquad (4.51)$$

where

$$\beta_{K,U}(y) = \frac{Cov\left(f(S), U\min\{K,y\}\right)}{Var\left(f(S)\right)},$$

$$\beta_{D,K,U}(y) = \frac{Cov\left(f(S), \min\{D, UK, Uy\}\right)}{Var\left(f(S)\right)}$$

and

$$\beta_D(y) = \frac{Cov\left(f(S), \min\{D,y\}\right)}{Var\left(f(S)\right)}.$$

Then, by using the optimal $\alpha^*(y)$, the optimization problem is

$$\max_y E\left[u\left(CF_{\alpha^*(y)}(\mathbf{X}, S, y)\right)\right].$$

The hedged cash flow can also be represented as

$$CF(x,w,z\wedge y,t,y) = \begin{cases} CF_-(x,w,z\wedge y,t,y) = -\left(ce^{rT}-v\right)w(z\wedge y) & x \le w(z\wedge y) \\ \qquad +(s-v)\,x & \\ \qquad +\alpha^*(y)\,f(t) & \\ CF_+(x,w,z\wedge y,t,y) = \left(s+p-ce^{rT}\right)w(z\wedge y) & x \ge w(z\wedge y) \\ \qquad -px & \\ \qquad +\alpha^*(y)\,f(t) & \end{cases}$$

where

$$\begin{aligned} CF_-\left(w(z\wedge y), w, z\wedge y, t, y\right) &= CF_+\left(w(z\wedge y), w, z\wedge y, t, y\right) \\ &= \left(s - ce^{rT}\right)w(z\wedge y) + \alpha^*(y)\,f(t). \end{aligned}$$

Then, the objective function is

$$\begin{aligned}
& E\left[u\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)\right] \\
= & \int_0^\infty g_U(w)\,dw \left(\int_0^y g_{K|w}(z)\,dz \left(\int_0^{wz} E_{x,z,w}\left[u\left(CF_-(x,w,z,S,y)\right)\right] g_{D|zw}(x)\,dx \right.\right. \\
& \left. + \int_{wz}^\infty E_{x,z,w}\left[u\left(CF_+(x,w,z,S,y)\right)\right] g_{D|zw}(x)\,dx \right) \\
& + \int_y^\infty g_{K|w}(z)\,dz \left(\int_0^{wy} E_{x,z,w}\left[u\left(CF_-(x,w,y,S,y)\right)\right] g_{D|zw}(x)\,dx \right. \\
& \left.\left. + \int_{wy}^\infty E_{x,z,w}\left[u\left(CF_+(x,w,y,S,y)\right)\right] g_{D|zw}(x)\,dx \right)\right)
\end{aligned} \tag{4.52}$$

where $E_{x,z,w}[Y] = E[Y|D=x, K=z, U=w]$ denote a conditional expectation.

Theorem 18 *The optimal order quantity y^*satisfies*

$$\begin{aligned}
& \left(\beta'_{D,K,U}(y^*) - \frac{(ce^{rT}-v)}{(s+p-v)}\beta'_{K,U}(y^*)\right) \frac{E\left[f(S)\,u'\left(CF_{\alpha^*(y^*)}\left(\mathbf{X},S,y^*\right)\right)\right]}{E\left[Uu'\left(CF_{\alpha^*(y^*)}\left(\mathbf{X},S,y^*\right)\right)1_{\{K>y^*\}}\right]} \\
& + \frac{E\left[Uu'\left(CF_{\alpha^*(y^*)}\left(\mathbf{X},S,y^*\right)\right)1_{\{D\le Uy^*,K>y^*\}}\right]}{E\left[Uu'\left(CF_{\alpha^*(y^*)}\left(\mathbf{X},S,y^*\right)\right)1_{\{K>y^*\}}\right]} \\
= & \ \widehat{p}.
\end{aligned} \tag{4.53}$$

Proof. *The derivative of (4.52) with respect to y is*

$$\begin{aligned}
& \frac{d}{dy}E\left[u\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)\right] \\
= & \frac{d\alpha^*(y)}{dy}\int_0^\infty g_U(w)\,dw\left(\int_0^\infty g_{K|w}(z)\,dz\left(\int_0^\infty \mathbf{E}^1 g_{D|zw}(x)\,dx\right)\right) \\
& -\left(ce^{rT}-v\right)\int_0^\infty g_U(w)\,dw\left(\int_y^\infty g_{K|w}(z)\,dz\left(\int_0^{wy}\mathbf{E}^2 g_{D|zw}(x)\,dx\right)\right) \\
& +\left(s+p-ce^{rT}\right)\int_0^\infty g_U(w)\,dw\left(\int_y^\infty g_{K|w}(z)\,dz\left(\int_{wy}^\infty \mathbf{E}^3 g_{D|zw}(x)\,dx\right)\right) \\
= & \left(\left(ce^{rT}-v\right)\beta'_{K,U}(y) - (s+p-v)\,\beta'_{D,K,U}(y)\right)E\left[f(S)\,u'\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)\right] \\
& -\left(ce^{rT}-v\right)E\left[Uu'\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)1_{\{D\le Uy,K>y\}}\right] \\
& +\left(s+p-ce^{rT}\right)E\left[Uu'\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)1_{\{D>Uy,K>y\}}\right]
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{E}^1 &= E_{x,z,w}\left[f(S)\,u'\left(CF\left(x,w,\min\{z,y\},S,y\right)\right)\right], \\
\mathbf{E}^2 &= E_{x,z,w}\left[wu'\left(CF_-(x,w,y,S,y)\right)\right]
\end{aligned}$$

and

$$\mathbf{E}^3 = E_{x,z,w}\left[wu'\left(CF_+\left(x,w,y,S,y\right)\right)\right].$$

Then, the first order condition can be written as

$$\begin{aligned} g\left(y\right) &= \left(s+p-v\right)E\left[Uu'\left(\mathbf{CF}_{\alpha^*(y)}\right)1_{\{K>y\}}\right]\left(\frac{\left(s+p-ce^{rT}\right)}{\left(s+p-v\right)}-h\left(y\right)\right) \\ &= 0 \end{aligned} \tag{4.54}$$

where

$$\begin{aligned} h\left(y\right) &= \left(\beta'_{D,K,U}\left(y\right)-\frac{\left(ce^{rT}-v\right)}{\left(s+p-v\right)}\beta'_{K,U}\left(y\right)\right)\frac{E\left[f\left(S\right)u'\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)\right]}{E\left[Uu'\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)1_{\{K>y\}}\right]} \\ &\quad+\frac{E\left[Uu'\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)1_{\{D\leq Uy,K>y\}}\right]}{E\left[Uu'\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)1_{\{K>y\}}\right]} \end{aligned}$$

and $\mathbf{CF}_{\alpha^*(y)} = CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)$. *Note that* $u'>0$ *and* $P\{K>y\}>0$ *by our assumption, we observe that*

$$E\left[Uu'\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)1_{\{K>y\}}\right]>0.$$

So, the first order condition can be written as

$$\begin{aligned} &\left(\beta'_{D,K,U}\left(y\right)-\frac{\left(ce^{rT}-v\right)}{\left(s+p-v\right)}\beta'_{K,U}\left(y\right)\right)\frac{E\left[f\left(S\right)u'\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)\right]}{E\left[Uu'\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)1_{\{K>y\}}\right]} \\ &\quad+\frac{E\left[Uu'\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)1_{\{D\leq Uy,K>y\}}\right]}{E\left[Uu'\left(CF_{\alpha^*(y)}\left(\mathbf{X},S,y\right)\right)1_{\{K>y\}}\right]} \\ &= \frac{\left(s+p-ce^{rT}\right)}{\left(s+p-v\right)}. \end{aligned} \tag{4.55}$$

Similarly, the existence and uniqueness of the optimal order quantity depends on the structure of $h(y)$. *By supposing that* $h(y)$ *is increasing in* y *and* $h(0)\leq\widehat{p}\leq h(\infty)$, *(4.55) gives the optimality condition.* ■

The first special case results with an optimality condition as

$$\frac{E\left[Uu'\left(CF_{\alpha^*(y^*)}\left(\mathbf{X},S,y^*\right)\right)1_{\{D\leq Uy^*,K>y^*\}}\right]}{E\left[Uu'\left(CF_{\alpha^*(y^*)}\left(\mathbf{X},S,y^*\right)\right)1_{\{K>y^*\}}\right]}=\widehat{p}$$

which satisfies the optimality condition in (3.8). For the second special case, $u(x)=a+bx$, the optimality function becomes as

$$\frac{E\left[U1_{\{K>y^*,D\leq Uy^*\}}\right]}{E\left[U1_{\{K>y^*\}}\right]}+\left(\beta'_{D,K,U}\left(y^*\right)-\frac{\left(ce^{rT}-v\right)}{\left(s+p-v\right)}\beta'_{K,U}\left(y^*\right)\right)\frac{E\left[f\left(S\right)\right]}{E\left[U1_{\{K>y\}}\right]}=\widehat{p}$$

which is the same condition as Okyay et al. [42].

4.4.2 Hedging with Multiple Securities

Now, suppose that there are n derivative securities in the market. We begin with, the optimization problem,

$$\min_{\boldsymbol{\alpha}} Var\left(CF_{\boldsymbol{\alpha}}\left(\mathbf{X}, S, y\right)\right)$$

where $\mathbf{X} = (D, U, K)$. The optimal asset quantity in (4.7) can be updated as

$$\boldsymbol{\alpha}^*\left(y\right) = -\mathbf{C}^{-1}\boldsymbol{\mu}\left(y\right) \tag{4.56}$$

where

$$C_{ij} = Cov\left(f_i\left(S\right), f_j\left(S\right)\right)$$

and

$$\mu_i\left(y\right) = Cov\left(f_i\left(S_i\right), CF\left(D, K, U, y\right)\right)$$

that denotes the covariance between the financial securities and the cash flow.

Then, the optimal $\boldsymbol{\alpha}^*\left(y\right)$ is used to maximize the utility of expected hedged cash flow. Then, the optimization problem is

$$\max_{y} E\left[u\left(CF_{\boldsymbol{\alpha}^*(y)}\left(\mathbf{X}, S, y\right)\right)\right]$$

where the hedged cash flow is

$$\begin{aligned} CF_{\boldsymbol{\alpha}^*(y)}\left(\mathbf{X}, S, y\right) &= -\left(ce^{rT} - v\right)U\min\left\{K, y\right\} + \left(s + p - v\right)\min\left\{D, UK, Uy\right\} \\ &\quad -pD + \boldsymbol{\alpha}^*\left(y\right)\mathbf{f}\left(S\right). \end{aligned}$$

We define the hedged cash flow as

$$CF\left(x, z\wedge y, w, t, y\right) = \begin{cases} CF_-\left(x, z\wedge y, w, t, y\right) = -\left(ce^{rT} - v\right)w\left(z\wedge y\right) & x \leq w\left(z\wedge y\right) \\ \qquad +\left(s - v\right)x & \\ \qquad -\boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{f}\left(t\right) & \\ CF_+\left(x, z\wedge y, w, t, y\right) = \left(s + p - ce^{rT}\right)w\left(z\wedge y\right) & x \geq w\left(z\wedge y\right) \\ \qquad -px & \\ \qquad -\boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{f}\left(t\right) & \end{cases}$$

where

$$\begin{aligned} CF_-\left(w\left(z\wedge y\right), z\wedge y, w, t, y\right) &= CF_+\left(w\left(z\wedge y\right), z\wedge y, w, t, y\right) \\ &= \left(s - ce^{rT}\right)w\left(z\wedge y\right) - \boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{f}\left(t\right). \end{aligned}$$

Then, we can write the objective function as

$$\begin{aligned}
& E\left[u\left(CF_{\boldsymbol{\alpha}^*(y)}\left(\mathbf{X},S,y\right)\right)\right] \\
= & \int_0^\infty g_U\left(w\right)dw\left(\int_0^y g_{K|w}\left(z\right)dz\left(\int_0^{wz} E_{x,z,w}\left[u\left(CF_-\left(x,z,w,S,y\right)\right)\right]g_{D|zw}\left(x\right)dx\right.\right. \\
& \left.\left.\qquad + \int_{wz}^\infty E_{x,z,w}\left[u\left(CF_+\left(x,z,w,S,y\right)\right)\right]g_{D|zw}\left(x\right)dx\right)\right) \\
& + \int_0^\infty g\left(w\right)dw\left(\int_y^\infty g_{K|w}\left(z\right)dz\left(\int_0^{wy} E_{x,z,w}\left[u\left(CF_-\left(x,y,w,S,y\right)\right)\right]g_{D|zw}\left(x\right)dx\right.\right. \\
& \left.\left.\qquad + \int_{wy}^\infty E_{x,z,w}\left[u\left(CF_+\left(x,y,w,S,y\right)\right)\right]g_{D|zw}\left(x\right)dx\right)\right).
\end{aligned} \tag{4.57}$$

