KORTEWEG-DE VRIES EQUATION ON BOUNDED DOMAINS

by

Çağrı Hacıyusufoğlu

A Thesis Submitted to the
Graduate School of Sciences and Engineering
in Partial Fulfillment of the Requirements for
the Degree of Master of Science
in Mathematics
Koç University

September 2011

ABSTRACT

In this thesis, we study existence, uniqueness and stability results for the solutions of initial-boundary value problems for the Korteweg-de Vries equation on bounded domains. First, we give a proof of the existence and uniqueness of weak solutions in the case of periodic boundary conditions. We also give a proof for the existence of an absorbing set in the Sobolev space H^2. Then, we give a proof of the existence and uniqueness of regular solutions of a non-periodic initial-boundary value problem for the Korteweg-de Vries equation. In the non-periodic case, the exponential decay of solutions for small enough initial data is also shown.

ÖZET

Bu çalışmada, sınırlı aralıklarda Korteweg-de Vries denkleminin çözümleriyle ilgili, varlik, teklik ve stabilite sonuclarini anlamaya çalıştık. Ílk olarak, periyodik sınır koşulları altında, zayıf çözümlerin varlığını ve tekliğini gösterdik. Ayrıca, bu çözümler için H^2 Sobolev uzayında bir soğurucu kümenin varlığını da ispatladık. Daha sonra, Korteweg-de Vries denklemi için, periyodik olmayan bir baslangicsinir probleminin düzgün çözümlerinin varlığını ve tekliğini österdik. Periyodik olmayan durumda, çözümlerin, yeterince küçük başlangıç koşulları altında, exponensiyel olarak azaldığını da ispatladık.

ACKNOWLEDGEMENTS

Firslty, I want to thank to my advisor Prof. Varga Kalantarov for his supports and guidance during my studies. Also, I would like to thank to my officemates for the great environment they created. Lastly, I want to thank TUBÍTAK for their financial support.

Contents

1 Introduction

Korteweg-de Vries -henceforth we will say KdV equation- equation was formulated as a model for one-directional water waves of small amplitude in shallow water. The equation first appeared in the paper of J. Boussinesq [2] and it was named for D.J. Korteweg and G. de Vries after they had studied on it in [6]. KdV equation we will consider is as follows

$$u_t(x,t) + u(x,t)u_x(x,t) + u_{xxx}(x,t) = 0,$$

where x is the space variable and t is the time variable. It is an interesting non-linear equation in some respects. Firstly, its solutions can be explicitly found using inverse scattering method which we will not touch upon in this study. Moreover, this equation can be formulated as a Lax equation

$$\frac{dL}{dt} = LP - PL,$$

where L is the Sturm-Liouville operator and P is another time-dependent operator. In [8], Lax observed that any equation that can be observed as a Lax equation for some time-dependent operators L, P shares many features of the KdV equation. Also, from this formulation, it follows that KdV equation has infinitely many conservation laws only three of which we will use for our estimates.

In this thesis, we will study the existence, uniqueness and stability of solutions of KdV equations on bounded domains. Note that, if we have periodic boundary conditions, multiplying the equation by u and integrating over the domain we find

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 = 0,$$

which implies that in the periodic case we have no decay of solutions. However, in many real situations, we have dissipation and external force and we can get stability for the solutions in that case. In this respect, in the following chapter, based on the works of Temam [11] and Ghidaglia [5], we will study the following damped KdV equation with periodic boundary conditions:

$$u_t + uu_x + u_{xxx} + \gamma u = f, \qquad x, t \in \mathbf{R},$$

$$u(x + L, t) = u(x, t), \qquad x, t \in \mathbf{R}, L > 0,$$

$$u(x, 0) = u_0(x), \qquad \forall x \in \mathbf{R}.$$

Then, in the last chapter we will study the KdV equation under non-periodic boundary conditions based on the work of Larkin [7] which is stated as follows:

$$u_t + uu_x + u_{xxx} = 0, \qquad x \in (0, 1), t > 0,$$

$$u(x, 0) = u_0(x), \qquad x \in (0, 1),$$

$$u(0, t) = u(1, t) = u_x(1, t) = 0, \qquad t > 0.$$

Some of the other results for KdV equation on bounded domains are obtained in [4] and [12].

Before we begin, we will state some definitions and theorems that we will use in our work.

1.1 Function spaces

Here we give the definitions of function spaces we use in the study of initial-boundary value problems for the Korteweg-de Vries equation. In the following, Ω will denote an open interval in $\mathbf{R}$.

Definition 1.1. *A Banach space is a complete normed vector space. If the norm is induced by an inner product then it is called a Hilbert space. The space of square* *integrable functions on Ω will be denoted by $L^2(\Omega)$ and $L^2(\Omega)$ is a Hilbert space with the following inner product:*

$$\langle u, v\rangle = \int_\Omega u(x)v(x)\,dx \qquad u, v \in L^2(\Omega).$$

Lastly, $\|\cdot\|$ will denote the norm on $L^2(\Omega)$.

Basic spaces that we use in the study of our problems are Sobolev spaces. In order to define these spaces we first need the notion of weak derivative. Let $C_0^\infty(\Omega)$ be the space of compactly supported functions on Ω. Then for $u \in C^1(\Omega)$ and for any test function $\eta \in C_0^\infty(\Omega)$, following equality holds:

$$\int_\Omega u\eta_x\,dx = -\int_\Omega u_x\eta\,dx.$$

If we take $u \in L^1_{loc}(\Omega)$, i.e. if u is locally integrable, then the left hand side of the equation is still meaningful. Also, it is possible to have another function $v \in L^1_{loc}(\Omega)$ such that above identity holds with u_x replaced by v, i.e. the following identity holds:

$$\int_\Omega u\eta_x\,dx = -\int_\Omega v\eta\,dx. \qquad \forall \eta \in C_0^\infty(\Omega).$$

Then we call v a *weak derivative* of u. It can be shown that weak derivative is unique so that if u is differentiable, then the usual derivative coincides with the weak derivative. We can generalize the above discussion as follows:

Definition 1.2. *Let α be a positive integer and D^α denote the α^{th} derivative. A function $v \in L^1_{loc}(\Omega)$ is called the α^{th} weak derivative of a function $u \in L^1_{loc}(\Omega)$ if for all $\eta \in C_0^\infty(\Omega)$, the following equality holds:*

$$\int_\Omega uD^\alpha\eta\,dx = (-1)^\alpha \int_\Omega v\eta\,dx.$$

This definition can be generalized to the case $\Omega \subset \mathbf{R}^n$, but since we will study only in dimension one, this is enough for our purposes. Now we can define Sobolev spaces.

Definition 1.3. *Let $0 \leq \alpha, k < \infty$ be integers. Then the Sobolev space $H^k(\Omega)$ is the space of all functions in $L^1_{loc}(\Omega)$ such that the function itself and all the α^{th}-weak derivatives, where $1 \leq \alpha \leq k$, belong to $L^2(\Omega)$. If $u \in H^k(\Omega)$ we define its norm as*

$$\|u\|_{H^k(\Omega)} = \left(\sum_{\alpha \leq k} \|D^\alpha u\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Moreover, $H^k(\Omega)$ a Hilbert space with the following inner product:

$$\langle u, v \rangle = \sum_{\alpha \leq k} \int_\Omega D^\alpha u(x) D^\alpha v(x)\, dx.$$

Following theorem is true only in dimension one.

Theorem 1.4. *For all $u \in H^1(\Omega)$, there exists $\widetilde{u} \in C(\overline{\Omega})$ such that $u = \widetilde{u}$ almost everywhere and*

$$\widetilde{u}(a) - \widetilde{u}(b) = \int_a^b u_x(x)\, dx \qquad \forall a,b \in \Omega.$$

This theorem also implies that functions belonging to $H^1(\Omega)$ are bounded if Ω is bounded, a fact that we will use very often.

It can be shown that $C^\infty(\overline{\Omega})$ is dense in $H^k(\Omega)$ from which it follows that $H^k(\Omega)$ is dense in $H^m(\Omega)$ for $k > m$. We will denote the closure of the $C_0^\infty(\Omega)$ in $H^k(\Omega)$ with $H_0^k(\Omega)$. Indeed, $H_0^k(\Omega)$ is the space of functions u in $H^k(\Omega)$ for which $D^\alpha u$ is zero on the boundary where $0 \leq \alpha \leq (k-1)$. We will also need the dual space of $H_0^k(\Omega)$.

Definition 1.5. *The space $H^{-k}(\Omega)$ will denote the dual of $H_0^k(\Omega)$. It can be identified with the completion of $L^2(\Omega)$ with respect to the norm*

$$\|u\|_{H^k(\Omega)} = sup_{v \in H_0^k(\Omega)} \frac{|\langle u,v \rangle|}{\|v\|_{H^k(\Omega)}},$$

where $\langle \cdot,\cdot \rangle$ is the inner product on $L^2(\Omega)$.

For a complete treatment of Sobolev spaces see [3].

The equations we will consider depend on time. If, for example, $u(x,t)$ is a solution for an equation where t is the time variable and x is the space variable, then we can consider the function $u(x,t)$ as a function of t with values in a Banach space. This motivates the following definition.

Definition 1.6. *Let X be a Banach space. The space $L^p(0,T;X)$ consists of all measurable functions $u:(0,T)\to X$ such that $\|u\|_{L^p(0,T;X)}<\infty$ where*

$$\|u\|_{L^p(0,T;X)}=\begin{cases}\left(\int_0^T \|u(t)\|_X^p\,dt\right)^{1/p}, & \text{if } 1\le p<\infty\\ ess\ sup_{t\in(0,T)}\|u(t)\|_X, & \text{if } p=\infty\end{cases}$$

and the space $C([0,T];X)$ consists of all continuous functions $u:[0,T]\to X$ with the norm

$$\|u\|_{C([0,T];H)}=max_{t\in[0,T]}\|u(t)\|_X$$

All the spaces in the above definition are Banach spaces. The space $L^2(0,T;X)$ is also an Hilbert space if X is an Hilbert space, where the inner product for u, $v\in L^2(0,T;X)$ is defined as

$$\langle u,v\rangle=\int_0^T\langle u(t),v(t)\rangle_X\,dt.$$

Similar to the real valued case, $L^q(0,T;X)$ is continuously embedded into $L^p(0,T;X)$ for $1\le p\le q\le\infty$ since $(0,T)$ is bounded. Moreover, if X is continuously embedded in Y, then $L^q(0,T;X)$ is continuously embedded into $L^p(0,T;Y)$. Lastly, we will need the definition of weak derivative for vector valued functions on $(0,T)$.

Definition 1.7. *A function $v\in L^1_{loc}(0,T;X)$ is said to be a weak derivative of a function $u\in L^1_{loc}(0,T;X)$ if for all $\eta\in C_0^\infty(0,T)$, the following equality holds:*

$$\int_0^T u(t)\frac{d\eta}{dt}(t)\,dt=-\int_0^T v(t)\eta(t)\,dt.$$

1.2 Compact embedding, weak and weak-star convergence

Definition 1.8. *Let X and Y be normed vector spaces. Then a continuous operator $T : X \to Y$ is called compact if, for every bounded set $B \subset X$, $T(B)$ is precompact, i.e. every sequence in $T(B)$ has a cauchy subsequence. If X is continuously embedded in Y, i.e. there is a continuous injection from X to Y, and the injection is compact, then we say X is compactly embedded into Y.*

In analysis, one of the ways of proving existence theorems is to create an appropriate sequence which is bounded and then try to extract a subsequence such that its limit works for us. However, if X is an infinite dimensional Banach space, we know that a bounded sequence may not have a convergent subsequence. Hence, in order to use compactness arguments when X is infinite dimensional, we need somewhat weaker definitions of convergence. In the following, X' will denote the dual of X.

Definition 1.9.

1. *Let X be a Banach space. Then we say that a sequence $u_n \in X$ converges to some $u \in X$ weakly, if for all $f \in X'$, $f(u_n)$ converges to $f(u)$. We will denote this by $u_n \rightharpoonup u$.*

2. *Let X be the dual of a Banach space Y. Then we say that a sequence $u_n \in X$ converges to some $u \in X$ weakly star, if for all $v \in Y$, $u_n(v)$ converges to $u(v)$. We will denote this by $u_n \overset{*}{\rightharpoonup} u$.*

If u_n converges to u in norm, then we will say u_n converges to u strongly and we will denote this by $u_n \to u$. Weakly convergent and weakly star convergent sequences are bounded and their limits are unique. Also, if X is a dual of another space, then we have three types of convergence in X; strong convergence, weak convergence, and weak-star convergence. It is obvious that strong convergence implies weak convergence, and weak convergence implies weak-star convergence. If X is reflexive, for instance a Hilbert space, then weak convergence and weak-star convergence are equivalent. Following theorem clarify the effectiveness of these definitions.

Theorem 1.10.

1. *Let X be a reflexive Banach space. Then every bounded sequence in X has a weakly convergent subsequence.*

2. *Let X be the dual of another separable Banach space Y. Then every bounded sequence in X has a weakly star convergent subsequence.*

For a proof, see [3].

Note also that, if T is a strongly continuous operator, then T is weakly continuous, and if the latter is true, then T is weakly star continuous. Using the first implication, we can show the following theorem easily.

Theorem 1.11. *Let X, Y be Banach spaces and $T : X \to Y$ be a linear compact operator. Then $u_n \rightharpoonup u$ in X implies that $Tu_n \to Tu$ in Y.*

Proof. Let $T : X \to Y$ be a compact operator and let $u_n \rightharpoonup u$ in X. Then $Tu_n \rightharpoonup Tu$ in Y. If Tu_n were not convergent, then, since u_n is a bounded sequence and T is a compact operator, Tu_n would have two different subsequences which converge strongly to two different limits. Since strong convergence imply weak convergence, this would imply that Tu_n has two different subsequences converging weakly to two different limits which is not possible. □

We will consider our time-dependent solutions as functions in the spaces $L^p(0, T; X)$ where X is an appropriate Banach space. In order to apply weak convergence methods, we need the duality relations of these spaces.

Theorem 1.12. *If $1 \leq p < \infty$ and if X is reflexive or X' is separable, then $(L^p(0,T;X))' \approx L^{p'}(0,T;X')$. In addition, if $1 < p < \infty$ and if X is reflexive, then $L^p(0,T;X)$ is also reflexive.*

Lastly we give the definition of the dual paring for the spaces $L^p(0,T;X)$.

Definition 1.13. *For $u \in L^p(0,T;X)$ and $v \in (L^p(0,T;X))' \approx (L^{p'}(0,T;X'))$, dual paring is given by the following formula:*

$$v(u) = \int_0^T \langle v(t),u(t)\rangle_{X',X} dt,$$

where $\langle \cdot, \cdot \rangle_{X',X}$ is the dual pairing between X and X'.

1.3 Some useful inequalities and embedding theorems

1. *Schwarz inequality*

   Let H be a Hilbert space. Then for any u, v in H we have

$$|\langle u, v\rangle| \leq \|u\|\|v\|.$$

2. *Young's inequality*

   Let $1 < p,q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for $a,b > 0$ we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

   Observe that writing $ab = \epsilon a \frac{1}{\epsilon} b$ for $\epsilon > 0$, we get

$$ab \leq \frac{\epsilon^p a^p}{p} + \frac{b^q}{\epsilon^q q}.$$

   We will use this inequality with $p = q = 2$ except few cases.

