ON A RESOLUTION OF SOME NON-ISOLATED HYPERSURFACE SINGULARITIES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF SCIENCES AND ENGINEERING
OF
KOÇ UNIVERSITY

BY

GÜLEN ÇEVİK

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

SEPTEMBER 2011

Approval of the thesis:

ON A RESOLUTION OF SOME NON-ISOLATED HYPERSURFACE SINGULARITIES

submitted by **GÜLEN ÇEVİK** in partial fulfillment of the requirements for the degree of **Master of Science in Mathematics , Koç University**:

Examining Committee Members:

Assist. Prof. Sinan Ünver
Department of Mathematics, Koç University ________________

Assoc. Prof. Meral Tosun
Department of Mathematics, Galatasaray University ________________

Assoc. Prof. Tolga Etgü
Department of Mathematics, Koç University ________________

Assoc. Prof. Burak Özbağcı
Department of Mathematics, Koç University ________________

Assist. Prof. Pınar Mete
Department of Mathematics, Balıkesir University ________________

Date: ________________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: GÜLEN ÇEVİK

Signature :

ABSTRACT

ON A RESOLUTION OF SOME NON-ISOLATED HYPERSURFACE SINGULARITIES

Çevik, Gülen

M.S, Department of Mathematics

Supervisor : Assist. Prof. Sinan Ünver

Co-Supervisor : Assoc. Prof. Meral Tosun

September 2011, 52 pages

In 1970s, F. Ehlers [6] and A. Kouchnirenko [13] defined the class of nondegenerate hypersurface singularities and computed some invariants of these singularities from the Newton polyhedron. The notion of nondegeneracy is extended to complete intersection singularities by A. Khovanskii [12]. In 1993, in his book M.Oka gave an algorithm to construct a resolution of a nondegenerate complete intersection singularity by using toric modification associated with Newton polyhedron. In this thesis we apply Oka's algorithm to find explicitly the minimal resolution graph of ADE singularities and we show that the same algorithm gives the minimal resolution graph of some non-isolated hypersurface singularities. We also give an example that the minimal resolution graph of degenerate non-isolated singularities can be obtained by Newton polyhedron and toric modification.

Keywords: surface singularities, Newton polyhedron, toric modification

ÖZET

BAZI AYRIK OLMAYAN HİPERYÜZEY TEKİLLİKLERİNİN ÇÖZÜMLEMESİ ÜZERİNE

Çevik, Gülen
Yüksek Lisans, Matematik Bölümü
Tez Yöneticisi : Yard. Doç. Dr. Sinan Ünver
Ortak Tez Yöneticisi : Doç. Dr. Meral Tosun
Eylül 2011, 52 sayfa

1970'lerde, F.Ehlers [6] ve A. Kouchnirenko [13] belirli dejenere olmayan hiperyüzey tekillik sınıfı tanımlamış ve bu tekilliklerin bazı değişmezlerini Newton polihedronunu kullanarak hesaplamıştır. A. Khovanskii, [12], dejenere olmama tanımını tam kesişme tekilliklerine genişletmiştir. 1993 yılında ise M. Oka kitabında torik modifikasyon vasıtası ile dejenere olmayan tam kesişme tekilliklerinin çözümlemesinin inşasını tarif etmiştir. Bu tezde M. Oka'nın tarif ettiği algoritma kullanılarak ADE tipindeki tekilliklerin minimal çözümleme çizgesi bulunmuş ve aynı zamanda bazı ayrık olmayan hiperyüzey tekilliklerinin aynı algoritma kullanılarak minimal çözümleme çizgesinin bulunabileceği gösterilmiştir. Buna ek olarak bazı dejenere ayrık olmayan tekilliklerin minimal çözümleme çizgesinin Newton polihedronu ve torsal modifikasyon kullanılarak elde edilebileceği bir örnek ile gösterilmiştir.

Anahtar Kelimeler: yüzey tekillikleri, Newton poligonu, torsal modifikasyon

To my family

ACKNOWLEDGMENTS

I would like to express my sincere thanks to my advisors Assist. Prof. Sinan Ünver and Assoc. Prof. Meral Tosun for their great supervision and continuous support. I am especially very grateful to Assoc. Prof. Meral Tosun for introducing me to this wonderful topic. Without her endless support, guidance and patience, I would not have finished this thesis.

I would also like to thank Assoc. Prof. Tolga Etgü, Assoc. Prof. Burak Özbağcı and Assist. Prof. Pınar Mete for their valuable participation in my thesis committee.

I am very thankful to all my friends, especially Pınar Kılıçer, for their motivation and friendship.

My deepest gratitude is to my family, for their endeavors, love and support through my entire life. Their presence made everything easier. I would especially like to thank my sister, Selen Çevik Baranoğlu, for her lasting faith in me and last-minute corrections.

Finally, I would like to express my sincere gratitude to Ayberk Zeytin for his unlimited support, love, affection and encouragement throughout my preparation of the thesis. Without him, I could never made it this far. Thank you for always being with me...

TABLE OF CONTENTS

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

In singularity theory the main problem is to find a non-singular algebraic variety, called resolution of singularities which is a sequence of blowing up maps together with normalization. The existence of resolution of singularities in dimension 2 over $\mathbb{C}$ proved by Robert J. Walker in 1935 [18] and in 1964, Japanese mathematician Heisuke Hironaka proved that any algebraic variety over a field of characteristic zero admits a resolution of singularities. Hironaka was awarded a Fields Medal for this work in 1970. However, the existence of a resolution does not help to determine explicitly the process of resolution, especially in higher dimension; a curve singularity can be resolved by a sequence of blowing up or normalization. The surface singularities can be isolated or non-isolated and their resolution can be rather complicated. There are some methods which enables us to rather overcome the difficulty of the resolution process. The simplest singularities of surface are of type ADE, of type quotient, of type simple elliptic, of type rational,... etc. These are classified according to the similarity of their resolution process. One of the important class of surface singularities are Newton non-degenerate singularities which are introduced, in hypersurface case, by F. Ehlers [6] and A. Kouchnirenko [13]. The importance of these singularities is coming from the fact that there is an explicit construction of resolution which depends only on Newton boundary (see below).

Given any polynomial f, let $V = \mathbb{V}(f)$ be an algebraic variety over an affine n-space. The Newton polyhedron of f is defined as the intersection of finitely many rational halfspaces. In 1980s, Mutsuo Oka give an explicit way to construct resolution of hypersurface and complete intersection singularities. We call this construction Oka's algorithm. Our aim here is to describe Newton non-degeneracy condition for non-isolated hypersurface singularities. To do this, we consider a class of some non-isolated hypersurface singularities and we construct their resolution via "Newton polyhedron" by applying Oka's algorithm.

In Chapter 2 of this work we recall some basic definitions and results on the subject. In Chapter 3 we first construct explicitly the minimal resolution of ADE singularities which appears in [15]. Then we construct a resolution of some non-isolated singularities of hypersurfaces. It is well known that an isolated surface singularity is obtained as the normalization of a non-isolated hypersurface singularity. We were planning to establish a relation between the Newton polyhedron of our non-isolated hypersurface singularity and the Newton polyhedron of its normalization but this will appear as a forthcoming work. In Chapter 4 we give the resolution of some surface singularities via toric modification. We obtain an example of non-isolated hypersurface singularity which is degenerate and which admits a resolution by toric modification.

CHAPTER 2

PRELIMINARIES

2.1 Cones and Fans

Let M be an integral lattice and N be its dual integral lattice. Let $M_{\mathbb{R}} := M \otimes \mathbb{R}^n$ and $N_{\mathbb{R}} := N \otimes \mathbb{R}^n$ be corresponding n-dimensional real vector spaces. As $N_{\mathbb{R}}$ is the dual space of $M_{\mathbb{R}}$, a point $w = (w_1, \cdots, w_n)$ in $N_{\mathbb{R}}$ defines a linear functional on $M_{\mathbb{R}}$. Namely $w : M_{\mathbb{R}} \to \mathbb{R}$ such that $m \mapsto w(m) = \langle w, m \rangle$ where $m \in M_{\mathbb{R}}$.

Definition 2.1.1. *A polyhedral cone in $N_{\mathbb{R}}$ is the set*

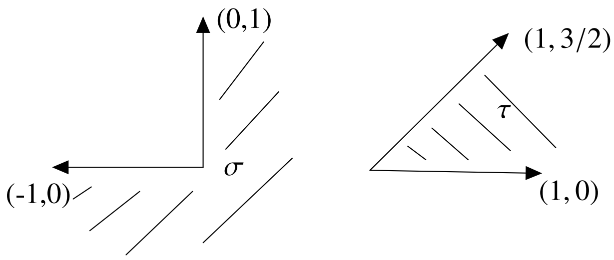
$$Cone(w_1, \ldots, w_n) = \{\textstyle\sum_{i=1}^n r_i w_i \colon w_i \in N_{\mathbb{R}}, r_i \in \mathbb{R}_{\geq 0}\}$$

The finite set of vectors $\{w_1, \ldots, w_n\}$ is called *generators* of the cone. We will consider w_i's as primitive vectors, that is the coordinates of w are coprime. If there is no confusion, σ will be used instead of $Cone(w_1, \ldots, w_n)$.

The *dimension* of σ is the dimension of the smallest subspace containing σ. A cone is said to be *rational* if it has integral generators. A cone is said to be *strictly convex* if it does not contain any positive dimensional vector subspace.

Remark 2.1.2. *A polyhedral cone is in fact* **convex** *which means that for any $a, b \in \sigma$ implies $ta + (1-t)b \in \sigma$ for all $0 \leq t \leq 1$.*

Example 2.1.3. The cone $\sigma = \langle(-1, 0), (0, 1)\rangle \subseteq N_{\mathbb{R}}$ is a rational cone but it is not strictly convex since it contains a line passing through the origin. The cone $\tau = \langle(1, 0), (1, 3/2)\rangle$ is strictly convex cone but not rational.



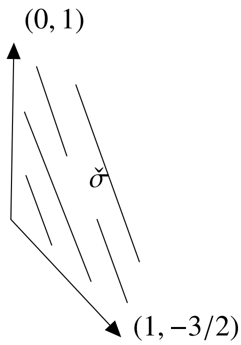
Throughout this thesis the word 'cone' will stand for a strictly convex rational polyhedral cone.

Definition 2.1.4. *The dual $\check{\sigma}$ of σ is the set*

$$\check{\sigma} = \{m \in M_{\mathbb{R}} : \langle w, m\rangle \geq 0 \ \forall w \in \sigma\}$$

Remark 2.1.5. *If σ is a cone then so is $\check{\sigma}$.*

Example 2.1.6. The dual $\check{\sigma}$ of σ above will give again σ itself. However the dual cone of τ is the following:



For a fixed $m \in M_{\mathbb{R}}$, we define the hyperplane

$$H_m = \{w \in N_{\mathbb{R}} : \langle w, m\rangle = 0\}$$

and the half-space

$$H_m^+ = \{w \in N_{\mathbb{R}} : \langle w, m\rangle \geq 0\}.$$

If $\sigma \cap H_m \neq \emptyset$ and $\sigma \subseteq H_m^+$ with $m \in \check{\sigma} - \{0\}$ then H_m is said to be the *supporting hyperplane* of the cone σ.

Remark 2.1.7. *σ can also be expressed as the intersection of the half-spaces H_m^+ for any $m \in \check{\sigma}$.*

Definition 2.1.8. *A face τ of σ is*

$$\tau = \sigma \cap H_m = \{w \in \sigma : \langle w, m \rangle = 0\}$$

for some $m \in \check{\sigma}$.

It is denoted by $\tau \preceq \sigma$. If $m = 0$ then $\sigma \cap H_m = \sigma$. So σ is a face of itself. If $\tau \neq \sigma$ then we call τ *proper face* and write $\tau \prec \sigma$.

A *vertex* of σ is a face of dimension 0. An *edge* of σ is a face of dimension 2. A *facet* of σ is a face of codimension 1.

Definition 2.1.9. *Let $\tau \preceq \sigma \subseteq N_{\mathbb{R}}$. The dual $\check{\tau}$ of τ is the set $\check{\tau} = \check{\sigma} \cap \tau^{\perp}$ where $\tau^{\perp} := \{m \in M_{\mathbb{R}} : \langle w, m \rangle = 0 \ \forall w \in \tau\}$.*

Proposition 2.1.10. *Let τ be a face of σ and $\check{\tau} = \check{\sigma} \cap \tau^{\perp}$. Then;*

i. $\check{\tau}$ is a face of $\check{\sigma}$.

ii. The map $\tau \mapsto \check{\tau}$ is a bijective inclusion-reversing correspondence between the faces of σ and the faces of $\check{\sigma}$.

iii. dim $\check{\tau}$ + dim $\tau = n$.

Example 2.1.11. Let $\sigma = \langle (1,0), (1,2) \rangle \subseteq N_{\mathbb{R}}$ and τ_1, τ_2 be facets of σ. Then dual cone $\check{\sigma}$ is generated by $\langle (2,-1), (0,1) \rangle$. The dual faces $\check{\tau}_1 = \check{\sigma} \cap \tau_1^{\perp}$ and $\check{\tau}_2 = \check{\sigma} \cap \tau_2^{\perp}$ are facets of $\check{\sigma}$.

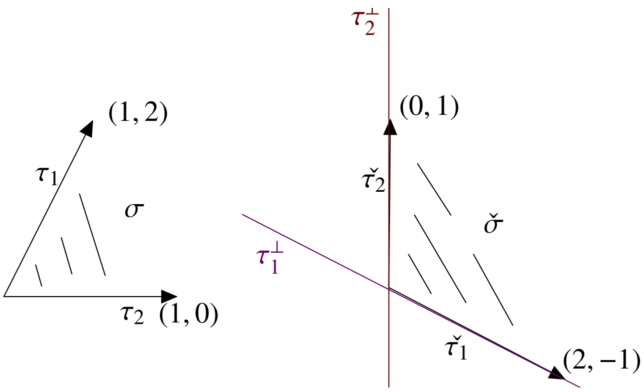


Figure 2.1: The cone σ and its dual $\check{\sigma}$

Note that *dim* τ_i + *dim* $\check{\tau}_i = 2$ for $i = 1, 2$.

Definition 2.1.12. *Let $\sigma = \langle w_1, \ldots, w_k \rangle$ be a cone in $N_{\mathbb{R}}$. The **determinant** of σ,denoted $det(w_1, \ldots, w_k)$, is the greatest common divisor of $k \times k$ minors of the integral $n \times k$ matrix. If $n = k$ then the determinant of σ is the usual determinant. If there is no ambiguity, we denote $det(\sigma)$.*

Remark 2.1.13. *By applying elementary row operations one can always multiply the determinant by -1. To avoid ambiguity, from now on we will consider determinant to be a positive integer.*

Example 2.1.14. Let $\sigma = \langle (3,2,1), (1,2,3) \rangle \subseteq N_{\mathbb{R}}$. Then

$$det(\sigma) = gcd(\left(\begin{smallmatrix} 3 & 2 \\ 1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)) = gcd(4, 4, 8) = 4.$$

Definition 2.1.15. *Let σ be a cone in $N_{\mathbb{R}}$. Then*

i. *σ is **simplicial** if the generating set $\{w_1, \ldots, w_k\}$ of σ is linearly independent over $\mathbb{R}$.*
ii. *σ is **regular** if the generators form a basis for the lattice $N \simeq \mathbb{Z}^n$. In other words if $det(w_1, \ldots, w_n) = 1$.*

Example 2.1.16. The cone $\sigma_1 = \langle (1,2), (3,1) \rangle \subseteq N_{\mathbb{R}}$ is simplicial because the generators are linearly independent, however it is not regular as $det((1,2),(3,1)) = 5 \neq 1$. The cone $\sigma_1 = \langle (1,0), (1,1), (0,1) \rangle$ is not simplicial because $(1,1) = (1,0) + (0,1)$ hence it is not regular.

