Perturbative Analysis of Spectral Singularities for Complex Potentials having a Compact Support

by

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This is to certify that I have examined this copy of a master's thesis by

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To my parents...

ABSTRACT

Spectral singularities are energies at which the reflection and transmission coefficients of a complex scattering potential diverge. Therefore they can be associated with a type of resonance states that have real energies and zero width. After theoretically determining the location of spectral singularities, experimental observations can confirm this concept as a new type of resonance effect and give approval to its possible application in optics and laser physics.

In this thesis we use perturbation theory to study scattering properties of complex potentials and the spectral singularities of relevant non-Hermitian Hamiltonians. We give a theoretical descriptions of spectral singularities for a single and double Dirac delta function potentials with complex coupling constants as the perturbation parameter and then generalize this analysis for the system of many Dirac delta function potentials (Dirac comb). We prove that the perturbation analysis for Dirac comb Model is indeed exact. At the end we consider the scattering problem for a barrier potential of arbitrary shape using perturbation method. Then we consider a set of solutions of Maxwell's equations traveling inside an infinite slab gain medium and convert their wave equation into the Schrödinger equation of the barrier potential. We use this result to determine the optical spectral singularities and then discuss an experimental set up to explore their physical aspects.

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Saber Rostamzadeh

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

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Chapter 1

INTRODUCTION

The attempts to unify the various well-known theories of nature have a long history in theoretical physics. Quantum Mechanics and the Theory of Relativity are the main cornerstones of modern physics. There is a huge amount of research on the direction to unify these two successful theories. The unification of Quantum Mechanics with Special theory of Relativity is one of the first discoveries of this kind which succeeded by different Quantum Field theories and Standard Model. There appear problems on the way to continue this program when trying to unify General theory of Relativity with Quantum Mechanics. The incompatibility of Quantum Mechanics with General Relativity is one of the main obstacles in building a complete theory. Over the years, there appeared theories which could partially satisfy this unification agenda, but none of these has been successful as an ultimate theory.

As a result of these efforts, some investigations have motivated the study of different generalizations of QM and GR. A recent attempt in this direction is to adapt all the postulates of quantum mechanics except the Hermiticity of the Hamiltonian operators, [1, 2]. These theories so-called $\mathcal{PT}$-Symmetric Quantum Mechanics have been proved to be the special case of what is known as *Pseudo-Hermitian Quantum Mechanics*, [4, 9, 10, 11]. The use of Non-Hermitian operators has different applications in nuclear and atomic physics, [5], and more specifically in quantum optics and complex crystals, [6], where they are traditionally employed in the effective description of physical systems displaying decay or dissipative behavior.

The main characteristic feature of this class of operators is that a generic non-Hermitian operator has complex eigenvalues whose imaginary part may be associated with decay rates. This property is not however common to all non-Hermitian operators. There is a category of non-Hermitian operators that, similar to Hermitian operators, have a real spectrum, [1, 2, 3]. The fact that some non-Hermitian operators have real spectrum leads to the conjecture that one can use these operators to construct unitary quantum systems.

Most of the problems that we meet in quantum mechanics are formulated in terms of so-called eigenvalue equations. What we interpret as the solutions, to the eigenvalue equation, are the eigenvalues and the corresponding eigenvectors of the physical observable. The concept of the spectrum of an operator is connected to the operator's eigenvalue problem.

Eigenvalues of an operator determine the basic characteristics of the operator. Consider a linear operator A acting in a finite dimensional vector space V. In general, if the eigenvalue problem for a matrix (matrix representation of A) has distinct solutions (nonrepeated eigenvalues) then the matrix is diagonalizable. It turns out that the real symmetric matrices (square matrices that are equal to their transpose), and more generally normal matrices (those that commute with their Hermitian conjugate) are always diagonalizable, [7]. Hermitian matrices can be understood as the complex extension of real symmetric matrices. They are clearly normal therefore a Hermitian matrix is similar to a diagonal matrix, [4, 7].

For Hermitian operators having a discrete spectra, diagonalizability is equivalent to the existence of an orthonormal basis consisting of the eigenvectors of the operator, [8]. This is commonly known as *completeness*. For non-Hermitian operators, however, diagonalizability of H means the existence of a basis $\mathcal{B}^\dagger$ consisting of the eigenvectors of the adjoint operator $H^\dagger$ that is *biorthonormal* to some basis $\mathcal{B}$ consisting of the eigenvectors of H, [4, 12]. The lack of such a pair of biorthonormality may be caused by the presence of exceptional points or spectral singularities.

Exceptional points are points on the space of parameters of the operator at which the eigenvalues together with their eigenvectors coalesce. It turns out that the coalescing causes loss of differentiability in the eigenvalues and loss of continuity in the eigenvectors, [13, 14]. For a Non-Hermitian operator with a continuous spectrum there is another obstruction for non-diagonalizability, namely a spectral singularity.

Though, spectral singularities are impossible for Hermitian Hamiltonians, they are rather typical for non-Hermitian Hamiltonians. In particular for a Schrödinger operator with a complex scattering potential, which involves a spectral singularity, for each point of the spectrum there corresponds two linearly independent eigenfunctions, but it is impossible to construct a biorthonormal eigensystem for the operator. The concept of spectral singularity was discovered by mathematicians and was studied thoroughly in mathematic literature, [15, 16, 17, 18, 19, 20, 21, 27], since the 1950's, but their physical meaning was understood quite recently, [12, 21, 25, 26, 28, 33, 34].

Chapter 2

SPECTRAL SINGULARITIES

Let $\mathcal{H}(= L^2)$ be a Hilbert space with an inner product $< .|. >$ and $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a linear operator with domain $D(A)$ dense in $\mathcal{H}$. The spectrum of A is related to the equation

$$L_\lambda \psi = 0, \qquad L_\lambda := A - \lambda I. \tag{2.1}$$

where I stands for the identity operator, $\psi \in D(A)$ and λ is a given complex number.

Definition 1. $\lambda \in \mathbb{C}$ is said to be a regular point(value) of the operator A if the inverse operator to L_λ, [52],:

- exists,
- is a bounded[1] linear operator (bounded on range of L_λ),
- is defined on a dense subspace of $\mathcal{H}$ (in another word the range of L_λ is dense in $\mathcal{H}$).

Definition 2. The set of all regular points of the operator A is called the resolvent set of A and denoted by $\rho(A)$, i.e.,[52],

$$\rho(A) = \{\lambda \in \mathbb{C} | \lambda \text{ is a regular point of } A\}.$$

Definition 3. The set $\sigma(A) = \mathbb{C} \setminus \rho(A)$, i.e., the complement of $\rho(A)$ in the complex plane $\mathbb{C}$, is called the spectrum of the operator A. Therefore a point $\lambda \in \mathbb{C}$ belongs to the spectrum of A if and only if it is not a regular point. The spectrum of an operator can be split into three mutually disjoint subsets of point, continuous and residual spectra where each corresponds to the failure of one of the conditions of Definition 1, [52, 21, 23, 24]

Definition 4. Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator ($A = A^\dagger$). Then the spectrum of A is real and the well-known spectral theorem for self-adjoint operators states

[1] A linear operator is bounded if and only if it is continuous.

that there is an orthonormal basis consisting of the eigenvectors of A and we have the spectral decomposition of A as

$$A = \int_{-\infty}^{\infty} \lambda \, dE_\lambda, \tag{2.2}$$

where E_λ is the resolution of identity associated to A, [21, 23].[2]

The use of self-adjoint operators in Axiomatic Quantum Mechanics can generally be understood as follows.

Let S be a quantum mechanical system. The possible states of the system S are represented by unit vectors ψ called state vectors. These state vectors formally reside in a complex separable Hilbert space[3] $\mathcal{H}$. The physical quantities (observables) are described by particular self-adjoint operators acting on $\mathcal{H}$. If, for instance, a physical quantity a is described by means of an operator A the following holds.

(*i*) Suppose that the system S is in a certain state vector ψ and $\psi \in D(A)$. Then $\langle\psi|A\psi\rangle$ is the expectation value for the quantity a in the state ψ.

(*ii*) If E_λ denotes the *spectral function* of operator A (Def.4) then $\langle\psi|E_\lambda\psi\rangle$ is the probability that in the state ψ the value of the quantity a lies in the interval $(\lambda - \frac{d\lambda}{2}, \lambda + \frac{d\lambda}{2})$. More precisely $\langle\psi|E_\lambda\psi\rangle$ is the distribution function for the physical quantity a, so that

$$\langle\psi|A\psi\rangle = \int_{-\infty}^{\infty} d\lambda \langle\psi|E_\lambda\psi\rangle. \tag{2.3}$$

In particular, if $\psi_0 \in D(A)$ is a unit eigenvector of the operator A with corresponding eigenvalue λ_0 , then clearly, the quantity a in state ψ_0 takes the value λ_0 with probability equal to 1.

Suppose that A has a discrete spectrum consisting of its eigenvalues and there is a complete orthonormal system consisting of the eigenvectors of A so that $A\psi_n = \lambda_n\psi_n$. Then λ_n's are the probable values of the quantity a and a takes each of these values with certainty only in the corresponding state vectors ψ_n. By writing the arbitrary state of the system as

$$\psi = \sum_{n=1}^{\infty} c_n\psi_n, \quad c_n = \langle\psi_n|\psi\rangle, \tag{2.4}$$

[2]In measure theory this is known as projection-valued measure. For A being a Hermitian operator in a Hilbert space with eigenvalues λ_i and corresponding eigenvectors ψ_i we have $E_\lambda := |\psi_i\rangle\langle\psi_i|$, [23].

[3]A Hilbert space is separable if and only if admits a countable orthonormal basis, [4].

we can write

$$\langle\psi|A\psi\rangle = \sum_{n=1}^{\infty}\lambda_n|c_n|^2, \tag{2.5}$$

where $|c_n|^2$ is the probability that in the state vector ψ the quantity a is equal to λ_n.

For a given self-adjoint operator A the existence of the spectral family E_λ has a significant role in the formulation of Quantum Mechanics. In the theory of non-self-adjoint operators, it is very natural to look for a spectral family. It turns out that the presence of spectral singularities is a serious obstacle for constructing a reasonable spectral family for non-self-adjoint operators [21].

For instance, we consider the eigenvalue problem, (2.1), of the Hamiltonian operator H

$$H = -\frac{d^2}{dx^2} + v^{\vec{z}}(x), \tag{2.6}$$

where $x \in \mathbb{R}$ and $v^{\vec{z}}(x)$ is a complex valued potential such that $[v^{\vec{z}}(x)]^* = v^{\vec{z}^*}(x)$ where $\vec{z} = (z_1, \cdots, z_d)$. The eigenfunctions $\psi^{\vec{z}}_{\mathfrak{a}k}(x)$ of the operator H satisfy the

$$H\psi^{\vec{z}}_{\mathfrak{a}k}(x) = k^2\psi^{\vec{z}}_{ak}(x), \tag{2.7}$$

where $k \in \mathbb{R}^+$, $\mathfrak{a} = 1, 2$ being the degeneracy label and $\vec{z} := (z_1, z_2, \cdots, z_d)$.

The non-Hermitian Hamiltonian H is diagonalizable if $\psi^{\vec{z}}_{\mathfrak{a}k}(x)$ together with a set of eigenfunctions of $H^\dagger$, $\phi^{\vec{z}}_{\mathfrak{b}q}(x)$, form a complete biorthonormal system $\{\psi^{\vec{z}}_{\mathfrak{a}k}, \phi^{\vec{z}}_{\mathfrak{b}q}\}$, [4, 12]. This means that

$$< \psi^{\vec{z}}_{\mathfrak{a}k}|\phi^{\vec{z}}_{\mathfrak{b}q} >= \delta_{\mathfrak{a}\mathfrak{b}}\delta(k-q), \quad \sum_{\mathfrak{a}=1}^{2}\int_0^\infty dk|\psi^{\vec{z}}_{\mathfrak{a}k} >< \phi^{\vec{z}}_{\mathfrak{a}k}| = 1. \tag{2.8}$$

In view of (2.7) and (2.8), the spectral representation of the Hamiltonian is given by

$$H = \sum_{\mathfrak{a}=1}^{2}\int_0^\infty dk\ k^2|\psi^{\vec{z}}_{\mathfrak{a}k} >< \phi^{\vec{z}}_{\mathfrak{a}k}|. \tag{2.9}$$

The existence of such a general expression for the spectral family of Hamiltonian operator has a great advantage in study of quantum systems. In [12] it has been shown that such an expression does not always exist. One of the main obstacles of its existence is the presence of a spectral singularity.

There is a rather general theory of spectral singularities for differential operators of the form (2.6) implying that the spectral singularities are the real zeros of a certain class of analytic functions [12, 21, 27]. For the case that the space, in which the differential operators act, is $L^2(\mathbb{R})$, this analytic function is the Wronskian of the Jost solutions, [12, 22]. Jost solutions are solutions of the equation $H\psi = k^2\psi$ with the following asymptotical behavior at infinity, [12, 28],

$$\begin{cases} \psi_{k_-}(x) \to e^{-ikx} \quad \text{for} \quad (x \to -\infty), \\ \\ \psi_{k_+}(x) \to e^{ikx} \quad \text{for} \quad (x \to +\infty). \end{cases} \tag{2.10}$$

Their Wronskian reads

$$W\left[\psi_{k_-}, \psi_{k_+}\right](x) := \begin{vmatrix} \psi_{k_-}(x) & \psi_{k_+}(x) \\ \\ \psi'_{k_-}(x) & \psi'_{k_+}(x) \end{vmatrix}, \tag{2.11}$$

where $|A|$ stands for the determinant of the matrix A.

Abel's Theorem for a second order differential equations, [46], states that if the Wronskian of solutions of the the homogeneous differential equation is zero then those solutions are linearly dependent. This implies that the Jost solutions become linearly dependent for the case that k^2 gives a spectral singularity.

Since $|v(x)|$ decays rapidly as $|x| \to \infty$, the Schrödinger eigenvalue problem will have solutions which are linear combinations of the plane waves as $x \to \pm\infty$.

$$\psi_k^{\vec{z}}(x) \to A_\pm e^{ikx} + B_\pm e^{-ikx} \qquad (x \to \pm\infty), \tag{2.12}$$

where $A_\pm$ and $B_\pm$ are k dependence coefficients. The 2×2 matrix M relating the coefficients A_+, B_+ to A_-, B_-, is called transfer matrix of the potential, [44]. This is defined by

$$\begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = M \begin{pmatrix} A_- \\ B_- \end{pmatrix}. \tag{2.13}$$

The Jost solution of the eigenvalue problem (2.7) takes the form

$$\psi_{k+}^{\vec{z}}(x) = \begin{cases} e^{ikx} & x \to +\infty, \\ A_-^+ e^{ikx} + B_-^+ e^{-ikx} & x \to -\infty, \end{cases} \tag{2.14}$$

$$\psi_{k-}^{\vec{z}}(x) = \begin{cases} A_+^- e^{ikx} + B_+^- e^{-ikx} & x \to +\infty, \\ e^{-ikx} & x \to -\infty. \end{cases} \tag{2.15}$$

In view of (2.13), (2.14) and (2.15) we obtain

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = M \begin{pmatrix} A_-^+ \\ B_-^+ \end{pmatrix}, \tag{2.16}$$

$$\begin{pmatrix} A_+^- \\ B_+^- \end{pmatrix} = M \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.17}$$

They in turn imply

$$A_-^+ = \frac{M_{22}}{|M|}, \quad B_-^+ = -\frac{M_{21}}{|M|}, \quad A_+^- = M_{12}, \quad B_+^- = M_{22}, \tag{2.18}$$

where $|M|$ stands for the determinant of the matrix M. Therefore the Jost solutions are given by

$$\psi_{k+}^{\vec{z}}(x) = \begin{cases} e^{ikx} & x \to +\infty, \\ \frac{M_{22}}{|M|} e^{ikx} - \frac{M_{21}}{|M|} e^{-ikx} & x \to -\infty, \end{cases} \tag{2.19}$$

$$\psi_{k-}^{\vec{z}}(x) = \begin{cases} M_{12} e^{ikx} + M_{22} e^{-ikx} & x \to +\infty, \\ e^{-ikx} & x \to -\infty. \end{cases} \tag{2.20}$$

and their Wronskian becomes

$$W\left[\psi_{k_-}, \psi_{k_+}\right] = 2ikM_{22}(k) \qquad (x \to \infty), \tag{2.21}$$

$$W\left[\psi_{k_-}, \psi_{k_+}\right] = \frac{2ikM_{22}(k)}{|M|} \qquad (x \to -\infty). \tag{2.22}$$

Comparing (2.21) and (2.22) we directly obtain $|M| = 1$, [12]. Furthermore, because the real zeros of the Wronskian of the Jost solutions are the spectral singularities, we have:

Definition 5. Spectral singularities are points, k^2, of the real spectrum of the operator (2.6) that satisfy $M_{22}(k) = 0$, [12].

The solutions of the eigenvalue problem (2.7), corresponding to the scattering states, can be obtained by imposing physically motivated asymptotic boundary conditions. We interpret a plane wave e^{ikx} in the asymptotic region to the left of the interaction region as an incoming ($k > 0$) or outgoing ($k < 0$) wave, whereas in the asymptotic region to the right of the interaction region the interpretation is the opposite one. We construct two linearly independent eigenfunctions of the Hamiltonian H by imposing the following asymptotic conditions, [45],

$$\psi_k^l(x) \longrightarrow N_l \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{as} \quad x \to -\infty, \\ T^l e^{ikx} & \text{as} \quad x \to +\infty, \end{cases} \tag{2.23}$$

$$\psi_k^r(x) \longrightarrow N_r \begin{cases} T^r e^{-ikx} & \text{as} \quad x \to -\infty, \\ e^{-ikx} + R^r e^{ikx} & \text{as} \quad x \to +\infty, \end{cases} \tag{2.24}$$

where $R^{l,r}(k)$ and $T^{l,r}(k)$ are complex reflection and transmission amplitudes and $N_{l,r}$ are the normalization constants, [33]. The superscripts l and r refer to incidence from the asymptotic region on the left (l) and right (r) of the interaction region. Comparing (2.23) and (2.24) with (2.19) and (2.20), we conclude that ψ_k^l and ψ_k^r are respectively proportional to the Jost solutions ψ_{k+} and ψ_{k-} and since these solutions are linearly dependent at a spectral singularity, ψ_k^l and ψ_k^r become linearly dependent. Other consequences of these comparisons are the relations

$$\begin{aligned} T^l &= \tfrac{1}{M_{22}(k)}, & R^l &= -\tfrac{M_{21}(k)}{M_{22}(k)}, \\ T^r &= \tfrac{1}{M_{22}(k)}, & R^r &= \tfrac{M_{12}(k)}{M_{22}(k)}. \end{aligned} \tag{2.25}$$

As seen from (2.25), at a spectral singularity, where $M_{22}(k)$ vanishes, the coefficients $R^{l,r}(k)$ and $T^{l,r}(k)$ diverge. This condition is a characteristic property of resonance phenomena therefore spectral singularities may be interpreted as a peculiar type of resonance, [33].

A dynamic view of resonance may be as follows: a wave packet with a distribution of energy comes into a potential and gets captured in a quasi-bound state for a long time before getting scattered off eventually. But in the static view, a resonance state is defined as a solution of the stationary Schrödinger eigenvalue problem (2.7) with *out-going* waves or Sigert boundary condition, (Table 2.1), [29, 30, 31, 32].

Unlike ordinary resonance, the spectral singularity is a point of spectrum such that $k \in \mathbb{R}$ and therefore the associated eigenvalue (k^2) is real and positive. Therefore these resonances have zero width, [33, 34].

Table 2.1: definition of resonant, anti-resonant, bound, anti-bound states and spectral singularity in k-complex plane, [31].

