NON-ABELIAN MAGNETIC MONOPOLES AND DYONS

by

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This is to certify that I have examined this copy of a master's thesis by

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and that any and all revisions required by the final

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ABSTRACT

We first review Dirac's Abelian magnetic monopole solutions in Maxwell's electrodynamics. Then after a short discussion of the unified electroweak gauge field theories and the Higgs mechanism for mass generation we derived the non-Abelian magnetic monopole solutions of 't Hooft and Polyakov. We discussed the topological nature of such solutions. We calculated the total mass and the magnetic charge of these solutions and verified that they saturate the BPS bound. Dyon solutions are also given. Finally, we briefly comment on Montonen and Olive's non-Abelian duality conjecture.

ÖZET

Öncelikle Maxwell elektrodinamiği kapsamında Dirac'ın Abelyen manyetik monopol çözümlerini gözden geçirdik. Birleşik elektrozayıf ayar alan teorilerini ve kütle yaratımını sağlayan Higgs mekanizmasını kısaca tartıştıktan sonra 't Hooft ve Polyakov'un Abelyen-olmayan manyetik monopol çözümlerini çıkardık. Bu çözümlerin topolojik niteliklerine dikkat çektik. Taşıdıkları toplam kütle ve manyetik yükü hesaplayarak BPS sınırına ulaştıklarını kanıtladık. Dyon genellemelerini de verdik. Son olarak Montonen ve Olive'in Abelyen-olmayan dualite sanıtına kısaca değindik.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

INTRODUCTION

Maxwell's electromagnetic field equations unify the electric and magnetic phenomena that occur in the Nature. One of the laws of Maxwell which carries no name, may in fact be called Gauss' law for magnetism and states that there are no isolated magnetic monopoles. Maxwell's field equations are not symmetric in this respect. There are electric charges in nature but their counterparts, magnetic monopoles have never been observed yet [9]. On the other hand, in theoretical physics magnetic monopoles arise in many different contexts such as quark confinement, the problem of proton decay, in astrophysics and during the early evolution of the universe [8]. In Chapter:2, we start by showing the electric-magnetic duality symmetry of Maxwell equations in the absence of any source, because Dirac's very first idea of a magnetic monopole is closely connected with electric-magnetic duality symmetry in source-free electrodynamics. After defining the duality transformations in a compact form by composing electric and magnetic fields into a complex 6-vector we end up with an interchange of electric and magnetic fields. Energy-momentum tensor densities are invariant under a global duality transformation. In Chapter:2, we also describe the Dirac monopole solution [1]. Dirac realized in 1931 that symmetry in Maxwell's equations is retrieved when hypothetical magnetic charges and currents are included. To do this, he first considered a point-like magnetic charge in a Coulomb like magnetic field and then introduced the vector potential on the northern and southern hemispheres of the Gauss sphere separately. These vector potentials are connected by a gauge transformation along the equator of the sphere and implies Dirac's quantization condition thus proving electric charge quantization provided even a single magnetic monopole actually exists. Dirac's quantization condition is generalized later to dyons by Schwinger [3],[4] and Zwanziger [5]. In Chapter:3, we discuss the unification of electromagnetic and weak interactions in a gauge field theory known as electroweak theory. In particular the Georgi-Glashow model [14] which exhibits mass generation by spontaneous symmetry breakdown through the Higgs mechanism is discussed because this model possesses simple monopole-like solutions. In Chapter:4, non-Abelian magnetic monopole solutions discovered by 't Hooft [23] and Polyakov [24] are given that follow from a static , spherically symmetric ansatz [22] for the independent field variables. We discuss explicit solutions at the BPS limit. In this special case we derive the first order Bogomol'yni equations and obtain the BPS bound for the mass of the monopole. Generalization to non-Abelian dyons is taken up in Chapter:5. The same method of construction as for 't Hooft-Polyakov monopoles is followed in the case dyons and we again get a BPS bound for dyons. In the last Chapter:6, we introduce the electric-magnetic duality conjecture proposed by Montonen and Olive [30] according to which there should be two equivalent formulations of the same theory that are dual to each other where Noether-type electric and (topological) magnetic charges would be exchanged. We also comment on Witten's θ-term [34] added to the Lagrangian density of Yang-Mills-Higgs theory that allows the derivation of Dirac's quantization condition by a complex shift in the monopole sector of the theory. Montonen-Olive duality conjecture may be extended under the duality and projective transformations through this surface term. This extended version of the Montonen-Olive duality conjecture is called "S-duality" in the more recent literature. Chapter:7 covers a brief conclusion and a discussion of further questions.

Chapter 2

MAXWELL EQUATIONS AND DIRAC MONOPOLE

2.1 Maxwell Equations And Electromagnetic Duality

We start by considering the Maxwell's equations satisfied by an electric field $\vec{E}$ and a magnetic field $\vec{B}$ given in natural units ($c = 1$ and $\hbar = 1$) below:

$$\vec{\nabla} \cdot \vec{E} = \rho \quad , \quad \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J}$$
$$\vec{\nabla} \cdot \vec{B} = 0 \quad , \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0. \tag{2.1}$$

In differential form notation, a p-form on a manifold M is a totally antisymmetric covariant tensor

$$\omega = \frac{1}{p!}\omega_{i_1 i_2 \dots . i_p} dx_1^i \wedge dx_2^i \wedge \dots\dots \wedge dx_p^i. \tag{2.2}$$

Thus we introduce the magnetic field 2-form on the 3-dimensional Euclidean space

$$B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \tag{2.3}$$

and the electric field 1-form

$$E = E_x dx + E_y dy + E_z dz \tag{2.4}$$

and the homogeneous set of Maxwell's equations in differential form language read

$$dE + \dot{B} = 0 \; , \qquad dB = 0 \tag{2.5}$$

where $\cdot$ over a symbol denotes the time derivative $\frac{\partial}{\partial t}$ and d is the exterior derivative. Furthermore, combining E and B in the electromagnetic field 2-form over space-time as

$$F = B + E \wedge dt, \tag{2.6}$$

we may simplify the homogenous Maxwell equations to

$$dF = dB + (\dot{B} + dE) \wedge dt = 0. \tag{2.7}$$

To work out the inhomogeneous Maxwell's equations in differential form language we also need the Hodge star operation $* : \Omega^p(M) \rightarrow \Omega^{n-p}(M)$, where M is an n-manifold and Hodge star operation is a linear map from p-forms to $(n-p)$-forms such that for ω above, we have

$$*\omega = \frac{1}{p!(n-p)!} w_{i_1 i_2 \dots i_p} \epsilon^{i_1 i_2 \dots i_n} e^{p+1} \wedge \dots . \wedge e^n. \tag{2.8}$$

We can write F in component form in terms of Cartesian space-time coordinates x^μ : (t, x, y, z) as

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \tag{2.9}$$

so that the Hodge star of F becomes

$$*F = \frac{1}{4} F_{\mu\nu} \, \epsilon_{\lambda\rho}{}^{\mu\nu} dx^\lambda \wedge dx^\rho \tag{2.10}$$

where $\epsilon_{\mu\nu\lambda\rho}$ is the totally antisymmetric Levi-Civita symbol on space-time with $\epsilon_{0123} = 1$. We use $F_{k0} = E_k$, $F_{ij} = \epsilon_{ijk} B_k$ and write the Cartesian components of F and $*F$ in matrix form as

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad , \quad *F_{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}.$$

Inhomogeneous Maxwell equations contain a current density $\vec{j}$ and charge density ρ as well. In differential form language we introduce the current density 1-form on the Euclidean 3-space

$$j = j_x dx + j_y dy + j_z dz \tag{2.11}$$

and combine ρ and $\vec{j}$ into the space-time current density 1-form

$$J = j - \rho dt. \tag{2.12}$$

With these definitions at hand, the inhomogeneous Maxwell equations simplify to

$$d * F = *J. \tag{2.13}$$

In order to show the electric-magnetic duality of the source-free Maxwell equations, it will be convenient to introduce a complex field 2-form

$$\mathcal{F} = F + i * F. \tag{2.14}$$

Then the source-free Maxwell equations are further simplified to

$$d\mathcal{F} = 0. \tag{2.15}$$

This doesn't change under an arbitrary duality rotation

$$\mathcal{F} \to e^{-i\theta}\mathcal{F} \tag{2.16}$$

where θ is a constant angle. In fact F and $*F$ transform according to

$$F \to \cos\theta F + \sin\theta * F \quad , \quad *F \to \cos\theta * F - \sin\theta F \tag{2.17}$$

while the electric and magnetic field vectors go as

$$\vec{E} \to \cos\theta\vec{E} + \sin\theta\vec{B} \quad , \quad \vec{B} \to -\sin\theta\vec{E} + \cos\theta\vec{B}. \tag{2.18}$$

In the particular case when $\theta = \frac{\pi}{2}$, $F \to *F$ so that the electric and magnetic field vectors are exchanged according to $\vec{E} \to \vec{B} \quad , \quad \vec{B} \to -\vec{E}$ by such a duality rotation.

