Quantum Information Transfer Between Photon Fields Coupled to an Ensemble of Cyclic Quantum Systems

by

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This is to certify that I have examined this copy of a master's thesis by

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To Ghodsi, Hasan and Yalda
for the unconditional love

ABSTRACT

Abstract

We study the interaction of quantized electromagnetic (EM) fields with an ensemble of Delta-type three-level quantum systems driven by a classical EM field. This system is considered to be promising for quantum information applications, in particular for implementing quantum repeater protocols in long distance quantum communications. We further develop a critical review and generalizations of canonical (Fröhlich) Transformations, which is frequently used in the analysis of such complicated quantum optical models in terms of simpler effective models. Next, we construct a method to find time-evolution of the quantum fields, based on which we demonstrate quantum information transfer between the fields. Our results are applicable for any number of atoms in the ensemble and for large coupling of atoms to classical field, relative to the quantum fields. The previous studies assumed both the atom number and the relative classical coupling to be infinitely large. Thus we relaxed these conditions, which is physically too restrictive. Finally, we discuss possible further developments of this work, and suggest the other cases for which application of this method could be fruitful.

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Chapter 1

INTRODUCTION

Quantum computers are proven to be efficient in solving many problems in which classical computers are inefficient [16]. Furthermore, classical communication are theoretically not immune to eavesdropping, since classical information can be clonned in the middle of the transmission line. But this is also an advantage in long distance classical communications, since one can 'amplify' the signal by making several identical copies of bits. Quantum communcation, which is based on trasnmitting quantum systems (such as photons) and performing quantum operations, is proved to be ideally immune to eavasdropping for certain classes of communication protocols such as BB84 [17]. But, on the other hand, quantum no-cloning theorem [3], [2] makes it impossible to amplify quantum information in the classical sense, since one cannot make a perfect copy an unknown qubit. Therefore, another scenario is required, both to store and retrieve, and to *amplify* quantum information that is carried on by physical systems such as photons. However, the problem of storage and retrieval of quantum information can be addressed by the so-called quantum memories [18], which are basically electronagnetically controlled atomic ensembles interacting with photonic qubits. Although unknown quantum states cannot be cloned, but they can be transferred between two modes via entanglement. For applications in quantum communication, by means of such ensembles, we can create new versions of qubits in appropriate distances before loss or decoherence. These are called quantum repeater protocols. Since the two photon modes must be simultaneously coupled with quantum systems in such ensembles in a way to make the modes entangled, therefore the modes must cor-

respond to energy level differences which have overlap with one another. This means that one and multi-photon (two or more) transitions must coexist in such quantum systems. But this is against electric-dipole selection rules which is valid for quantum systems with inversion symmetry [1]. Quantum systems with inversion symmetry are characterized by Hamiltonians which commute with rotation operator. Therefore, such transitions are not allowed in systems with inversion symmetry. We can either use prototypes that lack inversion symmetry, such as semiconductors [20]-[25], or we can artificially break the symmetry of a symmetric Hamiltonian. One of the simplest candidates are three-level Δ-type systems with the so-called cyclic transitions.

Chapter 2

FRÖLICH TRANSFORMATION OF HAMILTONIANS IN INTERACTION PICTURE

Introduction

Different kinds of canonical transformations have been introduced and used in quantum theory. Quantum canonical transformations were firstly introduced by Born, Jordan and Heisenberg [9] in the context of Matrix Mechanics, as all possible transformations that preserve the commutation relation:

$$[p_i, q_i] = \frac{\hbar}{i} I, \tag{2.1}$$

where q_i and p_i are pairs of conjugate variables. This can be done by a similarity transformation of matrices associated with the conjugate variables. Since the works of Weyl and Dirac much attention has been given to unitary canonical transformations, which preserve the orthonormality of the basis of Hilbert Space in addition to the commutative structure. In this chapter we first review Interaction picture in details. Then we review and develop Frölich Transformation which is an approximate canonical transformation. Then we combine these two transformations to give a combined transformation. Next we develop an approximation scheme and explain how to use it to find approximate dynamics imposed by a many-body Hamiltonian.

2.1 Interaction Picture

Introduction

Interaction Picture involves an exact canonical transformation. Consider a quantum system, such as a single atom in the presence of other quantum systems and/or fields, with Hamiltonian H. The terms in H can be grouped into two: time-dependent (H^{iso}) and time-independent (H_t). If individual quantum systems were not interacting with external fields, the system would be isolated and time-independent (e.g., see [13]). But note that H^{iso} may also include time-independent external fields. If the system was isolated and initially found to be in state $\mid \Psi^{iso}_{(0)}\rangle$, it will evolve into $U^{iso}_{(t)} \mid \Psi^{iso}_{(0)}\rangle$ after time t, where $U^{iso}_{(t)} = e^{-iH^{iso}t}$. Now, imagine the same system being under the influence of external time-varying classical fields. This must be treated with the extra type of terms H_t in the Hamiltonian to get $H = H^{iso} + H_t$ and the atom will evolve into $\mid \Psi_{(t)}\rangle_S = U_{(t)} \mid \Psi_{(0)}\rangle_S$ after time t, where $U_{(t)} = e^{-iHt}$. Now, if we multiply the time-dependent state of the new system $\mid \Psi_{(t)}\rangle_S$ with $U^{iso\dagger}_{(t)} = e^{iH^{iso}t}$ to get a new state $\mid \Psi_{(t)}\rangle_I$, then, that part of time-evolution that is purely due to term H_{iso} is excluded from $\mid \Psi_{(t)}\rangle_I$. In other words, if $H_t = 0$, then $\mid \Psi_{(t)}\rangle_I = \mid \Psi_{(0)}\rangle_S$. Moreover, if $[H^{iso}, H_t] = 0$, then $\mid \Psi_{(t)}\rangle_I = e^{-iH_t t} \mid \Psi^{iso}_{(0)}\rangle$. These observations suggest that an equation that governs time-evolution of $\mid \Psi_{(t)}\rangle_I$ might be easier to solve. In order to find such an equation let us find the time-derivative of $\mid \Psi_{(t)}\rangle_I$:

$$
\begin{aligned}
i\partial_t \mid \Psi_{(t)}\rangle_I &= (i\partial_t U^{iso\dagger}_{(t)}) \mid \Psi_{(t)}\rangle_S + U^{iso\dagger}_{(t)}(i\partial_t \mid \Psi_{(t)}\rangle_S) \\
&= (i(\partial_t U^{iso\dagger}_{(t)})U^{iso} + U^{iso\dagger}_{(t)} H U^{iso}_{(t)})U^{iso\dagger}_{(t)} \mid \Psi_{(t)}\rangle_I.
\end{aligned}
\tag{2.2}
$$

Therefore, $\mid \Psi_{(t)}\rangle_I$ obeys a new eqaution,

$$
i\partial_t \mid \Psi_{(t)}\rangle_I = H^I \mid \Psi_{(t)}\rangle_I,
\tag{2.3}
$$

where

$$H^I = i(\partial_t U^{iso\dagger}_{(t)})U^{iso} + U^{iso\dagger}_{(t)}HU^{iso}_{(t)}$$
$$= U^{iso\dagger}_{(t)}H_t U^{iso}_{(t)} \tag{2.4}$$

We made the last step by recalling that the Hamiltonian of isolated system is time-independent, i. e., $\partial_t H_{iso} = 0$. The new *picture* is called Interaction or Dirac Picture. Moreover, for an arbitrary operator A^S in the Schrödinger picture, if we require it to be transferred as

$$A^I_{(t)} = U^{iso\dagger}_{(t)}A^S_{(t)}U^{iso}_{(t)}, \tag{2.5}$$

then, if $A^S_{(t)}|\Psi_{(t)}\rangle_S = |\Psi'\rangle_S$, we get $A^I_{(t)}|\Psi_{(t)}\rangle_I = |\Psi'_{(t)}\rangle_I$, where $|\Psi_{(t)}\rangle_I$ and $|\Psi'_{(t)}\rangle_I$ are $|\Psi_{(t)}\rangle_S$ and $|\Psi'_{(t)}\rangle_S$ in the interaction picture, respectively. Furthermore, the matrix elements are preserved, i. e.,

$$_I\langle a_{i(t)}|A^I_{(t)}|a_{j(t)}\rangle_I = {}_S\langle a_{i(t)}|A^S_{(t)}|a_{j(t)}\rangle_S. \tag{2.6}$$

The next result is a dynamical equation for operators in this picture, the counterpart of the Heisenberg equation. In order to find it, let us take the partial-imaginary-time derivative of (2.5):

$$i\partial_t A^I_{(t)} = -U^{iso\dagger}_{(t)}H^{iso}A^S_{(t)}U^{iso}_{(t)} +$$
$$U^{iso\dagger}_{(t)}A^S_{(t)}H^{iso}U^{iso}_{(t)} + U^{iso\dagger}_{(t)(t)}i\partial_t A^S_{(t)}U^{iso}_{(t)}$$
$$= U^{iso\dagger}_{(t)(t)}[A^S_{(t)}, H^{iso}]U^{iso}_{(t)} + U^{iso\dagger}_{(t)}i\partial_t A^S_{(t)}U^{iso}_{(t)}$$
$$= [A^I_{(t)}, H^{iso}] + U^{iso\dagger}_{(t)}i\partial_t A^S_{(t)}U^{iso}_{(t)}. \tag{2.7}$$

When A^S is time-independent, we get:

$$i\partial_t A^I_{(t)} = [A^I_{(t)}, H^{iso}] \tag{2.8}$$

In many problems of physical interest, H^I is time independent. Then, solving Eq. (2.3) becomes much easier than solving the original Schrödinger equation. However, it is very important to note that the Hamiltonian in the interaction picture, H^I, is NOT a similarity transformation of H, but of H_t. And although eigenvectors of H^I, i. e. $|\Psi_n\rangle_I$s, are just unitary transformations of the eigenvectors of H, i. e. $|\Psi_n\rangle_S$s, but H and H^I are NOT isospectral. Using (2.4), this can be seen as below:

$$
\begin{aligned}
H &= H_0 + H_t \\
&= H_0 + U^{iso}{}_{(t)} H^I U^{iso\dagger}{}_{(t)}.
\end{aligned}
\tag{2.9}
$$

Here, while the second term is isospectral with H^I, however, due to the presence of H_0, $[H, H^I] \neq 0$ unless $[H_0, H_t] = 0$, which is not hold in general. Therefore, they do not pose the same set of eigenstates; neither are they isospectral. In Sec. 2.1.1 we will prove a necessary theorem that helps using H^I obtained from (2.4) to find approximate time-evolution of states.

2.1.1 *Time-evolution of states acted upon by operators*

Note that at t = 0, we have

$$
\begin{aligned}
|\Psi_{(0)}\rangle_I &= |\Psi_{(0)}\rangle_S \tag{2.10} \\
&= |\Psi_{(0)}\rangle. \tag{2.11}
\end{aligned}
$$

A problem of practical application is finding the time-evolution of a state as it is defined as below:

$$
|\Psi_{(0)}\rangle_S = A_{(0)}|\Phi\rangle_{(0)}.
\tag{2.12}
$$

With a proper choice of $A_{(0)}$, the state above may represent any state. We mentioned that solving the Schördinger equation in interaction picture (2.3)is usually easier. Therefore, we first solve the problem in interaction picture and then move to Schödinger picture by a unitary transformation of $U^{iso} = e^{-iH^{iso}t}$. The following lemma shows how to find $|\Psi_{(t)}\rangle_S$ by using interaction picture.