Solutions of Eq.(2.7)	Re(k)	Im(k)	Re(k^2)	Im(k^2)	Riemann sheets
Resonant state	positive	negative	any	negative	second
Anti-resonant state	negative	negative	any	positive	second
Bound state	zero	positive	negative	zero	first
Anti-bound state	zero	negative	negative	zero	second
Spectral singularity	any	zero	positive	zero	on the real line

In the following sections we use the Definition 5 to identify spectral singularities of different potentials. A systematic approach to determine the spectral singularities is based on constructing the transfer matrix M for the potential and exploring the zeros of the entry M_{22} of M.

Determining the spectral singularities for complicated systems which are not analytically solvable is a difficult task. One way out of this situation is to use numerical treatments that in turn may lead to different type of errors. Another solution is to use some appropriate approximation schemes. There exists a mathematical approximation scheme, known as perturbation theory, for describing a complicated system in terms of a simpler one. The idea is to start with a simple system for which the solutions are known, and add an additional weak disturbance (perturbation). If the disturbance is not too large, various physical quantities associated with the perturbed system (e.g. its energy levels and eigenstates) can be expressed as "corrections" to those of the simple system.

Chapter 3

PERTURBATION THEORY

Consider the Hamiltonian operator

$$H = H_0 + V_0 + \varepsilon V, \qquad \varepsilon \in \mathbb{C}, \tag{3.1}$$

where

$$H_0 = -\frac{\hbar^2}{2m}\partial_X^2, \tag{3.2}$$

is the Hamiltonian of a free particle, V_0 and V are functions of X, and $\varepsilon \in \mathbb{C}$. The eigenvalue problem for H is given by

$$(H_0 + V_0 + \varepsilon V)\psi(X) = E\psi(X). \tag{3.3}$$

Let us introduce the dimensionless quantities

$$x := \frac{X}{\ell}, \qquad k^2 := \frac{2m\ell^2 E}{\hbar^2}, \qquad v(x) := \frac{2m\ell^2 V}{\hbar^2}, \qquad v_0(x) := \frac{2m\ell^2 V_0}{\hbar^2}, \tag{3.4}$$

where ℓ is an arbitrary length scale, and suppose that $v_0(x)$ and $v(x)$ are piecewise continuous functions that vanish outside an interval $[a, b]$. Then the eigenvalue problem (3.3) leads to the dimensionless Schrödinger equation

$$\partial_x^2\psi(x) + [k^2 - v_0(x)]\psi(x) = \varepsilon v(x)\psi(x). \tag{3.5}$$

To solve this equation we use the ansatz

$$\psi(x) = \sum_{j=0}^{\infty} \varepsilon^j \psi^{(j)}(x), \tag{3.6}$$

where $\psi^{(j)}$'s represent the j-th order perturbative correction. An approximate 'perturbative solution' is obtained by truncating the series, usually by keeping only the first two or three terms.

Applying perturbation ansatz (3.6) to Eq. (3.3), we obtain the non-Homogeneous equation

$$\partial_x^2\psi^{(j)}(x) + [k^2 - v_0(x)]\psi^{(j)}(x) = v(x)\psi^{(j-1)}(x), \tag{3.7}$$

that we can solve recursively according to

$$\psi^{(j)}(x) = \int_{-\infty}^{x} G(x, x')v(x')\psi^{(j-1)}(x')dx'. \tag{3.8}$$

Here $G(x, x')$ denotes the Green's function of the differential equation $L\psi^{(0)}(x) = 0$ where $L = \partial_x^2 + [k^2 - v_0(x)]$. Let $\psi_1^{(0)}(x)$ and $\psi_2^{(0)}(x)$ be a pair of linearly independent solutions of this differential equation. Then we can express G in the form

$$G(x, x') = \begin{cases} \dfrac{\begin{vmatrix} \psi_1^{(0)}(x') & \psi_2^{(0)}(x') \\ \psi_1^{(0)}(x) & \psi_2^{(0)}(x) \end{vmatrix}}{\begin{vmatrix} \psi_1^{(0)}(x') & \psi_2^{(0)}(x') \\ \partial_{x'}\psi_1^{(0)}(x') & \partial_{x'}\psi_2^{(0)}(x') \end{vmatrix}} & \text{for} \quad x \in [a, b], \\ \\ \dfrac{\sin k(x - x')}{k} & \text{for} \quad x \notin [a, b], \end{cases} \tag{3.9}$$

where the symbol $|A|$ stands for the determinant of a square matrix A.

One can use the recursion relation (3.8) to obtain a general formula for the corrections, namely

$$\psi^{(j>2)}(x) = \int_{-\infty}^{x} G(x, x_j)v(x_j)\left[\prod_{i=1}^{j-1} \int_{-\infty}^{x_{i+1}} G(x_{i+1}, x_i)v(x_i)\psi^{(0)}(x_1)\, dx_i\right] dx_j. \tag{3.10}$$

Here, according to the definition of $v_0(x)$, we choose the solutions of $L\psi^{(0)}(x) = 0$ such that

$$\psi_1^{(0)}(x) = \begin{cases} e^{ikx} & x < a, \\ \mathscr{R}_0(x) & a < x < b, \\ \mathscr{A}_0(k)e^{ikx} + \mathscr{B}_0(k)e^{-ikx} & x > b, \end{cases} \tag{3.11}$$

$$\psi_2^{(0)}(x) = \begin{cases} e^{-ikx} & x < a, \\ \mathscr{T}_0(x) & a < x < b, \\ \mathscr{C}_0(k)e^{ikx} + \mathscr{D}_0(k)e^{-ikx} & x > b, \end{cases} \tag{3.12}$$

where $\mathscr{R}_0(x)$ and $\mathscr{T}_0(x)$ are the solutions of $L\psi^{(0)}(x) = 0$ for $x \in [a, b]$, and $\mathscr{A}_0(k)$, $\mathscr{B}_0(k)$, $\mathscr{C}_0(k)$ and $\mathscr{D}_0(k)$ are arbitrary k dependent coefficients. It is appropriate to define the following mapping

$$\forall \psi_1, \psi_2 \in \mathcal{H}, \qquad (\psi_1, \psi_2) \mapsto \Psi := \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{3.13}$$

Using the two-component functions Ψ in our calculations helps us to easily compute the transfer matrix. Therefore we write the perturbation ansatz as

$$\Psi(x) = \sum_{n=0}^{N} \varepsilon^j \Psi^{(j)}(x), \tag{3.14}$$

where $\Psi(x)$ satisfies

$$\Psi^{(j)}(x) = \int_{-\infty}^{x} G(x,x')v(x')\Psi^{(j-1)}(x')dx'. \tag{3.15}$$

For the zeroth order correction, we write

$$x < a: \quad \Psi^{(0)}(x) = \begin{pmatrix} e^{ikx} \\ e^{-ikx} \end{pmatrix}, \tag{3.16}$$

$$a < x < b: \quad \Psi^{(0)}(x) = \begin{pmatrix} \mathscr{R}_0(x) \\ \mathscr{T}_0(x) \end{pmatrix}, \tag{3.17}$$

$$x > b: \quad \Psi^{(0)}(x) = \begin{pmatrix} \mathscr{A}_0(k)e^{ikx} + \mathscr{B}_0(k)e^{-ikx} \\ \mathscr{C}_0(k)e^{ikx} + \mathscr{D}_0(k)e^{-ikx} \end{pmatrix}. \tag{3.18}$$

For the first order correction we have

$$x < a: \quad \Psi^{(1)}(x) = 0, \tag{3.19}$$

$$\begin{aligned} a < x < b: \quad \Psi^{(1)}(x) &= \int_{a}^{x} G(x,x')v(x') \begin{pmatrix} \mathscr{R}_0(x') \\ \mathscr{T}_0(x') \end{pmatrix} dx' \\ &= \begin{pmatrix} \int_a^x G(x,x')v(x')\mathscr{R}_0(x')dx' \\ \\ \int_a^x G(x,x')v(x')\mathscr{T}_0(x')dx' \end{pmatrix} = \begin{pmatrix} \mathscr{R}_1(x) \\ \\ \mathscr{T}_1(x) \end{pmatrix}, \end{aligned} \tag{3.20}$$

$$\begin{aligned} x > b: \quad \Psi^{(1)}(x) &= \int_{a}^{b} G(x,x')v(x') \begin{pmatrix} \mathscr{R}_0(x') \\ \mathscr{T}_0(x') \end{pmatrix} dx' \\ &= \begin{pmatrix} \int_a^x \frac{\sin k(x-x')}{k} v(x')\mathscr{R}_0(x')dx' \\ \\ \int_a^x \frac{\sin k(x-x')}{k} v(x')\mathscr{T}_0(x')dx' \end{pmatrix} \\ &= \begin{pmatrix} \mathscr{A}_1(k)e^{ikx} + \mathscr{B}_1(k)e^{ikx} \\ \\ \mathscr{C}_1(k)e^{ikx} + \mathscr{D}_1(k)e^{ikx} \end{pmatrix}. \end{aligned} \tag{3.21}$$

Repeating this calculation we find for the j-th order correction

$$x < a: \qquad \Psi^{(j>0)}(x) = 0, \tag{3.22}$$

$$x > b: \qquad \Psi^{(j)}(x) = \begin{pmatrix} \mathscr{A}_j(k)e^{ikx} + \mathscr{B}_j(k)e^{ikx} \\ \mathscr{C}_j(k)e^{ikx} + \mathscr{D}_j(k)e^{ikx} \end{pmatrix}, \tag{3.23}$$

where explicit calculations yields

$$\mathscr{A}_{j>2}(k) = \frac{1}{2ik}\int_a^b e^{-ikx_j} v(x_j)\left[\prod_{i=1}^{j-1}\int_a^{x_{i+1}} G(x_{i+1}, x_i)v(x_i)\mathscr{R}_0(x_1)\, dx_i\right] dx_j, \tag{3.24}$$

$$\mathscr{B}_{j>2}(k) = \frac{-1}{2ik}\int_a^b e^{ikx_j} v(x_j)\left[\prod_{i=1}^{j-1}\int_a^{x_{i+1}} G(x_{i+1}, x_i)v(x_i)\mathscr{R}_0(x_1)\, dx_i\right] dx_j, \tag{3.25}$$

$$\mathscr{C}_{j>2}(k) = \frac{1}{2ik}\int_a^b e^{-ikx_j} v(x_j)\left[\prod_{i=1}^{j-1}\int_a^{x_{i+1}} G(x_{i+1}, x_i)v(x_i)\mathscr{T}_0(x_1)\, dx_i\right] dx_j, \tag{3.26}$$

$$\mathscr{D}_{j>2}(k) = \frac{-1}{2ik}\int_a^b e^{ikx_j} v(x_j)\left[\prod_{i=1}^{j-1}\int_a^{x_{i+1}} G(x_{i+1}, x_i)v(x_i)\mathscr{T}_0(x_1)\, dx_i\right] dx_j. \tag{3.27}$$

Using the perturbation ansatz (3.14) therefore we have

$$x < a: \qquad \Psi(x) = \begin{pmatrix} e^{ikx} \\ e^{-ikx} \end{pmatrix}, \tag{3.28}$$

$$x > b: \qquad \Psi(x) = \begin{pmatrix} \mathscr{A}(k,\varepsilon)e^{ikx} + \mathscr{B}(k,\varepsilon)e^{ikx} \\ \mathscr{C}(k,\varepsilon)e^{ikx} + \mathscr{D}(k,\varepsilon)e^{ikx} \end{pmatrix}, \tag{3.29}$$

where $\mathscr{A}(k,\varepsilon)$, $\mathscr{B}(k,\varepsilon)$, $\mathscr{C}(k,\varepsilon)$ and $\mathscr{D}(k,\varepsilon)$ have the form

$$\begin{gathered} \mathscr{A}(k,\varepsilon) := \sum_{n=0}^{N} \varepsilon^j \mathscr{A}_j(k), \quad \mathscr{B}(k,\varepsilon) := \sum_{n=0}^{N} \varepsilon^j \mathscr{B}_j(k), \\ \mathscr{C}(k,\varepsilon) := \sum_{n=0}^{N} \varepsilon^j \mathscr{C}_j(k), \quad \mathscr{D}(k,\varepsilon) := \sum_{n=0}^{N} \varepsilon^j \mathscr{D}_j(k). \end{gathered} \tag{3.30}$$

Applying the inverse mapping (3.13) and using (3.28) and (3.29), we can write

$$\begin{aligned} e^{ikx} \quad &\text{for} \quad x < a \quad \longrightarrow \quad \mathscr{A}(k,\varepsilon)e^{ikx} + \mathscr{B}(k,\varepsilon)e^{-ikx} \quad \text{for} \quad x > b, \\ e^{-ikx} \quad &\text{for} \quad x < a \quad \longrightarrow \quad \mathscr{C}(k,\varepsilon)e^{ikx} + \mathscr{D}(k,\varepsilon)e^{-ikx} \quad \text{for} \quad x < b. \end{aligned} \tag{3.31}$$

In view of the definition of transfer matrix given in (2.13), we then find

$$M\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathscr{A}(k,\varepsilon) \\ \mathscr{B}(k,\varepsilon) \end{pmatrix}, \qquad \text{and} \qquad M\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathscr{C}(k,\varepsilon) \\ \mathscr{D}(k,\varepsilon) \end{pmatrix}. \tag{3.32}$$

Therefore

$$M_{11} = \mathscr{A}(k,\varepsilon), \quad M_{12} = \mathscr{B}(k,\varepsilon), \quad M_{21} = \mathscr{C}(k,\varepsilon), \quad M_{22} = \mathscr{D}(k,\varepsilon). \tag{3.33}$$

We recall that in order to identify the spectral singularities the solutions of the equation $M_{22} = \mathscr{D}(k,\varepsilon) = 0$ should be determined. Therefore we need to compute $\mathscr{D}(k,\varepsilon)$.

In the rest of our analysis we employ the perturbative method of computing the transfer matrix that we developed in this section. In the following sections first we consider the case $v_0(x) = 0$ perturbed by Dirac delta potential $v(x) = \delta(x)$. We then extend this study to double and many Dirac delta potentials. Finally we consider a barrier potential perturbed by an arbitrary potential $v(x)$.

Chapter 4

SINGLE DIRAC DELTA POTENTIAL

In this section we consider the Hamiltonian of a free particle system ($v_0(x) = 0$) perturbed by the potential

$$v(x) = \delta(x). \tag{4.1}$$

Here we have

$$\Psi^{(0)}(x) = \begin{pmatrix} e^{ikx} \\ e^{-ikx} \end{pmatrix}, \qquad \forall x \in \mathbb{R}. \tag{4.2}$$

Calculating the first order correction for this system gives

$$\begin{aligned} \Psi^{(1)}(x) &= \int_{-\infty}^{x} G(x,x')\delta(x')\Psi^{(0)}(x')\,dx' \\ &= \frac{1}{2ik}\begin{pmatrix} e^{ikx} - e^{-ikx} \\ e^{ikx} - e^{-ikx} \end{pmatrix}\theta(x), \end{aligned} \tag{4.3}$$

where $G(x,x') = \frac{\sin k(x-x')}{k}$, is the Green's function of this system and $\theta(x)$ stands for the step function defined as

$$\theta(x) := \begin{cases} 1 & x \geq 0, \\ 0 & x < 0. \end{cases} \tag{4.4}$$

Similarly the second order correction has the form

$$\begin{aligned} \Psi^{(2)}(x) &= \int_{-\infty}^{x} G(x,x')\delta(x')\Psi^{(1)}(x')\,dx' \\ &= G(x,0)\Psi^{(1)}(0)\theta(x) \\ &= 0, \end{aligned} \tag{4.5}$$

Therefore the general solution is

$$\begin{aligned} \Psi(x) &= \Psi^{(0)}(x) + \varepsilon\Psi^{(1)}(x) \\ &= \begin{pmatrix} e^{ikx} \\ e^{-ikx} \end{pmatrix} + \frac{\varepsilon}{2ik}\begin{pmatrix} e^{ikx} - e^{-ikx} \\ e^{ikx} - e^{-ikx} \end{pmatrix}\theta(x). \end{aligned} \tag{4.6}$$

Using the definition of the θ-function in (4.4) we can express this result as

$$x<0: \qquad \Psi(x)=\begin{pmatrix} e^{ikx} \\ e^{-ikx} \end{pmatrix} \tag{4.7}$$

$$x>0: \qquad \Psi(x)=\begin{pmatrix} (1-\frac{\varepsilon}{2ik})e^{ikx}-\frac{\varepsilon}{2ik}e^{-ikx} \\ \frac{\varepsilon}{2ik}e^{ikx}+(1-\frac{\varepsilon}{2ik})e^{-ikx} \end{pmatrix}. \tag{4.8}$$

Therefore the transfer matrix reads

$$M=\begin{pmatrix} 1+\dfrac{\varepsilon}{2ik} & \dfrac{\varepsilon}{2ik} \\ -\dfrac{\varepsilon}{2ik} & 1-\dfrac{\varepsilon}{2ik} \end{pmatrix}. \tag{4.9}$$

By solving the equation $M_{22}=0$, we obtain a spectral singularity at $\varepsilon = 2ik$. This is consistent with the results of [36].

Chapter 5

DOUBLE DIRAC DELTA POTENTIAL

Consider a free particle system with the following potential

$$v(x) = \delta(x-\alpha) + h\delta(x+\alpha), \quad h \in \mathbb{C},\ \alpha \in \mathbb{R}. \tag{5.1}$$

Using

$$\Psi^{(0)}(x) = \begin{pmatrix} e^{ikx} \\ e^{-ikx} \end{pmatrix}, \quad \forall x \in \mathbb{R} \qquad \text{and} \quad G(x,x') = \frac{\sin k(x-x')}{k}, \tag{5.2}$$

we have

$$\begin{aligned}
\Psi^{(1)}(x) =& \int_{-\infty}^{x} G(x,x')v(x')\Psi^{(0)}(x')\, dx' \\
=& \int_{-\infty}^{x} G(x,x')[\delta(x'-\alpha) + h\delta(x'+\alpha)]\Psi^{(0)}(x')\, dx' \\
=& \int_{-\infty}^{x} G(x,x')\Psi^{(0)}(x')\delta(x'-\alpha)dx' + h\int_{-\infty}^{x} G(x,x')\Psi^{(0)}(x')\delta(x'+\alpha)dx' \\
=& G(x,\alpha)\Psi^{(0)}(\alpha)\theta(x-\alpha) + hG(x,-\alpha)\Psi^{(0)}(-\alpha)\theta(x+\alpha) \\
=& \frac{h}{2ik}\begin{pmatrix} e^{ikx} - (e^{-2ik\alpha}e^{-ikx}) \\ (e^{2ik\alpha}e^{ikx}) - e^{-ikx} \end{pmatrix}\theta(x+\alpha) + \frac{1}{2ik}\begin{pmatrix} e^{ikx} - (e^{2ik\alpha}e^{-ikx}) \\ (e^{-2ik\alpha}e^{-ikx}) - e^{-ikx} \end{pmatrix}\theta(x-\alpha),
\end{aligned} \tag{5.3}$$

which gives

$$\Psi^{(1)}(x) = \begin{cases} 0 & \text{for} \quad x < -\alpha, \\ \frac{h}{2ik}\begin{pmatrix} e^{ikx} - e^{-ik(x+2\alpha)} \\ e^{ik(x+2\alpha)} - e^{-ikx} \end{pmatrix} & \text{for} \quad |x| < \alpha, \\ \frac{1}{2ik}\begin{pmatrix} [1+h]e^{ikx} - [he^{-2ik\alpha} + e^{2ik\alpha}]e^{-ikx} \\ [he^{2ik\alpha} + e^{-2ik\alpha}]e^{ikx} - [1+h]e^{-ikx} \end{pmatrix} & \text{for} \quad x > \alpha. \end{cases} \tag{5.4}$$

