An Introduction to Heegaard Floer Homology

by

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
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To my beloved husband Hilmi

To my dearest parents Halime and Sedat

And to my two lovely sisters Minnet and Yağmur

ABSTRACT

Heegaard Floer homology for a closed, oriented three-manifold Y is defined using Heegaard diagrams and a certain holomorphic curve count in the spirit of Lagrangian Floer homology. For each $s \in Spin^c(Y)$, similar constructions give different versions of homology groups $\widehat{HF}(Y, s)$, $HF^\infty(Y, s)$, $HF^-(Y, s)$, and $HF^+(Y, s)$, each of which is an invariant of the underlying three-manifold Y. The theory also contains a knot invariant, a smooth four-manifold invariant, and a contact three-manifold invariant besides other things.

In this thesis, we focus on the definition of Heegaard Floer homology for a closed, oriented three-manifold Y and the necessary topological tools to define it. The basics of knot Floer homology, an invariant of oriented, nullhomologous knots and links in closed, oriented three-manifolds are also discussed. In addition, we briefly mention another invariant called Khovanov homology for oriented links L which seems to be related to knot Floer homology.

ÖZETÇE

Kapalı, yönlü, 3-boyutlu çokkatlılar için Heegaard Floer homoloji Heegaard diyagramlar ve bazı holomorf eğrilerin Lagrangian Floer homoloji teorisine benzer bir sayım kullanılarak tanımlanır. Her $s \in Spin^c$ yapısı için benzer bir inşa ile farklı homoloji grupları elde edilir: $\widehat{HF}(Y,s)$, $HF^{\infty}(Y,s)$, $HF^{-}(Y,s)$, $HF^{+}(Y,s)$ ve bunların herbiri üzerinde çalışılan 3-boyutlu çokkatlı için birer değişmezdir. Bu teori aynı zamanda düğüm değişmezi, düzgün 4-boyutlu çokkatlı değişmezi ve kontak 3-boyutlu çok katlı değişmezini de içerir.

Tezde kapalı, yönlü, 3-boyutlu çokkatlılar için Heegaard Floer homoloji tanımı ve bu tanımı verebilmek için gerekli topolojik kavramların üzerinde odaklanılmıştır. Kapalı, yönlü 3-boyutlu çokkatlılarda yönlü ve homoloji sınıfı sıfır olan düğüm ve linkler için bir değişmez olan düğüm Floer homolojinin temellerine değinilmiş ve ek olarak da düğüm Floer homoloji ile benzer özellikler gösteren ve yönlü linkler için tanımlanan Khovanov homolojiden bahsedilmiştir.

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Chapter 1

INTRODUCTION

The aim of this thesis is to understand the Heegaard Floer homology for closed, oriented three-manifolds. First we begin with the definition of Heegaard Floer homology following [29], [8], [28], and [23]. Then, we study the basics of knot Floer homology which is defined similar to Heegaard Floer homology based on [32], [8]. Further, we briefly review Khovanov homology, following [3], which is also a knot invariant which seems to be related to knot Floer homology.

Heegaard Floer homology is a three-manifold invariant introduced by Ozsváth and Szabó in 2000. Any closed, oriented three-manifold Y can be decomposed into 2 handlebodies sharing a common boundary, called Heegaard surface, which is a genus-g surface Σ with two sets of attaching circles $\{\alpha_1, .., \alpha_g\}$ and $\{\beta_1, .., \beta_g\}$ where each set is homologically linearly independent containing closed, embedded, disjoint curves on Σ. The symmetric product $Sym^g(\Sigma)$ associated to Σ is the space of unordered g-tuple of points of Σ. The attaching circles induce a pair of smoothly embedded g-dimensional tori in $Sym^{(}\Sigma)$ defined by $\mathbb{T}_\alpha = \alpha_1 \times ... \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times ... \times \beta_g$, then Heegaard Floer homology is defined using the finite set of intersection points of the totally real tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ as generators of the chain complex. With a fixed generic complex structure over the Heegaard surface, the boundary map of this chain complex is defined by counting the number of holomorphic disks in the moduli space of holomorphic curves connecting the intersection points.