Lemma 2.1.1. *If* $|\Psi_{(0)}\rangle_I = A_{(0)}|\Phi_{(0)}\rangle$, *then*

$$|\Psi_{(t)}\rangle_S \quad = \quad e^{-iH^{iso}} A_{(-t)}|\Phi_{(t)}\rangle_I, \text{ is the only solution.} \tag{2.13}$$

Here H^I is the interaction Hamiltonian given by (2.4), $|\Phi_{(t)}\rangle_I$ is time-evolution of $|\Phi_{(0)}\rangle$ in the interaction picture given by (2.3) and $A_{(-t)}$ is determined by the following equation:

$$i\partial_t A_{(-t)} \quad = \quad -[A_{(-t)}, H^I]. \tag{2.14}$$

Proof. Note that equation $\Psi_{(t)}\rangle_I = A_{(-t)}|\Phi_{(t)}\rangle_I$ satisfies (2.3):

$$
\begin{aligned}
i\partial_t(A_{(-t)} \mid \Phi_{(t)}\rangle_I) \quad &= \quad (i\partial_t A_{(-t)}) \mid \Phi_{(t)}\rangle_I + A_{(-t)} i\partial_t \mid \Phi_{(t)}\rangle_I \\
&= \quad (-[A_{(-t)}, H^I] + H^I) \mid \Phi_{(t)}\rangle_I \\
&= \quad H^I A_{(-t)} \mid \Phi_{(t)}\rangle_I
\end{aligned}
\tag{2.15}
$$

Therefore, since the solution of (2.3) is unique and $|\Psi_{(0)}\rangle_I = A_{(0)}|\Phi_{(0)}\rangle_I$, $\Psi_{(t)}\rangle_I$ is the only solution with this initial condition. And, $|\Psi_{(t)}\rangle_S = e^{-iH^{iso}}\Psi_{(t)}\rangle_I$. $\qquad\square$

Corollary: Let $|\Phi_{(0)}\rangle_I$ be an energy eigenstate of H_I the Hamiltonian in interaction picture, then

$$|\Psi_{(t)}\rangle_S = e^{-iH^{iso}t} A_{(-t)}|\Phi_{(0)}\rangle_I. \tag{2.16}$$

2.2 Fröhlich Transformation

Introduction

Flohlich Transformation is used to approximate Hamiltonians and finding approximate eigenvalues. Roughly speaking, a Fröhlich Transformation is obtained by unitarily transforming a Hamiltonian such that the first order term vanishes and terms up to second order in λ are kept. This makes the subsequent eigenvalues of second order in λ as well, with zeroth order equal to the respected eigenvalues of unperturbed Hamiltonian. Therefore, the resultant Hamiltonian, although not isospectral with the originial perturbed Hamiltonian, but is isospectral up to

second order with it. Firstly introduced by Fröhlich [5] in investigating electron-phonon interactions in metals, the method has been widely used in solid-state physics and quantum optics and generally where Hamiltonians are perturbed by a small perturbation (comparing to energy levels of the unperturbed Hamiltonian), with respect to which the Hamiltonian is analytic. Fröhlich Transformation shows its advantage in approximating operators in infinite dimensional Hilbert spaces, where it is much harder to deal with series in second order perturbation theory. We show that all valid Fröhlich Transformations of a Hamiltonian do not even commute with each other, which means they do not have the same set of eigenvectors. The question then arises: If they are not unique, what do they refer to? And do they have the same set of eigenvalues? The answer to the latter question becomes crucial in tensor-product and infinite-dimensional Hilbert space, wherein as we will show, one can easily find the transformation for one body problem and generalize it to many-body system by summing over indexes of generators of Fröhlich transformations, but one may wonder if other more sophisticated transformations can give different results (eigenvalues, etc.). We will answer to this and more questions in terms of theorems. Then we generalize Fröhlich method and the approximate spectral uniqueness theorem to perturbative Hamiltonians. After this section, we will apply the results obtained here to attack our problem.

2.2.1 Formulation

Take a Hilbert space $\mathscr{H}$ and consider the space of linear operators $\mathscr{L}$ acting on it. Let us take two arbitrary non-commutative Hermitian operators H^0 and $H^1 \in \mathscr{L}$.

$$H := H^0 + \lambda H^1, \lambda \in \mathbb{R} \tag{2.17}$$

Note that if perturbation parameter is complex, then we must have Hermitian conjugate of λH^1 as well. Consequently, we can keep madulus and take the phase factors inside H^1 by redefining it. Therefore, all the results ontained by assuming real perturbation parameters (one or more) will be valid for the case of complex perturbations as well. Now, consider a unitary transformation $U := exp(\lambda S)$, where

S is an anti-Hermitian operator on $\mathscr{H}$. Then, applyng Baker-Campbell-Hausdorff formula and rearranging terms of the same order gives:

$$
\begin{aligned}
H' &= U^\dagger H U && (2.18)\\
&= H^0 + \lambda H^1 + \frac{[H^0 + \lambda H^1, \lambda S]}{1!} + \frac{[[H^0 + \lambda H^1, \lambda S], \lambda S]}{2!} + O(\lambda^3)\\
&= H^0 + \lambda(H^1 + [H^0, S]) + \lambda^2([H^1, S] + \frac{[[H^0, S], S]}{2!}) + O(\lambda^3) && (2.19)
\end{aligned}
$$

The idea is to choose S such that the first order term in λ in H' vanishes. Thus this must hold:

$$
\exists S \in \mathscr{L}, \text{ such that } S = -S^\dagger \text{ and } H^1 + [H^0, S] = 0. \tag{2.20}
$$

We take the terms up to second order,

$$
H_{eff} = H^0 + \lambda(H^1 + [H^0, S]) + \frac{\lambda^2([H^1, S] + [[H^0, S], S])}{2!} \tag{2.21}
$$

Using the condition (2.20), H_{eff} will be simplified as below:

$$
H_{eff} = H^0 + \lambda^2 \frac{[H^1, S]}{2}. \tag{2.22}
$$

Also note that from Eq.s (2.18) and (2.21) we have:

$$
H_{eff} = U^\dagger H' U + \mathscr{O}(\lambda^3), \tag{2.23}
$$

Thereforem, if $|\Psi_n\rangle$ is an eigenstate of H, then

$$
|\Psi'_n\rangle = U^\dagger |\Psi_n\rangle \tag{2.24}
$$

is an exact eigenvector of H' given by (2.18) and also an approximate eigenvector of H_{eff} given by (2.21) up to second order in purturbative parameter.

2.2.2   On Existence and Uniqueness of Fröhlich Transformation

As we will show below, Fröhlich Transformation may not exist and when it exists it is not unique.

Example: For $H = \sigma_1 + \lambda(\frac{I_2}{2} - \sigma_2)$ no solution S exists. Thus, Fröhlich Transformation is not even possible. In general, for finite dimentional Hilbert spaces, $Tr(H_1)$ must be zero in order for (2.20) to have a solution S. Note that for infinite dimensional spaces, the condition $Tr(H^1) = 0$ is not necessary; but whether S exists or not depends on the choices of H^0 and H^1.

S is clearly not unique. If S is a solution of (2.20), then $S + \Lambda$ is also a solution iff $\Lambda^\dagger = -\Lambda$ and $[H^0, \Lambda] = 0$.

Example: The set of generators of all Fröhlich Transformations for $H = \sigma_1 + \lambda\sigma_2$. Using the basis that spans $\mathscr{M}(\mathbb{C}, 2)$ and consists of Pauli matrices and identity, it follows that $\{S | S = \alpha I_2 + \beta\sigma_2, \forall \alpha, \beta \in \mathbb{C}\}$ is the set of all solutions of (2.20).

Conclusion:

Fröhlich transformation may neither exist, nor may it be unique. In fact, if S_1 and S_2 each satisfies (2.20), then,

$$H_{eff1} = H^0 + \lambda^2 \frac{[H^1, S_1]}{2} \tag{2.25}$$

$$H_{eff2} = H^0 + \lambda^2 \frac{[H^1, S_2]}{2}, \tag{2.26}$$

from which it follows that

$$[H_{eff1}, H_{eff1}] = \frac{\lambda^2}{2}[H_0, [H^1, S_2 - S_1]] + \mathcal{O}(\lambda^4).$$

Although $[H_0, S_2 - S_1] = 0$, since we assumed $[H_0, H^1] \neq 0$, and that since satisfying (2.20) by S_1, S_2 imposes no further conditions on $S_2 - S_1$, thus $[H_0, [H^1, S_2 - S_1]]$ may not be zero for two possible solutions of (2.20). In other words, if S_1 is a solution of (2.20), there exists S_2 such that its corresponding effective Hamiltonian does not commute with that of S_1. One may wonder if all the possible solutions have the same set of eigenvalues up to 2nd order. Answer to this question becomes crucial especially in infinite dimensional composed systems where determining the set of all F Transformations is cumbersome.

2.2.3 Second-order Isospectrality of Fröhlich Transformations

Definition *Second-order Isospectrality:* Second-order isospectrality of H_{eff1} and H_{eff2} means if we keep the series of each eigenvalue in the eigenset of each operator up to second order, the resulting sets will be equal.

We will prove second-order isospectrality of two effective Hamiltonian resulted from two Fröhlich Transformation of the same Hamiltonian in a theorem immediately after the following lemma.

Lemma 2.2.1. *Let*

$$O_{1(\lambda)} \text{ and } O_{2(\lambda)} \in \mathscr{L},$$

be analytic operators with respect to a parameter λ around a point λ_0 and have discrete spectrum. Then the following holds

$$O_{1(\lambda_0)} = O_{2(\lambda_0)}$$

$$\frac{dO_1}{d\lambda}(\lambda_0) = \frac{dO_2}{d\lambda}(\lambda_0)$$

$$\frac{d^2O_1}{d\lambda^2}(\lambda_0) = \frac{d^2O_2}{d\lambda^2}(\lambda_0),$$

where $\frac{d^k O_j}{d\lambda^k}(\lambda_0)$ means k-th order derivative of $O_{j(\lambda)}$ with respect to λ at $\lambda = \lambda_0$, then $O_{1(\lambda)}$ and $O_{2(\lambda)}$ are second-order isospectral.

Proof. Take an arbitrary linear operator defined on $\mathscr{H}$ which is analytic with respect to a parameter λ around $\lambda = \lambda_0$, in the form below:

$$O_{(\lambda)} = O^{(0)} + (\lambda - \lambda_0)O^{(1)} + (\lambda - \lambda_0)^2 O^{(2)} + (\lambda - \lambda_0)^3 F_{(\lambda)},$$

where $F_{(\lambda)}$ is an analytic operator. Let's symbolically denote eigenbasis and the corresponding eigenset of $O^{(0)}$ by $\{|\Psi_n\rangle\}$ and $\{\epsilon_n\}$ respectively, and similarly the engenbasis and eigenset of $O_{(\lambda)}$ by $\{|\Psi'_n\rangle\}$ and $\{\epsilon'_n\}$, respectively. Then, we can

write

$$| \Psi'_n \rangle = \sum_m C_{nm(\lambda)} | \Psi_m \rangle \qquad (2.27)$$

$$\epsilon'_n = \sum_{k=0} \epsilon_{nk} \lambda^k, \text{ and} \qquad (2.28)$$

$$C_{nm(\lambda)} = \sum_{k=0} C_{nm}{}^{(k)} \lambda^k. \qquad (2.29)$$

Now consider solving $O_{(\lambda)} | \Psi' \rangle = \epsilon'_n | \Psi'_n \rangle$. After manipulating $O_{(\lambda)}$ and $\epsilon_{(\lambda)}$, it gives

$$(O^{(0)} + (\lambda - \lambda_0)O^{(1)} + (\lambda - \lambda_0)^2 O^{(2)} + (\lambda - \lambda_0)^3 F_{(\lambda)}) | \Psi'_n \rangle = (\sum_{k=0} \epsilon_{nk}(\lambda - \lambda_0)^k) | \Psi_n \rangle,$$

$$(2.30)$$

By solving the above equation order by order in $\lambda - \lambda_0$, we will get the following results:

$$\epsilon'^{(0)}_n = \epsilon^0_n \qquad (2.31)$$

$$\epsilon'^{(1)}_n = \langle \Psi_n | O^{(1)} | \Psi_n | \rangle \qquad (2.32)$$

$$\epsilon'^{(2)}_n = \sum_{m \neq n} \frac{| \langle \Psi_n | O^{(1)} | \Psi_m \rangle |^2}{\epsilon^0_n - \epsilon^0_m} + \langle \Psi_n | O^{(2)} | \Psi_n \rangle \qquad (2.33)$$

This completes the proof of Lemma. $\square$

Theorem 2.2.1. *Fröhlich Transformed Hamiltonians are second-order isospectral.*

Proof. Consider the equations (2.22) for two Fröhlich Transformed Hamiltonians H_{eff1} and H_{eff2}. Since they are sum of the few first temrs of unitary transformations, thus they are analytic with respect to the generators of transformations, too. So, Lemma (2.2.1) applies. Note that each of them are equal up to second order to the respective unitary transformation of the initial Hamiltonian H. Therefore, since all the unitary transformations of a the Hamiltonian H are isospectral, H, H_{eff1} and H_{eff2} are all second-order isospectral. Since the generators of Fröhlich Transformations S_1 and S_2, were chosen arbitrarily, all the Fröhlich Transformations of Hamiltonian H are second-order isospectral. This completes the proof. $\square$

In the next subsection we find a resolution for this problem by means of second-order isospectrality.