Next we compute the second order correction:

$$\begin{aligned}
\Psi^{(2)}(x) &= \int_{-\infty}^{x} G(x,x')[\delta(x'-\alpha)+h\delta(x'+\alpha)]\Psi^{(1)}(x')\,dx' \\
&= \int_{-\infty}^{x} G(x,x')\Psi^{(1)}(x')\delta(x'-\alpha)\,dx' + h\int_{-\infty}^{x} G(x,x')\Psi^{(1)}(x')\delta(x'+\alpha)\,dx' \\
&= G(x,\alpha)\Psi^{(1)}(\alpha)\theta(x-\alpha) + hG(x,-\alpha)\Psi^{(1)}(-\alpha)\theta(x+\alpha) \\
&= \frac{\sin k(x-\alpha)}{k}\left[\frac{1}{2ik}\begin{pmatrix}[1+h]e^{ikx}-[he^{-2ik\alpha}+e^{2ik\alpha}]e^{-ikx} \\ [he^{2ik\alpha}+e^{-2ik\alpha}]e^{ikx}-[1+h]e^{-ikx}\end{pmatrix}\right]_{x=\alpha}\theta(x-\alpha) \\
&\quad+\frac{\sin k(x+\alpha)}{k}\left[\frac{h}{2ik}\begin{pmatrix}e^{ikx}-e^{-ik(x+2\alpha)} \\ e^{ik(x+2\alpha)}-e^{-ikx}\end{pmatrix}\right]_{x=-\alpha}\theta(x+\alpha) \\
&= \frac{h\sin k(x-\alpha)}{2ik^2}\begin{pmatrix}e^{ik\alpha}-e^{-3ik\alpha} \\ e^{3ik\alpha}-e^{-ik\alpha}\end{pmatrix}\theta(x-\alpha),
\end{aligned} \tag{5.5}$$

which gives

$$\Psi^{(2)}(x) = \begin{cases} 0 & \text{for} \quad x<-\alpha, \\ 0 & \text{for} \quad |x|<\alpha, \\ \frac{-h}{4k^2}\begin{pmatrix}[1-e^{-4ik\alpha}]e^{ikx}+[e^{-2ik\alpha}-e^{2ik\alpha}]e^{-ikx} \\ [e^{2ik\alpha}-e^{-2ik\alpha}]e^{ikx}+[1-e^{4ik\alpha}]e^{-ikx}\end{pmatrix} & \text{for} \quad x>\alpha. \end{cases} \tag{5.6}$$

Finally we obtain the third order correction that turns out to vanish identically.

$$\begin{aligned}
\Psi^{(3)}(x) &= \int_{-\infty}^{x} G(x,x')[\delta(x'-\alpha)+h\delta(x'+\alpha)]\Psi^{(2)}(x')\,dx' \\
&= \int_{-\infty}^{x} G(x,x')\Psi^{(2)}(x')\delta(x'-\alpha)\,dx' + h\int_{-\infty}^{x} G(x,x')\Psi^{(2)}(x')\delta(x'+\alpha)\,dx' \\
&= G(x-\alpha)\Psi_2(\alpha)\theta(x-\alpha) + hG(x+\alpha)\Psi_2(-\alpha)\theta(x+\alpha) \\
&= \frac{\sin k(x-\alpha)}{k}\left[\frac{-h}{4k^2}\begin{pmatrix}[1-e^{-4ik\alpha}]e^{ikx}+[e^{-2ik\alpha}-e^{2ik\alpha}]e^{-ikx} \\ [e^{2ik\alpha}-e^{-2ik\alpha}]e^{ikx}+[1-e^{4ik\alpha}]e^{-ikx}\end{pmatrix}\right]_{x=\alpha}\theta(x-\alpha) \\
&= 0.
\end{aligned} \tag{5.7}$$

Therefore for a double Dirac delta potential there is no perturbative correction of order greater than 2. This leads us to conjecture the exactness of the perturbation theory for potentials given by a string of delta functions. We will prove that, indeed, for a potential consisting of n delta functions all perturbative corrections of order greater than n vanish.

Returning to our analysis of the double Dirac delta potential, we have

$$x < -\alpha : \qquad \Psi(x) = \Psi^{(0)}(x), \tag{5.8}$$

$$\begin{aligned} |x| < \alpha : \qquad \Psi(x) &= \Psi^{(0)}(x) + \varepsilon\Psi^{(1)}(x) + \varepsilon^2\Psi^{(2)}(x) \\ &= \begin{pmatrix} e^{ikx} \\ e^{-ikx} \end{pmatrix} + \varepsilon\frac{h}{2ki}\begin{pmatrix} e^{ikx} - e^{-ik(x+2\alpha)} \\ e^{ik(x+2\alpha)} - e^{-ikx} \end{pmatrix} \\ &= \begin{pmatrix} [1 + \frac{\varepsilon}{2ki}(1+h)]e^{ikx} + [-\frac{\varepsilon}{2ki}e^{-2ik\alpha}]e^{-ikx} \\ [\frac{\varepsilon}{2ki}e^{2ik\alpha}]e^{ikx} + [1 - \frac{\varepsilon}{2ki}(1+h)]e^{-ikx} \end{pmatrix}, \end{aligned} \tag{5.9}$$

$$\begin{aligned} x > \alpha : \qquad \Psi(x) =& \Psi^{(0)}(x) + \varepsilon\Psi^{(1)}(x) + \varepsilon^2\Psi^{(2)}(x) \\ =& \begin{pmatrix} e^{ikx} \\ e^{-ikx} \end{pmatrix} + \varepsilon\frac{1}{2ki}\begin{pmatrix} (1+h)e^{ikx} - (he^{-2ik\alpha} + e^{2ik\alpha})e^{-ikx} \\ (he^{2ki\alpha} + e^{-2ki\alpha})e^{ikx} - (1+h)e^{-ikx} \end{pmatrix} \\ &+ \varepsilon^2\frac{-h}{4k^2}\begin{pmatrix} (1 - e^{-4ik\alpha})e^{ikx} + (e^{-2ik\alpha} - e^{2ik\alpha})e^{-ikx} \\ (e^{2ik\alpha} - e^{-2ik\alpha})e^{ikx} + (1 - e^{4ik\alpha})e^{-ikx} \end{pmatrix} \\ =& \begin{pmatrix} \mathcal{A}_{11}e^{ikx} + \mathcal{A}_{12}e^{-ikx} \\ \mathcal{A}_{21}e^{ikx} + \mathcal{A}_{22}e^{-ikx} \end{pmatrix}, \end{aligned} \tag{5.10}$$

where

$$\begin{aligned} \mathcal{A}_{11} :&= 1 + \frac{\varepsilon}{2ki}(1+h) - \frac{\varepsilon^2 h}{4k^2}(1 - e^{-4ik\alpha}) \\ &= 1 - \omega_+ - \omega_- + \omega_+\omega_-(1 - e^{-4ik\alpha}), \end{aligned} \tag{5.11}$$

$$\begin{aligned} \mathcal{A}_{12} :&= \frac{-\varepsilon}{2ki}(he^{-2ik\alpha} + e^{2ik\alpha}) - \frac{\varepsilon^2 h}{4k^2}(e^{-2ik\alpha} - e^{2ik\alpha}) \\ &= \omega_- e^{-2ik\alpha} + \omega_+ e^{2ik\alpha} + \omega_-\omega_+(-2i\sin 2k\alpha), \end{aligned} \tag{5.12}$$

$$\begin{aligned} \mathcal{A}_{21} :&= \frac{\varepsilon}{2ki}(he^{2ik\alpha} + e^{-2ik\alpha}) - \frac{\varepsilon^2 h}{4k^2}(e^{2ik\alpha} - e^{-2ik\alpha}) \\ &= -\omega_- e^{2ik\alpha} - \omega_+ e^{-2ik\alpha} + \omega_-\omega_+(2i\sin 2k\alpha) \\ &= \mathcal{A}_{21}^*, \end{aligned} \tag{5.13}$$

$$\begin{aligned} \mathcal{A}_{22} :&= 1 - \frac{\varepsilon}{2ki}(1+h) - \frac{\varepsilon^2 h}{4k^2}(1 - e^{4ik\alpha}) \\ &= 1 + \omega_+ + \omega_- + \omega_+\omega_-(1 - e^{4ik\alpha}) \\ &= \mathcal{A}_{11}^*, \end{aligned} \tag{5.14}$$

$\omega_\pm := \frac{i\zeta_\pm}{2k}$, $\zeta_+ := \varepsilon$ and $\zeta_- = \varepsilon h$.

For this system the transfer matrix is given by

$$M = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{21} \\ \mathcal{A}_{12} & \mathcal{A}_{22} \end{pmatrix}. \tag{5.15}$$

This coincides with the analytic result given in [12]. Next we set

$$M_{22} = 1 + \frac{i\zeta_+}{2k} + \frac{i\zeta_-}{2k} - \frac{\zeta_+\zeta_-}{4k^2}(1 - e^{4ik\alpha}) = 0, \tag{5.16}$$

to obtain spectral singularities. For the special case where $\zeta_+ = 0$ and $\zeta := \zeta_-$ or $\zeta_- = 0$ and $\zeta := \zeta_+$, Eq. (5.16) gives

$$1 + \frac{i\zeta}{2k} = 0, \tag{5.17}$$

and it determines a spectral singularity located at $k = -i\zeta/2$ which is the case for a single Dirac delta potential.

Chapter 6

DIRAC COMB

In this section we consider the potential given by a string of Dirac delta potentials and explore the spectral singularities for such a system. Dirac comb is a distribution constructed from δ-functions, [37], and reads

$$v(x) = \text{Ш}(x) := \sum_{n=0}^{N-1} \delta(x - nl), \qquad \text{label of delta barrier} = n + 1. \tag{6.1}$$

To study the spectral singularities of the Dirac comb we apply the method developed in Chapter 3 to construct its transfer matrix.

By using the sampling property of Dirac comb, namely

$$G(x, x')\, \text{Ш}\,(x') = \sum_{n=0}^{N-1} G(x, nl)\delta(x' - nl), \tag{6.2}$$

and employing (3.8) and (3.13), for the the first order correction we write

$$\begin{aligned}\Psi^{(1)}(x) &= \int_{-\infty}^{x} G(x, x')\, \text{Ш}_l\,(x')\Psi^{(0)}(x')\, dx' \\ &= \sum_{n_1=0}^{N-1} G(x, n_1 l)\Psi^{(0)}(n_1 l)\theta(x - n_1 l),\end{aligned} \tag{6.3}$$

and the corresponding second order correction will be

$$\begin{aligned}\Psi^{(2)}(x) &= \int_{-\infty}^{x} G(x, x_2)\, \text{Ш}_l\,(x_2)\Psi^{(1)}(x_2) dx_2 \\ &= \int_{-\infty}^{x} G(x, x_2) \sum_{n_2=0}^{N-1} \delta(x_2 - n_2 l) \Big[\int_{-\infty}^{x_2} G(x_2, x_1) \sum_{n_1=0}^{n_2-1} \delta(x_1 - n_1 l)\Psi^{(0)}(x_1)\, dx_1 \Big] dx_2 \\ &= \int_{-\infty}^{x} \int_{-\infty}^{x_2} \sum_{n_2=0}^{N-1} \sum_{n_1=0}^{n_2-1} G(x, n_2 l)\delta(x_2 - n_2 l)G(x_2, n_1 l)\delta(x_1 - n_1 l)\Psi^{(0)}(x_1)\, dx_1\, dx_2 \\ &= \sum_{n_2=0}^{N-1} \sum_{n_1=0}^{n_2-1} G(x, n_2 l)G(n_2 l, n_1 l)\Psi^{(0)}(n_1 l)\theta(x - n_2 l).\end{aligned} \tag{6.4}$$

Similarly for the j-th order correction ($j > 2$) we get the following general recursion relation:

$$\begin{aligned}\Psi^{(j>2)}(x) &= \int_{-\infty}^{x_0} G(x, x_j)\, Ш_l\,(x_j) \left[\prod_{i=1}^{j-1} \int_{-\infty}^{x_{i+1}} G(x_{i+1}, x_i)\, Ш\,(x_i) \Psi^{(0)}(x_1)\, dx_i \right] dx_j \\ &= \int_{-\infty}^{x_0} \sum_{n_j=0}^{N-1} G(x, n_j l) \delta(x - n_j l) \times \\ &\quad \left[\prod_{i=1}^{j-1} \int_{-\infty}^{x_{i+1}} \sum_{n_i=0}^{n_{i+1}-1} G(x_{i+1}, n_i l) \delta(x_i - n_i l) \begin{pmatrix} e^{ikx_1} \\ e^{-ikx_1} \end{pmatrix} dx_i \right] dx_j \\ &= \sum_{n_j=0}^{N-1} G(x, n_j l) \prod_{i=1}^{j-1} \sum_{n_i=0}^{n_{i+1}-1} G(n_{i+1} l, n_i l) \begin{pmatrix} e^{ikx_1} \\ e^{-ikx_1} \end{pmatrix} \theta(x - n_j l), \end{aligned} \tag{6.5}$$

where we used, $\Psi^{(0)}(x) = \begin{pmatrix} e^{ikx} \\ e^{-ikx} \end{pmatrix}$, $\forall x \in \mathbb{R}$, and have $G(x, x') = \frac{\sin k(x - x')}{k}$. By defining $\chi := e^{2ikl}$, the first order correction takes the form

$$\begin{aligned}\Psi^{(1)}(x) &= \frac{1}{k} \sum_{n_1=0}^{N-1} \sin k(x - n_1 l) \begin{pmatrix} e^{ikn_1 l} \\ e^{-ikn_1 l} \end{pmatrix} \theta(x - n_1 l) \\ &= \frac{1}{2ik} \sum_{n_1=0}^{N-1} \begin{pmatrix} e^{ikx} - \chi^{n_1} e^{-ikx} \\ \chi^{-n_1} e^{ikx} - e^{-ikx} \end{pmatrix} \theta(x - n_1 l). \end{aligned} \tag{6.6}$$

In the same manner the second order correction reads

$$\begin{aligned}\Psi^{(2)}(x) =& \frac{1}{k} \sum_{n_2=0}^{N-1} \sin k(x - n_2 l) \Psi^{(1)}(n_2 l) \theta(x - n_2 l) \\ =& (\frac{1}{2ik})^2 \sum_{n_2=0}^{N-1} \sum_{n_1=0}^{n_2-1} \begin{pmatrix} (1 - \chi^{n_1 - n_2})[e^{ikx} - \chi^{n_2} e^{-ikx}] \\ (1 - \chi^{-(n_1 - n_2)})[-\chi^{-n_2} e^{ikx} + e^{-ikx}] \end{pmatrix} \theta(x - n_2 l), \end{aligned} \tag{6.7}$$

and the third order correction becomes

$$\begin{aligned}\Psi^{(3)}(x) =& (\frac{1}{2ik})^3 \sum_{n_3=0}^{N-1} \sum_{n_2=0}^{n_3-1} \sum_{n_1=0}^{n_2-1} \\ & \begin{pmatrix} [(1 - \chi^{n_1 - n_2})(1 - \chi^{n_2 - n_3})](e^{ikx} - \chi^{n_3} e^{-ikx}) \\ [(1 - \chi^{-(n_1 - n_2)})(1 - \chi^{-(n_2 - n_3)})](\chi^{-n_3} e^{ikx} - e^{-ikx}) \end{pmatrix} \theta(x - n_3 l). \end{aligned} \tag{6.8}$$

By repeating the same calculation, the j-th order correction is given by

$$\Psi^{(j)}(x) = \begin{pmatrix} \big[(F^+)_j\big] e^{ikx} - \big[(F^+)_j \chi^{n_j}\big] e^{-ikx} \\ \\ -\big[(F_-)_j \chi^{-n_j}\big] e^{ikx} + \big[(F_-)_j\big] e^{ikx} \end{pmatrix} \theta(x - n_j l), \tag{6.9}$$

where

$$(F^{+})_j := \Big(\frac{1}{2ik}\Big)^j \sum_{n_j=0}^{N-1} \sum_{n_{j-1}=0}^{n_j-1} \cdots \sum_{n_1=0}^{n_2-1} \Bigg(\prod_{i=1}^{j-1} \Big[1-\chi^{\pm(n_i-n_{i+1})}\Big] \Bigg), \tag{6.10}$$

$$(F_{-})_j := \Big(\frac{1}{-2ik}\Big)^j \sum_{n_j=0}^{N-1} \sum_{n_{j-1}=0}^{n_j-1} \cdots \sum_{n_1=0}^{n_2-1} \Bigg(\prod_{i=1}^{j-1} \Big[1-\chi^{\pm(n_i-n_{i+1})}\Big] \Bigg), \tag{6.11}$$

or in a more compact form

$$(F^{+}_{-})_j := \Big(\frac{1}{\pm 2ik}\Big)^j \sum_{n_j=0}^{N-1} \sum_{n_{j-1}=0}^{n_j-1} \cdots \sum_{n_1=0}^{n_2-1} \Bigg(\prod_{i=1}^{j-1} \Big[1-\chi^{\pm(n_i-n_{i+1})}\Big] \Bigg). \tag{6.12}$$

Now we will examine this expression for the system of double Dirac delta potential with the unit coupling constants

$$v(x) = \sum_{n_1=0}^{1} \delta(x-n_1 l) = \delta(x) + \delta(x-l). \tag{6.13}$$

Note that $\prod_i^{j<i} = 1$, and applying the definition (6.12) for the case $N=2$, $j=1$, we obtain

$$(F^{+}_{-})_1 = \frac{1}{\pm 2ik} \sum_{n_1=0}^{1} \prod_{i=1}^{0} = \frac{1}{\pm 2ik} \sum_{n_1=0}^{1} 1 = \frac{2}{\pm 2ik}, \tag{6.14}$$

$$(F^{+}_{-})_1 \chi^{\pm n_1} = \frac{1}{\pm 2ik} \sum_{n_1=0}^{1} \chi^{\pm n_1} = \frac{1}{\pm 2ik}(1+\chi^{\pm 1}), \tag{6.15}$$

therefore

$$\Psi^{(1)}(x) = \frac{1}{2ik} \begin{pmatrix} 2e^{ikx} - (1+\chi)e^{-ikx} \\ (1+\chi^{-1})e^{ikx} - 2e^{-ikx} \end{pmatrix}. \tag{6.16}$$

For the second order correction we have

$$\begin{aligned} (F^{+}_{-})_2 &= (\frac{1}{\pm 2ik})^2 \sum_{n2=0}^{1} \sum_{n_1=0}^{n_2-1} \prod_{i=1}^{1} \Big[1-\chi^{\pm(n_i-n_{i+1})}\Big] \\ &= (\frac{1}{\pm 2ik})^2 \sum_{n_1=0}^{0} (1-\chi^{\pm(-1)}) \\ &= (\frac{1}{\pm 2ik})^2 (1-\chi^{\mp 1}), \end{aligned} \tag{6.17}$$

$$(F^{+}_{-})_2 \chi^{n_2} = (\frac{1}{\pm 2ik})^2 (\chi^{\pm 1} - 1), \tag{6.18}$$

which results in

$$\Psi^{(2)}(x) =(\frac{1}{2ik})^2 \begin{pmatrix} (1-\chi^{-1})e^{ikx} + (1-\chi)e^{-ikx} \\ (1-\chi^{-1})e^{ikx} + (1-\chi)e^{-ikx} \end{pmatrix}. \tag{6.19}$$

To compute the third order correction we set $N = 2, j = 3$. But it turns out that

$$(F_-^+)_3 = 0, \tag{6.20}$$

and hence $\Psi^{(3)}(x) = 0$. Therefore we generally have

$$\begin{aligned}
\Psi(x) =& \Psi^{(0)}(x) + \varepsilon\Psi^{(1)}(x) + \varepsilon^2\Psi^{(2)}(x) \\
=& \begin{pmatrix} e^{ikx} \\ e^{-ikx} \end{pmatrix} + \frac{\varepsilon}{2ik} \begin{pmatrix} 2e^{ikx} - (1+\chi)e^{-ikx} \\ (1+\chi^{-1})e^{ikx} - 2e^{-ikx} \end{pmatrix} \\
&+ (\frac{\varepsilon}{2ik})^2 \begin{pmatrix} (1-\chi^{-1})e^{ikx} + (1-\chi)e^{-ikx} \\ (1-\chi^{-1})e^{ikx} + (1-\chi)e^{-ikx} \end{pmatrix}.
\end{aligned}$$

In view of $\chi = e^{2ikl}$, the Transfer matrix takes the form

$$M = \begin{pmatrix} 1 + \frac{\varepsilon}{ik} + (\frac{\varepsilon}{2ik})^2(1 - e^{-2ikl}) & \frac{\varepsilon}{2ik}(1 + e^{-2ikl}) + (\frac{\varepsilon}{2ik})^2(1 - e^{-2ikl}) \\ -\frac{\varepsilon}{2ik}(1 + e^{2ikl}) + (\frac{\varepsilon}{2ik})^2(1 - e^{2ikl}) & 1 - \frac{\varepsilon}{ik} + (\frac{\varepsilon}{2ik})^2(1 - e^{2ikl}) \end{pmatrix},$$

which is the transfer matrix of a system of double Dirac delta potential with delta barriers located at $x = 0$ and $x = l$ on the real line.