The remainder of this thesis is organized as follows: In Chapter 2, we define Heegaard decomposition and Heegaard diagrams for closed, oriented three-manifolds. The most important part of this chapter is the relation between two Heegaard diagrams representing the same three-manifold via finite sequence of Heegaard moves. Chapter 3 reviews the necessary topological tools to define Heegaard Floer homology. It begins with the symmetric product space, $Sym^g(\Sigma)$ and its topology and also discusses $Spin^c$ structures over three-manifolds. In Chapter 4 we give the analytic background including Gromov's compactness theorem and transversality and compectness theorems of moduli space of holomorphic disks, and most of these statements are mentioned without proof. Then in Chapter 5 we define Heegaard Floer homology groups for $b_1 = 0$ and $b_1 > 0$ separately since there are certain technical complications when $b_1 > 0$. Then we mention some properties of these homology groups and give some simple examples. In Chapter 6 we study the dependence of Heegaard Floer homology groups on the coherent orientation system, complex structure over Σ, and the path of nearly symmetric almost complex structure over $Sym^g(\Sigma)$. In fact they are also independent of the chosen Heegaard diagram and with this additional property they become an invariant for closed oriented three-manifolds. Moreover, we define actions on these homology groups when $b_1 = 0$ and $b_1 > 0$ separately which provide extra structures. Chapter 7 includes the basics of knot Floer homology. We mention some properties of these groups, for example its Euler characteristic is the Alexander-Conway polynomial, it is sensitive to mutation move of knots, and it detects genus and fiberedness of a knot in S^3. We also review the definition of Khovanov homology, another invariant for oriented knots in S^3 and compare the two in certain cases.

Heegaard Floer homology is an active area of research and we were not able to cover many parts of it. For example, there is a different description of Heegaard Floer homology of three-manifolds given by Lipshitz in [20] where he defines Floer homology groups using $\Sigma \times [0,1] \times \mathbb{R}$ instead of symmetric product space $Sym^g(\Sigma)$. In this setting the invariance of the homology groups can also be proved and it is equivalent to the original construction given by Ozsvath and Szabo that we discuss in this thesis. There are generalizations of Heegaard Floer homology to compact, oriented three-manifolds with boundary, [21], [15]. In addition, a contact structure on a closed, oriented three-manifold has an invariant, Heegaard Floer homology of Y which is described in [30].

Chapter 2

HEEGAARD DECOMPOSITIONS AND HEEGAARD DIAGRAMS

We study Heegaard decomposition and Heegaard diagrams for a closed, oriented three- manifold Y. We prove that every closed, oriented three-manifold admits a Heegaard decomposition into two pieces called handlebodies. We study that a Morse function on the manifold $f : Y \to \mathbb{R}$ also provides a Heegaard decomposition of the three-manifold. These two handlebodies can be glued to obtain the three-manifold back, however the key point is how to glue. Thus we define a set of attaching circles for each handlebody which show how to glue two handlebodies to obtain the three-manifold back. Then the triple which is called *Heegaard diagram* containing two sets of attaching circles and the genus-g surface, called the *Heegaard surface* which is the common boundary of these two handlebodies, determines a three-manifold. We see then the same manifold can admit many different Heegaard decompositions but they are related by three basic moves, called Heegaard moves: isotopy, handleslide, and stabilization. We give some examples of Heegaard decompositions and diagrams in the next section, and in the last section we see how to relate different Heegaard decompositions of the same three-manifold by Heegaard moves which is the main result.

2.1 Heegaard Decompositions and Diagrams

A *genus-g handlebody* U is diffeomorphic to a regular neighborhood of bouquet of g circles in $\mathbb{R}^3$ with boundary genus-g surface Σ_g. Let $\varphi : \partial U_0 \to \partial U_1$ be a homeomorphism. By gluing two handlebodies U_0 and U_1 along their common boundary, we mean to identify each point $x \in \partial U_0$ with $\varphi(x) \in \partial U_1$, and we obtain a three manifold Y. For every $x \in \partial U_0$, $x = \varphi(x) \in \partial U_1$ has a neighborhood in Y which can be obtained by gluing two half balls, neighborhoods of x in U_0 and $\varphi(x)$ in U_1 respectively, thus Y is a manifold without boundary. Heegaard decomposition of a closed, oriented three manifold Y into two handlebodies U_0, U_1 is

$$Y = U_0 \cup_{\Sigma_g} U_1$$

where Σ_g is closed, orientable surface of genus g and $\partial U_0 \simeq \Sigma_g \simeq \partial U_1$.