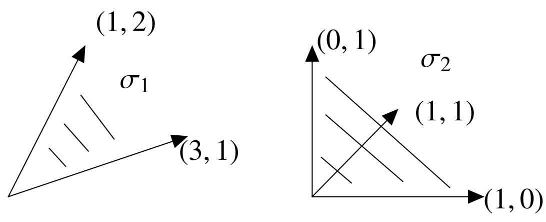


Figure 2.2: The cone σ_1 is simplicial but σ_2 is not

Example 2.1.17. The cone σ in Example 2.1.14 is simplicial but it is not regular as $det(\sigma) \neq 1$.

A *fan* Σ in $N_{\mathbb{R}}$ is a finite family of strictly convex rational polyhedral cones $\sigma \subseteq N_{\mathbb{R}}$ such that *i.* Each face of any cone σ is also a cone in Σ.

ii. Any intersection of two cones in Σ is a face of each cone.

Moreover, if Σ is a fan then the *support* of Σ is defined as $\bigcup_{\sigma\in\Sigma}\sigma$.

Definition 2.1.18. *Let Σ be a fan in $N_{\mathbb{R}}$. Then,*

i. *Σ is **simplicial** if every cone σ in Σ is simplicial.*

ii. *Σ is **regular** if every cone σ in Σ is regular.*

iii. *Σ is **complete** if its support $\bigcup_{\sigma\in\Sigma}\sigma = N_{\mathbb{R}}$.*

Example 2.1.19. Let $\Sigma_1 = \tau_1 \cup \tau_2$ and $\Sigma_2 = \sigma_1 \cup \sigma_2$ be two fans in $N_{\mathbb{R}}$(see below figure). Then Σ_1 is regular since each cone τ_1 and τ_2 are regular (as $det(\tau_1) = det(\tau_2) = 1$). The fan Σ_2 is not regular since σ_1 is not regular as $det(\sigma_1) = 2$. Here $v_1 = (1,0), v_2 = (1,1), v_3 = (0,1)$, $w_1 = (1,0), w_2 = (1,2), w_3 = (0,1)$.

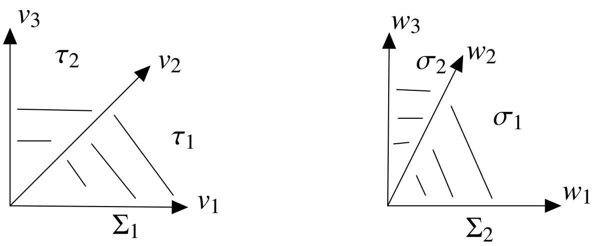


Figure 2.3: Regular fan Σ_1 and non-regular fan Σ_2

Note that the support of both fans is the positive quadrant ${N_{\mathbb{R}}}^+$. So both fans are not complete.

Definition 2.1.20. *We say that Σ^* is a refinement of a fan Σ if each cone of Σ^* is contained in a cone of Σ. If each cone of Σ is regular then Σ^* is called a refinement of a regular fan.*

Theorem 2.1.21 ([11])**.** *A rational fan can always be refined into a fan.*

2.2 Normalization of varieties

Let k be an algebraically closed field of characteristic 0. The set

$$\mathbb{A}^n(k) = \{\mathbf{z} = (z_1, \ldots, z_n) | z_i \in k\}$$

is called the *affine n-dimensional space* over k. Let f be a polynomial in $k[z_1, \ldots, z_n]$. The set

$$V = \mathbb{V}(f) = \{\mathbf{z} \in \mathbb{A}^n(k) : f(\mathbf{z}) = 0\}$$

is called a *hypersurface* in $\mathbb{A}^n(k)$. The set

$$V(f_1, \ldots, f_k) = \{\mathbf{z} \in \mathbb{A}^n : f_1(\mathbf{z}) = \cdots = f_k(\mathbf{z}) = 0\}$$

is called a *complete intersection variety* if the defining ideal of V is also generated by k functions.

Let V be an affine variety defined by f. A point $p \in V$ is called *singular point* if

$$(\frac{\partial f}{\partial z_j}(p))_{1 \leq i \leq k} = 0.$$

The set of singular points of V is called the *singular locus* and denoted by V_{Sing}. V is said to be *smooth* (or non-singular) if there is no singular point.

Remark 2.2.1. *The singular locus V_{Sing} is a subvariety of V.*

Example 2.2.2. Let $f(z_1, z_2) = z_2^2 - z_1^3$ then $(0, 0)$ is a singular point of $\mathbb{V}(f)$ because t $[\frac{\partial f}{\partial z_1} \ \frac{\partial f}{\partial z_2}] = [-3z_1^2 \ \ 2z_2]$ is 0 at the point $(0, 0)$.

A singular point p is said to be *isolated* if there is no other singularities in some small neighbourhood of p. If we can find other singularities in some small neighbourhood of p then p is called a *non-isolated* singular point.

Example 2.2.3. Consider $f(z_1, z_2, z_3) = z_1^2 + z_2^3 + z_3^4$, then $\mathbb{V}(f)$ has an isolated singularity at $(0, 0, 0)$.

Example 2.2.4. Let $f(z_1, z_2, z_3) = z_1^2 z_2^2 + z_2^3 z_3 + z_3^3$. $\mathbb{V}(f)$ has a singularity along z_1-axis because $[\frac{\partial f}{\partial z_1} \ \frac{\partial f}{\partial z_2} \ \frac{\partial f}{\partial z_3}] = [2z_1 z_2^2 \ \ 2z_1^2 z_2 + 3z_2^2 z_3 \ \ z_2^3 + 3z_3^2] = (0, 0, 0)$ along the line $z_2 = 0, z_3 = 0$. So f has a non-isolated singularity.

Let R be an integral domain with field of fraction K. Let f be any monic polynomial in $R[z]$. R is called *integrally closed* if every root $m/n \in K$ of f is in R. The set of elements of K which satisfies f is $\overline{R}$, integrally closure of R.

Let V be an affine variety. V is said to be *normal* at a point $p \in V$ if the local ring O_p is integrally closed. V is normal if it is *normal* at every point of V. As normality is a local property; [1] V is normal if its coordinate ring is normal.

Remark 2.2.5. *If V is a non-singular variety, then it is normal since regular local rings are normal.*

Example 2.2.6. $\mathbb{A}^n(k)$ is a normal variety since its coordinate ring is $\mathbb{C}[z_1, \cdots, z_n]$; recall that a UFD is a normal ring.

Example 2.2.7. Let $V = \mathbb{V}(z_2^2 - z_1^3)$. Then the coordinate ring of V is $\mathbb{C}[z_1, z_2]/(z_2^2 - z_1^3) \cong \mathbb{C}[t^2, t^3]$. The root of the monic polynomial $z_1^2 - t^2 \in \mathbb{C}[t^2, t^3][z_1]$ is t, but $t \notin \mathbb{C}[t^2, t^3]$. So $\mathbb{C}[t^2, t^3]$ is not a normal ring, hence V is not a normal variety.

If a variety is not normal we can normalize it:

Definition 2.2.8. *Let V be an affine variety with a coordinate ring O_V. Then W is called the normalization of V if $O_W = \overline{O_V}$ together with a birational map $f : W \to V$ where $\overline{O_V}$ is the integral closure of O_V.*

Example 2.2.9. In Example 2.2.7 we have seen that the variety V is not normal. Now we want to normalize the non-normal coordinate ring $\mathbb{C}[t^2, t^3]$. Observe that the only missing element in $\mathbb{C}[t^2, t^3]$ is t. Since t is a root of the equation $z_1^2 - t^2 \in \mathbb{C}[t^2, t^3][z_1]$, t and every element of $\mathbb{C}[t]$ is in the normalization of $\mathbb{C}[t^2, t^3]$. $\mathbb{C}[t]$ is UFD hence it is normal. Therefore $\mathbb{C}[t]$ is *normalization* of $\mathbb{C}[t^2, t^3]$ and $\mathbb{C}$ is the corresponding normal variety with birational map $f : \mathbb{C} \to V$ such that $t \mapsto (t^2, t^3)$.

2.3 Resolution of singularities

Let V be an affine variety in $\mathbb{A}^n(k)$. A *resolution* (or desingularization) of V at a point p is a map $\pi : \tilde{V} \to V$ such that

i. $\tilde{V}$ is a non-singular variety.

ii. π is proper map.

iii. $\tilde{V} - \pi^{-1}(p) \longrightarrow V - \{p\}$ is an isomorphism.

The inverse image of the point p by the map π, $\pi^{-1}(p)$, is called the *exceptional divisor*. A resolution is called *good* if the exceptional divisor $E := \pi^{-1}(p)$ is a normal crossing divisor, i.e if E consists of a finite union of smooth irreducible components,say $E_1, \ldots, E_k$, intersecting transversally.

A resolution is called *minimal* if any other resolution $\pi' : \tilde{V}' \to V$ factorizes by π.

$$\begin{array}{ccc} \tilde{V}' & \xrightarrow{\pi'} & V \\ & \searrow \quad \nearrow_{\pi} & \\ & \tilde{V} & \end{array}$$

Theorem 2.3.1. *[10] A resolution of a surface exists and it is obtained by a sequence of normalized blowing-ups of singular points.*

Suppose $\pi : \tilde{V} \to V$ be a good resolution and let $E = \bigcup_{i=1}^{k} E_i$ be the exceptional divisor. We associate a weighted graph $\mathcal{G}$ to the resolution π as follows:

i. A vertex v_i of $\mathcal{G}$ corresponds to each component $E_i = E(v_i)$ of E.

ii. Two vertices v_i and v_j are attached by an edge if and only if the intersection $E_i \cap E_j$ is non empty.

iii. Each vertex v_i is decorated by a weight $-m_i$ which is a self intersection number E_i^2.

The graph $\mathcal{G}$ is called the *resolution graph* of the resolution $\pi : \tilde{V} \to V$.

If in $\mathcal{G}$ the vertex of weight -1 does not exist or is of valency ≥ 3, then $\mathcal{G}$ is called the minimal resolution graph.

CHAPTER 3

RESOLUTION VIA NEWTON POLYHEDRON

3.1 Newton Polyhedron

In this section we introduce the notion of Newton polyhedron associated with a hypersurface. Then we define a polyhedral cone subdivision of the given Newton polyhedron.

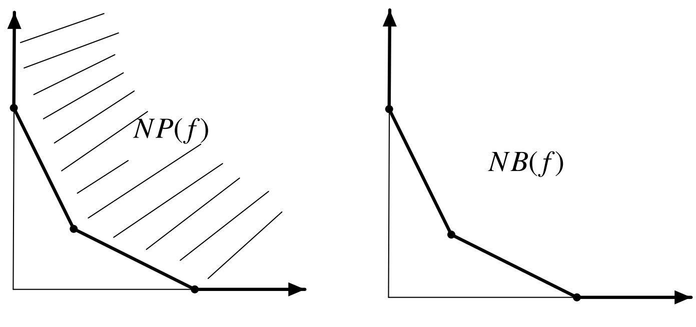
Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic function germ of n variables defined by $f(z) = \sum_\nu a_\nu z^\nu$ where $\nu = (\nu_1, \cdots \nu_n) \in \mathbb{Z}^n$ and $z^\nu = (z_1^{\nu_1} ... z_n^{\nu_n})$.

Definition 3.1.1. *The Newton polyhedron $NP(f)$ of f is the convex closure of*

$$\bigcup_{\nu \in supp(f)} \{\nu + \mathbb{R}^n_{\geqslant 0}\} \subseteq \mathbb{R}^n$$

where the support of f, denoted by supp(f), is defined as $\{\nu : a_\nu \neq 0\}$. The Newton boundary $NB(f)$ of f is the union of boundary faces of $NP(f)$.

Example 3.1.2. Let $f(z_1, z_2) = z_1^3 + z_1 z_2 + z_2^3$. We have $\text{supp}(f) = \{(3, 0), (1, 1), (0, 3)\}$, so the $NP(f)$ and the $NB(f)$ is as follows:



(a) Newton polyhedron of f (b) Newton boundary of f

Example 3.1.3. Let $f(z_1, z_2, z_3) = z_1^2 + z_2^3 + z_3^4$. Then $\operatorname{supp}(f) = \{(2,0,0), (0,3,0), (0,0,4)\}$. The Newton boundary of f contains one compact and three non-compact facets, three edges and three vertices.

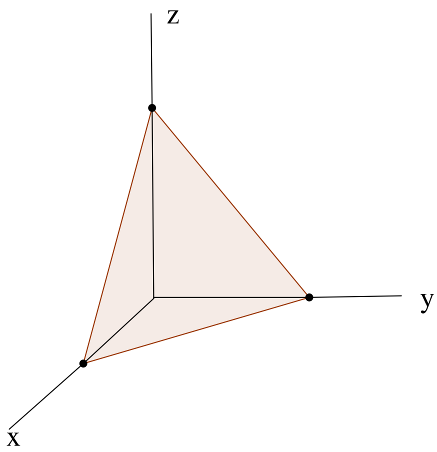


Figure 3.1: The Newton polyhedron of $z_1^2 + z_2^3 + z_3^4$

Definition 3.1.4. *A support function* $s\colon \check{\mathbb{R}}^n_{\geq 0} \to \mathbb{R}$ *associated with* $NP(f)$ *defined by*

$$s(w) = \min_{v \in NP(f)} \langle w, v \rangle$$

The support function takes integral values on the lattice N and it is piecewise linear in its domain.

By using the support function, one can define the Newton polyhedron in the following way:

$$NP(f) = \{v \in \mathbb{R}^n \colon \langle w, v \rangle \geq s(w)\ \forall w \in \check{\mathbb{R}}^n_{\geq 0}\}$$

Given a vector $w \in \check{\mathbb{R}}^n_{\geq 0}$. A *face* of $NP(f)$ with respect to w is defined as

$$F_w := \{v \in NP(f) \colon \langle w, v \rangle = s(w)\} \subseteq \mathbb{R}^n$$

Remark that w is the normal vector to the face F_w.

We define an equivalence relation on $\check{\mathbb{R}}^n_{\geq 0}$: $w \sim w'$ if and only if $F_w = F_{w'}$. In other words, any two element is in the same equivalence class if and only if they take their minimum values on the same face of $NP(f)$. Hence to each equivalence class of w

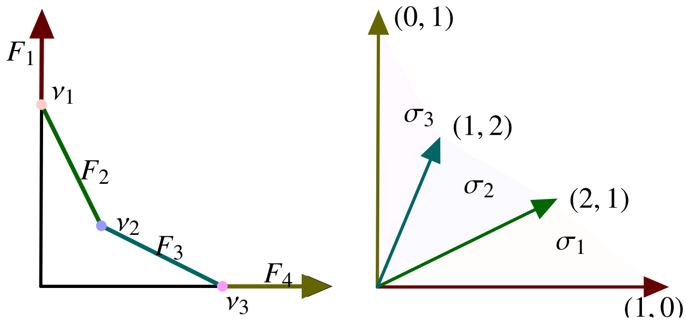
$$[w] := \{w \in \check{\mathbb{R}}^n_{\geq 0} : \langle w, v \rangle = s(w) \ \forall v \in NP(f)\}$$

we associate a cone, the inner product of any element of this cone is minimized by these v. These cones form a decomposition of $\check{\mathbb{R}}^n_{\geq 0}$. The collection of these cones constitutes a fan, denoted $DNP(f)$, called the *dual fan* of the Newton polyhedron. The elements of $DNP(f)$ is called *dual vectors*.

Remark 3.1.5. *There is a one to one correspondence between the faces of all dimensions of $NP(f)$ and the cones of $DNP(f)$.*

Example 3.1.6. Let $f(z_1, z_2) = z_1^3 + z_1z_2 + z_2^3 = 0$.

We have $\text{supp}(f) = \{v_1, v_2, v_3\} = \{(0,3), (1,1), (3,0)\}$. The $NP(f) \subseteq \mathbb{R}^2$ has three vertices v_1, v_2, v_3, two compact faces F_2, F_3 and two non-compact faces F_1, F_4 respectively.



