STARK'S CONJECTURES

and

HILBERT'S TWELFTH PROBLEM

by

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This is to certify that I have examined this copy of a master's thesis

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ABSTRACT

In this study, we state the principal Stark conjecture by defining Stark regulator which is an analogue of the regulator appearing in the Dirichlet Class Number Formula. The conjecture is independent of a choice of a set of places and a certain isomorphism of $Q[G]$-modules. We state Stark's refinement of this conjecture ('over $\mathbb{Z}$') for abelian L-functions with simple zeros at $s = 0$. This refinement predicts the existence of Stark units and we explain that the field generated over a totally real field k by the Stark units provides an answer to Hilbert's twelfth problem. We also express John Tate's reformulation for this refinement. Then, we give proofs of the conjecture in some simple cases and Stark's computational verification of the conjecture in a specific case. In the last chapter, we state the Rubin-Stark conjecture which is an extension of this conjecture which includes the case of abelian L-functions with higher order zeros at $s = 0$. We end by giving proofs of the conjecture in some cases and showing its relations between the Stark conjecture.

ÖZET

Bu çalışmada, ilk önce Stark regülatörlerini tanımlayarak, Stark varsayımını gösteriyoruz. Daha sonra, bu varsayımın değişmeli L-fonksiyonlarının $s = 0$ da tek sıfırlı olduğu durumlar için verilmiş varsayımını gösteriyoruz ve bu varsayımın özel bir durum için Hilbert'in 12. problemine çözüm sunduğunu gösteriyoruz. En son kısım da ise Rubin'in bu varsayımı genişlettiği durumu inceliyoruz.

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Contents

1 Introduction

Number theorists have been studying the fundamental interactions between the analytic and algebraic points of view. One of these interactions, probably the most crucial one, is the relationship between Dirichlet zeta function $\zeta_k(s)$ of a number field k and the invariants of k. When $k = \mathbb{Q}$ the Dedekind zeta function is exactly the same as the Riemann zeta function $\zeta_{\mathbb{Q}}(s)$ which also attracts mathematicians due to the relation between its zeros and the distribution of prime numbers.

Dirichlet proved the celebrated *class number formula*, which gives a formula for the residue of $\zeta_k(s)$ at $s = 1$:

$$\lim_{s\to 1}(s-1)\cdot\zeta_k(s) = \mathrm{Res}_{s=1}(\zeta_k(s)) = \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d_k|}}\frac{hR}{e}$$

where r_1 is the number of real and r_2 is the number of complex places in k, d_k is the discriminant, R is the regulator, h is the class number, e is the number of roots of unity in k. The regulator R is the determinant of $(r_1 + r_2 - 1)$-dimensional square matrix whose entiries are logarithms of the archimedean valuations of global units belonging to k. By using functional equation of the zeta function, Dirichlet also proved that the leading term of the Taylor expansion of the zeta function at $s = 0$ is:

$$\lim_{s\to 0} s^{1-r_1-r_2}\zeta_k(s) = -\left(\frac{hR}{e}\right).$$

In the 1970's, Stark gave the conjectural generalization of the above Dirichlet class number formula in the context of Artin L-functions. The Stark conjectures predict that the leading coefficient of an Artin L-function is the product of a type of the *Stark regulator* with an algebraic number. The Stark regulator is similar to the classical regulator as a determinant of global units.

In the case of abelian extension, Stark gave a refinement for the case of L-function with simple zero at $s = 0$. Stark's refined conjecture predicts the existence of the *Stark units* such that specific linear combinations of its archimedean valuations give the values of the derivatives of L-functions at $s = 0$.

Kronecker-Weber theorem states that every finite abelian extension of $\mathbb{Q}$ is a subfield of cyclotomic field. Hilbert's twelfth problem is an extension of Kronecker-Weber theorem which states that every finite abelian extension of $\mathbb{Q}$ is a subfield of cyclotomic field. In his twelwe problem, Hilbert suggested that any abelian extension of any base number field could be constructed via special values of complex analytic functions. In the case of totally real base field k, the field extension generated over k by the Stark units contains the maximal abelian extension of k. Thus, this provides an answer to Hilbert's twelfth problem.

In the 1990's, Rubin formulated an extension of this refinement of Stark conjecture which includes the case of the abelian L-function with multiple zeros at $s = 0$.

We begin this thesis by defining Artin L-functions and giving its properties. Then, we state the non-abelian Stark conjecture by defining Stark regulator. After, we prove that the Stark conjecture is true for L-functions with $r(\chi) = 1$ over $\mathbb{Q}$ and $r(\chi) = 0$ where $r(\chi)$ is the degree of vanishing of the relevant L-function at $s = 0$. Then, we give the refinement, noted above, for abelian L-functions with simple zeros at $s = 0$. We also provide an example of the abelian conjecture for a specific cubic base field k. Then, we show that for a specific case this refinement provides an answer for Hilbert's twelfth problem. We also state the Tate's reformulation of this refinement. Finally, we finish by stating Rubin's extension for the refined Stark Conjecture of abelian L-functions with multiple zeros at $s = 0$.

2 Some Notations and Definitions

In this thesis, the symbol k will denote an algebraic number field; that is, a finite extension of $\mathbb{Q}$ and K/k will denote a finite Galois extension of k with Galois group $G = \mathrm{Gal}(K/k)$. The group of roots of unity in k is denoted $\mu(k)$ and the number of roots of unity in k is denoted e_k.

The set S_∞ will denote the set of infinite (archimedean) places in k and the set S will be any finite set of places including all infinite places of k. We use the notations $\mathfrak{p}, \mathfrak{q}, \ldots$ for finite (non-archimedean) places and $v, v', \ldots$ for general (archimedean or non-archimedean) places in S. The set S_K will be the set of places of K lying above the places in the set S. We write $\wp$ for a place of K lying above the place $\mathfrak{p}$ and we write $w, w', \ldots$ for places of K lying above the places $v, v', \ldots$ of k.

The completions of the fields K and k will be denoted $k_\mathfrak{p}, K_\wp, k_v, K_w, \ldots$ with respect to the places $\mathfrak{p}, \wp, v, w, \ldots$. If w is a place of K lying above v, then the degree of local extension K_w/k_v is denoted $[w : v]$. The ring of integers of k will be denoted $\mathcal{O}_k$ or simply $\mathcal{O}$ and the group of fractional ideals of $\mathcal{O}_k$ will be denoted I_k.

For a place v of k the associated normalized valuation $|\cdot|_v$ on k is defined by $|0|_v = 0$ and, for $x \in k^*$:

$$|x|_v := \begin{cases} (\mathbf{N}\mathfrak{p})^{-\mathrm{ord}_\mathfrak{p}(x)} & \text{if } v = \mathfrak{p} \text{ is a finite place} \\ |x| = \pm x & \text{if } v \text{ is a real place} \\ x\overline{x} & \text{if } v \text{ is a complex place} \end{cases}$$

where $\mathrm{ord}_\mathfrak{p}(x) := \mathrm{ord}_\mathfrak{p}(x\mathcal{O}_k)$ is the exponent of $\mathfrak{p}$ in the prime ideal decomposition of $x\mathcal{O}_k$ and where $\mathbf{N}\mathfrak{p}$ is the size of the finite field $\mathcal{O}_k/\mathfrak{p}$.

With these normalized valuations, the product formula can be written as

$$\prod_v |x|_v = 1$$

where the product is taken over all places of k, [Neu, IV.-§1.]. Furthermore, if w is a place in K lying above the place v in k then, for $u \in k_v$, we have the equality

$$|u|_w = |u|_v^{[w:v]}.$$

When K/k is a Galois extension, for $x \in K$ and $\sigma \in G$, we have

$$|x^\sigma|_{\sigma w} = |x|_w.$$

The *S-integers* $\mathcal{O}_S$ is defined by

$$\begin{aligned} \mathcal{O}_S &= \{x \in k : x \in \mathcal{O}_\mathfrak{p}, \text{ for all places of k with } \mathfrak{p} \notin S\} && (2.1)\\ &= \bigcap_{\mathfrak{p} \notin S} \mathcal{O}_\mathfrak{p} && (2.2) \end{aligned}$$

where $\mathcal{O}_\mathfrak{p} = \{x \in k : |x|_\mathfrak{p} \leq 1\}$.

The group of fractional ideals of $\mathcal{O}_S$ is called *the ideal group* of $\mathcal{O}_S$ and is denoted by $I(\mathcal{O}_S)$. The collection of all principal fractional ideals of $\mathcal{O}_S$ forms a subgroup in $I(\mathcal{O}_S)$ which is denoted by $P(\mathcal{O}_S)$. The size of the ideal class group $\mathrm{Cl}(\mathcal{O}_S) = I(\mathcal{O}_S)/P(\mathcal{O}_S)$ of the Dedekind ring $\mathcal{O}_S$ is denoted by $h_{k,S}$ or, simply h_S. When $S = S_\infty$ we denote the size of the ideal class group by h_k. [Jan, I-4.].

If K/k is a Galois extension and $\mathfrak{p}$ is a finite place of k, $\mathfrak{p}$ splits in K into a product

$$\mathfrak{p} = (\wp_1 \cdots \wp_r)^e$$

and $efr = [K : k]$, where $f = f(\wp_i/\mathfrak{p})$ is the degree of residue class field extension and the integer $e = e(\wp_i/\mathfrak{p})$ is called the *ramification index* of $\wp$.

If $e > 1$, then we say that $\wp$ is *ramified* over $\mathfrak{p}$ or $\mathfrak{p}$ is ramified in K/k . If $e = 1$ then we say that $\mathfrak{p}$ is *unramified* in K/k.

The real place $v \in k$ is ramified in K/k if and only if there is a complex place of K lying above the place v, because in this case $e(w|v) = 2$. On the other hand, the complex place $v \in k$ is always unramified in K/k.

If there is a unique prime ideal $\wp$ lying above $\mathfrak{p}$ (so r=1) and the relative degree $f(\wp|\mathfrak{p}) = 1$, then we say that $\mathfrak{p}$ is *totally ramified* in K/k. In this case, $e(\wp|\mathfrak{p}) = [K : k]$.

If $e(\wp|\mathfrak{p}) = f(\wp|\mathfrak{p}) = 1$, then we say that $\mathfrak{p}$ is *totally split* (or splits completely) in K/k. Notice that there are exactly $r = [K : k]$ prime ideals of $\mathcal{O}_K$ lying above $\mathfrak{p}$.

For each place v of k, the Galois group $G = \mathrm{Gal}(K/k)$ acts transitively on the set of places w of K lying above the places v of k. The stabilizer of one of these w's is a subgroup of G, is called the *decomposition group* of w and is denoted by G_w. By the theory of finite fields, the Galois group of a finite field extension is cyclic and there exists a unique automorphism of $\mathcal{O}_K/\wp$ over $\mathcal{O}_k/\mathfrak{p}$ which generates the Galois group of the residue class field extension by sending x to $x^{\mathbf{N}\mathfrak{p}}$. For a finite place $\mathfrak{p}$, an element σ of $G_\wp$ induces an automorphism $\widetilde{\sigma}$ on the residue class field extension $(\mathcal{O}_K/\wp)/(\mathcal{O}_k/\mathfrak{p})$ and for $x \in \mathcal{O}_K$, $\widetilde{\sigma}$ is defined by

$$\widetilde{\sigma}(x + \wp) = \sigma(x) + \wp.$$

The mapping $\sigma \mapsto \widetilde{\sigma}$ is a homomorphism of $G_\wp$ into the Galois group of the residue class field extension. The kernel of this map is called *inertia group* of $\wp$ and is denoted by $I_\wp$. The orders of $G_\wp$ and $I_\wp$ equals to ef and e, respectively. [Jan, III-1.]. Any element of the coset of $I_\wp$ in $G_\wp$ is called the *Frobenius automorphism of* $\wp$ and is denoted by $\mathrm{Frob}_\wp$. If $\mathfrak{p}$ is unramified in K then the inertia group $I_\wp$ is trivial and $\mathrm{Frob}_\wp$ is uniquely determined as an element of the decomposition group $G_\wp$ of G. When K/k is abelian, we denote the decomposition group and inertia group by $G_\mathfrak{p}$ and $I_\mathfrak{p}$, respectively since in this case $G_\mathfrak{p}$ and $I_\mathfrak{p}$ does not depend on the place $\wp$ in K.

2.1 Some Facts from Finite Representation Theory

Let G be a finite group of order n. Then, the group algebra $\mathbb{C}[G]$ is a vector space of dimension n over $\mathbb{C}$ with the elements of G as basis. A *linear representation* of a group G in a vector space V over $\mathbb{C}$ is a group homomorphism

$$\rho : G \to \mathrm{GL}(V)$$

where $\mathrm{GL}(V)$ is the group of $\mathbb{C}$-linear automorphisms of V. Indeed, by representation of G we mean a finite dimensional left $\mathbb{C}[G]$-module V by action $\rho(\sigma)x$ (or simply σx) for $x \in V,\ \sigma \in G$. If $V = \mathbb{C}[G]$, the representation is called *regular* and G acts on V by left multiplication . If $V = \mathbb{C}$, the representation is called *trivial* and $\sigma x = x$ for all $\sigma \in G$ and $x \in V$.

Let K be a field of characteristic zero. If V is a K-vector space, we let $V_{\mathbb{C}}$ denote the $\mathbb{C}$-vector space $\mathbb{C} \otimes_K V$ (in terms of modules $\mathbb{C}[G] \otimes_{K[G]} V$) obtained from V by extending scalars from K to $\mathbb{C}$. If G is a finite group, each linear representation $\rho : G \to \mathrm{GL}(V)$ over the field K defines a linear representation

$$\rho_{\mathbb{C}} : G \to \mathrm{GL}(V) \to \mathrm{GL}(V_{\mathbb{C}})$$

over $\mathbb{C}$. A linear representation of G over $\mathbb{C}$ is said to be *realizable over* K if it is isomorphic to a representation of the form $\rho_{\mathbb{C}}$. (see [Ser, II., 12.1])

Let V and V' be vector spaces and $V \otimes_{\mathbb{C}} V'$ be the tensor product space over $\mathbb{C}$. Then we have, for $x \in V$, $x' \in V'$ and $\sigma \in G$,

$$\sigma(x \otimes x') = \sigma(x) \otimes \sigma(x').$$

The *dual* representation V^* of V is defined $\mathrm{Hom}(V, \mathbb{C})$.

A module is said to be *semisimple* if it is a direct sum of simple submodules. If K is a field of characteristic zero, the group algebra $K[G]$ is semisimple. To say that $K[G]$ is semisimple algebra is equivalent to saying that each $K[G]$-module V is semisimple. (see [Ser, I., 6.1])

The *character* of a representation ρ is the function $\chi_\rho = \chi_V : G \to \mathbb{C}$ given by $\chi_V(\sigma) = \mathrm{Tr}(\rho(\sigma))$, where $\mathrm{Tr}(\rho(\sigma))$ is the trace of the map $x \mapsto \sigma x$ of V into V. The *degree* of a character χ of a representation ρ is $\chi(1) = \dim V$. If V and V' are vector spaces over $\mathbb{C}$ then, the character of a direct sum is

$$\chi_{V \oplus_{\mathbb{C}} V'}(\sigma) = \chi_V(\sigma) + \chi_{V'}(\sigma);$$

the character of a tensor product is

$$\chi_{V \otimes_{\mathbb{C}} V'}(\sigma) = \chi_V(\sigma) \cdot \chi_{V'}(\sigma);$$

and the character of the dual representation is

$$\chi_{V^*}(\sigma) = \chi_V(\sigma^{-1}) = \overline{\chi}_V(\sigma)$$

Let $\{W_i\}$ be the irreducible representations of the finite group G. Let $V = U_1 \oplus \ldots \oplus U_s$ be a decomposition of V into a direct sum of irreducible representations. Then, V has a 'canonical decomposition' $V = \oplus_i V_i$ where V_i is the sum of those of the $U_1, \ldots, U_s$ which are isomorphic to W_i.

The characters of a finite group G are class functions (that is, χ is constant on conjugacy classes).

If ϑ and ϕ are the characters of a finite group G, we define *inner product*

$$\langle \vartheta, \phi \rangle_G = \frac{1}{|G|} \sum_{\sigma \in G} \vartheta(\sigma) \overline{\phi}(\sigma).$$

This is a real number since $\langle \vartheta, \phi \rangle_G = \overline{\langle \vartheta, \phi \rangle_G}$.

Let H be a subgroup of G. Let W be a representation of H over $\mathbb{C}$ with character χ. From W, we construct a representation of G called *induced representation*

$$\mathrm{Ind}_H^G W = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$$

where $\mathbb{C}[G]$ acts by multiplication on the left factor. The character of $\mathrm{Ind}_H^G W$ is written by $\mathrm{Ind}_H^G \chi$ and defined by

$$\mathrm{Ind}_H^G \chi(\sigma) = \frac{1}{|H|} \sum_{\substack{\tau \in G \\ \tau^{-1}\sigma\tau \in H}} \chi(\tau^{-1}\sigma\tau). \tag{2.3}$$

Theorem 2.1 (Frobenius Reciprocity)**.** *(see [Ser, II, 7.2]) Let H be a subgroup of a finite group G. If χ is a character of H and θ is a character of G, then*

$$\left\langle \mathrm{Ind}_H^G \chi, \theta \right\rangle_G = \langle \chi, \theta|_H \rangle_H.$$

Theorem 2.2. *(see [Das, Theorem A.12.4.]) Let χ be the character of an irreducible representation V of G over $\mathbb{C}$. Then there is an irreducible representation V' of G over $K(\chi)$ with character $\chi' = m\chi$, where m is the Schur index of χ' over $K(\chi)$. Furthermore, $\varphi = Tr_{K(\chi)/K}(\chi')$ is the character of an irreducible representation W of G over K, where $Tr_{K(\chi)/K}$ denotes the trace associated with the extension $K(\chi)/K$. Finally, $D = End_{K[G]}W$ is a division algebra with center $E \cong K(\chi)$ and $[D : E] = m^2$.*

Remark 2.3. *Every* $\chi \in \widehat{G}$ *extends by linearity to a* $\mathbb{C}$*-algebra homomorphism* $\mathbb{C}[G] \to \mathbb{C}$*. The* $\chi \in \widehat{G}$ *form a basis for the space of linear functionals* $\mathbb{C}[G] \to \mathbb{C}$*. Thus, if* x*,* $y \in \mathbb{C}[G]$ *and* $\chi(x) = \chi(y)$ *for all* $\chi \in \widehat{G}$*, then* $x = y$*.*

3 The non-abelian Stark Conjecture

In this section, most of the results are taken from [Tat] and [Das].

3.1 The Dedekind zeta-function

Let k be an algebraic number field and let S_∞ be the set of all infinite places. Define the Dedekind zeta-function for $Re(s) > 1$

$$\zeta_k(s) = \sum_{\mathfrak{U} \subseteq \mathcal{O}_k} \frac{1}{(\mathbf{N}\mathfrak{U})^s}$$

where the sum runs over the nonzero ideals $\mathfrak{U}$ of the ring of integers $\mathcal{O}_k$ of k and $\mathbf{N}\mathfrak{U}$ denotes the absolute norm of $\mathfrak{U}$. Furthermore, we define the Euler product for $Re(s) > 1$

$$\zeta_k(s) = \prod_{\mathfrak{p}} (1 - \mathbf{N}\mathfrak{p}^{-s})^{-1}$$

where $\mathfrak{p}$ ranges over all finite places in k.

Theorem 3.1 (Dirichlet Class Number Formula at $s = 1$)**.** *[Neu, V.-§2, Theorem 2.2] The Dedekind zeta-function* $\zeta_k(s)$ *has a simple pole at* $s = 1$ *with residue*

$$\lim_{s \to 1} (s-1) \cdot \zeta_k(s) = Res_{s=1}(\zeta_k(s)) = \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d_k|}} \frac{hR}{e}$$

where r_1 *is the number of real and* r_2 *is the number of complex places in* k*,* d_k *is the discriminant,* R *is the regulator,* h *is the class number,* e *is the number of roots of unity in* k*.*

Note that the regulator R is the determinant of $(r_1 + r_2 - 1)$-dimensional square matrix whose entiries are logarithms of the archimedean valuations of units belonging to k.

Proposition 3.2. *For $\chi = \mathbb{1}$, Artin L-function gives us the Dedekind zeta-function*

$$L_{K/k,S}(s, \mathbb{1}) = \zeta_k(s).$$

Proof. If $\rho : G \to \mathrm{GL}(\mathbb{C})$ is trivial representation, $\rho(\sigma) = 1$ for any $\sigma \in G$, then we have

$$\prod_{\mathfrak{p} \notin S} \det((1 - \mathrm{Frob}_{\wp} \mathbf{N}\mathfrak{p}^{-s})|\mathbb{C})^{-1} = \prod_{\mathfrak{p} \notin S} (1 - \mathbf{N}\mathfrak{p}^{-s})^{-1}.$$

□

Remark 3.3. *If we rearrange the functional equation of $L(s, \chi)$ as carried out in the Theorem 3.15, for $\chi = \mathbb{1}$, we get the functional equation for ζ_k*

$$\Lambda(1 - s, \mathbb{1}) = \Lambda(s, \bar{\mathbb{1}})$$

as $W(\mathbb{1}) = 1$, and

$$\Lambda(s, \mathbb{1}) = |d_k|^{s/2} \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{C}}(s)^{r_2} \zeta_k(s).$$

By using the functional equation for ζ_k, we may write the leading coefficient of the Taylor series at $s = 0$.

Theorem 3.4 (Dirichlet Class Number Formula at $s = 0$)**.** *The leading coefficient of the Taylor series of ζ_k at $s = 0$ is*

$$\lim_{s \to 0} s^{1-|S_\infty|} \zeta_k(s) = -\left(\frac{hR}{e}\right).$$

Proof. By using the functional equation for ζ_k, we get the equality

$$\Gamma((1-s)/2)^{r_1} \Gamma(1-s)^{r_2} \zeta_k(1-s) = |d_k|^{s-\frac{1}{2}} (2^{1-s} \pi^{1-2s})^{r_2} (\pi^{1/2-s})^{r_1} \Gamma(s/2)^{r_1} \Gamma(s)^{r_2} \zeta_k(s).$$

By the fact that $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1 and $\Gamma(n + 1) = n!$ when $n \in \mathbb{Z}_{\geq 0}$, $\Gamma(1/2) = (\pi)^{-1/2}$ and the Dirichlet Class Number Formula at $s = 1$, we can simplify the equality and at last we get our result as $s \to 0$. □

3.2 The function $\zeta_{k,S}$

Let S be a finite set of places including all infinite places of k. Define the general Dedekind zeta-function for $Re(s) > 1$ by

$$\zeta_{k,S}(s) = \sum_{(\mathfrak{U},S)=1} \frac{1}{(\mathbf{N}\mathfrak{U})^s}$$

where the sum runs over the nonzero integral ideals which are relatively prime to S. We write the Euler product expansion of $\zeta_{k,S}$ for $Re(s) > 1$

$$\zeta_{k,S}(s) = \prod_{\mathfrak{p} \notin S} (1 - N\mathfrak{p}^{-s})^{-1} = \prod_{\mathfrak{p} \in S - S_\infty} (1 - N\mathfrak{p}^{-s}) \cdot \zeta_k(s).$$

Definition 3.5 (S-regulator). *Let $u_1, \ldots, u_r$ be the set of generators of $\mathcal{O}_S{}^*/(\mathcal{O}_S{}^*)_{tors}$ with $r = |S| - 1$, and let us fix an arbitrary $v_0 \in S$ then the S-regulator of k is*

$$R_S(u_1, \ldots, u_r) = |\det_{\substack{1 \le i \le r \\ v \in (S - v_0)}} (\log |u_i|_v)|. \tag{3.1}$$

Theorem 3.6. *Let $\mathfrak{p} \notin S$ be a place in k and let $S' = S \bigcup \mathfrak{p}$. If m is the order of $\mathfrak{p}$ in the ideal class group of $\mathcal{O}_S$, then we have*

(i) $h_S = m \cdot h_{S'}$

(ii) $R_{S'} = m \cdot (\log \mathbf{N}\mathfrak{p}) \cdot R_S$

(iii) $\zeta_{k,S'}(s) \sim (\log \mathbf{N}\mathfrak{p}) \cdot s \cdot \zeta_{k,S}(s)$ *as* $s \to 0$

Proof. [Das]

(i) For the first assertion, we will show that the sequence

$$0 \to \langle [\mathfrak{p}] \rangle \longrightarrow \mathrm{Cl}(\mathcal{O}_S) \xrightarrow{\phi} \mathrm{Cl}(\mathcal{O}'_S) \to 0$$

is exact where $\langle [\mathfrak{p}] \rangle$ is the subgroup of $\mathrm{Cl}(\mathcal{O}_S)$ generated by the class of $\mathfrak{p}$. Because in the case of exactness of the above sequence, we get the first assertion, by definition $m = |\langle [\mathfrak{p}] \rangle|$.

Indeed, the map $I(\mathcal{O}_S) \to I(\mathcal{O}_{S'})$ is a natural surjection given by $\langle \mathfrak{U} \rangle \mapsto \langle \mathfrak{U}\mathcal{O}_{S'} \rangle$. Then composing with the projection map $I(\mathcal{O}_{S'}) \to \mathrm{Cl}(\mathcal{O}_{S'})$, we get the surjection $\phi : \mathrm{Cl}(\mathcal{O}_S) \to \mathrm{Cl}(\mathcal{O}'_S)$. Now, we wish to show that $\langle [\mathfrak{p}] \rangle$ is the kernel of the map ϕ. The ideal $\langle [\mathfrak{p}] \rangle$ generated by the class of $\mathfrak{p}$ is in $\ker(\phi)$. Conversely, for a given any element $\mathfrak{U} \in \ker(\phi)$ we may find an element $\beta \in K^*$ satisfies $\mathfrak{U}\mathcal{O}_{S'} = \beta\mathcal{O}_{S'}$. Hence, for all finite places $\mathfrak{q} \neq \mathfrak{p}$ of $\mathcal{O}_S$ we have $\mathrm{ord}_{\mathfrak{q}}(\mathfrak{U}) = \mathrm{ord}_{\mathfrak{q}}(\beta\mathcal{O}_S)$. Then we find $\mathfrak{U} = \mathfrak{p}^e\beta\mathcal{O}_S$ where $e = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{U}) - \mathrm{ord}_{\mathfrak{p}}(\beta\mathcal{O}_S)$. So, $\mathfrak{U}$ is in $\langle [\mathfrak{p}] \rangle$ and the proof of (i) follows.

(ii) Let $\{u_1, \ldots, u_r\}$ be the set of fundamental units in ${\mathcal{O}_S}^*$ with $r = |S| - 1$ (i.e., $u_1, \ldots, u_r$ be a set of generators of $(\mathcal{O}_S^*)^{\mathrm{free}}$). We have the following exact sequence,

$$0 \longrightarrow \mathcal{O}_S^* \longrightarrow \mathcal{O}_{S'}^* \stackrel{\mathrm{ord}_{\mathfrak{p}}}{\longrightarrow} m\mathbb{Z} \longrightarrow 0$$

Clearly, $\mathcal{O}_S^*$ injects into $\mathcal{O}_{S'}^*$ and $\mathrm{ord}_{\mathfrak{p}}$ is a surjection given by $\alpha \longmapsto m$ when $\alpha\mathcal{O}_S = \mathfrak{p}^m$.

If α is an element of $\mathcal{O}_{S'}^*$ with $\mathrm{ord}_{\mathfrak{p}}(\alpha) = m$ then α generates $\mathcal{O}_{S'}^*/\mathcal{O}_S^*$ and $\{u_1, \ldots, u_r, \alpha\}$ gives a basis for $({\mathcal{O}_{S'}}^*)^{\mathrm{free}}$.