Theorem 19 *The optimal order quantity* y^**satisfies*

$$\begin{aligned}
& \frac{\boldsymbol{\mu}\left(y^*\right)^{\mathbf{T}}\mathbf{C}^{-1}}{(s+p-v)}\frac{E\left[\mathbf{f}\left(S\right)u'\left(CF_{\boldsymbol{\alpha}^*(y^*)}\left(\mathbf{X},S,y^*\right)\right)\right]}{E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y^*)}\left(\mathbf{X},S,y^*\right)\right)1_{\{K>y^*\}}\right]} \\
& + \frac{E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y^*)}\left(\mathbf{X},S,y^*\right)\right)1_{\{D\le Uy^*,K>y^*\}}\right]}{E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y^*)}\left(\mathbf{X},S,y\right)\right)1_{\{K>y^*\}}\right]} = \widehat{p}.
\end{aligned} \tag{4.58}$$

Proof. *The first derivative of (4.57) is*

$$\begin{aligned}
& \frac{d}{dy}E\left[u\left(CF_{\boldsymbol{\alpha}^*(y)}\left(\mathbf{X},S,y\right)\right)\right] \\
= & \int_0^\infty g_U\left(w\right)dw\left(\int_0^y g_{K|w}\left(z\right)dz\left(\int_0^{wz}\mathbf{E}^1 g_{D|zw}\left(x\right)dx + \int_{wz}^\infty \mathbf{E}^2 g_{D|zw}\left(x\right)dx\right)\right. \\
& \left. + \int_y^\infty g_{K|w}\left(z\right)dz\left(\int_0^{wy}\mathbf{E}^3 g_{D|zw}\left(x\right)dx + \int_{wy}^\infty \mathbf{E}^4 g_{D|zw}\left(x\right)dx\right)\right) \\
= & -\boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}E\left[\mathbf{f}\left(S\right)u'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(\mathbf{X},S,y\right)\right)\right] \\
& -\left(ce^{rT}-v\right)E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(\mathbf{X},S,y\right)\right)1_{\{D\le Uy,K>y\}}\right] \\
& +\left(s+p-ce^{rT}\right)E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y)}\left(\mathbf{X},S,y\right)\right)1_{\{D>Uy,K>y\}}\right]
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{E}^1 &= E_{x,z,w}\left[\left(-\boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{f}\left(S\right)\right)u'\left(CF_-\left(x,z,w,S,y\right)\right)\right], \\
\mathbf{E}^2 &= E_{x,z,w}\left[\left(-\boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{f}\left(S\right)\right)u'\left(CF_+\left(x,z,w,S,y\right)\right)\right], \\
\mathbf{E}^3 &= E_{x,z,w}\left[\left(-\left(ce^{rT}-v\right)w-\boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{f}\left(S\right)\right)u'\left(CF_-\left(x,z,w,S,y\right)\right)\right]
\end{aligned}$$

and

$$\mathbf{E}^4 = E_{x,z,w}\left[\left(\left(s+p-ce^{rT}\right)w - \boldsymbol{\mu}(y)^{\mathbf{T}}\mathbf{C}^{-1}\mathbf{f}(S)\right)u'\left(CF_+(x,z,w,S,y)\right)\right].$$

Then, the first order condition can be written as

$$g(y) = (s+p-v)\,E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)1_{\{K>y\}}\right]\left(\frac{s+p-ce^{rT}}{s+p-v} - h(y)\right) \tag{4.59}$$

where

$$\begin{aligned} h(y) &= \frac{\boldsymbol{\mu}(y)^{\mathbf{T}}\mathbf{C}^{-1}}{(s+p-v)}\frac{E\left[\mathbf{f}(S)\,u'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)\right]}{E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)1_{\{K>y\}}\right]} \\ &\quad + \frac{E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)1_{\{D\le Uy,K>y\}}\right]}{E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)1_{\{K>y\}}\right]}. \end{aligned}$$

Note that $u'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right) > 0$ *and* $P\{K>y\} > 0$ *by our assumption, we observe that*

$$E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)1_{\{K>y\}}\right] > 0$$

and the first order condition is updated as

$$\begin{aligned} &\frac{\boldsymbol{\mu}(y)^{\mathbf{T}}\mathbf{C}^{-1}}{(s+p-v)}\frac{E\left[\mathbf{f}(S)\,u'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)\right]}{E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)1_{\{K>y\}}\right]} \\ &\quad + \frac{E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)1_{\{D\le Uy,K>y\}}\right]}{E\left[Uu'\left(CF_{\boldsymbol{\alpha}^*(y)}(\mathbf{X},S,y)\right)1_{\{K>y\}}\right]} \\ = &\ \frac{s+p-ce^{rT}}{s+p-v}. \end{aligned} \tag{4.60}$$

The existence and uniqueness of the optimal order quantity depends on the structure of $h(y)$, *the same as before. We suppose that* $h(y)$ *is increasing in* y. *If* $h(0) \le \widehat{p} \le h(\infty)$, *(4.60) gives the optimality condition.* ■

For the first special case, the optimality condition can be written as

$$\frac{E\left[Uu'\left(CF(\mathbf{X},S,y^*)\right)1_{\{D\le Uy^*,K>y^*\}}\right]}{E\left[Uu'\left(CF(\mathbf{X},S,y^*)\right)1_{\{K>y^*\}}\right]} = \widehat{p}$$

which clearly satisfies (3.8). For the second special case, the optimality function becomes

$$\frac{E\left[U1_{\{D\le Uy^*,K>y^*\}}\right]}{E\left[U1_{\{K>y^*\}}\right]} + \frac{\boldsymbol{\mu}(y^*)^{\mathbf{T}}\mathbf{C}^{-1}E\left[\mathbf{f}(S)\right]}{(s+p-v)\,E\left[U1_{\{K>y^*\}}\right]} = \widehat{p}$$

which is the same condition as Okyay et al. [42].

Chapter 5

NUMERICAL ILLUSTRATIONS

Up to now, we discuss the newsvendor problem within expected utility maximization framework. In Chapter 3, we analyze the newsvendor problem when there are risks associated with the uncertainty in demand as well as supply. Then, in Chapter 4, we consider the same problem when the randomness in demand and supply is correlated with the financial markets.

In this chapter, we demonstrate our results from Chapter 3 and Chapter 4 by using some illustrative numerical examples. The aim of this chapter is to investigate the effects of parameters on the decision variables. First, we consider a simple example to analyze how some important parameters affect the optimal order quantity. Then, we use the Monte Carlo method to simulate our models and comment on how utility theory and hedging influences the optimal decisions.

5.1 A Simple Example

In this section, to see the effects of some parameters on the optimal order quantity we consider an example similar to the one in Eeckhoudt et al. [18]. The aim of the newsvendor is to maximize the expected utility of the cash flow. Let's define the utility function as $u(x) = -e^{-\frac{1}{\beta}x}$ where β represents the newsvendor's degree of risk tolerance. Suppose that the newsvendor has no initial wealth, so that $z_0 = 0$. Moreover, no salvage or extra buying options exist, so that $v = 0$ and $p = 0$. He purchases each item with purchase cost c and sells it at sale price s. We first analyze the problem when there is no hedging option, and then when there is hedging opportunity.

5.1.1 Newsvendor Model without Hedging

For this simple example, we suppose that there is no hedging option. First, we suppose that the randomness only results from the demand. Secondly, we consider the case when the supply yield is also random. Thirdly, the supply capacity is random. Finally, we consider the case when both yield and capacity are random.

Random Demand

We first consider the case when the randomness results only from demand. A similar example is also discussed in Eeckhoudt et al. [18] and this part is only review of it. Let demand take two values so that $D \in \{0, M\}$, where the probability that demand is zero is p_1 and the probability that demand is M is p_2. The optimality condition for the standard newsvendor model in (3.8) can be updated for our example as

$$h(y^*) = \frac{E\left[u'(CF(D,y^*))\,1_{\{D\leq y^*\}}\right]}{E\left[u'(CF(D,y^*))\right]} = \frac{s-c}{s} = \widehat{p}$$

where the cash flow is

$$CF(D,y^*) = -cy^* + s\min\{D,y^*\}.$$

More explicitly, we can write

$$h(y) = \begin{cases} \frac{p_1}{p_1+p_2 e^{-\frac{1}{\beta}sy}} & 0 \leq y < M \\ 1 & y \geq M \end{cases}.$$

It is obvious that $h(y)$ is increasing in y. If $h(0) < \widehat{p} < h(M)$, then there exists a unique y^* that satisfies the optimality condition. However, if $h(0) \geq \widehat{p}$, we have $y^* = 0$; and if $h(M-) \leq \widehat{p}$, we have $y^* = M$. Therefore the optimal order quantity is

$$y^* = \begin{cases} 0 & p_2 \leq \frac{c}{s} \\ \frac{\beta}{s}\ln\left(\frac{p_2}{p_1}\left(\frac{s-c}{c}\right)\right) & \frac{c}{s} < p_2 < \frac{c}{c+(s-c)e^{-\frac{1}{\beta}sM}} \\ M & p_2 \geq \frac{c}{c+(s-c)e^{-\frac{1}{\beta}sM}} \end{cases}. \tag{5.1}$$

This characterization of the order quantity depends on the probability of demand. If p_2 is less than or equal to c/s, the decision maker orders nothing. If p_2 is larger than c/s and less than $c/\left(c+(s-c)e^{-\frac{1}{\beta}sM}\right)$, the decision maker orders $(\beta/s)\ln\left(\frac{p_2}{p_1}\left(\frac{s-c}{c}\right)\right)$ units. If p_2 is larger than or equal to $c/\left(c+(s-c)e^{-\frac{1}{\beta}sM}\right)$, the decision maker orders M units. Moreover, we can also conclude that the optimal order quantity is linear in β while p_2 is less than or equal to $c/\left(c+(s-c)e^{-\frac{1}{\beta}sM}\right)$ and greater than c/s. We observe that the optimal order quantity increases up to M as β increases and the decision maker orders at most M units which is logical because the demand can be at most M.

Thereafter, we update this example for the newsvendor model with random supply, which has not been considered before. The following analyses, therefore, are new.

Random Yield

Secondly, we suppose that beyond demand randomness, supply randomness exists in yield. That is, when y is ordered, Uy is received. For our example, in addition to demand, U also takes two values $U \in \{0, N\}$ where $0 < N \leq 1$. Moreover, U and D have a joint probability mass function (p.m.f). The joint p.m.f is

$D = 0$ and $U = 0$	with probability p_1
$D = 0$ and $U = N$	with probability p_2
$D = M$ and $U = 0$	with probability p_3
$D = M$ and $U = N$	with probability p_4

For this example, the optimality condition in (3.33) becomes

$$h\left(y^*\right) = \frac{E\left[Uu'\left(CF(D,U,y^*)\right)1_{\{D \leq Uy^*\}}\right]}{E\left[Uu'\left(CF(D,U,y^*)\right)\right]} = \frac{s-c}{s} = \widehat{p}$$

where the cash flow can be written as

$$CF\left(D,U,y^*\right) = -cUy^* + s\min\left\{D, Uy^*\right\}.$$

Using the discrete probability distribution, we can write

$$h\left(y\right) = \begin{cases} \frac{p_2}{p_2 + p_4 e^{-\frac{1}{\beta}sNy}} & 0 \leq y < \frac{M}{N} \\ 1 & y \geq \frac{M}{N} \end{cases}.$$

Again, it is obvious that $h\left(y\right)$ is increasing in y. If $h\left(0\right) < \widehat{p} < h\left(\frac{M}{N}\right)$,then there exists a unique y^* that satisfies the optimality condition. However, if $h\left(0\right) \geq \widehat{p}$, we have $y^* = 0$; and if $h\left(\frac{M}{N}-\right) \leq \widehat{p}$, we have $y^* = \frac{M}{N}$. The optimal order quantity can be characterized as

$$y^* = \begin{cases} 0 & \frac{p_4}{p_2+p_4} \leq \frac{c}{s} \\ \frac{\beta}{Ns}\ln\left(\frac{p_4}{p_2}\left(\frac{s-c}{c}\right)\right) & \frac{c}{s} < \frac{p_4}{p_2+p_4} < \frac{c}{c+(s-c)e^{-\frac{1}{\beta}sM}} \\ \frac{M}{N} & \frac{p_4}{p_2+p_4} \geq \frac{c}{c+(s-c)e^{-\frac{1}{\beta}sM}} \end{cases}. \tag{5.2}$$

It is clear from (5.2) that the optimal order quantity depends on probabilities p_2 and p_4. If

$$P\{D = M|U = N\} = \frac{p_4}{p_2 + p_4} \leq \frac{c}{s}$$

the decision maker orders nothing. Else if

$$P\{D = M|U = N\} \geq \frac{c}{c + (s-c)\, e^{-\frac{1}{\beta} sM}}$$

he orders M/N units. In other cases, the decision maker orders $(\beta/Ns) \ln \left(\frac{p_4}{p_2}\left(\frac{s-c}{c}\right)\right)$ units. We observe that the optimal order quantity is linear in β while $P\{D = M|U = N\}$ is less than $c/\left(c + (s-c)\, e^{-\frac{1}{\beta} sM}\right)$ and greater than c/s. The optimal order quantity increases up to M/N units as β increases. We also observe that the decision maker orders at most M/N units which is again logical.