(a) $NP(f)$ (b) $DNP(f)$

To compute $DNP(f)$ first we find normal vectors to the faces. The equations of the lines passing through F_2, F_3 are $2x + y - 3 = 0$ and $x + 2y - 3 = 0$ respectively. So the normal vector to F_2 is $(2, 1)$ and to F_3 is $(1, 2)$. The normal vectors to the non-compact faces F_1, F_4 are $(1, 0), (0, 1)$ respectively. As it can be seen in the figure these vectors form a decomposition of $\check{\mathbb{R}}^2_{\geq 0}$. Let $w \in \sigma_1$ then $w = a(2, 1) + b(1, 0) = (2a + b, a)$ for some $a, b > 0$. The inner products

$$\langle (2a + b, a), v_1 \rangle = 3a = s(w)$$

$$\langle (2a + b, a), v_2 \rangle = 3a + b$$

$$\langle (2a + b, a), v_3 \rangle = 6a + 3b$$

The elements of σ_1 take their minimum value on the vertex v_1. Similarly, v_2, v_3 minimizes the inner product of the elements of σ_2, σ_3 respectively. The dual fan of Newton polyhedron is $DNP(f) = \bigcup_{i=1}^3 \sigma_i$.

Example 3.1.7. Let $f(z_1, z_2, z_3) = z_1^2 + z_2^3 + z_3^4$.

We have $\text{supp}(f) = \{v_1 = (2,0,0), v_2 = (0,3,0), v_3 = (0,0,4)\}$. The $NP(f) \subseteq \mathbb{R}^3$ contains three vertices v_1, v_2, v_3, three edges $[v_1, v_2], [v_1, v_3], [v_2, v_3]$,and one compact facet F_P and three non-compact facets F_1, F_2, F_3 where $F_i = \mathbb{V}(z_i)$ for $i = 1, 2, 3$.

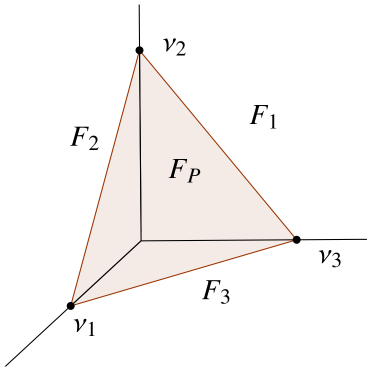


Figure 3.2: $NP(f)$

The normal vector to compact facet F_P is computed as follows: $v_1 - v_2 = (2, -3, 0)$ and $v_1 - v_3 = (2, 0, -4)$ then the perperdicular vector to both these vectors is:

$$\begin{pmatrix} i & j & k \\ 2 & -3 & 0 \\ 2 & 0 & -4 \end{pmatrix} = (12, 8, 6)$$

The normal vector to F_P is the primitive vector $P = (6, 4, 3)$. For the non-compact facets F_1, F_2, F_3 the normal vectors are $E_1 := (1, 0, 0)$, $E_2 := (0, 1, 0)$, $E_3 := (0, 0, 1)$ respectively. The dual fan $DNP(f) \subseteq \check{\mathbb{R}}^3_{\geq 0}$ is:

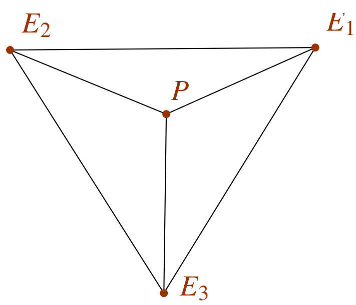


Figure 3.3: $DNP(f)$

The $DNP(f)$ has three 2-dimensional cones:

1. $\sigma_1 = \langle(6,4,3),(1,0,0)\rangle$
2. $\sigma_2 = \langle(6,4,3),(0,1,0)\rangle$
3. $\sigma_3 = \langle(6,4,3),(0,0,1)\rangle$

In what follows we will say that E_1, E_2, E_3 form the **canonical basis** for $\mathbb{R}^3$.

3.2 Minimal resolution graphs of ADE singularities via Newton Polyhedron

In this section we will consider a hypersurface in $\mathbb{A}^3(\mathbb{C})$ with a singularity of type ADE. We will first write the corresponding Newton polyhedron then we will construct a regular subdivision of this polyhedron. We will see that there exist a suitable regular subdivision of Newton polyhedron which gives the minimal resolution graph of the singularity. We will call this construction as Oka's algorithm [15].

Theorem 3.2.1 ([3]). *Let f be an analytic function germ of n variables defined by $f(z) = \sum_\nu a_\nu z^\nu$ where $\nu = (\nu_1, \ldots, \nu_n)$ and $z^\nu = (z_1^{\nu_1} \ldots z_n^{\nu_n})$. The function f has an isolated singularity if and only if the Newton polyhedron $NP(f)$ satisfies the following additional properties:*

- ***i.*** $\{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\} \cap NP(f) = \emptyset$
- ***ii.*** *$NP(f)$ has a vertex on each coordinate plane.*
- ***iii.*** *For each coordinate axis, $NP(f)$ has a vertex at most one unit far from the axis.*

Proposition 3.2.2 ([15], p. 94). *Let Σ be a fan in $\check{\mathbb{R}}^n_{\geq 0}$. Let $\sigma = Cone(P,Q)$ be a 2-dimensional cone in Σ where P,Q are primitive integral vectors and $s := det\ \sigma$. If $s > 1$ then there exists an integral vector $P_1 \in \sigma$ such that $det(P,P_1) = 1$. We can write $P_1 = \dfrac{Q + s_1 P}{s}$ with $s_1 \in \mathbb{Z}_{\geq 0}$ satisfying $1 \leq s_1 < s$ where $s_1 = det(P_1, Q)$.*

Proof. We can assume that $P = (1,0,\ldots,0)$. Let $Q = (q_1, q_2, \ldots, q_n)$ be a primitive vector. Then $s = det(P,Q) = gcd(q_2, \ldots, q_n)$ and $gcd(s, q_1) = 1$. Let $R \in \sigma$ be an integral vector with $det(P,R) = 1$. So we can write $R = aP + bQ = (a + bq_1, q_2, \ldots, q_n)$ for some positive rational numbers a, b. Then bs, cs are integers as $det(P,R) = bs$ and $det(R,Q) = cs$ are integers. Hence the assumption $det(P,R) = 1$ implies that we can write $R = Q + \alpha P/s$ where $\alpha \geq 0$. As we know R is an integral vector, $s|\alpha + q_1$. Thus it has a unique solution s_1 with $0 < s_1 < s$ as $gcd(s, q_1) = 1$. □

By applying the proposition above to each cone in $DNP(f)$, we can obtain a regular subdivision of $DNP(f)$: If $s_1 > 1$, we apply again Proposition 3.2.2 to cone $\sigma_1 = Cone(R_1, Q)$. Then $R_2 = \dfrac{Q + s_2 R_1}{s_1}$ with $1 \leq s_2 < s_1 < s$. Hence by induction, we get primitive integral vectors $R_1, \ldots, R_k$ and unique integers $1 = s_k < \cdots < s_1 < s = s_0$ where $s_i := det\,(R_i, Q)$, $R_0 = P$ and $R_{k+1} = Q$. It can be written as $R_i = \dfrac{Q + s_i R_{i-1}}{s_{i-1}}$ with $det\,(R_{i-1}, R_i) = 1$. Then the decomposition $\{P, R_1, \ldots, R_k, Q\}$ of σ is called *regular subdivision* of σ.

If we generalize the Proposition 3.2.2 then we can find regular subdivision of a cone of dimension ≥ 3.

Proposition 3.2.3 ([15], p. 98)**.** *Let $\sigma = Cone(P_1, \ldots, P_{k+1})$ be a cone in $\mathbb{R}^n_{\geq 0}$ where $P_1, \ldots, P_{k+1}$ are primitive integral vectors and assume $det(P_1, \ldots, P_k) = 1$ and $s := det(P_1, \ldots, P_{k+1}) > 1$. Then there exists unique integers $1 \leq s_1, \ldots, s_k < s$ such that $R = (P_{k+1} + \sum_{i=1}^{k} s_i P_i)/s$ is an integral vector.*

Before giving the Oka's algorithm we need one more definition:

Definition 3.2.4. *Let $\sigma = Cone(P, Q)$ be a cone in $\check{\mathbb{R}}^n$. Let F_P be a compact face of $NP(f)$ and $dim\ F_P \cap F_Q = 1$. The integer $r(\sigma)$ is the number of integral points in the relative interior of $F_P \cap F_Q$.*

In general the resolution process of a singularity is a difficult task. By the following algorithm we construct the minimal resolution graph of ADE singularities. Then, we will apply this algorithm to find the minimal resolution graph of some non-isolated singularities.

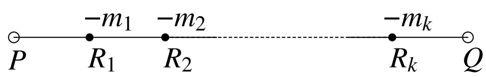
OKA'S ALGORITHM

Let f be an analytic function germ of n variables defined by $f(z) = \sum_\nu a_\nu z^\nu$ with isolated singularity. Let $\mathcal{F}$ be the set of all faces of $NP(f)$ and $\mathcal{F}_c$ be the set of compact faces of $NP(f)$. For any $F_P \in \mathcal{F}_c$, let $\mathcal{F}_P$ be the set of all faces of $NP(f)$ which are adjacent to F_P. We consider the normal vectors to the faces as vertices of a tree. The (dual) resolution graph $\mathcal{G}$ is obtained by the following construction:

Take a face $F_P \in \mathcal{F}_c$ and let F_Q be its adjacent face. Let $\sigma = Cone(P, Q)$ be a 2-dimensional cone and $s := det\ \sigma$. If $s > 1$ then by Proposition 3.2.2 there is a unique integer s_1 such that $1 \leq s_1 < s$ for which $R_1 = \dfrac{Q + s_1 P}{s}$ is an integral vector with $det\ (P, R_1) = 1$. Let $s_i := det\ (R_i, Q)$. We write $\dfrac{s}{s_i}$ as a continued fraction:

$$\frac{s}{s_i} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{\ddots - \cfrac{1}{m_k}}} \tag{3.1}$$

Then we connect the vectors P, Q by $r(\sigma) + 1$ copies of the following tree:



If $s = 1$ then P, Q are joined by an edge (setting $P_1 := Q$ and $s_1 = 0$) and put $r(\sigma) + 1$ copies of them. Then all the copies of the both end points are identified with P, Q. After that we omit the vectors (also the edges adjacent to them) which are not strictly positive, that is we delete the vectors which correspond to non-compact faces. Applying the same procedure for all adjacent faces of P what we obtain is the *resolution graph* $\mathcal{G}$ of f.

Let $P_1, \ldots, P_l$ be the adjacent vectors to P in regular $DNP(f)$. For any $F_{Q_i} \in \mathcal{F}_P$ with $dim\ F_P \cap F_{Q_i} = 1$, let $\sigma_i := Cone(P, Q_i)$ and $P_i \in \sigma_i$ for $i = 1, \ldots, l$. The *weight* of P, denoted $w(P)$, is computed by

$$w(P) = -\frac{\sum_{F_{Q_i} \in \mathcal{F}_P} (r(\sigma_i) + 1) P_i}{P} \tag{3.2}$$

Definition 3.2.5. *The integer $-m_i$ above is called weight of the vector P_i and for brevity we denote the continuous fraction defined in 3.1 by* $[m_1 : \ldots : m_k]$.

Now we will consider each of the singularities of type ADE and we will apply Oka's algorithm to find their resolution graph via Newton polyhedron.

Notation 1. We will denote $DNP_2(f)$ as the set of all 2-dimensional cones σ of $DNP(f)$ such that interior points of σ are strictly positive, i.e. all the components of the vector is > 0. We will not consider σ as a subset of $Cone(E_i, E_j)$ where E_i's are the canonical basis for $\mathbb{R}^n$. Moreover, we will use regular $DNP_2(f)$ to express regular subdivision of each 2-dimensional cones of $DNP(f)$ containing P.

*Notation 2.*In each graph the '•' vertices means that it is of weight −2.

Example 3.2.6 (The singularity E_6)**.** Consider $f(z_1, z_2, z_3) = z_1^2 + z_2^3 + z_3^4$.

The hypersurface $V = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid f(z_1, z_2, z_3) = 0\}$ has an isolated singularity at $(0, 0, 0) \in \mathbb{C}^3$. Its minimal resolution graph is

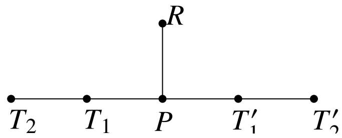


Figure 3.4: Resolution graph of E_6

Let us obtain this minimal resolution graph by using Oka's algorithm. The support of f is $\text{supp}(f) = \{(2, 0, 0), (0, 3, 0), (0, 0, 4)\}$. (See Figure 3.2 for $NP(f)$. Consider 2-dimensional cones in $DNP(f)$ as in Example 3.1.7. First we should find the regular subdivisions of these cones.

$det(\sigma_1) = det(P, E_1) = det\begin{pmatrix} 6 & 4 & 3 \\ 1 & 0 & 0 \end{pmatrix} = gcd(4, 3) = 1$. σ_1 is a 2-dimensional regular cone.

$det(\sigma_2) = det(P, E_2) = det\begin{pmatrix} 6 & 4 & 3 \\ 0 & 1 & 0 \end{pmatrix} = gcd(3, 6) = 3$. So σ_2 is not regular. In this case by Proposition 3.2.2:

$$T_1 := \frac{E_2 + s_1 P}{s} = \frac{(0, 1, 0) + 2(6, 4, 3)}{3} = (4, 3, 2)$$
$$T_2 := \frac{E_2 + s_2 T_1}{s_1} = \frac{(0, 1, 0) + (4, 3, 2)}{2} = (2, 2, 1)$$

where $s = 3, s_1 = 2, s_2 = 1$

$det(\sigma_3) = det(P, E_3) = det\begin{pmatrix} 6 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix} = gcd(4, 6) = 2$. So σ_3 is not regular and by Proposition 3.2.2 we have:

$$R = \frac{E_3 + s_1 P}{s} = \frac{(0, 0, 1) + (6, 4, 3)}{2} = (3, 2, 2)$$

where $s = 2, s_1 = 1$.

The regular $DNP_2(f)$ is as follows:

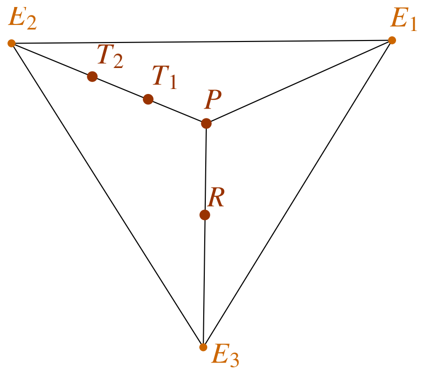
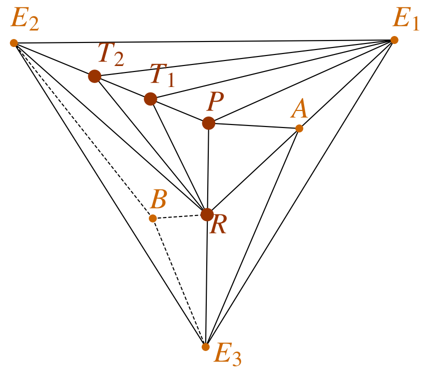


Figure 3.5: Regular $DNP(f)$

In fact we constitute the regular $DNP(f)$ if we add the vectors A, B as follows:



where $A = (2, 1, 1), B = (1, 1, 1)$ are the vectors that we should add for the cone $Cone(R, E_1)$ and $Cone(R, E_2, E_3)$ respectively. As we mention, to find the resolution graph of f we only focus on the regular subdivision of each two dimensional cones containing P.