2.3   Combination of Transformations as an Approximate Picture

In the next chapter, we will apply the mathematical tools reviewed and further developed in this chapter to the case of atomic ensemble in interaction with external fields. There, transforming the Hamiltonian to interaction picture is followed by a time-independent Fröhlich Transformation, which makes the model solvable. We would like to find the relation between time-dependent states that satisfy a Schördinger equation with Fröhlich transformed Hamiltonian in interaction picture and the corresponding Schrödinger picture states. If H^I is Hamiltonian H in the ineraction picture, then from Eq.s (2.18) and (2.21), Fröhlich Transformation of it gives

$$H_{eff}{}^I \;\; = \;\; U^\dagger H^I U + \mathcal{O}(\lambda^3), \tag{2.34}$$

Trivially, for a time-independent Fröhlich Transformation defined by (2.21), U is time-independent. Now, let $|\Psi_{(0)}\rangle_S \;=\; A_{(0)}|\Phi_{(0)}\rangle$ denote the initial state in the Schrödinger picture. Then, because of spectral theorem, one can expand every element $|\Psi\rangle$ of the Hilbert space of interaction picture Hamiltonian H^I by its eigenvectors $|\Psi_n\rangle$:

$$|\Psi\rangle = \sum_n C_n |\Psi_n\rangle. \tag{2.35}$$

Substituting $|\Psi_n\rangle$ from (2.24) we get:

$$\begin{aligned} |\Psi\rangle \;\; &= \;\; \sum_n C_n U |\Psi'_n\rangle \\ &= \;\; U \sum_n C_n |\Psi'_n\rangle, \end{aligned} \tag{2.36}$$

where $|\Psi'_n\rangle$s are exact eigenvectors of H' and approximate eigenvectors of H_{eff}. Starting by classifying the terms in Hamiltonian as in Interaction Picture in the

form $H = H^{iso} + H_t$, let us define a combined unitary transformation $T = U^\dagger e^{iH^{iso}t}$ as below:

$$T_{(t)} : |\Psi_{(t)}\rangle_S \longmapsto |\Psi_{(t)}\rangle_C, \text{ and} \tag{2.37}$$

$$T_{(t)} : O^S_{(t)} \longmapsto O^C = T_{(t)}^\dagger O^S T_{(t)}, \ t > 0, \tag{2.38}$$

$$, \tag{2.39}$$

where S and C are used to denote the same ket or operator in Schrödinger and Combined representations, respectively. Two unitarily equivalent representations indicate the same physics because the expectation value of operators are the same, that is:

$$_S\langle\Psi|O^S|\Phi\rangle_S = {}_C\langle\Psi|O^C|\Phi\rangle_C, \tag{2.40}$$

for all pairs of $|\Psi\rangle_S$, $|\Phi\rangle_C$ and $|\Psi\rangle_S$, $|\Psi\rangle_C$ in $\mathscr{H}$ and connected via (2.39). Let us see if we can find an equation similar to Schrödinger equation (first-order in time) that governs the dynamics of $|\Psi_{(t)}\rangle_C$. Taking the time derivative of it we find the Hamiltonian in the new representation is given as below:

$$i\partial_t|\Psi_{(t)}\rangle_C = H^C|\Psi_{(t)}\rangle_C, \text{ with} \tag{2.41}$$

$$H^C = U^\dagger e^{-iH^{iso}t} H_t e^{iH^{iso}t} U. \tag{2.42}$$

The following Lemma and its corollary gives exact time-evolution of kets in C Picture.

Lemma 2.3.1. *Let* $|\Psi_{(0)}\rangle_C = A_{(0)}|\Phi_{(0)}\rangle \in \mathscr{H}$, *then*

$$|\Psi_{(t)}\rangle_S = e^{-iH^{iso}t}U A_{(-t)}|\Phi_{(t)}\rangle_C, \text{ is the only solution,} \tag{2.43}$$

where H^C *is given by (2.42),* $|\Phi_{(t)}\rangle_C$ *is time-evolution of* $|\Phi_{(0)}\rangle$ *as given by (2.41) and* $A_{(-t)}$ *is determined by the follwing equation:*

$$i\partial_t A_{(-t)} = -[A_{(-t)}, H^C]. \tag{2.44}$$

Proof. Since $|\Psi_{(0)}\rangle_C = A_{(0)}|\Phi_{(0)}\rangle_C \in \mathscr{H}$, with $A_{(0)}$ being an operator in C representation and $|\Phi_{(0)}\rangle_C$ an arbitrary ray in $\mathscr{H}$, then we have:

$$|\Psi_{(t)}\rangle_C = A_{(-t)}|\Phi_{(t)}\rangle_C \qquad (2.45)$$

where,

$$i\partial_t A_{(t)} = [A_{(t)}, H^C] \qquad (2.46)$$

$$\text{and} \quad |\Phi_{(t)}\rangle_C \text{ satisfies (2.41).}$$

To get back the state in Schrödinger representation we perform inverse of the transformation defined in (2.37):

$$|\Psi_{(t)}\rangle_S = T^\dagger_{(t)}|\Psi_{(t)}\rangle_C \qquad (2.47)$$

$$\Rightarrow |\Psi_{(t)}\rangle_S = e^{-iH^{iso}t}UA_{(-t)}U^\dagger e^{iH^{iso}t}|\Phi_{(t)}\rangle_S, \qquad (2.48)$$

with the initial state:

$$|\Psi_{(0)}\rangle_S = UA_{(0)}|\Phi_{(0)}\rangle_S \qquad (2.49)$$

$\square$

Note that any initial state can be written as $A_{(0)}|\Phi_{(0)}\rangle_C$ in the C representation, which indicates the same physical system as $UA_{(0)}|\Phi_{(0)}\rangle_S$. Depending on the choice of T, finding time-evolution in C representation is normally easier wherever interaction picture and Fröhlich Transformations are both useful.

2.3.1 Time-evolution of kets in the approximate picture

A unitary tranformation to interaction picture followed by a Fröhlich Transformation, which equals to U up to second order in the perturbative parameters, is equivalent to affecting T up to second order in perturbative parameters. Since all the equations above from (2.37) to (2.49) hold order by order in purturbative parameters, it means they also hold for a transformation consisting of an interaction picture followed by a Fröhlich Transformation. In other words, if we replace

H^C with Fröhlich Transformation of an Hamiltonian in interaction picture, then (2.45), (2.46), (2.3.1) and (2.49) hold up to second order in perturbative parameters.

Corollary: Let $|\Phi_{(0)}\rangle$ be an energy eigenstate of $H_{eff}{}^I$ the Fröhlich Transformation of Hamiltonian in interaction picture, then

$$|\Psi_{(t)}\rangle_S = e^{-iH^{iso}t}UA_{(-t)}|\Phi_{(0)}\rangle, \tag{2.50}$$

up to second order in perturbation parameters, wherein $A_{(t)}$ satisfies (2.46) with H^C replaced by $H_{eff}{}^I$. In other words, (2.50) is the satisfies the Schrödinger equation with H at time t up to second order in the perturbation parameters. Proof: Let $|\Psi_{(0)}\rangle_C$ in (2.3.1) be an eigenvector of H^C replaced by $H_{eff}{}^I$ in . Then $|\Psi_{(t)}\rangle_C = e^{-i\gamma}|\Psi_{(0)}\rangle_C$ up to the phase $e^{-i\gamma}$. Note that (2.50) is the a unique solution of Schrödinger equation, since as we showed in section (2.2.2) Fröhlich transformations if exist are not unique for each Hamiltonian. But due to theorem (2.2.1) Fröhlich transformed Hamiltonians have unique eigenvalues up to second order and therefore, time-evolution given in (2.50) is unique up to second order in perturbation paramters for any chosen Fröhlich transformation.

Chapter 3

Δ-TYPE ATOMS COUPLED TO QUANTUM AND CLASSICAL FIELDS

3.1 Constructing the Hamiltonian

Single quantum systems with Δ-type transitions have been studied in Refs. [26]-[34]. A quantum systems with Δ-type transitions can be realized by different kinds of physical systems, including atoms whose symmetry is broken by applying a coherent radio frequency field [35]-[38], chiral molecules[30]-[34] and superconducting circuits[39]-[40]. It has been shown that by coupling two photon modes to an ensemble of Δ-type atoms with an electromagnetic field in resonance with transitions between the two excited states, one can swap the quantum information between two modes. Therefore, it provides a general model for physical realization of quantum repeaters and memories [18], which are the key component in implementing large distance quantum communication technologies[41]. The Hamiltonian of a system consisting of one Δ-type atom coupled to two photon modes with frequencies ω_a and ω_b and driven by a monochromatic electromagnetic field with frequency ω_ν, can be written as below[14]:

$$
\begin{aligned}
H &= \omega_a a^\dagger a + \omega_b b^\dagger b + \omega_{ab}\sigma_{aa} + \omega_{cb}\sigma_{cc} + \Omega' e^{-i\omega_\nu t} e^{iK_\nu . \mathbf{r}}\sigma_{ac} + g_a a e^{iK_a . \mathbf{r}}\sigma_{ab} + \\
&\quad g_b b e^{iK_b . \mathbf{r}}\sigma_{cb} + h.c.
\end{aligned}
\tag{3.1}
$$

Later on, we solve the problem for one atomic and many atomic case by requiring resonance condition, that is requiring the frequency of the classical field to be equal to that of energy difference between the excited states, that is $\omega_\nu = \omega_a - \omega_b$ (See Figure (**??**)). The reason for that is we want to achieve maximum coupling between photon fields to achieve maximum rate of state transfer between them.
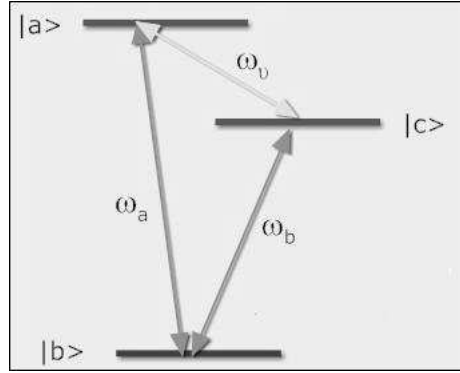
Figure 3.1: A Δ-type atom and the allowed transitions

Hamiltonian for the Atomic Ensemble

Putting an index j on Hamiltonian of jth atom and summing over j gives the Hamiltonian of the Atomic ensemble+photon fields driven by a classical field:

$$
\sum_{j=1}^{N} H^{(j)} = \omega_a a^\dagger a + \omega_b b^\dagger b + \sum_{j=1}^{N} \omega_{ab}\sigma_{aa}^{(j)} + \sum_{j=1}^{N} \omega_{cb}\sigma_{cc}^{(j)} + \sum_{j=1}^{N} \Omega' e^{-i\omega_\nu t} e^{iK_\nu \cdot \mathbf{r}} \sigma_{ac}^{(j)}
$$

$$
+ \sum_{j=1}^{N} g_a a e^{iK_a \cdot \mathbf{r}} \sigma_{ab}^{(j)} + \sum_{j=1}^{N} g_b b e^{iK_b \cdot \mathbf{r}} \sigma_{cb}^{(j)} + h.c. \tag{3.2}
$$

This operator is defined on the space made by tensor product Hilbert space of atoms and Fock space of the two photon fields, that is
$\mathscr{H} = \mathscr{F}_a \otimes \mathscr{F}_b \otimes_{j=1}^{N} \mathscr{H}_j$. And,

$$
\sigma_{xy}^{(j)} = \otimes_{k=1}^{j} I_k \otimes |x_j\rangle\langle y_j| \otimes_{l=j+1}^{N} I_l \otimes I_a \otimes I_b, \tag{3.3}
$$

where N is the number of atoms, I_d is the Identity operator on Hilbert space $\mathscr{H}_d$ of number dth atom, and I_a and I_b are Identity operators on Fock spaces of photon fields. The Hamiltonian above was approximately solved in the interaction picture by first applying Group Contraction method and then finding the limit of infinite Rabi frequency of the classical field, previously [14]. Here, we first give a review of the former results and then in Sec. 3.2 we will solve it by means of a C Transformation as defined in Sec. 3.3.