Let us now define the electromagnetic drive 3-forms

$$\tau_\mu := \frac{1}{2}\left(\iota_{\partial_\mu} F \wedge *F - F \wedge \iota_{\partial_\mu} * F\right) = T_{\mu\nu} * dx^\nu \tag{2.19}$$

where the interior product operator ι_X with respect to a vector field X is a derivation $\iota_X : \Omega^p_M \to \Omega^{p-1}_M$. In Cartesian coordinates, the components of the electromagnetic energy-momentum tensor turn out to be

$$T^\mu{}_\nu = F^\mu{}_\lambda F^\lambda{}_\nu + \frac{1}{4}\eta^\mu{}_\nu F_{\lambda\rho}F^{\lambda\rho}. \tag{2.20}$$

We have in particular,

$$\begin{aligned} T^0{}_0 &= F^0_k F^k{}_0 + \frac{1}{4}(F_{\mu\nu}F^{\mu\nu}) \\ &= |\vec{E}|^2 + \frac{1}{2}(-|\vec{E}|^2 + |\vec{B}|^2) \\ &= \frac{1}{2}(|\vec{B}|^2 + |\vec{E}|^2) \end{aligned} \tag{2.21}$$

that is the electromagnetic energy density. Similarly,

$$T^j{}_0 = F^j{}_k F^k{}_0 = (\vec{E} \times \vec{B})_j \tag{2.22}$$

that is the Poynting vector associated with the electromagnetic momentum density. Finally,

$$\begin{aligned} T^j{}_k &= F^j{}_0 F^0{}_k + F^j{}_l F^l{}_k + \frac{1}{4}\eta^j{}_k (F_{\mu\nu}F^{\mu\nu}) \\ &= E_j E_k + \epsilon_{jlm}\epsilon_{lkn} B_m B_n + \frac{1}{2}\delta_{jk}(-|\vec{E}|^2 + |\vec{B}|^2) \\ T^j{}_k &= -\frac{1}{2}\delta_{jk}(|\vec{E}|^2 + |\vec{B}|^2) + E_j E_k + B_j B_k \ . \end{aligned} \tag{2.23}$$

Now let us look at the duality transformation of the electromagnetic drive 3-forms τ_μ. For this we work out the duality transformation of the terms $\iota_{\partial_\mu} F \wedge *F$ and $F \wedge \iota_{\partial_\mu} *F$ separately. Using the duality rotation law (2.17) we get

$$\begin{aligned} \iota_{\partial_\mu} F \wedge *F &\rightarrow \cos^2\theta \iota_{\partial_\mu} F \wedge *F - \sin\theta\cos\theta \iota_{\partial_\mu} * F \wedge *F \\ &+ \cos\theta \sin\theta \iota_{\partial_\mu} F \wedge F - \sin^2\theta \iota_{\partial_\mu} * F \wedge F, \end{aligned}$$

and

$$\begin{aligned} F \wedge \iota_{\partial_\mu} * F &\rightarrow \cos^2\theta F \wedge \iota_{\partial_\mu} * F - \sin\theta\cos\theta * F \wedge \iota_{\partial_\mu} * F \\ &+ \cos\theta F \wedge \iota_{\partial_\mu} F - \sin^2\theta * F \wedge \iota_{\partial_\mu} F. \end{aligned} \tag{2.24}$$

Then, since $*F \wedge *F = -F \wedge F$, we prove that the drive 3-forms are duality invariant:

$$\begin{aligned} \tau_\mu &\rightarrow \frac{\cos^2\theta}{2}(\iota_{\partial_\mu} F \wedge *F - F \wedge \iota_{\partial_\mu} * F) + \frac{\sin^2\theta}{2}(-\iota_{\partial_\mu} * F \wedge F + *F \wedge \iota_{\partial_\mu} F) \\ &= \frac{1}{2}\left(\iota_{\partial_\mu} F \wedge *F - \iota_{\partial_\mu} * F \wedge F\right) = \tau_\mu. \end{aligned} \tag{2.25}$$

It is possible to conclude then that the electromagnetic energy-momentum tensor remains invariant under a duality transformation. The following argument is usually provided. For $\theta = \frac{\pi}{2}$ the electric-magnetic duality is nothing but an interchange of electric and magnetic fields according to

$$\vec{E} \rightarrow \vec{B} \qquad \vec{B} \rightarrow -\vec{E}. \tag{2.26}$$

Both the energy density $\frac{1}{2}(|\vec{E}|^2 + |\vec{B}|^2)$ and the Poynting vector $\vec{E} \times \vec{B}$ are invariant under a duality transformation at this particular angle. We generalize this argument to an arbitrary duality rotation here. However, the invariance of the energy-momentum tensor does not mean by itself that Maxwell theory has an electric-magnetic duality symmetry in the Noether sense [6]. To justify this last statement we look at the duality transformation of the Maxwell action ($J = 0$)

$$I[F, \mu] = \int_M \left(\frac{1}{2} F \wedge *F + dF \wedge \mu\right) \tag{2.27}$$

where μ is a Lagrange multiplier 1-form that imposes the Bianchi identity $dF = 0$ as a constraint. We know that for $\theta = \frac{\pi}{2}$, $F \wedge *F \to -F \wedge *F$, thus the action changes sign under this duality transformation and this is sufficient to see that the duality symmetry is not a Noether symmetry. In general it is not difficult to show that

$$F \wedge *F \to \cos 2\theta F \wedge *F - \sin 2\theta F \wedge F. \tag{2.28}$$

In order to deal with the case with sources we must introduce locally the electromagnetic potential 1-form A such that

$$F = dA. \tag{2.29}$$

In fact, we may write

$$A = \Phi dt + \vec{A} \cdot d\vec{x} \tag{2.30}$$

where Φ is the electric (scalar) potential and $\vec{A}$ is the magnetic (vector) potential. Maxwell action written in terms of A reads up to a closed form ($J \neq 0$)

$$\begin{aligned} I[A] &= \int_M \left(\frac{1}{2} dA \wedge *dA + *J \wedge A \right) \\ &= \int_M \left(\frac{1}{2} A \wedge d * dA - A \wedge *J \right). \end{aligned} \tag{2.31}$$

The variation of this action now with respect to A gives

$$\delta_A I[A] \quad = \quad \int_M (\delta A \wedge d * dA - \delta A \wedge *J) \tag{2.32}$$

so that the inhomogeneous Maxwell equations become

$$d * dA = *J. \tag{2.33}$$

This may be further sharpened by exploiting the $U(1)$ gauge symmetry of the Maxwell action under

$$iA \to giAg^{-1} + gdg^{-1} \quad , \quad iF \to giFg^{-1} \tag{2.34}$$

where $g = e^{-i\alpha} \in U(1)$. Electromagnetic fields are gauge invariant, however, electromagnetic potentials are defined up to an arbitrary gauge function because $A \to A + d\alpha$. This freedom allows one to fix a gauge by requiring, for example, the Lorenz gauge condition $*d * A = 0$. Then the inhomogeneous Maxwell equations turn into the wave equation

$$\Delta A \equiv d * d * A + *d * dA = J \tag{2.35}$$

where the Laplace-Beltrami operator

$$\Delta \equiv d * d * + * d * d = \partial_\mu \partial^\mu = -\frac{\partial^2}{\partial^2 t} + \vec{\nabla}^2. \tag{2.36}$$