Remark 3.2.7. *The integers $r(\sigma_1) = r(\sigma_3) = 0$ and $r(\sigma_2) = 1$. The fact that $r(\sigma_2) = 1$, we copy T_1, T_2 to the right side of P.*

After copying and identifying all end points we obtain the following tree:

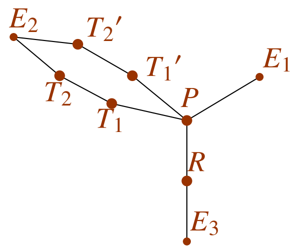


Figure 3.6: Tree

The weights of each vector in the graph above is calculated as follows:

$$\begin{aligned} w(P) &= -[(r(\sigma_1)+1)E_1 + (r(\sigma_2)+1)T_1 + (r(\sigma_3)+1)R]/P \\ &= -[1(1,0,0)) + 2(4,3,2) + 1(3,2,2)]/(6,4,3) \\ &= -(12,8,6)/(6,4,3) = -2 \end{aligned}$$

The weights m_1, m_2 for the vectors T_1 and T_2 is calculated by the equation 3.1 we have $\frac{s}{s_1} = \frac{3}{2} = 2 - \frac{1}{2}$. Hence $[m_1 : m_2] = [2 : 2]$. The weight of R is also $[2]$. So all the weights are -2. After omitting the vectors E_1, E_2, E_3 which correspond to the non-compact faces of $NP(f)$ we get the minimal resolution graph given in Figure 3.4 by the singularity defined by f.

Example 3.2.8 (The singularity E_7)**.** Consider $f(z_1, z_2, z_3) = z_1^2 + z_2^3 + z_2 z_3^3$.

The hypersurface $V = \mathbb{V}(f)$ has an isolated singularity at $(0,0,0) \in \mathbb{C}^3$. The minimal resolution graph of f is

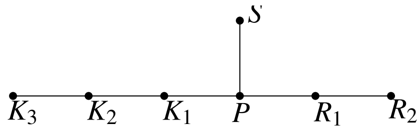


Figure 3.7: Resolution graph of E_7

We have $\mathrm{supp}(f) = \{(2,0,0), (0,3,0), (0,1,3)\}$. The $NP(f)$ has one compact facet F_P and four non-compact facets F_1, F_2, F_3, F_Q as follows:

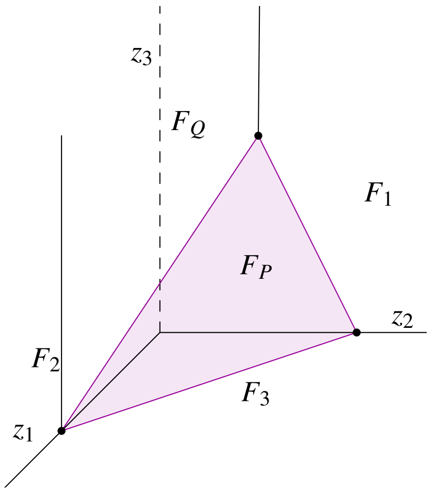


Figure 3.8: Newton polyhedron $NP(f)$

The normal vector of the compact facet F_P is $P = (9,6,4)$. The normal vectors of the non-compact facets F_1, F_2, F_3, F_Q are E_1, E_2, E_3, $Q = (1,2,0)$ respectively. The $DNP_2(f)$ has four cones:

1. $\sigma_1 :=< (9,6,4),(1,0,0) >$
2. $\sigma_2 :=< (9,6,4),(0,1,0) >$
3. $\sigma_3 :=< (9,6,4),(0,0,1) >$
4. $\sigma_4 :=< (9,6,4),(1,2,0) >$

Let us refine $DNP_2(f)$ into a regular $DNP_2(f)$:

$det(\sigma_1) = det(P, E_1) = 2$. This says that we add one vertex $S = (5,3,2)$. The weight of S is [2].

$det(\sigma_2) = det(P, E_2) = 1$. So σ_2 is regular.

$det(\sigma_3) = det(P, E_3) = 3$. So σ_3 is not regular. We add $R_1 = (6,4,3)$ and $R_2 = (3,2,2)$. The weights of R_1 and R_2 are $[2:2]$.

$det(\sigma_4) = det(P, Q) = 4$. Hence we add $K_1 = (7,5,3)$, $K_2 = (5,4,2)$, $K_3 = (3,3,1)$. The weights of K_1, K_2, K_3 are $[2:2:2]$.

The regular $DNP_2(f)$:

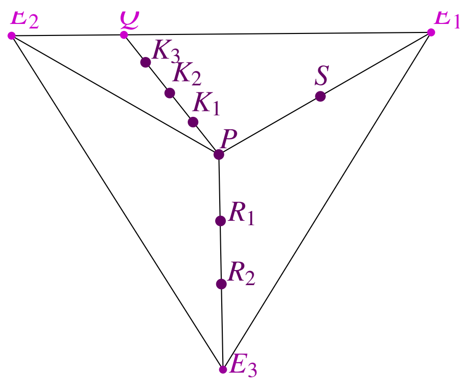


Figure 3.9: Regular $DNP(f)$

Remark 3.2.9. *The integers* $r(\sigma_1) = r(\sigma_3) = r(\sigma_4) = 0$.

Let us take out the bold points from the figure above and note that the points form the resolution graph of our singularity E_7.

It only remains to check the weight of P. So $w(P) = -[(r(\sigma_1)+1)S + (r(\sigma_3)+1)R_1 + (r(\sigma_4)+$

$1)K_1]/P = -(18, 12, 8)/(9, 6, 4) = -2$. Hence this graph is the minimal resolution graph of E_7 singularity which is given in Figure 3.7.

Example 3.2.10 (The singularity E_8)**.** Consider $f(z_1, z_2, z_3) = z_1^2 + z_2^3 + z_3^5$.

The hypersurface $V = \mathbb{V}(f)$ has an isolated singularity at $(0, 0, 0) \in \mathbb{C}^3$. The minimal resolution of f is the following

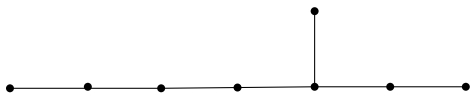


Figure 3.10: Resolution graph of E_8

The support of f is $\text{supp}(f) = \{(2, 0, 0), (0, 3, 0), (0, 0, 5)\}$. The Newton polyhedron $NP(f)$ has one compact facet F_P and three non-compact facets F_1, F_2, F_3 as follows:

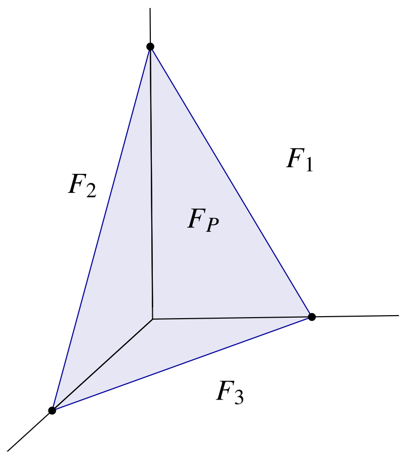


Figure 3.11: Newton polyhedron $NP(f)$

For the compact facet F_P the dual vector is $P = (15, 10, 6)$. For the non-compact facets F_1, F_2, F_3 the dual vectors are E_1, E_2, E_3 respectively. The $DNP(f)$ has three 2-dimensional cones:

1. $\sigma_1 =< (15, 10, 6), (1, 0, 0) >$
2. $\sigma_2 =< (15, 10, 6), (0, 1, 0) >$
3. $\sigma_3 =< (15, 10, 6), (0, 0, 1) >$

The regular $DNP_2(f)$ is:

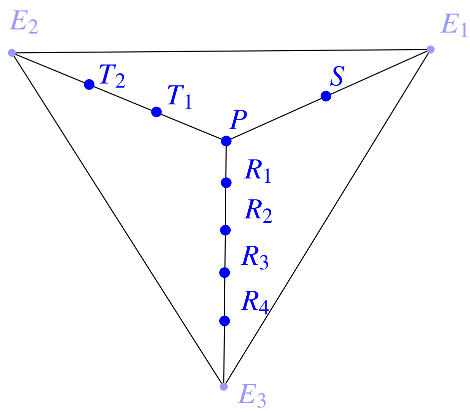


Figure 3.12: Regular $DNP(f)$

where $S = (8,5,3)$, $T_1 = (10,7,4)$, $T_2 = (5,4,2)$, $R_1 = (12,8,5)$, $R_2 = (9,6,4)$, $R_3 = (6,4,3)$, $R_4 = (3,2,2)$.

Remark 3.2.11. *The integers* $r(\sigma_1) = r(\sigma_2) = r(\sigma_3) = 0$.

Note that the bold points on the subdivision gives exactly the minimal resolution graph (see Figure 3.10) of our singularity E_8 as the weights of all the vertices are -2.

Example 3.2.12 (The singularity D_n)**.** Consider $f(z_1,z_2,z_3) = z_1^2 + z_2^2 z_3 + z_3^{n-1}$.

The hypersurface $V = \mathbb{V}(f)$ has an isolated singularity at $(0,0,0) \in \mathbb{C}^3$ which has the following minimal resolution graph

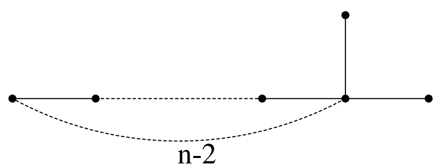


Figure 3.13: Resolution graph of D_n

We have $\text{supp}(f) = \{(2,0,0),(0,2,1),(0,0,n-1)\}$. The $NP(f)$ has one compact facet F_P and four non-compact facets F_1, F_2, F_3, F_Q as follows:

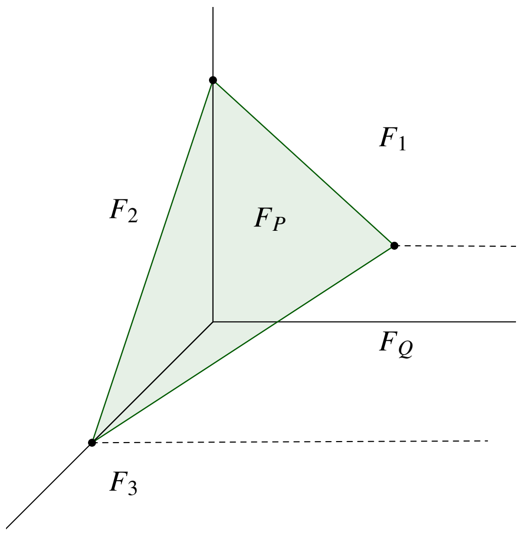


Figure 3.14: Newton polyhedron $NP(f)$

For the compact facet F_P the dual vector is $P = (n-1, n-2, 2)$. For the non-compact facets F_1, F_2, F_3, F_Q the dual vectors are E_1, E_2, E_3, $Q = (1, 0, 2)$ respectively. The $DNP(f)$ has four 2-dimensional cones:

1. $\sigma_1 =< (n-1, n-2, 2), (1, 0, 0) >$
2. $\sigma_2 =< (n-1, n-2, 2), (0, 1, 0) >$
3. $\sigma_3 =< (n-1, n-2, 2), (0, 0, 1) >$
4. $\sigma_4 =< (n-1, n-2, 2), (1, 0, 2) >$.

Let us consider 2-dimensional cones in $DNP(f)$.

$$det(\sigma_1) = \begin{cases} 2, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

We add $S = \dfrac{E_1 + P}{2} = (\dfrac{n}{2}, \dfrac{n-2}{2}, 1)$ with a weight [2] if n is even.

$$det(\sigma_2) = \begin{cases} 2, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even} \end{cases}$$

We add $T = \dfrac{E_2 + P}{2} = (\dfrac{n-1}{2}, \dfrac{n-1}{2}, 1)$ with a weight [2] if n is odd.

$det(\sigma_3) = 1$ for all n. So σ_3 is regular.

$det(\sigma_4) = n-2$ for all n. We add $n-3$ vertices $K_1, \dots, K_{n-3}$ with weights $[2 : \cdots : 2]$ where $K_1 = \dfrac{Q+(n-3)P}{n-2} = (n-2, n-3, 2)$.

The regular $DNP_2(f)$ is as follows:

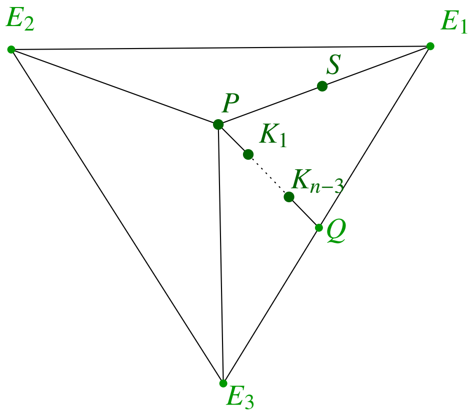


Figure 3.15: $DNP(f)$ if n is even

Remark 3.2.13. *If n is even, the integers $r(\sigma_1) = 1$ and $r(\sigma_2) = r(\sigma_4) = 0$. Hence we copy S to the other side of P. If n is odd the integers $r(\sigma_1) = r(\sigma_4) = 0$ and $r(\sigma_2) = 1$. Hence we copy T to the other side of P.*

The weights of the vertices is -2 because if n is even (similarly if n is odd) then $w(P) = -[(r(\sigma_1)+1)S + (r(\sigma_2)+1)E_2 + (r(\sigma_4)+1)K_1]/P = -(2n-2, 2n-4, 4)/(n-1, n-2, 2) = -2$. Hence after deleting the vectors which are not strictly positive we get graph given in Figure 3.13 which is the exactly the minimal rersolution graph of f.

Example 3.2.14 (The singularity A_n)**.** Consider $f(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3^{n+1}$.

The hypersurface $V = \mathbb{V}(f)$ has an isolated singularity at $(0,0,0) \in \mathbb{C}^3$. The minimal resolution graph of f is

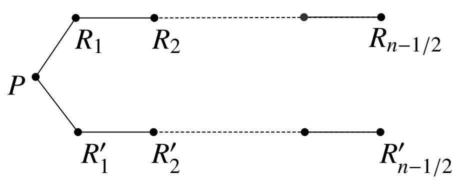


Figure 3.16: Resolution graph of A_n

We have $\mathrm{supp}(f) = \{(2,0,0), (0,2,0), (0,0,n+1)\}$. The $NP(f)$ has one compact facet F_P and three non-compact facets F_1, F_2, F_3 as follows:

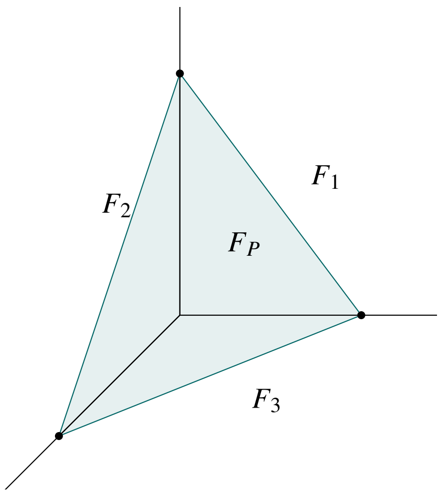


Figure 3.17: Newton polyhedron $NP(f)$

For the compact facet F_P the orthogonal vector is $P = (n+1, n+1, 2)$. For the non-compact facets F_1, F_2, F_3 the normal vectors are E_1, E_2, E_3 respectively. The corresponding dual fan $DNP_2(f)$ has three cones:

1. $\sigma_1 :=< (n+1, n+1, 2), (1, 0, 0) >$
2. $\sigma_2 :=< (n+1, n+1, 2), (0, 1, 0) >$
3. $\sigma_3 :=< (n+1, n+1, 2), (0, 0, 1) >$.

Let us check whether each cone in $DNP_2(f)$ is regular. We have two cases for P:

Case 1. n is odd, i.e $n + 1 = 2k$, and $P = (k, k, 1)$

$det(\sigma_1) = det(\sigma_2) = 1$. So σ_1, σ_2 are regular.

$det(\sigma_3) = k$. We add further $k - 1$ vertices $R_1, \cdots, R_{k-1}$ where $R_1 = E_3 + (k-1)P/k = (k-1, k-1, 1)$. The weights are $[2 : \ldots : 2]$ since $s = k, s_1 = k - 1$.