3. *Compactness theorem*

   Suppose that Ω is a bounded interval. If $0 \leq n < m$, where n and m are integers, then the embedding of $H^m(\Omega)$ into $H^n(\Omega)$ is compact. For a proof, see [9].

4. *Ehrling's inequality*

   Let Ω be a bounded interval and $u \in H^m(\Omega)$. Then for all $\epsilon > 0$ and $0 \leq j \leq m$, there exists $C(\epsilon, j) > 0$ such that

   $$\|D^j u\|_{L^2(\Omega)} \leq K \left(\epsilon \|D^m u\|_{L^2(\Omega)} + C(\epsilon, j)\|u\|_{L^2(\Omega)} \right).$$

   For a proof, see [1].

5. *Agmon's inequality*

   Let $u \in H^1(0, L)$ and u be periodic. Then the following inequality holds:

   $$\sup_{x\in(0,L)}|u(x)| \leq \|u\|^{1/2}_{L^2(0,L)} \left(2\|u_x\|_{L^2(0,L)} + L^{-1}\|u\|_{L^2(0,L)} \right)^{1/2}.$$

6. *Interpolation inequality*

   In order to state this inequality, we need to define Sobolev spaces H^s for $s \in \mathbf{R}$. We will need this at one instance only so we will define it briefly. For this we will first define the spaces $[X, Y]_\theta$. Suppose that X and Y are Hilbert spaces such that X is dense in Y and the embedding is compact. By Riesz Representation theorem, the inner product on X induces an onto isomorphism from X to dual of X which we call A. Then considering A as an operator from $D(A) \subset V$ onto H, the inverse of A is a compact operator from H to H which follows from X being compactly embedded into Y and the fact that composition of a continuous function with a compact function is compact. Since the inner product is symmetric, A is also self-adjoint. Then using the spectral theory for compact self-adjoint operators, we can define the powers A^s of A. We define the interpolation space $[X,Y]_\theta$ to be the domain of $A^{(1-\theta)/2}$ for $0 \le \theta \le 1$ which is equipped with the following inner product

$$\langle u, v\rangle_{[X,Y]_\theta} = \langle A^{(1-\theta)/2}u, A^{(1-\theta)/2}v\rangle_Y.$$

and the norm on $[X,Y]_\theta$ satisfies

$$\|u\|_{[X,Y]_\theta} \le \|u\|_X^{1-\theta}\|u\|_Y^{\theta}. \tag{1}$$

Now let Ω be a bounded interval. Then, taking $X = H^m(\Omega)$ and $Y = H^0(\Omega)$ in the definition of $[X,Y]_\theta$ (which we can do due to compactness theorem above), for $0 \le s \le m$ we can define $H^s(\Omega)$ as an interpolation between $H^m(\Omega)$ and $H^0(\Omega)$:

$$H^s(\Omega) = \left[H^m(\Omega), H^0(\Omega)\right]_\theta.$$

where $s = (1-\theta)m$.

Let $m > 0$ be an integer and $s > 0$ be real number. Then

$$\|u\|_{H^s(\Omega)} \le \|u\|_{H^m(\Omega)}^{1-\theta}\|u\|_{H^0(\Omega)}^{\theta},$$

where $s = (1-\theta)m$ which follows from (1). For the complete treatment of Interpolation spaces, see [9].

7. *Embedding theorem*

   Let $s \ge 0$ be a real number and $\Omega \subset \mathbf{R}$ be an open interval. Then for $s \le 1/2$ and $1/q = 1/2 - s$ we have $H^s(\Omega) \subset L^q(\Omega)$ and the embedding is continuous.

1.4 Auxiliary theorems

Following theorems will be used while proving existence and uniqueness theorems.

Theorem 1.14. *Let X be a Banach space. Then for $u, u^* \in L^1(0,T;X)$, u^* is the weak derivative of u if and only if for all $\phi \in C_0^\infty(0,T)$ and for all $v \in X'$ following equality holds:*

$$\int_0^T \langle u(t), v\rangle \phi_t \, dt = -\int_0^T \langle u^*(t), v\rangle \phi \, dt.$$

For a proof, see [10].

Theorem 1.15. *Let $X \subset Y \subset Z$ be Banach spaces where X and Y are reflexive. Assume that embeddings are continuous where the embedding $X \hookrightarrow Y$ is also compact. For any $1 < p_0, p_1 < \infty$. Let*

$$W = \left\{ u | u \in L^{p_0}(0,T;X), u_t \in L^{p_1}(0,T;Z) \right\}.$$

Then the embedding $W \hookrightarrow L^{p_0}(0,T;Y)$ is compact.

For a proof, see [13].

Theorem 1.16. *Let V, H, V' be three Hilbert spaces such that each space is included and dense in the following one, where V' is the dual of V. If a function u belongs to $L^2(0,T;V)$ and its derivative u' belongs to $L^2(0,T;V')$, then u belongs to $C([0,T];H)$ and the following equality holds on $(0,T)$:*

$$\frac{d}{dt}\|u(t)\|_H^2 = 2\langle u'(t), u(t)\rangle_{V',V}.$$

For a proof, see [10].

Theorem 1.17. *Let X be a reflexive space. If*

$$\begin{cases} u_n \rightharpoonup u & \text{in } L^2(0,T;X), \\ u_n' \rightharpoonup u' & \text{in } L^2(0,T;X), \end{cases}$$

then

$$u_n(0) \rightharpoonup u(0) \ \text{ in } X.$$

For a proof, see [13].

Theorem 1.18. *Let T be a self-adjoint compact operator on a separable Hilbert space H. Then there exist eigenfunctions of T that form an orthonormal basis of H.*

For a proof, see [3].

2 Damped KdV equation with periodic boundary conditions

In this chapter we consider the following damped KdV equation

$$u_t + uu_x + u_{xxx} + \gamma u = f, \tag{2}$$

under the periodic boundary conditions

$$u(x+L,t) = u(x,t), \qquad \forall x \in \mathbf{R}, \forall t \in \mathbf{R}, \tag{3}$$

where $L > 0$ is given, and we have the initial condition

$$u(x,0) = u_0(x), \qquad \forall x \in \mathbf{R}. \tag{4}$$

Here $f = f(x)$ is a given forcing term and $\gamma \in \mathbf{R}$ is a given number.

Notation

In this chapter $\Omega = (0, L)$ and $H_L^k(\Omega)$ will denote the subspace of $H^k(\Omega)$ which consists of periodic functions. For $u = u(x,t)$ we will use $u(t)$ as u can be seen as a function of t with values in a Banach space. Also the following notation will be used:

$$|u(t)|_j = \left\|D^j u(x,t)\right\|^2_{L^2(0,L)}, \qquad D^j = \frac{\partial^j}{\partial x^j}.$$

In general, we will not write the t argument and write only $|u|_j$ for $|u(t)|_j$. Lastly $\|\cdot\|_\infty$ will denote the L^∞ norm and $H^{-k}(\Omega)$ will denote the dual of $H_L^k(\Omega)$.

2.1 Existence and Uniqueness

Theorem 2.1. *For $\gamma \in \mathbf{R}$, $f \in H_L^2(\Omega)$, and $u_0 \in H_L^2(\Omega)$, there exists a unique solution u of* (2)-(4) *from the class*

$$u \in L^\infty(0,T; H_L^2(\Omega) \cap C([0,T], L^2(\Omega)), \qquad \forall T > 0.$$

Proof. Let $\epsilon \in (0,1)$. Also let f_ϵ and $u_{0\epsilon}$ be the approximations in $H_L^2(\Omega)$ of f and u_0 by smooth (C^∞ and L-periodic) functions. The existence of a solution for the following regularization of (2)-(4)

$$\frac{du_\epsilon}{dt} + u_\epsilon Du_\epsilon + D^3 u_\epsilon + \gamma u_\epsilon + \epsilon D^4 u_\epsilon = f_\epsilon, \tag{5}$$

$$u_\epsilon(0) = u_{0\epsilon}, \tag{6}$$

from the class

$$u_\epsilon \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T,H_L^4(\Omega)), \qquad \frac{du_\epsilon}{dt} \in L^2(0,T;L^2(\Omega)),$$

is known and we assume it here. Our aim is to get estimates for the solutions u_ϵ which are independent of ϵ and then passing to a convergent subsequence in the weak sense such that limit works for us. We will implicitly use the fact $u_\epsilon \in L^2(0,T,H_L^4(\Omega))$ in the following estimates.

Estimates for the norms $|u_\epsilon|_0$, $|u_\epsilon|_1$, and $|u_\epsilon|_2$ (for the notation see the beginning of this chapter) will be obtained separately as follows:

Estimate 1

Multiplying (2) by $2u_\epsilon$ and integrating over Ω we get

$$\begin{aligned}\frac{d}{dt}|u_\epsilon|_0^2 + 2\gamma|u_\epsilon|_0^2 + 2\epsilon\left|D^2u_\epsilon\right|_0^2 &= 2\int_\Omega f_\epsilon u_\epsilon\,dx \\ &\le 2|f_\epsilon|_0|u_\epsilon|_0 \\ &\le |\gamma||u_\epsilon|_0^2 + \frac{1}{|\gamma|}|f_\epsilon|_0^2.\end{aligned}$$

So

$$\frac{d}{dt}|u_\epsilon|_0^2 + 2\epsilon\left|D^2u_\epsilon\right|_0^2 \le 3|\gamma||u_\epsilon|_0^2 + \frac{1}{|\gamma|}|f_\epsilon|_0^2, \tag{7}$$

and this implies

$$\frac{d}{dt}|u_\epsilon|_0^2 \le 3|\gamma||u_\epsilon|_0^2 + \frac{1}{|\gamma|}|f_\epsilon|_0^2.$$

Now let $T > 0$. Multiplying the inequality above by $e^{-3|\gamma|t}$ and integrating over $(0,t)$ for $t \in (0,T)$ we get

$$
\begin{aligned}
&|u_\epsilon(t)|_0^2\, e^{-3|\gamma|t} \leq |u_{0\epsilon}|_0^2 + \frac{1}{3\gamma^2}|f_\epsilon|_0^2 \\
&\Rightarrow |u_\epsilon(t)|_0^2 \leq |u_{0\epsilon}|\, e^{3|\gamma|t} + \frac{e^{3|\gamma|t}}{3\gamma^2}|f_\epsilon|_0^2 \leq^a c,
\end{aligned} \tag{8}
$$

where we have used in (a) the fact that $u_{0\epsilon}$ and f_ϵ are convergent in $H^2(\Omega)$ so that the norms $|f_\epsilon|_0$ and $|u_{0\epsilon}|_0$ are uniformly bounded with respect to $\epsilon \in (0,1)$ and $t \in (0,T)$. Therefore, c depends on T but does not depend on $\epsilon \in (0,1)$ and $t \in (0,T)$. Hence inserting (8) into (7) and integrating over $(0,t)$ again, it follows that

$$
|u_\epsilon(t)|_0^2 + \epsilon \int_0^t \left|D^2 u_\epsilon(s)\right|_0^2 \, ds \leq c, \tag{9}
$$

for all $t \in (0,T)$ and $\epsilon \in (0,1)$ where $c > 0$ is a generic constant which we will use for later estimates as well.

Estimate 2

This time multiplying (2) by $-2D^2u_\epsilon - u_\epsilon^2$ and integrating over Ω, the terms coming from $u_\epsilon D u_\epsilon$ and $D^3 u_\epsilon$ cancel and we get

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \left((Du_\epsilon)^2 - \frac{u_\epsilon^3}{3}\right) dx + 2\gamma \int_\Omega \left((Du_\epsilon)^2 - \frac{u_\epsilon^3}{2}\right) dx \\
&+ 2\epsilon \int_\Omega \left(\left(D^3 u_\epsilon\right)^2 + u_\epsilon D u_\epsilon D^3 u_\epsilon\right) dx = \int_\Omega \left(2Df_\epsilon D u_\epsilon - f_\epsilon u_\epsilon^2\right) dx.
\end{aligned} \tag{10}
$$

Let $\varphi(u_\epsilon) = \int_\Omega (Du_\epsilon^2 - \frac{u_\epsilon^3}{3})\, dx$. We first try to get the following estimate:

$$
\frac{d}{dt}\varphi(u_\epsilon) + 2\gamma\varphi(u_\epsilon) + \epsilon|u_\epsilon|_3^2 \leq |\gamma|\varphi(u_\epsilon) + c. \tag{11}
$$

For this, we add $\frac{\gamma}{3}\int u^3\, dx$ to the second term in (10) and subtract it again from the equation which gives

$$\frac{d}{dt}\varphi(u_\epsilon) + 2\gamma\varphi(u_\epsilon) + 2\epsilon|u_\epsilon|_3^2 \leq \frac{\gamma}{3}\int_\Omega u_\epsilon^3\,dx - 2\epsilon\int_\Omega \left(u_\epsilon Du_\epsilon D^3 u_\epsilon\right)\,dx$$
$$+\int_\Omega \left(2Df_\epsilon Du_\epsilon - f_\epsilon u_\epsilon^2\right)\,dx. \qquad (12)$$

Now we will estimate the terms on the right hand side separately. Firstly

$$\begin{aligned}\left|\int_\Omega u_\epsilon^3\,dx\right| &\leq \|u_\epsilon\|_\infty |u_\epsilon|_0^2 \\ &\leq^a |u_\epsilon|_0^{5/2}\left(2|u_\epsilon|_1 + L^{-1}|u_\epsilon|_0\right)^{1/2} \\ &\leq^b \sqrt{2}\,|u_\epsilon|_0^{5/2}|u_\epsilon|_1^{1/2} + L^{-1/2}|u_\epsilon|_0^3 \\ &\leq^c |u_\epsilon|_1^2 + \frac{3}{4}|u_\epsilon|_0^{10/3} + L^{-1/2}|u_\epsilon|_0^3, \qquad (13)\end{aligned}$$

where we have used Agmon's inequality in (a), the fact that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ in (b), and Young's inequality with $p = 4$ and $q = 4/3$ in (c). Then using the estimate above we get

$$\begin{aligned}\varphi(u_\epsilon) &\geq |u_\epsilon|_1^2 - \frac{1}{3}\left|\int u_\epsilon^3\,dx\right| \\ &\geq \frac{2}{3}|u_\epsilon|_1^2 - \frac{1}{4}|u_\epsilon|_0^{10/3} - \frac{1}{3L^{1/2}}|u_\epsilon|_0^3,\end{aligned}$$

which implies that

$$|u_\epsilon|_1^2 \leq \frac{3}{2}\varphi(u_\epsilon) + \frac{3}{8}|u_\epsilon|_0^{10/3} + \frac{1}{2L^{1/2}}|u_\epsilon|_0^3. \qquad (14)$$

Inserting the above inequality into (13) and taking into account (9) it follows that

$$\frac{\gamma}{3}\int u_\epsilon^3\,dx \leq \frac{|\gamma|}{2}\varphi(u_\epsilon) + c. \qquad (15)$$