2.2 Dirac Monopole

The duality symmetry between electricity and magnetism for the source-free Maxwell equations cannot be maintained in general because electric charges occur in nature while magnetic monopoles are absent. We are also aware that the electric charges have a discrete character as evidenced by the famous Millikan oil drop experiment. Dirac [1] realized first in 1931 that the electric-magnetic duality symmetry could be restored and the electric charge quantization may be explained by assuming hypothetical magnetic charges and currents. To this end, he considered a point-like magnetic charge that generates a Coulomb-like magnetic field

$$\vec{B} = \vec{\nabla} \times \vec{A} = g_m \frac{\vec{x}}{r^3}. \tag{2.37}$$

Then

$$\vec{\nabla} \cdot \vec{B} = 4\pi g_m \delta^{(3)}(\vec{x}), \tag{2.38}$$

that is, there is a point magnetic charge of magnitude g_m located at the origin. The corresponding vector potential can be defined everywhere except along a curve that starts from the origin and extends towards infinity along which the potential goes singular. This gives rise to the so-called Dirac string singularity which is nothing but a single magnetic flux line that starts from the origin and extends to infinity. This should not be a surprise as the vector potential for a magnetic charge cannot be given globally because the atlas of a Gauss sphere S^2 that encloses the magnetic charge can be covered by at least two coordinate charts. Suppose we divide the Gauss sphere surrounding a single magnetic monopole into two overlapping hemispheres and introduce the vector potentials on the northern and southern hemispheres separately. Let us define these vector potentials in spherical polar coordinates (r, θ, φ). The vector potential on the northern hemisphere

$$\vec{A}_N = \frac{g_m}{4\pi r} \frac{\cos\theta - 1}{\sin\theta} \hat{e}_\varphi \tag{2.39}$$

is well defined over all space except along the half-line $\theta = \pi$; so the S-pole is not covered. On the southern hemisphere

$$\vec{A}_S = \frac{g_m}{4\pi r} \frac{\cos\theta + 1}{\sin\theta} \hat{e}_\varphi \tag{2.40}$$

which is singular when $\theta = 0$; so the N-pole is not covered. These two potentials are related by a gauge transformation on the overlap region, that is, on the equatorial belt at $\theta = \frac{\pi}{2}$. In fact,

$$\vec{A}_S - \vec{A}_N = \frac{g_m}{2\pi r \sin\theta}\hat{e}_\varphi \quad . \tag{2.41}$$

In terms of differential forms

$$A_S = A_N + \frac{g_m}{2\pi} d\varphi. \tag{2.42}$$

Note that the electromagnetic field of a Dirac monopole itself has no string singularity since

$$F_S = F_N. \tag{2.43}$$

Now the condition that the gauge transformation above must be single-valued implies Dirac's quantization condition:

$$\frac{g_m g_e}{2\pi} = n \in \mathbb{Z} \tag{2.44}$$

from which Dirac concluded [2] that the existence of even a single magnetic monopole in nature would require that all electric charges come quantized and vice versa.

The Dirac quantization condition can be generalized to dyons that are hypothetical point objects that carry both electric and magnetic charges envisaged for the first time by Schwinger [3] [4] in 1961. Suppose we take two dyons with (g_{e_1}, g_{m_1}) and (g_{e_2}, g_{m_2}), respectively. Then the quantization of the total electromagnetic field angular momentum

$$\int d^3x \;\; \vec{x} \times (\vec{E} \times \vec{B}) \tag{2.45}$$

will leads to [5]

$$\frac{g_{e_1} g_{m_2} - g_{e_2} g_{m_1}}{2\pi} = n \in \mathbb{Z}. \tag{2.46}$$

This generalized quantization condition for dyons is called the Dirac-Zwanziger-Schwinger quantization condition [7].

Chapter 3

ELECTROWEAK UNIFICATION

3.1 Electroweak Theory

We discussed that in Maxwell's theory the electric-magnetic duality relates to the existence of a compact $U(1)$ group. However, this is not the same as the $U(1)_{EM}$ gauge group associated with the gauge invariance of electromagnetism. The unification of weak sub-nuclear forces with electromagnetic forces is achieved in the standard electroweak theory in analogy with QED (quantum electrodynamics). QED gives a quantum field theoretical description of interactions between the electromagnetic field and electrically charged particles of matter. In the world of weak interactions two new symmetries are needed to describe the weak interactions between the leptons, that is, elementary particles that carry weak isospin and weak hypercharge Y. These two symmetries lead to two symmetry groups; $SU(2)_I$ for isospin and $U(1)_Y$ for hypercharge as expected. The corresponding $SU(2)_I \times U(1)_Y$ gauge theory [10],[11] requires four massless carrier particles to mediate the unified electroweak interactions; two of which are electrically charged while the other two are electrically neutral. Since weak sub-nuclear forces are short ranged, they should be carried by massive intermediate bosons while the long ranged electromagnetic forces are carried by the massless photon. This means that the $SU(2)_I \times U(1)_Y$ gauge symmetry of the electroweak theory must be broken by some mechanism that gives mass to three intermediate bosons while the residual $U(1)_{EM}$ gauge symmetry is associated with the photon exchanged in electromagnetic interactions[12]. The assignment of masses to the intermediate bosons by hand will induce the required symmetry breaking, however, such an explicit symmetry breaking destroys the renormalizability and hence the predictive power of the electroweak unified theory [13]. Spontaneous symmetry breakdown was suggested as an alternative to escape these undesirable features [15][16],[17]. Spontaneous symmetry breaking coupled with a local gauge invariance provides a mass generation mechanism that does not destroy renormalizability of a quantum field theory. This mechanism is called the Higgs Mechanism[18],[19],[20].

When electromagnetic subgroup $U(1)_{EM}$ is diagonally embedded into the semi-simple gauge group $SU(2)_I \times U(1)_Y$ that is spontaneously broken down by the Higgs mechanism; this unification possesses topological magnetic monopole solutions. We will discuss below the Higgs mechanism that generates masses in the bosonic sector of the standard electroweak theory. The fermionic sector with Yukawa couplings between the Higgs field and fermions that generate lepton masses will not be given.

3.2 Higgs Mechanism and Spontaneous Symmetry Breaking

The Standard Electroweak Theory is provided with a symmetry breaking mechanism. We restrict our attention to the bosonic sector of the $SU(2)_I \times U(1)_Y$ gauge theory for which the action is given by[10]

$$I = \int_{M^4} \left[\frac{1}{2} dA \wedge *dA + \frac{1}{2} Tr(\mathbf{F} \wedge *\mathbf{F}) + \frac{1}{2} Tr(\nabla_A \Phi \wedge *\nabla_A \Phi) + V(|\Phi|) * 1 \right] \tag{3.1}$$

where iA is the hypercharge potential 1-form, $\mathbf{A}$ is the $SU(2)$ Lie algebra (with basis T_a) valued potential 1-form with $\mathbf{F} = d\mathbf{A} + [\mathbf{A}, \mathbf{A}]$. The Higgs scalar Φ is a complex isodoublet

$$\Phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} \tag{3.2}$$

which under $SU(2)_I \times U(1)_Y$ transforms as

$$\Phi \to e^{-g\theta T - \frac{i}{2} g' \theta} \Phi. \tag{3.3}$$

Here the $U(1)_Y$ gauge group has a coupling constant g' while the $SU(2)_I$ gauge group has coupling constant g. The $\frac{1}{2}$ factor in the gauge transformation law comes from the Gell-Mann-Nishijima relation [13] between the electric charge Q and the third component of isospin I_3 and hypercharge Y:

$$Q = I_3 + \frac{1}{2} Y. \tag{3.4}$$

Thus the exterior covariant derivative of the Higgs field will be given by

$$\nabla_A \Phi = d\Phi + g\mathbf{A}\Phi + i\frac{g'}{2} A\Phi \tag{3.5}$$

where $\mathbf{A} = A^a \frac{t_a}{2i}$ with Pauli matrices t_a.