The regular $DNP_2(f)$ is:

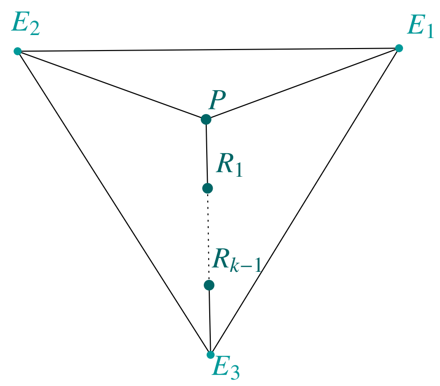


Figure 3.18: $DNP(f)$

The weight of P is $w(P) = -2$. So the weights of each vertices is -2. Remark that the integers $r(\sigma_1) = r(\sigma_2) = r(\sigma_3) = 1$. Thus we copy the corresponding branches and we obtain the graph given in Figure 3.16 which is the minimal resolution graph of A_{2k-1}. singularity.

Case 2. n is even, i.e $n = 2k$, and $P = (2k+1, 2k+1, 2)$

$det(\sigma_1) = det(\sigma_2) = 1$. So σ_1, σ_2 are regular.

$det(\sigma_3) = k$. We have additionally k vertices $R_1, \cdots, R_k$ where $R_1 = \dfrac{E_3 + kP}{2k+1} = (k, k, 1)$ with weights $[3 : 2 : \cdots : 2]$.

The regular $DNP_2(f)$ is:

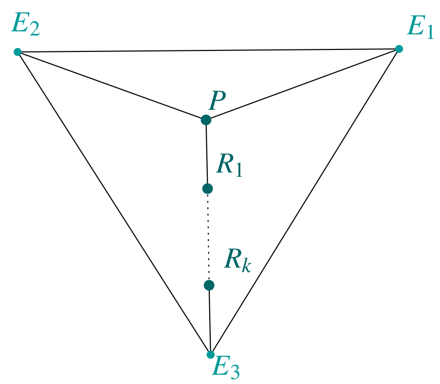
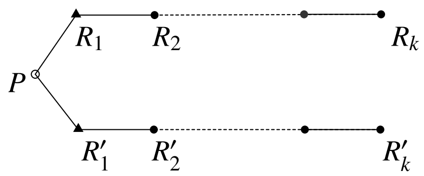


Figure 3.19: $DNP(f)$

The integers $r(\sigma_1) = r(\sigma_2) = 0$ and $r(\sigma_3) = 1$. So we copy $R_1, \ldots, R_k$ to the other side of P. Note that $w(P) = -1$. We obtain the following graph from the $DNP(f)$:

Here the weight of vertex '∘' is -1, '•' is -2 and '▲' is -3. The graph is not minimal. If



we blow down the -1 vertex once, we obtain the following graph in Figure 3.16 which is the minimal resolution graph of A_{2k}. This example shows us that the resolution graph obtained by this method may not be the minimal resolution graph. However, after successive blow downs one can obtain the minimal resolution graph.

3.3 Minimal resolution graphs of some non-isolated hypersurface singularities via Newton Polyhedron

In this section we will deal with non-isolated hypersurface singularities. An isolated surface singularity is obtained by normalization of a non-isolated hypersurface singularity. Here we consider some examples of non-isolated hypersurface singularities and we construct the minimal resolution graph of these singularities by using Oka's algorithm.

Notation. The weight of vertex '▲' is (-3) and of the vertex '•' is (-2).

Example 3.3.1 (The singularity NE_6). Consider $f(z_1, z_2, z_3) = z_1^2 z_2^2 + z_2^3 z_3 + z_3^3$.

The partial derivatives of f; $\frac{\partial f}{\partial z_1} = 2z_1 z_2^2$, $\frac{\partial f}{\partial z_2} = 2z_1^2 z_2 + 3z_2^2 z_3$, $\frac{\partial f}{\partial z_3} = z_2^3 + 3z_3^2$. The hypersurface $V = \mathbb{V}(f)$ has a non-isolated singularity along z_1 axis. Its minimal resolution is

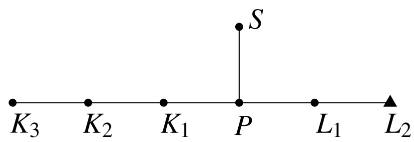


Figure 3.20: Resolution graph of NE_6

We have $\text{supp}(f) = \{(2, 2, 0), (0, 3, 1), (0, 0, 3)\}$. The $NP(f)$ has one compact facet F_P and five non-compact facets $F_1, F_2, F_3, F_{Q_1}, F_{Q_2}$ as following:

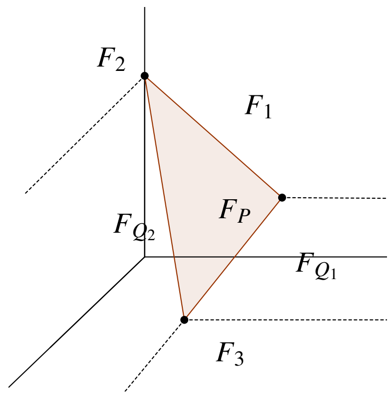


Figure 3.21: Newton polyhedron $NP(f)$

The dual vector P to the compact face F_P is $(5,4,6)$. For the non-compact faces $F_1, F_2, F_3, F_{Q_1}, F_{Q_2}$ the dual vectors are E_1, E_2, E_3, $Q_1 = (1,0,2)$, $Q_2 = (0,3,2)$ respectively. The $DNP_2(f)$ has three cones: $\sigma_1 :=< (5,4,6),(1,0,0) >, \sigma_2 :=< (5,4,6),(0,3,2) >, \sigma_3 :=< (5,4,6),(1,0,2) >$.

Let us find regular subdivision of each cone in $DNP_2(f)$.

$det(\sigma_1) = det(P, E_1) = gcd(4,6) = 2$. We have $S = \dfrac{E_1 + P}{2} = (3,2,3)$ with weight [2].

$det(\sigma_2) = det(P, Q_1) = gcd(8,4,4) = 4$. We have $K_1 = \dfrac{Q_1 + 3P}{4} = (4,3,5)$, $K_2 = \dfrac{Q_1 + 2K_1}{3} = (3,2,4)$, $K_3 = \dfrac{Q_1 + K_2}{2} = (2,1,3)$ with weights [2 : 2 : 2].

$det(\sigma_3) = det(P, Q_2) = gcd(15,10,10) = 5$. We have $L_1 = \dfrac{Q_2 + 3P}{5} = (3,3,4)$, $L_2 = \dfrac{Q_2 + L_1}{2} = (1,2,2)$ with weights [2 : 3] respectively.

The regular $DNP_2(f)$ is:

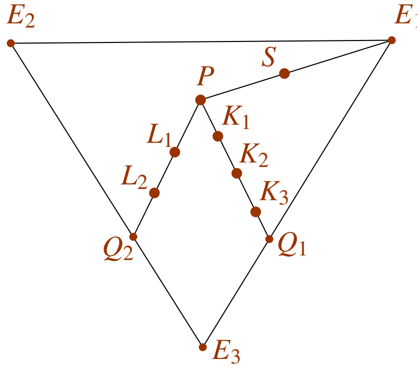


Figure 3.22: $DNP(f)$ of NE_6

The integers $r(\sigma_1) = r(\sigma_2) = r(\sigma_3) = 0$. So we will not copy any branch. Also, note that

$w(P) = -(L_1 + K_1 + S)/P = -2$. By taking out the bold points from the figure above we have the minimal resolution of f given in Figure 3.20.

Example 3.3.2 (The singularity $N1E_7$)**.** Consider $f(z_1, z_2, z_3) = z_1^2 z_2 z_3 + z_2^4 + z_3^3$.

The hypersurface $V = \mathbb{V}(f)$ has a non-isolated singularity along z_1 axis. Its minimal resolution is

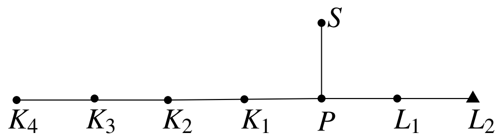


Figure 3.23: Resolution graph of $N1E_7$

We have $\mathrm{supp}(f) = \{(2, 1, 1), (0, 4, 0), (0, 0, 3)\}$. The $NP(f)$ has one compact facet F_P and five non-compact facets $F_1, F_2, F_3, F_{Q_1}, F_{Q_2}$ as follows:

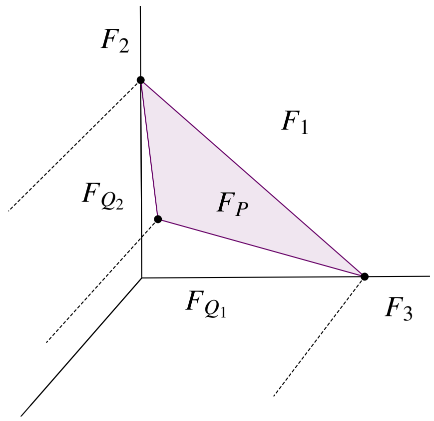


Figure 3.24: Newton polyhedron $NP(f)$

The orthogonal vector P to the compact face F_P is $(5, 6, 8)$. For the non-compact faces $F_1, F_2, F_3, F_{Q_1}, F_{Q_2}$ the dual vectors are E_1, E_2, E_3, $Q_1 = (0, 1, 3), Q_2 = (0, 2, 1)$ respectively. The $DNP_2(f)$ has three cones: $\sigma_1 :=< (5, 6, 8), (1, 0, 0) >$, $\sigma_2 :=< (5, 6, 8), (0, 1, 3) >$, $\sigma_3 :=< (5, 6, 8), (0, 2, 1) >$.

$det(\sigma_1) = det(P, E_1) = 2$. We have $S = (3, 3, 4)$ with weight [2].

$det(\sigma_2) = det(P, Q_1) = 5$. We have further 4 vertices $K_1 = (4, 5, 7)$, $K_2 = (3, 4, 6)$, $K_3 = (2, 3, 5)$, $K_4 = (1, 2, 4)$ with weights [2 : 2 : 2 : 2] respectively.

$det(\sigma_3) = det(P, Q_2) = 5$. We have $L_1 = (3, 4, 5), L_2 = (1, 2, 2)$ with weight [2 : 3] respectively.

The regular $DNP_2(f)$ is:

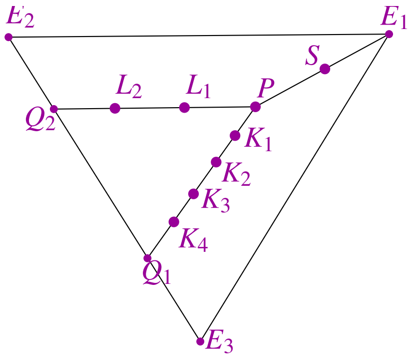


Figure 3.25: Regular $DNP(f)$ of $N1E_7$

Remark that the integers $r(\sigma_1) = r(\sigma_2) = r(\sigma_3) = 0$. Moreover, observe that $w(P) = -(S + L_1 + K_1)/P = -2$. After taking out the bold points we obtain the minimal resolution graph given in Figure 3.23.

Example 3.3.3 (The singularity $N2E_7$)**.** Consider $f(z_1, z_2, z_3) = z_1^2 z_2^2 + z_2^5 + z_3^3$. The hypersurface $V = \mathbb{V}(f)$ has a non isolated singularity along z_1 axis. Its minimal resolution graph is

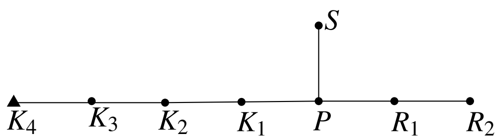


Figure 3.26: Resolution graph of $N2E_7$

We have $\text{supp}(f) = \{(2, 2, 0), (0, 5, 0), (0, 0, 3)\}$. The $NP(f)$ has one compact facet F_P and four non-compact facets F_1, F_2, F_3, F_Q as follows:

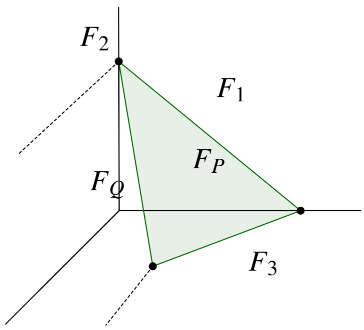


Figure 3.27: Newton polyhedron $NP(f)$

For the compact facet F_P the dual vector is $P = (9,6,10)$. For the non-compact facets F_1, F_2, F_3, F_Q the dual vectors are E_1, E_2, E_3, $Q = (0,3,2)$ respectively. The $DNP(f)$ three 2-dimensional cones: $\sigma_1 :=< (9,6,10),(1,0,0) >$, $\sigma_2 :=< (9,6,10),(0,0,1) >$, $\sigma_3 :=< (9,6,10),(0,3,2) >$.

$det(\sigma_1) = det(P, E_1) = 2$. We have $S = (5,3,5)$ with weight [2].

$det(\sigma_2) = det(P, E_3) = 3$. We have $R_1 = (6,4,7), R_2 = (3,2,4)$ with weights [2 : 2].

$det(\sigma_3) = det(P, Q) = 9$. We have $K_1 = (7,5,8)$, $K_2 = (5,4,6)$, $K_3 = (3,3,4)$, $K_4 = (1,2,2)$ with weights [2 : 2 : 2 : 3] respectively.

The regular $DNP_2(f)$ is:

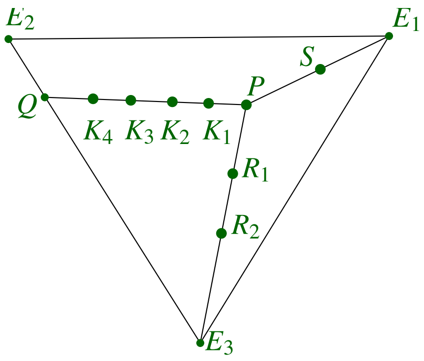


Figure 3.28: Regular $DNP(f)$ of $N2E_7$

The integers $r(\sigma_1) = r(\sigma_2) = r(\sigma_3) = 0$. The weight of P is $w(P) = -2$. Let us take out the bold points and what we obtain is same figure which is given in Figure **??**.

Example 3.3.4 (The singularity ND_n)**.** Consider $f(z_1, z_2, z_3) = z_3^3 + z_1 z_3^2 + z_2^{n+3} z_3 + z_1^2 z_2^{2n+2}$.

The hypersurface $V = \mathbb{V}(f)$ has a non-isolated singularity along z_1 axis. Its minimal resolution is

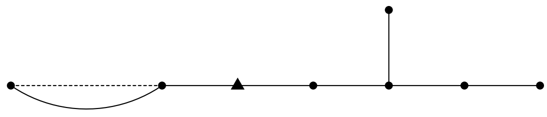


Figure 3.29: Resolution graph of ND_n

We have that $\text{supp}(f) = \{(1,0,2),(0,n+3,1),(0,0,3),(2,2n+2,0)\}$. The $NP(f)$ has two compact facet F_{P_1}, F_{P_2} and five non-compact facets $F_1, F_2, F_3, F_{Q_1}, F_{Q_2}$ as follows:

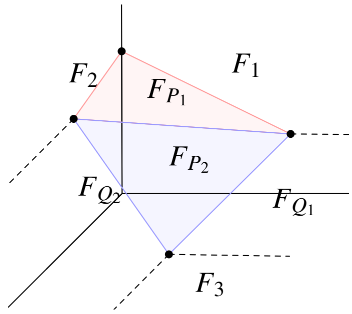


Figure 3.30: Newton polyhedron $NP(f)$

The orthogonal vectors P_1,P_2 to the compact facet F_{P_1}, F_{P_2} is $(n+3,2,n+3)$, $(4,3,3n+5)$ respectively. For the non-compact facets $F_1, F_2, F_3, F_{Q_1}, F_{Q_2}$ the orthogonal vectors are E_1, E_2, E_3, $Q_1 = (1,0,2)$, $Q_2 = (0,1,n+1)$ respectively.