3.2 Approach I: Group Contraction

We can get rid of time dependence by transferring (3.2) to Interaction Picture and imposing resonance condition. The resonace condition is simply requiring the frequency of classical field to be equal to that of the energy difference of excited states, here a and c. To do so, following Sec. 2.1, we distinguish the terms in (3.1) in the way below:

$$
H^{iso} = \omega_a a^\dagger a + \omega_b b^\dagger b + \sum_{j=1}^{N} \omega_{ab} \sigma_{aa}{}^{(j)} + \sum_{j=1}^{N} \omega_{cb} \sigma_{cc}{}^{(j)} \tag{3.4}
$$

$$
H_t = \sum_{j=1}^{N} \Omega' e^{-i\omega_\nu t} e^{iK_\nu \cdot \mathbf{r}} \sigma_{ac}{}^{(j)} + \sum_{j=1}^{N} g_a a e^{iK_a \cdot \mathbf{r}} \sigma_{ab}{}^{(j)} +
$$

$$
\sum_{j=1}^{N} g_b b e^{iK_b \cdot \mathbf{r}} \sigma_{cb}{}^{(j)} + h.c.. \tag{3.5}
$$

With this division, transforming H to Interaction Picture and requiring the resonance condition $\omega_\nu = \omega_a - \omega_b$ gives the following Hamiltonian:

$$
H = \sum_{j=1}^{N} (\Omega' \sigma_{ac}{}^{(j)} + g_a a \sigma_{ab}{}^{(j)} + g_b b \sigma_{cb}{}^{(j)} + h.c.) \tag{3.6}
$$

Note that, we assume the photon fields are homogeneously coupled to atoms, that is coefficients g_a, g_b is the same for all atoms. This is for reducing decoherence. In his section, in order to solve (3.6), we apply Group Contraction. Following [19], we define a set of collective operators:

$$
T = \sum_{j=1}^{N} \sigma_{ac}{}^{(j)}, \ A = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \sigma_{ba}{}^{(j)}
$$

$$
C = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \sigma_{bc}{}^{(j)}, \ T^z = \sum_{j=1}^{N} (\sigma_{aa}{}^{(j)} - \sigma_{cc}{}^{(j)}), \tag{3.7}
$$

Rewriting the Hamiltonian we get

$$
H = \Omega' T^\dagger + g_a \sqrt{N} a T^\dagger + g_b \sqrt{N} b C^\dagger + h.c. \tag{3.8}
$$

A dynamical symmetry imposed by a Lie algebra lies in the above equation. The Lie algebra can be characterized by generators A, C, T, which in the limit of large N and low exitations, obey the following rules [15]:

$$[A, C^\dagger] = [A, C] = 0, \ [A, A^\dagger] = I, \ [A, T^\dagger] = C, \ [A, T] = 0$$
$$[C, C^\dagger] = I, \ [C, T^\dagger] = 0, \ [C, T] = A, \ [T^\dagger, T] = T^z, \tag{3.9}$$

where $I = \otimes_{l=1}^{N} I_l$ is the identity operator for the Hilbert space of atoms. Low excitations means that the ratio of atoms in the excited state to the total number of atoms is small, which implies that $< C^\dagger C > \ll 1$ and $< A^\dagger A > \ll 1$. Note that since A and C commute, they represent two independent bosonic collective modes. Following [19] we consider the case where coulings of atoms to the photon modes are different only in phase, that is $g_a = g e^{i\phi_a}$, $g_b = g e^{i\phi_b}$. Also we have $\Omega' = \Omega e^{i\zeta}$, with $\zeta = |arg(\Omega')|$ and $\Omega \in \mathcal{R}$. By absorbing these phases in the atomic kets involved in (3.8), and replacing g with $\frac{g_N}{\sqrt{N}}$, we get:

$$H = \Omega e^{i\varphi} T^\dagger + g_N a T^\dagger + g_N b C^\dagger + h.c., \tag{3.10}$$

where $\varphi = \zeta + \phi_c - \phi_a$. To diagonalize (3.10), we write it in terms of Polariton Operators as below:

$$H = \sum_{k=1}^{4} \epsilon_k D_k^\dagger D_k, \ \text{with} \tag{3.11}$$

$$D_{1,2} = \frac{\sin(\theta)}{\sqrt{2}}(a \pm b e^{i\varphi}) + \frac{\cos(\theta)}{\sqrt{2}}(C e^{i\varphi} \pm A), \tag{3.12}$$

$$D_{3,4} = \frac{\cos(\theta)}{\sqrt{2}}(a \pm b e^{i\varphi}) - \frac{\sin(\theta)}{\sqrt{2}}(C e^{i\varphi} \pm A), \tag{3.13}$$

$$and$$

$$\epsilon_1 = -\epsilon_2 = \frac{\Omega + \sqrt{\Omega^2 + 4g_N^2}}{2}, \tag{3.14}$$

$$\epsilon_3 = -\epsilon_4 = \frac{\Omega + \sqrt{\Omega^2 - 4g_N^2}}{2}, \ \text{and} \tag{3.15}$$

$$\theta = arctan\frac{2g_N}{\Omega + \sqrt{\Omega^2 + 4g_N^2}} \tag{3.16}$$

To obtain the above form, note that (3.10) is a bilinear in operators; thus one must look for a linear combination of them as Polariton Operators each with 4 unknown coefficients. By requiring the bosonic relations

$$[D_k, D_k^\dagger] = I, \tag{3.17}$$

$$[D_k, D_j] = [D_k, D^\dagger{}_j] = 0, \text{ for } k \neq j, \tag{3.18}$$

one can obtain 4 equations by substituting (3.10) in the form below:

$$[D_k, H] = D_k, \; k = 1, ..4. \tag{3.19}$$

Then the eigenstates of (3.11) can be written as

$$|\Psi_{lmnk}\rangle = |l, m, n, k\rangle_{D_1, D_2, D_3, D_4} = \frac{1}{\sqrt{l!m!n!k!}} D^\dagger{}_1^l D^\dagger{}_2^m D^\dagger{}_3^n D^\dagger{}_4^k |0\rangle, \tag{3.20}$$

in which $|0\rangle = |0, 0\rangle_{a,b} \otimes |b\rangle$ and $|0, 0\rangle_{a,b}$ is the vacuum state of photon modes and $|b\rangle = \otimes_j^N |b\rangle_j$ the ground state of atoms. Then the eigenvalues can be readily found:

$$\epsilon_{lmnk} = (l - m)\epsilon_1 + (n - k)\epsilon_3. \tag{3.21}$$

3.2.1 Time-Evolution of Fock States

Using the approximate Hamiltonian written in terms of Polariton Operators (3.10), one can find the time-evolution of Polariton Operators:

$$i\partial_t D_k = [D_k, H]$$

$$= \epsilon_k D_k$$

$$\rightarrow D_k = e^{-i\epsilon_k t} D_{k(0)}. \tag{3.22}$$

Let us say we want to find time-evolution of an initially direct product Fock state that is given as below in the Schordinger Pictre:

$$|\Psi_{(0)}\rangle_I = \frac{1}{\sqrt{n_a! n_b!}} a^{\dagger n_a} b^{\dagger n_b} |0, 0\rangle \otimes |b\rangle. \tag{3.23}$$

Since $|0, 0\rangle \otimes |b\rangle$ is an eigenstate of both (3.6) and (3.11), by using Lemma (2.1.1) and its corollary, we can write

$$|\Psi_{(t)}\rangle = \frac{1}{\sqrt{n_a! n_b!}} a^{\dagger n_a}{}_{(-t)} b^{\dagger n_b}{}_{(-t)} |0, 0\rangle \otimes |b\rangle, \tag{3.24}$$

Here, note that the 4 equations in (3.12)-(3.13) hold at all times. So by converting them

$$a = \frac{1}{\sqrt{2}}[(D_1 + D_2)Sin(\theta) + (D_3 + D_4)Cos(\theta)], \tag{3.25}$$

$$b = \frac{1}{\sqrt{2}}[(D_1 - D_2)Sin(\theta) + (D_3 - D_4)Cos(\theta)], \tag{3.26}$$

and substituting $D_{k(t)}$ from (3.22), we can find, operators $a_{(t)}$ and $b_{(t)}$ in terms of $D_{k(t)}$ that are nothing but time dependent phases times $D_{k(0)}$s. The latter can be written back in terms of $a_{(0)}$ and $b_{(0)}$ (by means of (3.12) - (3.13)) to get $a_{(t)}$ and $b_{(t)}$ in terms of $a_{(0)}$, $b_{(0)}$, $A_{(0)}$ and $C_{(0)}$:

$$\begin{aligned} |\Psi_{(t)}\rangle = \ &\frac{1}{\sqrt{n_a! n_b!}}(F^a{}_{a(t)}a^\dagger{}_{(0)} + F^a{}_{b(t)}b^\dagger{}_{(0)} + F^a{}_{A(t)}A^\dagger{}_{(0)} + F^a{}_{C(t)}C^\dagger{}_{(0)})^{n_a} \\ &(F^b{}_{a(t)}a^\dagger{}_{(0)} + F^b{}_{b(t)}b^\dagger{}_{(0)} + F^b{}_{A(t)}A^\dagger{}_{(0)} + F^b{}_{C(t)}C^\dagger{}_{(0)})^{n_b}|0,0\rangle \otimes |b\rangle \end{aligned}$$

$$\tag{3.27}$$

with coefficients F_α found as below [19]:

$$F^a{}_{a(t)} = F^b{}_{b(t)} = \cos(\epsilon_1 t)\sin^2(\theta) + \cos(\epsilon_3 t)\cos^2(\theta), \tag{3.28}$$

$$F^a{}_{b(t)} = F^b{}_{a(t)} = -i(\sin(\epsilon_1 t)\sin^2(\theta) + \sin(\epsilon_3 t)\cos^2(\theta)), \tag{3.29}$$

$$F^a{}_{A(t)} = F^A{}_{a(t)} = -i\sin(\theta)\cos(\theta)(\sin(\epsilon_1 t) - \sin(\epsilon_3 t)), \tag{3.30}$$

$$F^a{}_{C(t)} = F^C{}_{a(t)} = \sin(\theta)\cos(\theta)(\cos(\epsilon_1 t) - \cos(\epsilon_3 t)), \tag{3.31}$$

and

$$F^b{}_{A(t)} = F^A{}_{b(t)} = (\cos(\epsilon_1 t) - \cos(\epsilon_3))\sin(\theta)\cos(\theta), \tag{3.32}$$

$$F^b{}_{C(t)} = F^C{}_{b(t)} = -i\sin(\theta)\cos(\theta)(\sin(\epsilon_1 t) - \sin(\epsilon_3 t)), \tag{3.33}$$

$$F^A{}_{A(t)} = F^C{}_{C(t)} = \cos(\epsilon_1 t)\cos^2(\theta) + \cos(\epsilon_3)\sin^2(\theta), \tag{3.34}$$

$$F^A{}_{C(t)} = F^C{}_{A(t)} = -i(\sin(\epsilon_1 t)\cos^2(\theta) + \sin(\epsilon_3 t)\sin^2(\theta)). \tag{3.35}$$

Note that (3.27) shows that in general photon fields are entangled with all atoms. But we are interested in cases, if any, in which the state of photon fields is separeble

from that of atoms. This in fact can be realized in (3.27) by taking the limit of $|\Omega| \longrightarrow \infty$. Taking the limit $\Omega \longrightarrow +\infty$ we get $\theta = 0$ from (3.16) and

$$|\Psi_{(t)}\rangle = \frac{1}{\sqrt{n_a! n_b!}} (a^\dagger_{(0)} \cos(\epsilon_1 t) - i b^\dagger_{(0)} \sin(\epsilon_1 t))^{n_a} (-i a^\dagger_{(0)} \sin(\epsilon_1 t) b^\dagger_{(0)} \cos(\epsilon_1 t))^{n_b} |0, 0\rangle \otimes |b\rangle.$$

(3.36)

Taking the limit $\Omega \longrightarrow -\infty$ we get $\theta = \frac{\pi}{2}$ from (3.16) and

$$|\Psi_{(t)}\rangle = \frac{1}{\sqrt{n_a! n_b!}} (a^\dagger_{(0)} \cos(\epsilon_3 t) - i b^\dagger_{(0)} \sin(\epsilon_3 t))^{n_a} (-i a^\dagger_{(0)} \sin(\epsilon_3 t) b^\dagger_{(0)} \cos(\epsilon_3 t))^{n_b} |0, 0\rangle \otimes |b\rangle.$$

(3.37)

The results above give the time-evolution of intially direct product state of photon fields and atoms. Note that the condition of infinitely large Rabi frequency made in deriving (3.36) and (3.37) is not physically realizable. Increasing it beyond some point can destroy the sample. In the next part we introduce our approach in tackling this problem, with which we confirm the result found here but without the physically unrealizable assumption.