Random Capacity

Thirdly, we suppose that the capacity of the supplier is also random. In other words, when y is ordered, $\min\{K, y\}$ is received. In addition to demand, we also suppose that capacity takes two values, $K \in \{0, M\}$ for computational simplicity. Moreover, K and D have a joint distribution function given by

$D = 0$ and $K = 0$	with probability p_1
$D = 0$ and $K = M$	with probability p_2
$D = M$ and $K = 0$	with probability p_3
$D = M$ and $K = M$	with probability p_4

.

The optimality condition for the newsvendor problem with random capacity in (3.56) can be written as

$$h(y^*) = \frac{E\left[u'\left(CF(D, K, y^*)\right) 1_{\{D \leq y^*, K > y^*\}}\right]}{E\left[u'\left(CF(D, K, y^*)\right) 1_{\{K > y^*\}}\right]} = \frac{s-c}{s} = \widehat{p}$$

where the cash flow is

$$CF(D, K, y^*) = -c \min\{K, y^*\} + s \min\{D, K, y^*\}.$$

Using the discrete probability distribution, we can write

$$h(y) = \begin{cases} \frac{p_2}{p_2 + p_4 e^{-\frac{1}{\beta} sy}} & 0 \leq y < M \\ 1 & y \geq M \end{cases}.$$

It is obvious that $h(y)$ is increasing in y. If $h(0) < \widehat{p} < h(M)$, then there exists a unique y^* that satisfies the optimality condition. However, if $h(0) \geq \widehat{p}$, we have $y^* = 0$ and if $h(M-) \leq \widehat{p}$, we have $y^* = M$. Therefore the optimal order quantity is

$$y^* = \begin{cases} 0 & \frac{p_4}{p_2+p_4} \leq \frac{c}{s} \\ \frac{\beta}{s} \ln\left(\frac{p_4}{p_2}\left(\frac{s-c}{c}\right)\right) & \frac{c}{s} < \frac{p_4}{p_4+p_2} < \frac{c}{c+(s-c)e^{-\frac{1}{\beta}sM}} \\ M & \frac{p_4}{p_2+p_4} \geq \frac{c}{c+(s-c)e^{-\frac{1}{\beta}sM}} \end{cases}. \quad (5.3)$$

The characterization of the optimal order quantity depends on the probability

$$P\{D = M | K = M\} = \frac{p_4}{p_4 + p_2}.$$

Moreover, the same remarks for random yield are also valid for random capacity. The only difference is that the decision maker orders at most M units.

Random Yield and Capacity

Finally, we suppose that the randomness results from random demand, random yield and random capacity. The quantity received from ordering y unit is $U \min\{D, K\}$. All of them take two values, $D \in \{0, M\}$, $U \in \{0, N\}$ and $K \in \{0, M\}$. The joint discrete probability distribution function is

$D = 0, U = 0, K = 0$	with probability p_1
$D = 0, U = 0, K = M$	with probability p_2
$D = 0, U = N, K = 0$	with probability p_3
$D = 0, U = N, K = M$	with probability p_4
$D = M, U = 0, K = 0$	with probability p_5
$D = M, U = 0, K = M$	with probability p_6
$D = M, U = N, K = 0$	with probability p_7
$D = M, U = N, K = M$	with probability p_8

Therefore, the optimality condition for this example is

$$h(y^*) = \frac{E\left[Uu'\left(CF(D, K, U, y^*)\right) 1_{\{D \leq Uy^*, K > y^*\}}\right]}{E\left[Uu'\left(CF(D, K, U, y^*)\right) 1_{\{K > y^*\}}\right]} = \widehat{p}$$

where

$$CF(D, K, U, y^*) = -cU \min\{K, y^*\} + s \min\{D, UK, Uy^*\}.$$

It can be more explicitly written as

$$h\left(y\right)=\begin{cases}\frac{p_4}{p_4+p_8e^{-\frac{1}{\beta}sNy}} & 0\leq y<\frac{M}{N}\\ 1 & y\geq\frac{M}{N}\end{cases}.$$

Again, it is obvious that $h\left(y\right)$ is increasing in y. If $h\left(0\right)<\widehat{p}<h\left(\frac{M}{N}\right)$, then there exists a unique y^* that satisfies the optimality condition. However, if $h\left(0\right)\geq\widehat{p}$, we have $y^*=0$ and if $h\left(\frac{M}{N}-\right)\leq\widehat{p}$, we have $y^*=M$. Therefore the optimal order quantity can be written as

$$y^*=\begin{cases}0 & \frac{p_8}{p_4+p_8}\leq\frac{c}{s}\\ \frac{\beta}{Ns}\ln\left(\frac{p_8}{p_4}\left(\frac{s-c}{c}\right)\right) & \frac{c}{s}<\frac{p_8}{p_4+p_8}<\frac{c}{c+(s-c)e^{-\frac{1}{\beta}sM}}\\ \frac{M}{N} & \frac{p_8}{p_4+p_8}\geq\frac{c}{c+(s-c)e^{-\frac{1}{\beta}sM}}\end{cases}. \tag{5.4}$$

We can also make the same remarks for this optimal order quantity characterization. The optimal order quantity depends on the probability

$$P\left\{D=M|U=N,K=M\right\}=\frac{p_8}{p_4+p_8}$$

and linearly increases up to M/N as β increases.

We will now discuss all of the results obtained up to this point with a numerically illustrative example. We suppose that M is 100 and N is 0.5. Moreover, the probabilities for a standard model, model with random yield, model with random capacity, and model with random yield and capacity are respectively

$\boldsymbol{p}_1=\left[0.25,0.75\right]$,

$\boldsymbol{p}_2=\left[0.10,0.15,0.35,0.40\right]$,

$\boldsymbol{p}_3=\left[0.09,0.16,0.15,0.60\right]$,

$\boldsymbol{p}_4=\left[0.01,0.09,0.05,0.10,0.10,0.25,0.10,0.30\right]$.

First we want to analyze the effect of the risk-tolerance parameter. Suppose that the decision maker sells newspapers at $s=28\ TL$ and buys at $c=20\ TL$. For each problem, we show that the optimal order quantity linearly increases up to a maximum value $y_{\max}$ under some condition depending on the probabilities. Now, we redefine these conditions depending on the risk tolerance values. Then, we obtain a critical value of risk tolerance $\widehat{\beta}$ that defines the optimal order quantity as

$$y^*=\begin{cases}C\beta & \beta<\widehat{\beta}\\ y_{\max} & \beta\geq\widehat{\beta}\end{cases}$$

where C is the slope. For the risk tolerance values greater than $\widehat{\beta}$, the optimal order quantity takes the largest value $y_{\max}$ that it can take. For example, when supply randomness results from both random yield and capacity, the slope is

$$C = \frac{\ln\left(\frac{p_8}{p_4}\left(\frac{s-c}{c}\right)\right)}{Ns} = 0.01302$$

and while the risk tolerance is less than or equal to

$$\widehat{\beta} = \frac{sM}{\ln\left(\frac{p_8}{p_4}\left(\frac{s-c}{c}\right)\right)} = 15357.48$$

the optimal order quantity is linear. When the risk tolerance value is larger than $\widehat{\beta}$, the optimal order quantity equals

$$y_{\max} = \frac{M}{N} = 200.$$

Table 5.1 summarizes the values of C, $\widehat{\beta}$ and $y_{\max}$ for all cases where Model 1-4 represent Standard, Random Yield, Random Capacity and Random Yield and Capacity models, respectively. Note that for all models, the order quantity increases as risk-tolerance increases.

	Model 1	**Model 2**	**Model 3**	**Model 4**
C	0.00651	0.00461	0.01448	0.01302
$\widehat{\beta}$	15357.48	43384.94	6905.65	15357.48
$y_{\max}$	100	200	100	200

Table 5.1: The slope, the critical risk-tolerance level, and the maximum order quantity

Secondly, we analyze the effect of the sale price on the optimal order quantity. We suppose that $\beta = 1000$ and $c = 20\ TL$. Figure 5.1 demonstrates the optimal order quantities for different sales price values from 20 to 100 TL. We can conclude that as the sales price increases the optimal order quantity rapidly increases up to a level and then slowly decreases. As we state before, the effect of sale price is not monotone.

Finally, we investigate the effect of c on the optimal order quantity. We suppose that $\beta = 1000$ and $s = 28\ TL$. For order cost values from 1 to 25 TL, Figure 5.2 depicts the optimal order quantity values. We observe that the order quantity decreases as order cost increases. Up to a low c value, the decision maker orders as much as he can. Moreover, after some large value of c, the decision maker orders nothing.

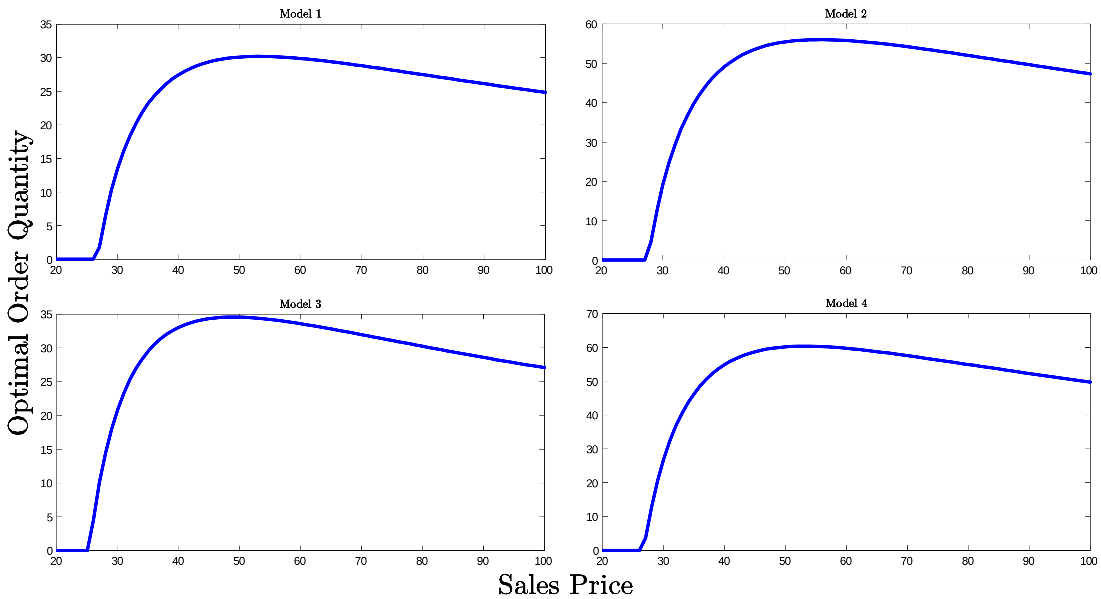

Figure 5.1: The Effect of Sales Price on the Optimal Order Quantity

Up to this point, we analyzed the newsvendor model without the hedging opportunity by using a simple example. Now, we want to replicate the same example for the newsvendor model with financial hedging opportunity.

5.1.2 Newsvendor Model with Financial Hedging

Now, we suppose that the randomness in demand and supply is correlated with the financial markets. First, we suppose that the randomness only results from the demand. Then, we consider the situations where the supply is also random.