Next we write down the Higgs potential

$$\begin{aligned} V(|\Phi|) &= -\frac{1}{2}\mu^2|\Phi|^2 + \frac{\lambda}{4}|\Phi|^4 + V_0 \\ &= \frac{\lambda}{4}(|\Phi|^2 - \Phi_0^2)^2 \end{aligned} \tag{3.6}$$

where λ and μ^2 are real constants. We introduce a cosmological constant $V_0 = \frac{\lambda}{4}\Phi_0^4 = \frac{\mu^4}{4\lambda}$ to complete the above expression to a square. We always choose $\lambda > 0$ to make sure that the total action is bounded from below. The critical points of the above Higgs potential then depend on the sign of μ^2: If $\mu^2 < 0$, $|\Phi_0| = v_c^1 = 0$ is the unique critical point that is an isolated, stable minimum. If $\mu^2 > 0$ on the hand, as it is here, the critical points bifurcate. $|\Phi_0| = v_c^1 = 0$ becomes an unstable local maximum. We get an infinite family of critical points $|\Phi_0| = v_c^2 = \frac{\mu^2}{\lambda}$ which now make-up the locus of our local minima. Then the Higgs potential takes the shape of a Mexican hat. If we suppose the system is static, minimum energy state, i.e. the vacuum, does not correspond to a unique value of Φ since the ground state of the system has infinite degeneracy. In that case we are free to choose any of one of these. Consequently the Higgs vacuum in this case will have a $U(1)_H$ symmetry corresponding to our freedom of the choice of a particular ground state.

The important point here is that the symmetries of the action are not displayed by the vacuum state. For a theory where the vacuum state has less symmetry than the Lagrangian density, we say that a spontaneous symmetry breakdown has occurred. Glashow was the first to propose the idea of spontaneously breaking the electroweak gauge group into electromagnetic gauge subgroup by introducing a multiplet of scalar fields with a potential in the Lagrangian. In 1967 Weinberg and Salam applied the Higgs mechanism to the $SU(2) \times U(1)$ gauge theory [16] to generate the gauge boson masses. The general idea is that weak interactions should be mediated by W bosons which are to begin with massless. Then Higgs fields are introduced with a non-vanishing vacuum expectation value and the resulting symmetry breaking gives masses to gauge bosons.

We will demonstrate the mass generation mechanism in the bosonic sector of the Standard Electroweak Theory by looking at the Yang-Mills-Higgs equations that are obtained by varying the Weinberg-Salam action with respect to $\mathbf{A}$,A and Φ:

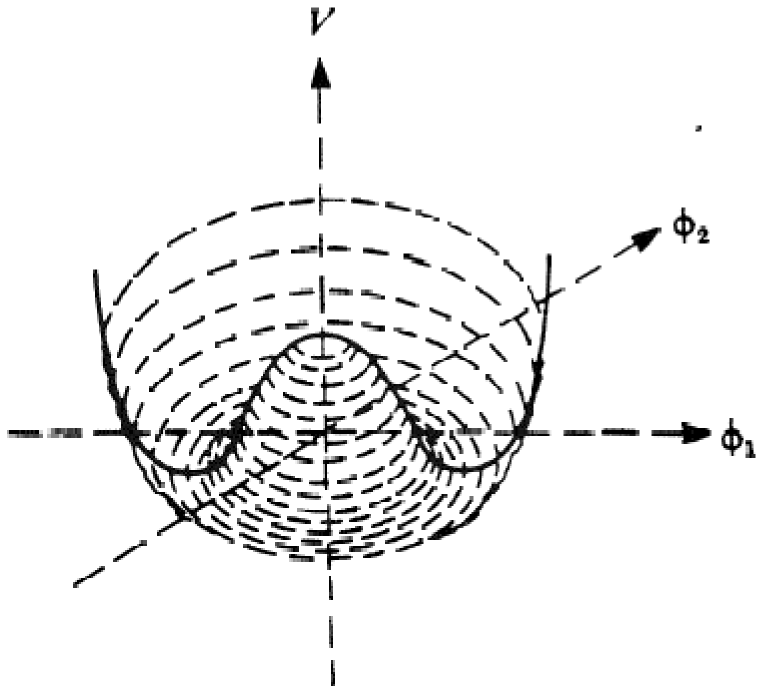

Figure 3.1: Higgs Potential

$$\nabla_A * \mathbf{F} + \frac{g}{2} * ((\nabla_A \Phi)^\dagger \frac{t_a}{2i}\Phi - \Phi^\dagger \frac{t_a}{2i}\nabla_A \Phi) T_a = 0, \tag{3.7}$$

$$d * dA - i\frac{g'}{4} * ((\nabla_A \Phi)^\dagger \Phi - \Phi^\dagger \nabla_A \Phi) = 0, \tag{3.8}$$

$$\nabla_A(*\nabla_A \Phi) - \frac{\partial V}{\partial \Phi^\dagger} * 1 = 0. \tag{3.9}$$

A stationary vacuum solution is given by the Minkowski space-time metric $g = \eta$, $\mathbf{A} = 0$, $A = 0$ and

$$\Phi_0 = \begin{pmatrix} 0 \\ v \end{pmatrix}$$

so that $|\Phi_0|^2 = v^2$. To break the symmetry group an effective linearization about the vacuum is required [21]:

$$g = \eta + \epsilon^2 \hat{g} \quad , \quad \mathbf{A} = \epsilon \hat{\mathbf{A}} \quad , \quad A = \epsilon \hat{A} \quad , \quad \Phi = \begin{pmatrix} \epsilon \hat{\phi_+} \\ v \end{pmatrix}. \tag{3.10}$$

This linearization of the $\mathbf{A}$ and A field equations equips the W-bosons defined by $W^{\pm} =$

Field	Mass	Charge
A_γ	0	0
$W^\pm$	$M_W = g\frac{v}{2}$	$\pm e$
Z^0	$M_Z = \sqrt{g^2 + g'^2}\frac{v}{2}$	0
ϕ_H	$M_H = 2v\sqrt{2\lambda}$	0

Table 3.1: Perturbative Boson Spectrum after the Higgs Mechanism

$\hat{A}_1 \pm i\hat{A}_2$ with mass

$$M_W^2 = \frac{v^2}{4}g^2 \tag{3.11}$$

and the Z-boson defined by $Z^0 = g\hat{\mathbf{A}}_3 - g'\hat{A}$ with mass

$$M_Z^2 = \frac{v^2}{4}(g^2 + g'^2) \tag{3.12}$$

and the photon field $A_\gamma = g'\hat{\mathbf{A}}_3 + g\hat{A}$ remains massless

$$M_\gamma^2 = 0. \tag{3.13}$$

The $\hat{\phi}_+$ excitation is not independent and can be determined in terms of $SU(2)$ potentials and appropriate boundary conditions [21]. There will be just a single real scalar boson ϕ_H left behind with mass that is determined from $\hat{\phi}_+$ excitations as

$$M_H^2 = 8\lambda v^2. \tag{3.14}$$

To summarize, in the Weinberg-Salam-Glashow model, we start with four massless gauge fields A_1, A_2, A_3 and A, and two complex scalar fields ϕ_+ and ϕ_0. These add up to 12 physical degrees of freedom. We end up with certain linear combinations of these gauge fields identified with the electrically charged weak bosons $W^\pm$ and the neutral weak boson Z^0 that acquire masses through the Higgs mechanism and a massless photon A_γ plus a single real scalar Higgs field ϕ_H that is left behind. These also add up to 12 physical degrees of freedom. Therefore the Higgs mechanism of mass generation does not involve any gain or loss in the physical (bosonic) degrees of freedom; these are just shifted between gauge fields and scalar fields in an appropriate way.