The following Theorem, stating the exactness of perturbation for a finite Dirac comb. It simplifies the calculation of the wave functions and the construction of the transfer matrix.

Theorem. For a finite Dirac comb, as a finite series of N δ-function potential, perturbation corrections of order higher than N vanish (see Appx. B for the proof).

This Theorem implies that for the potential (6.1), $\Psi^{(j>N)}(x) = 0$. Using this theorem and the perturbation ansatz the wave function is

$$\begin{aligned}
\Psi(x) =& \sum_{j=0}^{N} \varepsilon^j \Psi^{(j)}(x) \\
=& \sum_{j=0}^{N} \varepsilon^j \begin{pmatrix} [(F^+)_j]e^{ikx} - [(F^+)_j\chi^{n_j}]e^{-ikx} \\ [-(F_-)_j\chi^{-n_j}]e^{ikx} + [(F_-)_j]e^{-ikx} \end{pmatrix},
\end{aligned} \tag{6.21}$$

and

$$M = \sum_{j=0}^{N} \varepsilon^j \begin{pmatrix} (F^+)_j & -(F_-)_j \chi^{-n_j} \\ \\ -(F^+)_j \chi^{n_j} & (F_-)_j \end{pmatrix}. \tag{6.22}$$

Consequently to identify the spectral singularities we look for the zeros of M_{22} given by

$$\begin{aligned} M_{22}(k,\varepsilon) &= \sum_{j=0}^{N} \varepsilon^j (F_-)_j \\ &= \sum_{j=0}^{N} \big(\frac{\varepsilon}{-2ik}\big)^j \sum_{n_j=0}^{N-1} \sum_{n_{j-1}=0}^{n_j-1} \cdots \sum_{n_1=0}^{n_2-1} \prod_{i=1}^{j-1} (1-\chi^{-(n_i-n_{i+1})}). \end{aligned} \tag{6.23}$$

Using (6.23), we see that M_{22} is a polynomial function in ε. For a given N (the number of delta functions), computing this polynomial function is a trivial task. Considering the first order perturbation we find

$$(F_-)_1 = \frac{1}{2ik} \sum_{n_1=0}^{N-1} 1 = \frac{N}{2ik}. \tag{6.24}$$

In this case for determining the spectral singularities we write

$$M_{22} = 1 - \frac{N\varepsilon}{2ik} = 0 \qquad \Longrightarrow \qquad k = \frac{N\varepsilon}{2i}. \tag{6.25}$$

This intriguing result, implies that up to the first order the Dirac comb with N delta function and identical coupling constants acts as a single delta function with coupling constant $N\varepsilon$.

So far we have considered the Dirac comb as a chain of delta functions with identical coupling constants. More generally, the Dirac comb for a chain of delta functions having different coupling constants reads

$$v(x) = \sum_{n=0}^{N-1} \epsilon_n \delta(x - nl), \tag{6.26}$$

where $\epsilon_n \in \mathbb{C}$. Specifying the location of spectral singularities for this potential is a difficult problem. But, the qualitative consideration can provide some understanding of the nature of these points. Recall that

$$M_{22} = \sum_{j=1}^{N} \varepsilon^j (\mathfrak{F}_-)_j, \tag{6.27}$$

where

$$(\mathfrak{F}^+_-)_j := \Big(\frac{1}{\pm 2ik}\Big)^j \sum_{n_j=0}^{N-1} \sum_{n_{j-1}=0}^{n_j-1} \cdots \sum_{n_1=0}^{n_2-1} \Big(\prod_{i=1}^{j} \epsilon_{n_i}\Big) \Big(\prod_{i=1}^{j-1} \Big[1 - \chi^{\pm(n_i-n_{i+1})}\Big]\Big). \tag{6.28}$$

Up to the first order, the spectral singularities for this system is given by

$$k = \frac{\varepsilon}{2i} \sum_{n_1=0}^{N-1} \epsilon_{n_1}. \tag{6.29}$$

Here, our results may be extended to explore the spectral singularities of both systems (6.1) and (6.26) up to any order of interest.

From another point of view a Dirac comb can be considered as a system of single delta functions. Therefore the transfer matrix of the whole system is the product of the transfer matrix of these single delta functions with identical coupling constants,

$$\begin{aligned} \mathbb{M} &= \mathbb{M}_n(\varepsilon)\mathbb{M}_{n-1}(\varepsilon)\cdots\mathbb{M}_1(\varepsilon) \\ &= \mathbb{N}(\varepsilon)\mathbb{M}_1(\varepsilon) \\ &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \end{aligned} \tag{6.30}$$

where $\alpha, \beta, \gamma, \delta$ are all functions of the variable ε. M_{ij}'s are the entries of the transfer matrix for the single delta function located at $x = \alpha$ with constant ε given by

$$\begin{pmatrix} 1 + \frac{\varepsilon}{2ik} & \frac{\varepsilon}{2ik} e^{-2ik\alpha} \\ -\frac{\varepsilon}{2ik} e^{2ik\alpha} & 1 - \frac{\varepsilon}{2ik} \end{pmatrix}. \tag{6.31}$$

Thus

$$\begin{aligned} \mathbb{M}_{22} &= \gamma M_{12} + \delta M_{22} \\ &= (\gamma a + \delta c) + (\gamma b + \delta d)\varepsilon, \end{aligned} \tag{6.32}$$

where we set $M_{12} = a + b\varepsilon,\ M_{22} = c + d\varepsilon$. $\mathbb{M}_{22} = 0$ yields an expression, e.g., $\mathscr{F}_n(k, \varepsilon) = 0$, where $\mathscr{F}_n$ is a polynomial function. This describes the surfaces in the space of coupling constants on which the spectral singularities reside.

Chapter 7

PERTURBING THE BARRIER POTENTIAL

In this section we consider the perturbation over the barrier potential $v_0(x) = \xi$, having compact support in the interval $[-\alpha, \alpha]$, given by $v(x)$ defined in the same interval. The zeroth order correction is given by the solutions of the Schrödinger equation (3.5) if $\varepsilon = 0$. In view of (3.10), The Green's function for this equation is

$$\mathrm{G}(x, x') = \begin{cases} \frac{\sin k(x-x')}{k} & x < -\alpha, \\ \frac{\sin k'(x-x')}{k'} & |x| < \alpha, \\ \frac{\sin k(x-x')}{k} & x > \alpha, \end{cases} \tag{7.1}$$

where $k' := \sqrt{k^2 - \xi}$. The zeroth order correction for the wave function is therefore

$$\Psi^{(0)}(x) = \begin{cases} \begin{pmatrix} e^{ikx} \\ e^{-ikx} \end{pmatrix} & \text{for} \quad (x < -\alpha), \\ \\ \begin{pmatrix} A(k,\omega)e^{ik'x} + A(k,-\omega)e^{-ik'x} \\ A(-k,\omega)e^{ik'x} + A(-k,-\omega)e^{-ik'x} \end{pmatrix} & \text{for} \quad (|x| < \alpha), \\ \\ \begin{pmatrix} M_{11}e^{ikx} + M_{21}e^{-ikx} \\ M_{12}e^{ikx} + M_{22}e^{-ikx} \end{pmatrix} & \text{for} \quad (x > \alpha), \end{cases} \tag{7.2}$$

where

$$A(k, \omega) := \frac{1}{2}(1 + \frac{1}{\omega})e^{-ik(1-\omega)\alpha}, \tag{7.3}$$

and M_{ij} are the entries of the matrix

$$M^{(0)} = \begin{pmatrix} e^{-2i\alpha k}\frac{(1+\omega)^2 e^{2i\alpha k\omega} - (1-\omega)^2 e^{-2i\alpha k\omega}}{4\omega} & \frac{i(\omega^2-1)}{2\omega}\sin(2\alpha k\omega) \\ -\frac{i(\omega^2-1)}{2\omega}\sin(2\alpha k\omega) & e^{2i\alpha k}\frac{(1+\omega)^2 e^{-2i\alpha k\omega} - (1-\omega)^2 e^{2i\alpha k\omega}}{4\omega} \end{pmatrix}, \tag{7.4}$$

where $\omega := \frac{k}{k'}$, (Appx. B).

In light of (7.1) and (7.2), the first order correction has the form

$x < -\alpha$:

$$\Psi^{(1)}(x) = \int_{-\infty}^{-\alpha} G(x,x')v(x')\Psi^{(0)}(x')dx' = 0, \tag{7.5}$$

$|x| < \alpha$:

$$\Psi^{(1)}(x) = \int_{-\alpha}^{x} G(x,x')v(x')\Psi^{(0)}(x')dx', \tag{7.6}$$

$x > \alpha$:

$$\begin{aligned}\Psi^{(1)}(x) &= \int_{-\alpha}^{\alpha} G(x,x')v(x')\Psi^{(0)}(x')dx' \\ &= \int_{-\alpha}^{\alpha} \frac{\sin k(x-x')}{k} v(x') \begin{pmatrix} A(k,\omega)e^{i\omega kx'} + A(k,-\omega)e^{-i\omega kx'} \\ A(-k,\omega)e^{i\omega kx'} + A(-k,-\omega)e^{-i\omega kx'} \end{pmatrix} dx' \\ &= \int_{-\alpha}^{\alpha} \begin{pmatrix} \partial_{x'}\Delta(-k,\omega,x')e^{ikx} + \partial_{x'}\Theta(k,\omega,x')e^{-ikx} \\ \partial_{x'}\Theta(-k,\omega,x')e^{ikx} + \partial_{x'}\Delta(k,\omega,x')e^{-ikx} \end{pmatrix} dx' \\ &= \begin{pmatrix} \mathcal{T}(-k,\omega)e^{ikx} + \mathcal{R}(k,\omega)e^{-ikx} \\ \mathcal{R}(-k,\omega)e^{ikx} + \mathcal{T}(k,\omega)e^{-ikx} \end{pmatrix}, \end{aligned} \tag{7.7}$$

where

$$\partial_{x'}\Delta(k,\omega,x') := \frac{-v(x')}{2ik}\left[A(-k,\omega)e^{ik(1+\omega)x'} + A(-k,-\omega)e^{ik(1-\omega)x'}\right], \tag{7.8}$$

$$\partial_{x'}\Theta(k,\omega,x') := \frac{-v(x')}{2ik}\left[A(k,\omega)e^{ik(1+\omega)x'} + A(k,-\omega)e^{ik(1-\omega)x'}\right], \tag{7.9}$$

$$\mathcal{T}(k,\omega) := \int_{-\alpha}^{\alpha} \partial_{x'}\Delta(k,\omega,x')dx', \tag{7.10}$$

$$\mathcal{R}(k,\omega) := \int_{-\alpha}^{\alpha} \partial_{x'}\Theta(k,\omega,x')dx'. \tag{7.11}$$

Similar formulas can be written for the case when $k \to -k$. The valid solution up to first order in ε is given by:

For $x < -\alpha$:

$$\Psi(x) = \Psi_0(x) = \begin{pmatrix} e^{ikx} \\ e^{-ikx} \end{pmatrix}, \tag{7.12}$$

For $x > \alpha$:

$$\Psi(x) = \begin{pmatrix} M_{11}^{(0)} e^{ikx} + M_{21}^{(0)} e^{-ikx} \\ M_{12}^{(0)} e^{ikx} + M_{22}^{(0)} e^{-ikx} \end{pmatrix} + \varepsilon \begin{pmatrix} \mathcal{T}(-k,\omega) e^{ikx} + \mathcal{R}(k,\omega) e^{-ikx} \\ \mathcal{R}(-k,\omega) e^{ikx} + \mathcal{T}(k,\omega) e^{-ikx} \end{pmatrix}$$
$$= \begin{pmatrix} \left[M_{11}^{(0)} + \varepsilon \mathcal{T}(-k,\omega)\right] e^{ikx} + \left[M_{21}^{(0)} + \varepsilon \mathcal{R}(k,\omega)\right] e^{-ikx} \\ \\ \left[M_{12}^{(0)} + \varepsilon \mathcal{R}(-k,\omega)\right] e^{ikx} + \left[M_{22}^{(0)} + \varepsilon \mathcal{T}(k,\omega)\right] e^{-ikx} \end{pmatrix}. \tag{7.13}$$

Therefore the transfer matrix has the form

$$M = \begin{pmatrix} M_{11}^{(0)} + \varepsilon \mathcal{T}(-k,\omega) & M_{12}^{(0)} + \varepsilon \mathcal{R}(-k,\omega) \\ M_{21}^{(0)} + \varepsilon \mathcal{R}(k,\omega) & M_{22}^{(0)} + \varepsilon \mathcal{T}(k,\omega) \end{pmatrix}$$
$$= M_{\xi}^{(0)} + \varepsilon M^{(1)}, \tag{7.14}$$

where

$$M^{(1)} := \begin{pmatrix} \mathcal{T}(-k,\omega) & \mathcal{R}(k,\omega) \\ \mathcal{R}(-k,\omega) & \mathcal{T}(k,\omega) \end{pmatrix}. \tag{7.15}$$

According to (7.14),

$$M_{22}(k,\xi) = M_{22}^{(0)}(k,\xi) + \varepsilon \mathcal{T}(k,\xi), \tag{7.16}$$

where $k \in \mathbb{R}$ and $\xi \in \mathbb{C}$. $M_{22}(k,\xi) = 0$ will give the locations of spectral singularities. To continue we assume the following set of approximations

$$k = k^{(0)} + \varepsilon k^{(1)}, \qquad \xi = \xi^{(0)} + \varepsilon \xi^{(1)}. \tag{7.17}$$

$$\xi, \xi^{(0)}, \xi^{(1)} \in \mathbb{C}, \quad k, k^{(0)}, k^{(1)}, \varepsilon \in \mathbb{R}, \qquad \text{and let} \qquad \xi^{(1)} = \xi_r^{(1)} + i\xi_i^{(1)}, \tag{7.18}$$

where $(k,\xi) \in \Omega$ and $(k^{(0)}, \xi^{(0)}) \in \Omega_0$, so that

$$\Omega_0 := \left\{(k^{(0)}, \xi^{(0)}) \middle| M_{22}^{(0)}(k^{(0)}, \xi^{(0)}) = 0\right\}, \quad \Omega := \left\{(k,\xi) \middle| M_{22}(k,\xi) = 0\right\}, \tag{7.19}$$

where Ω_0 is the set of all spectral singularities for the barrier potential ($\varepsilon = 0$) and Ω is the set of spectral singularities for the case ($\varepsilon \neq 0$).

Ref. [40], gives a complete analysis of the set Ω_0. If we define $\frac{\xi^{(0)}}{k^{(0)2}} := \rho^{(0)} + i\sigma^{(0)}$, then $\rho^{(0)}$ and $\sigma^{(0)}$ satisfy

$$\sinh^2 \left\{ [n\pi - g_1(\rho^{(0)}, \sigma^{(0)})] \sqrt{1 - \frac{2(1-\rho^{(0)})}{g_2(\rho^{(0)}, \sigma^{(0)})}} \right\} = \frac{2g_2(\rho^{(0)}, \sigma^{(0)})}{\rho^{(0)2} + \sigma^{(0)2}}, \tag{7.20}$$

$$g_1(\rho^{(0)},\sigma^{(0)}) := \cos^{-1}\left[\frac{1-\sqrt{(1-\rho^{(0)})^2+\sigma^{(0)2}}}{\sqrt{\rho^{(0)2}+\sigma^{(0)2}}}\right],$$
$$g_2(\rho^{(0)},\sigma^{(0)}) := 1-\rho^{(0)}+\sqrt{\rho^{(0)2}+\sigma^{(0)2}}.$$

These define an infinite set of curves in the $\rho-\sigma$ plane. Each value of $(\rho^{(0)},\sigma^{(0)})$ on one of these curves gives a spectral singularity with the corresponding wave number

$$k^{(0)} = \frac{n\pi - g_1(\rho^{(0)},\sigma^{(0)})}{\alpha\sqrt{2g_2(\rho^{(0)},\sigma^{(0)})}}. \tag{7.21}$$

Substituting (7.17) in (7.16) and neglecting terms of order ε^2 and higher, we set

$$\begin{aligned}
M_{22}(k,\xi) =& M_{22}^{(0)}(k,\xi) + \varepsilon\mathcal{T}(k,\xi)\\
=& M_{22}^{(0)}(k^{(0)}+\varepsilon k^{(1)},\xi^{(0)}+\varepsilon\xi^{(1)}) + \varepsilon\mathcal{T}(k^{(0)}+\varepsilon k^{(1)},\xi^{(0)}+\varepsilon\xi^{(1)})\\
=& M_{22}^{(0)}(k^{(0)},\xi^{(0)}) + \varepsilon\Big[k^{(1)}\partial_k M_{22}^{(0)}(k,\xi) + \xi^{(1)}\partial_\xi M_{22}^{(0)}(k,\xi) + \mathcal{T}(k^{(0)},\xi^{(0)})\\
&+ \varepsilon k^{(1)}\partial_k\mathcal{T}(k,\xi) + \varepsilon\xi^{(1)}\partial_\xi\mathcal{T}(k,\xi)\Big]_{(k^{(0)},\xi^{(0)})}\\
=& \varepsilon\Big[k^{(1)}\partial_k M_{22}^{(0)}(k,\xi)\big|_{(k^{(0)},\xi^{(0)})} + \xi^{(1)}\partial_\xi M_{22}^{(0)}(k,\xi)\big|_{(k^{(0)},\xi^{(0)})} + \mathcal{T}(k^{(0)},\xi^{(0)})\Big]\\
&+ M_{22}^{(0)}(k^{(0)},\xi^{(0)}).
\end{aligned}$$

Since $M_{22}^{(0)}(k^{(0)},\xi^{(0)}) = 0$, $M_{22}(k,\xi)=0$ requires

$$k^{(1)}\partial_k M_{22}^{(0)}(k,\xi)\big|_{(k^{(0)},\xi^{(0)})} + \xi^{(1)}\partial_\xi M_{22}^{(0)}(k,\xi)\big|_{(k^{(0)},\xi^{(0)})} + \mathcal{T}(k^{(0)},\xi^{(0)}) = 0. \tag{7.22}$$

In view of (7.4) and (7.18) and introducing

$$\mathring{\mathcal{P}}_r + i\mathring{\mathcal{P}}_i := \partial_k M_{22}^{(0)}(k,\xi)\big|_{(k^{(0)},\xi^{(0)})}, \tag{7.23}$$
$$\mathring{\mathcal{Q}}_r + i\mathring{\mathcal{Q}}_i := \partial_\xi M_{22}^{(0)}(k,\xi)\big|_{(k^{(0)},\xi^{(0)})}, \tag{7.24}$$
$$\mathring{\mathcal{T}}_r + i\mathring{\mathcal{T}}_i := \mathcal{T}(k^{(0)},\xi^{(0)}), \tag{7.25}$$

we can express Eq.(7.22) as the following system of linear equations in three unknowns $k^{(1)}$, $\xi_r^{(1)}$ and $\xi_i^{(1)}$.