Now, assume $M_S = (\log |u_i|_{\mathfrak{q}})_{\substack{1 \leq i \leq r \\ \mathfrak{q} \in (S - v_0)}}$ for fixed $v_0 \in S$ then $R_S = |\det(M_S)|$. So, the regulator $R_{S'}$ is the absolute value of the determinant

$$M_{S'} = \det \left(\begin{array}{c|c} M_S & * \\ \hline 0 & \log |\pi|_{\mathfrak{p}} \end{array} \right)$$

since $v_{\mathfrak{q}}(\alpha) = 0$ for all $\mathfrak{q} \neq \mathfrak{p}$ in S'. Hence,

$$R_{S'} = |\log |\pi|_{\mathfrak{p}}| \cdot R_S = m \cdot (\log \mathbf{N}\mathfrak{p}) \cdot R_S.$$

(iii) We may write the following equality by defnition,

$$\zeta_{k,S'}(s) = (1 - \mathbf{N}\mathfrak{p}^{-s}) \cdot \zeta_{k,S}(s).$$

Then, taking the limits as $s \to 0$ gives the desired result.

□

Corollary 3.7. *The Dirichlet class number formula can be generalized as follows*

$$\lim_{s\to 0} s^{1-|S|}\zeta_{k,S} = -\left(\frac{h_S R_S}{e}\right)$$

Proof. Let S be a finite set of primes of k. Assume that the set of finite places $S - S_\infty = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and m_i is the order of $\mathfrak{p}_i$ in the class group of $\mathcal{O}_S$ Then, by the Theorem 3.4 and Theorem 3.6-(iii), we get

$$\begin{aligned}\zeta_{k,S}(s) &\sim (\prod \log \mathbf{N}\mathfrak{p}_i) \cdot s^{|S-S_\infty|} \cdot \left(-\frac{hR}{e}\right) \cdot s^{|S_\infty|-1} \\ &\sim (\prod \log \mathbf{N}\mathfrak{p}_i) \cdot \left(-\frac{\prod m_i \cdot h_S R_S}{\prod \log \mathbf{N}\mathfrak{p}_i \prod m_i \cdot e}\right) \cdot s^{|S|-1} \\ &\sim \left(-\frac{h_S R_S}{e}\right) \cdot s^{|S|-1}.\end{aligned}$$

□

3.3 Abelian L-functions

Suppose that K/k is a finite abelian extension and χ is a character of Galois group $G = \mathrm{Gal}(K/k)$. Let S be the finite set of places which contains all infinite places of k, as well as all finite places of k which ramify in K. Define L-function for $\mathrm{Re}(s) > 1$ by

$$L_S(s,\chi) = L_{K/k,S}(s,\chi) = \sum_{(\mathfrak{U},S)=1} \frac{\chi(\mathrm{Frob}_{\mathfrak{U}})}{(\mathbf{N}\mathfrak{U})^s}$$

where the sum runs over the nonzero integral ideals which are relatively prime to S.

Define the partial zeta function of K/k associated to $\sigma \in G$ as, for $\mathrm{Re}(s) > 1$,

$$\zeta_S = \zeta_{K/k,S}(s,\sigma) = \sum_{\substack{(\mathfrak{U},S)=1 \\ \mathrm{Frob}_{\mathfrak{U}}=\sigma}} (\mathbf{N}\mathfrak{U})^{-s}. \tag{3.2}$$

Theorem 3.8 (Siegel)**.** *If K/k is an abelian extension and $\sigma \in \mathrm{Gal}(K/k)$, then $\zeta_S(0,\sigma)$ is a rational number.*

Siegel's proof of this theorem can be found in [Sie].

Remark 3.9. *The* L*-function* L_S *and the partial zeta function* ζ_S *can be meromorphically continued to the entire complex plane. Furthermore, these functions have the following relations*

$$L_S(s,\chi) = \sum_{\sigma \in G} \chi(\sigma)\zeta_S(s,\sigma)$$

and

$$\zeta_S(s,\sigma) = \frac{1}{|G|}\sum_{\chi \in \widehat{G}} \overline{\chi}(\sigma) L_S(s,\chi).$$

The Euler product of the abelian L-function for $\mathrm{Re}(s) > 1$ is defined by

$$L_S(s,\chi) = \prod_{\mathfrak{p} \notin S} (1 - \chi(\mathrm{Frob}_{\mathfrak{p}})\mathbf{N}\mathfrak{p}^{-s})^{-1}$$

where $\mathbf{N}\mathfrak{p} = |\mathcal{O}_k/\mathfrak{p}|$.

3.4 Artin L-functions

Let K/k be a Galois extension with the group $G = \mathrm{Gal}(K/k)$. Let $\mathfrak{p}$ be a place in k and $\wp$ be the place in K lying above $\mathfrak{p}$ with ramification index e. We define

$$\chi(I_\wp) = \sum_{\tau \in I_\wp} \chi(\tau)$$

where $I_\wp = \sum_{\tau \in I_\wp} \tau \in \mathbb{Z}[G]$. For an integer $m \geq 1$ and the element $\mathrm{Frob}_\wp \in G_\wp/I_\wp$, we define

$$\chi(\mathfrak{p}^m) = \frac{1}{e}\chi((\mathrm{Frob}_\wp)^m I_\wp)$$

and this can also be viewed as the value of χ on the element

$$\frac{1}{e}\sum_{\tau \in I_\wp} \mathrm{Frob}_\wp^m \tau.$$

We now define the Artin L-function $L_{K/k}(s,\chi)$ by defining its logarithm, following [Lan, XII-§2],

$$\log L_{K/k}(s,\chi) = \sum_{\mathfrak{p},m} \frac{\chi(\mathfrak{p}^m)}{m\mathbf{N}\mathfrak{p}^{ms}}.$$

For $\mathrm{Re}(s) > 1$, the Artin L-function is then the exponent of this logarithm and the following definition express the Euler product representation of Artin L-function, [Neu, V-§4, Prop. 4.4].

Definition 3.10. *Let K/k be any Galois extension with Galois group $G = \mathrm{Gal}(K/k)$ and S be the finite set of places contains all infinite places of k. The element* $\mathrm{Frob}_\wp \in G_\wp/I_\wp$ *is an endomorphism of the fixed module $V^{I_\wp}$. Then, for Re$(s) > 1$, we define*

$$L_{K/k,S}(s,\chi) = \prod_{\mathfrak{p}\notin S} \det((1-\mathrm{Frob}_\wp \mathbf{N}\mathfrak{p}^{-s})|V^{I_\wp})^{-1} \tag{3.3}$$

where $\mathfrak{p}$ is a finite place in k and $\wp$ is the place in K lying above the place $\mathfrak{p}$. Since the elements $\mathrm{Frob}_\wp$ *are conjugate, the value of 'characteristic polynomial' of* $\mathrm{Frob}_\wp$, *which is written as $\mathfrak{p}$-factor of the Euler product, does not depend on the choice of $\wp$.*

Proposition 3.11. *Let K/k be a finite Galois extension with the group G and let S be a finite set of places of k containing infinite places. Then Artin L-function satisfies the following properties:*

(i) Additivity: If χ_1, χ_2 are the characters of G, then

$$L_{K/k,S}(s,\chi_1+\chi_2) = L_{K/k,S}(s,\chi_1)\cdot L_{K/k,S}(s,\chi_2);$$

(ii) Inflation: If $K' \supset K \supset k$ is a bigger Galois extension with the group $G' =$ $\mathrm{Gal}(K'/k)$ *and χ is a character of G, denote* $\mathrm{Infl}\chi$ *is a character $G' \to G \xrightarrow{\chi} \mathbb{C}$, then*

$$L_{K/k,S}(s,\chi) = L_{K'/k,S}(s,\mathrm{Infl}\chi);$$

(iii) Induction: If H is the subgroup of G and $K^H = F$ is the fixed field of H and χ is the character of H, then

$$L_{K/k,S}(s,\mathrm{Ind}_H^G\chi) = L_{K/F,S_F}(s,\chi);$$

Proof. The proof of (i) is trivial since the character χ appears linearly in the logarithmic definition of L-function. The proof of (iii) may be found in [Neu, V-§4]. Now, we will prove (ii), following [Lan, XII-§2].

Let $K' \supset K \supset k$ be a finite Galois extension with the group $H = \mathrm{Gal}(K'/K)$. Let $\wp'$, $\wp$ be places in K' and K, respectively, lying above $\mathfrak{p}$ in k. Let $G_{\wp'}$ be the decomposition group and $I_{\wp'}$ be the inertia group of the place $\wp'$ in G'. We first show that

$$G_{\wp} \cong G_{\wp'}H/H$$
$$I_{\wp} \cong I_{\wp'}H/H.$$

When we restrict an element of $G_{\wp'}H$ into K this restriction leaves $\wp$ fixed, since the restriction to K of an element of $G_{\wp'}$ is in $G_{\wp}$. Conversely, let λ be an element in G' and its restriction to K be in $G_{\wp}$. Since λ maps $\wp'$ on another divisor of $\wp$ and H permutes such divisors transitively, then there exists an element $\gamma \in H$ such that $\lambda\gamma \in G_{\wp'}$. So $\lambda \in G_{\wp'}H$, and this proves that

$$G_{\wp} \cong G_{\wp'}H/H.$$

By the same way, we can show that

$$I_{\wp} \cong I_{\wp'}H/H.$$

Since the restriction homomorphisms $G_{\wp'} \to G_{\wp}$ and $I_{\wp'} \to I_{\wp}$ from K' to K are surjective, we have

$$G_{\wp} \cong G_{\wp'}/(G_{\wp'} \cap H)$$

and

$$I_{\wp} \cong I_{\wp'}/(I_{\wp'} \cap H).$$

The value of $\mathrm{Infl}\chi$ on an element of G' depends only its conjugacy class mod H, by definition, so we can see $\mathrm{Infl}\chi$ as equal to χ mod H. Lastly, if $\mathrm{Frob}_{\wp'}$ is a Frobenius element of $\wp'$ in $G_{\wp'}$ then $\mathrm{Frob}_{\wp'}|_K = \mathrm{Frob}_{\wp}$ is a Frobenius element in $G_{\wp}$, by the above isomorphisms. It follows that

$$\chi(\mathfrak{p}^m) = \frac{1}{e}\sum_{\tau \in I_{\wp}} \chi(\mathrm{Frob}_{\wp}^m \tau) = \frac{1}{e'}\sum_{\tau \in I_{\wp'}} \chi(\mathrm{Frob}_{\wp'}^m \tau)$$

and hence the value $\chi(\mathfrak{p}^m)$ does not depend on the field K'. So, the result follows from the logarithmic definition of the Artin L-function. □

Theorem 3.12 (Brauer)**.** *[Ser, 10.5] Each character of a finite group G can be written as a $\mathbb{Z}$-linear combination of characters induced from characters of elementary subgroups.*
In particular,

$$\chi = \sum n_i \cdot \mathrm{Ind}_{H_i}^{G} \chi_i$$

where $n_i \in \mathbb{Z}$; χ_i be the one-dimensional character of $H_i \subset G$ and χ be the character of G.

Let k_i be the fixed field of H_i, then by the addition and induction properties of Artin L-function and the Brauer's theorem 3.12, we have

$$\begin{aligned} L_{K/k,S}(s,\chi) &= L_{K/k,S}(s, \sum_i n_i \mathrm{Ind}_{H_i}^{G} \chi_i) \\ &= \prod_i L_{K/k,S}(s, \mathrm{Ind}_{H_i}^{G} \chi)^{n_i} && \text{(Additivity)} \\ &= \prod_i L_{K/k_i,S_{k_i}}(s, \chi_i)^{n_i} && \text{(Induction)} \end{aligned}$$

Remark 3.13. *The function $L_{K/k,S}$ can be meromorphically continued to the entire complex plane, [Mar].*

3.4.1 The Functional Equation for Artin L-function

In this section, we will state the functional equation of Artin L-functions for $S = S_\infty$ and we denote $L(s,\chi) = L_{K/k,S_\infty}(s,\chi)$.

Now, we define the following functions to define local factors for L-function at the infinite places of k. Set:

$$\begin{aligned} \Gamma_{\mathbb{R}}(s) &= \pi^{-s/2}\Gamma(s/2); \\ \Gamma_{\mathbb{C}}(s) &= \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s). \end{aligned} \tag{3.4}$$

Let v be a real place of k and w be the place of K lying above v. Let us define $n_+ = \dim V^{G_w}$ and $n_- = \mathrm{codim} V^{G_w}$.

Note that $\Gamma(s)$ is Euler's gamma function with a simple pole at $s = 0$ with residue 1. We also note that $\Gamma(n+1) = n!$ if $n \in \mathbb{Z}_{\geq 0}$, [Apo, 12.2].

Now, we define the local factors of L-function;

$$L_v(s,\chi) = \begin{cases} \Gamma_{\mathbb{C}}(s)^{\chi(1)} & \text{if } v \text{ is a complex place} \\ \Gamma_{\mathbb{R}}(s)^{n_+}\Gamma_{\mathbb{R}}(s+1)^{n_-} & \text{if } v \text{ is a real place} \end{cases}.$$

In particular, we conclude that if v is complex place then L_v has a pole of order $\chi(1) = \dim V$ at $s = 0$ and if v is real place then L_v has a pole of order $n_+ = \dim V^{G_w}$ at $s = 0$.

Now, let us define

$$\Gamma_\chi(s) = \prod_{v|\infty} L_v(s,\chi) = \prod_{v:\ real} L_v(s,\chi) \cdot \prod_{v:\ complex} L_v(s,\chi) \tag{3.5}$$

Then by definition of Γ_χ, it has a pole at $s = 0$ with order

$$\begin{aligned} \text{ord}_{s=0}\Gamma_\chi(s) &= \sum_{v:\ \text{real}} n_+ + \sum_{v:\ \text{complex}} \chi(1) \\ &= \sum_{v:\ \text{real}} \dim V^{G_w} + \sum_{v:\ \text{complex}} \dim V \\ &= \sum_{v:\ \text{real}} \dim V^{G_w} + \sum_{v:\ \text{complex}} \dim V^{G_w} \\ &= \sum_{v|\infty} \dim V^{G_w}. \end{aligned} \tag{3.6}$$

The third equality is because G_w is trivial when v is a complex place of k.

Note that $\Gamma_\chi(s)$ is non-zero at $s = 1$ because $\Gamma(s)$ is non-zero at $s = 1$.

Definition 3.14. *Let $\mathfrak{p}$ be a finite place of k, let us fix $\wp$ of K lying above $\mathfrak{p}$. Let $I_\wp = G_0 \supset G_1 \supset G_2 \supset \ldots$ be the sequence of ramification groups where the G_i are normal subgroups of $G_\wp$, [SLF, ch. IV]. For $|G_i| = g_i$, define*

$$f(\chi,\mathfrak{p}) = \sum_{i=0}^{\infty} \frac{g_i}{g_0} \text{codim}\, V^{G_i}.$$

If χ is a character of G, then $f(\chi,\mathfrak{p})$ is a non-negative rational number [SLF, ch. VI-§2]. If $\mathfrak{p}$ is unramified in K/k then $G_0 = I_\mathfrak{p}$ is trivial and $\text{codim}\, V^{G_i} = 0$. *Hence, $f(\chi,\mathfrak{p}) = 0$. Then, we define $\mathfrak{f}(\chi)$, the Artin conductor of χ, by*

$$\mathfrak{f}(\chi) = \prod_{\mathfrak{p}} \mathfrak{p}^{f(\chi,\mathfrak{p})},$$

where $\mathfrak{p}$ runs over all finite places of k.

Theorem 3.15. *[Tat, 0.-§6] Let the notation be as above. Then, the completed L-function, for $S = S_\infty$, is defined by*

$$\Lambda(s,\chi) = \{|d_k|^{\chi(1)} \mathrm{N}\mathfrak{f}(\chi)\}^{s/2} \cdot \prod_{v|\infty} L_v(s,\chi) \cdot L(s,\chi).$$

Then, $\Lambda(s,\chi)$ can be extended to a meromorphic function on the complex plane satisfing the functional equation

$$\Lambda(1-s,\chi) = W(\chi) \cdot \Lambda(s,\bar{\chi})$$

where $|d_k|$ is the absolute discriminant of $k/\mathbb{Q}$, $\mathrm{N}\mathfrak{f}(\chi) > 0$ is the absolute norm of $\mathfrak{f}(\chi)$, and the Artin root number $W(\chi) \in \mathbb{C}^*$ is a constant with absolute value 1.

This functional equation helps us to find the order of the zero of the L-function at $s = 0$:

$$\mathrm{ord}_{s=0}\,\Lambda(s,\chi) = \mathrm{ord}_{s=0}\,\Gamma_\chi(s) + \mathrm{ord}_{s=0}\,L(s,\chi) = \sum_{v \in S_\infty} \dim V^{G_w} + \mathrm{ord}_{s=0}\,L(s,\chi)$$

$$\mathrm{ord}_{s=0}\,\Lambda(1-s,\bar{\chi}) = \mathrm{ord}_{s=1}\,\Gamma_{\bar{\chi}}(s) + \mathrm{ord}_{s=1}\,L(s,\bar{\chi}) = \mathrm{ord}_{s=1}\,L(s,\bar{\chi}).$$

Hence, when $S = S_\infty$, the order of the zero of L-function at $s = 0$ is

$$\mathrm{ord}_{s=0}L(s,\chi) = -\dim V^G + \sum_{v \in S_\infty} \dim V^{G_w} \tag{3.7}$$

as $\mathrm{ord}_{s=1}L(s,\bar{\chi}) = -\dim V^G$.

3.5 The Stark Regulator

Notation 3.16. *For any finite set of places with $S_\infty \subset S$, we write the Taylor series of Artin L-function at $s = 0$*

$$L_S(s,\chi) = c_S(\chi) s^{r_S(\chi)} + O(s^{r_S(\chi)+1})$$

Now, we will determine the multiplicity $r_S(\chi)$, that is, the order of the zero of $L_S(s,\chi)$ at $s=0$.

Let $Y_{S,K}$ be the finitely generated abelian group with basis S_K. Note that $G = \mathrm{Gal}(K/k)$ acts on S_K by permuting the places w of K lying above the places $v \in S$. In particular, if we choose $v \in S$ and fixed a place $w \in S_K$ lying above v, we write

$$Y_{K,S} = Y = \bigoplus_{v\in S}\left(\bigoplus_{w|v}\mathbb{Z}w\right) \cong \bigoplus_{v\in S}\mathrm{Ind}_{G_w}^{G}\mathbb{Z} \tag{3.8}$$

where the decomposition group G_w acts trivially on $\mathbb{Z}$.

Then, the augmentation map

$$\mathrm{aug}_K : Y \longrightarrow \mathbb{Z} : \sum_{w\in S_K} n_w \cdot w \longmapsto \sum_{w\in S_K} n_w \tag{3.9}$$

is surjective G-module homomorphism with kernel

$$X_{K,S} = X = \left\{\sum_{w\in S_K} n_w \cdot w \in Y : \sum_{w\in S_K} n_w = 0\right\} \tag{3.10}$$

Then, we have the following short exact sequence

$$0 \longrightarrow X \longrightarrow Y \overset{\mathrm{aug}_K}{\longrightarrow} \mathbb{Z} \longrightarrow 0.$$

and tensoring the above exact sequence with $\mathbb{C}$ over $\mathbb{Z}$, we have the short exact sequence of $\mathbb{C}[G]$-modules

$$0 \longrightarrow \mathbb{C}X \longrightarrow \mathbb{C}Y \longrightarrow \mathbb{C} \longrightarrow 0$$

Since $\mathbb{C}[G]$-modules are semi-simple, the above exact sequence splits. In particular, $\mathbb{C}Y = \mathbb{C}X \bigoplus \mathbb{C}$ as $\mathbb{C}[G]$-modules.

If we denote the characters of $\mathbb{C}X$ and $\mathbb{C}Y$ by χ_X and χ_Y, respectively, then we have

$$\chi_Y = \chi_X + \mathbb{1}_G. \tag{3.11}$$

Since Y is the free abelian group generated by the places w lying above the places $v \in S$, we may write the characters χ_Y and χ_X as

$$\chi_Y = \sum_{v \in S} \mathrm{Ind}_{G_w}^G \mathbb{1}_{G_w}$$
$$\chi_X = \sum_{v \in S} \mathrm{Ind}_{G_w}^G \mathbb{1}_{G_w} - \mathbb{1}_G.$$

Note that χ_X and χ_Y takes rational values.

Remark 3.17. *Suppose $k \subset L \subset K$ with L/k Galois. Let $H \subset G$ be the subgroup fixing L and $Y_{K,S}$, $Y_{L,S}$ and $X_{K,S}$, $X_{L,S}$ be defined as in (3.8) and (3.10).Then, we have the natural embedding*

$$Y_{L,S} \hookrightarrow Y_{K,S} : w_L \mapsto \sum_{w|w_L} [w : w_L] w_0 = \sum_{h \in H} h w_0$$

where $[w : w_L]$ is the degree of local extension K_w/L_{w_L} and w_0 is an arbitrary fixed place of K lying above w_L. Then, we find that $X_{L,S} = \mathrm{N}_H X_{K,S}$ where $\mathrm{N}_H = \sum_{h \in H} h \in \mathbb{Z}[G]$. We do not say in general $X_{L,S} = X_{K,S}^H$ however $\mathrm{N}_H X_{K,S}$ has finite index in $X_{K,S}$ then we have $E \otimes_{\mathbb{Z}} X_{L,S} = E \otimes_{\mathbb{Z}} X_{K,S}^H$ for any field E of characteristic zero.

Proposition 3.18. *If χ is the character of $\mathbb{C}[G]$-module with finite $\mathbb{C}$ dimension V, then*

$$\mathrm{ord}_{s=0} L_S(s, \chi) := r_S(\chi) = \sum_{v \in S} \dim V^{G_w} - \dim V^G = \langle \chi, \chi_X \rangle_G$$
$$= \dim_{\mathbb{C}}(\mathrm{Hom}_G(V^*, \mathbb{C}X))$$

where $V^* = \mathrm{Hom}(V, \mathbb{C})$

Proof. Since $\mathrm{Hom}_G(V^*, \mathbb{C}X) \cong (V \otimes \mathbb{C}X)^G$ has the character $\chi \cdot \chi_X$ and χ_X takes only rational values,we have

$$\dim_{\mathbb{C}}(\mathrm{Hom}_G(V^*, \mathbb{C}X)) = \langle \chi \cdot \chi_X, \mathbb{1}_G \rangle_G = \langle \chi, \overline{\chi}_X \rangle_G = \langle \chi, \chi_X \rangle_G$$

So, the last equality holds.

Now, we prove the second equality by using Frobenius Reciprocity

$$\begin{aligned}
\langle \chi, \chi x \rangle_G &= \left\langle \chi, \sum_{v \in S} \mathrm{Ind}_{G_w}^{G} \mathbb{1}_{G_w} - \mathbb{1}_{G_w} \right\rangle_G \\
&= \sum_{v \in S} \left\langle \chi, \mathrm{Ind}_{G_w}^{G} \mathbb{1}_{G_w} \right\rangle_G - \langle \chi, \mathbb{1}_G \rangle_G \\
&= \left\langle \sum_{v \in S} \mathrm{Res}_{G_w}^{G} \chi, \mathbb{1}_{G_w} \right\rangle_G - \dim V^G \\
&= \sum_{v \in S} \dim V^{G_w} - \dim V^G
\end{aligned}$$

It remains to show that $r_S(\chi)$ is equal to the second expression. By using Brauer's theorem 3.12, we can write χ as a $\mathbb{Z}$-linear combination of monomial characters so, we may assume that χ is a monomial character.

If $\chi = \mathbb{1}_G$, then we have $L_S(s, \mathbb{1}_G) = \zeta_{k,S}(s)$. Furthermore, we obtain from Corollary 3.7 that

$$r_S(\chi) = |S| - 1 = \sum_{v \in S} \dim V^{G_w} - \dim V^G$$

since $\dim V^{G_w} = 1$ and $\dim V^G = 1$ when $\chi = \mathbb{1}_G$.

If $\chi \neq \mathbb{1}_G$ and χ is one dimensional, then $V^G = \{0\}$. If $S = S_\infty$ the equality holds as we stated before in (3.7). Now, assume that S is a finite set of places including all infinite places in k. Then we have

$$L_{K/k,S}(s, \chi) = \prod_{v \in S - S_\infty} \det((I - \mathrm{Frob}_w \mathbf{N}v^{-s})|V^{I_w}) \cdot L_{S_\infty}(s, \chi).$$

Now, we claim that

$$\mathrm{ord}_{s=0} \det((I - \mathrm{Frob}_w \mathbf{N}v^{-s})|V^{I_w}) = \dim V^{G_w}.$$

If $\lambda_1, \lambda_2, \ldots, \lambda_r$ is the eigenvalues of Frob_w on V^{I_w} then

$$\det((I - \mathrm{Frob}_w \mathbf{N}v^{-s})|V^{I_w}) = \prod_{i=1}^{r} (1 - \frac{\lambda_i}{\mathbf{N}v^s})$$

and we thus, write

$$\begin{aligned}\mathrm{ord}_{s=0}\det((I-\mathrm{Frob}_w\mathbf{N}v^{-s})|V^{I_w}) &= \mathrm{ord}_{s=0}\prod_{i=1}^{r}(1-\frac{\lambda_i}{\mathbf{N}v^s})\\ &= \#\{i:\lambda_i=1\}\\ &= \dim(V^{I_w})^{\mathrm{Frob}_w=1}\\ &= \dim V^{G_w}.\end{aligned}$$

We, therefore, see that

$$\begin{aligned}\mathrm{ord}_{s=0}L_{K/k,S}(s,\chi) &= \sum_{v\in S-S_\infty}\dim V^{G_w}+\mathrm{ord}_{s=0}L_{S_\infty}(s,\chi)\\ &= |\{v\in S-S_\infty:\chi(G_w)=1\}|+\mathrm{ord}_{s=0}L_{S_\infty}(s,\chi)\\ &= \sum_{v\in S}\dim V^{G_w}\end{aligned}$$

The second equality is because V is one-dimensional and $\dim V^{G_w}$ is 1 when the character χ is trivial on G_w and 0 otherwise. Similarly, the third equality is satisfied because $\mathrm{ord}_{s=0}L_{S_\infty}(s,\chi)=r_{S_\infty}(\chi)=\sum_{v\in S_\infty}\dim V^{G_w}-\dim V^G$ by (3.7) and $\dim V^G=0$ in the case of the non-trivial character χ.