3.3 Georgi-Glashow Model

We will also consider the Georgi-Glashow model [14] of electroweak forces which is not realistic because it doesn't have a neutral weak boson, but it is simple enough to exhibit the topological properties of the non-Abelian magnetic monopoles in a more transparent way. Georgi-Glashow model is a $SU(2)$ gauge theory with spontaneous symmetry breaking down to $U(1)$. The action is written in terms of a Yang-Mills potential 1-form

$$A = A^a_\mu dx^\mu \otimes T_a \tag{3.15}$$

and an isovector Higgs field

$$\phi = \phi^a T_a \tag{3.16}$$

in the following way:

$$I = \int_M \left[\frac{1}{2} Tr(F \wedge *F) + \frac{1}{2} Tr(\nabla_A \phi \wedge *\nabla_A \phi) + V(|\phi|) * 1 \right] \tag{3.17}$$

where the Yang-Mills field 2-form

$$F = dA + A \wedge A = \frac{1}{2} F^a_{\mu\nu} dx^\mu \wedge dx^\nu \otimes T_a \tag{3.18}$$

and the gauge covariant derivative

$$\nabla_A \phi = d\phi + A\phi - \phi A. \tag{3.19}$$

$V(|\phi|)$ is the Higgs potential. The oriented volume element is fixed by

$$*1 = dt \wedge dx \wedge dy \wedge dz. \tag{3.20}$$

$\{T_a \ , \ a = 1, 2, 3\}$ are the Lie algebra generators which satisfy the $su(2)$ Lie algebra

$$[T_a, T_b] = \epsilon_{abc} T_c \tag{3.21}$$

and subject to the normalization

$$Tr(T_a T_b) = \frac{1}{2} \delta_{ab}. \tag{3.22}$$

Explicitly we have

$$iT_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad , \quad iT_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad iT_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We may also give the Yang-Mills field 2-form alternatively as

$$F^a = dA^a + \frac{1}{2}\epsilon_{abc}A^b \wedge A^c. \tag{3.23}$$

Under a local $SU(2)$ gauge transformation A and ϕ transform according to

$$A \to gAg^{-1} + gdg^{-1} \quad , \quad \phi \to g\phi g^{-1} \tag{3.24}$$

where $g = e^{i\theta^a T_a}$ and $\theta^a T_a \in su(2)$ is an arbitrary Lie algebra element. Thus the Yang-Mills field and the exterior derivative transform similarly as

$$F \to gFg^{-1} \quad , \quad \nabla_A \phi \to g\nabla_A \phi g^{-1}. \tag{3.25}$$

Finally the variation of the Georgi-Glashow action yields the coupled Yang-Mills-Higgs field equations

$$d * F - *F \wedge A + A \wedge *F \quad = \quad \phi \; * \nabla_A \phi - *\nabla_A \phi \; \phi \tag{3.26}$$

$$\nabla_A * \nabla_A \phi \quad = \quad \frac{\partial V}{\partial \phi} * 1 \tag{3.27}$$

which we are going to solve in the next chapter.

Chapter 4

NON-ABELIAN MONOPOLES

4.1 ’t Hooft-Polyakov Monopole

In the models for electroweak interactions where the gauge symmetry is enlarged to a non-Abelian Lie group symmetry, the field equations are shown to admit static magnetic monopole solutions that are essentially topological by construction. If such theories are correct, which we think they are, then magnetic monopoles must exist in nature. The theoretical possibility of getting monopoles of this type was discovered independently by ’t Hooft [23] and Polyakov [24] in 1974. They considered a class of static, spherically symmetric solutions of the $SO(3)$ Yang-Mills-Higgs field equations derived in the previous chapter in the context of Georgi-Glashow model in particular. ’t Hooft-Polyakov monopoles are based on the Wu-Yang ansatz [22]

$$A_0^a = 0 \qquad , A_k^a = \epsilon_{akj} x^j f(r) \qquad , \phi^a = x^a h(r) \tag{4.1}$$

where functions $f(r)$ and $h(r)$ of the radial coordinate r are to be determined. We will plug this ansatz into the Yang-Mills-Higgs equations

$$\begin{aligned} \nabla_A * F &= \phi \, * \nabla_A \phi - * \nabla_A \phi \, \phi, && (4.2) \\ \nabla_A * \nabla_A \phi &= \frac{\partial V}{\partial \phi} * 1. && (4.3) \end{aligned}$$

We write the Yang-Mills potential 1-form as

$$A = \epsilon_{akj} f(r) x^j dx^k T_a \tag{4.4}$$

and calculate its exterior derivative

$$dA = (\epsilon_{akj} f(r) dx^j \wedge dx^k + \epsilon_{akj} \frac{f'(r)}{r} x^j x^l dx^l \wedge dx^k) T_a. \tag{4.5}$$

We also calculate

$$A \wedge A = \frac{1}{2} (\epsilon_{akj} f(r) x^j dx^k \wedge \epsilon_{cln} x^n dx^l) \otimes [T_b, T_c]. \tag{4.6}$$

Since $[T_b, T_c] = \epsilon_{abc} T_a$ and using the identity $\epsilon_{cln}\epsilon_{cab} = \delta_{la}\delta_{nb} - \delta_{lb}\delta_{na}$ we get

$$A \wedge A = (\frac{1}{2}\epsilon_{bkj} f^2(r) x^j x^a dx^b \wedge dx^k) T_a. \tag{4.7}$$

Therefore the Yang-Mills 2-forms reduce to

$$\begin{aligned} F^a &= dA^a + \frac{1}{2}\epsilon^a{}_{bc} A^b \wedge A^c \\ &= \epsilon_{akj} f(r) dx^j \wedge dx^k + \epsilon_{akj}\frac{f'(r)}{r} x^j x^l dx^l \wedge dx^k + \frac{1}{2}\epsilon_{bkj} f^2(r) x^j x^a dx^b \wedge dx^k. \end{aligned} \tag{4.8}$$

We use the Hodge star identity $*(dx^1 \wedge dx^2) = dx^3$ and its cyclic permutations and obtain

$$*F^a = -f(r)dx^a + \frac{f'(r)}{2} x^n x^a dx^n - \frac{f'(r)}{r^2} dx^a + \frac{f^2(r)}{2} x^n x^a dx^n. \tag{4.9}$$

In order to work out the right hand side of the Yang-Mills-Higgs field equations we also need the covariant exterior derivative

$$\begin{aligned} \nabla_A \phi &= d\phi + A\phi - \phi A \\ = (\nabla_A \phi)^a T_a &= d\phi^a T_a + A^b \phi^c T_b T_c - \phi^c A^b T_c T_b \\ &= (d\phi^a + \epsilon^a{}_{bc} A^b \phi^c) T_a. \end{aligned} \tag{4.10}$$

Putting in the Wu-Yang ansatz into the above expression we obtain

$$(\nabla_A \phi)^a = h(r) dx^a + \frac{h'(r)}{r} x^a x^b dx^b + f(r)h(r) x^a x^k dx^k - f(r)h(r) r^2 dx^a. \tag{4.11}$$

It is convenient at this stage to re-name the arbitrary functions according to

$$f(r) = \frac{1 - K(r)}{r^2} \qquad , \qquad h(r) = \frac{H(r)}{r^2}. \tag{4.12}$$

Then the Yang-Mills-Higgs field equations reduce to the following system of second order ordinary differential equations:

$$r^2 \frac{d^2 K}{dr^2} = K H^2 + K(K^2 - 1), \tag{4.13}$$

$$r^2 \frac{d^2 H}{dr^2} = 2K^2 H + \lambda H (H^2 - r^2). \tag{4.14}$$

We should solve these subject to the boundary conditions i) as $r \to 0$

$$K(r) \to 1 \qquad , \qquad H(r) \to 0, \tag{4.15}$$

and ii) as $r \to \infty$

$$K(r) \to 0 \qquad , \qquad \frac{H(r)}{r} \to 1. \tag{4.16}$$

It is not difficult to perform numerical integration to get solutions for particular values of the Higgs coupling constant λ. We restrict attention below to the analytical solution obtained for the specific value $\lambda = 0$ [26].