For the second term on the right hand side of (12) we have

$$\begin{aligned}\left|2\int \left(u_\epsilon Du_\epsilon D^3 u\epsilon\right)\,dx\right| &\leq \|u_\epsilon\|_{L^4}\|Du_\epsilon\|_{L^4}\left|D^3u_\epsilon\right|_0 \\ &\leq^a c\,\|u_\epsilon\|_{H^{1/4}}\|Du_\epsilon\|_{H^{1/4}}|u_\epsilon|_3, \qquad (16)\end{aligned}$$

where we have used the embedding $L^4 \hookrightarrow H^{1/4}$ in (a). Note that $\|Du_\epsilon\|_{H^{1/4}} \leq \|u_\epsilon\|_{H^{5/4}}$ which is actually a generalization of the fact

$$\|Du\|_{H^n} \leq \|u\|_{H^{n+1}},$$

where n is an integer. Then using the interpolation inequality we get

$$\begin{aligned}
&\|u_\epsilon\|_{H^{1/4}} \leq \|u_\epsilon\|_{H^3}^{1/12} |u_\epsilon|_0^{11/12} \\
&\|Du_\epsilon\|_{H^{1/4}} \leq \|u_\epsilon\|_{H^{5/4}} \leq \|u_\epsilon\|_{H^3}^{5/12} |u_\epsilon|_0^{7/12}.
\end{aligned}$$

Inserting these into (16) we obtain

$$\left|2 \int \left(u_\epsilon D u_\epsilon D^3 u \epsilon\right) \, dx\right| \leq c \|u_\epsilon\|_{H^3}^{1/2} |u_\epsilon|_0^{3/2} |u_\epsilon|_3. \tag{17}$$

Now observe that there exists $c > 0$ such that for all $u \in H_L^3(\Omega)$, $\|u\|_{H^3} \leq c\,(|u|_0 + |u|_3)$ which follows from the following reasoning:

$$\begin{aligned}
&|u|_1^2 = -\int u D^2 u \, dx \leq |u|_0 \, |u|_2 \leq \frac{|u|_0^2}{2} + \frac{|u|_2^2}{2}, \\
&|u|_2^2 = -\int Du D^3 u \, dx \leq |u|_1 \, |u|_3 \leq \frac{|u|_1^2}{2} + \frac{|u|_3^2}{2}, \\
&\Rightarrow |u|_1^2 + |u|_2^2 \leq |u|_0^2 + |u|_3^2.
\end{aligned} \tag{18}$$

Hence

$$\begin{aligned}
\left|2 \int \left(u_\epsilon D u_\epsilon D^3 u_\epsilon\right) \, dx\right| &\leq c \, |u_\epsilon|_0^2 \, |u_\epsilon|_3 + c \, |u_\epsilon|_0^{3/2} \, |u_\epsilon|_3^{3/2} \\
&\leq^a |u_\epsilon|_3^2 + c \, |u_\epsilon|_0^4 + c \, |u_\epsilon|_0^6 \\
&\leq^b |u_\epsilon|_3^2 + c,
\end{aligned} \tag{19}$$

where we have used Young's inequality in (a), and the inequality (9) in (b). For the last term in (12) we have

$$\left|\int \left(2Df_\epsilon Du_\epsilon - f_\epsilon u_\epsilon^2\right)\, dx\right| \leq 2\left|f_\epsilon\right|_1 \left|u_\epsilon\right|_1 + ||f_\epsilon||_\infty \left|u_\epsilon\right|_0^2$$

$$\leq \frac{|\gamma|}{3}\left|u_\epsilon\right|_1^2 + \frac{3}{|\gamma|}\left|f_\epsilon\right|_1^2 + \|f_\epsilon\|_\infty \left|u_\epsilon\right|_0^2$$

$$\leq^a \frac{|\gamma|}{2}\varphi(u_\epsilon) + c. \tag{20}$$

where we have used in (a) the inequality (14) and the fact that $\left|f_\epsilon\right|_1$ and $\left|f_\epsilon\right|_\infty$ are uniformly bounded with respect to ϵ which follows from the convergence of f_ϵ in H_L^2 and Agmon's inequality. Inserting (15), (19) and (20), into (12), we obtain the inequality (11).

Now the inequality

$$-2\gamma\varphi(u_\epsilon) \leq 2|\gamma|\varphi(u_\epsilon) + c$$

is obvious for $\gamma \leq 0$ and is equivalent to the fact that $\varphi(u_\epsilon)$ is bounded from below when $\gamma \geq 0$ which is true due to (14) and (9). Therefore, (11) can be written as

$$\frac{d}{dt}\varphi(u_\epsilon) + \epsilon|u_\epsilon|_3^2 \leq 3|\gamma|\varphi(u_\epsilon) + c.$$

Multiplying the inequality above by $e^{-3|\gamma|t}$ and integrating over $(0.t)$ we get

$$\varphi(u_\epsilon(t)) + \epsilon\int_0^t |u_\epsilon(s)|_3^2\, ds \leq \varphi(u_{0\epsilon})e^{3|\gamma|t} + \frac{c}{3|\gamma|}e^{3|\gamma|t}. \tag{21}$$

It follows from (13) that

$$\varphi(u_{0\epsilon}) \leq \frac{4}{3}\left|u_{0\epsilon}\right|_1^2 + \frac{1}{4}\left|u_{0\epsilon}\right|_0^{10/3} + \frac{1}{3L^{1/2}}\left|u_{0\epsilon}\right|_0^3.$$

Since $u_{0\epsilon}$ is convergent in $H^2(\Omega)$, $\left|u_{0\epsilon}\right|_0$ and $\left|u_{0\epsilon}\right|_1$ are bounded. Hence, from the inequality (21), it follows that

$$\varphi(u_\epsilon(t)) + \epsilon\int_0^t |u_\epsilon|_3^2(s)\, ds \leq c.$$

Taking (14) into account we obtain the following inequality

$$|u_\epsilon(t)|_1^2 + \epsilon \int_0^t |u_\epsilon|_3^2(s)\, ds \le c, \tag{22}$$

for $0 \le t \le T$, where c depends on T but does not depend on ϵ.

Estimate 3

To estimate $|u_\epsilon|_2$, we multiply (2) with $M(u_\epsilon) = 2D^4 u_\epsilon + \frac{5}{3} Du_\epsilon^2 + \frac{10}{3} u_\epsilon D^2 u_\epsilon + u^3$. For $\frac{d}{dt} u_\epsilon M(u_\epsilon)$, from the third term of $M(u_\epsilon)$, there comes

$$\begin{aligned}\frac{10}{3}\int_\Omega \frac{d}{dt} u_\epsilon u_\epsilon D^2 u_\epsilon \, dx &= -\frac{10}{3}\int_\Omega \frac{d}{dt} u_\epsilon (Du_\epsilon)^2 - \frac{5}{3}\int_\Omega u_\epsilon \frac{d}{dt}(Du_\epsilon)^2 \, dx \\ &= -\frac{5}{3}\frac{d}{dt}\int_\Omega u_\epsilon (Du_\epsilon)^2 \, dx - \frac{5}{3}\int_\Omega \frac{d}{dt} u_\epsilon (Du_\epsilon)^2 \, dx.\end{aligned}$$

Remaining terms in $\frac{d}{dt} u_\epsilon M(u_\epsilon)$ are obvious to calculate. Then we have

$$\frac{d}{dt} u_\epsilon M(u_\epsilon) = \int_\Omega \left(\left(D^2 u_\epsilon\right)^2 - \frac{5}{3} u_\epsilon (Du_\epsilon)^2 + \frac{5}{36} u_\epsilon^4 \right) dx.$$

After calculating $\gamma u_\epsilon M(u_\epsilon)$ similarly, and observing that

$$\int_\Omega M(u_\epsilon) \frac{5}{3} (Du_\epsilon)^2 \, dx + \int_\Omega M(u_\epsilon) \frac{10}{3} u_\epsilon D^2 u_\epsilon \, dx = 0,$$

if we take

$$\varphi(u_\epsilon) = \int_\Omega \left(\left(D^2 u_\epsilon\right)^2 - \frac{5}{3} u_\epsilon (Du_\epsilon)^2 + \frac{5}{36} u_\epsilon^4 \right) dx,$$

we can write the resulting identity as follows:

$$\begin{aligned}\frac{d}{dt}\varphi(u_\epsilon) + 2\gamma\varphi(u_\epsilon) + 2\epsilon \int_\Omega \left(D^4 u_\epsilon\right)^2 dx + \epsilon \int_\Omega D^4 u_\epsilon \left(\frac{5}{3}(Du_\epsilon)^2 + \frac{10}{3} u_\epsilon D^2 u_\epsilon + u^3 \right) dx \\ = \int_\Omega \left(2D^2 f_\epsilon D^2 u_\epsilon + \frac{5}{3}(Du_\epsilon)^2 f_\epsilon + \frac{10}{3} u_\epsilon D^2 u_\epsilon f_\epsilon + \frac{5}{9} u_\epsilon^3 f_\epsilon \right) dx \\ + \gamma \int_\Omega \left(\frac{5}{3} u_\epsilon (Du_\epsilon)^2 - \frac{5}{18} u_\epsilon^4 \right) dx. \end{aligned} \tag{23}$$

Now we will estimate the terms in (23). Note that applying Agmon's inequality in (9) and (22), we infer that $\|u_\epsilon\|_\infty$ is uniformly bounded with respect to $t \in (0, T)$

and $\epsilon \in (0,1)$. We will use this fact for the following estimates.

First using similar arguments as in (19) we have

$$\begin{aligned}
&\left|\int_\Omega D^4 u_\epsilon \left(\frac{5}{3}(Du_\epsilon)^2 + \frac{10}{3}u_\epsilon D^2 u_\epsilon + u^3\right) dx\right| \\
&\le c\left(\|Du_\epsilon\|_{L^4}^2 + \|u_\epsilon\|_\infty |u_\epsilon|_2 + \|u_\epsilon\|_\infty^2 |u_\epsilon|_0\right)|u_\epsilon|_4 \\
&\le c\left(\|Du_\epsilon\|_{H^{1/4}}^2 + |u_\epsilon|_2 + 1\right)|u_\epsilon|_4 \\
&\le c\,|u_\epsilon|_0^{11/8}\|u_\epsilon\|_4^{5/8}|u_\epsilon|_4 + c\,|u_\epsilon|_0^{1/2}\|u_\epsilon\|_4^{1/2}|u_\epsilon|_4 \,. \qquad (24)
\end{aligned}$$

Using the same idea in (18) we can bound H_L^4 norm with the norms $|\cdot|_0^2$ and $|\cdot|_4^2$. Hence using this and Young's inequality we can deduce from (24) that

$$\left|\int_\Omega D^4 u_\epsilon \left(\frac{5}{3}(Du_\epsilon)^2 + \frac{10}{3}u_\epsilon D^2 u_\epsilon + u^3\right) dx\right| \le |u_\epsilon|_4^2 + c. \qquad (25)$$

Also

$$\left|\gamma \int_\Omega \left(\frac{5}{3}u_\epsilon (Du_\epsilon)^2 - \frac{5}{18}u_\epsilon^4\right) dx\right| \le c\,\|u_\epsilon\|_\infty |u_\epsilon|_1^2 + c\,\|u_\epsilon\|_\infty^2 |u_\epsilon|_0^2 \le c. \qquad (26)$$

Lastly

$$\begin{aligned}
&\left|\int_\Omega \left(2D^2 f_\epsilon D^2 u_\epsilon + \frac{5}{3}(Du_\epsilon)^2 f_\epsilon + \frac{10}{3}u_\epsilon D^2 u_\epsilon f_\epsilon + \frac{5}{9}u_\epsilon^3 f_\epsilon\right) dx\right| \\
&\le |f_\epsilon|_2 |u_\epsilon|_2 + c\,\|f_\epsilon\|_\infty |u_\epsilon|_1^2 + c\,|f_\epsilon|_1 \|u_\epsilon\|_\infty |u_\epsilon|_1 + c\,|f_\epsilon|_0 |u_\epsilon|_0 \|u_\epsilon\|_\infty^2 \\
&\le |\gamma|\,|u_\epsilon|_2^2 + \frac{1}{|\gamma|}|f_\epsilon|_2^2 + c \le |\gamma|\,|u_\epsilon|_2^2 + c. \qquad (27)
\end{aligned}$$

Inserting (25), (26), and (27) into (23) we get

$$\frac{d}{dt}\varphi(u_\epsilon) + 2\gamma\varphi(u_\epsilon) + \epsilon\,|u_\epsilon|_4^2\, dx \le |\gamma|\,|u_\epsilon|_2^2 + c. \qquad (28)$$

Observe that

$$|\varphi(u_\epsilon)| \le \left|D^2 u_\epsilon\right|_2^2 + c\,|u_\epsilon|_1^2 + c\,\|u_\epsilon\|_\infty^2 |u_\epsilon|_0^2 \le |u_\epsilon|_2^2 + c \le \varphi(u_\epsilon) + c.$$

So, using (28), it follows that

$$\frac{d}{dt}\varphi(u_\epsilon) + 3\epsilon\,|u_\epsilon|_4^2 \leq 3|\gamma|\varphi(u_\epsilon) + c,$$

from which we conclude as in *Estimate 1* and *Estimate 2* that

$$|u_\epsilon(t)|_2^2 + \epsilon\int_0^t |u_\epsilon|_4^2(s)\,ds \leq c, \tag{29}$$

for $0 \leq t \leq T$ where c does not depend on ϵ but depends on T.

Passing to the limit

First we will show that $\frac{du_\epsilon}{dt}$ remains bounded in $L^2(0,T;H^{-1}(\Omega))$. By (5)

$$\frac{du_\epsilon}{dt} = -u_\epsilon Du_\epsilon - D^3u_\epsilon + \gamma u_\epsilon - \epsilon D^4 u_\epsilon + f_\epsilon. \tag{30}$$

Then the result follows from the following implications:

1. $u_\epsilon Du_\epsilon$ remains bounded in $L^\infty(0,T;H^{-1}(\Omega))$: First note that

$$\begin{aligned}\|u_\epsilon Du_\epsilon\|_{H^{-1}(\Omega)} &\leq^a \|u_\epsilon Du_\epsilon\|_{L^2(\Omega)}\\ &\leq \|u_\epsilon\|_\infty \|Du_\epsilon\|_\infty\,,\end{aligned}$$

   where we have used in (a) the definition of H^{-1} norm. Also using (9), (22), (29) and Agmon's inequality, we can bound $\|u_\epsilon\|_\infty$ and $\|Du_\epsilon\|_\infty$ with respect to $t \in (0,T)$ and ϵ from which the result follows.

2. D^3u_ϵ remains bounded in $L^\infty(0,T;H^{-1}(\Omega))$: Since

$$\left\|D^3u_\epsilon\right\|_{H^{-1}(\Omega)} = sup_{v\in H^1_L(\Omega)}\frac{|\langle D^3u_\epsilon, v\rangle|}{\|v\|_{H^1(\Omega)}},$$

   and

$$\frac{|\langle D^3u_\epsilon, v\rangle|}{\|v\|_{H^1(\Omega)}} = \frac{|\langle D^2u_\epsilon, Dv\rangle|}{\|v\|_{H^1(\Omega)}} \leq \frac{|\langle D^2u_\epsilon, Dv\rangle|}{\|Dv\|_{L^2(\Omega)}} \leq^a \left\|D^2u_\epsilon\right\|_{L^2(\Omega)},$$

   where we have used in (a) the fact that D^2u_ϵ can be seen as a functional on $L^2(\Omega)$. Hence from (29) the result follows.