□

Corollary 3.19. *If V is one-dimensional, then the order of vanishing of $L_{K/k,S}(s,\chi)$ is*

$$r_S(\chi)=\begin{cases}|S|-1 & \text{if } \chi=\mathbb{1}_G\\ |\{v\in S:\chi(G_w)=1\}| & \text{if } \chi\neq\mathbb{1}_G\end{cases}.$$

Definition 3.20 (S-units)**.** *Let S be the finite set of primes that includes all infinite primes of k. The set of S-units of the Dedekind ring $\mathcal{O}_{k,S}$ is*

$$U_{k,S}=\mathcal{O}_{k,S}^*=\{x\in k^*:|x|_v=1 \text{ for all } v\notin S\}.$$

We generally use the notation U for S_K-units in K. Now, for the set U of S_K-units we define a $\mathbb{Z}[G]$-module homomorphism

$$\begin{aligned}\lambda=\lambda_K:U &\longrightarrow \mathbb{R}X\\ u &\longmapsto \sum_{w\in S_K}\log|u|_w\cdot w.\end{aligned}$$

This map is well-defined due to the product formula [Jan, I-§5], indeed

$$\sum_{w\in S}\log|u|_w = 0 \;\Leftrightarrow\; \log(\prod_{w\in S}|u|_w) = 0 \;\Leftrightarrow\; \prod_{w\in S}|u|_w = 1.$$

Theorem 3.21 (Unit Theorem)**.** *[Lan, V.-§1] Let K be a number field and S_K be a finite set of places of K including all infinite places. Then $U^{free} = U/U_{tors}$ is a free abelian group on $|S_K|-1$ generators. Equivalently, if $\{u_1,\ldots,u_r\}$ is a basis of a free abelian group U^{free} then a S_K-unit u can be uniquely written as*

$$u = wu_1^{a_1}\cdots u_r^{a_r}$$

for some root of unity $w\in\mu(K)$ and integers a_i.

Furthermore, the kernel of λ is exactly the group of the roots of unity $\mu(K)$ in K and the image of λ is a lattice of full rank $|S_K|-1$. Thus, $U/\mu(K)$ is a free abelian group on $|S_K|-1$ generators and $1\otimes\lambda : \mathbb{R}U \longrightarrow \mathbb{R}X$ is an isomorphism of $\mathbb{R}[G]$-modules, [Lan, V.-§1].

Lemma 3.22. *Suppose that K/k is a finite Galois extension. Recall the embedding $j : X_{k,S} \hookrightarrow X_{K,S}$ shown in Remark 3.17. Then, the diagram*

$$\begin{array}{ccc} U_K & \xrightarrow{\lambda_K} & \mathbb{R}X_{K,S} \\ \uparrow & & \uparrow j \\ U_k & \xrightarrow[\lambda_k]{} & \mathbb{R}X_{k,S} \end{array}$$

commutes.

Proof. If $u\in U_k$, then

$$\begin{aligned}\lambda_K(u) = \sum_{w\in S_K}\log|u|_w w &= \sum_{v\in S}\sum_{\sigma G_w\in G/G_w}\log|u|_v^{[w:v]}\sigma w \\ &= \sum_{v\in S}\sum_{\sigma G_w\in G/G_w}|G_w|\log|u|_v\sigma w \\ &= \sum_{v\in S}\log|u|_v j(v) = j(\lambda_k(u))\end{aligned}$$

since

$$j(v) = \sum_{w|v} [w:v]w = \sum_{w|v} |G_w| w = \sum_{\sigma G_w \in G/G_w} |G_w| \sigma w.$$

□

Since $\lambda : \mathbb{R}U \to \mathbb{R}X$ is an isomorphism, $\mathbb{R}U$ and $\mathbb{R}X$ have the same characters. Since tensoring a $\mathbb{Q}[G]$-module with $\mathbb{R}$ over $\mathbb{Q}$ does not change characters, the characters of $\mathbb{Q}U$ and $\mathbb{Q}X$ are equal. Hence, they are isomorphic as $\mathbb{Q}[G]$-modules, but not canonically. Let $f : \mathbb{Q}X \to \mathbb{Q}U$ be a $\mathbb{Q}[G]$ isomorphism. By extending scalars from $\mathbb{Q}$ to $\mathbb{C}$, we get an isomorphism $f : \mathbb{C}X \to \mathbb{C}U$. Any isomorphism $f : \mathbb{C}X \to \mathbb{C}U$ is said to be defined over $\mathbb{Q}$ if it is obtained as above. Thus, $\lambda \circ f$ gives an automorphism of $\mathbb{C}X$.

Let V be a finite dimensional $\mathbb{C}[G]$-module with a character χ and $V^* = \mathrm{Hom}_{\mathbb{C}[G]}(V, \mathbb{C}[G]) \cong \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Then, $\lambda \circ f$ induces a $\mathbb{C}$-linear automorphism under the functor $\mathrm{Hom}_G(V^*, \_)$

$$(\lambda \circ f)_V : \mathrm{Hom}_G(V^*, \mathbb{C}X) \longrightarrow \mathrm{Hom}_G(V^*, \mathbb{C}X) \qquad (3.12)$$
$$\phi \longmapsto \lambda \circ f \circ \phi.$$

The *Stark regulator* is defined to be the determinant of this automorphism:

$$R(\chi, f) = \det((\lambda \circ f)_V). \qquad (3.13)$$

Remark 3.23. *Let $K = k$ and $S = \{v_0, \ldots, v_r\}$. So, we have the trivial Galois group and character for this trivial extension. For a fixed place $v_0 \in S$, we may write X as*

$$X = \bigoplus_{i=1}^{r} \mathbb{Z}(v_i - v_0).$$

Let $f : X \hookrightarrow U$ be an injection given by $f(v_i - v_0) = u_i$ for $u_i \in U$. Note that the unit $u_i \in U/\mu(k)$ since X is a finitely generated torsion-free abelian group and f is an injection. By complexifying, we get an isomorphism $f : \mathbb{C}X \to \mathbb{C}U$. We also have

$$\mathrm{Hom}_G(V^*, \mathbb{C}X) = \mathrm{Hom}_G(\mathbb{C}, \mathbb{C}X) = \mathbb{C}X.$$

Thus,

$$(\lambda \circ f)_V = \lambda \circ f = \lambda(u_i) = \sum_{v \in S} \log |u_i|_v v = \sum_{j=1}^{r} \log |u_i|_{v_j} (v_j - v_0)$$

and we could write the last equality because $\sum_{j=1}^{r} \log |u_i|_{v_j} v_0 = 0$ *by the product formula. Therefore, the matrix for* $\lambda \circ f$ *with respect to our basis* $\{v_i - v_0 : 1 \leq i \leq r\}$ *of* $\mathbb{C}X$ *and the basis* $\{u_1, \ldots, u_r\}$ *of* $U/\mu(k)$ *is* $(\log |u_i|_{v_j})_{i,j}$. *The Stark regulator* $R(\mathbb{1}, f)$ *and the* S*-regulator* R_S *as defined in Definition 3.5, have the following relation*

$$R(\mathbb{1}, f) = \det(\lambda \circ f) = \det(\log |u_i|_{v_j})_{i,j} = \pm R_S[U : f(X)\mu(k)] = R_S \frac{[U : f(X)]}{e_k}$$

where $e_k = |\mu(k)|$. *Note that* $[U : f(X)\mu(k)]$ *gives us the index of the subgroup of* $U/\mu(k)$ *generated by* u_i. *The last equality comes from the fact that* $f(X) \cap \mu(k) = 0$.

We conclude that the Stark regulator differs from R_S by a factor which is a *rational* number when $\chi = \mathbb{1}$.

3.6 The non-abelian Stark conjecture

Let K/k be a finite Galois extension with $G = \mathrm{Gal}(K/k)$. Let χ be a character of finite dimensional representation of G over $\mathbb{C}$ and let $f : \mathbb{Q}X \to \mathbb{Q}U$ be a $\mathbb{Q}[G]$-module isomorphism.

Conjecture 3.24 (Stark). *Let us define* $A(\chi, f) = \frac{R(\chi, f)}{c(\chi)} \in \mathbb{C}$ *where* $c(\chi)$ *is the leading coefficient of the Taylor series of the Artin* L*-function* $L(s, \chi)$. *Then, for any* $\alpha \in \mathrm{Aut}(\mathbb{C})$

$$A(\chi, f)^\alpha = A(\chi^\alpha, f). \tag{3.14}$$

Equivalently,

$$\begin{cases} A(\chi, f) \in \mathbb{Q}(\chi) \\ A(\chi, f)^\alpha = A(\chi^\alpha, f) \text{ for all } \alpha \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}) \end{cases}. \tag{3.15}$$

These two statements of the conjecture are equivalent. Suppose the statement (3.14) holds. Then, for any $\alpha \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q}(\chi))$ we have $A(\chi, f)^\alpha = A(\chi^\alpha, f) = A(\chi, f)$ and hence $A(\chi, f) \in \mathbb{Q}(\chi)$. Since $\mathrm{Aut}(\mathbb{C}) \to \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$ is a surjection, $A(\chi, f)^\alpha = A(\chi^\alpha, f)$ for all $\alpha \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$. The converse implication is trivial.

Here, $\mathbb{Q}(\chi)$ is the subfield of $\mathbb{C}$ generated by its values of $\chi(\sigma)$, $\sigma \in G$. Note that each $\chi(\sigma)$ is a sum of roots of unity, so that $\mathbb{Q}(\chi)$ is an abelian extension of $\mathbb{Q}$.

Let E be a field of characteristic zero and $\chi : G \longrightarrow E$ be a character of a representation $G \longrightarrow \mathrm{GL}_E(V)$, where V be a finite dimensional vector space over E. Let f be any $E[G]$-module isomorphism $EX \longrightarrow EU$. For any $\alpha \in \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{C})$, we get the $\mathbb{C}[G]$-module $V^\alpha = \mathbb{C} \otimes_{E,\alpha} V$ whose character is $\chi^\alpha = \alpha \circ \chi$. The corresponding Artin L-function of $\alpha \circ \chi$ of $\mathrm{Gal}(K/k)$ is $L(s, \chi^\alpha)$. Also, we define $f^\alpha : \mathbb{C}X \longrightarrow \mathbb{C}U$ by extension of scalars of f by means $(\alpha \otimes 1) \circ f : X \to \mathbb{C}U$ and this map induces an endomorphism $(\lambda \circ f^\alpha)_{V^\alpha}$ of the $\mathbb{C}[G]$-module $\mathrm{Hom}_G((V^\alpha)^*, \mathbb{C}X)$. Note that the determinant of $(\lambda \circ f^\alpha)_{V^\alpha}$ is the Stark regulator $R(\chi^\alpha, f^\alpha)$, which is independent of realization V of χ over E.

Conjecture 3.25 (Deligne)**.** *There exist an element $A(\chi, f) \in E$ such that for all $\alpha : E \hookrightarrow \mathbb{C}$, we have*

$$R(\chi^\alpha, f^\alpha) = A(\chi, f)^\alpha \cdot c(\chi^\alpha).$$

Note that if $E = \mathbb{C}$ then we have $f : \mathbb{C}X \longrightarrow \mathbb{C}U$ and $f^\alpha = f$ for all $\alpha \in \mathrm{Aut}(\mathbb{C})$. As wee see that Conjecture 3.24 is a special case of Conjecture 3.25. Conversely, we will show that Conjecture 3.24 implies Conjecture 3.25 and then, we will conclude that these conjectures are equivalent. This equivalence shows that Stark's conjecture is independent of the choice of f.

3.6.1 Independence of f

Definition 3.26.

Let χ be a character of a $\mathbb{C}[G]$-module V. For each $\mathbb{C}[G]$-endomorphism θ of $\mathbb{C}X$, we define $\delta(\chi, \theta)$ to be the determinant of the automorphism θ_V of $\mathrm{Hom}_G(V^*, \mathbb{C}X)$ induced by θ. In fact, δ is independent of the realization V of χ.

We can write the Stark regulator in terms of δ

$$R(\chi, f) = \delta(\chi, \lambda \circ f)$$

where $\lambda \circ f : \mathbb{C}X \to \mathbb{C}X$ is an automorphism induced by $f : \mathbb{C}X \to \mathbb{C}U$ defined over $\mathbb{Q}$.

Proposition 3.27. *The determinant δ satisfies the following properties:*

(i) For characters $\chi_1,\ \chi_2$ of a finite group $G = \mathrm{Gal}(K/k)$, we have

$$\delta(\chi_1 + \chi_2, \theta) = \delta(\chi_1, \theta) \cdot \delta(\chi_2, \theta).$$

(ii) If H is the subgroup of $G = \mathrm{Gal}(K/k)$ with the character χ and θ is considered as $\mathbb{C}[H]$-module of $\mathbb{C}X$, then we have

$$\delta(\mathrm{Ind}_H^G \chi, \theta) = \delta(\chi, \theta).$$

(iii) Let H be a normal subgroup of $G = \mathrm{Gal}(K/k)$ and $E \subset K$ be its fixed field. If χ is the character of $\mathrm{Gal}(E/k)$ and $\theta|_{(\mathbb{C}X)^H}$ is considered as $\mathbb{C}[G/H]$-module of $(\mathbb{C}X)^H$ then, we have

$$\delta(\mathrm{Infl}\chi, \theta) = \delta(\chi, \theta|_{(\mathbb{C}X)^H}).$$

(iv) For $\mathbb{C}[G]$-endomorphisms $\theta,\ \theta'$ of $\mathbb{C}X$,

$$\delta(\chi, \theta \circ \theta') = \delta(\chi, \theta)\delta(\chi, \theta').$$

(v) For any $\alpha \in \mathrm{Aut}(\mathbb{C})$, θ^α is the $\mathbb{C}[G]$-endomorphism of $\mathbb{C} \otimes_{\mathbb{C},\alpha} \mathbb{C}X$ and we have

$$\delta(\chi, \theta)^\alpha = \delta(\chi^\alpha, \theta^\alpha).$$

Proof.

(i) This is clear, because we may write the representation of the sum of two characters in a block matrix is so the determinant of the block matrix is equal to the multiplications of the determinants of these two matrices.

(ii) For any representation V of the subgroup H of G and $\mathbb{C}[G]$-module $\mathbb{C}X$, there is an isomorphism,

$$\mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{Ind}_H^G V^*, \mathbb{C}X) \cong \mathrm{Hom}_{\mathbb{C}[H]}(V^*, \mathbb{C}X)$$

where on the right hand side $\mathbb{C}X$ is considered as a H-module, [Ser, 7.2, pg.54]. Then the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{C}[H]}(V^*, \mathbb{C}X) & \xrightarrow{(\lambda\circ f)_V} & \mathrm{Hom}_{\mathbb{C}[H]}(V^*, \mathbb{C}X) \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{Ind}_H^G V^*, \mathbb{C}X) & \xrightarrow{(\lambda\circ f)_{\mathrm{Ind}_H^G V}} & \mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{Ind}_H^G V^*, \mathbb{C}X) \end{array}$$

commutes as $\mathrm{Ind}_H^G V^* \cong (\mathrm{Ind}_H^G V)^*$. So, the result follows.

(iii) Let V be a representation of $\mathrm{Gal}(E/k)$ with the character χ. Then, there is an isomorphism

$$\mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{Infl}V^*, \mathbb{C}X) \cong \mathrm{Hom}_{\mathbb{C}[G/H]}(V^*, (\mathbb{C}X)^H).$$

where on the right hand side $\mathbb{C}X$ is considered as a G/H-module, [Ser]. Then, the following diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{C}[G/H]}(V^*, \mathbb{C}X) & \xrightarrow{(\theta|_{(\mathbb{C}X)^H})_V} & \mathrm{Hom}_{\mathbb{C}[G/H]}(V^*, \mathbb{C}X) \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{Infl}V^*, \mathbb{C}X) & \xrightarrow{\theta_{\mathrm{Infl}V}} & \mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{Infl}V^*, \mathbb{C}X) \end{array}$$

commutes. Hence, the result follows.

(iv) This is clear, since $(\theta \circ \theta')_V = \theta_V \circ \theta'_V$.

(v) Let $\alpha \in \mathrm{Aut}\mathbb{C}$. Then $V^\alpha = (\mathbb{C}\otimes_{\mathbb{C},\alpha} V)$ is a realization of χ^α. For the determinant of θ_V on $\mathrm{Hom}_{\mathbb{C}[G]}(V^*, \mathbb{C}X) \cong V \otimes \mathbb{C}X$, the determinant of $\theta^\alpha_{V^\alpha}$ is defined on $\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}\otimes_{\mathbb{C},\alpha} V^*, \mathbb{C}\otimes_{\mathbb{C},\alpha} \mathbb{C}X)$. Then, the following diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}\otimes_{\mathbb{C},\alpha} V^*, \mathbb{C}\otimes_{\mathbb{C},\alpha} \mathbb{C}X) & \xrightarrow{\theta^\alpha_{V^\alpha}} & \mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}\otimes_{\mathbb{C},\alpha} V^*, \mathbb{C}\otimes_{\mathbb{C},\alpha} \mathbb{C}X) \\ \downarrow\cong & & \downarrow\cong \\ \mathbb{C}\otimes_{\mathbb{C},\alpha} \mathrm{Hom}_{\mathbb{C}[G]}(V^*, \mathbb{C}X) & \xrightarrow{(\theta_V)^\alpha} & \mathbb{C}\otimes_{\mathbb{C},\alpha} \mathrm{Hom}_{\mathbb{C}[G]}(V^*, \mathbb{C}X) \end{array}$$

commutes, since

$$V^\alpha \otimes \mathbb{C}X \cong (V \otimes \mathbb{C}X)^\alpha,$$

by sending $(\gamma \otimes v) \otimes x$ to $\gamma \otimes (v \otimes x)$. So, we have the equality of determinants.

□

Proposition 3.28. *If Conjecture 3.25 with $E = \mathbb{C}$ is true for a particular choice of an isomorphism $f_o : \mathbb{C}X \xrightarrow{\sim} \mathbb{C}U$, then it is true for all $f : \mathbb{C}X \xrightarrow{\sim} \mathbb{C}U$. In particular, Conjecture 3.25 implies Conjecture 3.24 with $E = \mathbb{C}$.*

Proof. We have the equality $R(\chi, f) = \delta(\chi, \theta)$ where $\theta = \lambda \circ f$ for a $\mathbb{C}[G]$-module isomorphism $f : \mathbb{C}X \to \mathbb{C}U$. Now, let us take $\theta = f_o^{-1} \circ f : \mathbb{C}X \longrightarrow \mathbb{C}X$. For any complex character χ of G, we have the equalities

$\delta(\chi, \lambda \circ f_o \circ f_o^{-1} \circ f) = \delta(\chi, \lambda \circ f_o)\delta(\chi, f_o^{-1} \circ f)$ by Proposition 3.27, (iv);

$R(\chi, f) = R(\chi, f_o)\delta(\chi, f_o^{-1} \circ f)$ by definition of δ;

$A(\chi, f) = A(\chi, f_o)\delta(\chi, \theta)$.

where $A(\chi, f_o)$ satisfies Conjecture 3.25 for all $\alpha \in \mathrm{Aut}(\mathbb{C})$.
Then, by Proposition 3.27,

$$\begin{aligned} A(\chi, f)^\alpha &= A(\chi, f_o)^\alpha \delta(\chi, \theta)^\alpha \\ &= \frac{R(\chi^\alpha, f_o^\alpha)}{c(\chi^\alpha)} \cdot \delta(\chi^\alpha, \theta^\alpha) \\ &= \frac{\delta(\chi^\alpha, \lambda \circ f_o^\alpha)}{c(\chi^\alpha)} \cdot \delta(\chi^\alpha, (f_o^{-1} \circ f)^\alpha) \\ &= \frac{\delta(\chi^\alpha, \lambda \circ f_o^\alpha \circ (f_o^{-1})^\alpha \circ f^\alpha)}{c(\chi^\alpha)} \\ &= \frac{\delta(\chi^\alpha, \lambda \circ f^\alpha)}{c(\chi^\alpha)} = \frac{R(\chi^\alpha, f^\alpha)}{c(\chi^\alpha)} \end{aligned}$$

So, for any $f : \mathbb{C}X \to \mathbb{C}U$, Conjecture 3.25 holds.
Since $f^\alpha = f$ with $E = \mathbb{C}$, we have

$$A(\chi, f)^\alpha = \frac{R(\chi^\alpha, f^\alpha)}{c(\chi^\alpha)} = \frac{R(\chi^\alpha, f)}{c(\chi^\alpha)} = A(\chi^\alpha, f)$$

So, Conjecture 3.25 implies Conjecture 3.24. □

Proposition 3.29. *Conjecture 3.25 and Conjecture 3.24 are equivalent.*

Proof. For $E = \mathbb{C}$ we have shown that Conjecture 3.25 implies Conjecture 3.24. Now, we will show the converse implication so assume that Conjecture 3.24 holds. Let E be finitely generated over $\mathbb{Q}$ and $f : EX \to EU$ be an isomorphism of $E[G]$-modules. Let V be a representation of $G = \mathrm{Gal}(K/k)$ over E with a character χ. Let us fix an injection $\alpha : E \hookrightarrow \mathbb{C}$ and consider $f^\alpha : \mathbb{C}X \to \mathbb{C}U$. By our assumption that Stark Conjecture holds and by Proposition prop Deligne to Stark, we may assume that Conjecture 3.25 holds for f^α. Thus, $A(\chi^\alpha, f^\alpha) \in \mathbb{C}$ and for any $\gamma \in \mathrm{Aut}(\mathbb{C})$ we have

$$R((\chi^\alpha)^\gamma, (f^\alpha)^\gamma) = A(\chi^\alpha, f^\alpha)^\gamma c((\chi^\alpha)^\gamma).$$

If γ is any automorphism fixing $\alpha(E) \subset \mathbb{C}$, then $A(\chi^\alpha, f^\alpha)^\gamma = A(\chi^\alpha, f^\alpha)$. By our assumption $A(\chi, f)^\alpha = A(\chi^\alpha, f^\alpha)$ thus, $A(\chi, f)^\alpha$ is fixed by γ and $A(\chi, f) \in E$. Moreover, for any injection $\beta : E \hookrightarrow \mathbb{C}$ we may find $\gamma \in \mathrm{Aut}(\mathbb{C})$ such that $\gamma \circ \alpha = \beta$. Then

$$A(\chi, f)^\beta = A(\chi^\alpha, f^\alpha)^\gamma = \frac{R((\chi^\alpha)^\gamma, (f^\alpha)^\gamma)}{c((\chi^\alpha)^\gamma)} = \frac{R(\chi^\beta, f^\beta)}{c(\chi^\beta)}$$

as desired. So, Conjecture 3.25 holds. □

3.6.2 Independence of S

In this section we prove that the conjecture 3.24 is independent the choice of the set of places S. First, we will state some properties of $A(\chi, f)$.

Proposition 3.30. *With the notations in Conjecture 3.24, $A(\chi, f)$ satisfies the following properties.*

(i) For characters $\chi_1,\ \chi_2$ of a finite group $G = \mathrm{Gal}(K/k)$, we have

$$A(\chi_1 + \chi_2, f) = A(\chi_1, f) \cdot A(\chi_2, f).$$

(ii) If H is the subgroup of $G = \mathrm{Gal}(K/k)$ with the character χ then we have

$$A(\mathrm{Ind}_H^G \chi, f) = A(\chi, f).$$

(iii) Let H be a normal subgroup of $G = \mathrm{Gal}(K/k)$ and $E \subset K$ be its fixed field. If χ is the character of $\mathrm{Gal}(E/k)$ then we have

$$A(\mathrm{Infl}\chi, f) = A(\chi, f|_{(\mathbb{C}X)^H}).$$

(iv) $\overline{A(\chi, f)} = A(\overline{\chi}, f)$.

Proof. The first three assertions follow from Proposition 3.11 and Proposition 3.27. Furthermore, the inclusion $X_E \hookrightarrow X_K = X$ induces the equality $(\mathbb{C}X)^H = \mathbb{C}X_E$ and so we might have stated the third equality as $A(\mathrm{Infl}\chi, f) = A(\chi, f|_{\mathbb{C}X_E})$.For the last assertion, the statement (v) of Proposition 3.27 gives us $\overline{R(\chi, f)} = R(\overline{\chi}, f)$ since $\lambda \circ f$ is defined over $\mathbb{R}$. By the fact that complex conjugation is continuous, we say that $\overline{L_S(\overline{s}, \chi)} = L_S(s, \overline{\chi})$ and hence $c(\overline{\chi}) = \overline{c(\chi)}$. □

Lemma 3.31.

(i) If Stark Conjecture is true for every finite Galois extension $K/\mathbb{Q}$, then it is true in general;

(ii) If Stark Conjecture is true for every abelian extensions K/k, then it is true for every finite Galois extensions.

Proof.

(i) Assume that K/k is Galois with a character χ and L is the Galois closure of $\mathbb{Q}$ over K with the Galois group $G = \mathrm{Gal}(L/\mathbb{Q})$. Assume that S is the finite set of places containing the archimedean places in K. Then, let us restrict the places of S onto $\mathbb{Q}$ and take the places of L lying over them. For any $\mathbb{C}[G]$-module isomorphism $f : \mathbb{C}X_L \longrightarrow \mathbb{C}U_L$ over $\mathbb{Q}$, we have

$$A(\mathrm{Ind}_{G'}^{G}\mathrm{Infl}\chi, f) = A(\mathrm{Infl}\chi, f) = A(\chi, f|_{\mathbb{C}X_K})$$

by Proposition 3.30. So, if Stark Conjecture is true for the Galois extension $L/\mathbb{Q}$, then it is true for any Galois extension K/k.

(ii) Let L/k be an arbitrary Galois extension. By Brauer Theorem 3.12, we can write the character χ of $\mathrm{Gal}(L/k)$ as

$$\chi = \sum n_i \mathrm{Ind}_{H_i}^{G} \chi_i$$

where $n_i \in \mathbb{Z}$; χ_i is the one-dimensional character of $H_i \subset G$. Then we can write

$$A(\chi, f) = \prod_i A(\chi_i, f)^{n_i}$$

by additivity of $A(\chi, f)$. So, if Stark's Conjecture holds for one dimensional characters, it is true for all χ. So, we can take χ to be a one dimensional character: and we can also take $L^{\ker \chi}/k$ which is an abelian extension in L/k which satisfies Stark Conjecture as we assumed. Then, by the inflation property of $A(\chi, f)$, Stark Conjecture also holds for L/k.

□

Remark 3.32. *Stark Conjecture is true for the trivial character* $\mathbb{1}$*. Remark 3.23 says that when* $K = k$*,*

$$R(\mathbb{1}, f) = R_S \frac{[U : f(X)]}{e_k} \in \mathbb{Q}$$

and by Class Number Formula at $s = 0$*,*

$$c(\mathbb{1}) = -\frac{h_S R_S}{e_k}.$$

Therefore,

$$A(\mathbb{1}, f) == \frac{[U : f(X)]}{h_S}$$

which is also in $\mathbb{Q}$*. By Inflation property of* $A(\chi, f)$*, Stark Conjecture is true for the trivial character* $\mathbb{1}$ *for any extension of* k*.*

Proposition 3.33. *Stark's Conjecture is independent of the set of places* S*.*

Proof. It will be enough to prove this statement for one dimensional characters due to Lemma 3.31. Now, assume that S is a finite set of places with $S \supset S_\infty$ and let us define $S' = S \cup \{\mathfrak{p}\}$ for a place $\mathfrak{p} \notin S$. The conjecture is true for the set

S if and only if it is true for S'. Let $f : \mathbb{C}X_S \to \mathbb{C}U_{S_K}$ and $f' : \mathbb{C}X_{S'} \to \mathbb{C}U_{S'_K}$ be $\mathbb{C}[G]$-module isomorphisms. We know that $f'|_{\mathbb{C}X_S} = f$ by semi-simplicity and $r'(\chi) = r(\chi) + \dim V^{G_\wp}$ where $\wp \in S'_K$ is the place lying above $\mathfrak{p}$ and $G_\wp \subset G$ is the decomposition group of $\wp$. We may assume that K/k is abelian since we may replace K with $K^{\mathrm{Ker}\chi}$ due to the inflation property in Proposition 3.30. Since we assumed that K/k is abelian, we may write $G_\mathfrak{p}$ instead of $G_\wp$ because $G_\wp$ does not depend on the choice of $\wp$.

Now, we divide the proof into two cases;

Case 1: $\chi(G_\mathfrak{p}) \neq 1$

Since V is one dimensional, $\dim V^{G_\mathfrak{p}} = 0$, and so $r'(\chi) = r(\chi)$.