In order to appreciate the topological nature of this particular solution, let us first define the magnetic charge and the rest mass of the above static field configurations. The magnetic charge is given by the surface integral of the magnetic field on a Gauss sphere S^2_∞ set at spatial infinity:

$$g_m = \int_{S^2_\infty} \vec{B} \cdot da = \int_{S^2_\infty} \phi^a \vec{B}^a \cdot d\vec{a} \tag{4.17}$$

where

$$B^a_i = \frac{1}{2}\epsilon_{ijk} F^a_{jk}. \tag{4.18}$$

Thus the magnetic charge here is

$$g_m = \int_{S^2_\infty} da^i \epsilon_{ijk} F^a_{jk} = 4\pi. \tag{4.19}$$

For static solutions, the total energy integral is just the rest mass of the magnetic monopole. Thus to define the mass we should first write the total Hamiltonian of the system that can be obtained from the Yang-Mills-Higgs Lagrangian by a Legendre transformation

$$\int dt \mathcal{H} = \sum_\alpha \int dt \dot{\mathcal{Q}}_\alpha P_\alpha - \int dt \mathcal{L} \tag{4.20}$$

where $\mathcal{Q}_\alpha$ denotes the generalized fields of the system and $\mathcal{P}_\alpha$ are the corresponding field momenta. For the Yang-Mills-Higgs system we consider, the total (field) Hamiltonian is given by the following spatial volume integral

$$\mathcal{H} = \int_{M^3} \left[\frac{Tr}{2}(F \wedge *^3 F) + \frac{Tr}{2}(\nabla_A \phi \wedge *^3 \nabla_A \phi) + V *^3 1 \right]. \tag{4.21}$$

Putting in the Wu-Yang ansatz in the above integral, the total Hamiltonian that coincides with the mass of the magnetic monopole is given by

$$\mathcal{M} = 4\pi \int_{M^3} \frac{dr}{r^2} \left[r^2 \left(\frac{dK}{dr} \right)^2 K^2 H^2 + \frac{1}{2} \left(r\frac{dH}{dr} - H \right)^2 + \frac{1}{2}(K^2 - 1)^2 + \frac{\lambda}{4}(H^2 - r^2)^2 \right]. \tag{4.22}$$

Now, we go to the specific limit $\lambda \to 0$, since we are looking for analytical solutions. To obtain static, spherically symmetric solutions in this limit, let us first note that the Hamiltonian integral

$$\mathcal{H} = \int_{M^3} \left[\frac{Tr}{2} (F \wedge *^3 F + \nabla_A \phi \wedge *^3 \nabla_A \phi) \right] \tag{4.23}$$

is positive definite. Then we consider the bound [27]

$$\int_{M^3} \frac{Tr}{2} (F \mp *\nabla_A \phi) \wedge *(F \mp *\nabla_A \phi) \geq 0 \tag{4.24}$$

Opening up the parantheses, we organize terms to read

$$\int_{M^3} \frac{Tr}{2} (F \wedge *F + \nabla_A \phi \wedge *\nabla_A \phi) \geq \pm \int_{M^3} Tr(F \wedge \nabla_A \phi). \tag{4.25}$$

The right hand side turns into a surface integral by Stokes' theorem:

$$\int_{M^3} Tr(F \wedge \nabla_A \phi) = \int_{M^3} d(Tr(F\phi)) = \int_{S^2_\infty} Tr(F\phi). \tag{4.26}$$

Thus we obtain a lower bound for the mass of the monopole given by the expression

$$\mathcal{M} \geq 4\pi |g_m|, \tag{4.27}$$

called the BPS (Bogomol'nyi-Prasad-Sommerfield) bound [26],[27],[28]. We note that the equality will be attained by field configurations that satisfy

$$F = \pm * \nabla_A \phi \tag{4.28}$$

These first order field equations are called the Bogomol'nyi equations [27]and their solutions correspond to field configurations with minimal mass.

Substituting the Wu-Yang ansatz into the Bogomol'nyi equation, we obtain a coupled system of first order ordinary differenntial equations

$$r \frac{dK}{dr} = -KH, \tag{4.29}$$

$$r \frac{dH}{dr} = H + (1 - K^2). \tag{4.30}$$

It is not difficult to verify that the integrability condition of the Bogomol'nyi equations are just the second order Yang-Mills-Higgs equations in the limit $\lambda \to 0$. We now solve the Bogomol'nyi equations in terms of elementary hyperbolic functions as

$$K(r) = \frac{r}{\sinh r} \qquad , \qquad H(r) = r \coth r - 1. \tag{4.31}$$

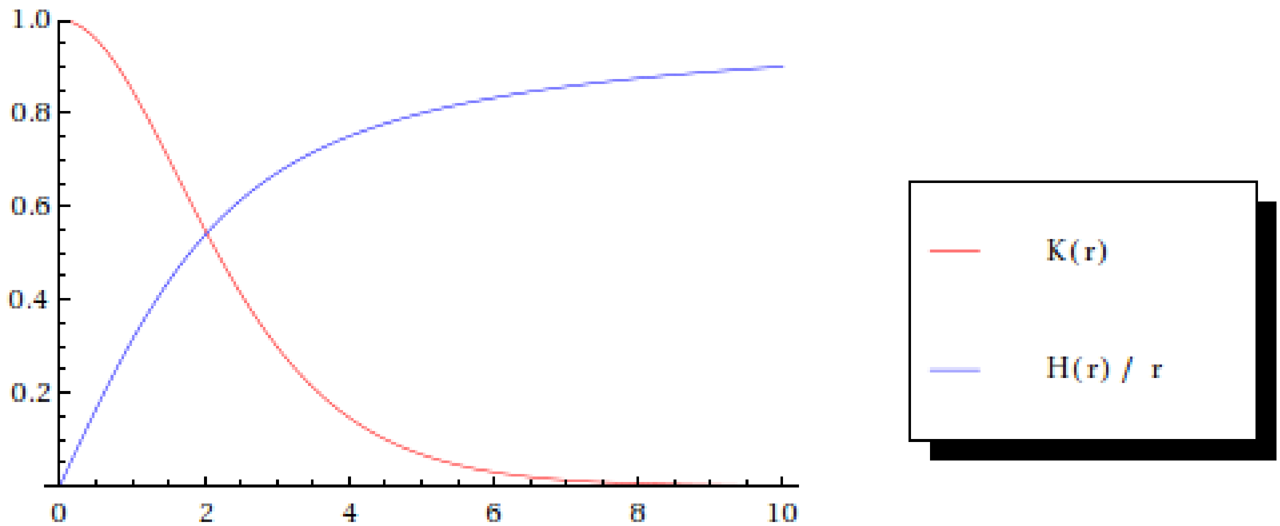

Figure 4.1: 't Hooft-Polyakov Monopole at BPS limit

These solutions were first obtained by Prasad-Sommerfield [26] by trial and error and later re-derived by Bogomol'nyi [27].

To saturate the Bogomol'nyi bound, we require that the potential $V(|\phi|)$ vanishes. However, for spontaneous symmetry breaking to make sense with this vanishing potential we still impose the boundary condition $\phi^a\phi^a \to v^2$ as $r \to \infty$. This is the condition that defines the Higgs vacuum of the system. The constant v can be interpreted as the vacuum expectation value of the Higgs boson. Since ϕ is an isovector, the unit vector $\hat{\phi}$ describes a sphere in the field space (isospace). The Higgs vacuum thus defines a sphere S^2_H in the 3-dimensional isospace. Polyakov calls this radial configuration a "hedgehog" configuration. On the other hand, solutions of classical field equations in the limit $r \to \infty$ map the vacuum manifold S^2_H onto the boundary S^2_∞ of the actual 3-dimensional space. Thus we have established a linear map between two 2-spheres

$$\phi : S^2_H \to S^2_\infty \tag{4.32}$$

that is characterized by its winding number, that is, by an integer that counts the number of times one sphere is wound around the other sphere [29]. In our case, where we impose the condition $\phi^a\phi^a \to v^2$ with vanishing potential at infinity, the winding number of the mapping turns out to be $n = 1$. Therefore our magnetic monopole solution carries just a single unit of magnetic charge: it is a "1-monopole solution". Higher degree ($n \neq 1$) magnetic monopole solutions are not easy to obtain.