3.3   Approach II: C Transformation

In order to find the Fröhlich Transformation involved in the C Transformation of the Hamiltonian for the atomic ensemble, it is much easier to first find it for one atomic Hamiltonian and then 'guess' the Fröhlich Transformation for many-body Hamiltonian of the atomic ensemble. Then, theorem (2.2.1) assures the uniqueness.

3.3.1   Solving the Hamiltonian for One Atom

As it was previously defined, a C Transformation consists of Tranformation to Interaction Picture followed by a Fröhlich Transformation. The transformation to Interaction Picture was done for Approach I. Transforming H to Interaction Picture and requiring the resonance condition $\omega_\nu = \omega_a - \omega_b$ led to (3.6). Setting N=1

in (3.6), gives the following Hamiltonian:

$$H = \Omega' \sigma_{ac} + g_a a \sigma_{ab} + g_b b \sigma_{cb} + h.c. \tag{3.38}$$

This operator is defined on the space made by tensor product Hilbert space of atom and Fock space of the two photon fields, that is $\mathscr{H} = \mathscr{F}_a \otimes \mathscr{F}_b \otimes \mathscr{H}$. And,

$$\sigma_{xy} = I_a \otimes I_b \otimes | x \rangle \langle y |, \tag{3.39}$$

where N is the number of atoms, I is the Identity operator on Hilbert space $\mathscr{H}$, and I_a and I_b are Identity operators on Fock spaces of photon fields. Using a Fröhlich Transformation we can approximate the Hamiltonian for $|\Omega'| \gg |g_b|, | g_a |$. Thus, by distinguishing terms in the following way:

$$H^1 = g_a a \sigma_{ab} + g_b b \sigma_{cb} + h.c., \text{ and}$$

$$H^0 = \Omega' \sigma_{ac} + h.c..$$

we can write Eq. (2.20) to find S:

$$g_a a \sigma_{ab} + g_b b \sigma_{cb} + h.c. + [\Omega' \sigma_{ac} + h.c., S] = 0.$$

Here the perturbative parameters are absorbed in S. Considering the equation above as zero operator Θ, each box in the following table is obtained from $\langle x | \Theta | y \rangle$ as projections of Θ onto the Fock space of photons:

		From Right		
		$\mid a \rangle$	$\mid b \rangle$	$\mid c \rangle$
From Left	$\langle a \mid$	$S_{ac} = e^{i\xi} S_{ca}$	$S_{cb} = -\frac{g_a}{\Omega'} a$	$S_{cc} = S_{aa}$
	$\langle b \mid$	$S_{bc} = \frac{g_a^*}{\Omega'^*} a^\dagger$	$0 = 0$	$S_{ba} = \frac{g_b^*}{\Omega'} b^\dagger$
	$\langle c \mid$	$S_{cc} = S_{aa}$	$S_{ab} = -\frac{g_b}{\Omega'^*} b$	$S_{ac} = e^{i\xi} S_{ca}$

$\Omega' = \Omega e^{i\varsigma}$, $\zeta = |arg(\Omega')|$ and $\Omega \in \mathscr{R}$

Note that in the table above Ω can take both positve and negative values. Since S must be anti-Hermitian, there was no need to derive the off-diagonal entries in the table. However, note that these are all the necessary restrictions on operator S. Therefore, we can choose $S_{aa} = 0$, $S_{ca} = 0$ to get:

$$S = -\frac{g_a}{\Omega'}a\sigma_{cb} - \frac{g_b}{\Omega'^*}b\sigma_{ab} - h.c..\tag{3.40}$$

It satisfies (2.20) and from Theorem 2.2.1, we know that all the other effective Hamiltonians that can be derived from other possible choices are second-order isospectral with the effective Hamiltonian obtained by inserting S into (2.22):

$$
\begin{aligned}
H_{eff} &= H_0 + \frac{1}{2}[H_1, S]\\
&= \Omega'\sigma_{ac} + h.c. + \frac{1}{2}[g_a a\sigma_{ab} + g_b b\sigma_{cb} + h.c., -\frac{g_a}{\Omega'}a\sigma_{cb} - \frac{g_b}{\Omega'^*}b\sigma_{ab} - h.c.]\\
&= \Omega'\sigma_{ac} + \Omega'^*\sigma_{ca} + \frac{1}{2}\{[g_a a\sigma_{ab} + g_b b\sigma_{cb} + g_a{}^* a^\dagger\sigma_{ba} + g_b{}^* b^\dagger\sigma_{bc}, -\frac{g_a}{\Omega'}a\sigma_{cb}\\
&\quad -\frac{g_b}{\Omega'^*}b\sigma_{ab}] + h.c.\}
\end{aligned}\tag{3.41}
$$

Avoiding irritation of calculations, in the last step above we exploited the identity $[A + A^\dagger, B - B^\dagger] = [A + A^\dagger, B] + h.c..$ The result is:

$$
\begin{aligned}
H_{eff} &= \Omega'\sigma_{ac} + \Omega'^*\sigma_{ca} + \{\frac{|g_a|^2}{2\Omega'}aa^\dagger\sigma_{ca} + \frac{|g_b|^2}{2\Omega'^*}bb^\dagger\sigma_{ac} + \frac{g_b g_a^*}{2\Omega'^*}a^\dagger b(\sigma_{aa} - \sigma_{bb})\\
&\quad +\frac{g_a g_b^*}{2\Omega'}ab^\dagger(\sigma_{cc} - \sigma_{bb}) + h.c.\}.
\end{aligned}\tag{3.42}
$$

After rearranging terms we get:

$$
\begin{aligned}
H_{eff} &= \Omega'\sigma_{ac} + \Omega'^*\sigma_{ca} + aa^\dagger(\frac{|g_a|^2}{2\Omega'^*}\sigma_{ac} + \frac{|g_a|^2}{2\Omega'}\sigma_{ca}) + bb^\dagger(\frac{|g_b|^2}{2\Omega'^*}\sigma_{ac} + \frac{|g_b|^2}{2\Omega'}\sigma_{ca})\\
&\quad + \left(r'a^\dagger b + r'^* ab^\dagger\right)\left(\sigma_{aa} + \sigma_{cc} - 2\sigma_{bb}\right),
\end{aligned}
$$

$$\tag{3.43}$$

where $r' = \frac{g_b g_a^*}{2\Omega'^*}$. The Hamiltonian is invariant under the change of operators $a \leftrightarrow b$ together with change of the their coupling constants to the atoms $g_a \leftrightarrow g_b$. To make it more clear one can rewrite it in the following form:

$$
\begin{aligned}
H_{eff} &= \left(\Omega' \sigma_{ac} + \Omega'^* \sigma_{ca}\right) + \left(aa^\dagger + bb^\dagger\right)\left(s' \sigma_{ac} + s'^* \sigma_{ca}\right) \\
&\quad + \left(r' a^\dagger b + r'^* ab^\dagger\right)\left(\sigma_{aa} + \sigma_{cc} - 2\sigma_{bb}\right) + \left(aa^\dagger - bb^\dagger\right)\left(s'' \sigma_{ac} + s''^* \sigma_{ca}\right)
\end{aligned}
$$

(3.44)

where $s' = \frac{g^2}{2\Omega'^*}$, $g^2 = \frac{|g_a|^2 + |g_b|^2}{2}$, $s'' = \frac{\Delta^2}{2\Omega'^*}$, $\Delta = \frac{|g_a|^2 - |g_b|^2}{2}$. In fact there are several ways to factorizing common operator factors, but in the above form all the terms commute except the last two. As far as applications in quantum communication are concerned, we are interested in making the two photon modes a and b entangled. And as we will show this can be done by a Hamiltonian without the last term, too. This is equivalent to requiring amplitude of the couplings to be equal, that is $\mid g_a \mid = \mid g_b \mid = g$, which is equivalent to set $\Delta = 0$. This simplifying condition is physically achievable. After setting $\Delta = 0$ we get:

$$
\begin{aligned}
H_{eff} &= \left(\Omega' \sigma_{ac} + \Omega'^* \sigma_{ca}\right) + \left(aa^\dagger + bb^\dagger\right)\left(s' \sigma_{ac} + s'^* \sigma_{ca}\right) + \\
&\quad \left(r' a^\dagger b + r'^* ab^\dagger\right)\left(I - 3\sigma_{bb}\right),
\end{aligned}
$$

(3.45)

where $I = \sigma_{aa} + \sigma_{cc} + \sigma_{bb}$ is identity operator of the Hilbert space of atom. Since all the terms in H_{eff} are commuting, therefore they are simultaneously diagonalizable. Thus, we shall proceed by finding a common basis and the corresponding eigenset for each operator and then adding up the eigenvalues for each state to get the eigenvalues of H_{eff}.

The eigenvectors and corresponding eigenvalues of operator $P = \alpha' \sigma_{ac} + \alpha'^* \sigma_{ca}$

with $\alpha' = \alpha e^{i\zeta}$, $\alpha \in \mathscr{R}$, $\zeta \in [0, \pi)$ are:

$$
\begin{aligned}
|\psi_{-1}\rangle &= \frac{1}{\sqrt{2}}(|c\rangle - e^{i\zeta}|a\rangle), \quad \lambda_{-1} = -\alpha, \\
|\psi_0\rangle &= |b\rangle, \quad \lambda_0 = 0, \\
|\psi_1\rangle &= \frac{1}{\sqrt{2}}(|c\rangle + e^{i\zeta}|a\rangle), \quad \lambda_1 = \alpha,
\end{aligned}
\tag{3.46}
$$

Symbol α' may denote s', or Ω'. Note that $\zeta = |arg(\alpha')|$ and $arg(s') = arg(\Omega')$, with arg(z) giving the principle value of z. The corresponding eigenvalues of operator $(I - 3\sigma_{bb})$ are $\lambda_{\pm 1} = 1$, $\lambda_0 = -2$.