Let us consider the Heegaard decomposition of S^3 into genus-0 handlebodies which is done in a unique way. Genus-0 handlebody is just a solid ball with boundary S^2, so if we attach two solid balls along their common boundary we obtain get S^3. Similarly, genus-1 Heegaard decomposition of S^3 is given by gluing two solid tori along T^2, since complement of a solid torus in S^3 is also a solid torus.

Lens space, which is a closed, oriented three-manifold denoted as $L(p,q)$ with $(p,q) = 1$ and $1 \leq q < p$ has a genus-1 Heegaard decomposition as follows. Lens spaces can be described by a free $\mathbb{Z}/p$ action on S^3 as follows. Consider S^3 in $\mathbb{C}^2$ and define the free $\mathbb{Z}/p$ action given by γ as $\Gamma(z,w) = (e^{2\pi/p}z, e^{2\pi q/p}w)$. Solid tori in S^3 are preserved by the action, so they remain as solid tori in $L(p,q)$ giving genus-1 Heegaard decomposition of $L(p,q)$. The detailed explanation of this and more examples will be given next section.

Next we prove the existence of Heegaard decomposition for closed, oriented 3-manifolds.

Theorem 2.1.1. *Any orientable, closed three manifold Y admits a Heegaard decomposition.*

Proof. Take a triangulation K of Y. By a *triangulation* we mean the presentation of a three manifold as a finite union of tetrahedra whose pairwise intersection is either a common face, or a common edge, or a common vertex, or a void. We assume that all three manifolds can be triangulated, see [4]. Let us describe *barycentric subdivision* K' of K. A median of a triangle divides it into 6 small triangles. Medians of each face of tetrahedron in the triangulation of Y divides the tetrahedron into 24 small tetrahedra. The tetrahedra are called the first barycentric subdivision K' of K. Take barycentric subdivision of K' to obtain second barycentric subdivision K'' of K'. Then define U_0 as a union of tetrahedra of second barycentric subdivision K'' of K having common points with the set of edges of K and U_1 as the closure of $Y - U_0$. More explicitly, take a tetrahedron Δ from K. The parts belonging to U_1 can be visualized as 4 solid cylinders attached to a solid sphere along the boundary such that on each face of Δ, the soles correspond to the following picture below.

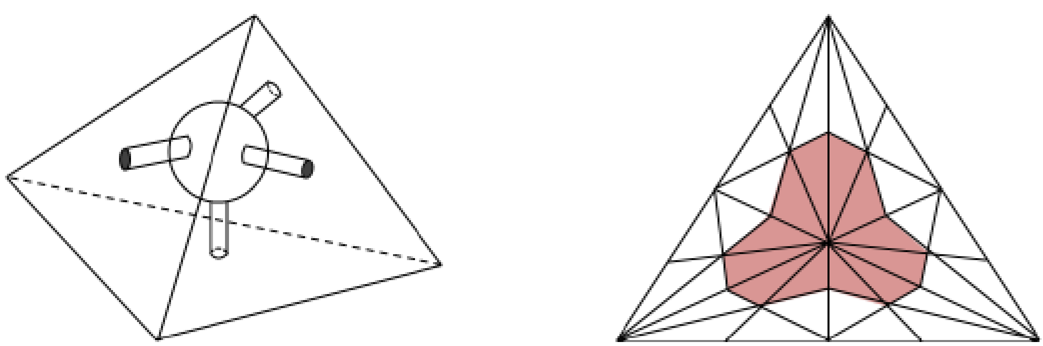


Figure 2.1: Illustration of a piece of handlebodies and soles [34]

Attach those pieces along their soles to form U_1. U_0 corresponds to joining solid spheres centered at vertices of K by solid cylinders along the edges of K. By the above description we obtain U_0 and U_1 by gluing solid spheres and cylinders. Note that a cylinder joins parts of the same handlebody in oriented or nonoriented way and if at some step we have a nonoriented gluing then three manifold Y will be nonoriented. If we continue gluing step by step we will get the handlebodies U_0 and U_1. Note that U_0 and U_1 are homeomorphic as they have the same boundary genus-g surface Σ_g and the same number of handles.