The $DNP_2(f)$ has five cones: $\sigma_1 :=< (n+3,2,n+3),(1,0,0) >$, $\sigma_2 :=< (n+3,2,n+3),(0,1,0) >$, $\sigma_3 :=< (n+3,2,n+3),(4,3,3n+5) >$, $\sigma_4 :=< (4,3,3n+5),(1,0,2) >$, $\sigma_5 :=< (4,3,3n+5),(0,1,n+1) >$.

We have two cases:

Case 1. n is odd, i.e $n = 2k+1$, then $P_1 = (k+2,1,k+2)$, $P_2 = (4,3,6k+8)$, $Q_2 = (0,1,2k+2)$

$det(\sigma_1) = det(P_1, E_1) = 1$. So σ_1 is regular.

$det(\sigma_2) = det(P_1, E_2) = k+2$. We have further $k+1$ vertices $T_1, \cdots, T_{k+1}$ where $T_1 = E_2 + (k+1)P_1/k+2 = (k+1,1,k+1)$ with weights $[2 : \cdots : 2]$.

$det(\sigma_3) = det(P_1, P_2) = 3k+2$. We have $k+1$ vertices $U_1, \ldots, U_{k+1}$ where $U_1 = P_2 + (3k-1)P_1/3k+2 = (k+1,1,k+3)$, $U_{k+1} = P_1 + (2k+1)P_2/3k+2 = (3,2,4k+5)$ with weights

[2 : 2 : ⋯ : 3 : 2] respectively.

$det(\sigma_4) = det(P_2, Q_1) = 3$.We have $K_1 = (3, 2, 4k + 6)$ and $K_2 = (2, 1, 2k + 3)$ with weights [2 : 2].

$det(\sigma_5) = det(P_2, Q_2) = 2$. We have $L = (2, 2, 4k + 5)$ with weight -2.

The regular $DNP_2(f)$ is:

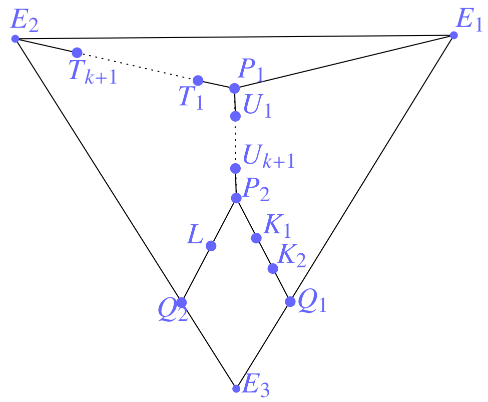


Figure 3.31: Regular $DNP(f)$ of ND_{2k+1}

The weights of the vertices $w(P_1) = -2$ and $w(P_2) = -2$. Hence we have the minimal resolution graph in Figure 3.29 after deleting the non-strictly convex cones.

Case 2. n is even, i.e $n = 2k$, then $P_1 = (2k+3, 2, 2k+3)$, $P_2 = (4, 3, 6k+5)$, $Q_2 = (0, 1, 2k+1)$.

$det(\sigma_1) = det(P_1, E_1) = 1$. So σ_1 is regular.

$det(\sigma_2) = det(P_1, E_2) = 2k + 3$. We have $k + 1$ vertices $T_1, \cdots, T_{k+1}$ where $T_1 = E_2 + (k + 1)P_1/2k + 3 = (k + 1, 1, k + 1)$ with weights [3 : 2 : ⋯ : 2] .

$det(\sigma_3) = det(P_1, P_2) = 6k + 1$. We have $k + 1$ vertices $U_1, \ldots, U_{k+1}$ where $U_1 = P_2 + (3k - 1)P_1/6k + 1 = (k + 1, 1, k + 2)$ and $U_{k+1} = P_1 + 4kP_2/6k + 1 = (3, 2, 4k + 3)$ with weights [3 : 2 : ⋯ 2 : 3 : 2] respectively.

$det(\sigma_4) = det(P_2, Q_1) = 3$. We have $K_1 = (3, 2, 4k + 4)$, $K_2 = (2, 1, 2k + 3)$ with weights [2 : 2].

$det(\sigma_5) = det(P_2, Q_2) = 2$. We have $L = (2, 2, 4k + 3)$ with a weight -2.

The regular $DNP_2(f)$ is same as the previous one(see Figure 3.31). The weights of the vertices

$w(P_1) = -1$ and $w(P_2) = -2$. Hence we can obtain the following graph: The weight of the

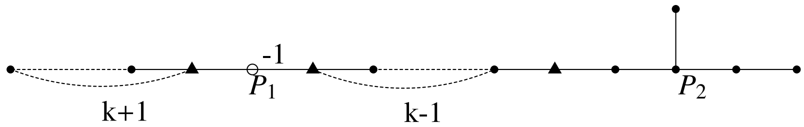


Figure 3.32: Resolution graph of ND_{2k}

vertex '▲' is (-3), '•' is (-2), '∘' is (-1). The minimal resolution graph can be obtained after one blowing down.

Remark 3.3.5. *In both cases the integers* $r(\sigma_1) = r(\sigma_2) = r(\sigma_3) = r(\sigma_4) = r(\sigma_5) = 0$.

Example 3.3.6 (The singularity NG_n)**.** Consider $f(z_1, z_2, z_3) = z_3^3 + z_1^{p+2} z_2 z_3 + z_1^2 z_2^3$ for $n = 3p + 1$.

The hypersurface $V = \mathbb{V}(f)$ has a non-isolated singularity.Its minimal resolution graph is the following:

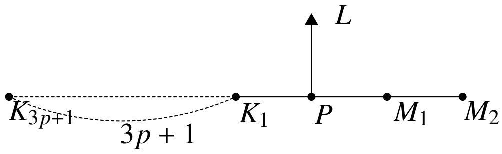


Figure 3.33: Resolution graph of NG_{3p+1}

We have $\mathrm{supp}(f) = \{(0, 0, 3), (p + 2, 1, 1), (2, 3, 0)\}$. The $NP(f)$ has one compact facet F_P and six non-compact facets $F_1, F_2, F_3, F_{Q_1}, F_{Q_2}, F_{Q_3}$ as follows:

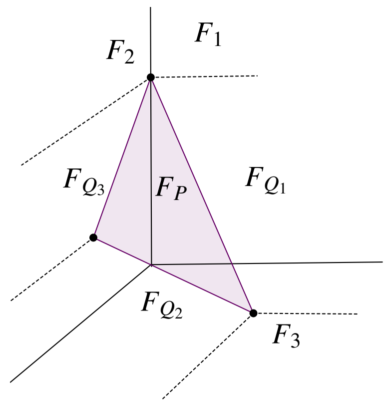


Figure 3.34: Newton polyhedron $NP(f)$

For the compact facet F_P the dual vector P is $(3, 3p+2, 3p+4)$. For the non-compact facets $F_1, F_2, F_3, F_{Q_1}, F_{Q_2}, F_{Q_3}$ the dual vectors are E_1, E_2, E_3, $Q_1 = (3, 0, 2)$, $Q_2 = (0, 1, 2)$, $Q_3 = (0, 2, 1)$ respectively. The $DNP(f)$ has three 2-dimensional cones: $\sigma_1 :=< (3, 3p+2, 3p+4), (3, 0, 2) >$, $\sigma_2 :=< (3, 3p+2, 3p+4), (0, 1, 2) >$, $\sigma_3 :=< (3, 3p+2, 3p+4), (0, 2, 1) >$.

Let us check whether each 2-dimensional cone in $DNP(f)$ is regular.

$det(\sigma_1) = det(P, Q_1) = 3p+2$. So we need $3p+1$ vertices with $K_1 = Q_1 + (3p+1)P/3p+2 = (3, 3p+1, p+1)$ with weights $[2 : \cdots : 2]$.

$det(\sigma_2) = det(P, Q_2) = 3$. We have $L = (1, p+1, p+2)$ with a weight $[3]$.

$det(\sigma_3) = det(P, Q_3) = 3$. We have $M_1 = (2, 2p+2, 2p+3)$ and $M_2 = (1, p+2, p+2)$ with weights $[2 : 2]$.

The regular $DNP_2(f)$ is:

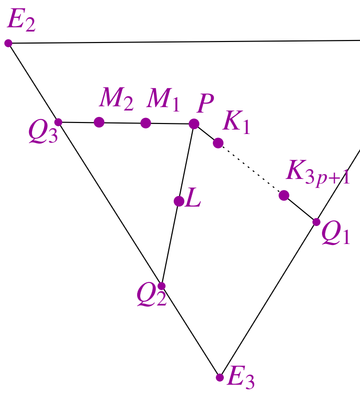


Figure 3.35: regular $DNP(f)$ of NG_{3p+1}

The weight of P is $w(P) = -2$, hence by taking out the inner points we get the minimal resolution graph.

Example 3.3.7 (The singularity $NC_{m,n}$)**.** The minimal resolution graph of $(i), (ii)$ is

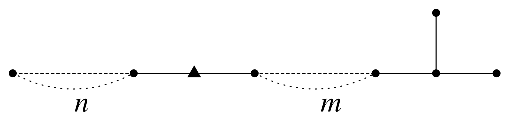


Figure 3.36: Resolution graph of $NC_{m,n}$

i. Consider the hypersurface $f(z_1, z_2, z_3) = z_1 z_2^2 z_3 + z_1^{m+3} z_3 + z_1 z_2^{p+4} + 2 z_1^{m+3} z_2^{p+2} + z_1^{2m+5} z_2^{p} + z_3^3$

for $n = 3p + 1$. The hypersurface $V = \mathbb{V}(f)$ has a non-isolated singularity.

$$\mathrm{supp}(f) = \{(1,2,1), (m+3,0,1), (1,p+4,0), (m+3,p+2,0), (2m+5,p,0), (0,0,3)\}$$

The $NP(f)$ has two compact facet F_{P_1}, F_{P_2} and five non-compact facets $F_1, F_2, F_3, F_{Q_1}, F_{Q_2}$ as follows: For the compact facets F_{P_1}, F_{P_2} the dual vectors $P_1 = (2, m+2, m+3)$,$P_2 = (2p+$

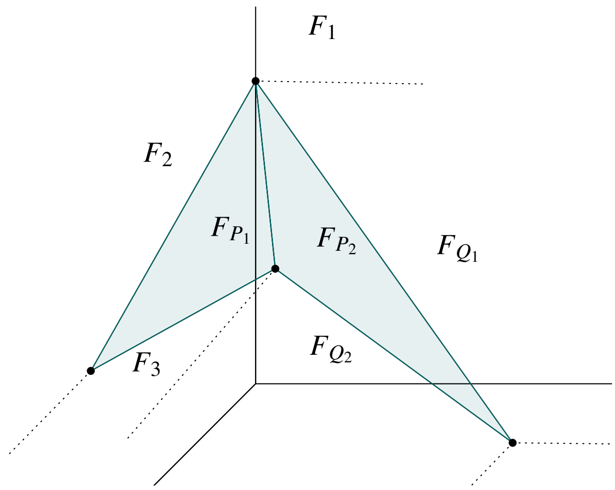


Figure 3.37: Newton polyhedron $NP(f)$

$2, 1, p+2)$. For the non-compact facets $F_1, F_2, F_3, F_{Q_1}, F_{Q_2}$ the dual vectors are E_1, E_2, E_3, $Q_1 = (3,0,1), Q_2 = (0,1,p+2)$ respectively. The $DNP(f)$ has five 2-dimensional cones: $\sigma_1 :=< (2, m+2, m+3), (0,1,0) >$, $\sigma_2 :=< (2, m+2, m+3), (0,0,1) >$, $\sigma_3 :=< (2, m+2, m+3), (2p+2, 1, p+2) >$, $\sigma_4 :=< (2p+2, 1, p+2), (3,0,1) >$, $\sigma_5 :=< (2p+2, 1, p+2), (0, 1, p+2) >$.

$$det(\sigma_1) = det(P_1, E_2) = \begin{cases} 2, & \text{if } m \text{ is odd} \\ 1, & \text{if } m \text{ is even} \end{cases}$$

We have $T = (3,2,4)$ with weight [2] if m is odd.

$$det(\sigma_2) = det(P_1, E_3) = \begin{cases} 1, & \text{if } m \text{ is odd} \\ 2, & \text{if } m \text{ is even} \end{cases}$$ We have $R = (1, k+1, k+2)$ with weight [2] if m is even.

$det(\sigma_3) = det(P_1, P_2) = mp + 2p + m + 1$. We have $m + p$ vectors $U_1, \ldots, U_{m+p}$ where $U_1 = (2, m+1, m+2)$ and $U_{m+p} = (2p, 1, p+1)$ with weights $[2 : \ldots : 2 : 3 : 2 : \ldots : 2]$ respectively.

$det(\sigma_4) = det(P_2, Q_1) = 1$.

$det(\sigma_5) = det(P_2, Q_2) = 2p + 2$. So we have $2p + 1$ vectors $L_1, \ldots, L_{2p+1}$ with weights $[2 : \ldots : 2]$ where $L_1 = (2p + 1, 1, p + 2)$.

The regular $DNP_2(f)$ is:

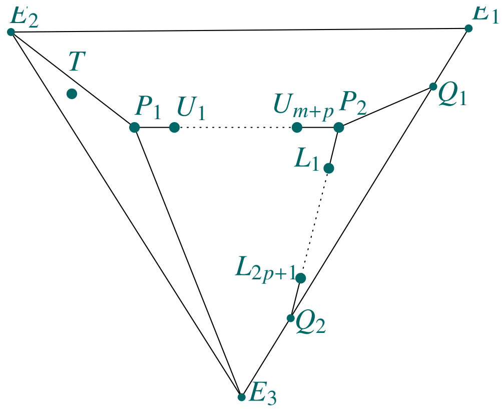


Figure 3.38: Regular $DNP(f)$ of $NC_{m,n}$

Remark 3.3.8. *The integers $r(\sigma_1) = 1$ if n is odd.(respectively $r(\sigma_2) = 1$ if n is even.) and* $r(\sigma_3) = 0$.

The weights $w(P_1) = w(P_2) = -2$. The bold points from the $DNP(f)$ form the graph given in Figure 3.36 for $n = 3p + 1$.

ii. Consider the hypersurface $f(z_1, z_2, z_3) = z_1 z_2^2 z_3 + z_1^{m+3} z_3 + z_2^{n+5} + 2z_1^{m+2} z_2^{n+3} + z_1^{2m+4} z_2^{n+1} + z_3^3$ for $n \not\equiv 1 \pmod 3$.

The hypersurface $V = \mathbb{V}(f)$ has a non-isolated singularity.

$$\text{supp}(f) = \{(1, 2, 1), (m + 3, 0, 1), (0, n + 5, 0), (m + 2, n + 3, 0), (2m + 4, n + 1, 0), (0, 0, 3)\}$$

The $NP(f)$ has two compact facets F_{P_1}, F_{P_2} and four non-compact facets F_1, F_2, F_3, F_Q as follows:

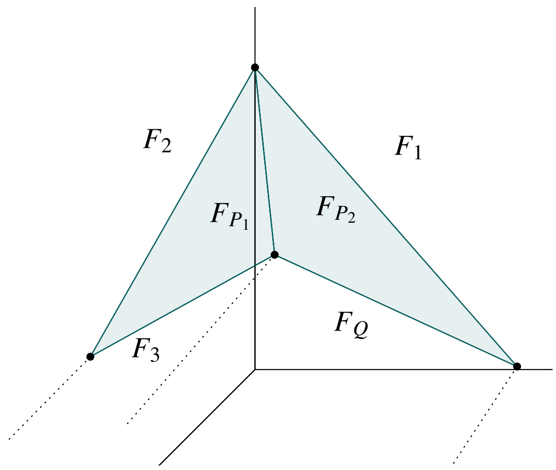


Figure 3.39: Newton polyhedron $NP(f)$

For the compact facets F_{P_1}, F_{P_2} the dual vectors $P_1 = (2, m+2, m+3)$,$P_2 = (2n+4, 3, n+5)$. For the non-compact facets F_1, F_2, F_3, F_Q the dual vectors are E_1, E_2, E_3, $Q = (0, 1, n+3)$ respectively. The $DNP_2(f)$ has five cones: $\sigma_1 :=< (2, m+2, m+3), (0, 1, 0) >$, $\sigma_2 :=< (2, m+2, m+3), (0, 0, 1) >$, $\sigma_3 :=< (2, m+2, m+3), (2n+4, 3, n+5) >$, $\sigma_4 :=< (2n+4, 3, n+5), (1, 0, 0) >$, $\sigma_5 :=< (2n+4, 3, n+5), (0, 1, n+3) >$.

$$det(\sigma_1) = det(P_1, E_2) = \begin{cases} 2, & \text{if } m \text{ is odd} \\ 1, & \text{if } m \text{ is even} \end{cases}$$

We have $T = (1, k+2, k+2)$ with weight [2] if m is odd.

$$det(\sigma_2) = det(P_1, E_3) = \begin{cases} 1, & \text{if } m \text{ is odd} \\ 2, & \text{if } m \text{ is even} \end{cases}$$

We have $R = (1, k+1, k+2)$ with weight [2] if m is even.