Random Demand

For our example, both demand and $f(S)$ take two values, $D \in \{0, M\}$ and $f(S) \in \{-L, L\}$. We suppose that $f(S)$ is either $-L$ or L for computational simplicity. They have a joint

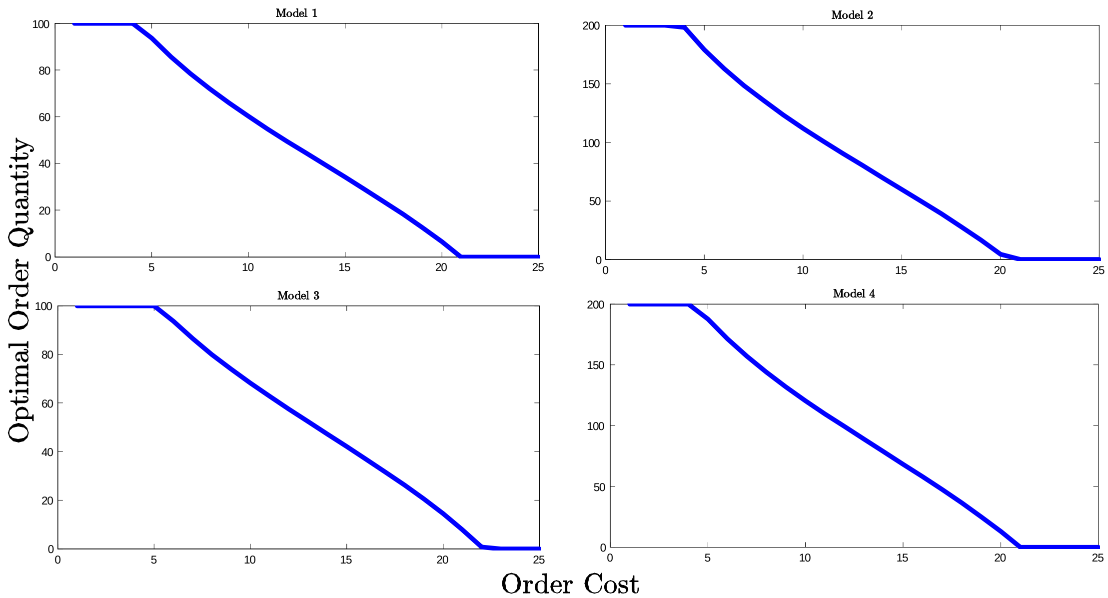

Figure 5.2: The Effect of Order Cost on the Optimal Order Quantity

distribution function

$f(S) = -L,\ D = 0$	with probability p_1
$f(S) = -L,\ D = M$	with probability p_2
$f(S) = L,\ D = 0$	with probability p_3
$f(S) = L,\ D = M$	with probability p_4

.

To make sure that the expected financial gain is zero, we suppose that

$$E[f(S)] = -(p_1 + p_2) L + (p_3 + p_4) L = 0 \tag{5.5}$$

so that

$$Var(f(S)) = L^2. \tag{5.6}$$

We always assume (5.5) and (5.6) in the remainder of this section. The optimal asset quantity for a single asset with random demand in (4.9) can be updated for this example as

$$\alpha^*(y) = -s\beta_D(y)$$

where

$$\beta_D(y) = \frac{Cov(f(S), \min\{D, y\})}{Var(f(S))}.$$

It can be easily calculated that

$$\begin{aligned} Cov(f(S), \min\{D, y\}) &= E[f(S)\min\{D, y\}] - E[f(S)]E[\min\{D, y\}] \\ &= E[f(S)\min\{D, y\}] \\ &= p_1(-L\min\{0, y\}) + p_2(-L\min\{M, y\}) \\ &\quad +p_3(L\min\{0, y\}) + p_4(L\min\{M, y\}) \\ &= (p_4 - p_2)Ly. \end{aligned}$$

Hence, the optimal asset quantity equals

$$\alpha^*(y) = -s\frac{(p_4 - p_2)Ly}{L^2} = -s\frac{(p_4 - p_2)}{L}y.$$

We observe that the sign of the optimal asset quantity depends on the sign of the $(p_4 - p_2)$. We know that

$$\begin{aligned} Cov(f(S), D) &= E[f(S)D] - E[f(S)]E[D] \\ &= E[f(S)D] \\ &= (p_4 - p_2)LM. \end{aligned}$$

If $(p_4 - p_2)$ is positive, $f(S)$ and D are positively correlated and if $(p_4 - p_2)$ is negative, $f(S)$ and D are negatively correlated. Therefore, we can conclude that if $f(S)$and D are positively correlated, the sign of the optimal asset quantity is negative and then the optimal decision is short-selling. If $f(S)$and D are negatively correlated, the sign of the optimal asset quantity is positive and then the optimal decision is buying.

The optimality condition in (4.14) is also updated as

$$\frac{E\left[u'(CF(D,S,y^*))1_{\{D\leq y^*\}}\right] + \beta'_D(y^*)E[f(S)u'(CF(D,S,y^*))]}{E[u'(CF(D,S,y^*))]} = \frac{s-c}{s}.$$

It is clear that

$$\beta'_D(y) = \frac{(p_4 - p_2)}{L}$$

and the cash flow is

$$\begin{aligned} CF_{\alpha^*(y)}(D, S, y^*) &= -cy^* + s\min\{D, y^*\} + \alpha^*(y)f(S) \\ &= -cy^* + s\min\{D, y^*\} - s\frac{(p_4 - p_2)}{L}y^*f(S). \end{aligned}$$

As the utility function is $u(x) = -e^{-\frac{1}{\beta}x}$, the optimality condition can be updated as

$$\begin{aligned} &((p_2 - p_4) s + c) p_1 \left(e^{-\frac{1}{\beta}(p_4 - p_2) s y^*} \right) \\ &\quad + ((p_2 - p_4 - 1) s + c) p_2 \left(e^{-\frac{1}{\beta}(p_4 - p_2 + 1) s y^*} \right) \\ &\quad\quad + ((p_4 - p_2) s + c) p_3 \left(e^{-\frac{1}{\beta}(p_2 - p_4) s y^*} \right) \\ &\quad\quad\quad + ((p_4 - p_2 - 1) s + c) p_4 \left(e^{-\frac{1}{\beta}(p_2 - p_4 + 1) s y^*} \right) = 0. \end{aligned} \tag{5.7}$$

By letting $C^* = y^*/\beta$, (5.7) becomes

$$a_1 e^{-b_1 C^*} + a_2 e^{-b_2 C^*} + a_3 e^{-b_3 C^*} + a_4 e^{-b_4 C^*} = 0 \tag{5.8}$$

where $a_1 = ((p_2 - p_4) s + c) p_1$, $a_2 = ((p_2 - p_4 - 1) s + c) p_2$, $a_3 = ((p_4 - p_2) s + c) p_3$, $a_4 = ((p_4 - p_2 - 1) s + c) p_4$, $b_1 = (p_4 - p_2) s$, $b_2 = (p_4 - p_2 + 1) s$, $b_3 = (p_2 - p_4) s$ and $b_4 = (p_2 - p_4 + 1) s$. We can easily conclude that if there exists a solution to (5.8), it is independent of β. Then, the optimal order quantity y^* is linear in β. To analyze the effect of the risk-tolerance parameter, we solve it for a specific example. We assume that the newsvendor sells at $s = 28\ TL$ and buys at $c = 20\ TL$. Moreover, the probabilities are $\mathbf{p} = [0.15, 0.35, 0.10, 0.40]$. Let M be 100, then the optimality condition (5.8) becomes

$$2.79e^{-1.4C^*} - 3.29e^{-29.4C^*} + 2.14e^{1.4C^*} - 2.64e^{-26.6C^*} = 0. \tag{5.9}$$

Multiplying both sides of (5.9) by $e^{-1.4C^*}$, we obtain

$$2.79e^{-2.8C^*} - 3.29e^{-30.8C^*} - 2.64e^{-28C^*} + 2.14 = 0.$$

Moreover, by letting $x = e^{-2.8C^*}$, the optimality condition is

$$r(x) = 2.79x - 3.29x^{11} - 2.64x^{10} + 2.14 = 0$$

where $r(x)$ is a polynomial and the problem is to find the root of r.

Note that

$$r'(x) = 2.79 - 36.19x^{10} - 26.4x^9$$

and

$$r''(x) = -361.9x^9 - 237.6x^8 \le 0$$

for $x \ge 0$. Therefore, r is concave. Since

$$r(0) = 2.14 > 0$$

and

$$r'(0) = 2.79 > 0$$

there may exist only one positive root x^* that satisfies $r(x^*) = 0$. By using solve function of Scientific Workplace we found x^* 0.982 and so $C^* = 0.0066$. Therefore, we observe that the optimal order quantity equals

$$y^*(\beta) = 0.0066\beta. \tag{5.10}$$

The optimal order quantity is clearly linear in β. We can conclude that as the risk tolerance parameter increases, the optimal order quantity increases.

Random Yield

Now, we suppose that there is supply randomness in yield. For our example, demand, $f(S)$ and yield all take two possible values, $D \in \{0, M\}$, $f(S) \in \{-L, L\}$ and $U \in \{0, N\}$. Moreover, the joint distribution is

$f(S) = -L,\ D = 0,\ U = 0$	with probability p_1
$f(S) = -L,\ D = 0,\ U = N$	with probability p_2
$f(S) = -L,\ D = M,\ U = 0$	with probability p_3
$f(S) = -L,\ D = M,\ U = N$	with probability p_4
$f(S) = L,\ D = 0,\ U = 0$	with probability p_5
$f(S) = L,\ D = 0,\ U = N$	with probability p_6
$f(S) = L,\ D = M,\ U = 0$	with probability p_7
$f(S) = L,\ D = M,\ U = N$	with probability p_8

The optimal asset quantity for a single asset with random demand in (4.27) can be updated as

$$\alpha^*(y) = c\beta_U y - s\beta_{D,U}(y)$$

where

$$\beta_U = \frac{Cov(f(S), U)}{Var(f(S))}$$

and

$$\beta_{D,U}(y) = \frac{Cov(f(S), \min\{D, Uy\})}{Var(f(S))}.$$

It is easy to show that

$$\begin{aligned} Cov\left(f\left(S\right),U\right) &= E\left[f\left(S\right)U\right]-E\left[f\left(S\right)\right]E\left[U\right] \\ &= E\left[f\left(S\right)U\right] \\ &= p_2\left(-L\right)N+p_4\left(-L\right)N+p_6\left(L\right)N+p_8\left(L\right)N \\ &= \left(p_6+p_8-p_2-p_4\right)LN \end{aligned}$$

and

$$\begin{aligned} Cov\left(f\left(S\right),\min\left\{D,Uy\right\}\right) &= E\left[f\left(S\right)\min\left\{D,Uy\right\}\right]-E\left[f\left(S\right)\right]E\left[\min\left\{D,Uy\right\}\right] \\ &= E\left[f\left(S\right)\min\left\{D,Uy\right\}\right] \\ &= \left(p_8-p_4\right)LNy. \end{aligned}$$

Hence, the optimal asset quantity is

$$\begin{aligned} \alpha^*\left(y\right) &= \left(p_6+p_8-p_2-p_4\right)c\left(\frac{N}{L}\right)y-\left(p_8-p_4\right)s\left(\frac{N}{L}\right)y \\ &= \left(\left(p_6+p_8-p_2-p_4\right)c-\left(p_8-p_4\right)s\right)\left(\frac{N}{L}\right)y. \end{aligned}$$

The optimality condition in (4.29) can be also updated as

$$\frac{E\left[Uu'\left(CF\left(\mathbf{X},S,y^*\right)\right)1_{\{D\leq Uy^*\}}\right]+\left(\beta'_{D,U}\left(y^*\right)-\frac{c}{s}\beta_U\right)E\left[f\left(S\right)u'\left(CF\left(\mathbf{X},S,y^*\right)\right)\right]}{E\left[Uu'\left(CF\left(\mathbf{X},S,y^*\right)\right)\right]}=\frac{s-c}{s}$$

where

$$\beta'_{D,U}\left(y^*\right)=\frac{\left(p_8-p_4\right)N}{L}$$

and

$$\beta_U=\frac{\left(p_6+p_8-p_2-p_4\right)N}{L}.$$

Moreover, the cash flow is

$$\begin{aligned} CF_{\alpha^*(y)}CF\left(\mathbf{X},S,y^*\right) &= -cUy^*+s\min\left\{D,Uy^*\right\}+\alpha^*\left(y\right)f\left(S\right) \\ &= -cUy^*+s\min\left\{D,Uy^*\right\} \\ &\quad +\left(\left(p_6+p_8-p_2-p_4\right)c-\left(p_8-p_4\right)s\right)\left(\frac{N}{L}\right)y^*f\left(S\right). \end{aligned}$$

As the utility function is $u(x) = -e^{-\frac{1}{\beta}x}$, the optimality condition can be updated as

$$\left\{\begin{array}{c} \left(\begin{array}{c}(p_4-p_8)s\\+(p_6+p_8-p_2-p_4)c\end{array}\right)p_1Ne^{-\frac{1}{\beta}(((p_2+p_4-p_6-p_8)c+(p_8-p_4)s)Ny^*)} \\ +\left(\begin{array}{c}(p_4-p_8)s\\+(p_6+p_8-p_2-p_4+1)c\end{array}\right)p_2Ne^{-\frac{1}{\beta}(((p_2+p_4-p_6-p_8-1)c+(p_8-p_4)s)Ny^*)} \\ +\left(\begin{array}{c}(p_4-p_8)s\\+(p_6+p_8-p_2-p_4)c\end{array}\right)p_3Ne^{-\frac{1}{\beta}(((p_2+p_4-p_6-p_8)c+(p_8-p_4)s)Ny^*)} \\ +\left(\begin{array}{c}(p_4-p_8-1)s\\+(p_6+p_8-p_2-p_4+1)c\end{array}\right)p_4Ne^{-\frac{1}{\beta}(((p_2+p_4-p_6-p_8-1)c+(p_8-p_4+1)s)Ny^*)} \\ +\left(\begin{array}{c}(p_8-p_4)s\\-(p_6+p_8-p_2-p_4)c\end{array}\right)p_5Ne^{-\frac{1}{\beta}(((p_6+p_8-p_2-p_4)c-(p_8-p_4)s)Ny^*)} \\ +\left(\begin{array}{c}(p_8-p_4+)s\\-(p_6+p_8-p_2-p_4-1)c\end{array}\right)p_6Ne^{-\frac{1}{\beta}(((p_6+p_8-p_2-p_4-1)c-(p_8-p_4)s)Ny^*)} \\ +\left(\begin{array}{c}(p_8-p_4)s\\-(p_6+p_8-p_2-p_4)c\end{array}\right)p_7Ne^{-\frac{1}{\beta}(((p_6+p_8-p_2-p_4)c-(p_8-p_4)s)Ny^*)} \\ +\left(\begin{array}{c}(p_8-p_4-1)s\\-(p_6+p_8-p_2-p_4-1)c\end{array}\right)p_8Ne^{-\frac{1}{\beta}(((p_6+p_8-p_2-p_4-1)c-(p_8-p_4-1)s)Ny^*)} \end{array}\right\} = 0.$$

Again, by letting $C^* = y^*/\beta$, this equation take the form

$$a_1e^{-b_1C^*} + a_2e^{-b_2C^*} + a_3e^{-b_3C^*} + a_4e^{-b_4C^*} + a_5e^{-b_5C^*} + a_6e^{-b_6C^*} + a_7e^{-b_7C^*} + a_8e^{-b_8C^*} = 0.$$

We can easily conclude that the solution is independent of β. Therefore, the optimal order quantity y^* is linear in β.