$$\begin{cases} k^{(1)}\mathring{\mathcal{P}}_r + \xi_r^{(1)}\mathring{\mathcal{Q}}_r - \xi_i^{(1)}\mathring{\mathcal{Q}}_i = -\mathring{\mathcal{T}}_r,\\ k^{(1)}\mathring{\mathcal{P}}_i + \xi_r^{(1)}\mathring{\mathcal{Q}}_i + \xi_i^{(1)}\mathring{\mathcal{Q}}_r = -\mathring{\mathcal{T}}_i,\end{cases} \tag{7.26}$$

where the mathring on $\mathcal{T}$, $\mathcal{P}$ and $\mathcal{Q}$ shows that these quantities have been evaluated at the point $(k^{(0)}, \xi^{(0)})$. The system of equations, written above, can be put in the following form

$$\begin{cases} \Big[\mathring{\mathcal{P}}_i\mathring{\mathcal{Q}}_r - \mathring{\mathcal{P}}_r\mathring{\mathcal{Q}}_i\Big]\xi_r^{(1)} + \Big[\mathring{\mathcal{P}}_i\mathring{\mathcal{Q}}_i + \mathring{\mathcal{P}}_r\mathring{\mathcal{Q}}_r\Big]\xi_i^{(1)} = \Big[-\mathring{\mathcal{P}}_i\mathring{\mathcal{T}}_r + \mathring{\mathcal{P}}_r\mathring{\mathcal{T}}_i\Big], \\ \\ k^{(1)} = -\dfrac{\mathring{\mathcal{T}}_i + \mathring{\mathcal{Q}}_i\xi_r^{(1)} + \mathring{\mathcal{Q}}_r\xi_i^{(1)}}{\mathring{\mathcal{P}}_i}. \end{cases} \tag{7.27}$$

Next we define $\rho + i\sigma := \frac{\xi}{k}$ and employ (7.17) and (7.18) to write

$$\begin{aligned} \rho + i\sigma &= \frac{\xi^{(0)} + \varepsilon\xi^{(1)}}{[k^{(0)} + \varepsilon k^{(1)}]^2} \\ &= \frac{\xi^{(0)} + \varepsilon\xi^{(1)}}{k^{(0)2}}\Big[1 + \varepsilon\frac{k^{(1)}}{k^{(0)}}\Big]^{-2} \\ &= \Big[(\rho^{(0)} + i\sigma^{(0)}) + \varepsilon\frac{\xi^{(1)}}{k^{(0)2}}\Big]\Big[1 - 2\varepsilon\frac{k^{(1)}}{k^{(0)}}\Big] \\ &= \Big[\big(1 - 2\varepsilon\frac{k^{(1)}}{k^{(0)}}\big)\rho^{(0)} + \varepsilon\frac{\xi_r^{(1)}}{k^{(0)2}}\Big] + i\Big[\big(1 - 2\varepsilon\frac{k^{(1)}}{k^{(0)}}\big)\sigma^{(0)} + \varepsilon\frac{\xi_i^{(1)}}{k^{(0)2}}\Big], \end{aligned} \tag{7.28}$$

therefore

$$\begin{cases} \rho = \big(1 - 2\varepsilon\frac{k^{(1)}}{k^{(0)}}\big)\rho^{(0)} + \varepsilon\frac{\xi_r^{(1)}}{k^{(0)2}}, \\ \sigma = \big(1 - 2\varepsilon\frac{k^{(1)}}{k^{(0)}}\big)\sigma^{(0)} + \varepsilon\frac{\xi_i^{(1)}}{k^{(0)2}}. \end{cases} \tag{7.29}$$

The system (7.29) gives the spectral singularities of the perturbed barrier potential based on the spectral singularities of the unperturbed system, namely, $\rho^{(0)}$ and $\sigma^{(0)}$. For instance, we compute the spectral singularities in case $k = k^{(0)}$ meaning that $k^{(1)} = 0$. This choice helps us compare the shifts in the spectral singularity curves of the barrier potential due to the perturbation in different values of $k = k^{(0)}$. In the following (Table.7.1, Fig.7.1) we give the numerical values of these points for the barrier potential perturbed by the potential $v(x) = \cosh x$ having compact support in interval $[-1/2, 1/2]$ for perturbation strength $\varepsilon = 0$(unperturbed barrier potential)$, 0.1, 0.01$, and 0.001.

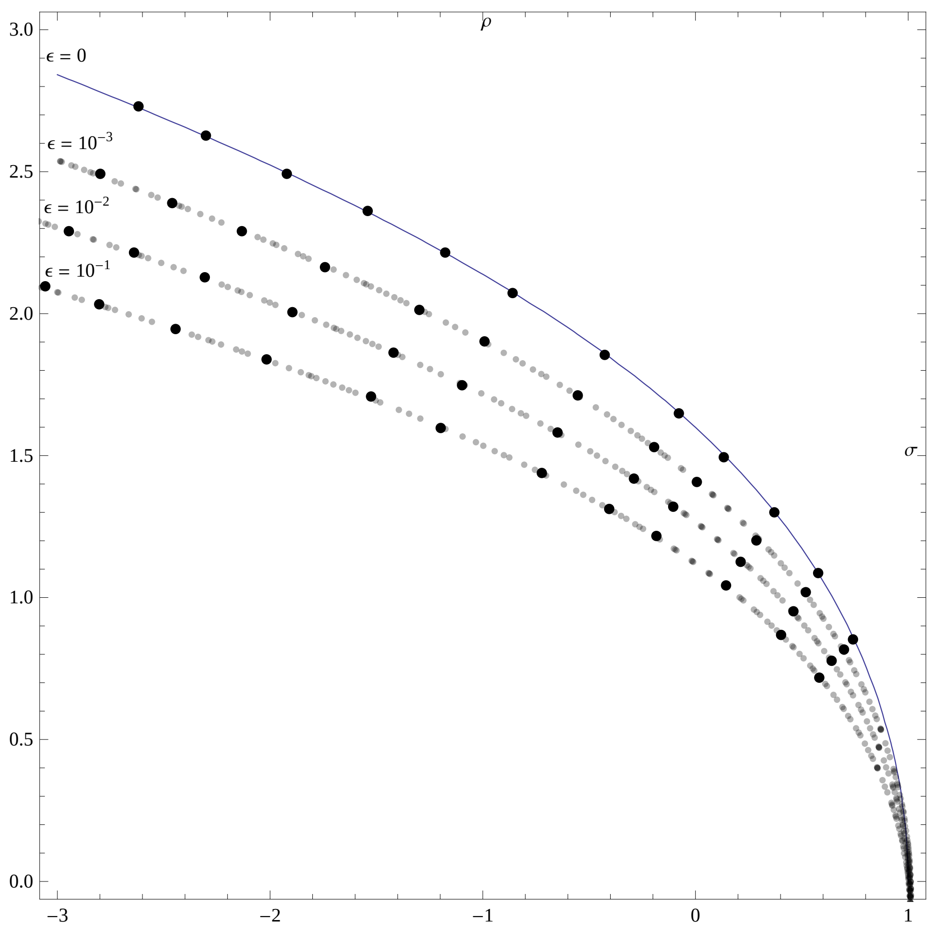

Figure 7.1: The shift in the curve of the spectral singularities of the unperturbed barrier potential. These shifts are due to the perturbation potential $v(x) = \cosh x$ with different values of the perturbation strength ε.

$\varepsilon = 0$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-1}$
ρ=0.99999	0.99998	0.99996	0.99984
σ=0.00299	0.00290	0.00219	0.00018
k=25.75943			
0.76992	0.76943	0.76509	0.72169
0.81192	0.80993	0.70209	0.51366
1.33227			
0.50513	0.50261	0.47995	0.25335
1.16609	1.16165	1.12170	0.72218
1.01173			
0.27778	0.26949	0.19496	-0.55036
1.38606	1.37545	1.27942	1.09987
0.86662			
-2.13748	-2.13742	-2.13691	-2.13178
2.57002	2.56656	2.53543	2.22237
0.37871			
-2.37650	-2.37505	-2.36202	-2.23172
2.64863	2.64431	2.60584	2.22062
0.35885			
-3.13771	-2.32129	-3.35344	-3.67601
2.88151	2.87914	2.85711	2.63724
0.30723			

Table 7.1: Spectral singularities of barrier potential perturbed by $v(x) = \cosh x$ for the different values of the perturbation parameter. $\varepsilon = 0$ corresponds to the case of unperturbed barrier potential.

Chapter 8

OPTICAL SPECTRAL SINGULARITIES OF AN INFINITE SLAB GAIN MEDIUM

For optical realization of spectral singularities of the complex barrier potential (8.5), we use a special set up consisting of two infinite planar slab aligned along the Y-axis and the region between the planes confined to $|X| < a$ is filled with a material with reflective index $\mathfrak{n}$ acting as a gain medium(Fig. 8.1). We propose that this set up can be applied to generate lasers provided that we tune the parameters of the system to admit a spectral singularities. Interestingly, our set up does not include any mirror which are important part of typical optical systems generating lasers.

Practically, a laser consists of a gain medium inside a highly reflective optical cavity (an arrangement of mirrors surrounding the gain medium). The gain medium should be pumped[5] to achieve population inversion. Due to stimulated emission of excited states and by locating two highly refractive mirrors to bounce the emitted light back and forth to reach high amplification, one can obtain a highly amplified beam, known as a laser, [47, 48, 49].

Generally, optical cavities acts as a Fabry-Perot interferometer and adds a phase (path) difference $\delta = 2\mathfrak{n}\vec{k}\cdot\vec{l}$ to each reflected beam inside the gain medium, where $\vec{k}$ and $\vec{l}$ stand for the wave number and the length of the gain medium, respectively. Since the refractive index of the gain medium is a complex number $\mathfrak{n} = \eta + i\kappa$, the phase shift is

$$e^{i\delta} = e^{-2k\kappa l}e^{2ik\eta l}. \tag{8.1}$$

Hence, while traveling through the gain medium, the beam is amplified by a factor of $e^{-2k\kappa l}$. In this manner by continuous reflections of the beam inside the gain medium one can gain a much larger amplification of the incident wave, [48]. This is how the ordinary lasers work. The spectral singularities, however suggest an alternative way to obtain this amplification effect without using mirrors.

[5] The act of energy transfer from an external source into the gain medium of a laser is called laser pumping.

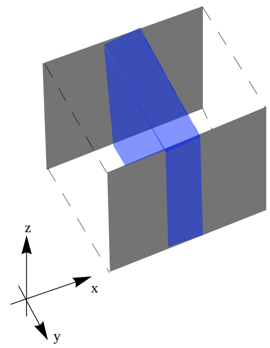

Figure 8.1: Scheme of an infinite planar slab filled with a dielectric acting as a gain medium (blue part).

In Appendix C we show that the study of the dynamics of electromagnetic beams inside an infinite planar slab is related to the problem of scattering for the complex barrier potentials. Consider the following set of solutions of Maxwell's equations which are EM fields in a medium with a complex refractive index $\mathfrak{n}$,

$$\vec{E}(X,t) = Ee^{-i\omega t}\psi(X)\hat{e_Y},$$
$$\vec{B}(X,t) = -i\omega^{-1}Ee^{-i\omega t}\psi'(X)\hat{e_Z}, \tag{8.2}$$

where $\hat{e_Y}$ and $\hat{e_Z}$ are unit vectors in Y and Z direction, respectively. The electromagnetic wave equation reads

$$\nabla^2\vec{E} = \frac{\mathfrak{n}^2}{c^2}\partial_t^2\vec{E}, \qquad \nabla^2\vec{B} = \frac{\mathfrak{n}^2}{c^2}\partial_t^2\vec{B}. \tag{8.3}$$

Substituting (8.2) in (8.3) produces the following equation which is analogous to the dimensionless time-independent Schrödinger equation:

$$-\psi''(x) + v_0(x)\psi(x) = k^2\psi(x), \tag{8.4}$$

where

$$x := \frac{X}{\ell}, \qquad k^2 = \ell^2\frac{\omega^2}{c^2}, \qquad v_0(x) = \begin{cases} k^2(1-\mathfrak{n}^2) & |x| < \alpha, \\ 0 & |x| > \alpha, \end{cases} \tag{8.5}$$

where $v_0(x)$ is a complex barrier potential. The solutions of Schrödinger equation for this potential have the form

$$\psi(x) = \begin{cases} A_1e^{ikx} + B_1e^{-ikx} & \text{for} \quad (x < -\alpha), \\ A_2e^{i\mathfrak{n}kx} + B_2e^{-i\mathfrak{n}kx} & \text{for} \quad (|x| < \alpha), \\ A_3e^{ikx} + B_3e^{-ikx} & \text{for} \quad (x < -\alpha), \end{cases} \tag{8.6}$$

where $A_\nu, B_\nu \in \mathbb{C}$ and $\nu \in \{1, 2, 3\}$.

The mathematical meaning of spectral singularities associates them with a zero-width resonance phenomena, [33, 34]. This fact indicates that if the frequency of an incoming electromagnetic wave is tuned so that $\omega \longrightarrow \omega_\star = k_\star^2 c^2$, where $k_\star^2$ is a spectral singularity, the amplitude of the outgoing waves will diverge. Therefore

sending waves of frequency $\omega \approx \omega_\star$ will induce outgoing (transmitted and reflected) waves of considerably enhanced amplitude (Fig.8.2).

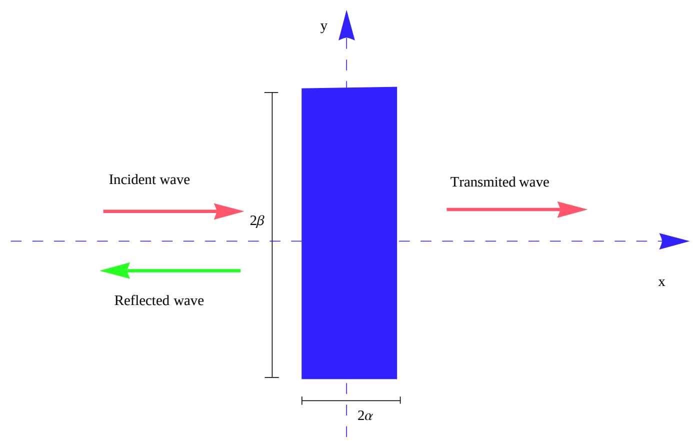

Figure 8.2: Cross-section of the gain medium in $X - Y$ plane and the interaction of the incident wave with the medium.

The refractive index of the gain medium, that is constructed by doping a host medium of refractive index n_0, has the form (Appx. C)

$$\mathfrak{n}^2 = n_0^2 + \frac{\hat{\omega}_p^2}{1 + \hat{\omega}^2 - i\hat{\gamma}\hat{\omega}}, \tag{8.7}$$

where $\hat{\omega} = \frac{\omega}{\omega_0}$, $\hat{\omega}_p = \frac{\omega_p}{\omega_0}$, $\hat{\gamma} = \frac{\gamma}{\omega_0}$, [43, 50, 51]. Since $\mathfrak{n} = \eta + i\kappa$, $\mathfrak{n}^2 = (\eta^2 - \kappa^2) + 2i\kappa\eta$. In

$\hat{\omega} = 1$ we have

$$\begin{cases} \eta^2 - \kappa^2 = n_0^2, \\ 2\kappa\eta = -\dfrac{\hat{\omega}_p}{\hat{\gamma}\hat{\omega}}. \end{cases} \tag{8.8}$$

Solving this system for η yields

$$\begin{aligned} \hat{\omega}_p^2 &= 2\kappa\hat{\gamma}\hat{\omega}\sqrt{n_0^2 + \kappa^2} \\ &= -\frac{\hat{\gamma} n_0 \lambda_0 \mathfrak{g}}{2\pi\hat{\omega}}\sqrt{1 + (\frac{\lambda_0 \mathfrak{g}}{4\pi n_0})^2}. \end{aligned} \tag{8.9}$$

where $\mathfrak{g} = -2\kappa k$ is the gain coefficient of the gain medium at resonance frequency and $\lambda_0 = \frac{2\pi c}{\omega_0}$, (Appx C). In terms of dimensionless quantities we have

$$\hat{\omega}_p^2 = -\frac{\hat{\gamma} n_0 g}{k_0\hat{\omega}}\sqrt{1 + (\frac{g k_0}{2 n_0})^2}, \quad \text{where} \quad g = \mathfrak{g}\ell,\ k_0 = \frac{2\pi\ell}{\lambda_0}. \tag{8.10}$$

The gain coefficient depends on the material that we use as gain medium. To continue we consider the following two cases of gain medium:

(i) **Uniform Gain Medium**

In this case we suppose that the gain coefficient $\mathfrak{g}$ maintains a constant value throughout the gain medium. Ref. [38] gives the complete discussion for this special type of gain medium where the spectral singularities of the complex barrier potential and their associated zero-width resonance appear at lasing threshold $\mathfrak{g}_t$. In other words, $\mathfrak{g} = \mathfrak{g}_t$ is only the necessary condition for observing a spectral singularity, [38].

(ii) **Non-Uniform Gain Medium**

In general the gain coefficient $\mathfrak{g}$ is a function of the space. Generally, when an electromagnetic wave travels through a medium it undergoes exponential decay as described by Beer-Lambert law, [41]. Therefore the intensity of the EM beam entering to a medium in x direction is given by

$$I(\mathrm{x}) = I_0 e^{-\mathfrak{a}X}, \tag{8.11}$$

where $\mathfrak{a}$ is the attenuation coefficient of the medium. This is a natural property of any gain medium to somehow attenuate any beam entering and passing through it. For our setup, the energy (pumping) intensity inside the gain medium will have an exponentially decaying behavior. Therefore some part of the energy gets absorbed by the medium and thus the effective gain will be $\mathfrak{g}_{\text{eff}}(X) = \mathfrak{g}(X) - \mathfrak{a}$, where $\mathfrak{g}(X)$ is the gain inside the gain medium. For instance, having single pumping gives the following effective gain

$$\begin{aligned}\mathfrak{g}_{\text{eff}}(X) &= \beta e^{-\mu a} e^{-\mu X} - \mathfrak{a} \\ &= (\mathfrak{g}_0 + \mathfrak{a}) e^{-\mu X} e^{-\mu a} - \mathfrak{a} \qquad \text{for} \qquad |X| < a, \end{aligned} \tag{8.12}$$

where μ is the gain decay, [39]. We used the boundary condition $\mathfrak{g}_{\text{eff}}(-a) =: \mathfrak{g}_0$ to obtain $\beta = (\mathfrak{g}_0 + \mathfrak{a})$, and $\mathfrak{g}_0$ is the initial gain. Using double pumping one obtains a *cosh* like pattern for the gain inside the medium therefore, [39], we have

$$\mathfrak{g}_{\text{eff}}(X) = \beta \cosh \mu X - \mathfrak{a} \qquad \text{for} \qquad |X| < a. \tag{8.13}$$

The maximum $\mathfrak{g}_0$ happens in the boundary of the gain medium. Therefore $\mathfrak{g}_{\text{eff}}(\pm a) = \mathfrak{g}_0$ which gives $\beta = \dfrac{\mathfrak{g}_0 + \mathfrak{a}}{\cosh \mu a}$. Thus

$$\mathfrak{g}_{\text{eff}}(X) = \left[\frac{\mathfrak{g}_0 + \mathfrak{a}}{\cosh \mu a}\right] \cosh \mu X - \mathfrak{a} \qquad \text{for} \qquad |X| < a. \tag{8.14}$$

Using this gain coefficient and noting that $\frac{\lambda_0 \mathfrak{g}}{4\pi} \ll n_0$, the refractive index (8.7) can be written as

$$\begin{aligned}\mathfrak{n}^2(\mathfrak{g}_{\text{eff}}) &= n_0^2 + \frac{\left(-\dfrac{\hat{\gamma} n_0 \lambda_0 \mathfrak{g}_{\text{eff}}}{2\pi\hat{\omega}}\right)}{\hat{\omega}^2 - 1 + i\hat{\gamma}\hat{\omega}} \sqrt{1 + (\frac{\lambda_0 \mathfrak{g}_{\text{eff}}}{4\pi n_0})^2} \\ &= n_0^2 + \frac{\left(-\dfrac{\hat{\gamma} n_0 \lambda_0 \mathfrak{g}_0}{2\pi\hat{\omega}}\right)}{\hat{\omega}^2 - 1 + i\hat{\gamma}\hat{\omega}} + \frac{\left(-\dfrac{\hat{\gamma} n_0 \lambda_0 (\mathfrak{g}_0 + \mathfrak{a})}{2\pi\hat{\omega}}\right)}{\hat{\omega}^2 - 1 + i\hat{\gamma}\hat{\omega}} \left(\frac{\cosh \mu X}{\cosh \mu a} - 1\right) \\ &= \mathfrak{n}^2(\mathfrak{g}_0) + \varepsilon \mathfrak{t} \Big(1 + \frac{\mathfrak{g}_0}{\mathfrak{a}}\Big) f(x), \end{aligned} \tag{8.15}$$

where we have introduced

$$\mathfrak{n}^2(\mathfrak{g}_0) := n_0^2 - \frac{\hat{\gamma} n_0 \lambda_0 \mathfrak{g}_0}{2\pi\hat{\omega}(\hat{\omega}^2 - 1 + i\hat{\gamma}\hat{\omega})}, \tag{8.16}$$

$$\varepsilon := \frac{n_0^2 \hat{\mu}^2}{2}, \tag{8.17}$$

$$\mathfrak{t} := \frac{\hat{\gamma} \lambda_0 \mathfrak{a}}{2\pi n_0 (1 - \hat{\omega}^2 - i\hat{\gamma}\hat{\omega})}, \tag{8.18}$$

$$f(x) := \frac{2}{\hat{\mu}^2} \left(\frac{\cosh \hat{\mu} x}{\cosh \hat{\mu} \alpha} - 1\right), \tag{8.19}$$

$\hat{\mu} = \mu\ell$, $\alpha = \frac{a}{\ell}$ and $x = \frac{X}{\ell}$ and ℓ is an arbitrary length scale.