We can decompose $X_{S'}$ as

$$X_{S'} \cong X \bigoplus (\bigoplus_{\wp|\mathfrak{p}} \mathbb{Z}\wp)$$

where $(\bigoplus_{\wp|\mathfrak{p}} \mathbb{Z}\wp) = \mathbb{Z}[G/G_\mathfrak{p}]$. Then,

$$\mathrm{Hom}_G(V^*, \mathbb{C}X_{S'}) \cong \mathrm{Hom}_G(V^*, \mathbb{C}X) \oplus \mathrm{Hom}_G(V^*, \mathbb{C}[G/G_\mathfrak{p}]).$$

However, $\mathrm{Hom}_G(V^*, \mathbb{C}[G/G_\mathfrak{p}]) = 0$ since G acts on $G_\mathfrak{p}$ trivially. So,

$$\mathrm{Hom}_G(V^*, \mathbb{C}X_{S'}) \cong \mathrm{Hom}_G(V^*, \mathbb{C}X).$$

Thus, $R_S(\chi, f) = R_{S'}(\chi, f)$ since $f'|_{\mathbb{C}X_S} = f$. We also have

$$L_{S'}(s, \chi) = L_S(s, \chi) \cdot (1 - \chi(\mathrm{Frob}_\mathfrak{p})\mathbf{N}\mathfrak{p}^{-s}).$$

If $\chi(I_\mathfrak{p}) \neq 1$, then $c_{S'}(\chi) = c_S(\chi)$ and so $A_{S'}(\chi, f) = A_S(\chi, f)$. If $\chi(I_\mathfrak{p}) = 1$, then $c_{S'}(\chi) = c_S(\chi)(1 - \chi(\mathrm{Frob}_\mathfrak{p}))$. Therefore,

$$A_{S'}(\chi, f) = A_S(\chi, f)(1 - \chi(\mathrm{Frob}_\mathfrak{p})).$$

and

$$\begin{aligned}(A_{S'}(\chi, f))^\alpha &= (A_S(\chi, f))^\alpha(1 - \chi(\mathrm{Frob}_\mathfrak{p}))^\alpha \\ &= A_S(\chi^\alpha, f)(1 - \chi^\alpha(\mathrm{Frob}_\mathfrak{p})) \\ &= A_{S'}(\chi^\alpha, f).\end{aligned}$$

The result follows.

Case 2: $\chi(G_{\mathfrak{p}}) = 1$

Since $\chi(G_{\mathfrak{p}}) = 1$ and χ is faithful character then $\mathrm{Frob}_{\mathfrak{p}} = 1$ and $\dim V^{G_{\mathfrak{p}}} = 1$. Then, we have $r^{'}(\chi) = r(\chi) + 1$ and $L_{S'}(s,\chi) = L_S(s,\chi)\cdot(1-\mathbf{N}\mathfrak{p}^{-s})$. Thus, $c_{S'}(\chi) = c_S(\chi)\cdot\log(\mathbf{N}\mathfrak{p})$.

Furthermore, we have the following exact sequence

$$0 \longrightarrow \mathcal{O}^*_{K,S_K} \longrightarrow \mathcal{O}^*_{K,S'_K} \longrightarrow \bigoplus_{\wp|\mathfrak{p}} \mathbb{Z}\wp$$

and by tensoring with $\mathbb{Q}$, we get

$$0 \longrightarrow \mathbb{Q}\mathcal{O}^*_{K,S_K} \longrightarrow \mathbb{Q}\mathcal{O}^*_{K,S'_K} \longrightarrow \bigoplus_{\wp|\mathfrak{p}} \mathbb{Q}\wp \longrightarrow 0$$

$(\bigoplus_{\wp|\mathfrak{p}} \mathbb{Q}\wp) = \mathbb{Q}[G]$ since $G_{\mathfrak{p}} = 1$.

Suppose that $\pi\mathcal{O}_{K,S_K} = \wp^h$ for some positive $h \in \mathbb{Z}$ and $\pi \in K$. By semi simplicity, this sequence splits and we get

$$\mathbb{Q}\mathcal{O}^*_{K,S'_K} \cong \mathbb{Q}\mathcal{O}^*_{K,S_K} \oplus \mathbb{Q}[G]\pi.$$

By similar reasoning we find that

$$\mathbb{Q}X_{S'} \cong \mathbb{Q}X_S \bigoplus \mathbb{Q}[G]x$$

where $x = \wp - \frac{1}{|G|}\sum_{\sigma\in G}\sigma w$ for an archimedean place $w \in K$.

Now, fix an $\mathbb{Q}[G]$-module isomorphism $f : \mathbb{Q}X_S \tilde{\longrightarrow} \mathbb{Q}\mathcal{O}^*_{K,S_K}$. Let $j : \mathbb{Q}[G]x \longrightarrow \mathbb{Q}[G]\pi$ be the $\mathbb{Q}[G]$-module isomorphism which sends x to π. Since Stark's conjecture is independent from the choice of $f^{'}$ we may assume that $f' = f \oplus \iota : \mathbb{Q}X_{S'} \longrightarrow \mathbb{Q}\mathcal{O}^*_{K,S'_K}$.

$$\lambda' \circ (f\oplus\iota)(x^\sigma) = \lambda'(\pi^\sigma) = \sum_{w\in S'_K} |\pi^\sigma|_w \cdot w = \log|\pi^\sigma|_{\sigma\wp}\sigma\wp$$

since π's are all units for $\mathcal{O}_{K,S_K}$, namely $\sum_{w\in S_K}|\pi^\sigma|_w \cdot w = 0$.

Then,

$$[\lambda\circ(f\oplus\iota)] = \left(\begin{array}{c|c} [\lambda\circ f] & * \\ \hline 0 & h\log(\mathbf{N}\mathfrak{p})I_{|G|} \end{array}\right)$$

where $I_{|G|}$ is $|G| \times |G|$ identity matrix. Finally we have,

$$[\lambda \circ (f \oplus \iota)]_V = \left(\begin{array}{c|c} [\lambda \circ f]_V & * \\ \hline 0 & h \log(\mathbf{N}\mathfrak{p}) \end{array}\right)$$

Thus $R_{S'}(\chi, f') = h \log(\mathbf{N}\mathfrak{p}) R_S(\chi, f)$, so

$$\frac{A_{S'}(\chi, f')}{A_S(\chi, f)} = h \in \mathbb{Q}$$

and so which is independent of χ. ☐

Remark 3.34. *With the assumptions and notations in Remark 3.23, we have shown that*

$$R(\mathbb{1}, f) = R_S \frac{[U : f(X)]}{e_k}.$$

By definition of $A(\mathbb{1}, f)$,

$$A(\mathbb{1}, f) = \pm \frac{R(\mathbb{1}, f)}{c(\mathbb{1})} = \frac{[U : f(X)]}{h_S} \in \mathbb{Q}$$

where h_S is the class number of the ring $\mathcal{O}_S$ of S integers of k. To conclude this result we assumed $K = k$, but the inflation property of $A(\chi, f)$ we can generalize the result. So, Stark Conjecture is true for the trivial character.

3.7 The Cases $r(\chi) = 0$ and $r(\chi) = 1$

In this section, we analyze the cases $r(\chi) = 0$ and $r(\chi) = 1$. In the first case, we reduce the extension to the abelian case and then we use Seigel theorem 3.8, to indicate Conjecture 3.24 is true when $r(\chi) = 0$. In the second case, at first we express the truth of Conjecture 3.24 for $r(\chi) = 1$.

3.7.1 The Case $r(\chi) = 0$

We have proved that Conjecture 3.24 is independent from the choice of the set of places. Furthermore, if $r_S(\chi) = 0$ then this implies $r_{S_\infty}(\chi) = 0$ and thus, we can assume $S = S_\infty$. When $r(\chi) = 0$, the conjecture becomes $L(0, \chi^\alpha) = L(0, \chi)^\alpha$ for all

$\alpha \in \mathrm{Aut}\mathbb{C}$ as $R(\chi, f) = 1$ and $L_S(0, \chi) \neq 0$. To reduce the conjecture to the abelian case, we will need a stronger version of Brauer's Theorem stated by Serre [[CoL], App.].

Remark 3.35. *[Das]The commutator group $G' = [G, G]$ of G is normal subgroup of G. Furthermore, G/G' is abelian and $[G : G']$ gives us the order of the group of 1-dimensional representations of G.*

Lemma 3.36. *Let G be a finite group with center C, and χ be an iredducible character of G. The restriction of χ to C is a multiple of a 1-dimensional character ψ_χ of C, and we can write*

$$\chi = \sum_i n_i \mathrm{Ind}_{H_i}^G \chi_i$$

where for each i, H_i is a subgroup of G containing C, χ_i is the character of H_i whose restriction to C is ψ_χ and $n_i \in \mathbb{Z}$.

Proof. [Das] For any subgroup H and the center C of G, the commutator group $[CH, CH]$ is equal to $[H, H]$ since any element in C commutes with the elements of H. Denote $\widehat{CH}$ (resp., $\widehat{H}$) as the group of 1-dimensional representations of CH (respectively, H). By Remark 3.35, we have

$$|\widehat{CH}| = [CH : [H, H]] = [CH : H][H : [H, H]] = [CH : H]|\widehat{H}|.$$

Therefore, for each element of $\widehat{H}$ there are exactly $[CH : H]$ elements in $\widehat{CH}$ which restrict to it. Suppose that $\chi_{i,H} \in \widehat{CH}$ for $i = 1, .., [CH : H]$ have restriction equal to χ_H. Frobenius Reciprocity provides that

$$\left\langle \mathrm{Ind}_H^{CH} \chi_H, \chi_{i,H} \right\rangle_{CH} = \langle \chi_H, \chi_{i,H}|_H \rangle_H = \langle \chi_H, \chi_H \rangle_H = 1.$$

Therefore, as $\chi_{i,H}$ appears as a summand of $\mathrm{Ind}_H^{CH} \chi_H$. As $\mathrm{Ind}_H^{CH} \chi_H$ is a $[CH : H]$-dimensional character, we find

$$\mathrm{Ind}_H^{CH} \chi_H = \sum_1^{[CH:H]} \chi_{i,H}$$

and

$$\text{Ind}_H^G \chi_H = \sum_{1}^{[CH:H]} \text{Ind}_{CH}^G \chi_{i,H}. \tag{3.16}$$

By combining (3.16) with Brauer's Theorem, we can write

$$\chi = \sum_i n_i \text{Ind}_{H_i}^G \chi_i$$

where $n_i \in \mathbb{Z}$, $H_i \supset C$ and χ_i is one dimensional.

Now, let V be a realization of χ. Consider V as a representation of $C \subset G$ and W be an irreducible component of the decomposition of V. Since W is irreducible and C is abelian, W must be one dimensional. Consider the subspace W'

$$W' = \sum_{g \in G} gW \subset V.$$

Since W' is G-stable and V is irreducible, we have $W' = V$. Thus, as a representation of C we have $V = \sum gW$ and $\chi|_C = m_\chi \psi_\chi$ where $m_\chi \in \mathbb{Z}$ and ψ_χ to be the character of W. Furthermore, for an irreducible character χ, Frobenius Reciprocity provides that

$$\left\langle \chi, \text{Ind}_{H_i}^G \chi_i \right\rangle_G = \langle \chi|_{H_i}, \chi_i \rangle_{H_i}. \tag{3.17}$$

Since χ is irreducible and with (3.17), χ is a summand of $\text{Ind}_{H_i}^G \chi_i$ only if $\chi|_{H_i}$ is a summand of χ_i. This occurs only if $\chi|_C = m_\chi \psi_\chi$ with $\psi_\chi = \chi_i|_C$. Thus, we have

$$\chi = \sum_{\chi_i|_C = \psi_\chi} n_i \text{Ind}_{H_i}^G \chi_i.$$

Thus, this gives the desired result. □

Theorem 3.37. *Conjecture 3.24 is true if* $r(\chi) = 0$.

Proof. Since Conjecture 3.24 is independent from the choice of S, we can take $S = S_\infty$. Furthermore, by replacing K with $K^{\text{Ker}\chi}$, we can assume that χ is a faithful character and the extension is abelian. We can also take χ to be an irreducible character since $r(\chi)$ behaves additively under direct sums. In particular, if χ is written as a linear combination of irreducible characters χ_i, then $r(\chi) = 0$ if and only if $r(\chi_i) = 0$.

We can also assume $\chi \neq \mathbb{1}$ because we have proved that Conjecture 3.24 is true for $\chi = \mathbb{1}$ as $A(\mathbb{1}, f)$ is a rational number. The zero rank of L-function of $\chi \neq \mathbb{1}$ implies $\dim V^{G_w} = 0$ for each archimedean place w of K. In particular, k is totally real and K is totally complex field. If $G_w = \{1, \tau_w\}$ for a complex place w of K, $\tau_w^2 = 1$ hence τ_w acts as -1 on V. Since V is faithful representation, all τ_w are equal to the same element, say τ, in G. This implies that K is totally imaginary quadratic field of a totally real field K^τ. Furthermore, $G_w = \{1, \tau\}$ is contained in the center of C since for any $\sigma \in G$ we have

$$\sigma\tau\sigma^{-1} = \sigma\tau_w\sigma^{-1} = \tau_{\sigma w} = \tau.$$

By Lemma 3.36 we have

$$\chi = \sum_{H_i} n_i \mathrm{Ind}_{H_i}^G \chi_i$$

where $n_i \in \mathbb{Z}$, $C \subset H_i$, χ_i is 1-dimensional character of H_i and $\chi_i|_C = \psi_\chi$ with the definition of ψ_χ as in Lemma 3.36. Thus,

$$\chi_i(\tau) = \psi_\chi(\tau) = \frac{\chi(\tau)}{\chi(1)} = -1$$

and we obtain

$$L_{K/K^{H_i}}(0, \chi_i) \neq 0$$

since $K^{H_i} \subset K^\tau$ is totally real. We therefore have

$$L_{K/k}(0, \chi) = \prod_i L_{K/K^{H_i}}(0, \chi_i)^{n_i}.$$

We may take $K = K^{\mathrm{Ker}\chi_i}$ to reduce the problem about general L-functions to the problem about L-functions corresponding to abelian extensions with faithful characters χ_i. Furthermore, it will be enough to prove the proposition for $L_{K/K^{H_i}}(0, \chi_i)$ thus, we can assume $k = K^{H_i}$ and $\chi = \chi_i$. Then, we can write this abelian L-function as a linear combination of partial zeta function

$$L_{K/k}(s, \chi) = \sum_{\sigma \in G} \chi(\sigma)\zeta(0, \sigma).$$

Siegel's theorem 3.8 implies $\zeta(0,\sigma)$ is a rational number for all $\sigma \in G$, thus $L(0,\chi)^\alpha = L(0,\chi^\alpha)$ for any $\alpha \in \mathrm{Aut}\mathbb{C}$. □

3.7.2 The Case $r(\chi) = 1$

Since the conjecture is independent of the choice of S we do not make any assumption for S. We may suppose χ is irreducible since we can write χ as a direct sum of irreducible characters with $r(\theta_i) = 0$ and an irreducible character with $r(\theta) = 1$. As we have shown in (3.11), the values of χ_X are rational integers and so $r(\chi) = r(\chi^\alpha)$ for all $\alpha \in$Aut$\mathbb{C}$.

Definition 3.38. *Let χ be an irreducible character. Then the idempotent element of $\mathbb{C}[G]$ is*

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma^{-1})\sigma$$

and acts as projection onto χ-component in the canonical decomposition of any $\mathbb{C}[G]$-module, (see [Ser, §2]).

If $\Gamma = \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$ and $a \in \mathbb{Q}(\chi)$ we define

$$\pi(a,\chi) = \sum_{\gamma \in \Gamma} a^\gamma L'_S(0,\chi^\gamma) e_{\overline{\chi}^\gamma} \ \in \ \mathbb{C}[G]. \tag{3.18}$$

If $r(\chi) > 1$ then $L'_S(0,\chi) = 0$ and if $r(\chi) = 0$ then $e_\chi \mathbb{Q}X = 0$ as $\langle \chi, \chi_X \rangle = r(\chi) = 0$. In particular, $\mathbb{Q}X$ has no subrepresentations isomorphic to χ when $r(\chi) = 0$. So, $\pi(a,\chi)\mathbb{Q}X = 0$ unless $r(\chi) = 1$. Note that multiplication by $\pi(a,\chi)$ is a G-homomorphism $\mathbb{C}X \to \mathbb{C}X$. Suppose that $\sum_{w \in S_K} a_w w$ is the image of an element of X under $\pi(a,\chi)$. Thus, the coefficients a_w are $\mathbb{Q}(\chi)$-linear combinations of $L'_S(0,\chi^\gamma)$ for $\gamma \in \Gamma$. Therefore, if there is a unit $\epsilon \in U$ such that $\lambda(\epsilon) = \sum_{w \in S_K} \log|\epsilon|_w w = \sum_{w \in S_K} a_w w$ then the logarithms of the valuation of ϵ are equal to these linear combinations of $L'_S(0,\chi^\gamma)$. Thus, we try to understand the intersection between $\pi(a,\chi)\mathbb{Q}X$ and $\lambda\mathbb{Q}U$.

Let V be a realization of χ over $\mathbb{C}$. Then by Theorem 2.2, there is an irreducible representation V' of G over $\mathbb{Q}(\chi)$ with character $\chi' = m\chi$ where m is the Schur index of χ' over $\mathbb{Q}(\chi)$. Thus,

$$\langle \chi', \chi_X \rangle = \langle m\chi, \chi_X \rangle = m \langle \chi, \chi_X \rangle = m$$

and this implies that V' (respectively, $\mathbb{C} \otimes_{\mathbb{Q}(\chi)} V' \cong mV$) can be seen as a subrepresentation of $\mathbb{Q}(\chi)X$ (respectively, $\mathbb{C} \otimes_{\mathbb{Q}(\chi)} \mathbb{Q}(\chi)X \cong \mathbb{C}X$). Hence,

$$m \leq \langle \chi, \chi_X \rangle = r(\chi) = 1.$$

Thus, χ is realizable over $\mathbb{Q}(\chi)$. Furthermore, Theorem 2.2 shows that

$$\varphi = \mathrm{Tr}_{\mathbb{Q}(\chi)/\mathbb{Q}}(\chi) = \sum_{\gamma \in \Gamma} \chi^\gamma$$

is the character of an irreducible representation W of G over $\mathbb{Q}$. This implies that W is the simple $\mathbb{Q}[G]$-module with character φ. By the similar arguements to the proof above that $m = 1$ we can show that $\langle \varphi, \chi_X \rangle = 1$, thus the multiplicity of W in $\mathbb{Q}X$ is 1. Then, we write X_W (respectively, U_W) is the unique $\mathbb{Q}[G]$-submodule of $\mathbb{Q}X$ (respectively $\mathbb{Q}W$) isomorphic to W.

Proposition 3.39. *Let $a \in \mathbb{Q}(\chi)^*$ and let χ be the irreducible character of G over $\mathbb{C}$ with $r(\chi) = 1$. The following statements are equivalent:*

(i) $\pi(a, \chi)\mathbb{Q}X \cap \lambda\mathbb{Q}U \neq \{0\}$ *in* $\mathbb{C}X$*;*

(ii) $\pi(a, \chi)\mathbb{Q}X = \lambda U_W$ *in* $\mathbb{C}X$*;*

(iii) the Stark conjecture is true for χ*.*

Proof. *(i)* ⇔ *(ii)* The canonical decomposition of $\mathbb{Q}X$ is

$$\mathbb{Q}X \cong X_W \oplus \bigoplus_i W_i$$

and so we can write $\mathbb{C}X$ as

$$\mathbb{C}X \cong \bigoplus_{\gamma \in \Gamma} V^\gamma \oplus (W_i)_\mathbb{C}$$

where $(W_i)_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Q}} (\bigoplus_i W_i)$. Since $(W_i)_{\mathbb{C}}$ does not have any subrepresentation isomorphic to V^γ, e_{χ^γ} annihilates each $(W_i)_{\mathbb{C}}$. So, $\pi(a,\chi)\mathbb{Q}X = \pi(a,\chi)X_W$ and which is $\{0\}$ or a simple $\mathbb{Q}[G]$-module isomorphic to W. Thus, if $\pi(a,\chi)\mathbb{Q}X \cap \lambda\mathbb{Q}U \neq \{0\}$ and since λ is an isomorphism, this implies that $\lambda^{-1}(\pi(a,\chi)\mathbb{Q}X) \cap \mathbb{Q}U \neq \{0\}$ and it must be a simple $\mathbb{Q}[G]$-submodule U_W of $\mathbb{Q}U$. Hence, $\pi(a,\chi)\mathbb{Q}X = \lambda U_W$ and so equivalence of (i) and (ii) is proven.

(ii) ⇒ *(iii)* Let us decompose $\mathbb{Q}X$ as $\mathbb{Q}X \cong \mathbb{Q}X_W \oplus \mathbb{Q}X'$ and $\mathbb{Q}U$ as $\mathbb{Q}U \cong \mathbb{Q}U_W \oplus \mathbb{Q}U'$. Since $\mathbb{Q}X \cong \mathbb{Q}U$ and $X_W \cong U_W$ then $X' \cong U'$ by semisipmlicity of $\mathbb{Q}[G]$. Similarly, by extending scalars from $\mathbb{Q}$ to $\mathbb{C}$, we have

$$\mathbb{C}X \cong \mathbb{C}X_W \oplus \mathbb{C}X' \quad \text{and} \quad \mathbb{C}U \cong \mathbb{C}U_W \oplus \mathbb{C}U'.$$

Define

$$f(a,\chi) := \begin{cases} \lambda^{-1}(\pi(a,\chi)) & \text{on } \mathbb{C}X_W \\ 1 \otimes f' & \text{on } \mathbb{C}X' \end{cases}$$

where $f' : X' \to U'$ is any $\mathbb{Q}[G]$-isomorphism and $\lambda^{-1} : \mathbb{C}X_W \to \mathbb{C}U_W$ is an isomorphism and $\pi(a,\chi)$ represents multiplication by $\pi(a,\chi)$ which is also a G-homomorphism. Since $\dim_{\mathbb{C}}(\mathrm{Hom}_G((V^\gamma)^*, \mathbb{C}X)) = \langle \chi^\gamma, \chi_X \rangle = r(\chi^\gamma) = r(\chi) = 1$, $\mathrm{Hom}_G((V^\gamma)^*, \mathbb{C}X)$ is one dimensional and $\lambda \circ f(a,\chi)$ acts on $\mathrm{Hom}_G((V^\gamma)^*, \mathbb{C}X)$ as $\pi(a,\chi)$ acts on $(V^\gamma)^*$; namely by multiplication by the complex number $a^\gamma L_S'(0,\chi^\gamma)$. Therefore, we have

$$A(\chi^\gamma, f(a,\chi)) = \frac{R(\chi^\gamma, f)}{c(\chi^\gamma)} = \frac{a^\gamma L_S'(0,\chi^\gamma)}{L_S'(0,\chi^\gamma)} = a^\gamma \tag{3.19}$$

for all $\gamma \in \Gamma$.

Thus, if (ii) is true, then we have $\lambda^{-1}(\pi(a,\chi))\mathbb{Q}X = \mathbb{Q}U_W$ and $f(a,\chi)$ is a $\mathbb{Q}[G]$-isomorphism. The formula (3.19) is then the expression of the Stark conjecture for χ.

(iii) ⇒ *(i)* Suppose that the Stark Conjecture 3.24 is true. Then by Theorem 3.29, and the formula (3.19) we deduce that, for all $\alpha,\ \beta \in \mathrm{Aut}(\mathbb{C})$,

$$A(\chi^\alpha, f(a,\chi)^\beta) = A(\chi^{\beta^{-1}\alpha}, f(a,\chi))^\beta = (a^{\beta^{-1}\alpha})^\beta = a^\alpha = A(\chi^\alpha, f(a,\chi)).$$

Therefore, the determinant of $\lambda \circ f(a,\chi)^\beta$ and $\lambda \circ f(a,\chi)$ on the one dimensional space $\mathrm{Hom}_G((V^\alpha)^*, \mathbb{C}X)$ are equal. Thus, these automorphisms are equal on the subrepresentation of $\mathbb{C}X$ isomorphic to V^α and since this is true for all α, they are equal on $\mathbb{C}X_W$. Since $\lambda \circ f(a,\chi)^\beta$ and $\lambda \circ f(a,\chi)$ coincide on $\mathbb{C}X_W$ for all β and since f' is defined on over $\mathbb{Q}$ on X' we deduce that $f(a,\chi)$ is defined over $\mathbb{Q}$. Therefore,

$$\pi(a,\chi)\mathbb{Q}X = \pi(a,\chi)\mathbb{Q}X_W \subset \lambda U_W$$

and since $\pi(a,\chi)$ act as multiplication by $a \cdot L'_S(0,\chi) \neq 0$ on $\mathbb{Q}X$, this implies that (i) is true. □

Let Ψ be a finite set of irreducible characters of G satisfies $r(\chi) = 1$ such that

- $\mathbb{1}_G \notin \Psi$
- if $\chi \in \Psi$ then, for all $\alpha \in \mathrm{Aut}(\mathbb{C})$, $\chi^\alpha \in \Psi$.

Assume Stark Conjecture 3.24 is true for all $\chi \in \Psi$. Proposition 3.39 implies that

$$\sum_{\chi\in\Psi} a_\chi L'_S(0,\chi) e_{\overline{\chi}} X \subset \lambda\mathbb{Q}U = \mathbb{Q}\lambda U \tag{3.20}$$

for any family $(a_\chi)_{\chi\in\Psi}$ of elements of $\mathbb{C}$ satisfies $a_{\chi^\alpha} = (a_\chi)^\alpha$ for all $\chi \in \Psi$ and $\alpha \in \mathrm{Aut}(\mathbb{C})$. Since $\mathbb{1}_G \notin \Psi$, the trivial representation is annihilated by each $e_{\overline{\chi}}$ with $\chi \in \Psi$. Therefore, we can replace X by Y in the inclusion (3.20). Given any place $w \in S_K$ lying above the place $v \in S$, we can write

$$m \sum_{\chi\in\Psi} a_\chi L'_S(0,\chi) e_{\overline{\chi}} w = \lambda(\epsilon)$$

where m is a nonzero integer and ϵ is a unit in U. This ϵ is called a *Stark* unit. When the integer m is fixed, the unit ϵ is uniquely determined up to a root of unity.

In the next section, we will state the refinement of the Stark Conjecture for an abelian L-function with simple zeros by using the Stark units.

4 The rank one abelian refined Stark conjecture

4.1 Notations

From now, we suppose that K/k is an abelian extension with Galois group G.