3. γu_ϵ is bounded in $L^\infty(0,T;H^{-1}(\Omega))$: It follows from (9).

4. Lastly, by (29), we have

$$\left\|\epsilon D^4 u_\epsilon\right\|_{L^2(\Omega)} = \epsilon \left\|D^4 u_\epsilon\right\|_{L^2(\Omega)} \leq^a \sqrt{\epsilon}\left\|D^4 u_\epsilon\right\|_{L^2(\Omega)} \leq c,$$

where we have used in (a) the fact that $\epsilon \in (0,1)$. Hence $\epsilon D^4 u_\epsilon$ remains bounded in $L^2(0,T;H^{-1}(\Omega))$.

Now we have that u_ϵ is bounded in $L^\infty(0,T;H^2(\Omega))$ and $\frac{du_\epsilon}{dt}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. Then by Theorem 1.10 there exists $u \in L^\infty(0,T;H^2(\Omega))$, $u^* \in L^2(0,T;H^{-1}(\Omega))$ and a subsequence of u_ϵ which we denote also by u_ϵ, such that

$$u_\epsilon \overset{*}{\rightharpoonup} u \quad \text{in} \quad L^\infty(0,T;H^2(\Omega)), \tag{31}$$

and

$$\frac{du_\epsilon}{dt} \rightharpoonup u^* \quad \text{in} \quad L^2(0,T;H^{-1}(\Omega)). \tag{32}$$

We now show that $u^* = \frac{du}{dt}$ –the derivative is to be understood in the weak sense– in $L^2(0,T;H^{-1}(\Omega))$. According to Theorem 1.14 this is equivalent to the fact that for all $\phi \in C_0^\infty(0,T)$ and $v \in H_L^1(\Omega)$ the following equality holds:

$$\int_0^T \langle v\phi_t(t), u(t)\rangle_{H_L^1,H^{-1}}\, dt = -\int_0^T \langle v\phi(t), u^*(t)\rangle_{H_L^1,H^{-1}}\, dt. \tag{33}$$

Since

$$\begin{aligned}
\int_0^T \langle v\phi_t(t), u(t)\rangle_{H_L^1,H^{-1}}\, dt &=^a \lim_{\epsilon\to 0}\int_0^T \langle v\phi_t(t), u_\epsilon(t)\rangle_{H_L^1,H^{-1}}\, dt \\
&= -\lim_{\epsilon\to 0}\int_0^T \langle v\phi(t), \frac{du_\epsilon}{dt}(t)\rangle_{H_L^1,H^{-1}} \\
&=^b -\int_0^T \langle v\phi(t), u^*(t)\rangle_{H_L^1,H^{-1}}\, dt,
\end{aligned}$$

where we have used the definition of (31) with the fact that $v\phi_t \in L^\infty(0,T;H^1(\Omega)) \to L^1(0,T;H^{-2}$ in (a) and (32) in (b), the equality (33) is proved.
Considering the term $u_\epsilon Du_\epsilon$, we will first show that u_ϵ goes to u strongly in $L^2(0,T;H^1_L(\Omega))$. Using the embedding $L^\infty(0,T;H^2_L(\Omega)) \hookrightarrow L^2(0,T;H^2_L(\Omega))$ it follows from (31) that

$$u_\epsilon \rightharpoonup u \ \text{ in } \ L^2(0,T;H^2_L(\Omega)). \tag{34}$$

Now take $X = H^2_L(\Omega)$, $Y = H^1_L(\Omega)$, $Z = H^{-1}(\Omega)$ and $p_0 = p_1 = 2$ in Theorem 1.15. Then from W being continuously embedded in $L^2(0,T;H^2_L(\Omega))$ –where W is defined as in Theorem 1.15– and (34), it follows that $u_\epsilon \rightharpoonup u$ also in W. Hence from Theorem 1.15 and Theorem 1.11, we conclude that u_ϵ goes to u strongly in $L^2(0,T;H^1_L(\Omega))$. Now using this, we will show that $u_\epsilon Du_\epsilon$ converges weakly to uDu in $L^2(0,T;H^{-1}(\Omega))$, i.e. for all $v \in L^2(0,T;H^1_L(\Omega))$ the following holds:

$$\int_0^T \langle u_\epsilon(t)Du_\epsilon(t), v(t)\rangle_{H^{-1},H^1_L}\,dt \to \int_0^T \langle u(t)Du(t), v(t)\rangle_{H^{-1},H^1_L}\,dt \qquad \text{as } \epsilon \to 0. \tag{35}$$

We have

$$\begin{aligned}
&\int_0^T \left|\langle u_\epsilon(t)Du_\epsilon(t) - u(t)Du(t), v(t)\rangle_{H^{-1},H^1_L}\right|\,dt \\
&\le \int_0^T \|v\|_{H^1(\Omega)} \|u_\epsilon Du_\epsilon - uDu\|_{H^{-1}(\Omega)}\,dt \\
&\le^a \int_0^T \|v\|_{H^1(\Omega)} |u_\epsilon Du_\epsilon - uDu_\epsilon|_0\,dt + \int_0^T \|v\|_{H^1(\Omega)} |uDu_\epsilon - uDu|_0\,dt \\
&\le \int_0^T \|Du_\epsilon(t)\|_\infty |u_\epsilon - u|_0\,dt + \int_0^T \|u(t)\|_\infty |Du_\epsilon - Du|_0\,dt,
\end{aligned} \tag{36}$$

where we have used in (a) the fact that $\|\cdot\|_{H^{-1}(\Omega)} \le |\cdot|_0$ and the triangle inequality. Note that $\|Du_\epsilon(t)\|_\infty$ is bounded. Also, since $u \in L^\infty(0,T;H^2_L(\Omega))$, by Agmon inequality $\|u(t)\|_\infty$ is bounded. Hence it follows from (36) and u_ϵ being convergent to u strongly in $L^2(0,T;H^1_L(\Omega))$ that (35) holds. Also D^3u_ϵ being weakly convergent to $D_3 u$ follows easily from the definitions. Now it remains to prove that $\epsilon D^4 u_\epsilon$ converges weakly to 0 weakly in $L^2(0,T;H^{-1}(\Omega))$. So let $v \in L^2(0,T;H^1_L(\Omega))$. Then

$$\begin{aligned}\int_0^T \langle \epsilon D^4 u_\epsilon(t), v(t)\rangle_{H^{-1},H^1_L}\, dt &= \int_0^T \langle \epsilon D^3 u_\epsilon(t), Dv(t)\rangle_{L^2}\, dt \\ &\le \sqrt{\epsilon}\int_0^T \sqrt{\epsilon}\, |u_\epsilon(t)|_3\, |v(t)|_1\, dt \\ &\le^a \sqrt{\epsilon} c \qquad (37)\end{aligned}$$

where we have used the inequality (22) in (a). Note that here c depends on v and T but not on ϵ. Then $\sqrt{\epsilon}c$ tends to 0 as $\epsilon \to 0$ from which the result follows. Hence, we can pass to the limit in $L^2(0,T;H^{-1}(\Omega)$ as ϵ goes to 0, to get

$$\frac{du}{dt} + uDu + D^3 u + \gamma u = f, \qquad (38)$$

where the equality is understood to be hold in $H^{-1}(\Omega)$ for $t \in (0,T)$. Note that since the term $D^4 u_\epsilon$ disappeared we have

$$\frac{du}{dt} \in L^\infty(0,T;H^{-1}(\Omega))$$

Now taking $V = H^1_L(\Omega)$ and $H = L^2(\Omega)$ in Theorem 1.16, we get

$$u \in C([0,T];L^2(\Omega))$$

Lastly, it follows from Theorem 1.17 that $u_\epsilon(0)$ converges weakly to $u(0)$ in $H^{-1}(\Omega)$. Since u_ϵ converges to u_0 strongly, we have $u(0) = u_0$.

Uniqueness

Let $u, v \in L^\infty(0,T;H^2_L(\Omega)$ satisfy (38) with $u(0) = v(0) = u_0$. If $w = u - v$, then w satisfies

$$\frac{dw}{dt} + D^3 w + \gamma w = -uDu + vDv, \qquad (39)$$

$$w(0) = 0. \qquad (40)$$

Note that by Theorem 1.16 we have

$$\frac{1}{2}\frac{d}{dt}|w|_0^2 = \langle \frac{dw}{dt}, w\rangle_{H^{-1},H_L^1},$$

which implies the following identity.

$$\frac{1}{2}\frac{d}{dt}|w|_0^2 + \gamma|w|_0^2 = -\int_\Omega (uDu - vDv)w\,dx.$$

Note that

$$\int_\Omega (uDu - vDv)w\,dx = \int_\Omega (wDw - D(vw)w\,dx = \int_\Omega \frac{1}{2}w^2 Dv\,dx.$$

So

$$\begin{aligned}\frac{d}{dt}|w|_0^2 &= -2\gamma|w|_0^2 - \frac{1}{2}\int_\Omega w^2 Dv\,dx \\ &\le \frac{1}{2}\|Du\|_\infty|w|_0^2 + 2|\gamma||w|_0^2 \\ &\le c|w|_0^2.\end{aligned}$$

Hence, multiplying this by e^{-ct} and integrating over $(0,t)$, we get

$$|w(t)|_0^2 \le |w(0)|e^{ct}.$$

Using the fact that $w(0) = 0$, uniqueness follows. □

2.2 Existence of an absorbing set

In this section we will assume that $\gamma > 0$.

Using the existence and uniqueness result in the previous section we can define for each $t \in \mathbf{R}^+$ the nonlinear mapping $S(t) : H^2(\Omega) \to H^2(\Omega)$ which is defined as

$$u_0 \Rightarrow u(t) \equiv S(t)u_0.$$

We will prove the existence of a *bounded absorbing set* for $\{S(t)\}$,i.e. there exists a closed ball C such that for every bounded set B in H_L^2, there exists $T(B)$ such that

$$S(t)B \subset C, \qquad \forall t \geq T(B).$$

In other words we will prove the following theorem.

Theorem 2.2. *Let $\gamma > 0$ and $f \in H_L^2$ be given. There exists a constant $\rho = \rho(L, \gamma, |f|_0)$ such that for every $R > 0$, there exists $T(R)$ such that*

$$|S(t)u_0|_0 \leqslant \rho_2, \forall u_0 \in H_L^2, \qquad |u_0|_0 \leqslant R, \forall t \geqslant T(R).$$

For the proof we will use the same multipliers as in Section 1. We will recall them for easy referencing. Multiplying (2) by $2u$ and integrating over Ω we get

$$\frac{d}{dt}\int_\Omega u^2\,dx + \int_\Omega \left(2\gamma u^2 - 2fu\right)\,dx = 0. \tag{41}$$

Now multiplying (2) by $-2u_{xx} - u^2$ and integrating over Ω again, the terms coming from uu_x and u_{xxx} cancel and we get

$$\frac{d}{dt}\int_\Omega \left(u_x^2 - \frac{u^3}{3}\right)\,dx + \int_\Omega \left(2\gamma\left(u_x^2 - \frac{u^3}{2}\right) + fu^2 - 2f_x u_x\right)\,dx = 0. \tag{42}$$

Lastly we multiply (2) by $M(u) = \frac{18}{5}u_{xxxx} + 6uu_{xx} + 3u_x^2 + u^3$. Note that this is the same multiplier in the previous section but it is multiplied by $9/5$ here. Doing the same calculations as we did before we get

$$\begin{aligned}&\frac{d}{dt}\int\left\{\frac{9}{5}u_{xx}^2 - 3uu_x^2 + \frac{u^4}{4}\right\}dx\\&+\int\left\{\gamma\left(\frac{18}{5}u_{xx}^2 - 9uu_x^2 + u^4\right) + \frac{18}{5}f_{xx}u_{xx} - 3u_x^2 f - uf_xu_x + fu^3\right\}dx = 0.\end{aligned} \tag{43}$$

Proof. (*Proof of Theorem 2.2*) From (41) we get

$$\frac{d}{dt}|u|_0^2 + 2\gamma|u|_0^2 \leqslant 2|f|_0^2|u|_0^2 \leqslant \gamma|u|_0^2 + \frac{1}{\gamma}|f|_0^2,$$

which implies

$$\frac{d}{dt}\left|u\right|_0^2+\gamma\left|u\right|_0^2\leqslant\frac{1}{\gamma}\left|f\right|_0^2.$$

Multiplying the above inequality by $e^{\gamma t}$ and integrating over $(0,t)$ we get

$$\left|u\right|_0^2 e^{\gamma t}\leqslant\left|u_0\right|_0^2+\frac{e^{\gamma t}-1}{\gamma^2}\left|f\right|_0^2,$$

and so

$$\left|S(t)u_0\right|_0^2\leqslant\left|u_0\right|_0^2 e^{-\gamma t}+\left|f\right|_0^2\left(1-e^{-\gamma t}\right)/\gamma^2. \tag{44}$$

Hence

$$\left|S(t)u_0\right|_0^2\leqslant 2\left|f\right|_0^2/\gamma,\qquad t\geqslant T_0(u_0),$$

where

$$T_0(u_0)=\frac{1}{\gamma}Log\frac{\gamma\left|u_0\right|_0^2}{\left|f\right|_0^2}.$$