Combining all these results and using (8.4) and (8.5), we relate the dynamics of EM waves inside a gain medium with non-uniform gain coefficient (8.14) to a Schrödinger equation given by the

$$H = H_0 + v_0(x) + \varepsilon v(x), \tag{8.20}$$

where H_0 is the Hamiltonian of the free particle and we define

$$v_0(x) := \begin{cases} \zeta = k^2[1 - \mathfrak{n}^2(\mathfrak{g}_0)] & |x| < \alpha, \\ 0 & |x| > \alpha, \end{cases} \tag{8.21}$$

$$v(x) := \begin{cases} k^2\mathfrak{t}\Big(1 + \frac{\mathfrak{g}_0}{\mathfrak{a}}\Big) f(x) & |x| < \alpha, \\ 0 & |x| > \alpha. \end{cases} \tag{8.22}$$

The Hamiltonian (8.20) describes the perturbation of the barrier potential $v_0(x)$. At the end of Chapter 7, we gave a graphical result showing how the spectral singularities of the barrier potential shift due to this perturbation. Therefore an infinite slab non-uniform gain medium with the gain coefficient (8.14), is an ideal model to observe the spectral singularities resonance effects of the perturbed barrier potential. Next, we use (8.21) to define

$$\frac{\zeta}{k^2} = 1 - \mathfrak{n}^2(\mathfrak{g}_0) := \rho_e + i\sigma_e. \tag{8.23}$$

we then find

$$\rho_e := 1 - n_0^2 - \frac{\hat{\omega}_p^2(1-\hat{\omega}^2)}{(1-\hat{\omega}^2) + \hat{\gamma}^2\hat{\omega}^2}, \sigma_e := -\frac{\hat{\omega}_p^2\hat{\gamma}\hat{\omega}}{(1-\hat{\omega}^2) + \hat{\gamma}^2\hat{\omega}^2}. \tag{8.24}$$

Using the definition of the plasma frequency near resonance frequency, (8.10), namely $\hat{\omega}_p^2 := -\frac{\hat{\gamma}n_0 g}{k_0\hat{\omega}}$ and since $\hat{\omega} = k/k_0$, we obtain an expression for $\rho_e(k,g)$ and $\sigma_e(k,g)$ in terms of the dimensionless quantities:

$$\begin{aligned} \rho_e(k,g) &:= 1 - n_0^2 - \frac{\hat{\gamma}n_0 g(k^2-k_0^2)k_0^2}{[(k^2-k_0^2)^2 + \hat{\gamma}^2k^2k_0^2]k}, \\ \sigma_e(k,g) &:= \frac{\hat{\gamma}^2 n_0 g k_0^3}{(k^2-k_0^2)^2 + \hat{\gamma}^2k^2k_0^2}. \end{aligned} \tag{8.25}$$

Since $k = k^{(0)} + \varepsilon k^{(1)}$ and assuming $g = g^{(0)} + \varepsilon g^{(1)}$ we get

$$\begin{aligned} \rho_e(k,g) &:= \rho_e^{(0)} + \varepsilon(\Lambda_1 k^{(1)} + \Lambda_2 g^{(1)}) + O(\varepsilon^2), \\ \sigma_e(k,g) &:= \sigma_e^{(0)} + \varepsilon(\Lambda_3 k^{(1)} + \Lambda_4 g^{(1)}) + O(\varepsilon^2), \end{aligned} \tag{8.26}$$

where

$$\begin{aligned}\rho_e^{(0)} &:= 1 - n_0^2 - \frac{\hat{\gamma} n_0 g^{(0)} ({k^{(0)}}^2 - k_0^2) k_0^2}{[({k^{(0)}}^2 - k_0^2)^2 + \hat{\gamma}^2 {k^{(0)}}^2 k_0^2] k^{(0)}}, \\ \sigma_e^{(0)} &:= \frac{\hat{\gamma}^2 n_0 g^{(0)} k_0^3}{({k^{(0)}}^2 - k_0^2)^2 + \hat{\gamma}^2 {k^{(0)}}^2 k_0^2},\end{aligned} \tag{8.27}$$

and

$$\Lambda_1 := \frac{\hat{\gamma} n_0 k_0^2 g^{(0)} \Big[3{k^{(0)}}^6 - k_0^6 + {k^{(0)}}^2 k_0^2 \big[(-7 + \hat{\gamma}^2) {k^{(0)}}^2 + (5 + 3\hat{\gamma}^2) k_0^2 \big] \Big]}{-{k^{(0)}}^2 \big[({k^{(0)}}^2 - k_0^2)^2 + {k^{(0)}}^2 k_0^2 \hat{\gamma}^2 \big]^2}, \tag{8.28}$$

$$\Lambda_2 := \frac{\hat{\gamma} n_0 k_0^2 \Big[{k^{(0)}}^6 - k_0^6 + 3{k^{(0)}}^2 k_0^2 (\hat{\gamma}^2 - 1)(k_0^2 - {k^{(0)}}^2) \Big]}{k^{(0)} \big[({k^{(0)}}^2 - k_0^2)^2 + {k^{(0)}}^2 k_0^2 \hat{\gamma}^2 \big]^2}, \tag{8.29}$$

$$\Lambda_3 := -\frac{\hat{\gamma}^2 n_0 g^{(0)} k_0^3 k^{(0)} \big[4{k^{(0)}}^2 - 2k_0^2 \hat{\gamma}^2 - 4k_0^2 \big]}{\big[({k^{(0)}}^2 - k_0^2)^2 + {k^{(0)}}^2 k_0^2 \hat{\gamma}^2 \big]^2}, \tag{8.30}$$

$$\Lambda_4 := \frac{\hat{\gamma}^2 n_0 k_0^3}{({k^{(0)}}^2 - k_0^2)^2 + {k^{(0)}}^2 k_0^2 \hat{\gamma}^2}. \tag{8.31}$$

Next, we use the result of the calculation of spectral singularities of the perturbed barrier potential that we obtained in the end of Chapter 7 . If we solve (7.29) for $\xi_r^{(1)}$ and $\xi_i^{(1)}$ we get

$$\xi_r^{(1)} = \frac{{k^{(0)}}^2}{\varepsilon} \Big[\rho - \Big(1 - 2\varepsilon \frac{k^{(1)}}{k^{(0)}} \Big) \rho^{(0)} \Big], \qquad \xi_i^{(1)} = \frac{{k^{(0)}}^2}{\varepsilon} \Big[\sigma - \Big(1 - 2\varepsilon \frac{k^{(1)}}{k^{(0)}} \Big) \sigma^{(0)} \Big]. \tag{8.32}$$

To observe the optical spectral singularity we require $\rho_e = \rho$, $\sigma_e = \sigma$, $\rho_e^{(0)} = \rho^{(0)}$ and $\sigma_e^{(0)} = \sigma^{(0)}$, substituting (8.26) in (8.32) gives

$$\begin{aligned}\xi_r^{(1)} &= {k^{(0)}}^2 \Big[(\Lambda_1 - \tfrac{2\rho^{(0)}}{k^{(0)}}) k^{(1)} + \Lambda_2 g^{(1)} \Big], \\ \xi_i^{(1)} &= {k^{(0)}}^2 \Big[(\Lambda_3 - \tfrac{2\sigma^{(0)}}{k^{(0)}}) k^{(1)} + \Lambda_4 g^{(1)} \Big].\end{aligned} \tag{8.33}$$

Combining (8.33) with (7.27) results in a system of two linear equations with $k^{(1)}$ and $g^{(1)}$ as unknowns. By solving this system we obtain

$$k^{(1)} = \frac{\Gamma_4 - \Gamma_2}{\Gamma_1 \Gamma_4 - \Gamma_2 \Gamma_3}, \qquad g^{(1)} = \frac{\Gamma_1 - \Gamma_3}{\Gamma_1 \Gamma_4 - \Gamma_2 \Gamma_3}, \tag{8.34}$$

where we have defined

$$\Gamma_1 := \frac{\left[\mathring{\mathcal{P}}_i\mathring{\mathcal{Q}}_r - \mathring{\mathcal{P}}_r\mathring{\mathcal{Q}}_i\right](\Lambda_1 - \frac{2\rho^{(0)}}{k^{(0)}}) + \left[\mathring{\mathcal{P}}_i\mathring{\mathcal{Q}}_i + \mathring{\mathcal{P}}_r\mathring{\mathcal{Q}}_r\right](\Lambda_3 - \frac{2\sigma^{(0)}}{k^{(0)}})}{\mathring{\mathcal{P}}_i\mathring{\mathcal{T}}_r - \mathring{\mathcal{P}}_r\mathring{\mathcal{T}}_i}k^{(0)2}, \tag{8.35}$$

$$\Gamma_2 := \frac{\left[\mathring{\mathcal{P}}_i\mathring{\mathcal{Q}}_r - \mathring{\mathcal{P}}_r\mathring{\mathcal{Q}}_i\right]\Lambda_2 + \left[\mathring{\mathcal{P}}_i\mathring{\mathcal{Q}}_i + \mathring{\mathcal{P}}_r\mathring{\mathcal{Q}}_r\right]\Lambda_4}{\mathring{\mathcal{P}}_i\mathring{\mathcal{T}}_r - \mathring{\mathcal{P}}_r\mathring{\mathcal{T}}_i}k^{(0)2}, \tag{8.36}$$

$$\Gamma_3 := \frac{\mathring{\mathcal{Q}}_i(\Lambda_1 - \frac{2\rho^{(0)}}{k^{(0)}}) + \mathring{\mathcal{Q}}_r(\Lambda_3 - \frac{2\sigma^{(0)}}{k^{(0)}})}{\mathring{\mathcal{T}}_i}k^{(0)2}, \tag{8.37}$$

$$\Gamma_4 := \frac{\mathring{\mathcal{Q}}_i\Lambda_2 + \mathring{\mathcal{Q}}_r\Lambda_4}{\mathring{\mathcal{T}}_i}k^{(0)2}. \tag{8.38}$$

In the remaining of this section we locate the optical spectral singularities of the non-uniform gain medium corresponding to (8.14). First, we obtain the optical spectral singularities of the uniform gain medium, [26, 38, 39], then use the result of (8.34) to compute the shifts in the values of these spectral singularities due to the perturbations (decay of the gain coefficient as the beam enters the gain region). Comparing the result of similar calculation done by semi-classical analysis, [39], with the one that we obtained using perturbation theory, we find a very good agreement in the results of both approaches.

To continue we consider the following typical semiconductor gain medium, [38, 39],

$$\begin{array}{llll} n_0 = 3.4, & \lambda_0 = 1500\text{nm}, & \hat{\gamma} = 0.02, & \mathfrak{a} = 200\text{cm}^{-1}, \\ \mu \approx 0.1, & g_0 \approx 50\text{cm}^{-1}, & \ell = 2a \approx 300\mu\text{m}. & \end{array} \tag{8.39}$$

For this gain medium the perturbation parameter (8.17) is given by $\varepsilon \approx 0.0578$. Near resonance frequency ω_0 we have $\lambda = \lambda_0 \pm \delta$ therefore $\hat{\omega} = 1 \mp \frac{n|\delta|}{\lambda_0}$. Here $|\delta|$ will stay fixed during the experiment and it shows the distance between the wavelengths that we consider in the experiment and n is an integer number. For instance we pick $|\delta| = 10$ nm.

From (8.18) We can write $\mathfrak{t} = |\mathfrak{t}|e^{i\theta}$ where

$$|\mathfrak{t}| \approx \Re(\mathfrak{t}), \quad \theta = \tan^{-1}\frac{\Im(\mathfrak{t})}{\Re(\mathfrak{t})}, \quad \text{where} \quad \frac{\Im(\mathfrak{t})}{\Re(\mathfrak{t})} = \frac{\hat{\gamma}}{\hat{\omega} - \frac{1}{\hat{\omega}}} \approx \mp\frac{\hat{\gamma}\lambda_0}{2n|\delta|}. \tag{8.40}$$

At resonance frequencies, $k = k_0 := \frac{2\pi\ell}{\lambda_0}$, we find that $|t| \approx |k^{-1}| \approx 10^{-3}$. These numerical bounds make our perturbative calculation simpler since $k|t| \approx 1$, therefore from (8.19), (8.22) and (8.40) we have

$$v(x) = \frac{2ke^{i\theta}}{\hat{\mu}^2}(1 + \frac{\mathfrak{g}}{\mathfrak{a}})\Big(\frac{\cosh \hat{\mu}x}{\cosh \frac{\hat{\mu}}{2}} - 1\Big), \tag{8.41}$$

where $\theta = \tan^{-1}\left(\mp \frac{\hat{\gamma}\lambda_0}{2n|\delta|}\right)$. If $\lambda = \lambda_0$, then $n = 0$ and $e^{i\theta} = 1$.

For instance, we wish to investigate the effect of perturbation on the spectral singularities of the barrier potential with height ζ, given in (8.21). For the barrier potential, near resonance frequency we have the spectral singularity with $\lambda_0 = 1500$ nm and $\mathfrak{g} = 40.43$ cm^{-1} and therefore we will get $k^{(0)} = 0.4\pi \times 10^3$ and $g^{(0)} = 1.212$ as the zeroth order perturbative solutions. For this case we have $\zeta = k_0^2(\rho + i\sigma) = k_0^2[1 - n_0^2] + i[k_0 n_0 g^{(0)}] = -1.66757 \times 10^7 + i5178.35$ where $\rho = -10.56$ and $\sigma = 0.00327923$ and therefore $\mathfrak{n} = \sqrt{(1-\rho) - i\sigma} = 3.4 - i0.000482239$. Using $\lambda_0 = 1500$ nm we compute the Reflection and Transmission coefficients $|T|^2$ and $|R|^2$, where $T = \frac{1}{M_{22}}$ and $R = \frac{M_{12}}{M_{22}}$. These give the amplification factor for the emitted electromagnetic energy density and we obtain 1.99575×10^7 and 3.68948×10^9, respectively. In other words we obtain an amplification of the background electromagnetic energy density at these wavelengths by a factor of $|T|^2 + |R|^2 \approx 3.70944 \times 10^9$. For the Non-Uniform Gain Medium we obtain small deviations in the values of k and g and therefore respectively we can calculate the wavelength for which we can observe a spectral singularity with attained gain coefficient. For example for the background (unperturbed) wavelength $\lambda^{(0)} = \lambda_0 = 1500$ nm with the corresponding gain coefficient $\mathfrak{g}^{(0)} = 40.4$ cm^{-1} perturbative calculation shows that we have the following spectral singularity at resonance frequency

$$\lambda_\star = 1499.9987128 \text{ nm}, \qquad \mathfrak{g}_\star = 40.43852 \text{ cm}^{-1}. \tag{8.42}$$

By controlling the intensity of the pumping beam we can adjust $\mathfrak{g}_\star$ in order to produce laser at any of different wavelengths shown in Table 8.1. This turns out to be almost spaced in the range 1478.558-1527.686 nm, (Table 8.1).

$\hat{\mu} = 0$	
λ (nm)	$\mathfrak{g}_\star$ (cm^{-1})
1478.5584532	124.17655
1481.7366532	101.10027
1484.9276537	81.72387
1489.2034578	61.54675
1494.5824861	45.73409
1495.6624982	43.83658
1500.000000	**40.43827**
1504.3632897	43.82109
1507.6512094	50.93261
1510.9539613	61.80307
1512.0582176	60.32103
1515.3802631	82.51231
1519.8322983	110.20124
1527.6859891	175.59110

$\hat{\mu} = 0.1$	
λ (nm)	$\mathfrak{g}_\star$ (cm^{-1})
1478.5583396	124.17684
1481.7362614	101.10219
1484.9274626	81.72396
1489.2034076	61.54698
1494.5823078	45.73437
1495.6624393	43.83688
1499.9987128	**40.43852**
1504.3632721	43.82159
1507.6515230	50.93283
1510.9539204	61.80377
1512.0582039	60.32129
1515.3802308	82.51247
1519.8321630	110.20215
1527.6833077	175.59871

Table 8.1: Wave length λ of spectral singularities for g with $\hat{\mu} = 0.1$.

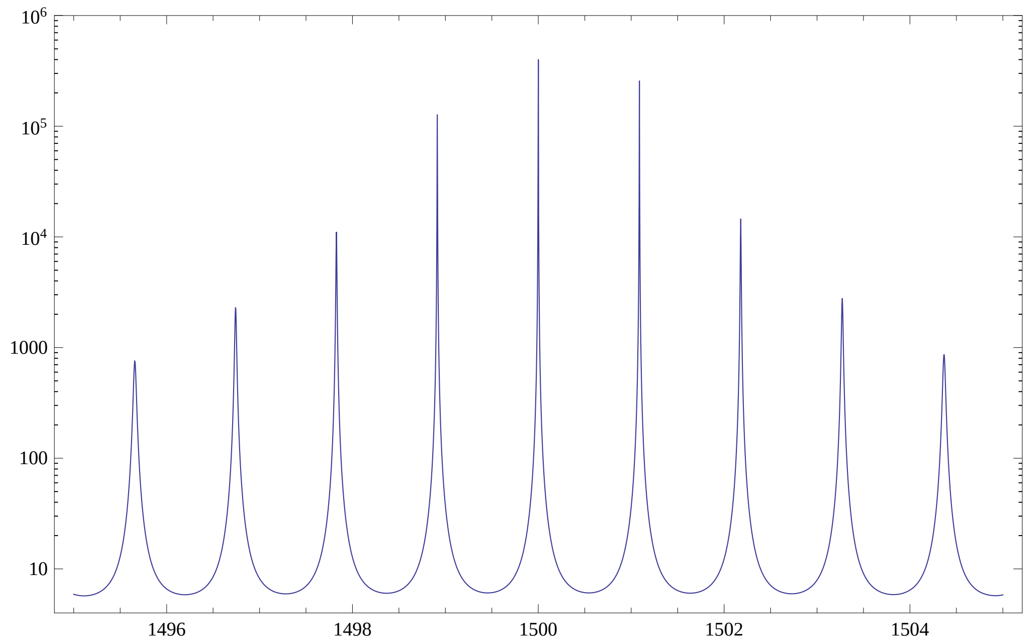

Figure 8.3: Logaritmic Graph of $(|T|^2 + |R|^2)$ as a function of wavelength λ with for $\ell = 300$ μm and $g = 40.43852$ cm^{-1}. $|T|^2$ and $|R|^2$ are Transmission and Reflection coefficients. The horizontal axis gives the frequencies and the vertical ax is the $\log(|T|^2 + |R|^2)$.