Lemma 4.1. *Suppose that T is a finite set of primes of k, containing*

(i) all archimedean places

(ii) all places ramified in K/k

(iii) all finite places dividing $e = e_K = |\mu(K)|$

Then we have the following equalities:

(1)

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K)) = \{a \in \mathbb{Z}[G] : \zeta^a = 1, \forall \zeta \in \mu(K)\} =< \mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p} >_{\mathfrak{p} \notin T}$$

where $\mathrm{Frob}_{\mathfrak{p}} = \mathrm{Frob}_{\mathfrak{p}}(K/k)$,

(2)

$$e = \mathrm{gcd}_{\mathfrak{p} \notin T}(\mathbf{N}\mathfrak{p} - 1).$$

Proof. If $\zeta \in \mu(K)$, then for $\mathfrak{p} \notin T$ and a place $\wp$ of K lying over $\mathfrak{p}$, we have $\zeta^{\mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p}} \equiv 1 (\mathrm{mod} \wp)$. Thus, $\zeta^{\mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p}}$ is a solution of the equation $x^e - 1 = 0$ in $\mathcal{O}_K/\wp$. Since $\wp$ does not divide e, by Hensel's Lemma we can lift this uniquely to a solution in $K_\wp$. Thus, $\zeta^{\mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p}} = 1$ and $< \mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p} >\subset \mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))$.
Now, suppose that $a \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))$. By Chebotarev Density Theorem (see [Neu, V.-§6]), any $\sigma \in G$ is a Frobenius element $\mathrm{Frob}_{\mathfrak{p}} = \mathrm{Frob}_{\mathfrak{p}}(K/k)$ for some $\mathfrak{p} \notin T$. Thus, any $a \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))$ can be written as

$$a = \sum_{\mathfrak{p} \notin T} a_{\mathfrak{p}} (\mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p}) + n, \quad n \in \mathbb{Z}. \tag{4.1}$$

Hence, $n \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))$ if only n is divisible by e. To finish the proof of (1), it suffices to show $e \in< \mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p} >_{\mathfrak{p} \notin T}$ so it is enough to show (2).

If $\mathfrak{p}$ splits completely in K/k, then $\mathrm{Frob}_{\mathfrak{p}} = 1$. So, we have $\mathbf{N}\mathfrak{p} - 1 \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))$ and $e|\mathbf{N}\mathfrak{p} - 1$.

Now, let us take

$$d = \gcd_{\substack{\mathfrak{p} \notin T \\ \mathrm{Frob}_{\mathfrak{p}}(K/k)=1}} (\mathbf{N}\mathfrak{p} - 1).$$

Let L be the abelian extension of K such that $L = K(\mu_d)$ and σ be an element of $\mathrm{Gal}(L/K) \subset \mathrm{Gal}(L/k)$ such that $\sigma = \mathrm{Frob}_{\mathfrak{p}}(L/k)$ for some $\mathfrak{p} \notin T$. Note that the restriction of σ to K/k is $\mathrm{Frob}_{\mathfrak{p}}(K/k)$. If $\mathrm{Frob}_{\mathfrak{p}}(K/k) = 1$ then $\zeta_d^{\mathrm{Frob}_{\mathfrak{p}}(L/k)} \equiv \zeta_d^{\mathbf{N}\mathfrak{p}} \equiv \zeta_d (\mathrm{mod} \wp)$. So, $\sigma = \mathrm{Frob}_{\mathfrak{p}}(L/k) = 1$. Since σ was an arbitrary element in $\mathrm{Gal}(L/k)$, we conclude that $\zeta_d \in K$ and $d|e$.

To summary, we have shown that $d|e$ and $e|\mathbf{N}\mathfrak{p} - 1$ hence this proves the second assertion. For the first assertion, we know that $e \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))$ and we have shown that $e = \gcd_{\mathfrak{p} \notin T}(\mathbf{N}\mathfrak{p} - 1)$, so $e \in < \mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p} >_{\mathfrak{p} \notin T}$. We conclude that $e|n$, and so $a \in < \mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p} >_{\mathfrak{p} \notin T}$. Hence, $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K)) \subset < \mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p} >$. So, the result follows. □

Let k^{ab} be the abelian closure of k containing K. For all intermediate field L, we define the canonical map $\sim : \mathcal{O}_L^* \longrightarrow \mathbb{Q}\mathcal{O}_L^*$ by $x \mapsto \widetilde{x} = 1 \otimes x$ where $\mathcal{O}_L^*$ is the group of S_L-units in L. The kernel of this homomorphism is the group of roots of unities $\mu(L)$ of L. Now, define

$$U_{K/k}^{ab} = \{\epsilon \in \mathcal{O}_K^* : K(\epsilon^{1/e})/k \text{ is abelian}\}.$$

Proposition 4.2. *For any $\sigma \in G$, choose $n_\sigma \in \mathbb{Z}$ such that $\zeta^\sigma = \zeta^{n_\sigma}$ for each $\zeta \in \mu(K)$. Then for $u \in \mathbb{Q}\mathcal{O}_K^*$, the following statements are equivalent:*

(i) There exists $\epsilon \in U_{K/k}^{ab}$ such that $\widetilde{\epsilon} = eu$,

(ii) There exists an abelian extension L/k such that $u \in \widetilde{L}$; equivalently, $u \in \widetilde{k^{ab}}$,

(iii) There exists a finite set of places T of k containing the archimedean places and the ramified places in K/k such that for $\mathfrak{p} \notin T$ there exists $\epsilon_{\mathfrak{p}} \in \mathcal{O}_K^*$ with $\epsilon_{\mathfrak{p}} \equiv 1 (\mathrm{mod}\ \mathfrak{p}\mathcal{O}_K)$ and $\widetilde{\epsilon_{\mathfrak{p}}} = u^{\mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p}}$,

(iv) For each $\sigma,\ \sigma' \in G$, there exists $\epsilon \in \mathcal{O}_K^*$ and $\alpha_\sigma,\ \alpha_{\sigma'} \in \mathcal{O}_K^*$ such that

$$\widetilde{\epsilon} = eu$$
$$\alpha_{\sigma'}^{\sigma - n_\sigma} = \alpha_{\sigma}^{\sigma - n_{\sigma'}}$$
$$\alpha_\sigma^e = \epsilon^{\sigma - n_\sigma}.$$

Proof. $(i) \implies (ii)$: Take $L = K(\epsilon^{1/e})/k$. Since $\epsilon \in U_{K/k}^{ab}$, then L/k is abelian. Hence, $u = \frac{1}{e}\widetilde{\epsilon} = \widetilde{\epsilon^{1/e}} \in \widetilde{\mathcal{O}_L^*}$.

$(ii) \implies (iii)$: Let $L \subset k^{ab}$ and T be a finite set of places of k satisfying the conditions of Lemma 4.1 for KL/k. For $\mathfrak{p} \notin T$, define $\epsilon_\mathfrak{p} = \alpha^{\mathrm{Frob}_\mathfrak{p} - \mathbf{N}\mathfrak{p}}$ where $\alpha \in \mathcal{O}_L^*$ and $\mathrm{Frob}_\mathfrak{p} = \mathrm{Frob}_p(KL/k)$. By assumption (ii) we can write $u = \widetilde{\alpha}$. For any $\tau \in \mathrm{Gal}(KL/K)$, we have $\epsilon_\mathfrak{p}^{\tau-1} = (\alpha^{\mathrm{Frob}_\mathfrak{p} - \mathbf{N}\mathfrak{p}})^{\tau-1} = (\alpha^{\tau-1})^{\mathrm{Frob}_\mathfrak{p} - \mathbf{N}\mathfrak{p}}$ and there exists $d \in \mathbb{Z}$ such that $\alpha^d \in \mathcal{O}_K^*$. Since $\tau \in \mathrm{Gal}(KL/K)$, τ fixes α^d. Hence $(\alpha^{\tau-1})^d = (\alpha^d)^{\tau-1} = 1$, and so $\alpha^{\tau-1}$ is KL root of unity. By Lemma 4.1, we have $\epsilon_\mathfrak{p}^{\tau-1} = 1$ and this implies that $\epsilon_\mathfrak{p} \in \mathcal{O}_K^*$. Furthermore, $\epsilon_\mathfrak{p} = \alpha^{\mathrm{Frob}_\mathfrak{p} - \mathbf{N}\mathfrak{p}} \equiv 1 (\mathrm{mod}\ \mathfrak{p}\mathcal{O}_{KL})$ and since $\epsilon_\mathfrak{p} \in K$, we have $\epsilon_\mathfrak{p} \equiv 1 (\mathrm{mod}\ \mathfrak{p}\mathcal{O}_K)$. Lastly, we have $\widetilde{\epsilon_\mathfrak{p}} = \widetilde{\alpha^{\mathrm{Frob}_\mathfrak{p} - \mathbf{N}\mathfrak{p}}} = u^{\mathrm{Frob}_\mathfrak{p} - \mathbf{N}\mathfrak{p}}$.

$(iii) \implies (iv)$: Let us consider T is defined as in Lemma 4.1 for K/k. Now, for $\mathfrak{p},\ \mathfrak{q} \notin T$ observe that

$$\widetilde{\epsilon_\mathfrak{p}^{(\mathrm{Frob}_\mathfrak{q} - \mathbf{N}\mathfrak{q})}} = \widetilde{\epsilon_\mathfrak{q}^{(\mathrm{Frob}_\mathfrak{p} - \mathbf{N}\mathfrak{p})}} (= u^{(\mathrm{Frob}_\mathfrak{p} - \mathbf{N}\mathfrak{p})(\mathrm{Frob}_\mathfrak{q} - \mathbf{N}\mathfrak{q})}).$$

Thus, we have

$$\zeta_{\mathfrak{p},\mathfrak{q}} \cdot \epsilon_\mathfrak{q}^{(\mathrm{Frob}_\mathfrak{p} - \mathbf{N}\mathfrak{p})} = \epsilon_\mathfrak{p}^{(\mathrm{Frob}_\mathfrak{q} - \mathbf{N}\mathfrak{q})}$$

for a root of unity $\zeta_{\mathfrak{p},\mathfrak{q}}$ which is congruent to 1 modulo $\mathfrak{p}$ and $\mathfrak{q}$ since $\epsilon_\mathfrak{p}$ and $\epsilon_\mathfrak{q}$ are congruent to 1 modulo $\mathfrak{p}$ and $\mathfrak{q}$ respectively, thus by Hensel's Lemma $\zeta_{\mathfrak{p},\mathfrak{q}} = 1$. In particular, we have

$$\epsilon_\mathfrak{p}^{(\mathrm{Frob}_\mathfrak{q} - \mathbf{N}\mathfrak{q})} = \epsilon_\mathfrak{q}^{(\mathrm{Frob}_\mathfrak{p} - \mathbf{N}\mathfrak{p})}.$$

Lemma 4.1 implies that we can find integers $b_\mathfrak{p}$ and $b_{\mathfrak{p},\sigma}$ for $\mathfrak{p} \notin T$ and $\sigma \in \mathrm{Gal}(K/k)$

such that

$$e = \sum_{\mathfrak{p} \notin T} b_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p})$$

$$\sigma - n_\sigma = \sum_{\mathfrak{p} \notin T} b_{\mathfrak{p},\sigma}(\mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p}).$$

Now, define

$$\epsilon = \prod_{\mathfrak{p} \notin T} \epsilon_{\mathfrak{p}}^{b_{\mathfrak{p}}} \text{ and } \alpha_\sigma = \prod_{\mathfrak{p} \notin T} \epsilon_{\mathfrak{p}}^{b_{\mathfrak{p},\sigma}}.$$

Then, we have

$$\widetilde{\epsilon} = \sum_{\mathfrak{p} \notin T} \widetilde{\epsilon_{\mathfrak{p}}}^{b_{\mathfrak{p}}} = \sum_{\mathfrak{p} \notin T} u^{b_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p})} = eu.$$

Furthermore,

$$\alpha_\sigma^e = \prod_{\mathfrak{p} \notin T} \epsilon_{\mathfrak{p}}^{eb_{\mathfrak{p},\sigma}} = \prod_{\mathfrak{p},\mathfrak{q} \notin T} \epsilon_{\mathfrak{p}}^{b_{\mathfrak{p},\sigma} b_{\mathfrak{q}}(\mathrm{Frob}_{\mathfrak{q}} - \mathbf{N}\mathfrak{q})} = \prod_{\mathfrak{p},\mathfrak{q} \notin T} \epsilon_{\mathfrak{q}}^{b_{\mathfrak{p},\sigma} b_{\mathfrak{q}}(\mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p})} = \epsilon^{\sigma - n_\sigma}.$$

Finally, for $\sigma, \sigma' \in \mathrm{Gal}(K/k)$ we have

$$\alpha_\sigma^{\sigma' - n_{\sigma'}} = \prod_{\mathfrak{p},\mathfrak{q} \notin T} \epsilon_{\mathfrak{p}}^{b_{\mathfrak{p},\sigma} b_{\mathfrak{q},\sigma'}(\mathrm{Frob}_{\mathfrak{q}} - \mathbf{N}\mathfrak{q})} = \prod_{\mathfrak{p},\mathfrak{q} \notin T} \epsilon_{\mathfrak{q}}^{b_{\mathfrak{p},\sigma} b_{\mathfrak{q},\sigma'}(\mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p})} = \alpha_{\sigma'}^{\sigma - n_\sigma}.$$

(iv) $\Longrightarrow$ *(i)*: Assume that the equalities in (iv) hold, then we need to show that $K(\epsilon^{1/e})/k$ is abelian. Let suppose η be the e^{th} root of ϵ. Let τ be an automorphism of $\overline{k}$ over k, so we get $\tau|_K \in \mathrm{Gal}(K/k)$ and write $\alpha_\tau \in \mathcal{O}_K^*$ and $n_\tau \in \mathbb{Z}$ for $\tau|_K$. Thus, we have

$$(\eta^\tau)^e = \epsilon^\tau = \epsilon^{n_\tau} \alpha_\tau^e = (\eta^{n_\tau} \alpha_\tau)^e.$$

Hence $\eta^\tau = \zeta \eta^{n_\tau} \alpha_\tau$ for some e^{th} root of unity $\zeta \in \mu(K)$. This implies that $\eta^\tau \in K(\eta)$ for all τ, so $K(\eta)/k$ is a Galois extension. Since $\zeta^\tau = \zeta^{n_\tau}$, we have

$$(\eta^{\tau - n_\tau})^{\tau' - n_{\tau'}} = (\zeta \alpha_\tau)^{\tau' - n_{\tau'}} = (\alpha_\tau)^{\tau' - n_{\tau'}} = (\alpha_{\tau'})^{\tau - n_\tau} = (\eta^{\tau' - n_{\tau'}})^{\tau - n_\tau}$$

for $\tau, \tau' \in \mathrm{Gal}(\overline{k}/k)$. Thus, $\eta^{\tau\tau'} = \eta^{\tau'\tau}$ and $K(\epsilon^{1/e})/k$ is abelian. □

Corollary 4.3. *In $\mathbb{Q}\mathcal{O}_{k^{ab}}^*$, we have $\mathbb{Q}\mathcal{O}_K^* \cap \widetilde{\mathcal{O}_{k^{ab}}^*} = \frac{1}{e}\widetilde{U_{K/k}^{ab}}$.*

Proof. Let ϵ be an element in $U^{ab}_{K/k}$. Then, this implies that $\epsilon \in \mathcal{O}^*_K$ and $\frac{1}{e}\widetilde{\epsilon} \in \mathbb{Q}\mathcal{O}^*_K$. Also, we have $\frac{1}{e}\widetilde{\epsilon} = \widetilde{\epsilon^{1/e}} \in \widetilde{\mathcal{O}^*_{k^{ab}}}$. Thus, $\frac{1}{e}\widetilde{U^{ab}_{K/k}} \subset \mathbb{Q}\mathcal{O}^*_K \cap \widetilde{\mathcal{O}^*_{k^{ab}}}$. Conversely, let us consider $u \in \mathbb{Q}\mathcal{O}^*_K$. If u has an image in $\mathbb{Q}\mathcal{O}^*_{k^{ab}}$ satisfying $u = \widetilde{\eta} \in \widetilde{\mathcal{O}^*_{k^{ab}}}$ for $\eta \in \mathcal{O}^*_{k^{ab}}$ then the second statement of Proposition 4.2 holds with $L = k(\eta)$. Then, by the first statement of the same proposition $eu \in \widetilde{U^{ab}_{K/k}}$. Thus, $u \in \frac{1}{e}\widetilde{U^{ab}_{K/k}}$. ☐

Corollary 4.4. *If L is an extension of k contained in K and $u \in \frac{1}{e}\widetilde{U^{ab}_{K/k}}$ in $\mathbb{Q}\mathcal{O}^*_K$, then $\mathrm{N}_{K/L}u \in \frac{1}{e_L}\widetilde{U^{ab}_{L/k}}$ in $\mathbb{Q}\mathcal{O}^*_L$.*

Proof. Let u be an element in $\frac{1}{e}\widetilde{U^{ab}_{K/k}}$ such that $u = \frac{1}{e}\widetilde{\epsilon} \Rightarrow eu = \widetilde{\epsilon}$ for $\epsilon \in U^{ab}_{K/k}$. Thus, u satisfies the first condition of Proposition 4.2. Now, consider a finite set T of places as defined in (iii) of Proposition 4.2. Then there exists $\epsilon_{\mathfrak{p}} \in \mathcal{O}^*_K$ with $\epsilon_{\mathfrak{p}} \equiv 1 (\mathrm{mod} \mathfrak{p}\mathcal{O}_K)$ and $\widetilde{\epsilon_{\mathfrak{p}}} = u^{\mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p}}$ where $\mathrm{Frob}_{\mathfrak{p}} = \mathrm{Frob}_{\mathfrak{p}}(K/k)$. Now, let us define $\epsilon^{'}_{\mathfrak{p}} = \mathrm{N}_{K/L}\epsilon_{\mathfrak{p}} \in \mathcal{O}^*_L$. Since $\epsilon^{\sigma}_{\mathfrak{p}} \equiv 1 (\mathrm{mod} \mathfrak{p}\mathcal{O}_K)$ for all $\sigma \in \mathrm{Gal}(K/L)$, we have

$$\epsilon^{'}_{\mathfrak{p}} = \mathrm{N}_{K/L}\epsilon_{\mathfrak{p}} = \prod_{\sigma} \sigma(\epsilon_{\mathfrak{p}}) = \epsilon_{\mathfrak{p}}^{\sum \sigma} \equiv 1 (\mathrm{mod} \mathfrak{p}\mathcal{O}_K)$$

$\epsilon^{'}_{\mathfrak{p}} \equiv 1 (\mathrm{mod} \mathfrak{p}\mathcal{O}_K)$ and this gives $\epsilon^{'}_{\mathfrak{p}} \equiv 1 (\mathrm{mod} \mathfrak{p}\mathcal{O}_L)$. Hence for $\sigma \in \mathrm{Gal}(K/L)$,

$$\widetilde{\epsilon^{'}_{\mathfrak{p}}} = \widetilde{\epsilon_{\mathfrak{p}}}^{\sum \sigma} = (u^{\mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p}})^{\sum \sigma} = (u^{\sum \sigma})^{\mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p}} = (N_{K/L}u)^{\mathrm{Frob}_{\mathfrak{p}} - \mathbf{N}\mathfrak{p}}$$

Since the condition (iii) of Proposition 4.2 satisfied for $N_{K/L}u \in \mathbb{Q}\mathcal{O}^*_L$, we conclude from (i) of Proposition 4.2 that $N_{K/L}u \in \frac{1}{e_L}\widetilde{U^{ab}_{L/k}}$. ☐

Now, we will make some definitions to state the rank one abelian Stark conjecture.

Definition 4.5. *(i) We define the Stickelberger element*

$$\Theta_S(s) = \Theta_{K/k,S}(s) = \sum_{\chi \in \widehat{G}} L_S(s, \chi) e_{\overline{\chi}}$$

which we view as a meromorphic function on $\mathbb{C}$ with values in $\mathbb{C}[G]$.

(ii) For every finite place $\mathfrak{p}$ of k, we define

$$\mathrm{F}_{\mathfrak{p}} = \frac{1}{|I_{\mathfrak{p}}|} \sum_{\tau \in \mathrm{Frob}_{\mathfrak{p}}} \tau^{-1} \in \mathbb{Q}[G]$$

where $\mathrm{Frob}_{\mathfrak{p}}$ *is the coset of $I_{\mathfrak{p}}$ in $G_{\mathfrak{p}}$ corresponding to Frobenius automorphism.*

Proposition 4.6. *The function Θ satisfies the following properties:*

(i) If $s \in \mathbb{C}$ and χ is a character of G, then we have

$$\chi(\Theta_S(s)) = L_S(s, \overline{\chi}).$$

(ii) For Re$(s) > 1$, we have

$$\Theta_S(s) = \prod_{\mathfrak{p} \notin S} (1 - \mathrm{F}_{\mathfrak{p}} \mathbf{N}\mathfrak{p}^{-s})^{-1}.$$

(iii) If S contains all ramified places of K/k, then for Re$(s) > 1$, we have

$$\Theta_S(s) = \sum_{(\mathfrak{U}, S)=1} \mathbf{N}\mathfrak{U}^{-s} \sigma_{\mathfrak{U}}^{-1} = \sum_{\sigma \in G} \zeta(s, \sigma) \sigma^{-1}$$

where $\zeta(s, \sigma)$ is the partial zeta function of $\sigma \in G$ as defined in (3.2).

Proof. (i) The definition of e_χ and the orthogonality relations of irreducible characters provide that

$$\chi(e_\psi) = \begin{cases} 0 & \text{if } \chi \neq \psi \\ 1 & \text{if } \chi = \psi \end{cases}.$$

So, the first assertion follows.

(ii)

$$\chi(\mathrm{F}_{\mathfrak{p}}) = \begin{cases} \chi(\mathrm{Frob}_{\mathfrak{p}}^{-1}) = \overline{\chi}(\mathrm{Frob}_{\mathfrak{p}}) & \text{if } \chi(I_{\mathfrak{p}}) = 1 \\ \frac{1}{|I_{\mathfrak{p}}|} \sum_{\tau \in \mathrm{Frob}_{\mathfrak{p}}} \chi(\tau^{-1}) = 0 & \text{otherwise} \end{cases}.$$

Hence, for Re$(s) > 1$

$$\begin{aligned} \chi\Big(\prod_{\mathfrak{p} \notin S} (1 - \mathrm{F}_{\mathfrak{p}} \mathbf{N}\mathfrak{p}^{-s})^{-1}\Big) = \prod_{\mathfrak{p} \notin S} (1 - \chi(\mathrm{F}_{\mathfrak{p}}) \mathbf{N}\mathfrak{p}^{-s})^{-1} &= \prod_{\mathfrak{p} \notin S} (1 - \overline{\chi}(\mathrm{Frob}_{\mathfrak{p}}) \mathbf{N}\mathfrak{p}^{-s})^{-1} \\ &= L_S(s, \overline{\chi}) \\ &= \chi(\Theta_S(s)). \end{aligned}$$

Note that $\widehat{G}$ form a basis for the space $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[G], \mathbb{C})$. For an element $\rho \in \mathbb{C}[G]$, we have the action $e_\chi \rho = \chi(\rho) e_\chi$ and $\mathbb{C}[G] = \oplus \mathbb{C} e_\chi$. It follows that for $\rho, \rho^{'} \in \mathbb{C}[G]$, $\chi(\rho) = \chi(\rho^{'})$ if and only if $\rho = \rho^{'}$. So, we have

$$\Theta_S(s) = \prod_{\mathfrak{p} \notin S} (1 - \mathrm{F}_{\mathfrak{p}} \mathbf{N}\mathfrak{p}^{-s})^{-1}.$$

(iii) Since S contains all ramified places and for unramified $\mathfrak{p}$, we have $\mathrm{F}_{\mathfrak{p}} = \mathrm{Frob}_{\mathfrak{p}}$. Then, by (ii) and the definition of $L_S(s, \chi)$ with respect to the partial zeta function we get the last assertion.

□

Corollary 4.7. *For $s \in \mathbb{C}$ and $\mathfrak{p} \notin S$ we have*

(i) $\Theta_{S \cup \{\mathfrak{p}\}}(s) = \Theta_S(s)(1 - \mathrm{F}_{\mathfrak{p}} \mathbf{N}\mathfrak{p}^{-s})$,

(ii) $\Theta^{'}_{S \cup \{\mathfrak{p}\}}(0) = \Theta^{'}_S(0)(1 - \mathrm{F}_{\mathfrak{p}}) + \log \mathbf{N}\mathfrak{p} \cdot \mathrm{F}_{\mathfrak{p}} \cdot \Theta_S(0)$.

Proof. These two assertions directly come from Proposition 4.6, (ii). □

Definition 4.8. *Let H be a subgroup of G and K' be the fixed field of H. Then there is a natural homomorphism*

$$\pi : \mathbb{C}[G] \to \mathbb{C}[G/H]$$

which is a projection from G to G/H. Also, any $\rho \in \mathbb{C}[G]$ gives by multiplication an endomorphism of the free $\mathbb{C}[H]$-module $\mathbb{C}[G]$. The determinant of this endomorphism is called the norm of $\rho \in \mathbb{C}[G]$. In particular, the homomorphism $\mathbb{C}[G] \to \mathrm{End}_H \mathbb{C}[G] : \rho \mapsto M_\rho$ induces the norm $\mathrm{N}(\rho) := \det(M_\rho)$.

Lemma 4.9. *Let $\widehat{\chi}$ be a character of G induced by the character $\chi \in \widehat{H}$. Then, $\widehat{\chi}$ can be written as a sum of characters $\{\psi\}$ in $\widehat{G}$ restrict to χ on H, namely*

$$\widehat{\chi} = \sum_{\substack{\psi \in \widehat{G} \\ \psi|_H = \chi}} \psi.$$

Then, we have

$$\chi \circ \mathrm{N} = \prod_{\substack{\psi \in \widehat{G} \\ \psi|_H = \chi}} \psi.$$

Proof. The first assertion directly comes from Frobenius Reciprocity,

$$\langle \widehat{\chi}, \psi \rangle_G = \langle \chi, \psi|_H \rangle_H$$

which is equal to 1 if $\psi|_H = \chi$, otherwise 0 valued.

For $\mu \in \widehat{H}$, let us fix $\widehat{\mu} \in \widehat{G}$ with $\widehat{\mu}|_H = \mu$. If $\phi \in \widehat{G/H}$ then we may write

$$\mathbb{C}[H] = \bigoplus_{\mu} \mathbb{C} e_{\mu,H} \cong \bigoplus_{\mu \in \widehat{H}} \mathbb{C} e_{\widehat{\mu}\phi}$$

and

$$\mathbb{C}[G] \cong \bigoplus_{\phi \in \widehat{G/H}} (\bigoplus_{\mu \in \widehat{H}} \mathbb{C} e_{\widehat{\mu}\phi}).$$

For $\rho \in \mathbb{C}[G]$ and fixed ϕ we have

$$\rho \sum_{\mu} \alpha_\phi e_{\widehat{\mu}\phi} = \sum_{\mu} \widehat{\mu}\phi(\rho) \alpha_\phi e_{\widehat{\mu}\phi} = \left(\sum_{\mu} \widehat{\mu}\phi(\rho) e_{\mu,H} \right) \left(\sum_{\mu} \alpha_\phi e_{\widehat{\mu}\phi} \right).$$

So

$$\mathrm{N}(\rho) := \det(M_\rho) = \prod_{\phi \in \widehat{G/H}} \left(\sum_{\mu} \widehat{\mu}\phi(\rho) e_{\mu,H} \right) = \sum_{\mu} \left(\prod_{\phi \in \widehat{G/H}} \widehat{\mu}\phi(\rho) \right) e_{\mu,H}.$$

Thus, we have

$$\chi \circ \mathrm{N}(\rho) = \prod_{\phi \in \widehat{G/H}} \widehat{\chi}\phi(\rho) = \prod_{\substack{\psi \in \widehat{G} \\ \psi|_H = \chi}} \psi(\rho).$$

□

Proposition 4.10. *With the above notations, if $k \subset K' \subset K$ with Galois group $H = \mathrm{Gal}(K'/k)$ and $G = \mathrm{Gal}(K/k)$. Denote $S_{K'}$ as the set of places of K' lying above the set S. Then, we have*

(i) $\Theta_{K'/k,S}(s) = \pi \Theta_{K/k,S}(s)$

(ii) $\Theta_{K/K',S_{K'}}(s) = \mathrm{N}\Theta_{K/k,S}(s)$.