For the H^1 estimate taking

$$\varphi(u)=\int\left(u_x^2-\frac{u^3}{3}\right)dx,$$

and

$$\xi(u)=\gamma|u|_1^2-\frac{2}{3}\gamma\int u^3\,dx+\int\left(fu^2-2f_xu_x\right)dx,$$

we can write (42) in the form

$$\frac{d\varphi(u(t))}{dt}+\gamma\varphi(u(t))=-\xi(u(t)).$$

Now we will estimate $\left|\int u^3\,dx\right|$ as follows:

$$\begin{aligned}\left|\int u^3\,dx\right| &\leqslant |u|_\infty|u|_0^2 \qquad (45)\\ &\leqslant^a |u|_0^{5/2}\left(2|u|_1+L^{-1}|u|_0\right)^{1/2}\\ &\leqslant^b \sqrt{2}|u|_0^{5/2}|u|_1^{1/2}+L^{-1/2}|u|_0^3\\ &\leqslant^c \frac{1}{2}|u|_1^2+\frac{2^{1/3}}{4}|u|_0^{10/3}+L^{-1/2}|u|_0^3.\end{aligned}$$

where we have used Agmon's inequality in (a), the fact that $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$ in (b), and Young's inequality with $p = 4$ and $q = 4/3$ in (c). Using this we estimate $-\xi(u)$ as follows:

$$\begin{aligned} -\xi(u) &\leqslant -\gamma|u|_1^2 + \frac{2}{3}\gamma\left|\int u^3\,dx\right| - \int \left(fu^2 - 2f_x u_x\right)\,dx \\ &\leqslant -\frac{2\gamma}{3}|u|_1^2 + \frac{2^{4/3}\gamma}{12}|u|_0^{10/3} + \frac{2\gamma L^{-1/2}}{3}|u|_0^3 + |f|_\infty|u|_0^2 + 2|f|_1|u|_1 \\ &\leqslant^a \frac{3}{2\gamma}|f|_1^2 + |f|_\infty|u|_0^2 + \frac{2^{4/3}\gamma}{12}|u|_0^{10/3} + \frac{2\gamma L^{-1/2}}{3}|u|_0^3, \end{aligned} \tag{46}$$

where we have used $2|f|_1|u|_1 \leqslant \frac{2\gamma}{3}|u|_1^2 + \frac{3}{2\gamma}|f|_1^2$ in (a). Now replacing $|u|_0^2$ with $|u|_0^2 e^{-\gamma t} + \frac{|f|^2}{\gamma^2}$ in the last inequlaity according to (44) and using the inequality $(a+b)^\alpha \leqslant 2^\alpha\left(a^\alpha + b^\alpha\right)$ we get

$$\frac{d\varphi(u(t))}{dt} + \gamma\varphi(u(t)) \leqslant K_1(u_0)e^{-\gamma t} + K_2, \tag{47}$$

where

$$K_1(u_0) = |f|_\infty|u_0|_0^2 + \frac{2^3\gamma}{12}|u_0|_0^{10/3} + \frac{2^{5/2}\gamma L^{-1/2}}{3}|u_0|_0^3,$$

and

$$K_2 = \frac{3}{2\gamma}|f|_1^2 + |f|_\infty|f|_0^2\gamma^{-2} + \frac{2^3}{12}|f|^{10/3}\gamma^{-7/3} + \frac{2^{5/2}L^{-1/2}}{3}|f|_0^3\gamma^{-2}.$$

Now multiplying (47) by $e^{\gamma t}$and integrating the resulting identity over $(0,t)$ we get

$$\varphi(u(t)) \leqslant \left(\varphi(u_0) + K_1(u_0)t\right)e^{-\gamma t} + K_2\left(1 - e^{-\gamma t}\right)\gamma^{-1}.$$

Hence from (7) it follows that

$$\varphi(u(t)) \leqslant \left\{\frac{3}{2}\left|u_0\right|_1^2 + \frac{2^{1/3}}{4}\left|u_0\right|_0^{10/3} + L^{-1/2}\left|u_0\right|_0^3 + K_1(u_0)t\right\}e^{-\gamma t} + K_2/\gamma. \tag{48}$$

Let $C(\left|u_0\right|_0, \left|u_0\right|_1) = \left\{\frac{3}{2}\left|u_0\right|_1^2 + \frac{2^{1/3}}{4}\left|u_0\right|_0^{10/3} + L^{-1/2}\left|u_0\right|_0^3 + K_1(u_0)t\right\}$. Then we have from the last inequality

$$\int u_x^2\,dx \leqslant C(\left|u_0\right|_0, \left|u_0\right|_1)e^{-\gamma t} + \int \frac{|u|^3}{3} + K_2/\gamma. \tag{49}$$

Hence using (46) and applying (44) again we can write (49) in the form

$$\frac{1}{2}\int u_x^2\,dx \leqslant C_1(|u_0|_0\,, |u_0|_1)e^{-\gamma t} + K_2/\gamma, \tag{50}$$

from which it follows similarly as in (44) the existence of an absorbing ball for the H^1-norm.

For the H^2-norm we write (43) in the following form:

$$\frac{d}{dt}\psi(u(t)) + \gamma\psi(u(t)) = -\eta(u(t)),$$

where $\psi(u)$ and $\eta(u)$ are defined as

$$\begin{aligned}
\psi(u) &= \frac{9}{5}|u|_2^2 + \int \left\{ \left(u^4/4\right) - 3uu_x^2 \right\}\,dx,\\
\eta(u) &= \frac{9}{5}\gamma\int |u|_2^2\,dx - 6\gamma\int uu_x^2\,dx + \frac{3\gamma}{4}\int u^4\,dx\\
&\quad + \frac{18}{5}\int f_{xx}u_{xx}\,dx + 2\int fu_x^2\,dx - \int uf_xu_x\,dx + \int fu^3\,dx.
\end{aligned}$$

Then

$$\begin{aligned}
\eta(u) \geqslant{}& \frac{9}{5}\gamma\int |u|_2^2\,dx - 6\gamma\int |u|u_x^2\,dx + \frac{3\gamma}{4}\int u^4\,dx\\
&- \frac{18}{5}\int |f_{xx}||u_{xx}|\,dx - 2\int |f|u_x^2\,dx - \int |u||f_x||u_x|\,dx - \int |f||u|^3\,dx.
\end{aligned} \tag{51}$$

We should estimate $-\eta(u(t))$ with terms depending only on f, $|u|_0$ and $|u|_1$ but not onto $|u|_2$. Then we can apply similar arguments as we used for the H^1-norm taking into account (44) and (50) to get

$$\frac{d\psi(u(t))}{dt} + \gamma\psi(u(t)) \leqslant R_1(u_0)e^{-\gamma t} + R_2, \tag{52}$$

where R_1 and R_2 are determined as K_1 and K_2 and R_2 contains only terms containing f and its derivatives. So first note that we can get rid of $|u|_2$ at the right hand side of (51) by using Young's inequality as follows:

$$\frac{18}{5}\int |f_{xx}||u_{xx}| \leqslant \frac{9\gamma}{5}|u_{xx}| + \frac{9}{5\gamma}|f_{xx}|^2.$$

Also

$$\int |u|u_x^2\,dx \leqslant |u|_\infty |u|_1^2 \tag{53}$$
$$\leqslant |u|_0^{1/2}\left(2|u|_1 + L^{-1}|u|_0\right)^{1/2}$$
$$\leqslant \sqrt{2}|u|_0^{1/2}|u|_1^{3/2} + L^{-1/2}|u|_0|u|_1^2$$
$$\leqslant |u|_0 + \frac{|u|_1^3}{2} + \frac{L^{-1}|u|_0^2}{2} + \frac{|u|_1^4}{2}.$$

The remaining terms can be bounded with $|u|_0$ and $|u|_1$ similarly. For example, we have

$$\int u^4\,dx \leqslant |u|_\infty^2 |u|_0^2,$$

and applying Agmon's inequality to $|u|_\infty^2$ the estimate follows. Now multiplying (52) by $e^{\gamma t}$ and integrating over $(0,t)$ we get as in (48)

$$\psi(u(t)) \leqslant C_2(|u_0|_0, |u_0|_1, |u_0|_2|)e^{-\gamma t} + R_2/\gamma.$$

From the last inequality it follows that

$$\frac{9}{5}|u|_2^2 \leqslant C_2(|u_0|_0, |u_0|_1, |u_0|_2|)e^{-\gamma t} + R_2/\gamma + 3\int |u|u_x^2\,dx.$$

Hence using (17) and applying (44) and (50) we conclude that

$$\frac{9}{5}|u|_2^2 \leqslant C_2(|u_0|_0, |u_0|_1, |u_0|_2|)e^{-\gamma t} + R_2/\gamma,$$

from which it follows the existence of absorbing ball for H^2-norm. □

As a final remark, since the constants K_2, R_2 and the constant in (44) only contains terms involving f and its derivatives, when we take $f = 0$ we get the exponential decay for the H^2-norm of the solution.

3 A non-periodic initial-boundary value problem for the KdV equation

In this chapter we will prove existence, uniqueness and stability for the solutions of the following problem:

$$u_t + uu_x + u_{xxx} = 0, \qquad x \in (0,1), t > 0, \tag{54}$$

$$u(x,0) = u_0(x), \qquad x \in (0,1), \tag{55}$$

$$u(0,t) = u(1,t) = u_x(1,t) = 0, \qquad t > 0. \tag{56}$$

For this, in the first section, using Galerkin method, we first prove the existence of the solutions for the following regularized problem:

$$u_t + uu_x + u_{xxx} + \nu(u_{xx} + u_{xxxx}) = 0, \qquad x \in (0,1), t > 0, \tag{57}$$

$$u(x,0) = u_0(x), \qquad x \in (0,1), \tag{58}$$

$$u(0,t) = u(1,t) = \nu u_{xx}(0,t) = u(1,t) + \nu u_{xx}(1,t) = 0, \qquad t > 0. \tag{59}$$

Then, in the second section, we prove the existence result for (54)-(56) passing to limit as ν tends to zero as we have done in the previous chapter. In this chapter $\|\cdot\|$ will denote the usual L^2 norm, $(\cdot,\cdot)$ will denote the inner product in L^2 and D_j will denote $\frac{\partial^j}{\partial x^j}$.

3.1 Solvability of the regularized problem

Theorem 3.1. *Let* $\nu > 0$ *and* $u_0 \in H^4(0,1) \cap H_0^1(0,1)$*;* $\nu u_{0xx}(0) = u_{0x}(1) + \nu u_{0xx}(1) = 0$*. Then there exists a unique solution to* (57)-(59) *from the class*

$$u \in C\left(0,T;\ H^2(0,1) \cap H_0^1(0,1)\right) \cap L^\infty\left(0,T;\ H^4(0,1) \cap H_0^1(0,1)\right),$$

and

$$u_t \in L^\infty\left(0,T;\ L^2(0,1)\right) \cap L^2\left(0,T;\ H^2(0,1) \cap H_0^1(0,1)\right).$$

First we need the following lemma.

Lemma 3.2. *Let V be the closure of the space of functions satisfying* (59) *in* $H^4(0,1)$*. Then for every $\nu > 0$ there exists eigenfunctions for the following problem*

$$\nu D_4 w = \mu w,$$

$$w(0) = w(1) = \nu w_{xx}(0) = w_x(1) + \nu w_{xx}(1) = 0,$$

that forms an orthogonal basis in V which is orthonormal in $L^2(0,1)$.

Proof. Let u and v be functions in $H^4(0,1)$ and satisfy the boundary conditions in the lemma. Then using integration by parts and boundary conditions we can show that

$$\nu(D_4 u, v) = \nu(u, D_4 v) \qquad \text{and} \qquad \nu(D_4 u, u) = \nu\|D_2 u\|^2 + u_x^2(1).$$

Observe that, if $\nu(D_4 u, u) = 0$, then $D_2 u = 0$. In this case, u is of the form $ax + b$ but since $u \in V$ we have $u = 0$. Then A is strictly positive and so invertible.
Note that since A is self-adjoint, its range is closed. Now taking the composition of A^{-1} with the injection of $H^4(0,1)$ into $L^2(0,1)$ which is compact, we can see A^{-1} as a compact operator from the range of A to $L^2(0,1)$. Then, by spectral theory, we know that there exist eigenfunctions of A^{-1} which form an orthonormal basis in the range of A and orthogonal basis in V. □

Let w_j denote the eigenfunctions in the above theorem. We construct approximate solutions to (57) -(59) in the form

$$u^N(x,t) = \sum_{i=1}^{n} g_j^N(t) w_j(x),$$

where $\big(g_1^N(t), g_2^N(t), \ldots, g_N^N(t)\big)$ is a solution of the following system of ordinary differential equations:

$$(u_t^N, w_j)(t) + (u^N Du^N, w_j)(t) + (D_3 u^N, w_j)(t) \\ + \nu(D_2 u^N, w_j)(t) + \nu(D_4 u^N, w_j)(t) = 0, \tag{60}$$

$$g_j^N(0) = (u_0, w_j) \qquad j = 1, \ldots, N. \tag{61}$$

We know from the theory of first order ordinary differential equations that for each N, there exists a solution $\big(g_1^N(t), g_2^N(t), \ldots, g_N^N(t)\big)$ on some interval $(0, T_N)$. In order to extend u^N to an arbitrary interval $(0,T)$ and pass to the limit as $N \to \infty$, we will estimate $u^N(t)$ with respect to N and $t \in (0,T)$.

Estimate 1

Since our aim is to pass to the limit as ν goes to 0 we will assume that $\nu \in (0,1)$. Multiplying (60) by $2g_j^N$ and summing over j we get

$$(u_t^N, 2u^N)(t) + (u^N Du^N, 2u^N)(t) + (D_3 u^N, 2u^N)(t) + \nu(D_2 u^N, 2u^N)(t) \\ + \nu(D_4 u^N, 2u^N)(t) = 0. \tag{62}$$

Considering the boundary conditions for w_j and applying integration by parts we have

$$(u^N Du^N, 2u^N)(t) = 0$$
$$(D_3 u^N, 2u^N)(t) = (Du^N(0,t))^2 - (Du^N(1,t))^2$$
$$(D_4 u^N, 2u^N)(t) = 2\left|\left|D_2 u^N(t)\right|\right|^2 - 2D_2 u^N(1,t) Du^N(1,t).$$

Inserting the above identities into (62) and adding $(-Du^N)^2(1,t)$ to both sides we get

$$\frac{d}{dt}\left|\left|u^N(t)\right|\right|^2 + (Du^N)^2(0,t) - 2Du^N(1,t)\left(Du^N(1,t) + \nu D_2 u^N(1,t)\right) + 2\nu\left|\left|D_2 u^N(t)\right|\right|^2 \\ = -2\nu(D_2 u^N, u^N)(t) - (Du^N(1,t))^2. \tag{63}$$

Observe that $Du^N(1,t)+\nu Du^N(1,t)$ is zero. Also using Schwarz and Young inequalities we get

$$-2\nu(D_2u^N,u^N)(t) \leqslant \nu\left|\left|D_2u^N(t)\right|\right|^2 + \nu\left|\left|u^N(t)\right|\right|^2.$$

Hence we have

$$\frac{d}{dt}\left|\left|u^N(t)\right|\right|^2 + (Du^N(0,t))^2 + \nu\left|\left|D_2u^N(t)\right|\right|^2 \leqslant \nu\left|\left|u^N(t)\right|\right|^2 \leqslant \left|\left|u^N(t)\right|\right|^2, \quad (64)$$

from which it follows that

$$\frac{d}{dt}\left|\left|u^N(t)\right|\right|^2 \leqslant \left|\left|u^N(t)\right|\right|^2.$$

Multiplying this inequality by e^{-t} and integrating over $(0,t)$ we obtain

$$\left|\left|u^N(t)\right|\right|^2 \leqslant e^t\left|\left|u^N(0)\right|\right|^2,$$

for t $\in (0,T)$. Inserting this into (64)and integrating over $(0,t)$ again, it follows that

$$\left|\left|u^N(t)\right|\right|^2 + \int_0^T (Du^N)^2(0,s)\,ds + \nu\int_0^T \left|\left|D_2u^N(s)\right|\right|^2\,ds$$
$$\leqslant (e^t+1)\left|\left|u^N(0)\right|\right|^2 + \leqslant^a C\left|\left|u_0\right|\right|^2, \quad (65)$$

where we used in (a) the fact that w_j are orthogonal in $L^2(0,1)$. Also $C>0$ does not depend on ν, N, and $t\in(0,T)$. C will be a generic constant which will be used for later estimates also. If it depends on ν, we will write $C(\nu)$.