An explicit solution for the optimal order quantity cannot be calculated directly. So, we solve it for a specific example to show the effect of the risk tolerance parameter. We assume that the newsvendor sells at $s = 28\ TL$ and buys at $c = 20\ TL$. Moreover, we suppose that the probabilities are $\mathbf{p} = [0.07, 0.08, 0.10, 0.25, 0.03, 0.07, 0.25, 0.15]$. Let M be 100 and N be 0.5. Therefore, the optimality condition is

$$\begin{array}{l} 0.051e^{(0.3)C^*} + 0.824e^{(10.3)C^*} - 0.925e^{-(3.7)C^*} \\ \quad -0.084e^{-(0.3)C^*} + 0.679e^{(9.7)C^*} - 0.645e^{-(4.3)C^*} = 0. \end{array}$$

By doing the same analysis before, we obtain the optimal order quantity as

$$y^*(\beta) = 0.0047\beta. \tag{5.11}$$

We can conclude that as the risk tolerance parameter increases, the optimal order quantity increases linearly.

Random Capacity

Now, we suppose that there is supply randomness in capacity. For our example, demand, $f(S)$ and capacity take two values, $D \in \{0, M\}$, $f(S) \in \{-L, L\}$ and $K \in \{0, M\}$. The joint distribution function is

$f(S) = -L,\ D = 0,\ K = 0$	with probability p_1
$f(S) = -L,\ D = 0,\ K = M$	with probability p_2
$f(S) = -L,\ D = M,\ K = 0$	with probability p_3
$f(S) = -L,\ D = M,\ K = M$	with probability p_4
$f(S) = L,\ D = 0,\ K = 0$	with probability p_5
$f(S) = L,\ D = 0,\ K = M$	with probability p_6
$f(S) = L,\ D = M,\ K = 0$	with probability p_7
$f(S) = L,\ D = M,\ K = M$	with probability p_8

The optimal asset quantity for a single asset with random demand in (4.40) can be updated as

$$\alpha^*(y) = \beta_K(y)\, c - \beta_{D,K}(y)\, s$$

where

$$\beta_K(y) = \frac{Cov\left(f(S), \min\{K, y\}\right)}{Var\left(f(S)\right)}$$

and

$$\beta_{D,K}(y) = \frac{Cov\left(f(S), \min\{D, K, y\}\right)}{Var\left(f(S)\right)}.$$

It follows that

$$\begin{aligned} Cov\left(f(S), \min\{K, y\}\right) &= E\left[f(S)\min\{K, y\}\right] - E\left[f(S)\right]E\left[\min\{K, y\}\right] \\ &= E\left[f(S)\min\{K, y\}\right] \\ &= p_2(-L)\,y + p_4(-L)\,y + p_6(L)\,y + p_8(L)\,y \\ &= (p_6 + p_8 - p_2 - p_4)\,Ly \end{aligned}$$

and

$$\begin{aligned} Cov\left(f\left(S\right),\min\left\{D,K,y\right\}\right) &= E\left[f\left(S\right)\min\left\{D,K,y\right\}\right]-E\left[f\left(S\right)\right]E\left[\min\left\{D,K,y\right\}\right] \\ &= E\left[f\left(S\right)\min\left\{D,K,y\right\}\right] \\ &= \left(p_8-p_4\right)Ly. \end{aligned}$$

Hence, the optimal asset quantity equals is

$$\begin{aligned} \alpha^*\left(y\right) &= \frac{Cov\left(f\left(S\right),\min\left\{K,y\right\}\right)}{Var\left(f\left(S\right)\right)}c-\frac{Cov\left(f\left(S\right),\min\left\{D,K,y\right\}\right)}{Var\left(f\left(S\right)\right)}s \\ &= \frac{\left(p_6+p_8-p_2-p_4\right)}{L}cy-\frac{\left(p_8-p_4\right)}{L}sy \\ &= \left(\left(p_6+p_8-p_2-p_4\right)c-\left(p_8-p_4\right)s\right)\left(\frac{1}{L}\right)y. \end{aligned}$$

The optimality condition in (4.42) can be also updated as

$$\frac{E\left[u'\left(\boldsymbol{CF}\right)1_{\{D\le y^*,K>y^*\}}\right]+\left(\beta'_{D,K}\left(y^*\right)-\frac{c}{s}\beta'_K\left(y^*\right)\right)E\left[f\left(S\right)u'\left(\boldsymbol{CF}\right)\right]}{E\left[u'\left(\boldsymbol{CF}\right)1_{\{K>y^*\}}\right]}=\frac{s-c}{s}$$

where

$$\beta'_{D,K}\left(y^*\right)=\frac{\left(p_8-p_4\right)}{L}$$

and

$$\beta'_K\left(y^*\right)=\frac{\left(p_6+p_8-p_2-p_4\right)}{L}.$$

Moreover, the cash flow is

$$\begin{aligned} CF_{\alpha^*(y)}CF\left(\mathbf{X},S,y^*\right) &= -c\min\left\{K,y^*\right\}+s\min\left\{D,K,y^*\right\}+\alpha^*\left(y\right)f\left(S\right) \\ &= -c\min\left\{K,y^*\right\}+s\min\left\{D,K,y^*\right\} \\ &\quad +\left(\left(p_6+p_8-p_2-p_4\right)c-\left(p_8-p_4\right)s\right)\frac{1}{L}y^*f\left(S\right). \end{aligned}$$

As the utility function is $u\left(x\right)=-e^{-\frac{1}{\beta}x}$, the optimality condition can be updated as

$$\left\{ \begin{array}{c} \left(\begin{array}{c} (p_4 - p_8)\, s \\ + (p_6 + p_8 - p_2 - p_4)\, c \end{array} \right) p_1 e^{-\frac{1}{\beta}((p_2+p_4-p_6-p_8)c+(p_8-p_4)s)y^*} \\ + \left(\begin{array}{c} (p_4 - p_8)\, s \\ + (p_6 + p_8 - p_2 - p_4 + 1)\, c \end{array} \right) p_2 e^{-\frac{1}{\beta}((p_2+p_4-p_6-p_8-1)c+(p_8-p_4)s)y^*} \\ + \left(\begin{array}{c} (p_4 - p_8)\, s \\ + (p_6 + p_8 - p_2 - p_4)\, c \end{array} \right) p_3 e^{-\frac{1}{\beta}((p_2+p_4-p_6-p_8)c+(p_8-p_4)s)y^*} \\ + \left(\begin{array}{c} (p_4 - p_8 - 1)\, s \\ + (p_6 + p_8 - p_2 - p_4 + 1)\, c \end{array} \right) p_4 e^{-\frac{1}{\beta}((p_2+p_4-p_6-p_8-1)c+(p_8-p_4+1)s)y^*} \\ + \left(\begin{array}{c} (p_8 - p_4)\, s \\ - (p_6 + p_8 - p_2 - p_4)\, c \end{array} \right) p_5 e^{-\frac{1}{\beta}((p_6+p_8-p_2-p_4)c-(p_8-p_4)s)y^*} \\ + \left(\begin{array}{c} (p_8 - p_4)\, s \\ - (p_6 + p_8 - p_2 - p_4 - 1)\, c \end{array} \right) p_6 e^{-\frac{1}{\beta}((p_6+p_8-p_2-p_4-1)c-(p_8-p_4)s)y^*} \\ + \left(\begin{array}{c} (p_8 - p_4)\, s \\ - (p_6 + p_8 - p_2 - p_4)\, c \end{array} \right) p_7 e^{-\frac{1}{\beta}((p_6+p_8-p_2-p_4)c-(p_8-p_4)s)y^*} \\ + \left(\begin{array}{c} (p_8 - p_4 - 1)\, s \\ - (p_6 + p_8 - p_2 - p_4 - 1)\, c \end{array} \right) p_8 e^{-\frac{1}{\beta}((p_6+p_8-p_2-p_4-1)c-(p_8-p_4-1)s)y^*} \end{array} \right\} = 0.$$

An explicit solution for the optimal order quantity cannot be obtained. So, we solve it for the same specific example. Let $\mathbf{p} = [0.07, 0.08, 0.10, 0.25, 0.02, 0.08, 0.05, 0.35]$ be the probabilities. Therefore, we again set $C^* = y^*/\beta$, the optimality condition is

$$\begin{aligned} & -0.136e^{-0.8C^*} + 1.536e^{19.2C^*} - 2.2e^{-8.8C^*} \\ & \quad +0.056e^{0.8C^*} + 1.664e^{20.8C^*} - 2.52e^{-7.2C^*} = 0. \end{aligned}$$

As before, the optimal order quantity is linear and it equals

$$y^*(\beta) = 0.0145\beta \tag{5.12}$$

where $\mathrm{y}_{\max} = 100$. Moreover, as the risk tolerance parameter increases, the optimal order quantity increases.

Random Yield and Capacity

Last, we suppose that there is supply randomness in both yield and capacity. For our example, demand, $f(S)$, yield and capacity take two values, $D \in \{0, M\}$, $f(S) \in \{-L, L\}$, $U \in \{0, N\}$ and $K \in \{0, M\}$. The joint distribution function is

$f(S) = -L,\ D = 0,\ U = 0,\ K = 0$	with probability p_1
$f(S) = -L,\ D = 0,\ U = 0,\ K = M$	with probability p_2
$f(S) = -L,\ D = 0,\ U = N,\ K = 0$	with probability p_3
$f(S) = -L,\ D = 0,\ U = N,\ K = M$	with probability p_4
$f(S) = -L,\ D = M, U = 0,\ K = 0$	with probability p_5
$f(S) = -L,\ D = M, U = 0,\ K = M$	with probability p_6
$f(S) = -L,\ D = 0,\ U = N,\ K = 0$	with probability p_7
$f(S) = -L,\ D = 0,\ U = N,\ K = M$	with probability p_8
$f(S) = L,\ D = 0,\ U = 0,\ K = 0$	with probability p_9
$f(S) = L,\ D = 0,\ U = 0,\ K = M$	with probability p_{10}
$f(S) = L,\ D = 0,\ U = N,\ K = 0$	with probability p_{11}
$f(S) = L,\ D = 0,\ U = N,\ K = M$	with probability p_{12}
$f(S) = L,\ D = M,\ U = 0,\ K = 0$	with probability p_{13}
$f(S) = L,\ D = M,\ U = 0,\ K = M$	with probability p_{14}
$f(S) = L,\ D = M,\ U = N,\ K = 0$	with probability p_{15}
$f(S) = L,\ D = M,\ U = N,\ K = M$	with probability p_{16}

The optimal asset quantity for a single asset with random demand in (4.51) can be updated as

$$\alpha^*(y) = \beta_{K,U}(y)\, c - \beta_{D,K,U}(y)\, s$$

where

$$\beta_{K,U}(y) = \frac{Cov\left(f(S), U \min\{K, y\}\right)}{Var\left(f(S)\right)}$$

and

$$\beta_{D,K,U}(y) = \frac{Cov\left(f(S), \min\{D, UK, Uy\}\right)}{Var\left(f(S)\right)}.$$

It follows that

$$\begin{aligned} Cov\left(f(S), U\min\{K,y\}\right) &= E\left[f(S)\, U \min\{K,y\}\right] \\ &= p_4(-L)\, Ny + p_8(-L)\, Ny + p_{12}(L)\, Ny + p_{16}(L)\, Ny \\ &= (p_{12} + p_{16} - p_4 - p_8)\, LNy \end{aligned}$$

and

$$\begin{aligned} Cov\left(f\left(S\right),\min\left\{D,UK,Uy\right\}\right) &= E\left[f\left(S\right)\min\left\{D,K,y\right\}\right] \\ &= p_8\left(-L\right)Ny + p_{16}\left(L\right)Ny \\ &= \left(p_{16}-p_8\right)LNy. \end{aligned}$$