$det(\sigma_4) = det(P_2, E_1) = 1.$

$det(\sigma_3) = det(P_1, P_2) = mn + 2n + 2m + 1$. We should separate into two cases depending on the value of n:

Case 1. If $n = 3p$, then $P_2 = (6p+4, 3, 3p+5)$, $Q = (0, 1, 3p+3)$.

$det(\sigma_3) = det(P_1, P_2) = 3pm + 2m + 6p + 1$. We have $m + p + 1$ vectors $U_1, \ldots, U_{m+p+1}$ where $U_1 = (2, m+1, m+2)$ and $U_{m+p+1} = (4p+2, 2, 2p+3)$ with weights $[2 : \ldots : 2 : 3 : 2 : \ldots : 2 : 3 : 2]$ respectively.

$det(\sigma_5) = det(P_2, Q) = 6p + 4$. We have $2p + 1$ vectors $K_1, \ldots, K_{2p+1}$ with weights $[4 : 2 : \ldots : 2]$ where $K_1 = (2p + 1, 1, p + 2)$.

The weights $w(P_1) = -2$ and $w(P_2) = -1$. So after two blow downs we get the minimal resolution graph as in Figure 3.36 for $n = 3p + 2$.

Case 2. If $n = 3p + 2$, then $P_2 = (6p + 8, 3, 3p + 7)$, $Q = (0, 1, 3p + 5)$.

$det(\sigma_3) = det(P_1, P_2) = 3pm + 4m + 6p + 1$. We have $m + p + 1$ vectors $U_1, \ldots, U_{m+p+1}$ where $U_1 = (2, m + 1, m + 2)$ and $U_{m+p+1} = (2p + 3, 1, p + 2)$ with weights $[2 : \ldots : 2 : 3 : 2 : \ldots : 2 : 4]$ respectively.

$det(\sigma_5) = det(P_2, Q) = 6p + 8$. We have $2p + 3$ vectors $K_1, \ldots, K_{2p+3}$ with weights $[2 : 3 : 2 : \ldots : 2]$ where $K_1 = (2p + 1, 1, p + 2)$. The weights of P_1,P_2 are $[2 : 1]$ respectively. Hence after two blow downs we get the minimal resolution graph as in Figure 3.36 for $n = 3p + 2$.

Remark 3.3.9. *In both cases the integers $r(\sigma_1) = 1$ if m is odd.(respectively $r(\sigma_2) = 1$ if m is even.) and $r(\sigma_i) = 0$ for $i = 3, 4, 5$.*

Example 3.3.10 (The singularity $NB_{m,n}$)**.** Consider $f(z_1, z_2, z_3) = z_1 z_3^2 + z_2^{2m+3} z_3 + z_1 z_2^{2m+3} + z_3^3$ for $n = 2m + 1$.

The hypersurface $V = \mathbb{V}(f)$ has a non-isolated singularity. Its minimal resolution graph is

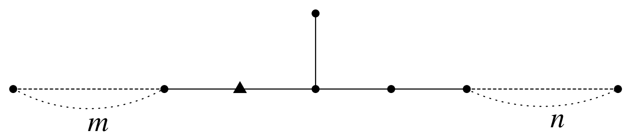


Figure 3.40: Resolution graph of $NB_{m,n}$

We have $\text{supp}(f) = \{(1, 0, 2), (0, 2m+3, 1), (1, 2m+3, 0), (0, 0, 3)\}$. The $NP(f)$ has one compact parallelogram facet F_P and five non-compact facets $F_1, F_2, F_3, F_{Q_1}, F_{Q_2}$ as follows:

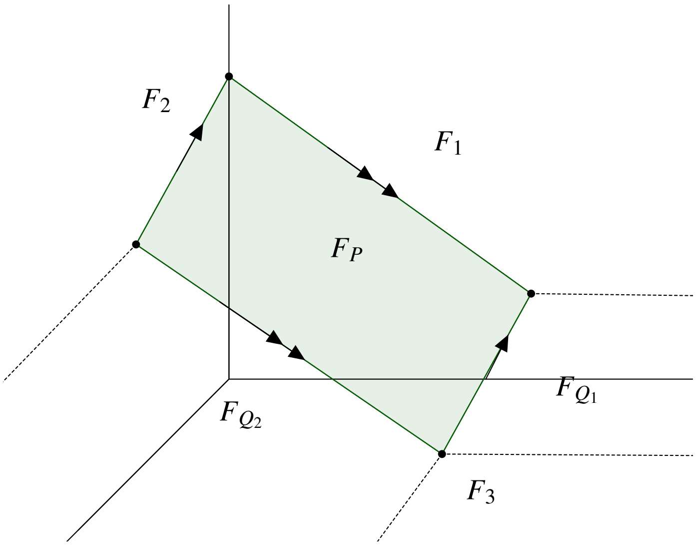


Figure 3.41: Newton polyhedron $NP(f)$

For the compact facet F_P the dual vector is $P = (2m+3, 2, 2m+3)$. For the non-compact facets $F_1, F_2, F_3, F_{Q_1}, F_{Q_2}$ the dual vectors are E_1, E_2, E_3, $Q_1 = (1, 0, 1)$, $Q_2 = (0, 2, 2p + 3)$ respectively. The $DNP(f)$ has four 2-dimensional cones: $\sigma_1 :=< (2m + 3, 2, 2m + 3), (1, 0, 0) >$, $\sigma_2 :=< (2m + 3, 2, 2m + 3), (0, 1, 0) >$, $\sigma_3 :=< (2m + 3, 2, 2m + 3), (1, 0, 1) >$, $\sigma_4 :=< (2m + 3, 2, 2m + 3), (0, 2, 2m + 3) >$.

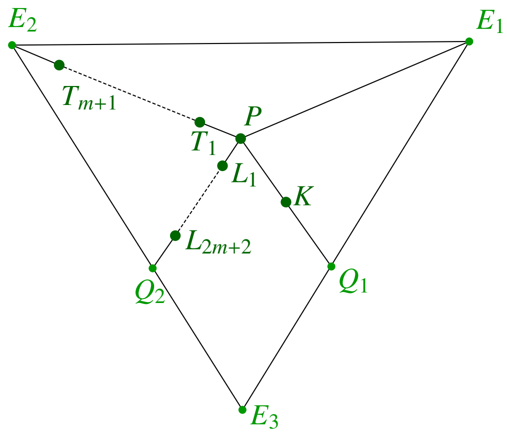
$det(\sigma_1) = det(P, E_1) = 1$.

$det(\sigma_2) = det(P, E_2) = 2m + 3$. We have $m + 1$ vectors $T_1, \ldots, T_{m+1}$ with weights $[3 : 2 : \ldots : 2]$ where $T_1 = (m + 1, 1, m + 1)$.

$det(\sigma_3) = det(P, Q_1) = 2$. We have one vector $K = (m + 2, 1, m + 2)$ with weight [2].

$det(\sigma_4) = det(P, Q_2) = 2m + 3$. So we have $2m + 2$ vectors $L_1, \ldots, L_{2m+2}$ with weights $[2 : \ldots : 2]$ where $L_1 = (2m + 2, 2, 2m + 3)$.

The regular $DNP_2(f)$ is:



Remark 3.3.11. *The integers* $r(\sigma_i) = 0$ *for all* $i = 1, 2, 3, 4$.

The weight of $w(P) = -2$. Hence we get the minimal resolution of $NB_{m,n}$ for $n = 2m + 1$.

CHAPTER 4

TORIC MODIFICATION

4.1 Non-degenerate singularities

The polynomial $f(\mathbf{z}) = \sum_{\nu} a_{\nu}\mathbf{z}^{\nu}$ is *non-degenerate* with respect to its Newton polyhedron if for every compact face F_w of $NP(f)$ the polynomial f_w defines a non-singular hypersurface of the torus $\mathbb{C}^{*n}$ where $f_w(\mathbf{z}) = \sum_{\nu \in F_w} a_{\nu}\mathbf{z}^{\nu}$ which is called a face function of f. Equivalently, for every compact face F_w of $NP(f)$ if the system of equations $\partial f_w/\partial z_1 = \partial f_w/\partial z_2 = \cdots = \partial f_w/\partial z_n = 0$ has no solution in $\mathbb{C}^{*n}$.

The class of non-degenerate singularities are important as they can be resolved by toric modifications.

Example 4.1.1. The ADE singularities given in preceding section are all non-degenerate singularities. For instance:

Let us consider Example 3.2.12. The face function f_P defined above equals f and

$$\frac{\partial f}{\partial z_1} = 2z_1 = 0, \quad \frac{\partial f}{\partial z_2} = 2z_2 z_3 = 0, \quad \frac{\partial f}{\partial z_3} = 4z_3^3 + z_2^2 = 0.$$

These three equations have no solution in $\mathbb{C}^{*3}$. Hence f defines a non-degenerate singularity with respect to its $NP(f)$.

Example 4.1.2. Some of the non-isolated singularities given in preceding section are degenerate. For instance:

1. In Example 3.2.12 the face function on F_{P_1} is $f_{P_1} = z_3^3 + z_1 z_3^2 + z_2^{n+3} z_3$ and,

$$\frac{\partial f}{\partial z_1} = z_3^2 = 0, \quad \frac{\partial f}{\partial z_2} = (n+3)z_2^{n+2} z_3 = 0, \quad \frac{\partial f}{\partial z_3} = 3z_3^2 + 2z_1 z_3 + z_2^{n+3} = 0$$

has no solution in $\mathbb{C}^{*3}$. So f is non-degenerate on F_{P_1}. However, the face function on F_{P_2} is $f_{P_2} = z_1z_3^2 + z_2^{n+3}z_3 + z_1^2z_2^{2n+2}$ and,

$$\frac{\partial f}{\partial z_1} = z_3^2 + 2z_1z_2^{2n+2} = 0, \quad \frac{\partial f}{\partial z_2} = (n+3)z_2^{n+2}z_3 + (2n+2)z_1^2z_2^{2n+1} = 0, \quad \frac{\partial f}{\partial z_3} = 2z_1z_3 + z_2^{n+3} = 0$$

has a solution in $\mathbb{C}^{*3}$. So f is degenerate on F_{P_2}. As f is not non-degenerate on every compact face of $NP(f)$, f is degenerate.

2. In Example 3.3.10 the face function $f_P = f$, and

$$\frac{\partial f}{\partial z_1} = z_2^{2p+2} + z_3^2 = 0, \quad \frac{\partial f}{\partial z_2} = (2p+3)z_2^{2p+2}z_3 + (2p+3)z_1z_2^{2p+2} = 0, \quad \frac{\partial f}{\partial z_3} = 2z_1z_3 + z_2^{2p+3} + 3z_3^2 = 0$$

has a solution $(-i, 1, i)$ in $\mathbb{C}^{*3}$. So f is degenerate.

All other non-isolated singularities given in preceding section is non-degenerate. Here we give only one example, the rest can be computed similarly.

Example 4.1.3. In Example 3.3.1 the face function $f_P = f$ and,

$$\frac{\partial f}{\partial z_1} = 2z_1z_2^2 = 0, \quad \frac{\partial f}{\partial z_2} = 3z_2^2z_3 + 2z_1^2z_2 = 0, \quad \frac{\partial f}{\partial z_3} = 3z_3^2 + z_2^3 = 0$$

has no solution in $\mathbb{C}^{*3}$. Hence f is non-degenerate.

4.2 Toric Modification

Let k be an algebraically closed field. Let $k^* = k - \{0\}$ and $(k^*)^n$ be the algebraic torus of dimension n, denoted by $\mathbb{T}^n$.

Definition 4.2.1. *A toric variety X is a normal algebraic variety of dimension n over k if it contains $\mathbb{T}^n$ as a Zariski open dense subset together with an action of $\mathbb{T}^n$ on the pair $\mathbb{T}^n \hookrightarrow X$ which is the standard torus action.*

We will deal with the construction of an affine toric variety by using combinatorial objects. Remember that σ is a strictly convex rational polyhedral cone (or simply cone) in $\check{\mathbb{R}}^n$.

A set S endowed with an operation + is called a *semigroup* if it is associative, commutative and it has identity element 0.

Remark 4.2.2. *If $\check{\sigma}$ is a cone in $\mathbb{R}^n$ then $S_\sigma := \check{\sigma} \cap M$ is a semigroup.*

Lemma 4.2.3 (Gordan P.,1873). *S_σ is finitely generated.*

Proof. Let $v_1, \ldots, v_n$ be the generators of $\check{\sigma}$. Each v_i is in $\check{\sigma} \cap M$. The set $K := \{\sum_{i=1}^{n} \lambda_i v_i \colon 0 \leq \lambda_i \leq 1 \ \forall i\}$ is compact and M is discrete. A discrete set on a compact set cannot contain any accumulation points. Therefore $K \cap M$ is finite.

Claim. *S_σ is generated by $K \cap M$.*

Take any element $v \in S_\sigma$. It can be written as $v = \sum_{i=1}^{n} r_i v_i \; r_i \in \mathbb{R}_{\geq 0}$. Let $\lfloor r_i \rfloor$ be the greatest integer less than r_i and let $[\![r_i]\!] = r_i - \lfloor r_i \rfloor$. Then

$$v = \sum_{i=1}^{n} (\lfloor r_i \rfloor + [\![r_i]\!]) v_i = \sum_{i=1}^{n} \lfloor r_i \rfloor v_i + \sum_{i=1}^{n} [\![r_i]\!] v_i$$

As $0 \leq [\![r_i]\!] < 1$ and $\sum_{i=1}^{n} \lfloor r_i \rfloor v_i \in S_\sigma$, the sum $\sum_{i=1}^{n} [\![r_i]\!] v_i \in K \cap M$. Further $v_i \in K \cap M$ (choose $\lambda_i = 1$ and $\lambda_j = 0$ for $j \neq i$). So $K \cap M$ generates S_σ.

□

Remark 4.2.4. *Any additive semigroup S determines a group ring $k[S]$ which is a k-algebra. Moreover any finitely generated k-algebra determines an affine variety. Hence $R_\sigma := k[S_\sigma]$ is a k-algebra such that for any f in R_σ, $supp(f) \subset \check{\sigma} \cap M$.*

Definition 4.2.5. *An affine toric variety X_σ associated to a cone σ is the spectrum of the finitely generated semigroup algebra R_σ.*

Definition 4.2.6. *Let X_σ be an affine algebraic variety of dimension n associated to a regular cone $\sigma = Cone(w^1, \ldots, w^n)$. The toric morphism is locally defined by*

$$\pi_\sigma : X_\sigma \to \mathbb{C}^n$$
$$\mathbf{y}_\sigma \mapsto \pi_\sigma(\mathbf{y}_\sigma) = (u_1, \ldots, u_n)$$

where $u_i = \prod_{j=1}^{n} y_{\sigma,j}^{w_i^j}$, $\mathbf{y}_\sigma = (y_{\sigma,1}, \ldots, y_{\sigma,n})$

Remark 4.2.7. *We can identify σ with the unimodular matrix (w_i^j).*

Toric modification associated with Newton polyhedron. Given a rational fan Σ in $\check{\mathbb{R}}^n$. Let $\mathcal{M}$ be the set of n-dimensional cones in Σ. If two cones σ and τ in $\mathcal{M}$ have a common face $\sigma\tau$;

then the affine algebraic varieties X_σ,X_τ can be glue along the corresponding variety $X_{\sigma\tau}$ by the gluing map $\pi_{\tau^{-1}\sigma}\colon X_\sigma \to X_\tau$ such that $y_\tau = \pi_{\tau^{-1}\sigma}(y_\sigma)$ where $\tau^{-1}\sigma$ is a unimodular matrix. This gives rise to an algebraic variety.