In order to find a mutual basis in which both $(aa^\dagger + bb^\dagger) = (a^\dagger a + b^\dagger b) + 2$ and $(r'a^\dagger b + r'^*ab^\dagger)$ are diagonal, we have to first find the basis in which the latter is diagonal, since it is not diagonal in the number basis in which the number operator is diagonal. We start by writing it in the following way:

$$
\begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} \begin{pmatrix} 0 & r' \\ r'^* & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}
$$

By choosing a unitary transformation $T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -e^{-i\varphi} & e^{-i\varphi} \end{pmatrix}$,

$$
T^\dagger \begin{pmatrix} 0 & r' \\ r'^* & 0 \end{pmatrix} T = \begin{pmatrix} -r & 0 \\ 0 & r \end{pmatrix} \text{ is diagonal, and}
$$

$$
\begin{aligned}
\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= T^\dagger \begin{pmatrix} a \\ b \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} a - be^{i\varphi} \\ a + be^{i\varphi} \end{pmatrix},
\end{aligned}
\tag{3.47}
$$

where $r = \frac{g^2}{2\Omega}$ and $\varphi = arg(r) = arg(\frac{g_b g_a^*}{2\Omega'^*})$ (Note that $r = s$). Rewriting our operators in terms of q_1 and q_2, we get:

$$
\begin{aligned}
(r'a^\dagger b + r'^*ab^\dagger) &= r\left(-q_1{}^\dagger q_1 + q_2{}^\dagger q_2\right) & (3.48) \\
(aa^\dagger + bb^\dagger) &= \left(q_1{}^\dagger q_1 + q_2{}^\dagger q_2 + 2\right). & (3.49)
\end{aligned}
$$

And hence from (3.45) we get:

$$
\begin{aligned}
H_{eff} &= \left(\Omega' \sigma_{ac} + \Omega'^* \sigma_{ca}\right) + \left(q_1{}^\dagger q_1 + q_2{}^\dagger q_2 + 2\right)\left(s' \sigma_{ac} + s'^* \sigma_{ca}\right) \\
&\quad + r\left(-q_1{}^\dagger q_1 + q_2{}^\dagger q_2\right)(I - 3\sigma_{bb}),
\end{aligned}
$$

(3.50)

Here, notice that the following identities hold:

$$
\begin{aligned}
[q_1, q_2] &= 0, \\
\left[q_1, q_2{}^\dagger\right] &= 0, \\
\left[q_1, q_1{}^\dagger\right] &= 1, \\
\left[q_2, q_2{}^\dagger\right] &= 1, \\
\left[q_k, q_k{}^{\dagger n}\right] &= nq_k{}^{\dagger n-1}, \; k = 1, 2
\end{aligned}
$$

(3.51)

Therefore, $\left(q_1{}^\dagger q_1 + q_2{}^\dagger q_2\right)$ is the number operator for the state $\mid \Psi_{lm}\rangle = \frac{q_1{}^{\dagger l} q_2{}^{\dagger m}}{\sqrt{l!m!}} \mid 0, 0\rangle$, where $\mid 0, 0\rangle$ is the vacuum state of tensor product of Fock spaces $\mathscr{F}_a \otimes \mathscr{F}_b$ and $\frac{1}{\sqrt{l!m!}}$ is a normalization factor. In other words:

$$
\begin{aligned}
\left(aa^\dagger + bb^\dagger\right) \mid \Psi_{lm}\rangle &= \left(q_1{}^\dagger q_1 + q_2{}^\dagger q_2 + 2\right) \mid \Psi_{lm}\rangle \\
&= (l + m + 2) \mid \Psi_{lm}\rangle
\end{aligned}
$$

(3.52)

And by using the identities (3.51) we get the desired result:

$$
\begin{aligned}
\left(ra^\dagger b + r^* ab^\dagger\right) \mid \Psi_{lm}\rangle &= \frac{g^2}{2\Omega}\left(-q_1{}^\dagger q_1 + q_2{}^\dagger q_2\right) \mid \Psi_{lm}\rangle \\
&= \frac{g^2}{2\Omega}(m - l) \mid \Psi_{lm}\rangle
\end{aligned}
$$

(3.53)

Consequently, summarizing the results:

$$
B = \{\mid \Psi_{nlm}\rangle \mid \; \mid \Psi_{nlm}\rangle = \mid \psi_n\rangle \otimes \mid \Psi_{lm}\rangle, \; n = -1, 0, 1 \text{ and } l, m \in \mathbb{N}\}
$$

(3.54)

is the basis in which H_{eff} is diagonal and,

$$
\begin{aligned}
E &= \{\epsilon_{nlm} \mid H_{eff} \mid \Psi_{nlm}\rangle = \epsilon_{nlm} \mid \Psi_{nlm}\rangle, \; n = -1, 0, 1 \text{ and } l, m \in \mathbb{N}\}, \\
\text{with } \epsilon_{nlm} &= n|\Omega'| + \frac{g^2}{2\Omega}[n(l + m + 2) + (m - l)(3|n| - 2)],
\end{aligned}
$$

(3.55)

is, according to theorem 2.2.1, the eigenset of the effective Hamiltonian in Interaction Picture up to second order in the perturbative parameter g. However, as it was concluded in Sec. 2.1, these are NOT the approximate eigenset of original Hamiltonian that we started with. But we can apply them to find time-evolution of states.

3.3.2 *Solving the Hamiltonian for Atomic Ensemble*

Here we complete a C Transformation on the Hamiltonian of an atomic ensemble in resonance condition by applying a Fröhlich Transformation on the Hamiltonian in Interaction Picture (3.6). Note that in (3.6), $[H^{(j)}, H^{(k)}] \neq 0$, $j \neq k$. So it is an interacting Hamiltonian and we need to approximate the Hamiltonian.

To find the effective Hamiltonian, we can distinguish two terns in each Hamiltonian as before:

$$H_0^{(j)} = \sum_{j=1}^{N} (\Omega' \sigma_{ac}^{(j)} + h.c.) \tag{3.56}$$

$$H_1^{(j)} = \sum_{j=1}^{N} (g_a a \sigma_{ab}^{(j)} + g_b b \sigma_{cb}^{(j)} + h.c.) \tag{3.57}$$

It can be readily seen that

$$[\sigma_{xy}^{(j)}, \sigma_{uv}^{(k)}] = (\delta_{jk}) (\delta_{uy}\sigma_{xv} - \delta_{xv}\sigma_{uy}), \tag{3.58}$$

Note that:

$$[H^{(j)}, H^{(k)}] = O_{(\lambda^2)} \neq 0, j \neq k, \text{ where } \lambda = g_a, g_b. \tag{3.59}$$

This result means that the Hamiltonian is not invariant under interchange of atoms. But, since the non-commutativity is of second order in perturbative parameter, the first candidate for S is the sum over individual generators $S^{(j)}$, that is:

$$S = \sum_{j=1}^{N} S^{(j)} = -\sum_{j=1}^{N} (\frac{g_a}{\Omega'} a \sigma_{cb}^{(j)} + \frac{g_b}{\Omega'^*} b \sigma_{ab}^{(j)}) - h.c.. \tag{3.60}$$

Interestingly, the sum above satisfies the Froclich condition (2.20) for (3.6):

$$\sum_{l=1}^{N} H_1^{(l)} + [\sum_{j=1}^{N} H_0^{(j)}, \sum_{k=1}^{N} S^{(k)}] = 0. \tag{3.61}$$

This is valid because

$$H_1^{(j)} + [H_0^{(j)}, S^{(k)}] = 0, \text{ if } j = k, \tag{3.62}$$

and the following identity holds:

$$[H_0^{(j)}, S^{(k)}] = 0, j \neq k. \tag{3.63}$$

Then the effective Hamiltonian is:

$$
\begin{aligned}
H_{eff} &= \sum_{j=1}^{N} H_0^{(j)} + \frac{1}{2}[\sum_{l=1}^{N} H_1^{(l)}, \sum_{k=1}^{N} S^{(k)}] \\
&= \sum_{j=1}^{N} H_0^{(j)} + \frac{1}{2}\sum_{j=1}^{N}[H_1^{(j)}, S^{(j)}] + \frac{1}{2}\sum_{l \neq k}^{N}[H_1^{(l)}, S^{(k)}] \\
&= \sum_{j=1}^{N} H_{eff}^{(j)} + \frac{1}{2}\sum_{l \neq k}^{N}[H_1^{(l)}, S^{(k)}],
\end{aligned} \tag{3.64}
$$

where $H_{eff}^{(j)}$ is effective Hamiltonian for one atom given by Eq. (3.50). The last term can be written as below:

$$
\begin{aligned}
\sum_{j \neq k}[H_1^{(j)}, S^{(k)}] &= \sum_{j \neq k}[g_a a \sigma_{ab}^{(j)} + g_b b \sigma_{cb}^{(j)} + g_a{}^* a^\dagger \sigma_{ba}^{(j)} + g_b{}^* b^\dagger \sigma_{bc}^{(j)}, -\frac{g_a}{\Omega'} a \sigma_{cb}^{(k)} \\
&\quad -\frac{g_b}{\Omega'^*} b \sigma_{ab}^{(k)}] + h.c. \\
&= \sum_{j \neq k}(\frac{|g_a|^2}{\Omega'}\sigma_{ba}^{(j)}\sigma_{cb}^{(k)} + \frac{|g_b|^2}{\Omega'^*}\sigma_{bc}^{(j)}\sigma_{ab}^{(k)} + h.c.), \tag{3.65}
\end{aligned}
$$

which gives:

$$
\begin{aligned}
H_{eff} &= \sum_{j=1}^{N}(\Omega'\sigma_{ac}^{(j)} + \Omega'^{*}\sigma_{ca}^{(j)}) + \left(q_1^{\dagger}q_1 + q_2^{\dagger}q_2 + 2\right)\sum_{j=1}^{N}(s'\sigma_{ac}^{(j)} + s'^{*}\sigma_{ca}^{(j)}) \\
&\quad + r\left(-q_1^{\dagger}q_1 + q_2^{\dagger}q_2\right)\left(NI - 3\sum_{j=1}^{N}\sigma_{bb}^{(j)}\right) \\
&\quad + \frac{1}{2}\sum_{l\neq k}\left(\frac{|g_a|^2}{\Omega'}\sigma_{ba}^{(l)}\sigma_{cb}^{(k)} + \frac{|g_b|^2}{\Omega'^{*}}\sigma_{bc}^{(l)}\sigma_{ab}^{(k)} + h.c.\right) \\
&= \sum_{j=1}^{N}(\Omega'\sigma_{ac}^{(j)} + \Omega'^{*}\sigma_{ca}^{(j)}) + \left(q_1^{\dagger}q_1 + q_2^{\dagger}q_2 + 2\right)\sum_{j=1}^{N}(s'\sigma_{ac}^{(j)} + s'^{*}\sigma_{ca}^{(j)}) \\
&\quad + r\left(-q_1^{\dagger}q_1 + q_2^{\dagger}q_2\right)\left(NI - 3\sum_{j=1}^{N}\sigma_{bb}^{(j)}\right) + \frac{1}{2}\sum_{l\neq k}\left(\frac{|g_a|^2}{\Omega'}\sigma_{ba}^{(l)}\sigma_{cb}^{(k)} + \frac{|g_a|^2}{\Omega'^{*}}\sigma_{ab}^{(l)}\sigma_{bc}^{(k)}\right) \\
&\quad + \frac{1}{2}\sum_{l\neq k}\left(\frac{|g_b|^2}{\Omega'}\sigma_{ba}^{(l)}\sigma_{cb}^{(k)} + \frac{|g_b|^2}{\Omega'^{*}}\sigma_{ab}^{(l)}\sigma_{bc}^{(k)}\right),
\end{aligned}
\tag{3.66}
$$

where $I = \otimes_{j=1}^{N}I_j$ is the identity operator of the atomic ensemble Hilbert space. We used $H^{(j)}$ with $\Delta = 0$. Therefore, by substituting $|g_a| = |g_b| = g$ and $s' = \frac{g^2}{2\Omega'^{*}}$ as before, we get:

$$
\begin{aligned}
H_{eff} &= \sum_{j=1}^{N}\left(\Omega'\sigma_{ac}^{(j)} + \Omega'^{*}\sigma_{ca}^{(j)}\right) + \left(q_1^{\dagger}q_1 + q_2^{\dagger}q_2 + 2\right)\sum_{j=1}^{N}\left(s'\sigma_{ac}^{(j)} + s'^{*}\sigma_{ca}^{(j)}\right) \\
&\quad + r\left(-q_1^{\dagger}q_1 + q_2^{\dagger}q_2\right)\left(NI - 3\sum_{j=1}^{N}\sigma_{bb}^{(j)}\right) + 2\sum_{l\neq k}\left(s'\sigma_{ab}^{(l)}\sigma_{bc}^{(k)} + s'^{*}\sigma_{ba}^{(l)}\sigma_{cb}^{(k)}\right).
\end{aligned}
\tag{3.67}
$$

The above Hamiltonian consists of 4 Hermitian terms. And the good news is that with the condition they do commute with each other and thus they are simultaneously diagonalizable!! Proof is straightforward but we would like to clarify the physical picture behind it. First let us understand the meaning of each term. The first three terms involve equivalent first order atomic processes between states $|a\rangle$ and $|c\rangle$ of each atom, while the last term involves second order processes between $|a\rangle$ and $|c\rangle$ of different atoms. Note that the very reason that atomic part of the first two terms and the last term in (3.66) do commute is that they invovle processes between the same two states $|a\rangle$ and $|c\rangle$, and second, the phase difference

between the "Go" processes $\sigma_{ca}{}^{(j)}$ and $\sigma_{ba}{}^{(l)}\sigma_{cb}{}^{(k)}$ and the "Return" processes $\sigma_{ac}{}^{(j)}$ and $\sigma_{ab}{}^{(l)}\sigma_{bc}{}^{(k)}$ that are involved in them are the same. An important consequence of simultaneously diagonalizability of the terms in (3.67) is that all the eigenstates are direct product of appropriate bases of $\mathscr{F}_a \otimes \mathscr{F}_b$ with those of $\otimes_{j=1}^{N}\mathscr{H}_j$. As we will see this disentangles the time-evolution of the subsystem consisting of photon fields from evolution of the atomic ensemble, when the latter is initially in an energy eigenstate.