□

Remark 2.1.2. Depending on the chosen triangulation K of Y, the same three manifold admits different Heegaard decompositions.

Consider a Heegaard decomposition of Y into genus-g handlebodies. There is an immediate question in mind how to obtain Y from these two handlebodies, and crucial part is the gluing. Which three-manifold we obtain depends on how we attach the handlebodies.

Definition 1. Let U be a genus-g handlebody. A set of *attaching circles* $\{\gamma_1, .., \gamma_g\}$ for U is a collection of closed embedded curves in the Heegaard surface $\Sigma_g = \partial U$ such that

- γ_i are disjoint from each other
- $\Sigma_g - \gamma_1 - .. - \gamma_g$ is connected
- γ_i bound disjoint embedded disks in U

Proposition 2.1.3. *$\Sigma - \gamma_1 - .. - \gamma_g$ is connected if and only if $\{[\gamma_1], .., [\gamma_g]\}$ are linearly independent in $H_1(\Sigma, \mathbb{Z})$.*

Definition 2. Take a genus-g Heegaard decomposition of Y with handlebodies U_0 and U_1. A compatible Heegaard diagram for Y is given by $(\Sigma_g, \alpha_1, .., \alpha_g, \beta_1, ..\beta_g)$ such that $\{\alpha_1, .., \alpha_g\}$ is a set of attaching circles for U_0, and $\{\beta_1, .., \beta_g\}$ is a set of attaching circles for U_1.

Any diagram $(\Sigma_g, \alpha_1, .., \alpha_g, \beta_1, ..\beta_g)$ with α and β curves satisfying the above definition determines in a unique way a Heegaard decomposition and thus a three-manifold. We can obtain three-manifold back as follows. Consider $\Sigma_g \times I$, where I is closed unit interval. Attach solid cylinders to each attaching circles α_i and do the same for β_j, then we will have two spherical regions as boundary of the resulting object. Add 2 solid balls to those boundaries, one to the boundary resulted in attaching handles to α_i and one to the other component of the boundary resulted in attaching handles to β_j. Then we get a three-manifold. In Section 2.3 we describe this construction like building blocks via Morse Theory.

Heegaard diagrams are unique modulo some basic moves. Thus the same three-manifold admits many different Heegaard diagrams. There are three Heegaard moves which do not change the three manifold : isotopy, handleslide, and stabilization.

- **Isotopy:** We can move the curves in the set of attaching circles $\{\gamma_1, .., \gamma_g\}$ in a one-parameter family such that the curves remain disjoint.

- **Handleslide:** We can choose two of the attaching circles (wlog) γ_1 and γ_2 and replace γ_1 with γ_1', where γ_1' is a simple closed curve on Σ_g disjoint from the set $\{\gamma_1, .., \gamma_g\}$ such that γ_1, γ_2, and γ_1' bound an embedded pair of pants in $\Sigma_g - \gamma_3 - .. - \gamma_g$, i.e., $[\gamma_1]$, $[\gamma_2]$, and $[\gamma_1']$ are linearly dependent in $H_1(\Sigma_g, \mathbb{Z})$.

- **Stabilization:** First two moves do not change the genus of the Heegaard surface. Stabilization move increases genus by 1 by taking the connected sum of Σ_g with a genus-1 surface. We choose α_{g+1} and β_{g+1} on the genus-1 surface intersecting transversally at a single point, and they are disjoint from the other α and β-curves. Then $\{\alpha_1, .., \alpha_g, \alpha_{g+1}\}$ and $\{\beta_1, .., \beta_g, \beta_{g+1}\}$ are the new set of attaching circles on $\Sigma_g \# T^2$.