Hence, the optimal asset quantity is

$$\begin{aligned} \alpha^*\left(y\right) &= \frac{Cov\left(f\left(S\right),U\min\left\{K,y\right\}\right)}{Var\left(f\left(S\right)\right)}c - \frac{Cov\left(f\left(S\right),\min\left\{D,UK,Uy\right\}\right)}{Var\left(f\left(S\right)\right)}s \\ &= \frac{\left(p_{12}+p_{16}-p_4-p_8\right)}{L}cNy - \frac{\left(p_{16}-p_8\right)}{L}sNy \\ &= \left(\left(p_{12}+p_{16}-p_4-p_8\right)c - \left(p_{16}-p_8\right)s\right)\left(\frac{N}{L}\right)y. \end{aligned}$$

The optimality condition in (4.42) can be also updated as

$$\frac{E\left[Uu'\left(\boldsymbol{CF}\right)1_{\{D\leq Uy^*,K>y^*\}}\right] + \left(\beta'_{D,K,U}\left(y^*\right) - \frac{c}{s}\beta'_{K,U}\left(y^*\right)\right)E\left[f\left(S\right)u'\left(\boldsymbol{CF}\right)\right]}{E\left[Uu'\left(\boldsymbol{CF}\right)1_{\{K>y^*\}}\right]} = \frac{s-c}{s}$$

where

$$\beta'_{K,U}\left(y\right) = \left(p_{12}+p_{16}-p_4-p_8\right)\left(\frac{N}{L}\right)$$

and

$$\beta'_{D,K,U}\left(y\right) = \left(p_{16}-p_8\right)\left(\frac{N}{L}\right).$$

The cash flow is

$$\begin{aligned} \boldsymbol{CF} &= -cU\min\left\{K,y^*\right\} + s\min\left\{D,UK,Uy^*\right\} \\ &\quad + \left(\left(\left(p_{12}+p_{16}-p_4-p_8\right)c - \left(p_{16}-p_8\right)s\right)\left(\frac{N}{L}\right)y^*\right)f\left(S\right). \end{aligned}$$

By doing some straightforward calculations, the optimality condition can be updated as

$$\left\{\begin{array}{c}
-\left(\left(p_{16}-p_{8}\right)s-c\left(p_{12}+p_{16}-p_{4}-p_{8}\right)\right)p_{1}Ne^{-\frac{1}{\beta}CF_{1}}\\
-\left(\left(p_{16}-p_{8}\right)s-c\left(p_{12}+p_{16}-p_{4}-p_{8}\right)\right)p_{2}Ne^{-\frac{1}{\beta}CF_{2}}\\
-\left(\left(p_{16}-p_{8}\right)s-c\left(p_{12}+p_{16}-p_{4}-p_{8}\right)\right)p_{3}Ne^{-\frac{1}{\beta}CF_{3}}\\
-\left(\left(p_{16}-p_{8}\right)s-c\left(p_{12}+p_{16}-p_{4}-p_{8}+1\right)\right)p_{4}Ne^{-\frac{1}{\beta}CF_{4}}\\
-\left(\left(p_{16}-p_{8}\right)s-c\left(p_{12}+p_{16}-p_{4}-p_{8}\right)\right)p_{5}Ne^{-\frac{1}{\beta}CF_{5}}\\
-\left(\left(p_{16}-p_{8}\right)s-c\left(p_{12}+p_{16}-p_{4}-p_{8}\right)\right)p_{6}Ne^{-\frac{1}{\beta}CF_{6}}\\
-\left(\left(p_{16}-p_{8}\right)s-c\left(p_{12}+p_{16}-p_{4}-p_{8}\right)\right)p_{7}Ne^{-\frac{1}{\beta}CF_{7}}\\
-\left(\left(p_{16}-p_{8}+1\right)s-c\left(p_{12}+p_{16}-p_{4}-p_{8}+1\right)\right)p_{8}Ne^{-\frac{1}{\beta}CF_{8}}\\
+\left(\left(p_{16}-p_{8}\right)s-c\left(p_{12}+p_{16}-p_{4}-p_{8}\right)\right)p_{9}Ne^{-\frac{1}{\beta}CF_{9}}\\
+\left(\left(p_{16}-p_{8}\right)s-c\left(p_{12}+p_{16}-p_{4}-p_{8}\right)\right)p_{10}Ne^{-\frac{1}{\beta}CF_{10}}\\
+\left(\left(p_{16}-p_{8}\right)s-c\left(p_{12}+p_{16}-p_{4}-p_{8}\right)\right)p_{11}Ne^{-\frac{1}{\beta}CF_{11}}\\
+\left(\left(p_{16}-p_{8}\right)s-c\left(p_{12}+p_{16}-p_{4}-p_{8}-1\right)\right)p_{12}Ne^{-\frac{1}{\beta}CF_{12}}\\
+\left(\left(p_{16}-p_{8}\right)s-c\left(p_{12}+p_{16}-p_{4}-p_{8}\right)\right)p_{13}Ne^{-\frac{1}{\beta}CF_{13}}\\
+\left(\left(p_{16}-p_{8}\right)s-c\left(p_{12}+p_{16}-p_{4}-p_{8}\right)\right)p_{14}Ne^{-\frac{1}{\beta}CF_{14}}\\
+\left(\left(p_{16}-p_{8}\right)s-c\left(p_{12}+p_{16}-p_{4}-p_{8}\right)\right)p_{15}Ne^{-\frac{1}{\beta}CF_{15}}\\
+\left(\left(p_{16}-p_{8}-1\right)s-c\left(p_{12}+p_{16}-p_{4}-p_{8}-1\right)\right)p_{16}Ne^{-\frac{1}{\beta}CF_{16}}
\end{array}\right\}=0$$

where

$CF_{1}=-\left(\left(p_{12}+p_{16}-p_{4}-p_{8}\right)c-\left(p_{16}-p_{8}\right)s\right)Ny^{*}$

$CF_{2}=-\left(\left(p_{12}+p_{16}-p_{4}-p_{8}\right)c-\left(p_{16}-p_{8}\right)s\right)Ny^{*}$

$CF_{3}=-\left(\left(p_{12}+p_{16}-p_{4}-p_{8}\right)c-\left(p_{16}-p_{8}\right)s\right)Ny^{*}$

$CF_{4}=-\left(\left(p_{12}+p_{16}-p_{4}-p_{8}+1\right)c-\left(p_{16}-p_{8}\right)s\right)Ny^{*}$

$CF_{5}=-\left(\left(p_{12}+p_{16}-p_{4}-p_{8}\right)c-\left(p_{16}-p_{8}\right)s\right)Ny^{*}$

$CF_{6}=-\left(\left(p_{12}+p_{16}-p_{4}-p_{8}\right)c-\left(p_{16}-p_{8}\right)s\right)Ny^{*}$

$CF_{7}=-\left(\left(p_{12}+p_{16}-p_{4}-p_{8}\right)c-\left(p_{16}-p_{8}\right)s\right)Ny^{*}$

$CF_{8}=-\left(\left(p_{12}+p_{16}-p_{4}-p_{8}+1\right)c-\left(p_{16}-p_{8}+1\right)s\right)Ny^{*}$

$CF_{9}=\left(\left(p_{12}+p_{16}-p_{4}-p_{8}\right)c-\left(p_{16}-p_{8}\right)s\right)Ny^{*}$

$CF_{10}=\left(\left(p_{12}+p_{16}-p_{4}-p_{8}\right)c-\left(p_{16}-p_{8}\right)s\right)Ny^{*}$

$CF_{11}=\left(\left(p_{12}+p_{16}-p_{4}-p_{8}\right)c-\left(p_{16}-p_{8}\right)s\right)Ny^{*}$

$CF_{12}=\left(\left(p_{12}+p_{16}-p_{4}-p_{8}-1\right)c-\left(p_{16}-p_{8}\right)s\right)Ny^{*}$

$CF_{13}=\left(\left(p_{12}+p_{16}-p_{4}-p_{8}\right)c-\left(p_{16}-p_{8}\right)s\right)Ny^{*}$

$CF_{14}=\left(\left(p_{12}+p_{16}-p_{4}-p_{8}\right)c-\left(p_{16}-p_{8}\right)s\right)Ny^{*}$

$CF_{15} = ((p_{12} + p_{16} - p_4 - p_8)\, c - (p_{16} - p_8)\, s)\, Ny^*$

$CF_{16} = ((p_{12} + p_{16} - p_4 - p_8 - 1)\, c - (p_{16} - p_8 - 1)\, s)\, Ny^*.$

An explicit closed form solution for the optimal order quantity cannot be obtained. So, we numerically solve it for the same specific example where the probabilities are

$$\mathbf{p} = [0.03, 0.04, 0.04, 0.04, 0.07, 0.03, 0.03, 0.22, 0.01, 0.02, 0.01, 0.06, 0.04, 0.21, 0.01, 0.14]\,.$$

Let $M = 100$, $N = 0.5$, $s = 28$ and $c = 20$. So, by letting $C^* = y^*/\beta$, the optimality condition is

$$\begin{aligned} &0.1248e^{0.52C^*} + 0.4208e^{10.52C^*} - 0.7656e^{-3.48C^*} \\ &\quad -0.156e^{-0.52C^*} + 0.5688e^{9.48C^*} - 0.6328e^{-4.52C^*} = 0. \end{aligned}$$

Similarly, the optimal order quantity is linear and it equals

$$y^*(\beta) = 0.0265\beta. \tag{5.13}$$

Moreover, as the risk tolerance parameter increases, the optimal order quantity increases linearly.

Up to this point, we consider a simple example to analyze how the risk tolerance affects the optimal order quantity. In the next section, we use the Monte Carlo method to simulate our models.

5.2 Simulation

In this section, we illustrate the properties of the problem by simulation. Our aim is to quantify the effects of the utility framework and financial hedging to compensate for demand and supply risks. As the base scenario, we take the setting of the example in Gaur and Seshadri [22] where the demand risk is hedged by a stock in the financial market. Let the initial stock price S_0 be \$660 and the interest rate be $r = 10\%$ per year. Assume that $T = 6$ months and that the return S_T/S_0 has a lognormal distribution under the risk-neutral measure with mean $\left(r - \frac{\sigma^2}{2}\right)T$ and standard deviation $\sigma\sqrt{T}$ where $\sigma = 20\%$ per year. That is,

$$\ln\left(\frac{S_T}{S_0}\right) \sim N\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right) = N\,(0.04, 0.14142)\,.$$

We assume that the market measure is the risk-neutral measure. Let the demand be $D = b\,S_T + \epsilon$ where $b = 10$ and ϵ has a normal distribution with mean zero and standard deviation σ_ϵ. Therefore, random demand is correlated with the financial market. The financial parameters are as follows: $s = 1$, $c = 0.6$, $p = -0.3$, and $v = 0.1$. Moreover, we suppose that the utility function is $u(x) = 800 - 100e^{-\frac{1}{\beta}x}$.

We consider three types of financial portfolios. The first portfolio consists of the future on the stock only and has the net payoff $f_1(S)$, the second portfolio consists of the call option on the stock with strike price κ only and has the net payoff $f_2(S)$. Finally, the third portfolio uses both instruments jointly and has the net payoffs $f_1(S)$ and $f_2(S)$. Therefore, the payoff of these derivative securities are

$$f_1\left(S\right) = S_T - e^{rT} S_0$$

and

$$f_2\left(S\right) = \max\left\{S_T - \kappa, 0\right\} - e^{rT} C$$

where κ is the strike price and C is the price of the call option at time 0. For this example, we assume that the strike price κ is y/b. We further suppose that the call price in the market does not provide any arbitrage opportunities so that

$$C = E\left[e^{-rT}\max\left\{S_T - \kappa, 0\right\}\right]$$

and

$$E\left[f_2\left(S\right)\right] = 0.$$

We can also calculate the call price by using Black-Scholes formula. That is

$$C = S_0 N(d_1) - \kappa e^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln(\frac{S_0}{\kappa}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

and N is the standard normal distribution.

We want to point out that all of our numerical calculations are done using Monte Carlo simulations throughout the remainder of this section. We use Matlab as a simulation tool. Cash flows are generated by using the simulated values of S, D, K, and U whenever needed.

We consider four different models: the random demand case and three generalizations, first with random capacity, second with random yield and third with random yield and capacity.