3.3.3 Time-Evolution of Fock States

Consider the subsystem of photon modes to be initially in Fock state, i.e., the number state and the state of atomic part be in one of the energy eigenstates of (3.66) as given below:

$$| \Psi_{(0)}\rangle = \frac{a^{\dagger n_a}b^{\dagger n_b}}{\sqrt{n_a!n_b!}} \, | \Psi_{\vec{\pi}00}\rangle, \tag{3.68}$$

for some n_a, $n_b \in \mathbb{N}$. In order to find $|\Psi_{(t)}\rangle_S$, the evolved state in the Schrödinger picture, we first find $|\Psi_{(t)}\rangle_I$, the evolved state in the Interaction Picture, and then transform it by an inverse-Fröhlich Transformation to the state prior to Fröhlich Transformation and then by another unitary transformation to the Schrödinger picture if necessary. According to Lemma 2.3.1,

$$|\Psi_{(t)}\rangle_I = \frac{(a^{\dagger n_a}b^{\dagger n_b})_{(-t)}}{\sqrt{n_a!n_b!}} \, | \Psi_{\vec{\pi}00(t)}\rangle_I. \tag{3.69}$$

Here since $| \Psi_{\vec{\pi}00(0)}\rangle_I$ is an eigenvector of H_{eff} (3.45), therefore, $| \Psi_{\vec{\pi}00(t)}\rangle_I$ indicates the same ray as $| \Psi_{\vec{\pi}00(0)}\rangle$. From (3.47) we get:

$$a^\dagger = \frac{q_2{}^\dagger + q_1{}^\dagger}{\sqrt{2}} \tag{3.70}$$

$$b^\dagger = e^{-i\varphi}\frac{q_2{}^\dagger - q_1{}^\dagger}{\sqrt{2}}, \tag{3.71}$$

which hold at all times. We will need two more results:

Lemma 3.3.1. *Let A and B be two operators defined on the same Hilbert space, then $(AB)_{(t)} = A_{(t)}B_{(t)}$.*

Proof. $A_{(t)} = e^{-iHt}Ae^{iHt}$ and $B_{(t)} = e^{-iHt}Be^{iHt}$, implies:

$$
\begin{aligned}
(AB)_{(t)} &= e^{-iHt}ABe^{iHt} \\
&= A_{(t)}B_{(t)}.
\end{aligned}
\tag{3.72}
$$

$\square$

Lemma 3.3.2. *If $A_{(t)}$ satisfies a Heisenberg Eq. (??) with $A_{(t)} = A$, then $(A^n)_{(t)} = (A_{(t)})^n$ satisfies the same Heisenberg equation.*

Proof. The proof is trivial. $A_{(t)} = e^{-iHt}Ae^{iHt}$ implies $A_{(t)}{}^n = e^{-iHt}A^n e^{iHt}$. $\square$

Therefore, the following holds:

$$
(a^{\dagger n_a}b^{\dagger n_b})_{(-t)} = a^{\dagger n_a}{}_{(-t)}b^{\dagger n_b}{}_{(-t)}
\tag{3.73}
$$

Now, we need to find reverse-time-evolution of $q_{1,2}$, by applying Lemma 2.3.1:

$$
I \otimes i\partial_t q_j = -[I \otimes q_j, H_{eff}], j = 1, 2,
\tag{3.74}
$$

where I is the identity operator for atomic Hilbert space and H_{eff} is given by (3.66). Also, the following identities can be derived by using (3.51):

$$
[q_j, (q_1{}^\dagger q_1 + q_2{}^\dagger q_2)] = q_j
\tag{3.75}
$$

$$
[q_j, (-q_1{}^\dagger q_1 + q_2{}^\dagger q_2)] = (-1)^j q_j, j = 1, 2
\tag{3.76}
$$

Using H_{eff} in the form given by (3.66) and the identities above, the right side of (3.74) can be found:

$$
-[I \otimes q_j{}^\dagger, H_{eff}] = \sum_{j=1}^{N}(s'\sigma_{ac}{}^{(j)} + s'^*\sigma_{ca}{}^{(j)})q_j{}^\dagger + \left(NI - 3\sum_{j=1}^{N}\sigma_{bb}{}^{(j)}\right)r(-1)^j q_j{}^\dagger.
$$

$$
\tag{3.77}
$$

Tracing out the atomic part we get:

$$\partial_t q_j{}^\dagger = i((-1)^j\phi - \beta)q_{j(t)}{}^\dagger, j = 1, 2,$$

$$\phi = Nr = \frac{Ng^2}{2\Omega}$$

$$= \frac{g^2 N}{2\Omega}, \text{ and } \beta = \frac{Nr}{3^{N-1}}$$

$$\Rightarrow q_j{}^\dagger_{(t)} = q_{j(0)}{}^\dagger e^{i((-1)^j\phi - \beta)t}$$

Putting these back to (3.70) and replacing $q^\dagger{}_{j(0)}$ from (3.47) in terms of $a^\dagger{}_{(0)}$ and $b^\dagger{}_{(0)}$ we will have:

$$a^\dagger_{(t)} = \frac{q_{2(0)}{}^\dagger e^{i(\phi - \beta)t} + q_{1(0)}{}^\dagger e^{i(-\phi - \beta)t}}{\sqrt{2}}$$

$$= e^{-i\beta t}(a^\dagger{}_{(0)}Cos(\phi t) + ie^{-i\varphi}b^\dagger{}_{(0)}Sin(\phi t)) \qquad (3.78)$$

$$b^\dagger{}_{(t)} = e^{-i\varphi}\frac{q_{2(0)}{}^\dagger e^{i(\phi - \beta)t} - q_{1(0)}{}^\dagger e^{i(-\phi - \beta)t}}{\sqrt{2}}$$

$$= e^{-i\beta t}(ie^{-i\varphi}a^\dagger{}_{(0)}Sin(\phi t) + e^{-2i\varphi}b^\dagger{}_{(0)}Cos(\phi t)) \qquad (3.79)$$

It is important to note that the lemma is also valid for time-evolution of any state resulting by a time-independent transformation, including the time-independent Fröchlich Transformation we performed. Inserting these above in (3.69) gives us what we are after (irrespective to phase):

$$|\Psi_{(t)}\rangle = \frac{(a^\dagger_{(0)}Cos(\phi t) + ie^{-i\varphi}b^\dagger_{(0)}Sin(\phi t))^{n_a}(ia^\dagger{}_{(0)}Sin(\phi t) + e^{-i\varphi}b^\dagger{}_{(0)}Cos(\phi t))^{n_b}}{\sqrt{n_a!n_b!}}|\Psi_{n00(0)}\rangle.$$

$$(3.80)$$

In fact, the Lie algebra associated with b and its commutation with a remains unchanged if we multiply it with a phase such as $-e^{i\varphi}$. Therefore, we can rewrite (3.80) in the following form:

$$|\Psi_{(t)}\rangle = \frac{(a^\dagger{}_{(0)}Cos(\phi t) - ib^\dagger{}_{(0)}Sin(\phi t))^{n_a}(ia^\dagger{}_{(0)}Sin(\phi t) - b^\dagger{}_{(0)}Cos(\phi t))^{n_b}}{\sqrt{n_a!n_b!}}|\Psi_{n00(0)}\rangle.$$

$$(3.81)$$

Note that in order to get the state in Schrödinger picture, we have to unitarily transform $|\Psi_{(t)}\rangle_I$ back, which means affecting the reverse of two affected subsequent unitary transformations or inverse C Transformation as defined in (2.37), in the following way:

$$|\Psi_{(t)}\rangle_S = e^S e^{-iH^{iso}t} A_{(-t)}|\Phi_{(0)}\rangle_I \tag{3.82}$$

The equation above is the same form of time-evolution found in [19] for the case $\frac{\Omega}{g^2} \longrightarrow +\infty$ with ϕ up to second order in perturbative parameter g (In the paper g is defined as $\frac{g_0}{\sqrt{N}}$, which gives $\phi = \frac{g_0^2}{2\Omega}$). As we showed in the previous section, in their paper, Li, et. al. [19], have imposed the condition $\frac{\Omega}{g_0^2} \longrightarrow \infty$ on the atomic part of the operators acting upon the ground state, and by doing so they have disetangled the photonic state vector from the atomic state vector. But first of all this is invalid since then ϕ becomes infinite too. Also, as we have shown, the condition $\frac{\Omega}{g_0^2} \longrightarrow \infty$ is not necessary to get the same time-evolution. The only requirement here is that $\frac{\Omega}{g^2}$ must be larger than one, in order for the approximate time-evolution to be identical to the exact time-evolution up to first order in $\frac{\Omega}{g^2}$ (e.i, second order in perturbation parameter g). In fact, the infinite coupling constant required by [19] is physically unachievable too, since the laser beam may destroy the sample if Ω becomes large enough.

3.3.4 Evolution of Entanglement between Initially Direct Product Fock States

We found that if the atomic ensemble is initially in an energy eigenstate, then the future state is given by (3.80). However, although in the interaction picture the photon states remain disentagled from the atomic ensemble up to second order in the perturbative parameter, but after t=0, the initially separable Fock states become non-separable and thus entangled. Since (3.80) is a pure state, we can quantify its entanglement by von Neumann entropy of either photonic modes [10], [11], [12] as below:

$$E = -Tr(\rho_a Log_2(\rho_a)), \qquad (3.83)$$

where with $\rho_{(t)} = |\Psi_{(t)}\rangle\langle\Psi_{(t)}|$ denoting the density matrix, ρ_a is the density matrix of photon field a and is obtained by taking partial trace over basis of $\mathcal{H}_b$:

$$\rho_{a(t)} = Tr_b(|\Psi_{(t)}\rangle\langle\Psi_{(t)}|). \qquad (3.84)$$

A note on periodicity of Entanglement

$q_{j(t)}$s are periodic operators with period of $\frac{2\pi|\Omega|}{g^2}$, therefore, independent from the number of initial photons in each mode, it seems that at times $t_\nu = 2\pi\nu\frac{|\Omega|}{g^2}$ ($\nu \in \mathbb{N}$) entanglement becomes zero, that is the state becomes equal to the initial seperable one, and the dynamics starts over. But in fact the period is shorter! To see this, let us take the phase i inside $b^\dagger$ by redefining b as $-ib$ in (3.81). Then we get:

$$|\Psi_{(t)}\rangle = \frac{(a^\dagger_{(0)}Cos(\phi t) - b^\dagger_{(0)}Sin(\phi t))^{n_a}(a^\dagger_{(0)}Sin(\phi t) + b^\dagger_{(0)}Cos(\phi t))^{n_b}}{\sqrt{n_a!n_b!}}|\Psi_{n00(0)}\rangle.$$

$$(3.85)$$

The form above suggests that $\rho_{(t+\tau)} = \rho_{(t)}$, where $\tau = \pi\frac{|\Omega|}{g^2}$. During this time, we observe that state of field a is swapped with state of field b. Note that in contrast to [19], N is a variable in the final result we found here. The periodicity is inversely proportional to N and might be adjusted depending on the application.

Time-Evolution of entanglement - Initial state $|1, 1\rangle$:

Substituting $|\Psi_{(t)}\rangle$ from (3.80) we will get:

$$\rho_{a(t)} = Cos^2(2\phi t)|1\rangle\langle 1| + \frac{1}{2}Sin^2(2\phi t)\left(|0\rangle\langle 0| + |2\rangle\langle 2|\right).$$

$$(3.86)$$

Substituting the above in (3.83), one can find time-evolution of entropy for initial state $|1, 1\rangle$:

$$E = -Cos^2(2\phi t)Log_2(Cos^2(2\phi t)) - Sin^2(2\phi t)Log_2(\frac{1}{2}Sin^2(2\phi t))$$

$$(3.87)$$

Figure 3.2 shows the function above, and Figures 3.3, 3.4 and 3.5 show it for other initial states.