Proof. (i) For $\chi \in \widehat{G/H}$, then by Proposition 4.6, (ii) and Proposition 3.11 we have

$$\begin{aligned}\chi(\Theta_{K'/k,S}(s)) = L_{K'/k,S}(s,\overline{\chi}) &= L_{K/k,S}(s,\mathrm{Infl}(\overline{\chi}))\\ &= \mathrm{Infl}\chi(\Theta_{K/k,S}(s))\\ &= \chi\circ\pi(\Theta_{K/k,S}(s))\\ &= \chi(\pi(\Theta_{K/k,S}(s))).\end{aligned}$$

The assertion follows from Remark 2.3.

(ii) Let $\chi \in \widehat{H}$, and let the notation be as in 4.9. Then by 4.6-(ii), 3.11, and 4.9 we have

$$\begin{aligned}\chi(\Theta_{K/K',S_{K'}}(s)) = L_{K/K',S_{K'}}(s,\overline{\chi}) &= L_{K/k,S}(s,\mathrm{Ind}\overline{\chi})\\ &= \prod_{\substack{\psi\in\widehat{G}\\ \psi|_H=\chi}} L_{K/k,S}(s,\overline{\psi})\\ &= \prod_{\substack{\psi\in\widehat{G}\\ \psi|_H=\chi}} \psi(\Theta_{K/k,S}(s))\\ &= \chi(\mathrm{N}(\Theta_{K/k,S}(s))).\end{aligned}$$

Thus, the result follows from Remark 2.3.

□

Notation 4.11. *For any place v of k, we define N_{G_v} as follows*

$$\mathrm{N}_{G_v} := \sum_{\sigma\in G_v}\sigma \in \mathbb{Z}[G]$$

where G_v is the decomposition group of v.

Proposition 4.12.

(i) If $|S| \geq 2$, for $v \in S$, we have $\mathrm{N}_{G_v}\cdot\Theta_S(0) = 0$.

(ii) If $|S| \geq 3$, for $v, v' \in S$ with $v \neq v'$ we have $\mathrm{N}_{G_v}\cdot NG_{v'}\cdot\Theta_S{}'(0) = 0$.

Proof.

(i) If $\chi(G_v) \neq 1$, then $\chi(\mathrm{N}_{G_v}) = 0$.
If $\chi(G_v) = 1$ with $\chi \neq \mathbb{1}$, then $\chi(\Theta_S(0)) = L(0, \overline{\chi}) = 0$.
If $\chi(G_v) = 1$ with $\chi = \mathbb{1}$, then $\chi(\Theta_S(0)) = \zeta(0) = 0$ as $|S| \geq 2$.
In each cases we get $\mathrm{N}_{G_v} \cdot \Theta_S(0) = 0$.

(ii) We only need to consider the case $\chi(G_v) = \chi(G_{v'}) = 1$ with $\chi \neq \mathbb{1}$. In this case $r_S(\chi) \geq 2$ so this leads to $L'_S(0, \overline{\chi}) = 0$. The other cases satisfied by (i) so the second assertion follows.

□

Remark 4.13.

(i) If $|S| \geq 2$ then $\Theta_S(0)Y = 0$ where Y is defined as in (3.8).

(ii) We always have $\Theta_S(0)X = 0$ where X is defined as in (3.10).

Proof.

(i) Let v be a place in k and $w \in S_K$ lying above v. Then, $\mathrm{N}_{G_v} w = |G_v| w$ as $\sigma \in G_v$, and so fixes w. Also, $\Theta_S(0)w = \Theta_S(0)\frac{\mathrm{N}_{G_v}}{|G_v|}w = 0$ by Proposition 4.12. Hence, $\Theta_S(0)Y = 0$.

(ii) If $|S| \geq 2$ then we are done by (i) as X is a submodule of Y.
If $|S| = v$, X is generated by $w - w^\sigma$ for some $w|v$ and $\sigma \in G$. Then

$$\Theta_S(0)w - w^\sigma = \Theta_S(0)\frac{\mathrm{N}_{G_v}}{|G_v|}(w - w^\sigma) = \left(\frac{\Theta_S(0)\mathrm{N}_{G_v}}{|G_v|}\right) \cdot e_{\mathbb{1}} \cdot (w - w^\sigma) = 0$$

since $e_{\mathbb{1}} \cdot (w - w^\sigma) = 0$. Thus, $\Theta_S(0)X = 0$.

□

4.2 The statement of the conjecture St$(K/k, S)$

The first form of the conjecture, stated by John Tate in [Tat], will be the reformulation of the second one which is the orginal statement given by Harold Stark.

We keep the notations of the subsection 5.1.

Conjecture 4.14 (First Form). *Suppose that K/k is an abelian extension, and S is the set of places of k which satisfies the following conditions:*

(i) S contains all archimedean places in k and non-archimedean places which ramify in K,

(ii) S contains at least one place which splits completely in K,

(iii) $|S| \geq 2$.

If S satisfies the above conditions, then we have

$$\Theta'_S(0)X_K \subset \frac{1}{e}\lambda(U^{ab}_{K/k}),$$

or equivalently,

$$\lambda^{-1}(\Theta'_S(0)X_K) \subset \widetilde{U_{k^{ab}}}$$

where X and λ are as defined in (3.10) and (3.20), respectively.

Remark 4.15. *The equivalence in the above conjecture follows from Corollary 4.3. Because, when we take an element in $\Theta'_S(0)X_K$ its inverse image under λ is always in $\widetilde{U^{ab}_{K/k}}$ and Corollary 4.3 says that it is also in $\widetilde{U_{k^{ab}}}$.*

Notation 4.16. *Let S be a set of places in k which satisfies the conditions in Conjecture 4.14. Let v be a place in S which splits completely in K, and let us fix an extension $w \in S_K$ lying above v.*
If $|S| \geq 3$, we define

$$U^{(v)} = \{u \in \mathcal{O}^*_{K,S_K} : |u|_{w'} = 1 \text{ for all } w' \nmid v\}.$$

If $S = \{v, v'\}$ and if w' is an extension above v', we define

$$U^{(v)} = \{u \in \mathcal{O}^*_{K,S_K} : |u|_{\sigma w'} = |u|_{w'} \text{ for all } \sigma \in G\}.$$

Conjecture 4.17 (Second Form)**.** *With notation as above, there exists a S-unit $\epsilon \in U_{K/k}^{ab} \cap U^{(v)}$ such that*

$$\log|\epsilon^\sigma|_w = -e\zeta_S'(0,\sigma) \text{ for all } \sigma \in G, \tag{4.2}$$

or equivalently,

$$L_S'(0,\chi) = -\frac{1}{e}\sum_{\sigma\in G}\chi(\sigma)\log|\epsilon^\sigma|_w \text{ for all } \chi \in \widehat{G}. \tag{4.3}$$

Remarks 4.18.

(i) The choice of place w lying above v does not affect the truth of the conjecture.

(ii) The S-unit ϵ, stated in the conjecture, is called a Stark unit.

(iii) ϵ is uniquely determined up to multiplication by a root of unity in K.

(iv) If G is cyclic and S contains only one place which splits completely in K, then such a unit ϵ, if it exists, generates K over k, i.e. $K = k(\epsilon)$. By Corollary 3.19, for any faithful character $\chi : G \to \mathbb{C}^*$ we have $r_S(\chi) = 1$ and therefore $L_S'(0,\chi) \neq 0$. If we have $\tau \in G$ such that $\epsilon^\tau = \epsilon$ then we replace $\epsilon^\sigma = \epsilon^{\tau\sigma}$ in the formula (4.3). It follows that

$$\begin{aligned} L_S'(0,\chi) &= -\frac{1}{e}\sum_{\sigma\in G}\chi(\sigma)\log|\epsilon^\sigma|_w \\ &= -\frac{1}{e}\sum_{\sigma\in G}\chi(\tau\sigma)\log|\epsilon^{\tau\sigma}|_w \\ &= -\frac{\chi(\tau)}{e}\sum_{\sigma\in G}\chi(\sigma)\log|\epsilon^\sigma|_w \\ &= \chi(\tau)L_S'(0,\chi). \end{aligned}$$

Then, $\chi(\tau) = 1$ hence $\tau = 1$ as χ is faithful. So, $K = k(\epsilon)$.

Theorem 4.19. *The conjecture St($K/k, S$) is true if S contains at least two places which split completely in K.*

Proof. Assume that v and v' are places which split completely in K. Let $|S| \geq 3$. Then by Corollary 3.19, $L'_S(0,\chi) = 0$ for all $\chi \in \widehat{G}$. Thus, for S-unit $\epsilon = 1$ Conjecture 4.3 is satisfied. Now, suppose $S = \{v,\ v'\}$. Then the rank of the group of S-units is 1, by Unit Theorem 3.21. Let η be the fundamental S-unit with $|\eta|_v > 1$. For $\chi = \mathbb{1}$, the Dirichlet class number formula 3.1 implies

$$L'_{K/k,S}(0,\mathbb{1}) = \zeta'_k(0) = -\frac{h_{k,S} \cdot \log|\eta|_v}{e_k}.$$

Since $\mu(k)$ is the subgroup of $\mu(K)$ and $h_{k,S}$ is a multiple of $[K:k]$ (see [Lan, VI,§1]), $m = \frac{e \cdot h_{k,S}}{e_k \cdot [K:k]} \in \mathbb{Z}$, we can define $\epsilon = \eta^m$. Since $\epsilon \in k$, it follows that $\epsilon \in U^{(v)}$. Furthermore, $K(\epsilon^{1/e})$ lies inside of the compositum $K(\eta^{1/e_k})$ of the abelian extension K/k and the Kummer extension $k(\eta^{1/e_k})/k$, so $K(\epsilon^{1/e})/k$ is abelian. Thus, $\epsilon \in U^{ab}_{K/k} \cap U^{(v)}$. Finally, since $\epsilon \in k$ and v splits completely (in fact, $|\epsilon|_v = |\epsilon^\sigma|_v = |\epsilon^\sigma|_v^{[w:v]} = |\epsilon^\sigma|_w$),

$$L'_S(0,\mathbb{1}) = -\frac{h_{k,S} \cdot \log|\eta|_v}{e_k} = -\frac{[K:k]}{e}\log|\epsilon|_v = -\frac{1}{e}\sum_{\sigma \in G}\mathbb{1}(\sigma)\log|\epsilon^\sigma|_w.$$

For $\chi \neq \mathbb{1}$, again by Corollary 3.19, we have $L'_S(0,\chi) = 0$ and

$$-\frac{1}{e}\sum_{\sigma \in G}\chi(\sigma)\log|\epsilon^\sigma|_w = -\frac{\log|\epsilon|_v}{e}\sum_{\sigma \in G}\chi(\sigma) = 0 = L'_S(0,\chi).$$

□

Remark 4.20. *St($k/k,S$) is true by Theorem 4.19, since all places in any set S (to be defined in Conjecture 4.14) split completely.*

Corollary 4.21.

(i) If S contains two complex places, then St($K/k,S$) is true.

(ii) If S contains a finite place which splits completely and k is not totally real, then St($K/k,S$) is true.

Proof. These two assertions are true because all complex places split completely and the second assertion implies that k contains at least one complex place. □

Theorem 4.22. *Conjecture 4.14 and Conjecture 4.17 are equivalent.*

Proof. `Conjecture 4.14 implies Conjecture 4.17:`

By assumption, the set of places S has at least two elements and contains at least one place splits completely in K/k. Let us fix $v \in S$ which splits completely in K. Let w, w' be places lying above v, v', respectively. Conjecture 4.14 says that for an element $(w - w') \in X_K$ there exists a S-unit $\epsilon \in U_{K/k}^{ab}$ such that

$$\Theta'_S(0)(w' - w) = \frac{1}{e}\lambda(\epsilon).$$

By the condition (iii) of Proposition 4.6 we have

$$\Theta'_S(0) = \sum_{\sigma \in G} \zeta'(0, \sigma)\sigma^{-1},$$

and so

$$\frac{1}{e}\sum_{w \in S_K} \log|\epsilon|_w w = \sum_{\sigma \in G} \zeta'(0, \sigma)\sigma^{-1}(w' - w).$$

On the other hand, we have

$$\begin{aligned} \Theta'_S(0)w &= \sum_{\sigma \in G} \zeta'(0, \sigma)\sigma^{-1}w \\ &= -\frac{1}{e}\sum_{w \in S_K} \log|\epsilon|_w w \\ &= -\frac{1}{e}\sum_{\sigma \in G} \log|\epsilon^\sigma|_w \sigma^{-1}w \end{aligned}$$

since $|\epsilon^\sigma|_w = |\epsilon|_{\sigma^{-1}w}$. Thus, the coefficient of $\sigma^{-1}w$ gives us

$$\zeta'(0, \sigma) = -\frac{1}{e}\log|\epsilon^\sigma|_w$$

for all $\sigma \in G$.

Now, If $|S| \geq 3$ then we know $\mathrm{N}_{G_v} \cdot \mathrm{N}_{G_{v'}} \cdot {\Theta_S}'(0) = 0$ by Proposition 4.12. Thus,

$${\Theta_S}'(0)w' = \frac{\mathrm{N}_{G_v} \cdot \mathrm{N}_{G_{v'}} \cdot {\Theta_S}'(0)}{|G_{v'}|}w' = 0$$

since $\mathrm{N}_{G_{v'}}w' = |G_{v'}|w'$ and $\mathrm{N}_{G_v} = 1$ as v splits completely in K. It follows that $\log|\epsilon^\sigma|_{w'} = 0$ for all places $w' \nmid v$ in S_K and ${\Theta_S}'(0)(w' - w) = -{\Theta_S}'(0)w$. So, $\epsilon \in U^{(v)}$ for $|S| \geq 3$.

If $|S| = 2$, then $\mathrm{N}_{G_v} \cdot \mathrm{N}_{G_{v'}} \cdot \Theta_S{}'(0) = a \cdot \mathrm{N}_{G_{v'}}$ for $a \in \mathbb{C}$. Then, we write $\Theta_S{}'(0)w' = \frac{a}{|G_{v'}|}\mathrm{N}_{G_{v'}}$. Therefore, $\log|\epsilon|_{\sigma^{-1}w'}$ is independent of σ, and so $\epsilon \in U^{(v)}$ for $|S| = 2$.

`4.17 implies 4.14:` Suppose that there exists a S-unit ϵ satisfying Conjecture 4.17. Since we have shown that for any w' not lying above v,

$$\Theta_S'(0)(w' - w) = \frac{1}{e}\lambda(\epsilon).$$

Then this implies $\Theta_S'(0)(w' - w) \in \frac{1}{e}\lambda(U_{K/k}^{ab})$. Finally, since X is generated over $\mathbb{Z}[G]$ by $(w' - w)$ we have the desired result

$$\Theta_S{}'(0)X \subset \frac{1}{e}\lambda(U_{K/k}^{ab}).$$

□

Proposition 4.23. *St($K/k, S$) implies St($K/k, S'$) for $S' \supset S$.*

Proof. Without loss of generality $S \neq S'$. Let us fix a place $v \in S$ which splits completely in K and choose $\mathfrak{p} \in S' - S$ which is unramified place because the set S satisfies the conditions in Conjecture 4.14. By Proposition 4.12, we have $\mathrm{N}_{G_v} \cdot \Theta_S(0) = 0$ which implies that $\Theta_S(0) = 0$ as v splits completely in K. Thus, by Corollary 4.7 and the fact thah $\mathrm{F}_\mathfrak{p} = \mathrm{Frob}_\mathfrak{p}$, we have

$$\Theta'_{S\cup\{\mathfrak{p}\}}(0) = \Theta'_S(0)(1 - \mathrm{Frob}_\mathfrak{p}) \ \in \Theta'_S(0)\mathbb{Z}[G].$$

Therefore,

$$\Theta'_{S\cup\{\mathfrak{p}\}}(0)X \subset \Theta'_S(0)\mathbb{Z}[G]X = \Theta'_S(0)X \subset \frac{1}{e}\lambda(U_{K/k}^{ab}).$$

□

Proposition 4.24. *If St($K/k, S$) is true then St($K'/k, S$) is also true for any intermediate field K' between K and k.*

Proof. By the embedding $X_{K',S} \hookrightarrow X_{K,S}$ as stated in Remark 3.17 and by Proposition 4.10, we have

$$\Theta'_{K'/k,S}(0)X_{K'} = (\pi\Theta'_{K/k,S}(0))X_{K'} \subset \Theta'_{K/k,S}(0)X_K \subset \frac{1}{e}\lambda(U_{K/k}^{ab})$$

and the result follows. □

Remark 4.25. *Since Tate's and Stark's refined Conjectures are equivalent, we sure to find a Stark unit $\epsilon' \in \mathcal{O}^*_{K',S}$ for Proposition 4.24. Suppose that v is a fixed place of k which splits completely in K and w is a place of K lying above v. Let $\epsilon \in U^{ab}_{K/k}$ be a Stark unit which satisfies Conjecture 4.17. By Corollary 4.4, if $u = \frac{1}{e}\widetilde{\epsilon} \in \widetilde{U^{ab}_{K/k}}$ then $N_{K/K'}u \in \frac{1}{e_{K'}}\widetilde{U^{ab}_{K'/k}}$. Hence there exists an $\epsilon' \in U^{ab}_{K'/k}$ such that*

$$\widetilde{(\epsilon')^{e/e_{K'}}} = eN_{K/K'}u = \widetilde{N_{K/K'}\epsilon}.$$

Therefore, for some root of unity ζ in K' we write $\epsilon'^{e/e_{K'}} = \zeta \cdot N_{K/K'}\epsilon$. Now, we show that $\epsilon' \in U^{(v)}$. If $|S| \geq 3$, then ϵ satisfies $|\epsilon|_{w'} = 1$ for all $w' \nmid v$ in S_K. Hence,

$$|\epsilon'|^{e/e_{K'}}_{w'_{K'}} = |N_{K/K'}\epsilon|_{w'} = |\epsilon^{\sum_\sigma}|_{w'} = 1$$

and $\epsilon' \in U^{(v)}$. Similarly, we can show that $\epsilon' \in U^{(v)}$ for $|S| = 2$. Now, denote $G = \mathrm{Gal}(K/k)$; $H = \mathrm{Gal}(K/K')$; $G/H = \mathrm{Gal}(K'/k)$. Let χ be a character of G/H and $\widehat{\sigma}$ be an element of G restrict to σ on G/H. Then, by Proposition 3.11

$$\begin{aligned}
L'_{K'/k,S}(0,\chi) &= L'_{K/k,S}(0,\mathrm{Infl}\chi) \\
&= -\frac{1}{e}\sum_{\gamma\in G}\mathrm{Infl}\chi(\gamma)\log|\epsilon|_w \\
&= -\frac{1}{e}\sum_{\sigma\in G/H}\chi(\sigma)\sum_{\tau\in H}\log|\epsilon^{\widehat{\sigma}\tau}|_w \\
&= -\frac{1}{e}\sum_{\sigma\in G/H}\chi(\sigma)\log|(\mathrm{N}_{K/K'}\epsilon)^{\widehat{\sigma}}|_w \\
&= -\frac{1}{e}\sum_{\sigma\in G/H}\chi(\sigma)\log|(\epsilon')^{\widehat{\sigma}}|^{e/e_{K'}}_{w_{K'}} \\
&= -\frac{1}{e_{K'}}\sum_{\sigma\in G/H}\chi(\sigma)\log|(\epsilon')^{\sigma}|_{w_{K'}}.
\end{aligned}$$

Hence, ϵ' satisfies Conjecture 4.17.

However, it is unclear whether ζ can be taken equal to 1. It is the case when $e = e_{K'}$, then we can simply have $\epsilon' = \mathbf{N}_{K/K'}\epsilon$.

Theorem 4.26. *St$(K/k,S)$ is true if $k = \mathbb{Q}$ or k is the imaginary quadratic field.*

Proof. Let $m \geq 3$ be an integer divisible by 4 or an odd integer and ζ be the primitive m-th root of unity. Let $k = \mathbb{Q}$ and $K = \mathbb{Q}(\zeta)^+$ denote the maximal totally real subfield of $\mathbb{Q}(\zeta)$, and $S = \{\infty, \mathfrak{p} : \mathfrak{p}|m\}$. Indeed, the set S has at least two elements and one (∞) is totally split in K since K is a totally real field. Corollary 3.19 implies that for any character χ of G, including the trivial one, we have $r(\chi) \geq 1$. We need to consider the case $r(\chi) = 1$ since in the other case the conjecture is trivial for $\epsilon = 1$. Note that the Galois group $G = \mathrm{Gal}(K/k)$ is canonically isomorphic to $(\mathbb{Z}/m\mathbb{Z})^*/\{\pm 1\}$. Denote σ_a as the K-automorphism corresponding to a (mod$m$) which is the restriction of the $\mathbb{Q}(\zeta)$-automorphism to K which sends ζ to ζ^a. So, we have $\sigma_a = \sigma_{-a}$ and the partial zeta function is written

$$\zeta(s, \sigma_a) = \sum_{\substack{n=1 \\ n \equiv \pm a (\mathrm{mod} m)}}^{\infty} |n|^{-s} = \sum_{\substack{n \in \mathbb{Z} \\ n \equiv a (\mathrm{mod} m)}} |n|^{-s}.$$

Consider $\mathbb{Q}(\zeta)$ injects into $\mathbb{C}$ by $\zeta = e^{2\pi i/m}$, and let us take

$$\epsilon = (1 - \zeta)(1 - \zeta^{-1}) = 2 - 2\cos(2\pi/m).$$

We have $|\epsilon|_w^{[w:v]} = |\epsilon|_v = 1$ for any $v \nmid \infty$. Thus, in both cases, namely $|S| \geq 3$ and $|S| = 2$, $\epsilon \in U^{(v)}$. We also have

$$\epsilon^{\sigma_a} = (1 - \zeta^a)(1 - \zeta^{-a}) = 2 - 2\cos(2\pi a/m).$$

When we calculate the derivative of partial zeta function at $s = 0$ (which is calculated by H.Stark in [StIV]), we find

$$\zeta'(0, \sigma_a) = -\frac{1}{2}\log(2 - 2\cos(2\pi a/m)) = \frac{1}{2}\log \epsilon^{\sigma_a}.$$

We therefore have, for any character χ of G

$$L(s, \chi) = \sum_{\sigma \in G} \chi(\sigma)\zeta(s, \sigma)$$

and

$$L'(0, \chi) = -\frac{1}{2}\sum_{\sigma \in G} \chi(\sigma) \log \epsilon^{\sigma}.$$

In particular, if w is an infinite place of K corresponding to the embedding of K in $\mathbb{R}$, ϵ^σ is positive, then we find that

$$L'(0,\chi) = -\frac{1}{2}\sum_{\sigma\in G}\chi(\sigma)\log|\epsilon^\sigma|_w.$$

Since K is totally real field, we can see $e = |\mu(K)| = 2$. Therefore, to prove Stark conjecture for K/k we need to show that $K(\epsilon^{1/2})$ is abelian over $\mathbb{Q}$.

Since

$$\epsilon = -e^{-2\pi i/m}(1-e^{2\pi i/m})^2,$$

we see that $\epsilon^{1/2}$ is abelian over $\mathbb{Q}$.

It remains to show in the case when $m = 2$. In this case, the extension is trivial; $K = \mathbb{Q}$, so the character is trivial; $\mathbb{1}$, and $S = \{\infty, 2\}$ and 2 is the only unit element (in fact, a fundamental unit) in $\mathcal{O}_{\mathbb{Q},S}$. By Class Number Formula at $s = 0$,

$$L'_S(0,\chi) = \zeta'_S(0) = -\frac{h_S R_S}{e} = -\frac{1}{2}\log|2|.$$

Indeed, $2^{1/2}$ is abelian over $\mathbb{Q}$. Thus, the conjecture is also satisfied in this case.

The proof of the quadratic imaginary case can be found in [StIV].