Chapter 5

NON-ABELIAN DYONS

So far we have considered 't Hooft-Polyakov monopoles that saturate the BPS bound with $A_0^a = 0$. In a more general case when $A_0^a \neq 0$, we can get dyon solutions that carry both magnetic and electric charges. In this case, we start from the ansatz [25]

$$A_0^a = x^a j(r) \quad , \quad A_k^a = \epsilon_{ajk} x^j f(r) \quad , \quad \phi^a = x^a h(r) \tag{5.1}$$

where we now have three functions $f(r), h(r)$ and $j(r)$ to be determined. We will follow the same steps that we used to construct the 't Hooft-Polyakov monopole, remembering that this time there is one more field equation coming from the A_0^a component. We now calculate

$$\begin{aligned} dA &= \Big[\frac{j'(r)}{r} x^a x^i dx^i \wedge dx^0 + j(r) dx^a \wedge dx^0 \\ &+ \epsilon_{ajk} \frac{f'(r)}{r} x^k x^i dx^i \wedge dx^j + \epsilon_{ajk} f(r) dx^k \wedge dx^j] T_a \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} A \wedge A &= \frac{1}{2}[\epsilon_{abc} A^b \wedge A^c] T_a \\ &= [j(r) f(r) r^2 dx^0 \wedge dx^a + j(r) f(r) x^a x^m dx^m \wedge dx^0 \\ &- \frac{1}{2}\epsilon_{bjk} x^k x^a f^2(r) dx^j \wedge dx^b] T_a \end{aligned} \tag{5.3}$$

so that

$$\begin{aligned} F &= \Big[\left(j(r)f(r) + \frac{j'(r)}{r}\right) x^a x^i dx^i \wedge dx^0 \\ &+ (j(r) - j(r) f(r) r^2) dx^a \wedge dx^0 \\ &+ \epsilon_{ajk} \frac{f'(r)}{r} x^k x^i dx^i \wedge dx^j + \epsilon_{ajk} f(r) dx^k \wedge dx^j \\ &- \frac{1}{2}\epsilon_{bjk} x^k x^a f^2(r) dx^j \wedge dx^b] T_a. \end{aligned} \tag{5.4}$$

The Hodge star of F turns out to be

$$
\begin{aligned}
*F &= \left[\frac{1}{2}\epsilon_{imn}\left(\frac{j'(r)}{r} + j(r)f(r)\right)x^a x^i dx^m \wedge dx^n \right. \\
&+ \frac{1}{2}\epsilon_{amn}(j(r) - j(r)f(r)r^2)dx^m \wedge dx^n + \frac{f'(r)}{r}x^m x^a dx^m \wedge^0 \\
&- \left. f'(r)rdx^a \wedge dx^0 - 2f(r)dx^a \wedge dx^0 + f^2(r)x^m x^a dx^m \wedge dx^0\right]T_a
\end{aligned} \tag{5.5}
$$

and its exterior derivative gives

$$
\begin{aligned}
d*F &= \left[\frac{1}{2}\epsilon_{imn}\left(\frac{j''(r)}{r} - \frac{j'(r)}{r^2} + j(r)f'(r) + f(r)j'(r)\right)\frac{x^a x^i x^l}{r}dx^l \wedge dx^m \wedge dx^n \right. \\
&+ \frac{1}{2}\epsilon_{imn}\left(\frac{j'(r)}{r} + j(r)f(r)\right)x^a dx^i \wedge dx^m \wedge^n \\
&+ \frac{1}{2}\epsilon_{imn}\left(\frac{j'(r)r}{r} + j(r)f(r)\right)x^i dx^a \wedge dx^m \wedge dx^n \\
&+ \frac{1}{2}\epsilon_{amn}(j'(r) - j'(r)f(r)r^2 - j(r)f'(r)r^2 - 2rj(r)f(r))\frac{x^l}{r}dx^l \wedge dx^m \wedge dx^n \\
&+ \frac{f'(r)}{r}x^m dx^a \wedge dx^m \wedge dx^0 - 2\frac{f'(r)}{r}x^l dx^l \wedge dx^a \wedge dx^0 \\
&- \left. (f'(r) + f''(r)\theta(r)\frac{x^l}{r}dx^l \wedge dx^a \wedge dx^0 + f^2(r)x^m dx^a \wedge dx^m \wedge dx^0\right]T_a
\end{aligned} \tag{5.6}
$$

We also calculate

$$
A \wedge *F = [\epsilon_{abc}A^b \wedge *F^c]T_a \tag{5.7}
$$

$$
= [\epsilon_{abc}j(r)x^b dx^0 \wedge *F^c + x^a f(r)dx^c \wedge *F^c - x^c f(r)dx^a \wedge *F^c]T_a \tag{5.8}
$$

$$
\begin{aligned}
&= \frac{1}{2}[(j^2(r) - j^2(r)f(r)r^2)x^n dx^0 \wedge dx^a \wedge dx^n \\
&+ \epsilon_{abc}\epsilon_{imn}\left(\frac{j'(r)f(r)}{r} + j(r)f^2(r)\right)x^c x^a x^i dx^c \wedge dx^m \wedge dx^n \\
&+ \epsilon_{abc}\epsilon_{cmn}(j(r)f(r) - j(r)f^2(r)r^2)x^a dx^c \wedge dx^m \wedge dx^n \\
&- \epsilon_{abc}\epsilon_{imn}(j'(r)f(r)r + j(r)f^2(r)r^2)x^i dx^a \wedge dx^m \wedge dx^n \\
&- \epsilon_{abc}\epsilon_{cmn}(j(r)f(r) - j(r)f^2(r)r^2)x^c dx^a \wedge dx^m \wedge dx^n \\
&+ 2f^2(r)x^c dx^a \wedge dx^c \wedge dx^0 - f^3(r)r^2 x^m dx^a \wedge dx^m \wedge dx^0]T_a
\end{aligned}
$$

with a similar expression obtained for $*F \wedge A$ that we do not write explicitly here. On the right hand side of Yang-Mills equations we will also insert

$$
\begin{aligned}
\phi * \nabla_A\phi - *\nabla_A\phi\ \phi &= [\epsilon_{abc}\phi^b * (\nabla_A\phi)^c]T_a \\
&= h^2(r)\,(1 - f(r))\,r^2 x^b dx^0 \wedge dx^b \wedge dx^a T_a.
\end{aligned} \tag{5.9}
$$

Now we plug all the above expressions into the Yang-Mills-Higgs field equations and re-define the undetermined functions

$$f(r) = \frac{(1-K(r))}{r^2} \quad , \quad h(r) = \frac{H(r)}{r^2} \quad , \quad j(r) = \frac{J(r)}{r^2}, \tag{5.10}$$

so that we obtain the following system of coupled second order ordinary differential equations

$$\begin{aligned} r^2 \frac{d^2K}{dr^2} &= K(K^2 - 1 + H^2 - J^2), \\ r^2 \frac{d^2J}{dr^2} &= 2JK^2, \\ r^2 \frac{d^2H}{dr^2} &= 2HK^2, \end{aligned} \tag{5.11}$$

that are written already in the Prasad-Sommerfield limit $\lambda \to 0$. We obtain an exact solution

$$K(r) = \frac{r}{\sinh r} \quad , \quad J(r) = \sinh\gamma(r\coth r - 1) \quad , \quad H(r) = \cosh\gamma(r\coth r - 1) \tag{5.12}$$

where γ is for the time being an arbitrary constant. We note that in the BPS limit, contribution of A_0^a component of the gauge field is similar to that of the Higgs field. Before giving the BPS bound for dyon solutions, we define the electric charge of a dyon in a way similar to the definition of magnetic charge given in the previous chapter:

$$\begin{aligned} g_e &= \int_{S^2_\infty} \vec{E} \cdot d\vec{a} = \int_{S^2_\infty} \phi^a \vec{E}^a \cdot d\vec{a} \\ &= \int_{S^2_\infty} [\vec{\nabla}\left(x^a \frac{J(r)}{r^2}\right)\phi^a - \vec{A}^a \phi^a] \cdot d\vec{a} \end{aligned} \tag{5.13}$$

Then one may generalise the mass bound for a dyon with a non-trivial contribution coming from the electric charge as [7]

$$\mathcal{M}_{dyon} \geq 4\pi\sqrt{g_m^2 + g_e^2} \tag{5.14}$$

This is the BPS bound for the dyon mass. It is a very important result and holds for any finite energy solution of the Yang-Mills-Higgs field equations. The BPS mass formula is universal and it is invariant under electric-magnetic duality.