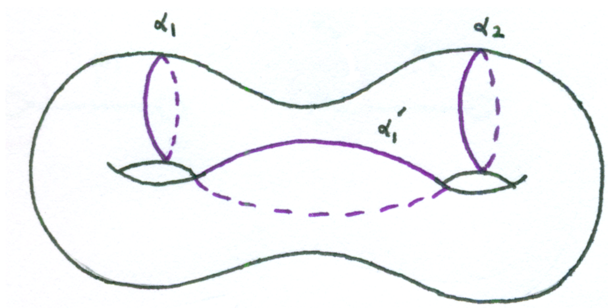


Figure 2.2: A handleslide

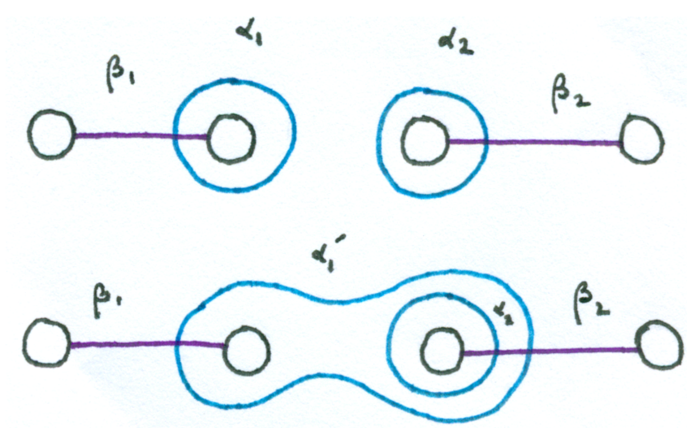


Figure 2.3: The same handleslide with different diagram style

In the following sections we will use pointed Heegaard diagrams where a basepoint $z \in \Sigma_g - \alpha - \beta$ is chosen on the Heegaard surface disjoint from α and β-curves. For pointed Heegaard diagrams $(\Sigma_g, \alpha, \beta, z)$ we can define pointed Heegaard moves as follows. We require that during the move z remains disjoint from α and β curves. For an isotopy z should be disjoint from the curves and for a handleslide z should not be in the pair of pants region.

2.2 Some Examples

Example.1: Let us study Heegaard splitting of S^3 into two solid tori. Consider $S^3 \subset \mathbb{C}^2$ and $S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ with handlebodies:

$$U_0 = \{(z, w) \in S^3 : |z| \leq |w|\}$$
$$U_1 = \{(z, w) \in S^3 : |z| \geq |w|\}$$

Observe that by definition of S^3, $|z| \leq |w|$ and $|z| \geq |w|$ are equivalent to $|z| \leq 1/2$ and $|z| \geq 1/2$ respectively. We will construct homeomorphisms to show that U_0 and U_1 are solid tori. For $x \in S^3 \subset \mathbb{C}^2$ it has coordinate representation $x = (ae^{i\alpha}, be^{i\beta})$ with $|x|^2 = a^2|e^{i\alpha}|^2 + b^2|e^{i\beta}|^2 = a^2 + b^2 = 1$. Note that U_0 is determined by $|z| \leq 1/2$, i.e., $a \leq 1/2$. Use coordinates (a, α, β) for the solid torus with $0 \leq a \leq 1/\sqrt{2}$ then the map from U_0 to solid torus sending $(ae^{i\alpha, be^{i\beta}}) \mapsto (a, \alpha, \beta)$ is a homeomorphism which is easily seen to be one-to-one and continuous. Similarly the map sending $(ae^{i\alpha, be^{i\beta}}) \mapsto (b, \beta, \alpha)$ is a homeomorphism from U_1 to solid torus. Note that the set

$$\Sigma_1 = \{(z, w) : |z| = |w| = 1/2\}$$

corresponds to the Heegaard surface. For this decomposition the attaching circles are

$$\alpha = \{1/\sqrt{2}(e^{i\alpha}, 1) : \alpha \in [0, 2\pi]\}$$
$$\beta = \{1/\sqrt{2}(1, e^{i\beta}) : \beta \in [0, 2\pi]\}$$

with unique intersection point, namely $\alpha \cap \beta = (1/\sqrt{2}, 1/\sqrt{2})$.