Throughout this chapter, we will compare the following eight different scenarios:

Scenario 1: Newsvendor does not use any portfolio and aims to maximize expected cash flow,

Scenario 2: Newsvendor uses the first portfolio (future) and aims to maximize expected cash flow,

Scenario 3: Newsvendor uses the second portfolio (call option) and aims to maximize expected cash flow,

Scenario 4: Newsvendor uses the third portfolio (future and call option) and aims to maximize expected cash flow,

Scenario 5: Newsvendor does not use any portfolio and aims to maximize the expected utility of the cash flow,

Scenario 6: Newsvendor uses the first portfolio (future) and aims to maximize the expected utility of the cash flow,

Scenario 7: Newsvendor uses the second portfolio (call option) and aims to maximize the expected utility of the cash flow,

Scenario 8: Newsvendor uses the third portfolio (future and call option) and aims to maximize the expected utility of the cash flow.

5.2.1 Random Demand

First we analyze the case where demand is the only source of uncertainty. We can define the unhedged cash flow at time T as

$$CF\left(D,y\right)=s\min\left\{D,y\right\}+v\max\left\{y-D,0\right\}-p\max\left\{D-y,0\right\}-ce^{rT}y$$

and the hedged cash flow at time T as

$$CF_{\boldsymbol{\alpha}^{*}(y)}\left(D,S,y\right)=CF\left(D,y\right)+\boldsymbol{\alpha}^{*}\left(y\right)\mathbf{f}\left(S\right).$$

We first suppose that the standard deviation of demand is $\sigma_{\epsilon}=600$ and the risk-tolerance parameter is $\beta=500$. We run our simulation for different order quantity values and generate $50,000$ instances to calculate the optimal portfolios. In each instance, we generate the stock price and demand. Then, we calculate the optimal portfolios by using the formulas obtained in Chapter 4. Finally, we generate another $50,000$ instances so that we obtain stock prices, demand quantities and profits. For all scenarios, we calculate the mean, the variance, and the coefficient of variance (CV) (the ratio of the standard deviation to the mean) of the cash flow for each order quantity.

Based on the mean of the cash flows, for scenarios 1-4, and the mean of the utility of the cash flows, for scenarios 5-9, we obtain the optimal order quantities approximately. Table 5.2 depicts the results for each scenario. Table 5.2 shows the variance reductions in the cash flows that are made possible by financial hedging. Let consider the variance reductions when both portfolios are used. The financial hedging provides variance reduction by 68.6% when we do not use the utility theory and by 66% when we use the utility theory. Then we analyze the effect of utility theory by comparing scenario 1 and scenario 5. The risk-averse decision maker orders less and so his expected gain is also less. However, the variance of the expected cash flow is reduced by 30%.

$\sigma_\epsilon = 600$	y^*	Mean	Variance	CV	Portfolio (α)
S1	5588	2435	181260	0.1748	—
S2	5587	2435	60561	0.1011	-3.5191
S3	5583	2435	66290	0.1058	-3.5761
S4	5587	2435	56966	0.0980	$-8.9780, 5.7328$
S5	4657	2401.4	127480	0.1487	—
S6	5086	2422.8	42432	0.0850	-3.1950
S7	5008	2418.9	41358	0.0841	-3.1745
S8	5164	2427.0	43355	0.0858	$-9.8903, 6.7478$

Table 5.2: The variances of the cash flows and the optimal investment amounts for random demand model when the standard deviation of demand error is 600

Moreover, by using the same framework, we analyze the problem for different demand variations. We again run our simulation by changing order quantity values when σ_ϵ is 0 and 300. We follow the same structure as before. The results are summarized in Table 5.3 for a perfect correlation between demand and the stock price, in Table 5.4 for a high degree of a correlation between demand and the stock price.

When the standard deviation of the demand error is zero (a perfect correlation between demand and the stock price), hedging with a portfolio of future and option eliminates the

$\sigma_\epsilon = 0$	y^*	Mean	Variance	CV	Portfolio (α)
S1	5804	2457	125410	0.1441	—
S2	5802	2457	7150	0.0344	-3.4823
S3	5800	2457	18470	0.0553	-3.5538
S4	5804	2457	0	0	$-9.00, 6.00$
S5	5235	2440.0	94726	0.1261	—
S6	5662	2455.4	4914	0.0285	-3.3467
S7	5468	2449.3	6426	0.0327	-3.2467
S8	5804	2458.1	0	0	$-9.00, 6.00$

Table 5.3: The variances of the cash flows and the optimal investment amounts for random demand model when the standard deviation of demand error is 0

variance of the cash flow and the variance of the utility of cash flow totally. When the standard deviation of the demand error is small ($\sigma_\epsilon = 300$) indicating a high degree of correlation between demand and the stock price, significant variance reductions are achieved, 89% for standard model and 87% for utility model. The reductions decrease when the correlation decreases since for $\sigma_\epsilon = 600$ the variance of the cash flow can be lowered considerably, 68.6% for standard model and 66% for utility model.

We also analyze the effect of risk-tolerance parameter β on the optimal order quantity and the variance. We again take the same example where $\sigma_\epsilon = 600$. We follow the same structure as before while simulating. The Table 5.5 depicts the optimal order quantities, mean the cash flows and variances of the cash flows and the optimal portfolios. From Table 5.5, we conclude that as risk-tolerance increases, the optimal order quantity increases. Moreover, from the variances of the cash flows, we can state that the hedging always reduces the variation. The variance reductions decrease as β increases.

As for the optimal portfolio structure, it is always optimal to sell the future since demand and stock price are positively correlated. On the other hand, in the optimal portfolio, the call option is bought when used as the second instrument along with the future, but is sold when it is used as the sole instrument. It is also interesting to note that using a portfolio consisting only of the future on the stock is very effective and achieves most of the variance reduction benefits. On the other hand, the call option serves to fine tune the portfolio along with the investment in the stock but is not as effective when used alone.

$\sigma_\epsilon = 300$	y^*	Mean	Variance	CV	Portfolio (α)
S1	5742	2451	139460	0.1523	—
S2	5740	2451	20553	0.0585	-3.4916
S3	5736	2451	29810	0.0705	-3.5601
S4	5744	2451	14855	0.0497	$-8.9135, 5.8281$
S5	5085	2430.1	103180	0.1322	—
S6	5513	2447.4	15294	0.0505	-3.2970
S7	5348	2441.3	15089	0.0503	-3.2273
S8	5640	2451.3	13554	0.0475	$-8.9600, 5.8849$

Table 5.4: The variances of the cash flows and the optimal investment amounts for random demand model when the standard deviation of demand error is 300

5.2.2 Random Yield

In this subsection, we briefly present an example for the random yield model. The functional form relating the stock price to the yield can take many forms and we take the following plausible example where $U = 1 - e^{-(1/S_0)(\gamma + S_T)}$. We take γ to be normally distributed with mean zero and standard deviation σ_γ. We take the same base scenario and use identical portfolio options. In the remaining of this chapter, we only consider the effect of financial hedging. Therefore, we fix the order quantity to $y^* = 7000$ and we consider only the first four scenarios. We first fix σ_ϵ to 600 and β to 500. Then, for different values of σ_γ (0, $200, 400$), by following the same structure, we calculate the means, variations, coefficient of variations and the optimal portfolios. The result are presented in Table 5.6.

Although the variance reductions decreases when σ_γ increases, we can conclude, from the variance values, that financial hedging provides considerable reductions in the variance for all scenarios.

Then, by considering the same example, we vary the standard deviations, σ_γ and σ_ϵ, together. The Table 5.7 reports the results of this experiment. We can conclude that when the standard deviations are smaller, the variance reduction is 94% for the standard models. When we further increase the standard deviation, we also obtain variance reductions, 76%, but less than before.

The same remarks for the optimal portfolios in previous subsection are also valid for this

	β	y^*	Mean	Variance	Portfolio (α)
S5	250	3850	2348.8	120680	—
	500	4650	2401.0	127340	—
	750	4950	2417.3	135830	—
S6	250	4500	2391.2	34444	-3.0469
	500	5100	2423.4	42766	-3.2010
	750	5250	2428.8	46896	-3.2753
S7	250	4450	2387.7	35286	-3.0415
	500	5000	2418.6	43532	-3.1712
	750	5200	2426.6	47319	-3.2693
S8	250	4000	2382.9	32700	$-14.2763, 11.2262$
	500	5150	2426.5	41017	$-9.9375, 6.7979$
	750	5350	2432.8	43391	$-9.3728, 6.1884$

Table 5.5: The variances of the cash flows and the optimal investment amounts for different risk-tolerance values when the standard deviation of demand error is 600

subsection.

5.2.3 Random Capacity

For the base example, we use the same assumptions. In addition, we assume the following relationship between the strike price and the capacity, $K = k\,S_T + \eta$ where $k = 9$ and η has a normal distribution with mean zero and standard deviation σ_η. For the examples in this subsection, we again fixed y^* to 7000 and β to 500. Note that as σ_ϵ and σ_η increase, the correlations between the demand and the market, and the capacity and the market weaken. At the same time, the correlation between the demand and the capacity also weakens. Table 5.8 reports the resulting variances as σ_ϵ and σ_η are varied together. It can be observed that, once again, significant reductions in variance can be achieved by hedging. The reductions are naturally most important when the market correlation is strong. For instance, the case $\sigma_\epsilon = \sigma_\eta = 300$ corresponds to high correlation with the market and the variance can be reduced by 92%. Even in the case when the correlations with the market are relatively

σ_γ	Scenario	Mean	Variance	CV	Portfolio(α)
0	S1	2395	135518	0.1537	—
	S2	2395	32688	0.0755	-3.25703
	S3	2395	88389	0.1242	-1.7302
	S4	2395	32678	0.0755	$-3.3122, 0.1032$
200	S1	2384	143524	0.1589	—
	S2	2384	37658	0.0814	-3.3048
	S3	2384	95544	0.1296	-1.7458
	S4	2384	37618	0.0813	$-3.4185, 0.2128$
400	S1	2349	173863	0.1775	—
	S2	2349	58096	0.1026	-3.4558
	S3	2349	122914	0.1492	-1.7990
	S4	2349	57862	0.1024	$-3.7315, 0.5159$

Table 5.6: The variances of the cash flows and the optimal investment amounts for different random yield models when the standard deviation of demand error is 600 and the order quantity is 7000

low ($\sigma_\epsilon = \sigma_\eta = 900$), the variance reduction is less but considerable, 44% for the standard models.

Then, we fix the demand-market correlation by letting $\sigma_\epsilon = 600$ and vary the capacity-market correlation by varying σ_η. Table 5.9 summarizes the results. As σ_η increases, reduction in variances decreases.

We again do the same remarks about the optimal portfolios as in the previous sections.

5.2.4 Random Yield and Capacity

In this last subsection, we want to analyze a combination of random yield and capacity models. The received order quantity is $U \min\{K, y\}$ where U and K are defined same as before. We take the same base scenario and use the identical portfolio options. We fix the standard deviations as $\sigma_\epsilon = 600$, $\sigma_\gamma = 200$ and $\sigma_\eta = 300$. We also fix the order quantity as $y^* = 7000$.Then, the resulting means, variations, coefficient of variations and the optimal portfolios are summarized in Table 5.10.

$\sigma_\gamma = \sigma_\epsilon$	Scenario	Mean	Variance	CV	Portfolio(α)
200	S1	2386	110752	0.1394	—
	S2	2386	6807	0.0346	-3.2746
	S3	2386	63221	0.1054	-1.7376
	S4	2386	6793	0.0345	$-3.3418, 0.1257$
400	S1	2353	148684	0.1639	—
	S2	2353	35452	0.0800	-3.4178
	S3	2353	98538	0.1334	-1.7847
	S4	2353	35275	0.0798	$-3.6577, 0.4490$

Table 5.7: The variances of the cash flows and the optimal investment amounts when the standard deviations of demand error and yield error vary together ($y = 7000$)

From Table 5.10, we can conclude that significant variance reductions for financial hedging are achievable, 78% for the models without the utility maximization objective and 78% for the models with the utility maximization objective

Moreover, the same remarks about optimal portfolio decisions as in the previous subsections are also valid.