Estimate 2

Substituting w_j by $\nu D_4\mu_j^{-1}w_j$ according to Lemma 3.2, multiplying with $g_j^N(t)$ and summing over j in (60) we get

$$(u_t^N,D_4u^N)(t) + (u^NDu^N,D_4u^N)(t) + (D_3u^N,D_4u^N)(t)$$
$$+\nu(D_2u^N,D_4u^N)(t) + \nu(D_4u^N,D_4u^N)(t) = 0. \quad (66)$$

We will estimate terms in (66) separately:

$$\begin{aligned}
I_1 &= (u_t^N, D_4u^N)(t) \\
&= -(Du_t^N, D_3u^N)(t) \\
&= (D_2u_t^N, D_2u^N)(t) - Du_t^N(1,t)D_2u^N(1,t) \\
&= \frac{1}{2}\frac{d}{dt}\left|\left|D_2u^N(t)\right|\right|^2 + \nu D_2u_t^N(1,t)D_2u^N(1,t) \\
&= \frac{1}{2}\frac{d}{dt}\left\{\left|\left|D_2u^N(t)\right|\right|^2 + \nu\left|D_2u^N(1,t)\right|^2\right\}.
\end{aligned}$$

For the second term in (66) we have

$$\begin{aligned}
I_2 &= (u^N Du^N, D_4u^N)(t), \\
&\geqslant -\left|\left|D_4u^N(t)\right|\right|\left|\left|u^N(t)Du^N(t)\right|\right|, \\
&\geqslant -\max_{x\in(0,1)}\left|u^N(t)\right|\left|\left|Du^N(t)\right|\right|\left|\left|D_4u^N(t)\right|\right|, \\
&\geqslant^a -\left|\left|Du^N(t)\right|\right|^2\left|\left|D_4u^N(t)\right|\right|,
\end{aligned}$$

where we have used in (a) the fact that $\max_{x\in(0,1)}\left|u^N(t)\right| \leqslant \left|\left|Du^N(t)\right|\right|$ which follows from the fact that $u^N(0,t) = 0$. Indeed, if $u(0) = 0$

$$|u(x)| = \left|\int_0^x Du(y)\,dy\right| \leq \left|\int_0^1 Du(y)\,dy\right| \leq \|Du\|.$$

Now observe that

$$\left|\left|Du^N(t)\right|\right|^2 = -\int_0^1 u^N(x,t)D_2u^N(x,t)\,dx \leqslant \left|\left|u^N(t)\right|\right|\left|\left|D_2u^N(t)\right|\right|, \qquad (67)$$

from which it follows that

$$\begin{aligned}
I_2 &\geqslant -\left|\left|u^N(t)\right|\right|\left|\left|D_2u^N(t)\right|\right|\left|\left|D_4u^N(t)\right|\right|, \\
&\geqslant^a -\frac{\nu}{8}\left|\left|D_4u^N(t)\right|\right|^2 - \frac{2}{\nu}\left|\left|D_2u^N(t)\right|\right|^2\left|\left|u^N(t)\right|\right|^2,
\end{aligned}$$

where we have used Young's inequality with $\epsilon = \nu/4$ in (a). Taking into account (65) we obtain

$$I_2 \geqslant -\frac{\nu}{8}\left|\left|D_4 u^N(t)\right|\right|^2 - C(\nu)\left|\left|D_2 u^N(t)\right|\right|^2.$$

For the third term

$$\begin{aligned}
I_3 &= (D_3 u^N, D_4 u^N)(t),\\
&\geqslant^a -\frac{\nu}{16}\left|\left|D_4 u^N(t)\right|\right|^2 - \frac{4}{\nu}\left|\left|D_3 u^N(t)\right|\right|^2,\\
&\geqslant^b -\frac{\nu}{16}\left|\left|D_4 u^N(t)\right|\right|^2 - \frac{4}{\nu}\left\{\epsilon\left|\left|D_4 u^N(t)\right|\right|^2 + C(\epsilon)\left|\left|u^N(t)\right|\right|^2\right\},
\end{aligned}$$

where we have used Young's inequality with $\epsilon = \nu/8$ in (a) and Ehrling's inequality in (b). Finally, taking $\epsilon = \frac{\nu^2}{64}$ and taking into account (65) , we find

$$I_3 \geqslant -\frac{\nu}{8}\left|\left|D_4 u^N(t)\right|\right|^2 - C(\nu).$$

Lastly

$$I_4 = \nu(D_2 u^N, D_4 u^N)(t) \geqslant -\frac{\nu}{8}\left|\left|D_4 u^N(t)\right|\right|^2 - 2\nu\left|\left|D_2 u^N(t)\right|\right|^2.$$

Substituting $I_1 - I_4$ into (65), we obtain

$$\begin{aligned}
\frac{d}{dt}\left\{\left|\left|D_2 u^N(t)\right|\right|^2 + \nu\left|D_2 u^N(1,t)\right|^2\right\} + \frac{5}{8}\nu\left|\left|D_4 u^N(t)\right|\right|^2&\\
\leqslant C(\nu)\left(\left|\left|D_2 u^N(t)\right|\right|^2 + 1\right),& \qquad (68)
\end{aligned}$$

from which it follows that

$$\frac{d}{dt}\left\{\left|\left|D_2 u^N(t)\right|\right|^2 + \nu\left|D_2 u^N(1,t)\right|^2\right\} \leqslant C(\nu)\left\{\left|\left|D_2 u^N(t)\right|\right|^2 + \nu\left|D_2 u^N(1,t)\right|^2\right\} + C(\nu).$$

Multiplying the above inequality by $e^{-C(\nu)t}$ and integrating over $(0,t)$ we have

$$\left\{\left|\left|D_2 u^N(t)\right|\right|^2 + \nu\left|D_2 u^N(1,t)\right|^2\right\} \leqslant C(\nu)\left\{\left|\left|D_2 u^N(0)\right|\right|^2 + \nu\left|D_2 u^N(1,0)\right|^2 + 1\right\}.$$

Inserting this into (68) and integrating over $(0,t)$ again, it follows that

$$\left|\left|D_2u^N(t)\right|\right|^2 + \nu\left|D_2u^N(1,t)\right|^2 \leq C(\nu)\left(\left|\left|u^N(0)\right|\right|^2_{H^4(0,1)} + 1\right), \tag{69}$$

where $C(\nu)$ does not depend on N and $t \in (0,T)$.

Estimate 3

Differentiating (60) with respect to t, multiplying by $2g^N_{jt}$ and summing over j we get

$$\begin{aligned}(u^N_{tt}, 2u^N_t)(t) + (u^N_t uD^N, 2u^N_t)(t) + (u^N Du^N_t, 2u^N_t)(t) + (D_3u^N, 2u^N_t)(t)\\ + \nu(D_2u^N, 2u^N_t)(t) + \nu(D_4u^N, 2u^N_t)(t) = 0.\end{aligned} \tag{70}$$

Observe that boundary conditions are invariant under taking derivative with respect to t. So we have the following identities:

$$\begin{aligned}(u^N_t Du^N, 2u^N_t)(t) &= -2(u^N Du^N_t, u^N_t)(t)\\ (D_3u^N_t, 2u^N_t)(t) &= (Du^N_t(0,t))^2 - (Du^N_t(1,t))^2\\ (D_4u^N, 2u^N)(t) &= 2\left|\left|D_2u^N_t(t)\right|\right|^2 - 2D_2u^N_t(1,t)Du^N_t(1,t).\end{aligned}$$

Inserting these into (70), adding $(-u^N_{xt}(1,t))^2$ to both sides and using the boundary condition $\left(Du^N_t(1,t) + \nu D_2u^N_t(1,t)\right) = 0$ we get

$$\begin{aligned}&\frac{d}{dt}\left|\left|u^N_t(t)\right|\right|^2 + (Du^N_t(0,t))^2 + 2\nu\left|\left|D_2u^N_t(t)\right|\right|^2 + 2\nu(D_2u^N_t, u^N_t)(t)\\ &= 2(u^N Du^N_t, u^N_t)(t) - (Du^N_t(1,t))^2.\end{aligned} \tag{71}$$

Continuing with the same reasoning as in I_3 of *Estimate 2:*

we have

$$\begin{aligned}(u^N Du^N_t, u^N_t)(t) &\leqslant \left|\left|u^N(t)\right|\right| \left|\left|Du^N_t(t)u^N_t(t)\right|\right|\\ &\leqslant \max_{x\in(0,1)}\left|u^N_t(t)\right| \left|\left|Du^N_t(t)\right|\right| \left|\left|u^N(t)\right|\right|\\ &\leqslant \left|\left|Du^N_t(t)\right|\right|^2 \left|\left|u^N(t)\right|\right|\\ &\leq^a C\left|\left|u_0\right|\right| \left|\left(u^N_t, D_2u^N_t\right)\right|,\end{aligned}$$

where we have used (65) and (67) in (a), and $C > 0$. Inserting the above inequality into (71) we get

$$\frac{d}{dt}\left|\left|u_t^N(t)\right|\right|^2 + 2\nu\left|\left|D_2 u_t^N(t)\right|\right|^2 \leqslant \left(C\left|\left|u_0\right|\right| + 2\nu\right)\left|(u_t^N, D_2 u_t^N)\right|.$$

Applying the Young's inequality to the right hand side with an appropriate ϵ we find

$$\frac{d}{dt}\left|\left|u_t^N(t)\right|\right|^2 + \nu\left|\left|D_2 u_t^N(t)\right|\right|^2 \leqslant C_3(\nu)\left|\left|u_t^N(t)\right|\right|^2. \tag{72}$$

Then from

$$\frac{d}{dt}\left|\left|u_t^N(t)\right|\right|^2 \leqslant C_3(\nu)\left|\left|u_t^N(t)\right|\right|^2,$$

it follows that, for $t \in (0, T)$

$$\left|\left|u_t^N(t)\right|\right|^2 \leqslant e^{C_3(\nu)t}\left|\left|u_t^N(0)\right|\right|^2. \tag{73}$$

In order to conclude *Estimate 3*, we need to estimate $\left|\left|u_t^N(0)\right|\right|^2$. So multiplying (60) by g_{jt}^N and summing over j we get

$$(u_t^N, u_t^N)(t) + (u^N Du^N, u_t^N)(t) + (D_3 u^N, u_t^N)(t) + \nu(D_2 u^N, u_t^N)(t) + \nu(D_4 u^N, u_t^N)(t) = 0.$$

Putting $t = 0$ gives

$$\left|\left|u_t^N(0)\right|\right|^2 \leqslant \left|\left|u_t^N(0)\right|\right|\left(\left|\left|u^N(0)Du^N(0)\right|\right| + \left|\left|D_2 u^N(0)\right|\right| + \left|\left|D_3 u^N(0)\right|\right| + \left|\left|D_4 u^N(0)\right|\right|\right).$$

Since

$$\left|\left|u^N(0)Du^N(0)\right|\right| \leqslant \max_{x\in(0,1)}\left|u^N(0)\right|\left|\left|Du^N(0)\right|\right| \leqslant \left|\left|Du^N(0)\right|\right|^2 \leqslant \left|\left|u^N(0)\right|\right|\left|\left|D_2 u^N(0)\right|\right|,$$

applying Young's inequality to the last term we obtain

$$\left|\left|u_t^N(0)\right|\right| \leqslant C\left|\left|u^N(0)\right|\right|_{H^4(0,1)} \leqslant C\left|\left|u_0\right|\right|_{H^4(0,1)}.$$

Integrating (72) over $(0,t)$ and taking into account (73) with the above inequality we get

$$\left|\left|u_t^N(t)\right|\right|^2 + \nu \int_0^t \left|\left|D_2 u_s^N(s)\right|\right|^2 ds \leqslant C(\nu) \left|\left|u_0\right|\right|_{H^4(0,1)}, \tag{74}$$

where $C(\nu)$ does not depend on N and $t \in (0,T)$.

Passing to the limit

By (65) and (69) we have $\|u^N\|$ and $\left\|D_2 u^N\right\|$ is bounded with respect to N and $t \in (0,T)$. Then by (67) $\left\|Du^N\right\|$ is also bounded. We now show that $\left\|D_3 u^N\right\|$ and $\left\|D_4 u^N\right\|$ are also bounded. From (66) we get

$$\begin{aligned}\nu \left\|D_4 u_N\right\|^2 \leq \left\|u_t^N\right\| \left\|D_4 u^N\right\| + \left\|u^N D u^N\right\| \left\|D_4 u^N\right\| \\ + \left\|D_3 u^N\right\| \left\|D_4 u^N\right\| + \nu \left\|D_2 u^N\right\| \left\|D_4 u^N\right\|.\end{aligned}$$

We will estimate the terms on right hand side as follows:

1. $\left\|u_t^N\right\| \left\|D_4 u^N\right\| \leq C(\epsilon) \left\|u_t^N\right\|^2 + \epsilon \left\|D_4 u^N\right\|^2$,
2. $\left\|u^N D u^N\right\| \left\|D_4 u^N\right\| \leq C(\epsilon) \left\|u^N D u^N\right\|^2 + \epsilon \left\|D_4 u^N\right\|^2 \leq C(\epsilon) \left\|D u^N\right\|^4 + \epsilon \left\|D_4 u^N\right\|^2$,
3. $\left\|D_3 u^N\right\| \left\|D_4 u^N\right\| \leq \epsilon \left\|D_4 u^N\right\|^2 \epsilon \left\|D_4 u^N\right\| + C(\epsilon) \left\|u^N\right\|$,
4. $\nu \left\|D_2 u^N\right\| \left\|D_4 u^N\right\| \leq C(\epsilon) \left\|D_2 u^N\right\|^2 + \epsilon \left\|D_4 u^N\right\|^2$,

where we have used in (2) the fact that $\max \left|u^N\right| \leq \left\|u_x^N\right\|$ and Ehrling's inequality in (3). Now taking $\epsilon = \nu/8$ it follows that $(\nu/2) \left\|D_4 u^N\right\|^2 \leq C$ since the remaining terms in (1)-(4) are bounded. Hence we have that u^N is bounded in $L^\infty(0,T; H^4(0,1)$. Also by (74) we have $u_t^N \in L^\infty\left(0,T;\; L^2(0,1)\right) \cap L^2\left(0,T;\; H^2(0,1) \cap H_0^1(0,1)\right)$. Then we can pass to the limit as N tends to infinity following similar arguments as we did in the previous chapter.

3.2 Existence and Uniqueness

Let u_ν denote the solution of regularized problem for each ν. Then for all $v \in L^2(0,1)$ we have

$$(u_{\nu t}, v)(t) + (u_\nu D u_\nu, v)(t) + (D_3 u_\nu, v)(t) + \nu (D_2 u_\nu, v)(t) + \nu (D_4 u_\nu, v)(t) = 0. \quad (75)$$

Existence theorem for the KdV equation is the following.