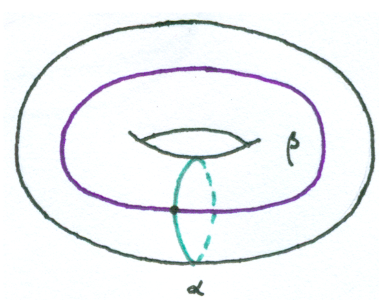


Figure 2.4: A Heegaard diagram for S^3

Example.2: [34] In this example we will decompose Lens space, denoted as $L(p, q)$, into two solid tori. Consider discrete group action on S^3 and let $(p, q) = 1$ with $p \geq 3$ and $S^3 \subset \mathbb{C}^2$ then there is a $\mathbb{Z}/p\mathbb{Z}$ action on S^3 as follows. Let $\mathbb{Z}/p\mathbb{Z} =< \sigma >$ and

$$\sigma(z, w) = (e^{\frac{2\pi i}{p}} z, e^{\frac{2\pi i q}{p}} w)$$

It has no fixed points thus this is a free action on S^3. Indeed, for $1 \leq k \leq p - 1$,

$$\sigma^k(z, w) = (e^{\frac{2\pi i k}{p}} z, e^{\frac{2\pi i q k}{p}} w) = (z, w)$$

we have $e^{\frac{2\pi i k}{p}} = 1$ and this implies $k/p \in \mathbb{Z}$ which is not true, so only identity element fixes points and the action is free. Therefore the quotient space $S^3/\sim$ is a 3-manifold [12] called Lens space $L(p, q)$. In order to understand genus-1 decomposition of $L(p, q)$, we need to understand what happens to two solid tori in the Heegaard decomposition of S^3 under the $\mathbb{Z}/p\mathbb{Z}$ action. Thus, let us consider the following cell decomposition of S^3:

- 0-cell: $(0, e^{\frac{2\pi i k}{p}})$, $0 \leq k \leq p - 1$
- 1-cell: $(0, e^{2\pi i \theta})$, $\frac{k}{p} < \theta < \frac{k+1}{p}$
- 2-cell: $(\rho e^{\frac{2\pi i k}{p}}, w)$, $0 < \rho \leq 1$, $|w| = \sqrt{(1 - \rho^2)}$
- 3-cell: $(\rho e^{\frac{2\pi i \theta}{p}}, w)$, $0 < \rho \leq 1$, $|w| = \sqrt{(1 - \rho^2)}, \frac{k}{p} < \theta < \frac{k+1}{p}$, $0 \leq k \leq p - 1$

There are p-cells in each dimension and under the $\mathbb{Z}/p\mathbb{Z}$ action same dimensional cells permute with each other. Indeed, for 0-cells, $\sigma(0, e^{\frac{2\pi i k}{p}}) = (0, e^{\frac{2\pi i (q+k)}{p}} w)$ but $(p, q) = 1$ and for $0 \leq k \leq p - 1$ we will obtain other 0-cells. Similarly for 1-cells, $\sigma(0, 2\pi i \theta) = (0, e^{2\pi i(\theta + \frac{q}{p})})$ for $\frac{k}{p} < \theta < \frac{k+1}{p}$ implies $\frac{k+q}{p} < \theta + \frac{q}{p} < \frac{k+q+1}{p}$ so we will get other 1-cells as k changes. Continuing in this manner we will see under the action, the same dimensional cells will permute each other. We identify each point $x \in S^3$ with $\sigma x, \sigma^2 x, .., \sigma^{p-1} x$ in the quotient, so every cell in the same dimension will be identified. It will induce a cell decomposition for $L(p, q)$ from S^3 containing one cell in each dimension.