Chapter 6

NON-ABELIAN ELECTRIC MAGNETIC DUALITY

6.1 Montonen-Olive Duality

In the previous chapters, we have introduced 't Hooft-Polyakov monopoles and looked at explicit solutions satisfying the BPS bound $M \geq 4\pi|g|$. Dirac [2] had argued that with the existence of magnetic monopoles, electric charges must come quantized:

$$g_e g_m = 2\pi\hbar n \quad n = 0, \pm 1, \pm 2, .. \tag{6.1}$$

If this is the case, a duality rotation through $\frac{\pi}{2}$ between the electric and magnetic field directions that was a symmetry of source-free Maxwell equations would still be valid in the presence of matter with magnetic monopoles included [30] as Dirac's quantization condition allows now the symmetry

$$g_e \to g_m \quad g_m \to -g_e. \tag{6.2}$$

This extended duality symmetry provides us a generalization to non-Abelian gauge groups instead of the Abelian gauge group $U(1)$ in classical Maxwell theory. As such a non-Abelian gauge group is constructed by an extension of the Abelian $U(1)$ group, Montonen and Olive argued that there should be a dual equivalent formulation of the same theory in which electric (Noether) and magnetic (topological) charges are exchanged [30],[31]. At this point let us take a look at the duality between perturbative and non-perturbative particle states in the Georgi-Glashow model (in the BPS limit) (See Table 6.1)
Montonen and Olive introduced a dual quantum field theory based upon formally the same Lagrangian as of Georgi-Glashow model (with $V(\Phi) = 0$) in which coupling constants are replaced by others [30]. In this supposedly equivalent dual theory the Noether charge which is the electromagnetic $U(1)$ charge will be interchanged with the magnetic monopole charge which was the topological charge and vice versa. At this dual field theory fundemental monopole fields will play the role of heavy gauge particles. Thus, magnetic charge now becomes the $SO(3)$ gauge coupling constant. In Montonen-Olive conjecture, the first point

State	Electric Charge	Magnetic Charge	Mass	Spin
$Photon$	0	0	0	1
$Higgs$	0	0	0	0
$W_\pm$	$g_e = \pm 1$	0	$a\lvert g_e\rvert$	1
$M_\pm$	0	$g_m = \pm\frac{4\pi}{g_e}$	$a\lvert g_m\rvert = \frac{4\pi a}{g_e}$	?

Table 6.1: Montonen-Olive Duality Spectrum

made towards the consistency of the dual theory was the similarity in the spectra of Noether (electric) and topological (magnetic) charges. Then, they also identified the mass of the heavy gauge particles as the mass of the monopoles in their role of gauge particles in the dual theory

$$M(g_e) = \frac{4\pi}{g_m^2} M(g_m). \tag{6.3}$$

The Lagrangians of two theories only differ by their coupling constants. It was the original Montonen-Olive conjecture that the Georgi-Glashow model i.e. the $SO(3)$ Yang-Mills-Higgs theory in the BPS limit, has an exact duality symmetry under an exchange of electric and magnetic fields made together with an exchange of coupling constants

$$g_e \to g_m = \pm\frac{4\pi\hbar}{g_e}. \tag{6.4}$$

It should be emphasized that so far there is no rigorous demonstration of the Montonen-Olive duality conjecture. This is not a symmetry in the usual sense of the word because it relates a theory with one set of coupling constants with a similar theory but with a different set of coupling constants.

In order to further develop this idea, Witten [33] has considered the following surface term added to the Lagrangian of Yang-Mills-Higgs theory:

$$L_\theta = -\frac{\theta g_e^2}{32\pi^2} F \wedge *F. \tag{6.5}$$

This so-called θ-term violates $\mathcal{CP}$ symmetry but not $\mathcal{C}$-symmetry alone. Since it is a total derivative the addition of the θ-term does not affect the classical field equations of the system. With the θ term in the Lagrangian in the presence of magnetic monopoles, the allowed values of electric charge get shifted in the monopole sector of the theory [34]. For example when θ term is there, the Dirac monopole acquires electric charge as well provided we consider duality rotations in a $U(1)$ subgroup of $SU(2)$. In the presence of θ term it is possible write down the Noether generator of this transformation. Since Φ must have the same value after a rotation of 2π we have

$$e^{2\pi in} = 1. \tag{6.6}$$

Thus the θ term modifies the Dirac's quantization condition ($g_e g_m = 4\pi n_m$)

$$q = en_e - \frac{e\theta}{2\pi} n_m. \tag{6.7}$$

This is called the Witten Effect [33] which explains the change in the induced electric charge of the BPS monopole by shifting $\theta \to \theta + 2\pi$.

The inclusion of a θ-term extends Montonen-Olive conjecture by allowing us to introduce a complex parameter;

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g_e^2} \tag{6.8}$$

Then a duality transformation

$$\tau \to \tau + 1 \tag{6.9}$$

defines the periodicity of θ with period 2π. The electro-magnetic duality transformation (Montonen-Olive duality transformation) takes the form

$$\tau \to -\frac{1}{\tau}. \tag{6.10}$$

In fact the above transformations belong to the group $SL(2, \mathbb{Z})$ of projective transformations

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{6.11}$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. In summary, in the presence of the θ-term Montonen-Olive electromagnetic duality symmetry will be extended to the group of projective transformations $SL(2, \mathbb{Z})$. This extended Montonen-Olive duality is called the "S-duality" [34] in recent literature.

Chapter 7

CONCLUSION

In this thesis, we have explicitly verified the non-Abelian magnetic monopole and dyon solutions in a Yang-Mills-Higgs theory. We first constructed in Chapter:4 static field monopole solutions known as the 't Hooft-Polyakov monopole. We looked at these solutions at BPS bound to get analytical solutions. We saw that magnetic monopole solutions at BPS bound has Coulomb-like behaviour at long distances. Later in Chapter:5, we modified the ansatz to include electric charge ($A_0^a \neq 0$) and got dyon solutions with both electric and magnetic charges. Finally, we briefly commented on the Montonen-Olive duality conjecture which claims the presence of two equivalent formulations of the same theory that are dual to each other and in which Noether and magnetic (topological) charges would be exchanged.

We were not able to cover all details on non-Abelian magnetic monopole and dyon solutions. We considered only the bosonic sector of a Yang-Mills-Higgs theory. This work can be extended by including the fermionic sector. In this case dyonic configurations are known to lead to a dynamical supersymmetry [32],[35],[36].

Another interesting direction to go may be to search for magnetic monopole and dyon configurations in higher rank gauge theories such as $SU(3)$ and others [37].

Finally we note that magnetic monopoles escaped detection as elementary particles so far [9]. Recently there are strong indications in condensed matter physics of emergent quasi-particles in a class of exotic magnets known as "spin ice" that resemble magnetic monopoles [38]. This may prove a productive area to work.

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