Theorem 3.3. *Let $u_0 \in H^3(0,1) \cap H_0^1(0,1)$ with $u_{0x}(1) = 0$. Then there exists a unique solution to the problem*

$$u_t + u u_x + u_{xxx} = 0 \qquad x \in (0,1), t > 0, \quad (76)$$

$$u(x,0) = u_0(x), \quad (77)$$

$$u(0,t) = u(1,t) = u_x(1,t) = 0 \qquad t > 0, \quad (78)$$

from the class

$$u \in L^\infty\left(0,T; H^3(0,1) \cap H_0^1(0,1)\right),$$

$$u_t \in L^\infty\left(0,T; L^2(0,1)\right) \cap L^2\left(0,T; H_0^1(0,1)\right).$$

Note that when $\nu = 0$ in the regularized problem, we have the KdV equation with the corresponding boundary values. What we will do is trying to pass to a subsequence of u_ν so that the limit is a solution for KdV. Indeed, this procedure will give us a weak solution and then we will show the regularity. For this we need to estimate u_ν where the constants of the estimate should not depend on ν. Following two lemmas will handle this problem.

Lemma 3.4. *For all $\nu \in (0,1/4)$ solutions of (57)–(59) satisfy the following inequality:*

$$||u_\nu(t)||^2 + \int_0^t \left|\left|D u_\nu^N(s)\right|\right|^2 ds \leqslant C\, ||u_0||^2,$$

where the constant C does not depend on ν.

Proof. Omitting the label ν, it follows from (65) that

$$||u(t)||^2 + \nu \int_0^t \left|\left|D_2 u^N(s)\right|\right|^2 ds \leqslant C\, ||u_0||^2, \tag{79}$$

where C does not depend on $\nu > 0$; $t \in (0,T)$. Taking in (75) $v = 2e^{\lambda x}u$ for $\lambda > 0$, and omitting the index ν we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(e^{\lambda x}, u^2\right)(t) - \frac{2\lambda}{3}\left(e^{\lambda x}, u^3\right)(t) \\
&+ 2\left(e^{\lambda x} D_3 u, u\right)(t) + 2\nu\left(e^{\lambda x} D_2 u, u\right)(t) + 2\nu\left(e^{\lambda x} D_4, u\right)(t) = 0,
\end{aligned} \tag{80}$$

where we have used the equality $2\left(e^{\lambda x} u D u, u\right)(t) = -\frac{2\lambda}{3}\left(e^{\lambda x}, u^3\right)(t)$. We will estimate the terms in (80) separately:

$$\begin{aligned}
I_1 &= -\frac{2\lambda}{3}\left(e^{\lambda x} u^3\right)(t), \\
&\geqslant -\frac{2\lambda e^{\lambda}}{3} \max_{x\in(0,1)} |u(x,t)|\, ||u(t)||^2 \\
&\geqslant -\frac{2\lambda e^{\lambda}}{3}\, ||Du(t)||\, ||u(t)||^2 \\
&\geqslant^a -\frac{\mu\lambda}{3}\left(e^{\lambda x}, u^2\right)(t) - \frac{e^{2\lambda}\lambda}{3\mu}\, ||u(t)||^4,
\end{aligned}$$

where, in (a), we have used Young's inequality with $\epsilon = \frac{\mu}{e^{\lambda}}$ and the fact that $e^{\lambda x} \geqslant 1$ for $x \in (0,1)$.

$$\begin{aligned}
I_2 &= 2\nu\left(e^{\lambda x} D_2 u, u\right)(t) \\
&\geqslant -2\nu \left|\left|e^{\frac{\lambda}{2}x} D_2 u(t)\right|\right| \left|\left|e^{\frac{\lambda}{2}x} u(t)\right|\right| \\
&\geqslant -\nu \left|\left|e^{\frac{\lambda}{2}x} D_2 u(t)\right|\right|^2 - \nu \left|\left|e^{\frac{\lambda}{2}x} u(t)\right|\right|^2 \\
&= -\nu\left(e^{\lambda x}, D_2 u^2\right)(t) - \nu\left(e^{\lambda x}, u^2\right)(t).
\end{aligned} \tag{81}$$

Also

$$
\begin{aligned}
I_3 &= 2\left(e^{\lambda x} D_3 u, u\right)(t) \\
&=^a (Du(0,t))^2 - e^{\lambda}(Du(1,t))^2 + 3\lambda\left(e^{\lambda x},(Du)^2\right)(t) - \lambda^3\left(e^{\lambda x},u^2\right)(t) \\
&\geqslant^b (Du(0,t))^2 - 2e^{\lambda}(Du(1,t))^2 + 3\lambda\left(e^{\lambda x},(Du)^2\right)(t) - \lambda^3\left(e^{\lambda x},u^2\right)(t), \qquad (82)
\end{aligned}
$$

where, in (a), we have used series of integration by parts and added $-e^{\lambda}(Du)^2(1,t)$ in (b). Lastly

$$
\begin{aligned}
I_4 &= 2\nu\left(e^{\lambda x} D_4 u, u\right)(t), \\
&= -2\nu e^{\lambda} u_x(1,t) D_2 u(1,t) + 2\nu\left(e^{\lambda x},(D_2 u)^2\right)(t) + 2\nu\lambda(Du)^2(1,t), \\
&- 2\nu\lambda(Du)^2(0,t) - 4\nu\lambda^2\left(e^{\lambda x},(Du)^2\right)(t) + \nu\lambda^4\left(e^{\lambda x},u^2\right)(t).
\end{aligned}
$$

Substituting $I_1 - I_4$ into (80) we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(e^{\lambda x},u^2\right)(t) + (1-2\nu\lambda)(Du(0,t))^2 + \lambda(3-\mu/3-4\nu\lambda)\left(e^{\lambda x},(Du)^2\right)(t) \\
&+ \nu\left(e^{\lambda x},(D_2 u)^2\right)(t) + (\nu\lambda^4 - \lambda^3 - \nu)\left(e^{\lambda x},u^2\right)(t) - \frac{e^{2\lambda}}{\lambda} 3\mu\,||u(t)||^4 \leqslant 0.
\end{aligned}
$$

Taking $\mu = 3$, $\lambda = 1$ and $\nu \in (0, 1/4)$, we reduce it to the inequality

$$
\begin{aligned}
&\frac{d}{dt}\left(e^{\lambda x},u^2\right)(t) + \left(e^{\lambda x},(Du)^2\right)(t) + \nu\left(e^{\lambda x},(D_2 u)^2\right)(t) \\
&\leqslant \left(e^{\lambda x},u^2\right)(t) + C\,||u(t))||^4 \\
&\leqslant C\left(||u(t))||^2 + ||u(t))||^4\right) \\
&\leqslant^a C\left(||u(0)||^2 + ||u(0)||^4\right), \qquad (83)
\end{aligned}
$$

where we have used (65) in (a). Since the inequality (65) does not depend on ν, C does not depend on $\nu \in (0, 1/4)$ either. Integrating (83) over $(0, t)$, we obtain

$$
\left(e^{\lambda x},u^2\right)(t) + \int_0^t \left(e^{\lambda x},(Du)^2\right)(s)\,ds + \nu\int_0^t \left(e^{\lambda x},(D_2 u)^2\right)(s)\,ds \leqslant C\,||u_0||^2 .
$$

Hence

$$||u_\nu(t)||^2 + \int\limits_0^t ||u_\nu(s)||^2 \, ds + \nu \int\limits_0^t ||D_2 u_\nu(s)||^2 \, ds \leqslant C \, ||u_0||^2 \, ,$$

for $t \in (0,T)$ where C depends on T, but does not depend on $\nu \in (0, 1/4)$. □

Lemma 3.5. *For all $\nu \in (0,1/4)$ u_ν satisfy the following inequality:*

$$||u_{\nu t}(t)||^2 + \int\limits_0^t ||D u_{\nu s}(s)||^2 \, ds + \int\limits_0^t ||D_2 u_{\nu s}(s)||^2 \, ds$$
$$\leqslant C \left(||u_0||^2_{H^3(0,1)\cap H_0^1(0,1)} + \nu \, ||D_4 u_0||^2 \right),$$

where the constant C does not depend on ν.

Proof. Differentiating (54) and taking inner product with $2e^{\lambda x} u_{\nu t}$ for $\lambda > 0$ and omitting the index ν we obtain

$$\frac{d}{dt}\left(e^{\lambda x}, u_t^2\right)(t) + 2\left(e^{\lambda x}\left(uDu\right)_t, u_t\right)(t) + 2\left(e^{\lambda x} D_3 u_t, u_t\right)(t)$$
$$+ 2\nu \left(e^{\lambda x} D_2 u_t, u_t\right)(t) + 2\nu\left(e^{\lambda x} D_4 u_t, u_t\right)(t) = 0. \tag{84}$$

We will estimate the terms in (84) separately. Since the boundary conditions are invariant under taking derivative replacing u with u_t in (82) we get

$$I_1 \geqslant \left(e^{\lambda x} D_3 u_t, u_t\right)(t) = (Du_t(0,t))^2 - 2e^{\lambda}(Du_t(1,t))^2 + 3\lambda\left(e^{\lambda x}, (Du_t)^2\right)(t) - \lambda^3\left(e^{\lambda x}, u_t^2\right)(t).$$

As we did in (81)

$$I_2 = 2\nu\left(e^{\lambda x} D_2 u_t, u_t\right)(t) \geqslant -\nu\left(e^{\lambda x}, (D_2 u_t)^2\right)(t) - \nu\left(e^{\lambda x}, u_t^2\right)(t).$$

Also

$$I_3 = 2\nu\left(e^{\lambda x} D_4 u_t, u_t\right)(t),$$
$$= -2\nu e^{\lambda} D u_t(1,t) D_2 u_t(1,t) + 2\nu\left(e^{\lambda x}, (D_2 u_t)^2\right)(t) + 2\nu\lambda (Du_t(1,t))^2,$$
$$- 2\nu\lambda(Du_t(0,t))^2 - 4\nu\lambda^2\left(e^{\lambda x}, (Du_t)^2\right)(t) + \nu\lambda^4\left(e^{\lambda x}, u_t^2\right)(t).$$

Lastly

$$I_4 = 2\left(e^{\lambda x}\left(uDu\right)_t, u_t\right)(t) = 2\left(e^{\lambda x}u_t^2, Du\right)(t) + \left(e^{\lambda x}u, D\left(u_t^2\right)\right)(t).$$

Since

$$2\left(e^{\lambda x}u_t^2, Du\right)(t) = -2\lambda\left(e^{\lambda x}u, u_t^2\right) - \left(e^{\lambda x}u, D\left(u_t^2\right)\right),$$

we have

$$I_4 = -2\lambda\left(e^{\lambda x}u, u_t^2\right) - 2\left(e^{\lambda x}u, u_tDu_t\right).$$

Note that

$$\begin{aligned}
-2\left(e^{\lambda x}u, u_tDu_t\right) &\geqslant -\delta\left|\left|e^{\frac{\lambda}{2}x}Du_t(t)\right|\right|^2 - \frac{1}{\delta}\left|\left|e^{\frac{\lambda}{2}x}uu_t\right|\right|^2 \\
&\geqslant -\delta\left(e^{\lambda x}, (Du_t)^2\right) - \frac{1}{\delta}max_{x\in(0,1)}u^2(x,t)\left(e^{\lambda x}, u_t^2\right) \\
&\geqslant^a -\delta\left(e^{\lambda x}, (Du_t)^2\right) - \frac{1}{\delta}\left|\left|Du(t)\right|\right|^2\left(e^{\lambda x}, u_t^2\right),
\end{aligned}$$

where we have used, in (a) the fact that $\max_{x\in(0,1)}|u(x,t)| \leqslant ||Du(t)||$ for $u \in H_0^1(0,1)$. Also with the same reasoning above

$$-2\lambda\left(e^{\lambda x}u, u_t^2\right) \geqslant -2\lambda t\left|\left|Du(t)\right|\right|\left(e^{\lambda x}, u_t^2\right)(t).$$

Hence

$$\begin{aligned}
I_4 &\geqslant -2\lambda\left|\left|Du(t)\right|\right|\left(e^{\lambda x}, u_t^2\right)(t) - \delta\left(e^{\lambda x}, (Du_t)^2\right) - \frac{1}{\delta}\left|\left|Du(t)\right|\right|^2\left(e^{\lambda x}, u_t^2\right), \\
&\geqslant -\delta\left(e^{\lambda x}, (Du_t)^2\right) - \left(2\lambda + \frac{1}{\delta}\right)\left(1 + \left|\left|Du(t)\right|\right|^2\right)\left(e^{\lambda x}, u_t^2\right)(t),
\end{aligned}$$

where $\delta > 0$ is an arbitrary constant. Substituting $I_1 - I_4$ into (84) we get

$$\begin{aligned}
&\frac{d}{dt}\left(e^{\lambda x}, u_t^2\right)(t) + \left[\lambda(3 - 4\nu\lambda) - \delta\right]\left(e^{\lambda x}, (Du_t)^2\right)(t) + \nu\left(e^{\lambda x}, (D_2u_t)^2\right)(t) \\
&- \left[\nu\left(1 + \lambda^4\right) + \lambda^3 + 2\lambda + \frac{1}{\delta}\right]\left(1 + \left|\left|Du(t)\right|\right|^2\right)\left(e^{\lambda x}, u_t^2\right)(t). \leqslant 0.
\end{aligned}$$

Since $\nu \in (0, 1/4)$, taking $\delta = 1/2$ and $\lambda = 1$ we obtain

$$\frac{d}{dt}\left(e^{\lambda x}, u_t^2\right)(t)+\left(e^{\lambda x}, (Du_t)^2\right)(t)+\nu\left(e^{\lambda x}, (D_2 u_t)^2\right)(t) \leqslant C\left(1 + ||Du(t)||^2\right)\left(e^{\lambda x}, u_t^2\right)(t)$$

Note that $u_{\nu t}(0) \leq C\left(\|u_{\nu 0}\|_{H^3(0,1)} + \nu \left\|D^4 u_{\nu 0}\right\|\right)$, so the result follows from the last inequality. $\square$

Lemmas 3.4 and 3.5 imply that

$$\begin{aligned} u_\nu &\text{ is bounded in } L^2(0,T;H_0^1(0,1)),\\ u_{\nu t} &\text{ is bounded in } L^2(0,T;H_0^1(0,1)),\\ \nu^{1/2} u_\nu &\text{ is bounded in } L^2(0,T;H^2(0,1)),\\ \nu^{1/2} u_{\nu t} &\text{ is bounded in } L^2(0,T;H^2(0,1)). \end{aligned}$$

Then by Theorem 1.16

$$u_\nu \in C([0,T];H_0^1(0,1)) \hookrightarrow L^\infty(0,T;H_0^1(0,1)), \tag{85}$$

$$\nu^{1/2} u_\nu \in C([0,T];H^2(0,1)) \hookrightarrow L^\infty(0,T;H^2(0,1)). \tag{86}$$

Proof. (*Theorem 3.3*) From (85) and (86), it follows that there exists a subsequence of u_ν which we denote also by u_ν and a function u such that

$$\begin{aligned} u_\nu &\to u \text{ strongly in } C(\overline{Q}),\\ u_\nu &\to u \text{ weakly-star in } L^\infty(0,T;H_0^1(0,1)),\\ u_{\nu t} &\to u_t \text{ weakly-star in } L^\infty(0,T;L^2(0,1)),\\ u_{\nu t} &\to u_t \text{ weakly in } L^2(0,T;H_0^1(0,1)),\\ \nu u_{\nu t} &\to 0 \text{ weakly-star in } L^\infty(0,T;L^2(0,1)), \end{aligned}$$

where $Q = (0,1) \times (0,T)$.