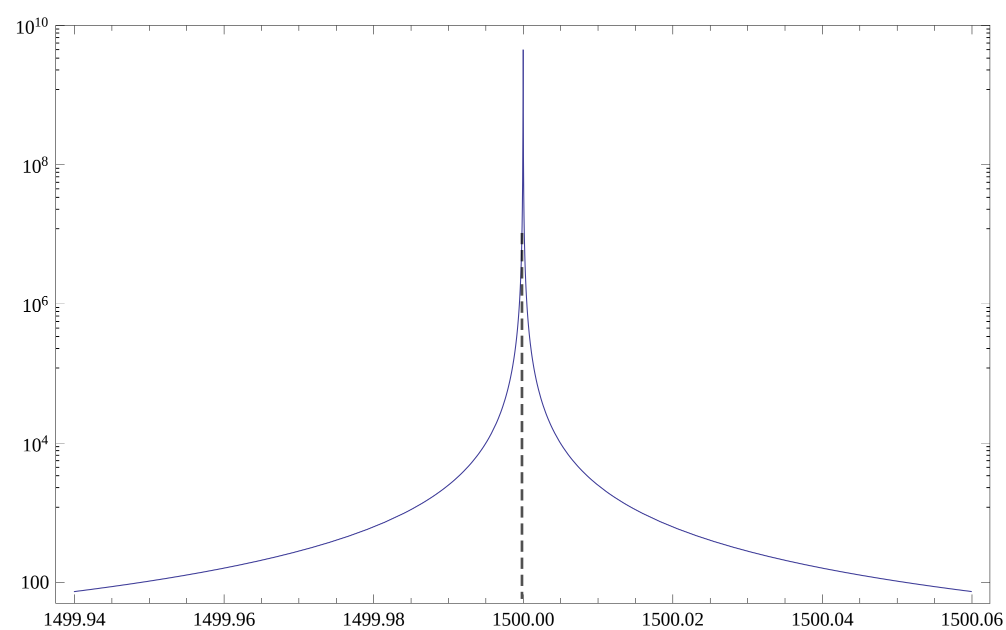

Figure 8.4: The highest peak in Logaritm Plot of $(|T|^2 + |R|^2)$ represents the spectral singularity (8.42) that occurs near $\lambda = 1500$ nm for a gain medium with $\ell = 300$ μm and $g = 40.43852$ cm^{-1} and $\hat{\mu} = 0.1$.

Chapter 9

CONCLUSION

Non-Hermitian Hamiltonians (with real spectrum) have provided insight in our understanding Quantum Mechanics. The non-Hermitian Hamiltonians with real energies first appeared in physics literature in the works of C. Bender involved replacing the condition of self-adjointness (Hermiticity) of the Hamiltonian with the, supposedly, weaker condition as $\mathcal{PT}$-symmetry. This gives rise to a class of complex non-Hermitian Hamiltonians having real and positive energies, [1]. This study brings forth the concept of $\mathcal{PT}$-symmetric Quantum Mechanics, [2]. A. Mostafazadeh, [9, 10, 11], by introducing the concept of Pseudo-Hermiticity proved that these non-Hermitian operators become Hermitian upon the redefinition of the inner product of the Hilbert space. The concept of Pseudo-Hermitian Quantum Mechanics broadens the formulation of Quantum Mechanics in the sense that it includes the $\mathcal{PT}$-symmetry Hamiltonians, [4].

Unfortunately, the spectral representation of non-Hermitian operators (with real spectrum) does not always exist and this is related to some deep properties of non-self adjoint operators, [15]. The spectral singularities appearing in the continuous spectrum of non-Hermitian Hamiltonians obstructs the spectral representation of the related Hamiltonian, [12]. These turn out to be related to a specific type of resonance phenomena, [33, 34, 40].

In this thesis we establish a perturbative method of identifying the spectral singularities. We first derive a general formula for the perturbative corrections of the eigenfunctions of the perturbed Hamiltonian. We assume that the potentials have compact support, i.e,vanish outside of an interval. Then we apply this method for a free particle perturbed by single and double Dirac delta potentials and determine the location of spectral singularities for these systems. Next, we use the exactness of the perturbation for the chain of Dirac delta potentials (Appx A) to locate the spectral singularities of this system. We then consider the problem of perturbing the barrier potential. We locate the spectral singularities of the perturbed system as a small shifts of the spectral singularities of the unperturbed barrier potential.

In the last section we discuss an experimental setup, [26, 33, 38, 40], which suggests that the spectral singularities of a barrier potential can be observed and verified experimentally using an appropriate infinite planar slab gain medium. Since the gain coefficient depends on the coordinates and decays inside the gain medium we consider a non-uniform gain medium that corresponds to a perturbed barrier potential where the perturbation is due to the decay of the gain coefficient. The spectral singularities of the non-uniform gain medium are determined as small shifts in the location of spectral singularities of the uniform gain medium. By the help of this sample we show how the amplification of electromagnetic beams can be achieved by just adjusting the parameters of the system, specially the gain coefficient $\mathfrak{g}$, so that it ensures a spectral singularity.

APPENDIX

Appendix A

EXACTNESS OF PERTURBATION FOR DIRAC-COMB

Theorem. *For a Dirac Comb, consisting of N Dirac delta potential, perturbative corrections of order higher than N vanish.*

Proof. Consider the potential given in (6.1)(a finite series of N Dirac delta barriers). Since the correction from each order is in a recursive relation with the correction of the previous order , we claim that

$$\Psi^{(N+1)}(x) = 0. \tag{A.1}$$

By this claim the orders higher than $N+1$ will not be generated therefore the Theorem will be proved. In lights of (6.9) and (6.12), this claim implies

$$(F_{-}^{+})_{(j=N+1)} = 0, \tag{A.2}$$

and correspondingly we need to prove that

$$\prod_{i=1}^{N} \left[1 - \chi^{\pm(n_i - n_{i+1})}\right] = 0, \quad \Longrightarrow \quad \exists\, i \in \mathfrak{N}\,;\; n_i = n_{i+1}, \tag{A.3}$$

and

$$\mathfrak{N} := \{1, \cdots, N\}. \tag{A.4}$$

In another words we have to show that the set $\{i \in \mathfrak{N}; n_i = n_{i+1}\}$ is not empty. This implies that if different labels of series in (6.12) have the same value, then the product (A.2) is zero. To simplify the form of the series (6.12), we start from the last series with label n_j whose values are elements of the following set

$$\mathfrak{N}_j = \{0, 1, 2, 3, \cdots, N-1\}, \quad n_j \in \mathfrak{N}_j. \tag{A.5}$$

By assigning a value for n_j (with the condition $n_1 < n_2 < \cdots < n_j \leq N-1$) the next series labels $n_{j-1}, n_{j-2}, \cdots, n_1$ should be chosen from the sets $\mathfrak{N}_{j-1}, \mathfrak{N}_{j-2}, \cdots, \mathfrak{N}_1$, respectively, given by definitions:

$$\mathfrak{N}_{j-1} = \{0, 1, 2, 3, \cdots, n_j - 1\}, \quad n_{j-1} \in \mathfrak{N}_{j-1}, \tag{A.6}$$

$$\mathfrak{N}_{j-2} = \{0, 1, 2, 3, \cdots, n_{(j-1)} - 1\}, \qquad n_{j-2} \in \mathfrak{N}_{j-2}, \tag{A.7}$$

$$\vdots$$

$$\mathfrak{N}_2 = \{0, 1, 2, \cdots, n_3 - 1\}, \qquad n_2 \in \mathfrak{N}_2, \tag{A.8}$$

$$\mathfrak{N}_1 = \{0, 1, 2, 3, \cdots, n_2 - 1\}, \qquad n_1 \in \mathfrak{N}_1, \tag{A.9}$$

and we have $\mathfrak{N}_j \subset \mathfrak{N}_{j-1} \subset \cdots \subset \mathfrak{N}_1$. In exact terms, to calculate sums appeared in expression of j-th correction (6.12), their labels need to be selected from the set $\mathfrak{N}_j$. More precisely we need j different and nonrepeated selection from the set $\mathfrak{N}_j$ in order to guarantee $n_1 \lneq n_2 \cdots \lneq n_{j-1} \lneq n_j$, and this implies a non-zero value for that correction. For calculating $\Psi^{(N+1)}(x)$, $N+1$ different labels must be selected form the set $\mathfrak{N}_j$ but since the cardinality of $\mathfrak{N}_j$ is less than $N+1$ therefore this selection will result in at least two similar values for different labels of the series

$$|\mathfrak{N}_j| = N < N + 1, \qquad \Longrightarrow \qquad \{i \in \mathfrak{N}; n_i = n_{i+1}\} \neq \varnothing, \tag{A.10}$$

and consequently this leads to $\prod_{i=1}^{N}(1 - \chi^{\pm(n_i - n_{i+1})}) = 0$ and as a result $(F_{-}^{+})_{(N+1)} = 0$ and finally $\Psi^{(N+1)}(x) = 0$. Because of the recursive nature of the perturbative corrections, the corrections from orders higher than $N+1$ will not be generated, $\Psi^{(j \gneq N)}(x) = 0$,(Fig. A.1)

□

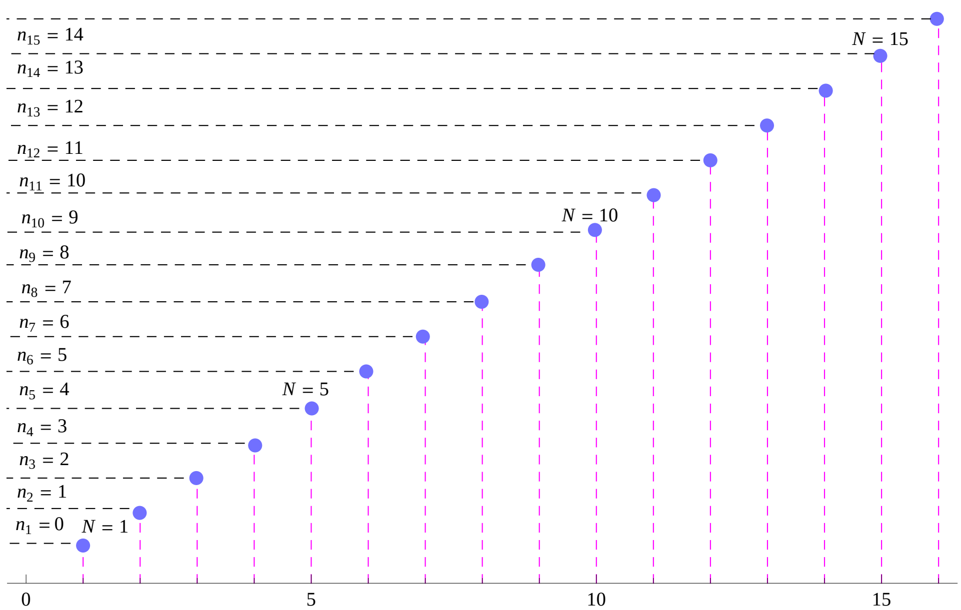

Figure A.1: Calculating the N-th correction for a Dirac-Comb with N delta barriers. n_i's are the labels of the series appearing in (6.12). Clearly for $(N+1)$-th correction one of the labels will be repeated $n_i = n_{i+1}$ and this will make the product (A.2)to vanish.

APPENDIX

Appendix B

TRANSFER MATRIX FOR THE COMPLEX BARRIER POTENTIAL

Consider a particle in a barrier potential given by the Hamiltonian $H = H_0 + \varepsilon$, where H_0 is the Hamiltonian of the free particle and ε is the height of the barrier potential. If the height of the barrier potential is small then we can use the formalism of Chapter 3 to solve this problem. For this purpose we consider the case where $v_0(x) = 0$ and $v(x) = 1$ having compact support in interval $[-\alpha, \alpha]$. Using (3.9) for the Green's function of this system we have: $G(x, x') = \frac{\sin k(x-x')}{k}$. The zeroth order correction is

$$\Psi^{(0)}(x) = \begin{pmatrix} e^{ikx} \\ e^{-ikx} \end{pmatrix}, \qquad \forall x \in \mathbb{R}. \tag{B.1}$$

For the first order correction we obtain

$x < -\alpha$:

$$\Psi^{(1)}(x) = 0, \tag{B.2}$$

$-\alpha < x < \alpha$:

$$\begin{aligned} \Psi^{(1)}(x) &= \int_{-\infty}^{x} G(x, x')\hat{v}(x')\Psi^{(0)}(x')dx' \\ &= \frac{1}{2ik}\int_{-\infty}^{x} (e^{ik(x-x')} - e^{-ik(x-x')}) \begin{pmatrix} e^{ikx'} \\ e^{-ikx'} \end{pmatrix} dx' \\ &= \frac{1}{2ik}\int_{-\alpha}^{x} \begin{pmatrix} e^{ikx} - e^{-ikx}e^{2ikx'} \\ e^{ikx}e^{-2ikx'} - e^{-ikx} \end{pmatrix} dx' \\ &= \frac{1}{2ik} \begin{pmatrix} (x + \alpha - \frac{1}{2ik})e^{ikx} + \frac{1}{2ik}e^{-2ik\alpha}e^{-ikx} \\ \frac{1}{2ik}e^{2ik\alpha}e^{ikx} - (x + \alpha + \frac{1}{2ik})e^{-ikx} \end{pmatrix}, \end{aligned} \tag{B.3}$$

$x > \alpha$:

$$\begin{aligned}\Psi^{(1)}(x) &= \frac{1}{2ik}\int_{-\infty}^{\alpha}(e^{ik(x-x')} - e^{-ik(x-x')})\begin{pmatrix} e^{ikx'} \\ e^{-ikx'}\end{pmatrix} dx' \\ &= \frac{1}{2ik}\int_{-\alpha}^{\alpha}\begin{pmatrix} e^{ikx} - e^{-ikx}e^{2ikx'} \\ e^{ikx}e^{-2ikx'} - e^{-ikx}\end{pmatrix} dx' \\ &= \frac{1}{2ik}\begin{pmatrix} 2\alpha e^{ikx} - \frac{\sin(2k\alpha)}{k}e^{-ikx} \\ \frac{\sin(2k\alpha)}{k}e^{ikx} - 2\alpha e^{-ikx}\end{pmatrix}. \end{aligned} \tag{B.4}$$

Similarly the second order corrections gives

$x < -\alpha$:

$$\Psi^{(2)}(x) = 0, \tag{B.5}$$

$x > \alpha$:

$$\begin{aligned}\Psi^{(2)}(x) &= \int_{-\alpha}^{\alpha} G(x,x')\hat{v}(x')\Psi^{(1)}(x')dx' \\ &= \int_{-\alpha}^{\alpha}\frac{\sin k(x-x')}{k}\frac{1}{2ik}\begin{pmatrix}(x+\alpha-\frac{1}{2ik})e^{ikx} + \frac{1}{2ik}e^{-2ik\alpha}e^{-ikx} \\ \frac{1}{2ik}e^{2ik\alpha}e^{ikx} - (x+\alpha+\frac{1}{2ik})e^{-ikx}\end{pmatrix} dx' \\ &= \frac{-1}{4k^2}\int_{-\alpha}^{\alpha}\begin{pmatrix}\mathcal{A}_k(x')e^{ikx} + \mathcal{B}_k(x')e^{-ikx} \\ \mathcal{B}_{-k}(x')e^{ikx} + \mathcal{A}_{-k}(x')e^{-ikx}\end{pmatrix} dx', \end{aligned} \tag{B.6}$$

where $\mathcal{A}_k(x')$ and $\mathcal{B}_k(x')$ is given by

$$\mathcal{A}_k(x') := (x' + \alpha - \frac{1}{2ik}) + \frac{e^{-2ik\alpha}}{2ik}e^{-2ikx'}, \tag{B.7}$$

$$\mathcal{B}_k(x') := -(x' + \alpha - \frac{1}{2ik})e^{2ikx'} - \frac{e^{-2ik\alpha}}{2ik}. \tag{B.8}$$

Using the relation

$$\int_{-\alpha}^{\alpha}(x' + \alpha - \frac{1}{2ik})e^{2ikx'}\, dx' = 2\alpha\frac{e^{2ik\alpha}}{2ik} - \frac{\sin(2k\alpha)}{ik^2}, \tag{B.9}$$

the second order correction will result in

$$\Psi^{(2)}(x) = (\frac{1}{2ik})^2\begin{pmatrix}\mathcal{C}_k e^{ikx} + \mathcal{D}_k e^{-ikx} \\ \mathcal{D}_{-k}e^{ikx} + \mathcal{C}_{-k}e^{-ikx}\end{pmatrix}, \tag{B.10}$$

where

$$\mathcal{C}_k := 2\alpha^2 - \frac{\alpha}{ik} + e^{-2ik\alpha}\frac{\sin(2k\alpha)}{2ik^2}, \tag{B.11}$$

$$\mathcal{D}_k := \frac{\sin(2k\alpha)}{ik^2} - 2\alpha\frac{\cos(2k\alpha)}{ik}. \tag{B.12}$$

Now for constructing the transfer matrix we need the scattering states in both sides of the interval where barrier potential has compact support. From (B.2) and (B.5) we have

$x < -a$:

$$\Psi(x) = 0, \tag{B.13}$$

and (B.4) and (B.10) result in:

$x > a$:

$$\begin{aligned}\Psi(x) &= \begin{pmatrix} e^{ikx} \\ e^{-ikx} \end{pmatrix} + \frac{\varepsilon}{2ik}\begin{pmatrix} 2\alpha e^{ikx} - \frac{\sin(2k\alpha)}{k}e^{-ikx} \\ \frac{\sin(2k\alpha)}{k}e^{ikx} - 2\alpha e^{-ikx} \end{pmatrix} \\ &+ (\frac{\varepsilon}{2ik})^2 \begin{pmatrix} \mathcal{C}_k e^{ikx} + \mathcal{D}_k e^{-ikx} \\ \mathcal{D}_{-k}e^{ikx} + \mathcal{C}_{-k}e^{-ikx} \end{pmatrix} \\ &= \begin{pmatrix} (1 + \frac{\varepsilon\alpha}{ik} - \frac{\varepsilon^2}{4k^2}\mathcal{C}_k)e^{ikx} + (-\varepsilon\frac{\sin(2k\alpha)}{2ik^2} - \frac{\varepsilon^2}{4k^2}\mathcal{D}_k)e^{-ikx} \\ (\varepsilon\frac{\sin(2k\alpha)}{2ik^2} - \frac{\varepsilon^2}{4k^2}\mathcal{D}_{-k})e^{ikx} + (1 - \frac{\varepsilon\alpha}{ik} - \frac{\varepsilon^2}{4k^2}\mathcal{C}_{-k})e^{-ikx} \end{pmatrix}.\end{aligned} \tag{B.14}$$

Therefore the transfer matrix is given by

$$\mathcal{M} = \begin{pmatrix} 1 + \frac{\varepsilon\alpha}{ik} - \frac{\varepsilon^2}{4k^2}\mathcal{C}_k & \varepsilon\frac{\sin(2k\alpha)}{2ik^2} - \frac{\varepsilon^2}{4k^2}\mathcal{D}_{-k} \\ -\varepsilon\frac{\sin(2k\alpha)}{2ik^2} - \frac{\varepsilon^2}{4k^2}\mathcal{D}_k & 1 - \frac{\varepsilon\alpha}{ik} - \frac{\varepsilon^2}{4k^2}\mathcal{C}_{-k} \end{pmatrix} \tag{B.15}$$

$\mathcal{C}_k$ and $\mathcal{D}_k$ are given by (B.11) and (B.20). Furthermore we can write

$$\begin{aligned}\mathcal{M}_{22}(k) :&= 1 - \frac{\varepsilon\alpha}{ik} - \frac{\varepsilon^2}{4k^2}(2\alpha^2 + \frac{\alpha}{ik} - e^{2ik\alpha}\frac{\sin(2k\alpha)}{2ik^2}) \\ &= \mathcal{M}_{11}(-k),\end{aligned} \tag{B.16}$$

$$\begin{aligned}\mathcal{M}_{12}(k) :&= \varepsilon\frac{\sin(2k\alpha)}{2ik^2} - \frac{\varepsilon^2}{4k^2}(-\frac{\sin(2k\alpha)}{ik^2} + 2a\frac{\cos(2k\alpha)}{ik}) \\ &= \mathcal{M}_{21}(-k) = -\mathcal{M}_{21}(k).\end{aligned} \tag{B.17}$$