□

4.3 A numerical example

Let $k = \mathbb{Q}(\beta)$ where $\beta = 3.079118864\ldots$ is one of the root of the polynomial

$$f(X) = X^3 - X^2 - 9X + 8 = (X-1)(X+3)(X-3) - 1.$$

The discriminant of this polynomial is $\triangle(\mathbb{Q}(\beta)/\mathbb{Q}) = 2597 = 7^2 \cdot 53$, hence the embeddings of this extension is all real and the ramified primes are only 7 and 53 in $\mathbb{Q}(\beta)$. The ring of integers of k is $\mathcal{O}_k = \mathbb{Z}[\beta]$ due to Stickelberger's criterian (see [Lan, III, §3]) (in fact, $\triangle(\mathbb{Q}(\beta)/\mathbb{Q}) = 2597 \equiv 1 (\text{mod } 4)$). And since we have

$$(\beta-1)(\beta+3)(\beta-3) = 1,$$

$\beta - 1$, $\beta + 3$, $\beta - 3$ are units in k, indeed, $\{\beta - 1, \beta - 3\}$ is the system of fundamental units. If one computes the Minkowski bound (see [Jan, Theorem 11.9]), we see that every ideal class $[\mathfrak{U}]$ in the class group $\mathrm{Cl}(\mathcal{O}_k)$ contains an ideal $\mathfrak{U}$ with $\mathrm{N}(\mathfrak{U}) \leq 11$. This means that we can generate $\mathrm{Cl}(\mathcal{O}_k)$ by classes $[\wp]$ with a prime ideal $\wp$ having norm $\mathrm{N}(\wp) \leq 11$. So to find these primes, it is necessary to describe $\mathfrak{p}\mathcal{O}_k$ when $\mathfrak{p}$ is an integral ideal < 11. For the primes $\mathfrak{p} = 2, 3, 5, 7, 11$ we use Theorem 7.6 and Proposition 7.7 in [Jan].

$$\begin{array}{lll} \mathfrak{p}=2 & x^3-x^2-9x+8 \equiv x(x^2+x+1) & \pmod 2 \\ \mathfrak{p}=3 & x^3-x^2-9x+8 \equiv (x+1)(x^2+x+2) & \pmod 3 \\ \mathfrak{p}=5 & x^3-x^2-9x+8 \equiv (x+1)(x^2+3x-2) & \pmod 5 \\ \mathfrak{p}=7 & x^3-x^2-9x+8 \equiv (x+2)(x^2-3x+4) & \pmod 7 \\ \mathfrak{p}=11 & x^3-x^2-9x+8 \equiv (x^3-x^2-9x+8) & \pmod{11} \end{array}.$$

Thus,

$$\begin{array}{l} 2\mathcal{O}_k = (2,\beta)(2,\beta^2+\beta+1) = \wp_2 \cdot \wp_2' \text{ with } \mathrm{N}(\wp_2)=2 \text{ and } \mathrm{N}(\wp_2')=2^2 \\ 3\mathcal{O}_k = (3,\beta+1)(3,\beta^2+\beta+2) = \wp_3 \cdot \wp_3' \text{ with } \mathrm{N}(\wp_3)=3 \text{ and } \mathrm{N}(\wp_3')=3^2 \\ 5\mathcal{O}_k = (5,\beta+1)(5,\beta^2+3\beta-2) = \wp_5 \cdot \wp_5' \text{ with } \mathrm{N}(\wp_5)=5 \text{ and } \mathrm{N}(\wp_5')=5^2 \\ 7\mathcal{O}_k = (7,\beta+2)(7,\beta^2-3\beta+4) = \wp_7 \cdot \wp_7' \text{ with } \mathrm{N}(\wp_7)=7 \text{ and } \mathrm{N}(\wp_7')=7^2 \\ 11\mathcal{O}_k = (11,\beta^3-\beta^2-9\beta+8) = \wp_{11} \text{ with } \mathrm{N}(\wp_{11})=11^3 \end{array}.$$

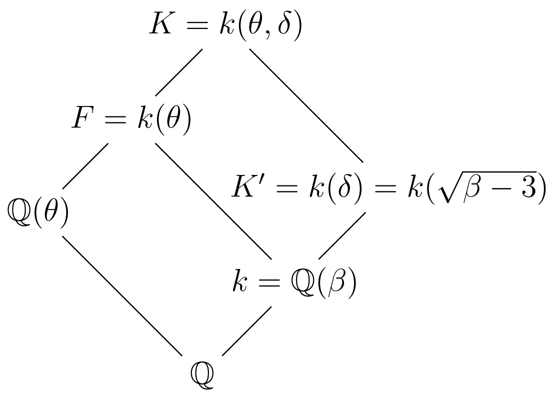
We have $[\wp_2] \cdot [\wp_2'] = 1$ since $2\mathcal{O}_k$ is principal. Similarly, we find that $\mathrm{Cl}(\mathcal{O}_k)$ is generated by the classes $[\wp_2]$, $[\wp_3]$, $[\wp_5]$, $[\wp_7]$ as $[\wp_{11}] = 1$. Now, assume that k, $\beta + k$ has minimum polynomial

$$(X-k)^3 - (X-k)^2 - 9(X-k) + 8.$$

So, $\mathrm{N}(\beta + k) = k^3 + k^2 - 9k - 8$ and one finds $\mathrm{N}(\beta - 2) = 6 = 2 \cdot 3$. Thus, $(\beta - 2)\mathcal{O}_k$ is the product of a prime with norm 2 and a prime with norm 3. In each case the prime is unique, so

$$(\beta - 2)\mathcal{O}_k = \wp_2 \cdot \wp_3.$$

This means that $[\wp_2] = [\wp_3]^{-1}$ in $\mathrm{Cl}(\mathcal{O}_k)$. By applying this method to other primes, we conclude that $\mathrm{Cl}(\mathcal{O}_k)$ is generated by $[\wp_2]$. Moreover, $\mathrm{N}(\beta) = 8$, and so $\beta\mathcal{O}_k = (\wp_2)^3$. Thus, the class number $h_k = 3$.



where $\theta = 2cos(2\pi/7) = \zeta_7 + \zeta_7^{-1}$ and $\delta = \frac{\beta+1-\sqrt{(\beta+1)^2-4}}{2}$ is the root of the equation

$$X^2 + (\beta + 1)X + 1 = 0.$$

The extension $\mathbb{Q}(\theta)/\mathbb{Q}$ is cyclic of degree 3 and satisfies the equation

$$X^3 + X^2 - 2X - 1 = 0.$$

The only ramified prime is 7 as the discriminant $\triangle(\mathbb{Q}(\theta)/\mathbb{Q}) = 7^2$ and the class number of $\mathbb{Q}(\theta) = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})^+$ is 1 since for all integral primes ≤ 7, $\mathfrak{p}\mathcal{O}_k$ is principal. The degree of the extension $\mathbb{Q}(\theta)/\mathbb{Q}$ is 3 and so the relative degree $f = 1$, or 3 but we know that 7 is a ramified prime. Hence, $f = 1$ and $\mathbb{Q}(\theta)/\mathbb{Q}$ is totally ramified. By the same reason, $\mathbb{Q}(\beta)/\mathbb{Q}$ is also totally ramified extension. Furthermore, the ramification of 7 disappears over k otherwise the extension $k(\theta)/k$ would be totally ramified and the ramification index of 7 in $k(\theta)/\mathbb{Q}$ would be 9 but the degree of a maximal totally ramified extension of $\mathbb{Q}$ is 6. Thus, F/k is an unramified extension. The extension K'/k is also unramifed since the discriminant of the equation for δ is

$$(\beta + 1)^2 - 4 = (\beta + 3)(\beta - 1)$$

which is unit in $\mathcal{O}_k$. Note that $\beta_2 = 0.878468\ldots$ and $\beta_3 = -2.957586\ldots$ are conjugates of β and the places ∞ (respectively ∞_2, ∞_3) are correspondent to the embedding $\beta \mapsto \beta$ (respectively, β_2, β_3) of k into $\mathbb{R}$. These real places of k splits completely in F since F is totally real field; in K', ∞ splits since the discriminant of the irreducible polynomial of δ is positive and ∞_2, ∞_3 ramifies since $(\beta_i+1)^2-4<0$ and hence the places ∞_2, ∞_3 are real places.

Now, we may consider the conjecture $\mathrm{St}(K/k,S)$ for $S=\{\infty,\ \infty_2,\ \infty_3\}$ since K/k is abelian and S satisfies the requirments of Conjecture 4.14. The group $G=\mathrm{Gal}(K/k)$ is cyclic of order 6. Let σ be the generator of G is given for the Artin symbol of the ideal $\wp_2=(\beta,2)$ of $\mathbb{Z}[\beta]$. Since we have $\mathbf{N}\wp_2=2$, σ acts on θ as

$$\theta^{\sigma}=2\cos\frac{4\pi}{7},\quad \theta^{\sigma^2}=2\cos\frac{6\pi}{7},\quad \theta^{\sigma^3}=\theta;$$

and σ acts non-trivially on K' because $\wp_2^3=(\beta)$, and $\beta_2\beta_3<0$. All places in S splits completely in F, then the Stark Conjecture $St(F/k,S)$ is true when we take $\epsilon=1$ as $r(\chi)>0$. According to Remark 4.25 we thus seek a unit ϵ of K such that $\mathbf{N}_{K/F}\epsilon=1$; to find such unit we will try to construct $\mathrm{Tr}_{K/F}\epsilon$. Then, we will try to find

$$\epsilon^{\sigma^j}=\exp(-2\zeta'(0,\sigma^j))\quad \text{for } j=0,\dots,5.$$

According to [StH], the values of $\zeta'(0,\sigma^j)$ follows:

$$2\zeta'(0,\sigma^0)=-2\zeta'(0,\sigma^3)=2.6229258798145494,$$
$$2\zeta'(0,\sigma)=-2\zeta'(0,\sigma^4)=-0.55674277199362199,$$
$$2\zeta'(0,\sigma^2)=-2\zeta'(0,\sigma^5)=-0.72668091960461237.$$

Construct $A=\mathrm{Tr}_{K/F}\epsilon=\epsilon+\epsilon^{\sigma^3}=\epsilon+\epsilon^{-1}$ then we have

$$A^{\sigma^0}=\exp(-2\zeta'(0,\sigma^0))+\exp(-2\zeta'(0,\sigma^3))\sim 13.84856\dots, \tag{4.4}$$
$$A^{\sigma}\sim 2.318052\dots,\ \text{and } A^{\sigma^2}\sim 2.5517158\dots.$$

We try to write A as a linear sum of $\{1,\beta,\beta^2\}$. Let $x_j=\mathrm{Tr}_{F/k}(A\cdot\theta^{\sigma^j})\in\mathbb{Z}[\beta]$ with $|x_j|_{\infty_2}<12$, $|x_j|_{\infty_3}<12$ and $x_0\sim 11.6392\dots$, $x_1\sim -7.1582\dots$, $x_2\sim -23.1993\dots$.

We can write x_j's as a linear sum of the elements $\{1,\beta,\beta^2\}$, namely, $x_0=[1,2,-4]$, $x_1=[0,-2,-1]$, $x_2=[-2,-3,5]$ where $[l,m,n]=l\beta^2+m\beta+n$. These x_j determines A; we find, as Stark,

$$A=\frac{-1}{7}([1,4,4]\theta+[2,8,1]\theta^{\sigma}+[4,9,-5]\theta^{\sigma^2}).$$

By construction, A satisfies (4.4).

Let ϵ be the smallest root of the equation $X^2 - AX + 1 = 0$. Then, we find that $F(\epsilon) = F(\sqrt{\beta - 3}) = K$, because

$$(\beta - 3)(A^2 - 4) = B^2$$

that means

$$\epsilon = \frac{\pm\sqrt{\beta - 3} - B}{2}$$

where $B \in F$ is given by

$$B = \frac{-1}{7}([1, -1, -6]\theta + [-1, 4, 5]\theta^{\sigma} + [0, 4, 1]\theta^{\sigma^2}).$$

In particular, $\epsilon^{\sigma^3} = \epsilon^{-1}$; so $|\epsilon|_v = 1$ for all places $w \nmid v$ in K. Hence, this shows that $\epsilon \in U^{(\infty)}$.

By construction, the numbers ϵ^{σ^j} and $\epsilon^{\sigma^{j+3}} = \epsilon^{-\sigma^j}$ are the roots of the equation $X^2 - A^{\sigma^j}X + 1 = 0$ since $\epsilon^{\sigma^j} + \epsilon^{\sigma^{j+3}} = A^{\sigma^j}$ and $\epsilon^{\sigma^j} \cdot \epsilon^{\sigma^{j+3}} = 1$. They are approximately equal to the values of $\exp(\mp 2\zeta'(0, \sigma^j))$. To find that

$$\epsilon^{\sigma^j} = \exp(-2\zeta'(0, \sigma^j))$$

for each j, as desired, it is sufficient to verify the signs of $\epsilon^{\sigma^{j+3}} - \epsilon^{\sigma^j}$ is $+, +, -$ respectively for $j = 0, 1, 2$. This results from

$$B = \sqrt{\beta - 3}(\epsilon^{\sigma^3} - \epsilon)$$

because $B > 0$, $B^{\sigma} < 0$, $B^{\sigma^2} < 0$ and $(-1)^j(\sqrt{\beta - 3})^{\sigma^j} > 0$.

It remains to see that $K(\sqrt{\epsilon})$ is abelian over k. In fact, we have $K(\sqrt{\epsilon}) = K(\sqrt{\beta - 1})$ and $\mathrm{Gal}(K(\sqrt{\epsilon})/k) \cong (\mathbb{Z}/6\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. This result follows from

$$(\sqrt{\epsilon} \pm \sqrt{\epsilon}^{-1})^2 = \epsilon + \epsilon^{-1} \pm 2 = A \pm 2$$

and

$$(\beta - 1)(A - 2) = C^2 \quad \text{hence} \quad \sqrt{\beta - 1} = C/(\sqrt{\epsilon} - \sqrt{\epsilon}^{-1})$$

where $C \in F$ is given by

$$C = \frac{-1}{7}([-1, 2, 1]\theta + [0, 3, -1]\theta^{\sigma} + [1, 2, 0]\theta^{\sigma^2}).$$

4.4 Application to Hilbert's Twelfth Problem

This part is based on the article by X.-F. Roblot, [Rob].

Let k be a totally real number field distinct from $\mathbb{Q}$ and let v be an infinite place of k. Let us define k^{Stark} as a subfield of $\mathbb{C}$ generetad over k by all the Stark units $\epsilon(L/k, w)$ where L/k runs through the finite abelian extensions of k in which v is totally split, and w is a places of L lying above v and Conjecture 4.17 is true for all such extensions L/k.

Theorem 4.27. *The maximal real abelian extension of k is contained in k^{Stark}. Equivalently, for any finite real abelian extension K/k, there exist Stark units $\epsilon_1, \dots, \epsilon_r$ such that $K \subset k(\epsilon_1, \dots, \epsilon_r)$.*

We will prove the theorem by proving the second assertion. Thus, we need to construct the Stark units $\epsilon_1, \dots, \epsilon_r$.

For a prime ideal $\mathfrak{p}$ of k we define an integer $r_{\mathfrak{p}}$ as follows: If $\mathfrak{p}$ does not divide 2 then $r_{\mathfrak{p}} = 2$, otherwise $r_{\mathfrak{p}} = n_{\mathfrak{p}} + 2$ where $n_{\mathfrak{p}}$ is the degree of the local extension $k_{\mathfrak{p}}/\mathbb{Q}_2$.

Proposition 4.28. *Let K/k be a finite abelian extension of totally real fields. Let v be an infinite place of k and T be the set of finite places of k such that for each place $\mathfrak{p}$ in T, the 2-rank of the decomposition group $G_{\mathfrak{p}}$ of $\mathfrak{p}$ in K/k is strictly less than $r_{\mathfrak{p}}$. Then, there exists a quadratic extension L/K verifying these three conditions:*

(i) The extension L/k is abelian,

(ii) All the infinite places of k except v become complex in L,

(iii) The finite places of K above T do not split in L/K.

Proof. Let $\mathfrak{p}$ be a finite place in T and fix a place $\wp$ in K dividing $\mathfrak{p}$. Let $s_{\mathfrak{p}}$ denote the 2-rank of the decomposition group $G_{\mathfrak{p}}$ of $\mathfrak{p}$ in K/k (i.e., $|G_{\mathfrak{p}}/{G_{\mathfrak{p}}}^2| = 2^{s_{\mathfrak{p}}}$). Since $|G_{\mathfrak{p}}|$ is the order of the extension $K_{\wp}/k_{\mathfrak{p}}$ we find that the number of quadratic extensions of $k_{\mathfrak{p}}$ in $K_{\wp}$ is $2^{s_{\mathfrak{p}}} - 1$. On the other hand, the number of quadratic extensions of $k_{\mathfrak{p}}$ is

$2^{r_\mathfrak{p}} - 1$. We assumed that $s_\mathfrak{p} < r_\mathfrak{p}$, thus there exists at least one quadratic extension, say $E_\beta/k_\mathfrak{p}$, which is not contained in $K_\wp$ and there exists a $\mathfrak{p}$-adic integer $x_\mathfrak{p}$ in $k_\mathfrak{p}$ such that $E_\beta = k_\mathfrak{p}(\sqrt{x_\mathfrak{p}})$. In particular, $x_\mathfrak{p}$ is not in $K_\wp$. For any finite places in T, we can find such $x_\mathfrak{p}$ in $k_\mathfrak{p}$ and let

$$m_\mathfrak{p} = v_\wp(x_\mathfrak{p}) + v_\wp(2) + 1$$

where $v_\wp$ denotes the valuation associated to $\wp$. Since $x_\mathfrak{p} \in k_\mathfrak{p}$ and K/k is a Galois extension, $m_\mathfrak{p}$ does not depend on the choice of $\wp$. By Approximation Theorem ([Jan, Theorem 1.1]), for any $\epsilon > 0$ we can find an algebraic integer $x \in k$ such that $|x - x_\mathfrak{p}|_v < \epsilon$ for each valuation v. This implies that there exists $x \in k$ and which satisfies

1. $v(x) > 0$
2. $v'(x) < 0$ for any infinite place v' of k distinct from v,
3. $x \equiv x_\mathfrak{p} (\mathrm{mod}\ \mathfrak{p}^{m_\mathfrak{p}})$ for any finite place $\mathfrak{p} \in T$.

We claim that $L = K(\sqrt{x})$ and L/K satisfies the conditions (i)-(iii). The extension L/k is abelian since L is the compositum of the two abelian extensions K/k and $k(\sqrt{x})$, hence the first condition is satisfied. Furthermore, all infinite places $v \neq v'$ become complex places whereas v remains real in L/k hence the second condition is also satisfied. Lastly, assume that $\mathfrak{p}$ is a finite place in T and splits in L/K. Denote by $\widetilde{\wp}$ and $\wp$ finite places in L and K, respectively, lying above $\mathfrak{p}$. The place $\mathfrak{p}$ splits in L/K means that $\wp$ splits in L/K, so the order of the decomposition group of $\wp$ is 1. Then, the local fields $L_{\widetilde{\wp}}$ and $K_\wp$ are the same, thus x is a square in $K_\wp$. Let us consider the polynomial $X^2 - x_\mathfrak{p}$ is in $K_\wp[x]$, $\sqrt{x}$ is a simple root of this polynomial modulo $\wp^{m_\mathfrak{p}}$. By Hensel's Lemma, it has a root in $K_\wp$ hence $x_\mathfrak{p}$ is a square in $K_\wp$. But this is not possible, since we know that $x_\mathfrak{p}$ is not a square in $K_\wp$ and thus $\mathfrak{p}$ does not split in L/K. □

Remark 4.29. *These three conditions are very important for the construction. The conditions (i)-(iii) allow us to apply Conjecture 4.17 to the extension L/k. The* *third condition is necessary to ensure that $L'_S(0, \chi)$ is not going to vanish for many characters χ, and so make sure that the Stark unit that we obtain is a generator of L.*

Now, we can prove Theorem 4.27.

Suppose that K/k is cyclic extension with Galois group G and let S be the finite set of places which contains all infinite places and ramifed places in K/k. We want to construct a quadratic extension L/K satisfying conditions (i)-(iii). To find Stark units and to apply Conjecture 4.17 to abelian extension L/K we have to ensure that $L'(s, \chi)$ is not going to vanish at $s = 0$. One way to do this is to guarantee that no finite places in S splits in L/K. By choosing S as minimal, the finite places in S are exactly the places that ramify in L/k. Let $\mathfrak{p}$ be a prime in S which ramifies in L/k. If $\mathfrak{p}$ is not ramified in K/k then $\mathfrak{p}$ must be ramified in L/K and thus $\mathfrak{p}$ does not split in L/K. On the other hand, if $\mathfrak{p}$ is ramified in K/k, then we want to sure that $\mathfrak{p}$ is not going to split in L/K and thus choose $\mathfrak{p}$ as an element of T. Thus, we choose T is the set of ramified places of k which ramify in K/k. For each finite places in k, 2-rank of its decomposition group in K/k is equal to 1 since K/k is cyclic. Thus, we can apply Proposition 4.28 and obtain a quadratic extension L/K verifying conditions (i)-(iii).

Let us fix an infinite place w lying above v in L and let $\epsilon = \epsilon(L/k, w)$ be the Stark unit in L satisfies the Conjecture 4.17. Since S contains all ramified primes in L/k not more, $r(\chi) = 1$ and so $L'_S(0, \chi) = 0$. Finally, we apply the following Theorem (see [StIII], Theorem 1.).

Theorem 4.30 (Stark)**.** *[StIII] Assume Stark Conjecture 4.17 is true for the extension L/k and G is the Galois group of the extension L/k. Let Γ be the quotient group $G/\{1, \tau\}$, so that Γ is the Galois group of K/k. Assume that for every character of χ of G with $\chi(\tau) = -1$, one has $L'(0, \chi) \neq 0$. Then, $L = \mathbb{Q}(\epsilon)$ and $K = \mathbb{Q}(\epsilon + \epsilon^{-1})$.*

When K/k is not cyclic, but abelian, we can write K as a compositum of cyclic extensions K_i/k for some i. Then, we can construct a quadratic extension L_i for each K_i hence for a fixed w_i, we find that $\epsilon_i = \epsilon_i(L_i/k,\ w_i)$ satisfies $K_i = k(\epsilon_i + \epsilon_i^{-1})$. The proof of Theorem 4.27 follows.

5 Rubin-Stark Conjecture

5.1 Basic Definitions and Facts

5.1.1 The Exterior Algebra

The following definitions and remarks can be seen in details in [SL, ch.XIX].

Let R be a commutative unital ring and $M,\ M'$ be R-modules.

Definition 5.1. *Let $f : M^{(r)} \to M'$ be an r-multilinear alternating map. We define the r^{th} exterior power of M*

$$\wedge^r M = T^r(M)/a_r$$

where a_r is the submodule of the tensor product $T^r(M)$ generated by the elements of type $x_1 \otimes \ldots \otimes x_r$ satisfying $x_i = x_j$ for some $i \neq j$. The elements in $\wedge^r M$ are denoted by $x_1 \wedge \ldots \wedge x_r$.

Definition 5.2. *We define the exterior algebra as the direct sum*

$$\wedge M = \bigoplus_{r=0}^{\infty} \wedge^r M$$

which has a $\mathbb{Z}$-graded R-algebra structure.

Facts 5.3. *(i) If $f : M \to M'$ is a R-module homomorphism we obtain a homomorphism of $\mathbb{Z}$-graded R-algebras*

$$\wedge(f) : \wedge M \to \wedge M'$$

which is such that for $x_1 \ldots x_r \in M$ we have

$$\wedge(f)(x_1 \wedge \ldots \wedge x_r) = f(x_1) \wedge \ldots \wedge f(x_r).$$

(ii) We write

$$f^{(r)} : \wedge^r M \to \wedge^r M'$$

for the restriction of $\wedge(f)$ on $\wedge^r M$.

(iii) We have $\wedge^0 M = R$ *and* $\wedge^1 M = M$ *and we use the convention that* $\wedge^r M = \{0\}$ *if* $r < 0$.

Proposition 5.4. *Let* M *be a free* R*-module of rank* n*. If* $n > r$ *then* $\wedge^r M = 0$*. Let* $\{v_1, \ldots, v_n\}$ *be the basis of* M *over* R*. If* $1 \leq r \leq n$*, then* $\wedge^r M$ *is free over* R*, and the elements*

$$v_{i_1} \wedge \ldots \wedge v_{i_r}, \qquad i_1 < \ldots < i_r$$

form a basis of $\wedge^r M$ *over* R*. We have*

$$\dim \wedge^r M = \binom{n}{r}.$$

Proof. See the proof of Proposition 1.1. in [SL, ch.XIX] □

5.1.2 $\mathbb{Z}[G]$-modules

Let G be a finite abelian group and $M,\ M'$ be $\mathbb{Z}[G]$-modules.

Definition 5.5. *A* $\mathbb{Z}[G]$-lattice *is a* $\mathbb{Z}[G]$*-module whose underlying* $\mathbb{Z}$*-module is a free on a finite number of generators.*

If M is a finitely generated $\mathbb{Z}[G]$-module we define its dual as $M^* = \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G])$. Since G is a finite group, we may also see M as a finitely generated $\mathbb{Z}$-module. Thus, M^* is a finitely generated free as a $\mathbb{Z}$-module. Hence, M^* is $\mathbb{Z}[G]$-lattice.

Proposition 5.6. *(i) If* M *is a* $\mathbb{Z}[G]$*-lattice then there is a canonical isomorphism*

$$M^{**} = M.$$

(ii) If

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of $\mathbb{Z}[G]$*-lattices, then*

$$0 \to (M'')^* \to (M)^* \to (M')^* \to 0$$

is also exact.

Proof. We have a canonical isomorphism of abelian groups

$$\mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G]) \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}).$$

This isomorphism takes $f \in \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G])$ to $\pi \circ f \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ where π is the ring homomorphism $\mathbb{Z}[G] \to \mathbb{Z}$ given by $\sum a_\sigma \sigma \mapsto \sum a_\sigma$. The inverse of this isomorphism sends $g \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ to the function in $\mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G])$ which sends $m \in M$ to $\sum_\sigma g(\sigma m)\sigma^{-1}$. Since M, M', M'' are $\mathbb{Z}[G]$-lattices, both assertions follow. ☐

Every $\phi \in M^{(*)} = \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G])$ induces a $\mathbb{Z}[G]$-homomorphism

$$\phi : \wedge^r M \to \wedge^{r-1} M$$

given by

$$\phi(m_1 \wedge \ldots \wedge m_r) = \sum_{i=1}^{r} (-1)^{i+1} \phi(m_i) m_1 \wedge \ldots \wedge m_{i-1} \wedge m_{i+1} \wedge \ldots \wedge m_r$$

for all $r \geq 1$. For $\phi_1, \ldots, \phi_k \in M^*$, we have the map

$$\begin{aligned} \wedge^k \mathrm{Hom}(M, \mathbb{Z}[G]) &\to \mathrm{Hom}(\wedge^r M, \wedge^{r-k} M) \\ \phi_1 \wedge \ldots \wedge \phi_k &\mapsto \phi_k \circ \ldots \circ \phi_1 \end{aligned} \tag{5.1}$$

for all $k \leq r$; when $k = r$ the map is

$$(\phi_1 \wedge \ldots \wedge \phi_r)(m_1 \wedge \ldots \wedge m_r) = \det(\phi_i(m_j)).$$

Definition 5.7. *Suppose M is a finitely generated $\mathbb{Z}[G]$-module and r is a nonnegative integer. By using the map (5.1), for $r = k$,*

$$\iota : \wedge^r \mathrm{Hom}(M, \mathbb{Z}[G]) \to \mathrm{Hom}(\wedge^r M, \mathbb{Z}[G])$$

we define

$$\begin{aligned} \wedge_0^r M &= (\iota(\wedge^r (M^*)))^* \subset \wedge^r M \otimes \mathbb{Q} \\ &= \{m \in \wedge^r M \otimes \mathbb{Q} : \phi_1 \wedge \ldots \wedge \phi_r(m) \in \mathbb{Z}[G], \text{ for every } \phi_1, \ldots, \phi_r \in M^*\}. \end{aligned}$$

Remark 5.8. *If ι is surjective then we immediately have $\wedge_0^r M = (\wedge^r M)^{**} = \overline{\wedge^r M}$ where $\overline{\wedge^r M}$ is the image of $\wedge^r M$ in $\wedge^r M \otimes \mathbb{Q}$.*

Proposition 5.9. *Suppose M is a $\mathbb{Z}[G]$-lattice and $r \geq 0$.*

(i) $\wedge_0^r M \supset \overline{\wedge^r M}$ *with finite index,*

(ii) $\wedge_0^r M = \wedge^r M$ *if $r \leq 1$.*

Proof. The first assertion comes from the definition of $\wedge_0^r M$. For the second one; If $r = 0$, $\wedge_0^0 M = \mathbb{Z}[G] = \wedge^0 M$ and if $r = 1$, $\wedge_0^1 M = M^{**} = \wedge^1 M$. □

Corollary 5.10. *Suppose M is a $\mathbb{Z}[G]$-lattice and $r \geq 1$. If $\phi \in M^*$, then ϕ induces the map*

$$\phi : \wedge_0^r M \to \wedge_0^{r-1} M.$$

Proof. Suppose $\phi_1, \ldots, \phi_{r-1} \in M^*$ and $m \in \wedge_0^r M$. Then we have the equality

$$\phi_1 \wedge \ldots \wedge \phi_{r-1} \wedge (\phi(m)) = \phi \wedge \phi_1 \wedge \ldots \wedge \phi_{r-1}(m) \in \mathbb{Z}[G]$$

for all $\phi_1, \ldots, \phi_{r-1}$. Hence, $\phi(m) \in \wedge_0^{r-1} M$.

In particular, by Proposition 5.9, if $\Phi \in \wedge^{r-1}\mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G])$, then

$$\phi_1 \wedge \ldots \wedge \phi_{r-1} : \wedge_0^r M \to \wedge_0^1 M = \overline{M} = M.$$

□

5.2 The statements of the conjectures

We define $\mathcal{O}_{K,S,T}^* = \{\alpha \in \mathcal{O}_{K,S}^* : \alpha \equiv 1 (\mathrm{mod}\ w) \text{ for all } w \in T\}$.