As a preliminary result we prove first the following theorem:

Theorem 3.6. *Let* $u_0 \in H^4(0,1) \cap H_0^1(0,1)$ *with* $u_{0x}(1) = 0$. *Then there exists* u *such that* $u \in C([0,T]; H_0^1(0,1))$, $u_t \in L^\infty(0,T; L^2(0,1)) \cap L^2(0,T; H_0^1(0,1))$ *and that satisfies the following:*

$$(u_t, v)(t) + (uu_x, v)(t) + (u_x, v_{xx})(t) = 0, u_x(1) = 0,$$

where $v(x,t) \in L^\infty(0,T; H^2(0,1)) \cap L^2(0,T; H_0^1(0,1)); v_x(0,t) = 0; t \in (0,T)$.

Proof. Due to the boundary conditions of u_ν and v, we can conclude from (75) that for all $v \in W$, the following identity holds:

$$(u_{\nu t}, v)(t) + (u_\nu u_{\nu x}, v)(t) + (u_{\nu x}, v_{xx})(t) + \nu (D_2 u_\nu, v)(t) + \nu (D_2 u_\nu, D_2 v)(t) = 0. \quad (87)$$

Passing to the limit as $\nu \to 0$, we obtain

$$(u_t, v)(t) + (uu_x, v)(t) + (u_x, v_{xx})(t) = 0$$

Also a function $u_0 \in H^4(0,1)$ in Theorem 3.1 satisfies

$$u_0(0) = u_0(1) = \nu u_{0xx}(0) = u_0(1) = u_{0x}(1) + \nu u_{0xx}(1) = 0.$$

When ν tends to zero we have

$$u_0(0) = u_0(1) = u_{0x}(1) = 0$$

and this completes the proof. □

Now let u be a function satisfying the properties in Theorem 3.6 and let, for a fixed $t \in (0,T)$, $F \in L^2(0,1)$ be defined as

$$F(x) = -u_t(x,t) - u(x,t)u_x(x,t) \qquad x \in (0,1). \quad (88)$$

Then for each $v \in W$, u satisfies the following conditions

$$(u_x, v_{xx})(t) = (F, v)(t), \quad (89)$$

$$u(0,t) = u(1,t) = u_x(1,t). \tag{90}$$

where $t \in (0,T)$.

Lemma 3.7. *Problem* (89) *and* (90) *has a unique solution in* $H_0^1(0,1)$.

Proof. It is enough to show that the only solution for the following equation is zero:

$$(u_x, v_{xx})(t) = 0,$$

where u satisfies the conditions in Theorem 3.6 and $v \in W$. Let

$$v(x,t) = (1-x)\int_0^x \int_0^s u(y,t)\,dy \qquad x \in (0,1).$$

Then, omitting t in the arguments we have

$$v_x(x) = -\int_0^x \int_0^s u(y)\,dy + (1-x)\int_0^x u(y)\,dy,$$

and

$$v_{xx}(x) = -2\int_0^x u(y)dy + (1-x)u(x),$$

from which it follows that $v \in W$. Substituting v into (89) (and taking $F = 0$) we have

$$0 = (u_x, v_{xx}) = -2\left(u_x, \int_0^x u(y)\,dy\right) + (u_x, (1-x)u) = 2(u,u) - \frac{1}{2}(x, \frac{d}{dx}u^2 = \frac{5}{2}\|u\|^2.$$

Thus $u = 0$. □

Now we will show that u is actually in $H^3(0,1)$. For this, let for any $F \in L^2(0,1)$, $w = w(x)$ be defined as follows:

$$w(x) = K_1 x + K_2 x^2 + \frac{1}{2}\int_0^x y^2 F(y)\,dy - x\int_0^x yF(y)\,dy + \frac{x^2}{2}\int_0^x F(y)\,dy.$$

When we take derivatives of w, the terms appearing with F(without the integral of F) cancel, so $w \in H^3(0,1)$. Also w clearly satisfies that $w(0) = 0$ and, lastly, given $F \in L^2(0,1)$ it is obvious that there exists K_1 and K_2 such that $w(1) = 0$ and $w_x(1) = 0$. Since $w_{xxx} = F(x)$ and w satisfy the boundary conditions of the problem (90), we have $w = u$, where u is the solution in Theorem 3.6. Hence, we proved the existence of regular solutions for (76)–(78) when $u_0 \in H^4(0,1) \cap H_0^1(0,1)$.

Note that in Theorem 3.3 we need $u_0 \in H^3(0,1) \cap H_0^1(0,1)$. For this, we observe that in Lemma 3.4 we need $u_0 \in L^2(0,1)$ and in Lemma 3.5, as $\nu \to 0$ we get

$$\|u_t\|^2 + \int_0^t \|u_{xs}(s)\|^2 \, ds \leq C\|u_0\|^2_{H^3(0,1)}.$$

Hence, approximating functions $u_0 \in H^3(0,1) \cap H_0^1(0,1)$ with $u_{0x}(1) = 0$ by functions $v \in H^4(0,1) \cap H_0^1(0,1)$ with $v_{0x}(1) = 0$, we prove the existence part of the Theorem 3.3.

Uniqueness

Now let u, v be two solutions. Then for $w = u - v$, we have

$$w_t + \frac{1}{2}\frac{d}{dx}u^2 - \frac{1}{2}\frac{d}{dx}v^2 + w_{xxx} = w_t + \frac{1}{2}\frac{d}{dx}(w(u+v)) + w_{xxx} = 0,$$

with $w(0) = w(1) = w_x(1) = 0$ and $w(x,0) = 0$. Multiplying the above identity by $e^{\lambda x}w$, integrating over $(0,1)$ and using the boundary conditions for w we get

$$\begin{aligned}\frac{d}{dt}\left(e^{\lambda x}, w^2\right)(t) - \left(e^{\lambda x}w_x, (u_1+u_2)w\right)(t) - \lambda\left(e^{\lambda x}, (u_1+u_2)w^2\right)(t) \\ + 2\left(e^{\lambda x}D_3 w, w\right)(t) = 0. \qquad (91)\end{aligned}$$

Now we will estimate the terms separately as follows: First note that $u \in C(\overline{Q})$. Then

$$\begin{aligned}
I_1 &= \left|\left(e^{\lambda x}w_x, (u+v)w\right)\right| \leq M\left(e^{\lambda x}, |ww_x|\right)(t) \\
&\quad \epsilon\left(e^{\lambda x}, w_x^2\right)(t) + C(\epsilon)\left(e^{\lambda x}, w^2\right)(t), \\
I_2 &= \lambda\left|\left(e^{\lambda x}, (u+v)w^2\right)(t)\right| \leq \lambda M\left(e^{\lambda x}, w^2\right)(t), \\
I_3 &= 2\left(e^{\lambda x}D^3 w, w\right)(t) = z_x^2(0,t) + 3\lambda\left(e^{\lambda x}, w_x^2\right)(t) - \lambda^3\left(e^{\lambda x}, w^2\right)(t).
\end{aligned}$$

Substituting I_1-I_3 into (91) we get

$$\frac{d}{dt}\left(e^{\lambda x}, w^2\right)(t) + (3\lambda - \epsilon)\left(e^{\lambda x}, w_x^2\right)(t) \leq C(\epsilon)\left(e^{\lambda x}, w^2\right)(t).$$

Taking $\lambda = 1$ and $\epsilon = 2$ and integrating over $(0, t)$ we get

$$\left(e^{\lambda x}, w^2\right)(t) \leq c\int_0^t \left(e^{\lambda x}, w^2\right)(s)\, ds,$$

from which it follows that $\|z(t)\| = 0$ for $t \in (0, T)$ □

3.3 Stability

Theorem 3.8. *There exists positive constants* $\lambda_0 \in (0,1)$ *and* K *such that if* $||u_0|| \leqslant 3/e$*, then strong solutions to* (2) - (4) *satisfy the following inequality:*

$$||u(t)||^2 \leqslant e^{\lambda_0}\, ||u_0||^2\, e^{-\chi_0 t},$$

where $\chi_0 = \lambda_0/(2e^{\lambda_0})$.

Proof. Multiplying (2) by u and integrating over $(0,1)$ we get

$$\frac{d}{dt}\, ||u(t)||^2 + 2\int_0^1 u^2 u_x\, ds + 2\int_0^1 u u_{xxx}\, ds = 0,$$

which gives, using (4)

$$\frac{d}{dt}\, ||u(t)||^2 + 2u_x^2(0,t) = 0.$$

Hence for all $t > 0$ we have

$$||u(t)|| \leqslant ||u_0||\,. \tag{92}$$

This time multiplying (2) by $e^{\lambda x}u$ for $\lambda \in (0,1)$, we get

$$\frac{d}{dt}\left(e^{\lambda x}, u^2\right)(t) + 2\left(e^{\lambda x}, u^2 u_x\right)(t) + 2\left(e^{\lambda x}, u u_{xxx}\right)(t). \tag{93}$$

Since

$$2\left(e^{\lambda x}, u^2 u_x\right)(t) = -\frac{2\lambda}{3}\left(e^{\lambda x}, u^3\right)(t),$$

and

$$2\left(e^{\lambda x} D_3 u, u\right)(t) = u_x^2(0,t) - e^{\lambda} u_x^2(1,t) + 3\lambda\left(e^{\lambda x}, u_x^2\right)(t), -\lambda^3\left(e^{\lambda x}, u^2\right)(t)$$

from (93) we get the following equality:

$$\frac{d}{dt}\left(e^{\lambda x}, u^2\right)(t) + 3\lambda\left(e^{\lambda x}, u_x^2\right)(t) - \frac{2\lambda}{3}\left(e^{\lambda x}, u^3\right)(t) - \lambda^3\left(e^{\lambda x}, u^2\right)(t) + u_x^2(0,t) = 0. \quad (94)$$

Now taking into account (92) we estimate

$$\begin{aligned}\left|\left(e^{\lambda x}, u^3\right)(t)\right| &\leqslant max_{x\in(0,1)}|u(x,t)|^2\left(e^{\lambda x}, |u|\right)(t),\\ &\leqslant e^{\lambda}\left||u_x(t)|\right|^2 \ \left||u_0|\right|^2 \leqslant e^{\lambda}\left||u_0|\right|\left(e^{\lambda x}, u_x^2\right)(t).\end{aligned}$$

Substituting the above inequality into (94) we get

$$\frac{d}{dt}\left(e^{\lambda x}, u^2\right)(t) + \lambda\left(3 - \frac{2e^{\lambda}}{3}\left||u_0|\right|\right)\left(e^{\lambda x}, u_x^2\right)(t) - \lambda^3\left(e^{\lambda x}, u^2\right)(t) \leqslant 0.$$

Since $0 < \lambda < 1$ and taking $||u_0|| < 3/e$, above inequality reduces to the following one:

$$\frac{d}{dt}\left(e^{\lambda x}, u^2\right)(t) + \lambda\left(e^{\lambda x}, u_x^2\right)(t) - \lambda^3\left(e^{\lambda x}, u^2\right)(t) \leqslant 0. \quad (95)$$

Observe that since $u \in H^3(0,1)$ and we are in one dimension u is bounded. Hence, $max_{x\in(0,1)}|u(x,t)| \geqslant ||u(t)||$ which gives

$$\left(e^{\lambda x}, u_x^2\right)(t) \geqslant \left||u_x(t)|\right|^2 \geqslant max_{x\in(0,1)}|u(x,t)|^2 \geqslant \left||u(t)|\right|^2 \geqslant \frac{1}{e^{\lambda}}\left(e^{\lambda x}, u^2\right)(t).$$

Then (95) becomes

$$\frac{d}{dt}\left(e^{\lambda x}, u^2\right)(t) + \lambda\left(e^{-\lambda} - \lambda^2\right)\left(e^{\lambda x}, u^2\right)(t) \leqslant 0.$$

We now need the following lemma:

Lemma 3.9. *There exists $\lambda_0 \in (0,1)$ that*

$$\frac{1}{e^{\lambda}} - \lambda^2 \geqslant \frac{1}{2e^{\lambda}},$$

for all $\lambda \in [0, \lambda_0]$.

Proof. The inequality we want to prove is equivalent to the following one:

$$f(\lambda) = 1 - 2e^{\lambda}\lambda^2 \geqslant 0, \qquad \lambda \in [0, \lambda_0].$$

Since f is continuous and $f(0) = 1$ there exists such $\lambda_0 \in (0,1)$. □

Then by this lemma, from the last inequality we get for all $\lambda \in [0, \lambda_0]$

$$\frac{d}{dt}\left(e^{\lambda x}, u^2\right)(t) + \frac{\lambda}{2e^{\lambda}}\left(e^{\lambda x}, u^2\right)(t) \leqslant 0.$$

Since $\chi(\lambda) = \lambda/(2e^{\lambda})$ is an increasing function of λ on $(0,1)$, we have

$$max_{\lambda\in[0,\lambda_0]}\chi(\lambda) = \frac{\lambda_0}{2e^{\lambda_0}} = \chi_0.$$

Now multiplying the last inequality by $e^{\chi_0 t}$ and taking $\lambda = \lambda_0$, we get

$$\frac{d}{dt}\left(e^{\chi_0 t}E(t)\right) \leqslant 0,$$

where $E(t) = \left(e^{\lambda_0 x}, u^2\right)(t)$. Hence we have

$$E(t) \leqslant E(0)e^{-\chi_0 t}.$$

which completes the proof.

□

References

[1] Adams, R. A. *Sobolev Spaces*, Academic Press, (1975).

[2] Boussinesq, J. "Essai sur la theorie des eaux courantes, Memoires presentes par divers savants.", *l'Acad. des Sci. Inst. Nat. France*, 23 (1877), 1-680.

[3] Brezis, H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, (2010).

[4] Bubnov, B. A. "General boundary value problems for the Korteweg-de Vries equation in a bounded domain", *Differential equations* 15 (1979), 26-31.

[5] Ghidaglia, J. M. "Weakly damped forced Korteweg-de Vries equations behave as a finite dimensional dynamical system in the long time", *J. Differential Equations* 74 (1988), 369-390.

[6] Korteweg D. J. and de Vries G. "On the change of form of long waves advancing in a rectangular channel and o a new type of long stationary wave", *Phil. Mag.* 39 (1895), 422-443.

[7] Larkin N. A. "Korteweg-de Vries and Kuramoto-Sivashinsky Equations in Bounded Domains", *J. Math. Anal. Appl.* 297 (2004), 169-185.

[8] Lax, P. "Integrals of nonlinear equations of evolution and solitary waves", *Comm. Pure Applied Math.* 21 (1968), 467-490.

[9] Lions, J.L. and Magenes E. *Nonhomogenous Boundary Value Problems and Application*, Springer-Verlag, (1972).

[10] Temam, R. *Navier-Stokes Equations, Theory and Numerical Analysis*, North-Holland, (1984).

[11] Temam R. *Infinite Dimensonal Dynamical Systems in Mechanics and Physics*, Springer, (1997).

[12] Ton, Bui "An initial-boundary-value problems for the Korteweg-de Vries equation", *J. Differ. Equations*, 25 (1977), 288-309.

[13] Zheng, S. *Nonlinear Evolution Equations*, Chapman and Hall/CRC, (2004).