Let $\bigsqcup_{\sigma\in\mathcal{M}} X_\sigma$ be the disjoint union . Define an equivalence relation on $\bigsqcup_{\sigma\in\mathcal{M}} X_\sigma$ as follows; take any points $y_\sigma \in X_\sigma$ and $y_\tau \in X_\tau$ and identify them if there exists a gluing map $\pi_{\tau^{-1}\sigma}\colon X_\sigma \to X_\tau$ such that $y_\tau = \pi_{\tau^{-1}\sigma}(y_\sigma)$.

Definition 4.2.8. *Let X_Σ be the quotient space of $\bigsqcup_{\sigma\in\mathcal{M}} X_\sigma$ by the equivalence relation defined above. We call X_Σ the toric variety associate with Σ. The map $\pi_\Sigma : X_\Sigma \to \mathbb{C}^n$ is called the toric modification with respect to Σ.*

Note that the variety X_Σ is covered by affine charts X_σ corresponding to n-dimensional cones σ of Σ. We have $\pi_\Sigma([y_\sigma]) = \pi_\sigma(y_\sigma)$ where $[y_\sigma]$ is the equivalence class of $y_\sigma \in X_\sigma$. In other words, the morphism π_Σ is defined by the restriction on each affine chart X_σ.

Theorem 4.2.9. *Let Σ be a fan in $N_\mathbb{R}$. Then X_Σ is non-singular if and only if Σ is a regular fan.*

Let $f(z) = \sum_\nu a_\nu z^\nu$ be any polynomial and let $\sigma = \langle w^1, \ldots, w^n\rangle$ be a regular cone in $\check{\mathbb{R}}^n_{\geq 0}$. We consider the behaviour of f under the toric map $\pi_\sigma : X_\sigma \to \mathbb{C}^n$ corresponding to σ. Recall that s is the support function associated $NP(f)$. We have

$$\begin{aligned} f \circ \pi_\sigma &= \sum_\nu a_\nu y_{\sigma,1}^{\langle w^1,\nu\rangle} \cdots y_{\sigma,n}^{\langle w^n,\nu\rangle} \\ &= y_{\sigma,1}^{s(w^1)} \cdots y_{\sigma,n}^{s(w^n)} \sum_\nu a_\nu y_{\sigma,1}^{\langle w^1,\nu\rangle - s(w^1)} \cdots y_{\sigma,n}^{\langle w^n,\nu\rangle - s(w^n)} \end{aligned}$$

The support function s determines a divisor on X_σ having the equation $y_{\sigma,1}^{s(w^1)} \cdots y_{\sigma,n}^{s(w^n)}$. We call it the *exceptional divisor* of the map π_σ. The strict transform $\tilde{f} = \sum_\nu a_\nu y_{\sigma,1}^{\langle w^1,\nu\rangle - s(w^1)} \cdots y_{\sigma,n}^{\langle w^n,\nu\rangle - s(w^n)}$.

4.3 Examples

In this section we first consider the toric modification of an ADE singularity, then of an example of a complete intersection singularity. Finally, we find out that there exist a resolution for the two examples of degenerate non-isolated singularities.

In the examples below we only do our computations in the chart π_{σ_1} instead of computing in all other charts π_{σ_i} because by construction of the toric modification, we will only observe the same phenomenon in a different chart.

Example 4.3.1. Consider the E_6 singularity given in Example 3.2.6.First we write down the map and then we will find the strict transform and exceptional divisor of the associated map. The $DNP(f)$ has four regular maximal dimensional cones:

1. $\sigma_1 = \langle (6,4,3),(4,3,2),(1,0,0)\rangle$
2. $\sigma_2 = \langle (6,4,3),(4,3,2),(3,2,2)\rangle$
3. $\sigma_3 = \langle (6,4,3),(2,1,1),(3,2,2)\rangle$
4. $\sigma_4 = \langle (6,4,3),(2,1,1),(1,0,0)\rangle$

Let us take the cone $\sigma_1 = \langle (6,4,3),(4,3,2),(1,0,0)\rangle$. We use y_i instead of y_{σ_i} for simplicity of notation. The toric modification of σ_1 is:

$$\begin{aligned} \pi_\sigma : z_1 &= y_1^6 y_2^4 y_3 \\ z_2 &= y_1^4 y_2^3 \\ z_3 &= y_1^3 y_2^2 \end{aligned}$$

Then, $\ f \circ \pi_\sigma = y_1^{12} y_2^8 (1 + y_2 + y_3^2)$.
The strict transform is $\tilde{f} = 1 + y_2 + y_3^2$ and exceptional divisor $\pi_{\sigma_3}^{-1}(0) = \{y_1 = 0\} \cup \{y_2 = 0\}$.

Example 4.3.2. Let $\mathbb{V}(f_1, f_2) = \{(z_1,z_2,z_3) \in \mathbb{C}^3 : f_1 = f_2 = 0\}$ be a complete intersection defined by the equations $f_1(z_1,z_2,z_3) = z_2^2 - z_1^3$ and $f_1(z_1,z_2,z_3) = z_3^2 - z_1^5 z_2$.

The Newton polyhedron of f_1, f_2, $NP(f_1, f_2)$, is defined by the Minkowski sum of the two Newton polyhedron $NP(f_1), NP(f_2)$. The Minkowski sum $NP(f_1) +_M NP(f_2) = \{v_1 + v_2 : v_1 \in NP(f_1), v_2 \in NP(f_2)\}$. Then, $NP(f_1)$ is the convex hull of $\{\{(3,0,0),(0,2,0)\} + \mathbb{R}^3\}$ and $NP(f_2)$ is the convex hull of $\{\{(5,1,0),(0,0,2)\} + \mathbb{R}^3\}$. Hence $NP(f_1, f_2)$ is the convex hull of $\{\{(8,1,0),(3,0,2),(5,3,0),(0,2,2)\} + \mathbb{R}^3\}$.

The $DNP(f_1, f_2)$ has four regular maximal dimensional cones in regular $DNP(f)$ as

1. $\sigma_1 = \langle (4,6,13),(2,3,7),(1,1,3)\rangle$
2. $\sigma_2 = \langle (4,6,13),(2,3,6),(1,1,3)\rangle$

3. $\sigma_3 = \langle (4,6,13),(2,3,7),(3,5,10)\rangle$
4. $\sigma_4 = \langle (4,6,13),(3,5,10),(2,3,6)\rangle$

Let us take $\sigma_1 = \langle (4,6,13),(2,3,7),(1,1,3)\rangle$. Then the toric modification is

$$\begin{aligned}\pi_\sigma : z_1 &= y_1^4 y_2^2 y_3\\ z_2 &= y_1^6 y_2^3 y_3\\ z_3 &= y_1^{13} y_2^7 y_3^3\end{aligned}$$

Then,

$$\begin{aligned}f_1 \circ \pi_\sigma &= y_1^{12} y_2^6 y_3^2 (1 - y_3)\\ f_2 \circ \pi_\sigma &= y_1^{26} y_2^{13} y_3^6 (y_2 - 1)\end{aligned}$$

The strict transforms are $\tilde{f}_1 = (1 - y_3)$ and $\tilde{f}_1 = (y_2 - 1)$ and exceptional divisor is

$$\pi_{\sigma_3}^{-1}(0) = \{y_1 = 0\} \cup \{y_2 = 0\} \cup \{y_3 = 0\}$$

.

Example 4.3.3. Consider the hypersurface in Example 3.3.10 with the equation $f(z_1, z_2, z_3) = z_1 z_3^2 + z_2^{2m+3} z_3 + z_1 z_2^{2m+3} + z_3^3$ for $n = 2m+1$. In previous section we showed that it has a degenerate singularity. The maximal dimensional cones of $DNP(f)$ are

1. $\sigma_1 = \langle (2m+3, 2, 2m+3), (m+1, 1, m+1), (2m+2, 2, 2m+3)\rangle$
2. $\sigma_2 = \langle (2m+3, 2, 2m+3), (2m+2, 2, 2m+3), (m+2, 1, m+2)\rangle$
3. $\sigma_3 = \langle (2m+3, 2, 2m+3), (m+1, 1, m+1), (m+2, 1, m+2)\rangle$

Let us take the regular cone $\sigma_1 = \langle (2m+3, 2, 2m+3), (m+1, 1, m+1), (2m+2, 2, 2m+3)\rangle$ of regular $DNP(f)$. The toric modification associated to σ_1 is

$$\begin{aligned}\pi_\sigma : z_1 &= y_1^{2m+3} y_2^{m+1} y_3^{2m+2}\\ z_2 &= y_1^2 y_2 y_3^2\\ z_3 &= y_1^{2m+3} y_2^{m+1} y_3^{2m+3}\end{aligned}$$

Then, $f \circ \pi_\sigma = y_1^{6m+9} y_2^{3m+3} y_3^{6m+8} (y_1 + 1 + y_2 y_3 + y_2)$.
The strict transform is $\tilde{f} = 1 + y_3 + y_2 + y_2 y_3$ and the exceptional divisor of the map is $\pi_{\sigma_2}^{-1}(0) = \{y_1 = 0\} \cup \{y_2 = 0\} \cup \{y_3 = 0\}$.

Example 4.3.4. Consider the hypersurface in Example 3.3.4 which has an equation $f(z_1, z_2, z_3) = z_1 z_3^2 + z_2^{2p+3} z_3 + z_1 z_2^{2p+3} + z_3^3$ for $n = 2p + 1$. In previous section we showed that it has a degenerate singularity. The maximal dimensional cones of $DNP(f)$ are

1. $\sigma_1 = \langle P_1, T_1, E_1 \rangle$
2. $\sigma_2 = \langle P_1, T_1, U_1 \rangle$
3. $\sigma_3 = \langle P_1, U_1, E_1 \rangle$
4. $\sigma_4 = \langle P_2, U_{k+1}, K_1 \rangle$
5. $\sigma_5 = \langle P_2, K_1, U_{k+1} \rangle$
6. $\sigma_6 = \langle P_2, L, K_1 \rangle$

where the vectors $P_1, P_2, K_1, L, T_1, U_1, U_{k+1}$ are given in Example 3.3.4.

Let us take the regular cone

$$\sigma_1 = \langle P, T_1, L_1 \rangle = \langle (2p + 3, 2, 2p + 3), (p + 1, 1, p + 1), (2p + 2, 2, 2p + 3) \rangle$$

. The toric modification associated to σ is

$$\begin{aligned} \pi_\sigma : z_1 &= y_1^{2p+3} y_2^{p+1} y_3^{2p+2} \\ z_2 &= y_1^2 y_2 y_3^2 \\ z_3 &= y_1^{2p+3} y_2^{p+1} y_3^{2p+3} \end{aligned}$$

Then, $f \circ \pi_\sigma = y_1^{6p+9} y_2^{3p+3} y_3^{6p+8} (y_1 + 1 + y_2 y_3 + y_2)$.
The strict transform is $\tilde{f} = 1 + y_3 + y_2 + y_2 y_3$ and the exceptional divisor of the map is $\pi_{\sigma_2}^{-1}(0) = \{y_1 = 0\} \cup \{y_2 = 0\} \cup \{y_3 = 0\}$.

4.4 Conclusion

In this work, we are interested in the polynomials of the form

$$z^3 + f(x, y) z^2 + g(x, y) z + h(x, y) = 0 \tag{4.1}$$

which give non-isolated surface singularity at $(0, 0, 0)$ in $\mathbb{C}^3$. We showed that a resolution of a non-isolated hypersurface singularity of the form (4.1) can be obtained by its Newton polyhedron.

Some of the singularities of the form (4.1) are special; for example the hypersurface defined in Example 3.3.4 is a degenerate singularity and its normalization is given by the following three equations:

$$
\begin{aligned}
f_1 &= z_3^4 - z_2^2 - z_1^2 z_2 \\
f_2 &= -z_0 z_3 + z_1^{n+1} z_2 \\
f_3 &= z_0 z_2 + z_0 z_1^2 - z_1^{n+1} z_3^3
\end{aligned}
$$

The normal surface defined by these three equations f_1, f_2, f_3 is non-degenerate.

As the next step we try to see the following question: does the singularities in the form (4.1) that are degenerate admit a resolution by toric modification.

We will establish a relation between the Newton polyhedron of non-isolated hypersurface singularity and the Newton polyhedron of its normalization as a forthcoming work.

REFERENCES

[1] Micheal F. Atiyah and Ian G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[2] Jean Paul Brasselet, *Introduction to toric varieties*, Publicações Matemáticas do IMPA, [IMPA Mathematical Publications], 23 Colóquio Brasileiro de Matemática, [23rd Brazilian Mathematics Colloquium], Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2001.

[3] Gábur Braun and András Némethi, *Invariants of Newton non-degenerate surface singularities*,Compos. Math. 143, no.4, 1003-1036, 2007.

[4] Zhijie Chen, Rong Du, Sheng-Li Tan, and Fei Yu, *Cubic equations of rational triple points of dimension two*, Stud. Adv. Math., 39, Amer. Math. Soc., 2007.

[5] David A. Cox, John B. Little, Hal Schenck, *Toric Varieties*, Graduate Studies in Mathematics, vol.124, American Mathematical Society, 2011.

[6] F. Ehlers, *Eine Klasse komplexer Mannigfaltigkeiten*, Math. Ann. 218, pp. 127-156, 1975.

[7] Günter Ewald, *Combinatorial Convexity and Algebraic Geometry*,Graduate Texts in Math. 168, Springer, New York, 1996.

[8] William Fulton, *Introduction to Toric Varieties*, Annals of Math. Studies 131, Princeton Univ. Press, Princeton, NJ, 1997.

[9] Robin Hartshorne, *Algebraic Geometry*,Graduate Texts in Math. 52, Springer, New York, 1977.

[10] Heisuke Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. Math. 79, pp. 109-326, 1964.

[11] George Kempf, Finn Faye Knudsen, David Mumford, Bernard Saint-Donat, *Torodial Embeddings*, Lecture Notes in Mathematics, no.339, Springer-Verlag, 1973.

[12] A.G. Khovanskii, *Newton polyhedra and the genus of complete intersections*, Funkts. Anal. Prilozhen., 12, No. 1 , pp. 51-61, 1977.

[13] A.G. Kouchnirenko, *Polyedres de Newton et Nombres de Milnor*,Inventiones math. 32, 1-32, 1976.

[14] Tadao Oda, *Convex Bodies and Algebraic Geometry*,Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 15, Springer, New York, 1988.

[15] Mutsuo Oka, *Non-Degenerate Complete Intersection Singularity*, Actualités Mathématiques, Hermann, Paris, 1997.

[16] Bernard Teissier, *Monomial ideals, binomial ideals, polynomial ideals*, Trends in commutative algebra, Math. Sci. Res. Inst. Publ., vol. 51, Cambridge Univ. Press, Cambridge, 2004, pp. 211-246.

[17] Bernard Teissier and Rebecca Goldin, *Resolving plane branch singularities with one toric morphism*, in 'Resolution of Singularities, a research textbook in tribute to Oscar Zariski', Birkhaüser, Progress in Math. No. 181, 2000, 315-340.

[18] Robert J. Walker, *Reduction of the Singularities of an Algebraic Surface*, The Annals of Mathematics 2nd Ser. 36(2), pp. 336-365, 1935.