So far we have derived the transfer matrix using perturbation theory. This matrix can be analytically evaluated using the solution of the dimensionless Schrödinger equation for the barrier potential on the real line. Solving the equation results in the following form of the solutions

$$\psi_I(x) = A_1 e^{ikx} + B_1 e^{-ikx}, \qquad \text{for} \quad x < -\alpha, \tag{B.18}$$

$$\psi_{II}(x) = A_2 e^{ik'x} + B_2 e^{-ik'x}, \qquad \text{for} \quad |x| < \alpha, \tag{B.19}$$

$$\psi_{III}(x) = A_3 e^{ikx} + B_3 e^{-ikx}, \qquad \text{for} \quad x > \alpha, \tag{B.20}$$

where $k'^2 = k^2 - \varepsilon$. The following boundary conditions

$$\begin{cases} \psi_I(x)\big|_{x=-\alpha} = \psi_{II}(x)\big|_{x=-\alpha}, \\ \\ \psi'_I(x)\big|_{x=-\alpha} = \psi'_{II}(x)\big|_{x=-\alpha}, \\ \\ \psi_{II}(x)\big|_{x=\alpha} = \psi_{III}(x)\big|_{x=\alpha}, \\ \\ \psi'_{II}(x)\big|_{x=\alpha} = \psi'_{III}(x)\big|_{x=\alpha}, \end{cases} \tag{B.21}$$

lead to the expressions

$$\begin{cases} A_1 e^{-ik\alpha} + B_1 e^{ik\alpha} = A_2 e^{-ik'\alpha} + B_2 e^{ik'\alpha}, \\ ikA_1 e^{-ik\alpha} - ikB_1 e^{ik\alpha} = ik'A_2 e^{-ik'\alpha} - ik'B_2 e^{ik'\alpha}, \end{cases} \tag{B.22}$$

$$\begin{cases} A_2 e^{ik'\alpha} + B_2 e^{-ik'\alpha} = A_1 e^{ik\alpha} + B_1 e^{-ik\alpha}, \\ ik'A_2 e^{ik'\alpha} - ik'B_2 e^{-ik'\alpha} = ikA_2 e^{ik\alpha} - ikB_2 e^{-ik\alpha}. \end{cases} \tag{B.23}$$

More precisely we have

$$\begin{cases} \frac{1}{2}(1+\frac{k}{k'})A_1 e^{-i(k-k')\alpha} + \frac{1}{2}(1-\frac{k}{k'})B_1 e^{i(k+k')\alpha} = A_2, \\ \frac{1}{2}(1-\frac{k}{k'})A_1 e^{-i(k+k')\alpha} + \frac{1}{2}(1+\frac{k}{k'})B_1 e^{i(k-k')\alpha} = B_2, \end{cases} \tag{B.24}$$

$$\begin{cases} \frac{1}{2}(1+\frac{k}{k'})A_3 e^{i(k-k')\alpha} + \frac{1}{2}(1-\frac{k}{k'})B_3 e^{-i(k+k')\alpha} = A_2, \\ \frac{1}{2}(1-\frac{k}{k'})A_3 e^{i(k+k')\alpha} + \frac{1}{2}(1+\frac{k}{k'})B_3 e^{-i(k-k')\alpha} = B_2. \end{cases} \tag{B.25}$$

In matrix language we can write

$$\begin{pmatrix} \frac{1}{2}(1+\frac{k}{k'})e^{-i(k-k')\alpha} & \frac{1}{2}(1-\frac{k}{k'})e^{i(k+k')\alpha} \\ \frac{1}{2}(1-\frac{k}{k'})e^{-i(k+k')\alpha} & \frac{1}{2}(1+\frac{k}{k'})e^{i(k-k')\alpha} \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}, \tag{B.26}$$

$$\begin{pmatrix} \frac{1}{2}(1+\frac{k}{k'})e^{i(k-k')\alpha} & \frac{1}{2}(1-\frac{k}{k'})e^{-i(k+k')\alpha} \\ \frac{1}{2}(1-\frac{k}{k'})e^{i(k+k')\alpha} & \frac{1}{2}(1+\frac{k}{k'})e^{-i(k-k')\alpha} \end{pmatrix} \begin{pmatrix} A_3 \\ B_3 \end{pmatrix} = \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}. \tag{B.27}$$

Thus we have

$$\begin{pmatrix} A_3 \\ B_3 \end{pmatrix} = M_1 M_2^{-1} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}, \tag{B.28}$$

where

$$M_1 := \begin{pmatrix} \frac{1}{2}(1+\frac{k}{k'})e^{-i(k-k')\alpha} & \frac{1}{2}(1-\frac{k}{k'})e^{i(k+k')\alpha} \\ \frac{1}{2}(1-\frac{k}{k'})e^{-i(k+k')\alpha} & \frac{1}{2}(1+\frac{k}{k'})e^{i(k-k')\alpha} \end{pmatrix}, \tag{B.29}$$

$$M_2 := \begin{pmatrix} \frac{1}{2}(1+\frac{k}{k'})e^{i(k-k')\alpha} & \frac{1}{2}(1-\frac{k}{k'})e^{-i(k+k')\alpha} \\ \frac{1}{2}(1-\frac{k}{k'})e^{i(k+k')\alpha} & \frac{1}{2}(1+\frac{k}{k'})e^{-i(k-k')\alpha} \end{pmatrix}. \tag{B.30}$$

Therefore the transfer matrix is

$$M := M_1 M_2^{-1} = \begin{pmatrix} \Im(k,k') & \wp(k,k') \\ \wp(-k,k') & \Im(-k,k') \end{pmatrix}, \tag{B.31}$$

where

$$\Im(k,k') := \frac{k'}{k}\Big(\frac{1}{4}(1+\frac{k}{k'})^2 e^{-2i(k-k')\alpha} - \frac{1}{4}(1-\frac{k}{k'})^2 e^{-2i(k+k')\alpha}\Big), \tag{B.32}$$

$$\wp(k,k') := ik'(1-\frac{k^2}{k'^2})\frac{\sin 2k'\alpha}{2k}. \tag{B.33}$$

Introducing

$$\omega := \frac{k'}{k}, \tag{B.34}$$

$$k' := \sqrt{k^2-\varepsilon}, \tag{B.35}$$

the matrix M can be written as

$$M = \begin{pmatrix} e^{-2i\alpha k}\frac{(1+\omega)^2 e^{2i\alpha k\omega} - (1-\omega)^2 e^{-2i\alpha k\omega}}{4\omega} & \frac{i(\omega^2-1)}{2\omega}\sin(2\alpha k\omega) \\ -\frac{i(\omega^2-1)}{2\omega}\sin(2\alpha k\omega) & e^{2i\alpha k}\frac{(1+\omega)^2 e^{-2i\alpha k\omega} - (1-\omega)^2 e^{2i\alpha k\omega}}{4\omega} \end{pmatrix}. \tag{B.36}$$

Using the following approximations

$$\begin{aligned} \omega := \frac{k'}{k} &= \sqrt{1-\frac{\varepsilon}{k^2}} \\ &= 1 - \frac{\varepsilon}{2k^2} - \frac{\varepsilon^2}{8k^4} + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{B.37}$$

$$\begin{aligned} \frac{1}{\omega} &= (1-\frac{\varepsilon}{k^2})^{-\frac{1}{2}} \\ &= 1 + \frac{\varepsilon}{2k^2} + \frac{3\varepsilon^2}{8k^4} + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{B.38}$$

$$\begin{aligned} k' &= \sqrt{k^2-\varepsilon} \\ &= k - \frac{\varepsilon}{2k} - \frac{\varepsilon^2}{8k^3} - \mathcal{O}(\varepsilon^3), \end{aligned} \tag{B.39}$$

the entries of matrix M can be expanded in powers of ε, i.e.,

$$\begin{aligned}
M_{22}(k,k') =& M_{11}(-k,k') \\
=& e^{2i\alpha k}\frac{(1+\omega)^2 e^{-2i\alpha k\omega} - (1-\omega)^2 e^{2i\alpha k\omega}}{4\omega} \\
=& \frac{e^{2i\alpha k}}{4}(1+\frac{\varepsilon}{2k^2}+\frac{3\varepsilon^2}{8k^4})\Big\{(2-\frac{\varepsilon}{2k^2}-\frac{\varepsilon^2}{8k^4})^2 e^{-2i\alpha k(1-\frac{\varepsilon}{2k^2}-\frac{\varepsilon^2}{8k^4})} \\
&-(\frac{\varepsilon}{2k^2}-\frac{\varepsilon^2}{8k^4})^2 e^{2i\alpha k(1-\frac{\varepsilon}{2k^2}-\frac{\varepsilon^2}{8k^4})}\Big\} \\
=& \frac{e^{2i\alpha k}}{4}(1+\frac{\varepsilon}{2k^2}+\frac{3\varepsilon^2}{8k^4})\Big\{(4-\frac{2\varepsilon}{k^2}-\frac{\varepsilon^2}{4k^4})e^{-2i\alpha k(1-\frac{\varepsilon}{2k^2}-\frac{\varepsilon^2}{8k^4})} \\
&-\frac{\varepsilon^2}{4k^4}e^{2i\alpha k(1-\frac{\varepsilon}{2k^2}-\frac{\varepsilon^2}{8k^4})}\Big\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{e^{2i\alpha k}}{4}\Big\{(4+\frac{\varepsilon^2}{4k^4})e^{-2i\alpha k(1-\frac{\varepsilon}{2k^2}-\frac{\varepsilon^2}{8k^4})} - \frac{\varepsilon^2}{4k^4}e^{2i\alpha k(1-\frac{\varepsilon}{2k^2}-\frac{\varepsilon^2}{8k^4})}\Big\} \\
&= \frac{e^{2i\alpha k}}{4}\Big\{e^{-2i\alpha k}e^{2i\alpha k(\frac{\varepsilon}{2k^2}+\frac{\varepsilon^2}{8k^4})} - \frac{i\varepsilon^2}{2k^4}\sin(2\alpha k(1-\frac{\varepsilon}{2k^2}-\frac{\varepsilon^2}{8k^4}))\Big\} \\
&= (1+\frac{i\alpha\varepsilon}{k}+\frac{i\alpha\varepsilon^2}{4k^3}-\frac{\alpha^2\varepsilon^2}{2k^2}) - \frac{i\varepsilon^2}{8k^4}\sin(2\alpha k)e^{2i\alpha k} \\
&= 1-\frac{\alpha\varepsilon}{ik}-\frac{\varepsilon^2}{4k^2}(2\alpha^2+\frac{\alpha}{ik}-e^{2i\alpha k}\frac{\sin(2\alpha k)}{2k^2}) \\
&= \mathcal{M}_{22}(k) = \mathcal{M}_{11}(-k),
\end{aligned} \tag{B.40}$$

and

$$\begin{aligned}
M_{12}(k,k') &= M_{21}(-k,k') \\
&= \frac{i(\omega^2-1)}{2\omega}\sin(2\alpha k\omega) \\
&= \frac{i}{2}(1+\frac{\varepsilon}{2k^2})(-\frac{\varepsilon}{k^2})\sin(2\alpha(k-\frac{\varepsilon}{2k}-\frac{\varepsilon^3}{8k^3})) \\
&= \frac{i}{2}(-\frac{\varepsilon}{k^2}-\frac{\varepsilon^2}{2k^4})[\sin(2\alpha k)\cos(-\frac{\alpha\varepsilon}{k})+\sin(-\frac{\alpha\varepsilon}{2k})\cos(2\alpha k)] \\
&= \frac{i}{2}(-\frac{\varepsilon}{k^2}-\frac{\varepsilon^2}{2k^4})[\sin(2\alpha k)-\frac{\alpha\varepsilon}{k}\cos(2\alpha k)] \\
&= \varepsilon(\frac{\sin(2\alpha k)}{2ik^2}) - \frac{\varepsilon^2}{4k^2}(-\frac{\sin(2\alpha)k}{ik^2}+2\alpha\frac{\cos(2\alpha k)}{ik}) \\
&= \mathcal{M}_{12}(k) = \mathcal{M}_{21}(-k) = -\mathcal{M}_{21}(k).
\end{aligned} \tag{B.41}$$

Therefore

$$M = \mathcal{M} + \mathcal{O}(\varepsilon^3), \tag{B.42}$$

which shows that the transfer matrix, we derived for the barrier potential with negligible height ε using perturbation method of Chapter 3, is the same as the one we obtain using analytic method at least up to second order.

APPENDIX

Appendix C

SCHRÖDINGER-LIKE EQUATION FOR THE DYNAMICS OF EM WAVES INSIDE DIELECTRIC WAVEGUIDES

Typically, laser gain media are made from various types of dielectric materials and these materials essentially have no significant conductivity under normal conditions. These materials, when effected by electric fields, conduct the field in a special manner. The electric charges do not follow through the material, instead, they shift slightly from their average equilibrium position causing *dielectric polarization* and accordingly generate internal fields. This as a result partially compensates the external fields. This is the general scheme of the response of dielectric materials to the applied electromagnetic fields, [43, 50, 51]. Here we will carry out a simple wave analysis which will lead to the explanation of optical properties of the laser media.

The polarization phenomena, happening in dielectric media due to the presence of external fields, is proportional to the distance of disposition of localized electric charges of each atoms. If we call P the macroscopic polarization then:

$$P = -Ne\chi, \tag{C.1}$$

where χ being the disposition and $-e$ is the charge of an electron and N is the number of charge density in a volume. Suppose that this disposition (polarization) is due to an non-static external electric filed E, [50, 51]. On the other hand, by assuming a simple Lorentz Oscillator Model of atoms [43], for balancing the interaction of external electric field with an electron and the restoring force (acting on the electron to return it back to its equilibrium position), we have

$$-eE = m\frac{d^2\chi}{dt^2} + m\gamma\frac{d\chi}{dt} + k\chi. \tag{C.2}$$

k is the restoring-force constant and γ the damping coefficient[7]. If we assume that non-

[7] $m\gamma\frac{d\chi}{dt}$ is the fractional damping term which is assumed to be proportional to the instantaneous velocity of the charge and γ is specific damping coefficient represents the rate at which the polarization decays after removing of the applied field.

static external field is harmonically varying by the form $E = E_0 e^{-iwt}$, where ω is the angular frequency, the disposition is also varying harmonically $x = x_0 e^{-iwt}$. By applying these assumptions into (C.2) and using the result in (C.1) we get

$$
\begin{aligned}
P &= \left(\frac{Ne^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}\right)E, \\
&= \epsilon_0 \chi E, \quad \text{and} \quad \chi = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}, \quad \omega_0 = \left(\frac{k}{m}\right)^{\frac{1}{2}} \quad \text{(resonance frequency).} \qquad \text{(C.3)}
\end{aligned}
$$

where $\omega_p^2 = \frac{Ne^2}{m\epsilon_0}$ is the plasma frequency. The source-free Maxwell's equations for an isotropic dielectric medium are

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{D} &= 0, \qquad & \vec{\nabla} \wedge \vec{E} &= -\partial_t \vec{B}, \\
\vec{\nabla} \cdot \vec{B} &= 0, \qquad & \vec{\nabla} \wedge \left(\frac{\vec{B}}{\mu}\right) &= -\partial_t \vec{D},
\end{aligned} \qquad \text{(C.4)}
$$

where μ and ϵ are magnetic[8] and electric permittivity of the medium, respectively, [50]. The electric displacement vector reads $\vec{D} = \epsilon \vec{E} = \epsilon_0 \vec{E} + \vec{P}$ and in view of (C.3) the permittivity of the dielectric media is $\epsilon = \epsilon_0(1+\chi)$. If we assume that the dielectric media (gain medium) is obtained by doping a host medium with permittivity $\epsilon_{\mathcal{H}}$ then the permittivity of the whole setup will be $\epsilon = \epsilon_{\mathcal{H}} + \epsilon_0 \chi$. It is appropriate to use the concept of refractive index instead of permittivity, formally, $\epsilon = \epsilon_0 \mathfrak{n}^2$. For the permittivity of the mentioned dielectric medium $\epsilon = \epsilon_0(\mathfrak{n}_{\mathcal{H}}^2 + \chi)$ where $\mathfrak{n}_{\mathcal{H}}$ is the reflective index of the host medium. By following some easy steps of vector analysis and using Maxwell's equations above we can construct differential equations describing the vector functions $\vec{E}$ or $\vec{B}$, namely as a wave equation. For instance, if A is either of the vectors $\vec{E}$ or $\vec{B}$, we obtain (using $c^{-2} = \mu_0 \epsilon_0$), [50, 51]

$$
\nabla^2 \vec{A} = \frac{\mathfrak{n}^2}{c^2} \partial_t^2 \vec{A}, \qquad \mathfrak{n}^2 = \mathfrak{n}_{\mathcal{H}}^2 + \chi = \mathfrak{n}_{\mathcal{H}}^2 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}. \qquad \text{(C.5)}
$$

This homogeneous wave equation can be solved by the method of separation of variables or Fourier Transform method due to a particular boundary conditions. The solution have the form $\vec{A} = \vec{A}_0 e^{i(\mathfrak{K}_Z Z - wt)}$ (a planar wave traveling in z direction) where the complex constant $\mathfrak{K}_Z$ is the wave vector (radians per meter). For a valid solution of the wave equation (C.5) the wave vector $\mathfrak{K}_Z$ and angular frequency must related to each other by the dispersion

[8]Since $\mu = \mu_0(1 + \chi_m)$, in a free-Magnetization media ($\chi_m = 0$) we have $\mu = \mu_0$.

relation $\mathfrak{K}_Z = \frac{\omega}{c}\mathfrak{n}$, thus

$$\begin{aligned}\mathfrak{K}_Z^2 &= \frac{\omega^2}{c^2}(\mathfrak{n}_{\mathcal{H}}^2 + \chi) \\ &= \frac{\omega^2}{c^2}\left(\mathfrak{n}_{\mathcal{H}}^2 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}\right).\end{aligned} \tag{C.6}$$

The form of the differential equation which describes the behavior of electromagnetic field as a wave equation might suggest an analogy with the concept of wave equation in Quantum Mechanics, [42]. Assume the following set of solutions of (C.5), [26, 38],

$$\begin{aligned}\vec{E}(X,t) &= Ee^{-i\omega t}\psi(X)\hat{e_Y}, \\ \vec{B}(X,t) &= -i\omega^{-1}Ee^{-i\omega t}\psi'(X)\hat{e_Z}.\end{aligned} \tag{C.7}$$

This set clearly satisfies Maxwell's equations. For instance, the electric field should be the solutions of the differential equation $\nabla^2\vec{E} = \frac{\mathfrak{n}^2}{c^2}\partial_t^2\vec{E}$. The vector Laplacian of the field simply has the form: $\nabla^2\vec{E} = \left(\nabla^2 E_X, \nabla^2 E_Y, \nabla^2 E_Z\right)$. Therefore

$$Ee^{-i\omega t}\frac{d^2\psi(X)}{dX^2}\hat{e_Y} = \frac{\mathfrak{n}^2}{c^2}(-\omega^2)Ee^{-i\omega t}\psi(X)\hat{e_Y},$$

or

$$\frac{d^2\psi(X)}{dX^2} = -\frac{\omega^2}{c^2}\mathfrak{n}^2\psi(X),$$

or

$$-\frac{d^2\psi(x)}{dx^2} + v(x)\psi(x) = k^2\psi(x). \tag{C.8}$$

This is the time-independent dimensionless Shrödinger equation[9] where we have defined the new variables:

$$x := \frac{X}{\ell}, \qquad k^2 = \ell^2\frac{\omega^2}{c^2}, \qquad v(x) = k^2(1-\mathfrak{n}^2). \tag{C.9}$$

[9]The same analysis can be repeated for the magnetic field $\vec{B}$.

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