$\sigma_\epsilon = \sigma_\eta$	Scenario	Mean	Variance	CV	Portfolio(α)
300	S1	2500	129629	0.1440	—
	S2	2500	10959	0.0419	-3.4989
	S3	2500	77189	0.1111	-1.8251
	S4	2500	10756	0.0415	$-3.7561, 0.4814$
600	S1	2459	202656	0.1831	—
	S2	2459	69219	0.1070	-3.7102
	S3	2459	143498	0.1541	-1.9385
	S4	2459	69020	0.1068	$-3.9644, 0.4757$
900	S1	2403.1	339815	0.2426	—
	S2	2402.7	189127	0.1810	-3.9427
	S3	2402.7	272720	0.2173	-2.0644
	S4	2402.8	188944	0.1809	$-4.1866, 0.4564$

Table 5.8: The variances of the cash flows and the optimal investment amounts for random capacity models when the standard deviations of demand error and capacity error vary together ($y = 7000$)

σ_η	Scenario	Mean	Variance	CV	Portfolio(α)
0	S1	2484.7	178184	0.1699	—
	S2	2484.4	54519	0.0940	-3.5718
	S3	2484.3	123385	0.1414	-1.8657
	S4	2484.4	54330	0.0938	$-3.8192, 0.4631$
300	S1	2477.2	185220	0.1737	—
	S2	2476.9	59146	0.0982	-3.6064
	S3	2476.8	129183	0.1451	-1.8867
	S4	2476.9	58979	0.0980	$-3.8392, 0.4358$
900	S1	2438.5	223040	0.1937	—
	S2	2438.2	79760	0.1158	-3.8446
	S3	2438.1	160377	0.1642	-1.9951
	S4	2438.3	79397	0.1156	$-4.1882, 0.6432$

Table 5.9: The variances of the cash flows and the optimal investment amounts for different standard deviation of capacity errors

Scenario	Mean	Variance	CV	Portfolio(α)
S1	2352.0	155739	0.1678	—
S2	2352.0	34967	0.0795	-3.5191
S3	2351.4	101383	0.1354	-1.8514
S4	2352.1	34912	0.0794	$-3.6528, 0.2511$

Table 5.10: The variances of the cash flows and the optimal investment amounts for a random yield and capacity model when the standard deviations of demand error, yield error and capacity error are 600, 200 and 300, respectively. ($y = 7000$)

Chapter 6

CONCLUSIONS

In this thesis, we discuss the single-period, single-item inventory problem when the decision-maker (newsvendor) is risk-averse. We use the expected utility theory framework to take a risk-sensitive position. The risks or uncertainties in our models result not only from random demand, but also from random supply. We divide our thesis into two main parts. In the first part, we consider inventory models with random supply when the objective is to maximize the expected utility of cash flow. In the second part, we consider the same problem when the randomness in demand and supply are correlated with financial markets.

The focus point of the first part in this thesis is inventory management using the expected utility theory framework. In this part, we consider that the randomness is generated by random demand as well as random supply. We analyze three different types of supply uncertainty: random yield, random capacity and random yield and capacity. In all cases, we find characterizations on the optimal order quantity by using the same critical ratio. However, these characterizations require certain properties and assumptions on the optimality conditions and the structure of the objective function. For the random demand and random yield cases, the objective function is concave, so we find simple and explicit characterizations for the optimal order quantity. For random capacity, and random yield and capacity cases, the concavity of the objective function is not satisfied. For these cases, we establish quasi-concavity of the objective function and then we find explicit characterizations for the optimal order quantity. In these cases, the existence and the uniqueness of the solution require certain assumptions. After reviewing the analyses for the effects of the risk aversion and other parameters on the optimal order quantity for the standard model, we discuss these effects for the model with random supply. However, for the models with random capacity and random yield and capacity, we can only state the effect of risk-aversion on the optimal order quantity under certain properties and assumptions.

In the second part of the thesis, we focus on the inventory models where the randomness in demand and supply is correlated with financial markets. We consider the opportunities of financial hedging to mitigate inventory risks. We analyze two types of portfolios, consisting of a single asset and multiple assets, for three types of random supply. In our context, the decision-maker needs to choose the financial portfolio and the order quantity. We provide a two step solution approach to this problem. In the first step, for any order quantity we find the optimal portfolio that minimizes the variance of the cash flow. Then, in the second step, by using the characterization for the optimal portfolio, we find the optimal order quantity that maximizes the expected utility of the cash flow. For all types of the problem, the objective of the first step is concave, and we easily find the explicit characterizations of the optimal order quantities depending on the correlation between the random variables of the inventory model and the financial variable. However, the objective functions in the second step are non-concave. For these cases, we establish quasi-concavity of the objective function and then we find explicit characterizations for the optimal order quantities. However, the existence and the uniqueness of the optimal order quantities in these cases require certain assumptions.

Some illustrative numerical examples on these models are presented. The effects of risk-tolerance and some other parameters on the optimal order quantities are examined. Moreover, we also analyze the effect of risk-sensitivity and financial hedging on the variance of the problem. We conclude that as risk-tolerance increases, the optimal order quantity also increases. We further observe that financial hedging reduces the variance of the problem significantly.

This line of research can be extended in several directions by future research. The model can involve multi-period, infinite-period, or multi-product. Bayesian models, random environment models, hidden Markov models, and mean-variance models are other suitable areas for extensions.

BIBLIOGRAPHY

[1] V. Agrawal and S. Seshadri, *Impact of uncertainty and risk aversion on price and order quantity in the newsvendor problem*, Manufacturing and Service Operations Management (2000), 410–423.

[2] ______, *Risk intermediation in supply chains*, IIE Transactions **32** (2000), 819–831.

[3] S. Ahmed, U. Çakmak, and A. Shapiro, *Coherent risk measures in inventory problems*, European Journal of Operational Research **182** (2007), 226–238.

[4] R. Akella, F.W. Ciarallo, and T.E. Morton, *A periodic review, production planning model with uncertain capacity and uncertain demand- optimality of extended myopic policies*, Management Science **40** (1994), 320–332.

[5] M. Anvari, *Optimality criteria and risk in inventory models: The case of the newsboy problem*, Journal of the Operational Research Society (1987), 625–632.

[6] K. Arifoğlu and S. Özekici, *Optimal policies for inventory systems with finite capacity and partially observed Markov-modulated demand and supply processes*, European Journal of Operational Research **204** (2010), 421–483.

[7] M. Bouakiz and M.J. Sobel, *Inventory control with exponential utility criterion*, Operations Research (1992), 603–608.

[8] G.P. Cachon and M.E. Schweitzer, *Decision bias in the newsvendor problem with a known demand distribution: Experimental evidence*, Management Science (2000), 404–420.

[9] R. Caldentey and M. Haugh, *Optimal control and hedging of operations in the presence of financial markets*, Mathematics of Operations Research (2006), 285–304.

[10] X. Chen, M. Sim, D. Simchi-Levi, and P. Sun, *Risk aversion in inventory management*, Operations Research **55** (2007), 828–842.

[11] T.C.E. Cheng, J. Li, S. Wang, and J. Wu, *Mean-variance analysis of the newsvendor model with stockout cost*, Omega (2009), 724–730.

[12] S.C. Chopra and M.S. Sodhi, *Managing risk to avoid supply-chain breakdown*, MIT Sloan Management Review **46** (2004), 53–61.

[13] L-K. Chu, J. Ni, Y. Shi, and Y. Xu, *Inventory risk mitigation by financial hedging*, Proceedings of the World Congress on Engineering and Computer Science (2009).

[14] M. Dada, N.C. Petruzzi, and L.B. Schwarz, *A newsvendor's procurement problem when suppliers are unreliable*, Manufacturing and Service Operations Management **9** (2007), 9–32.

[15] Y. Dallery, E. Sahin, and Y. Rekik, *A comprehensive analysis of the newsvendor model with unreliable supply*, OR Spectrum **29** (2007), 207–233.

[16] Q. Ding, L. Dong, and P. Kouvelis, *On the integration of production and financial hedging decisions in global markets*, Operations Research **55** (2007), no. 3, 470–489.

[17] K. Dowd, *Beyond value at risk: The new science of risk management*, John Wiley, New York, 1998.

[18] L. Eeckhoudt, C. Gollier, and H. Schlesinger, *The risk-averse (and prudent) newsboy*, Management Science **41** (1995), 786–794.

[19] A. Erdem and S. Özekici, *Inventory models with random yield in a random environment*, International Journal of Production Economics **78** (2002), 239–253.

[20] G. Gallego and H. Hu, *Optimal policies for Production/Inventory systems with finite capacity and Markov-modulated demand and supply processes*, Annals of Operations Research **126** (2004), 21–41.

[21] X. Gan, S.P. Sethi, and H. Yan, *Coordinations of suply chains with risk-averse agents*, Productions and Operations Management (2004), 135–149.

[22] V. Gaur and S. Seshadri, *Hedging inventory risk through market instruments*, Manufacturing and Service Operations Management **7** (Spring 2005), 103–120.

[23] Y. Gerchak and I.M Henig, *The structure of periodic review policies in the presence of random yield*, Operations Research **38** (1990), 634–643.

[24] Y. Gerchak, R.G. Vickson, and M. Parlar, *Periodic review production models with variable yield and uncertain demand*, IIE Transactions **20** (1988), 144.

[25] K. Jain and E. Silver, *The single period procurement problem where dedicated supplier capacity can be reserved*, Naval Research Logistics **42** (1995), 915–934.

[26] U. Jansson and A. Norrman, *Ericsson's proactive supply chain risk management approach after a serious sub-supplier accident*, International Journal of Physical Distribution and Logistics Management **34** (2004), 434–456.

[27] P. Jorion, *Value at risk*, McGraw-Hill, New York, 1997.

[28] U. Juttner, *Supply chain risk management: Understanding the business requirements from a practitioner perspective*, International Journal of Logistics Management **16** (2005), 120–141.

[29] F. Karaesmen, A. Özler, and B. Tan, *Multi-product newsvendor problem with value-at-risk considerations*, International Journal of Production Economics **117** (2009), 244–255.

[30] S. Karlin, *One stage model with uncertainty*, 1958, in K. J. Arrow, S. Karlin and H. Scarf (Eds.) Studies in the Mathematical Theory of Inventory and Production, pp. 109-143 Stanford: Stanford University Press.

[31] B. Kazaz, *Production planning under yield and demand uncertainty with yield-dependent cost and price*, Manufacturing and Service Operations Management **6** (2004), 209–224.

[32] G. Keller and A.H. Noori, *The lot-size reorder-point model with upstream, downstream uncertainty*, Decision Sciences **17** (1986), 285–291.

[33] B. Keren and J.S. Pliskin, *A benchmark solution for the risk-averse newsvendor problem*, European Journal of Operations Research **174** (2006), 1643–1650.

[34] O. Kharif, *Motorola's fuzzy camera-phone picture*, Business Week, 12 December 2003.

[35] S. Kumaraswamy and E. Sankarasubramanian, *Optimal ordering quantity to realize a pre-determined level of profit*, Management Science **29** (1983), 512–514.

[36] A.H. Lau and H. Lau, *Maximizing the probability of achieving a target profit in a two-product newsboy problem*, Decision Sciences **19** (1988), 392–408.

[37] H.S. Lau, *The newsboy problem under alternative optimization objectives*, Journal of the Operational Research Society (1980), 525–535.

[38] H.S. Lau, A.H.L. Lau, and J. Li, *A two-product newsboy problem with satisficing objective and independent exponential demands*, IIE transactions **23** (1991), 29–39.

[39] H.L. Lee and C.A. Yano, *Lot sizing with random yields: A review*, Operations Research **43** (1995), 311–334.

[40] J. Li, H. Lau, and A. Lau, *Some analytical results for a two-product newsboy problem*, Decision Sciences **21** (1990), 710–726.

[41] H.K. Okyay, F. Karaesmen, and S. Özekici, *Newsvendor models with dependent random supply and demand*, Tech. report, Koç University, Department of Industrial Engineering, Istanbul, Turkey, 2010.

[42] ———, *Hedging demand and supply risks in the newvendor model*, Tech. report, Koç University, Department of Industrial Engineering, Istanbul, Turkey, 2011.

[43] S. Özekici and M. Parlar, *Inventory models with unreliable suppliers in a random environment*, Annals of Operations Research **91** (1999), 123–136.

[44] M. Parlar and D. Wang, *Diversification under yield randomness in inventory models*, European Journal of Operation Research **66** (1993), 52–64.

[45] M. Parlar and Z.K. Weng, *Balancing desirable but conflicting objectives in the newsvendor problem*, IIE Transactions **35** (2003), no. 2, 131–142.

[46] J.W. Pratt, *Risk aversion in the small and in the large*, Econometrica **32** (1964), 122–36.

[47] W. Shih, *Optimal inventory policies when stockouts result from defective products*, International Journal of Production Research **18** (1980), 677–686.

[48] K. Simons, *Value-at-risk new approaches to risk management*, New England Economic Review (1996), 3–13.

[49] N.C. Suresh, C.X. Wang, and S. Webster, *Would a risk-averse newsvendor order less at a higher selling price?*, European Journal of Operational Research **196** (2008), 544–553.

[50] C.S. Tapiero, *Value at risk and inventory control*, European Journal of Operations Research (2005), 769–775.

[51] C.X. Wang and S. Webster, *The loss-averse newsvendor problem*, Omega **37** (2009), 93–105.

[52] S. Yang, J. Yang, and L. Abdel-Malek, *Sourcing with random yields and stochastic demand: A newsvendor approach*, Computers and Operations Research **34** (2007), 3682–3690.

VITA

Filiz SAYIN was born in Gaziantep on February 19, 1986. She received her B.Sc. degree in Industrial Engineering from Bilkent University, Ankara, in 2009. In September 2009, she started her M.Sc. degree and worked as a teaching and research assistant at Industrial Engineering Department of Koç University, Istanbul.