Hypotheses 5.11. *Suppose that S, T are disjoint finite set of places of k and r is a nonnegative integer.*

(i) *S contains all infinite and ramified places,*

(ii) *S contains at least r places split completely in K/k,*

(iii) $|S| \geq r+1$,

(iv) $\mathcal{O}^*_{K,S,T}$ *is torsion-free.*

Remark 5.12. *The conditions (ii) and (iii) guarantee that $s^{-r}\Theta_{S,T}(s)$ is holomorphic at $s=0$. The last condition is satisfied if T contains places of two different residue characteristics or a place of sufficiently large norm.*

We define $\lambda^{(r)} : \wedge^r \mathcal{O}^*_{K,S,T} \to \wedge^r \mathbb{R} \otimes X_S$ which is induced by

$$\lambda_{S,T} : \mathcal{O}^*_{K,S,T} \to \mathbb{R}X_S$$
$$\alpha \mapsto \sum_{w \in S_K} \log(|\alpha|_w) \cdot w$$

If $r \geq 0$ and $s^{-r}\Theta^{(r)}_{S,T}(0)$ is holomorphic at $s=0$, we define

$$\Theta^{(r)}_{S,T}(0) = \lim_{s\to 0} s^{-r}\Theta_{S,T}(s) = \sum_{\chi \in \widehat{G}} e_\chi \lim_{s\to 0} s^{-r} L_{S,T}(s,\overline{\chi}) = \sum_{\chi \in \widehat{G}} e_\chi L^{(r)}_{S,T}(0,\overline{\chi}) \in \mathbb{C}[G]$$

Conjecture 5.13 (St(K/k,S,T,r))**.** *If S, T, r satisfy Hypotheses 5.11, then*

$$\Theta^{(r)}_{S,T}(0) \wedge^r X_S \subset \lambda^{(r)}(\wedge^r_0 \mathcal{O}^*_{K,S,T}) \text{ in } \mathbb{R} \otimes X_S.$$

Conjecture 5.14 (St(K/k,S,T,r)$\otimes\mathbb{Q}$)**.** *If S, T, r satisfy Hypotheses 5.11, then*

$$\Theta^{(r)}_{S,T}(0) \wedge^r X_S \subset \mathbb{Q}\lambda^{(r)}(\wedge^r_0 \mathcal{O}^*_{K,S,T}) \text{ in } \mathbb{R} \otimes X_S.$$

For $w \in S_K$, we define $w^* \in Y^*_S$ as

$$w^*(w') = \sum_{\substack{\gamma w = w' \\ \gamma \in \mathrm{Gal}(K/k)}} \gamma, \qquad \text{for } w' \in S_K.$$

Let us fix w_i for each $v_i \in S$. If $\eta \in \wedge^r Y^*_S$, we define a regulator map

$$R_\eta : \wedge^r \mathcal{O}_{K,S,T} \xrightarrow{\lambda^{(r)}} \mathbb{R} \otimes \wedge^r X_S \xrightarrow{\eta} \mathbb{R}[G].$$

Lemma 5.15. *If $u_1, \ldots, u_r \in U_{S,T}$, $w_1, \ldots, w_r \in S_K$ and $\eta = w^*_1 \wedge \ldots \wedge w^*_r$, then*

$$R_\eta(u_1 \wedge \ldots \wedge u_r) = \det\Big(\sum_{\gamma \in \mathrm{Gal}(K/k)} \log|u_i^\gamma|_{w_j} \gamma^{-1}\Big).$$

Proof. By definition of the regulator, we have

$$\begin{aligned} R_\eta(u_1 \wedge \ldots \wedge u_r) &= \eta(\lambda(u_1) \wedge \ldots \wedge \lambda(u_r)) \\ &= \det(w_j^*(\lambda(u_i))) \end{aligned}$$

and

$$\begin{aligned} w_j^*(\lambda(u_i)) &= w_j^*(\sum_{\gamma \in \mathrm{Gal}(K/k)} \log |u_i|_{\gamma w_j} \gamma w_j) \\ &= \sum_{\gamma \in \mathrm{Gal}(K/k)} \log |u_i|_{\gamma w_j} \gamma \\ &= \sum_{\gamma \in \mathrm{Gal}(K/k)} \log |u_i^{\gamma^{-1}}|_{w_j} \gamma \\ &= \sum_{\gamma \in \mathrm{Gal}(K/k)} \log |u_i^{\gamma}|_{w_j} \gamma^{-1}. \end{aligned}$$

□

If Hypotheses 5.11 is satisfied, then $r(\chi) > r$ for all $\chi \in \widehat{G}$. Now, we define the $\mathbb{Z}[G]$-lattice $\Lambda_{S,T}$ in $\mathbb{Q} \wedge^r \mathcal{O}^*_{K,S,T}$ as

$$\Lambda_{S,T} = \{u \in \wedge_0^r \mathcal{O}^*_{K,S,T} : e_\chi u = 0 \text{ for every } \chi \text{ such that } r(\chi) > r\}.$$

Conjecture 5.16 (St(K/k,S,T,r)$'$)**.** *Suppose that Hypotheses 5.11 is satisfied and $v_1, \ldots, v_r \in S$ split completely in K/k. For each i, fix a place $w_i \in S_K$ and let $\eta = w_1^* \wedge \ldots \wedge w_r^*$. Then there is a unique $\epsilon_{S,T} \in \Lambda_{S,T}$ such that*

$$R_\eta(\epsilon_{S,T}) = \Theta_{S,T}^{(r)}(0)$$

equivalently,

$$\chi(R_\eta(\epsilon_{S,T})) = L_{S,T}^{(r)}(0, \overline{\chi}) \text{ for all } \chi \in \widehat{G}.$$

Conjecture 5.17 (St(K/k,S,T,r)$'\otimes\mathbb{Q}$)**.** *With hypotheses as in Conjecture 5.16, there is a unique $\epsilon_{S,T} \in \mathbb{Q}\Lambda_{S,T}$ such that*

$$R_\eta(\epsilon_{S,T}) = \Theta_{S,T}^{(r)}(0).$$

5.2.1 Relations between the conjectures

Let us fix $v_1, \ldots, v_r \in S$ which split completely in K/k and for each i fix a place w_i of K lying above v_i. Let S'_K be the set of places which do not lie above $v_1, \ldots, v_r$ in K. Note that S'_K is not empty, since $|S| \geq r+1$ by Hypotheses 5.11.

Lemma 5.18. *Let $w_1, \ldots, w_r$ be the places as above and $w \in S'_K$.*

(i) If $\chi \neq \mathbb{1}$ or if $|S| > r+1$ then $e_\chi \Theta^{(r)}_{S,T}(0)w = 0$ in $\mathbb{C}Y$,

(ii) Let $\boldsymbol{x} = (w_1 - w) \wedge \ldots \wedge (w_r - w) \in \wedge^r X_S$. Then,

$$\Theta^{(r)}_{S,T}(0) \wedge^r X_S = \mathbb{Z}[G]\Theta^{(r)}_{S,T}(0)\boldsymbol{x}.$$

Proof. (i) If $\chi \neq \mathbb{1}$ and $\chi(G_w) \neq \mathbb{1}$, then

$$e_\chi w = \frac{1}{|G|} \sum_{\gamma \in G_w} \sum_{\sigma G_w \in G/G_w} \chi(\sigma\gamma)\sigma\gamma w = \frac{1}{|G|} \sum_{\gamma \in G_w} \chi(\gamma) \sum_{\sigma G_w \in G/G_w} \chi(\sigma)\sigma w = 0.$$

If $\chi(G_w) = \mathbb{1}$ and $\chi \neq \mathbb{1}$ then $r(\chi) = |\{v \in S : \chi(G_w) = 1\}| \geq r+1$. Hence, $e_\chi \Theta^{(r)}_{S,T}(0) = e_\chi L^{(r)}_{S,T}(0, \overline{\chi}) = 0$.

If $\chi = \mathbb{1}$ and $|S| > r+1$ then $e_\chi \Theta^{(r)}_{S,T}(0) = e_{\mathbb{1}}\zeta^{(r)}(0) = 0$ since $r(\chi) = |S| - 1 > r$.

(ii) Suppose $|S| > r+1$. We can see X_S as

$$X_S \subset \sum_{i=1}^{r} \mathbb{Z}[G](w_i - w) + \sum_{w' \in S'_K} \mathbb{Z}[G]w'.$$

Thus, we write

$$\wedge^r X_S \subset \mathbb{Z}[G](w_1 - w) \wedge \ldots \wedge (w_r - w) + \sum_{\mathbf{y}} \alpha_{\mathbf{y}}\mathbf{y}$$

where $\mathbf{y}$ runs through $w'_1 \wedge \ldots \wedge w'_r$ and at least one of the $w'_i \in S_K$. For any $\chi \in \widehat{G}$ and for any element $\mathbf{z} \in \wedge^r X_S$,

$$e_\chi \Theta^{(r)}_{S,T}(0)\mathbf{z} = e_\chi \Theta^{(r)}_{S,T}(0)\alpha\mathbf{x}$$

where $\alpha \in \mathbb{Z}[G]$ as the component $w_i \in S'_K$ is annihilated by $e_\chi \Theta^{(r)}_{S,T}(0)$, by (i).
Hence, $\Theta^{(r)}_{S,T}(0) \wedge^r X_S \subset \mathbb{Z}[G]\Theta^{(r)}_{S,T}(0)(w_1 - w) \wedge \ldots \wedge (w_r - w)$.
Suppose $|S| = r + 1$. Then, X_S can be written in the form

$$X_S = \sum_{i=1}^{r} \mathbb{Z}[G](w_i - w) + I_G w$$

where I_G is the augmentation ideal of $\mathbb{Z}[G]$ (namely, kernel of the map $\mathbb{Z}[G] \to \mathbb{Z}$). Thus, we write

$$\wedge^r X_S = \mathbb{Z}[G](w_1 - w) \wedge \ldots \wedge (w_r - w) + \sum_{\mathbf{y}} \alpha_{\mathbf{y}} \mathbf{y} \wedge w$$

where $\alpha_{\mathbf{y}} \in I_G$. Hence, for any $\chi \in \widehat{G}$, $e_\chi \Theta^{(r)}_{S,T}(0)\alpha_{\mathbf{y}}\mathbf{y} \wedge w = 0$.
Converse inclusion also holds, so we have the equality.

□

Lemma 5.19. *Suppose $w_1, \ldots, w_r$ are as above and set $\eta = w_1^* \wedge \ldots \wedge w_r^* \in Y_S^*$.*

(i) η is injective on $\Theta^{(r)}_{S,T}(0)\mathbb{C} \wedge^r X_S = \mathbb{C}\lambda^{(r)}(\Lambda_{S,T})$,

(ii) R is injective on $\Lambda_{S,T} \otimes \mathbb{C} \to \mathbb{C}[G]$.

Proof. As we know, $r(\chi) = \dim_{\mathbb{C}} \mathrm{Hom}(V^*, \mathbb{C}X_S) = \dim_{\mathbb{C}} e_\chi \mathbb{C}X_S$. Hence:
If $r(\chi) > r$ then, $\dim_{\mathbb{C}} e_\chi \wedge^r \mathbb{C}X_S > 1$, and
if $r = r(\chi)$ then, $\dim_{\mathbb{C}} e_\chi \wedge^r \mathbb{C}X_S = 1$.
If $r(\chi) > r$ then, $\dim_{\mathbb{C}} e_\chi \Theta^{(r)}_{S,T}(0) \wedge^r \mathbb{C}X_S = 0$ by Lemma 5.18 (i), and
if $r = r(\chi)$ then, $\dim_{\mathbb{C}} e_\chi \Theta^{(r)}_{S,T}(0) \wedge^r \mathbb{C}X_S = 1$.
Since $\eta : \Theta^{(r)}_{S,T}(0) \wedge^r \mathbb{C}X_S \to \mathbb{C}[G]$, we can show the injectivity of η for the map $e_\chi \Theta^{(r)}_{S,T}(0) \wedge^r \mathbb{C}X_S \to e_\chi \mathbb{C}[G]$ as $\Theta^{(r)}_{S,T}(0) \wedge^r \mathbb{C}X_S$ is direct sum of these. Thus, it is enough to show the injectivity of η on $e_\chi \Theta^{(r)}_{S,T}(0) \wedge^r \mathbb{C}X_S$ for a character χ such that $r(\chi) = r$. For some $w \in S'_K$, let $\mathbf{x} = (w_1 - w) \wedge \ldots \wedge (w_r - w) \in X_S$. Then,

$$\eta(\mathbf{x}) = (w_1^* \wedge \ldots \wedge w_r^*)((w_1 - w) \wedge \ldots \wedge (w - w_r)) = \det(w_j^*(w_i - w)) = 1.$$

Hence, $\eta(e_\chi \Theta^{(r)}_{S,T}(0)\mathbf{x}) = L^{(r)}(0, \overline{\chi}) \neq 0$, and so η is injective on
$\Theta^{(r)}_{S,T}(0)(\wedge^r \mathbb{C}X_S) = \lambda^{(r)}(\Lambda_{S,T}) \otimes \mathbb{C}$. Since $\lambda : \mathcal{O}^*_{K,S,T} \otimes \mathbb{C} \to X \otimes \mathbb{C}$ is an isomorphism, $\lambda^{(r)}$ is also an isomorphism and $R = \eta \circ \lambda^{(r)}$ is so injective on $\Lambda_{S,T} \otimes \mathbb{C}$. □

Proposition 5.20.

Conjecture St($K/k, S, T, r$) is equivalent to Conjecture St($K/k, S, T, r$)$'$.

Proof. For some $w \in S'_K$, let $\mathbf{x} = (w_1 - w) \wedge \ldots \wedge (w_r - w) \in X_S$. Then

$$\begin{aligned}
\Theta^{(r)}_{S,T}(0) \wedge^r X_S \subset \lambda^{(r)}(\Lambda^r_0 \mathcal{O}^*_{K,S,T}) &\Leftrightarrow \Theta^{(r)}_{S,T}(0)\mathbf{x} \in \lambda^{(r)}(\Lambda^r_0 \mathcal{O}^*_{K,S,T}) \\
&\Leftrightarrow \Theta^{(r)}_{S,T}(0)\mathbf{x} \in \lambda^{(r)}(\Lambda_{S,T}) \\
&\Leftrightarrow \eta(\Theta^{(r)}_{S,T}(0)\mathbf{x}) \in \eta \circ \lambda^{(r)}(\Lambda_{S,T}) \\
&\Leftrightarrow \Theta^{(r)}_{S,T}(0) \in R(\Lambda_{S,T}) \\
&\Leftrightarrow \Theta^{(r)}_{S,T}(0) = R(\epsilon_{S,T}), \text{ where } \epsilon_{S,T} \in \Lambda_{S,T}.
\end{aligned}$$

The first equivalence comes from Lemma 5.18 (ii). The second equivalence is true since $\lambda^{(r)}$ is an isomorphism and $e_\chi \Theta^{(r)}_{S,T}(0)\mathbf{x} = 0$ when $r(\chi) > r$. The uniqueness of $\epsilon_{S,T}$ comes from the injectivity of R. □

Proposition 5.21. *Conjecture St($K/k, S, T, 1$) is equivalent to Conjecture 4.14.*

Proof. By Proposition 5.9 (ii), $\wedge^1_0 \mathcal{O}^*_{K,S,T} = \mathcal{O}^*_{K,S,T}$ and $\wedge^1 X_S = X_S$ thus St$(K/k, S, T, 1)$ implies that

$$\Theta'_{S,T}(0) X_S \subset \lambda_{S,T}(\mathcal{O}^*_{K,S,T})$$

for all T satisfying (5.11). We also have the relations

$$\Theta'_{S,T}(0) = \prod_{\mathfrak{p} \in T} (1 - \mathrm{Frob}_{\mathfrak{p}}^{-1} \mathbf{N}\mathfrak{p}), \quad \prod_{\mathfrak{p} \in T} (1 - \mathrm{Frob}_{\mathfrak{p}}^{-1} \mathbf{N}\mathfrak{p}) \mathcal{O}^*_{K,S} \subset \mathcal{O}^*_{K,S,T}$$

and so we get both implications by using these relations.

□

Proposition 5.22. *If $r(\chi) = r$, then Conjecture St($K/k, S, T, r$)$\otimes\mathbb{Q}$ is equivalent to Conjecture 3.24.*

Proof. Let $\Xi = \{\chi \in \widehat{G} : r(\chi) = r\}$ and suppose that Conjecture 3.24 holds for all $\chi \in \Xi$. Now, let us fix a $\mathbb{Q}[G]$-module isomorphism $f : \mathbb{Q}X \to \mathbb{Q}\mathcal{O}^*_{K,S,T}$ and denote the χ-th component of f as f_χ and which is

$$f_\chi : e_\chi(\mathbb{C}X) \to e_\chi(\mathbb{C}\mathcal{O}^*_{K,S,T}).$$

For any $\chi \in \Xi$ and $\alpha \in \mathrm{Aut}(\mathbb{C})$ we have

$$A(\chi, f) = \frac{\det(\lambda_{S,T} \circ f_\chi)}{L^{(r)}_{S,T}(0,\chi)}.$$

Now, define

$$\rho = \sum_{\chi \in \Xi} e_\chi A(\chi, f)^{-1}.$$

Note that $\rho \in \mathbb{Q}[G]$, since

$$\begin{aligned}\rho^\alpha &= (\sum_{\chi \in \Xi} e_\chi A(\chi, f)^{-1})^\alpha \\ &= \sum_{\chi \in \Xi} e_\chi^\alpha (A(\chi, f)^\alpha)^{-1} \\ &= \sum_{\chi \in \Xi} e_{\chi^\alpha} A(\chi^\alpha, f) = \rho\end{aligned}$$

as $e_{\chi^\alpha} = e_\chi$ and $A(\chi^\alpha, f) = A(\chi, f)^\alpha$ for all $\alpha \in \mathrm{Aut}(\mathbb{C})$.

$$\begin{aligned}\rho \sum_{\chi \in \Xi} e_\chi \det(\lambda_{S,T} \circ f_\chi) &= \sum_{\chi \in \Xi} e_\chi \det(\lambda_{S,T} \circ f_\chi) A(\chi, f)^{-1} \\ &= \sum_{\chi \in \Xi} e_\chi L^{(r)}_{S,T}(0, \overline{\chi}) = \Theta^{(r)}_{S,T}(0).\end{aligned}$$

If $\mathbf{x} \in \wedge^r X_S$ and $f^{(r)} : \mathbb{Q} \wedge^r X_S \to \mathbb{Q} \wedge^r \mathcal{O}_{K,S,T}$ denotes the map induced by f, then

$$\begin{aligned}\Theta^{(r)}_{S,T}(0)\mathbf{x} &= \rho \sum_{\chi \in \Xi} e_\chi \det(\lambda_{S,T} \circ f_\chi)\mathbf{x} \\ &= \rho \sum_{\chi \in \Xi} \lambda^{(r)} \circ f^{(r)}(e_\chi \mathbf{x}) \\ &= \lambda^{(r)}(f^{(r)}(\rho\mathbf{x})) \ \in \mathbb{Q}\lambda^{(r)}_{S,T}(\wedge^r \mathcal{O}^*_{K,S,T})\end{aligned}$$

Since $\wedge^r \mathcal{O}^*_{K,S,T}$ has finite index in $\wedge^r_0 \mathcal{O}^*_{K,S,T}$, Conjecture $\mathrm{St}(K/k, S, T, r) \otimes \mathbb{Q}$ is satisfied. The converse direction is similar. If $\lambda^{(r)}_{S,T}(f^{(r)}(\rho\mathbf{x})) \ \in \mathbb{Q}\lambda^{(r)}(\wedge^r \mathcal{O}^*_{K,S,T})$, then since $\lambda^{(r)}$ is injective, $\rho(\mathbf{x}) \in \mathbb{Q}(f^{(r)})^{-1}(\wedge^{(r)} \mathcal{O}^*_{K,S,T}) = \mathbb{Q} \wedge^{(r)} X_S$. Hence, $\rho \in \mathbb{Q}[G]$ and Conjecture 3.24 holds. □

Let $A_{k,S}$ denote the class group of $\mathcal{O}_{k,S}$; $A_{k,S,T}$ denote the quotient group of the fractional ideals of $\mathcal{O}_{k,S}$ prime to T by the subgroup of principal ideals with a generator congruent to 1 modulo all $\mathfrak{p} \in T$; $R_k = R_{k,S}$ denote the regulator of $\mathcal{O}_{k,S}$. Then, there exists an exact sequence

$$0 \to \mathcal{O}^*_{k,S,T} \to \mathcal{O}^*_{k,S} \to \prod_{\mathfrak{p}\in T}(\mathcal{O}_{k,S}/\mathfrak{p})^* \to A_{k,S,T} \to A_{k,S} \to 0. \tag{5.2}$$

We define

$$\zeta_{k,S,T}(s) = \zeta_{k,S}(s)\prod_{\mathfrak{p}\in T}(1-\mathbf{N}\mathfrak{p}^{1-s}).$$

Thus,

$$\begin{aligned}\zeta^{(r)}_{k,S,T}(0) &= \zeta^{(r)}_{k,S}(0)\prod_{\mathfrak{p}\in T}(1-\mathbf{N}\mathfrak{p})\\ &= \frac{|A_{k,S}|\cdot R_{k,S}}{|\mu(k)|}\prod_{\mathfrak{p}\in T}(1-\mathbf{N}\mathfrak{p})\end{aligned}$$

Since $\lambda_{k,S,T}$ is $\lambda_{k,S}$ composed with the inclusion $\mathcal{O}^*_{k,S,T} \hookrightarrow \mathcal{O}^*_{k,S}/\mu(k)$, the definition of $R_{k,S,T}$ gives

$$\begin{aligned}R_{k,S,T} &= [\mathcal{O}^*_{k,S,T} : \mathcal{O}^*_{k,S}/\mu(k)]\cdot R_{k,S}\\ &= \frac{[\mathcal{O}^*_{k,S,T} : \mathcal{O}^*_{k,S}]}{|\mu(k)|}\cdot R_{k,S}\end{aligned}$$

Then, by the exact sequence (5.2) we have

$$\frac{R_{k,S,T}}{R_{k,S}}|\mu(k)|\cdot|A_{k,S,T}| = \prod_{\mathfrak{p}\in T}(\mathbf{N}\mathfrak{p}-1)\cdot|A_{k,S}|,$$

and hence

$$\zeta^{(r)}_{k,S,T}(0) = -\frac{|A_{k,S}|\cdot R_{k,S}}{|\mu(k)|}\prod_{\mathfrak{p}\in T}(1-\mathbf{N}\mathfrak{p}) = \pm|A_{k,S,T}|\cdot R_{k,S,T}\ . \tag{5.3}$$

Proposition 5.23. *Suppose that S contains more then r places which splits completely in K/k. Then, St($K/k, S, T, r$)′ is true.*

Proof. If $|S| > r+1$ or S has more than r places split completely, then $\Theta^{(r)}_{S,T}(0) = 0$ as $\lim_{s\to 0} s^{-r}L_{S,T}(s,\chi) = 0$ for any $\chi \in \widehat{G}$. Hence, $\mathrm{St}(K/k, S, T, r)$ is trivially true.

If $|S| = r+1$, in which case all places in $S = \{v_1, \dots, v_{r+1}\}$ split completely in K/k, and $\chi \neq \mathbb{1}$ then $\Theta^{(r)}_{S,T}(0) = 0$. Hence by (5.3),

$$\Theta^{(r)}_{S,T}(0) = e_{\mathbb{1}}\zeta^{(r)}_{S,T}(0) = \pm\frac{\mathrm{N}_G}{|G|}\cdot|A_{k,S,T}|\cdot R_{k,S,T}$$

where $\mathrm{N}_G = \sum_{\sigma \in G} \sigma$.

Let $\{u_1, \ldots, u_r\}$ be basis of the free $\mathbb{Z}$-module $\mathcal{O}^*_{k,S,T}$. Then, define

$$\epsilon_{S,T} = \frac{|A_{k,S,T}|}{|G|^r} u_1 \wedge \ldots \wedge u_r \in (\wedge^r \mathcal{O}^*_{K,S,T})^G \otimes \mathbb{Q}.$$

If we fix $w_i|v_i$ in S_K, then for $\eta = w_1^* \wedge \ldots \wedge w_r^*$, Lemma 5.15 provides that

$$\begin{aligned} R_\eta(\epsilon_{S,T}) &= \frac{|A_{k,S,T}|}{|G|^r} \det\Big(\sum_{\gamma \in \mathrm{Gal}(K/k)} \log |u_i^\gamma|_{w_j} \gamma^{-1}\Big) \\ &= \pm \frac{|A_{k,S,T}|}{|G|^r} \cdot N_G^r \cdot R_{k,S,T} = \pm e_{\mathbb{1}} \cdot |A_{k,S,T}| \cdot R_{k,S,T} \\ &= \pm e_{\mathbb{1}} \zeta^{(r)}_{S,T}(0) = \Theta^{(r)}_{S,T}(0) \end{aligned}$$

Now, we need to show $\epsilon_{S,T} \in \wedge_{S,T}$. Let us take $\phi_1, \ldots, \phi_r \in \mathrm{Hom}(\mathcal{O}^*_{K,S,T}, \mathbb{Z}[G])$. Since $u_j \in (\mathcal{O}^*_{K,S,T})^G$ then $\phi_i(u_j) \in \mathbb{Z}[G]^G = \mathrm{N}_G \mathbb{Z}[G]$ and we have

$$\phi_1 \wedge \ldots \wedge \phi_r(\epsilon_{S,T}) = \frac{|A_{k,S,T}|}{|G|^r} \det(\phi_i(u_j)) \in N_G^r \mathbb{Z}[G]$$

where $N_G^r \mathbb{Z}[G] = |G|^{r-1} \mathrm{N}_G$. Hence,

$$\frac{|A_{k,S,T}|}{|G|^r} \det(\phi_i(u_j)) \in \frac{|A_{k,S,T}|}{|G|} \mathrm{N}_G \mathbb{Z}[G].$$

We assumed that $|S| = r + 1$ and all places split completely so K/k is an unramified extension. Thus, by class field theory $|G|$ divides $|A_{k,S}|$ and $|A_{k,S}|$ also divides $|A_{k,S,T}|$ which implies $\frac{|A_{k,S,T}|}{|G|} \in \mathbb{Z}$. Hence, $\epsilon_{S,T} \in \wedge_0^r \mathcal{O}^*_{K,S,T}$. Lastly, for $\chi \neq 1$, $e_\chi \epsilon_{S,T} = \frac{|A_{k,S,T}|}{|G|^r}(e_\chi u_1 \wedge \ldots \wedge u_r) = 0$ it follows that $\epsilon_{S,T} \in \wedge_{S,T}$. So, $\mathrm{St}(K/k, S, T, r)$ is true. □

Corollary 5.24. *Suppose S, T, r satisfies Hypotheses 5.11. Then St($K/k, S, T, r$) is true when $K = k$.*

Proof. When $K = k$, all places ramify and so $|S|$ contains more than $r + 1$ places. Hence, $\mathrm{St}(K/k, S, T, r)$ is true by Proposition 5.23. □

Proposition 5.25. *Suppose S, T, r satisfies Hypotheses 5.11 and $S' \supset S$ is a finite set of places of k disjoint from T. Then, S', T, r satisfies Hypotheses 5.11 and St($K/k, S, T, r$) implies St($K/k, S', T, r$).*

Proof. Suppose $\mathrm{St}(K/k, S, T, r)$ is true and $\epsilon_{S,T} \in \wedge_{S,T}$ which satisfies $R(\epsilon_{S,T}) = \Theta^{(r)}_{S,T}(0)$. Let $S' = S \cup \{\mathfrak{p}\}$ for a place $\mathfrak{p} \notin S \cup T$. Then,

$$\Theta^{(r)}_{S',T}(0) = (1 - \mathrm{Frob}_{\mathfrak{p}}^{-1})\Theta^{(r)}_{S,T}(0).$$

Let us take $\epsilon_{S',T} = (1-\mathrm{Frob}_{\mathfrak{p}}^{-1})\epsilon_{S,T} \in \wedge_{S',T}$ which can be taken since $(1-\mathrm{Frob}_{v}^{-1})\wedge_{S,T} \subset \wedge_{S',T}$. Hence,

$$R(\epsilon_{S',T}) = (1 - \mathrm{Frob}_{\mathfrak{p}}^{-1})R(\epsilon_{S,T}) = \Theta^{(r)}_{S',T}(0).$$

□

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