Time-Evolution of entanglement - Other initial state states:

In order to find the exact time-evolutions for initial states $|1, 2\rangle$ and $|2, 2\rangle$, since the equations get very long we have used symbolic computations in Matlab that can be found in Appendix A.
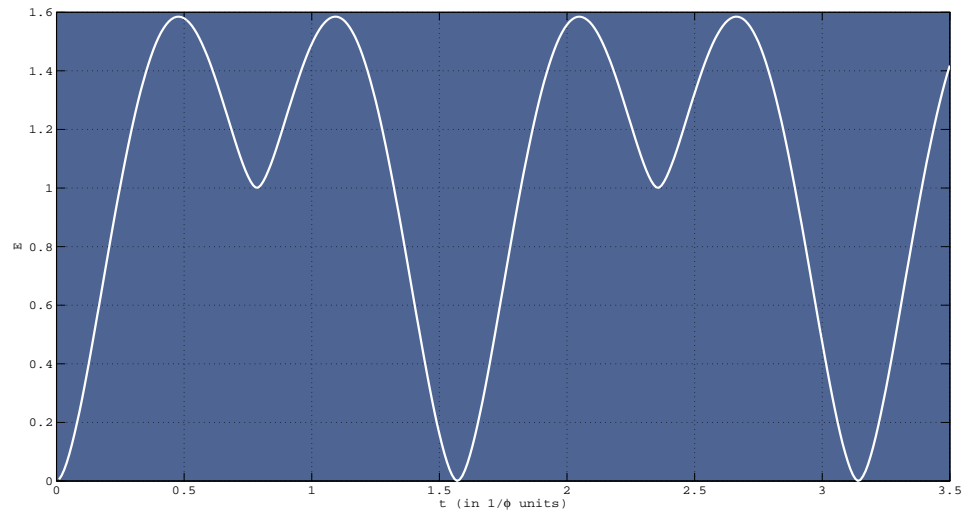
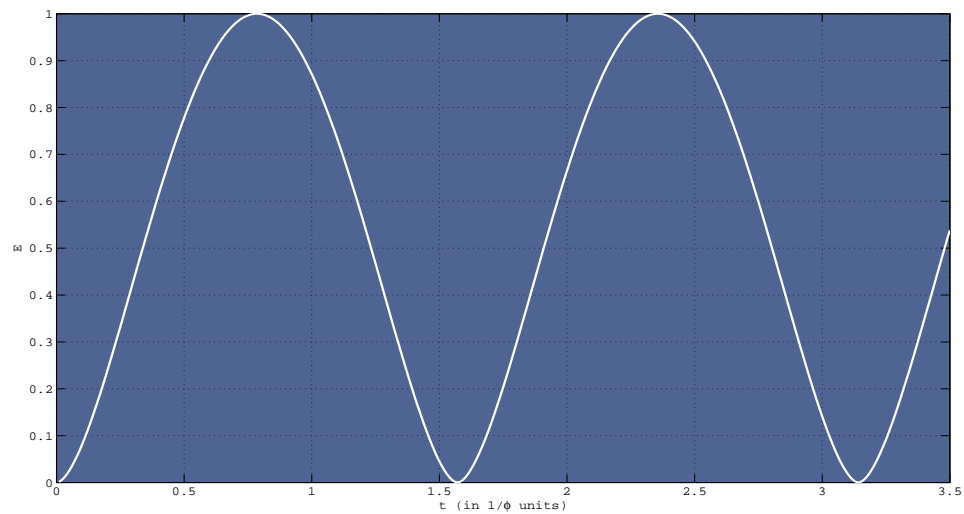Figure 3.2: Time-Evolution of Entanglement (von Neumann Entropy) for initial state $|1, 1\rangle$

Figure 3.3: Time-Evolution of Entanglement (von Neumann Entropy) for initial state $|0, 1\rangle$
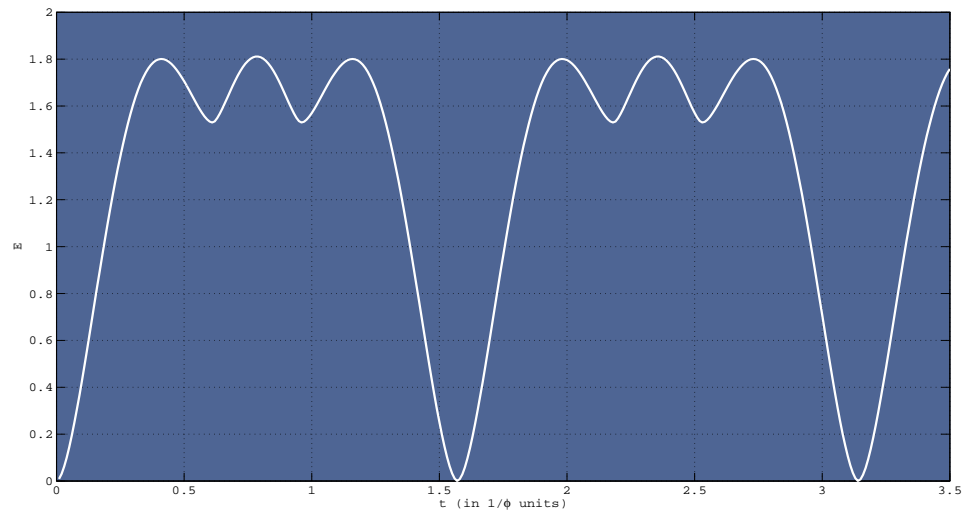
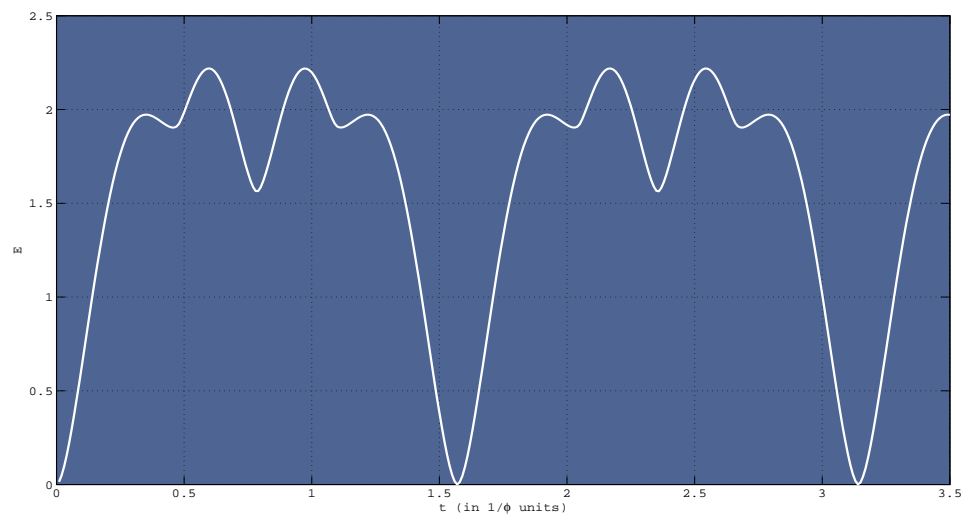Figure 3.4: Time-Evolution of Entanglement (von Neumann Entropy) for initial state$|1, 2\rangle$

Figure 3.5: Time-Evolution of Entanglement (von Neumann Entropy) for initial state$|2, 2\rangle$

Chapter 4

CONCLUSION

We introduced the notion of second-order isospectrality and developed the theory of Fröhlich transformations. So that in general case and especially for approximating many-body Hamiltonians, where there are many possible Fröhlich transformations, one can choose the simplest generator of transformation (S) and be sure that the eigenvalues would be the same with any other more complicated choice. Then we studied Fröhlich transformation as a combined transformation. We reviewed the previous studies done on interaction of quantized optical fields with ensemble of Δ-type quantum systems with the aim of group contraction. Finally, we applied our method developed based on the approximation scheme introduced. Note that Eq. (3.80) is equivalent to the time-evolution found with prevous approach (3.36) and (3.37) when the condition $\Omega \ll g$ is fully taken into consideration in time-evolution found in [19]. For positive Ω, $\epsilon_1 \longrightarrow \phi$ and for negative Ω, $\epsilon_3 \longrightarrow \phi$. But on the other hand, in our method we never assumed $\frac{\Omega}{g}$ to be infinitely large. We derived a decoupled time-evolution of the quantized fields from the atoms (3.80) with only assuming that $\Omega \ll g$. Moreover, note that in our method that lead to (3.80) we never assumed that N, the number of atoms in the ensemble, is infinitely large comparing to the number of excitations. So our method gave a result which is valid for any number of Δ-type quantum systems in the ensemble. N is a variable in our result, which means that it can be adjusted depending on the application. In fact, by increasing N periodicity decreases (3.85), thus using our result we can design an ensemble with enough big N such that the state will be transferred from one field to the other before an effective time after which decoherence happens. This work is finished on its own, but leaves practical

questions which depends on the particular system being considered. For example calculating the decoherence time. The latter depends on the properties of the ensemble being used and the type of environment. Moreover, one can apply our method to defferent kinds of systems. As we saw group contraction method is relied on the assumption that N is huge comparing to the number of atoms in the axcited state. In fact, the group contraction method is previously applied to other physical systems including ensemble of Λ-type quantum systems [42] in interaction with external fields, many-spin ensemble excitations [43], and other situations such as [44] and [45]. Their results is limited to the cases where N is huge. While we propose that applying out method, one can relax this condition and find results that are valid for any number of atoms per excitations.

Appendix A

CODES FOR SYMBOLIC COMPUTATIONS

The computations that lead to time-evolutions for initial conditions $|1, 2\rangle$ in Figure 3.4 and $|2, 2\rangle$ in Figure 3.2 were done by means of sybolic computations in Matlab. We supply the code as it can be used for further work. We partially used QOT package (quantum optics toolbox) that can be downloaded for free and must be added to the path in the begining of the code instead of 'qot address'. The code is not made for one-run and must be run step by step, according to the guidlines added below.

```
% This code is intended to find exact time evolution of two photon
% fields coupled to a cyclic atomic ensemble. Then the von Neumann
% entropy of one field is found as a measure of entanglement

% n_a is Number of photons in field a
% n_b is Number of photons in field b


clear;

%Set the path for qot folder below
addpath('qot_address');

% Set n_a, n_b the number of initial photons
n_a=1;
```

```matlab
n_b=2;


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


l=0;
k=0;
N=m+n+1;
aob=create(N);
id=eye(N);
a=full(aob(:,:));
am=kron(a,id);
bm=kron(id,a);
psi1=id(l+1,1:N).';
psi2=id(k+1,1:N).';
rhoi=kron(psi1,psi2)*kron(psi1,psi2)';


syms s;
u=(cos(s).*am-1i*sin(s).*bm)^m * (-1i*sin(s).*am+cos(s).*bm)^n
/sqrt(factorial(m)*factorial(n));
rhof=u*rhoi*u';
rhof1=0*id(N^2);
for r=1:N
    for p=1:N
        for t=1:N
            rhof1=rhof1+(kron(id(r,1:N),id(t,1:N))*rhof*kron(id(p,1:N)
                        id(t,1:N))') * (id(r,1:N)'*id(p,1:N));
        end
    end
end
```

%IN ORDER TO CANCELL OUT ANY REMAINED EXPRESSIONS WTIH i ,

%or NEGATIVE SIGNS THAT MAY BE INTERPRETED WRONGLY BY MATLAB:

```
diagvec1=diag(simplify(rhof1));
```

%At this point, by the input below Matlab will give a symbolic

%expression for ent:

```
ent=−diagvec1'*log2(diagvec1)
```

%Copy this and paste it inside of ent in the line below and continue

%the rest. By doing this, stars and slashes will be replaced by

%.∗ and ./, which are required by for drawing in Matlab.

```
  st='ent';

z=0;
for j=1:length(st)
    if (st(j)=='^')
        stf(j+z:j+z+1)='.^';
        z=z+1;
    elseif (st(j)=='*')
        stf(j+z:j+z+1)='.*';
        z=z+1;
    elseif (st(j)=='/')
        stf(j+z:j+z+1)='./';
        z=z+1;
    else
        stf(j+z)=st(j);
    end;
```

end

stf

```
%At this point , Matlab will give a modified output . Copy and paste it
% in front of @( s ) in the line below and end it with a semicolon .

Entropy = @( s )              ;
% And finally in order to draw the plot :

x = 0:.01:3.5;
plot ( x , Entropy ( x ))
```

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