$L(p, q)$ can also be obtained by taking one of the 3-cells and applying identifications on the boundary according to $\mathbb{Z}/p\mathbb{Z}$ action. Let us change coordinates from complex to real, since 3-cell$\in \mathbb{R}^3$ and use (x_1, x_2, x_3). The complex coordinated 3-cell is given as:

$$(\rho e^{\frac{2\pi i \theta}{p}}, w),\ 0 < \rho \leq 1,\ |w| = \sqrt{(1 - \rho^2)},\ \frac{k}{p} < \theta < \frac{k+1}{p},\ 0 \leq k \leq p - 1$$

then let $w = x_1 + i x_2$ and $x_3 = (2p\theta - 2k - 1)\rho$. Note that $|x_3| \leq \rho$ and ${x_1}^2 + {x_2}^2 + {x_3}^2 \leq 1$. Points on sphere if $x_3 > 0$ are assigned to points for which $\theta = \frac{k+1}{p}$ and if $x_3 < 0$ are assigned to points for which $\theta = \frac{k}{p}$. Also points with $\theta = \frac{k}{p}$ go to points with

$\theta = \frac{k+1}{p}$ under the action σ which corresponds to a rotation by $2\pi q/p$ angle about the origin. This says that 2-cells of upper hemisphere are identified with 2-cells of lower hemisphere producing $L(p, q)$.

Solid torus U_0 in S^3 corresponding to $|w|^2 \leq 1/2$ intersects with 3-cell, from which we obtain $L(p, q)$ under identifications coming from the $\mathbb{Z}/p\mathbb{Z}$ action, along a solid cylinder whose bases are spherical, and this can be seen geometrically. So under the $\mathbb{Z}/p\mathbb{Z}$ action identifying upper and lower bases results a solid torus. Other handlebody U_1 in S^3 corresponding to $|w|^2 \geq 1/2$ and intersects with 3-cell along a solid torus. Under boundary identifications this intersection also remains a solid torus. Therefore genus-1 handlebodies of S^3 remain as solid tori in $L(p, q)$ under the $\mathbb{Z}/p\mathbb{Z}$ action and this finishes the genus-1 Heegaard decomposition of $L(p, q)$.

It is complicated to represent a Heegaard diagram as a genus-g surface with α and β curves are drawn on it. To simplify, the following method is used. Consider the plane as S^2, the boundary of 0-handle. To obtain Heegaard surface draw g pairs of small disks on the plane, which correspond to attaching regions of 1-handles, then draw the α and β curves on plane. The convention is if a curve goes into one disk it comes out of the other disk of the pair and there is nothing nontrivial happens on the 1-handle. With this description we draw the following diagrams first in the described way then we attach 1-handles. Last we draw the genus-g surface with two sets of attaching circles.

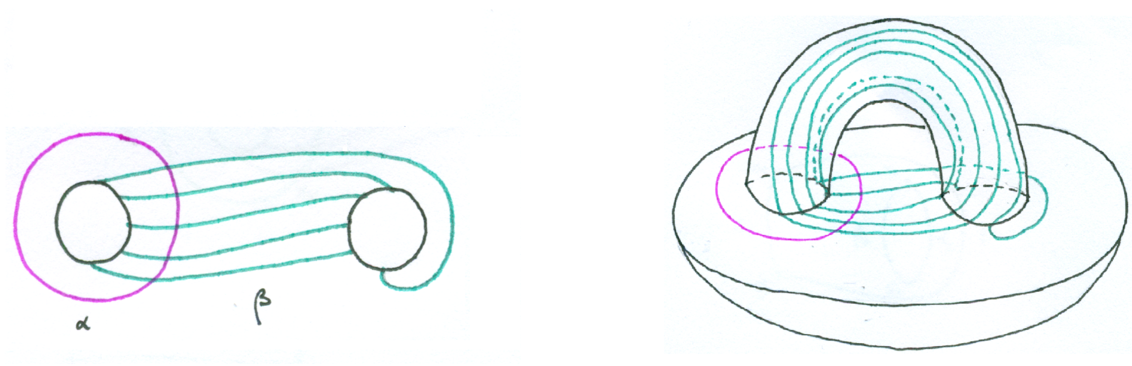


Figure 2.5: A Heegaard diagram for $L(5, 1)$

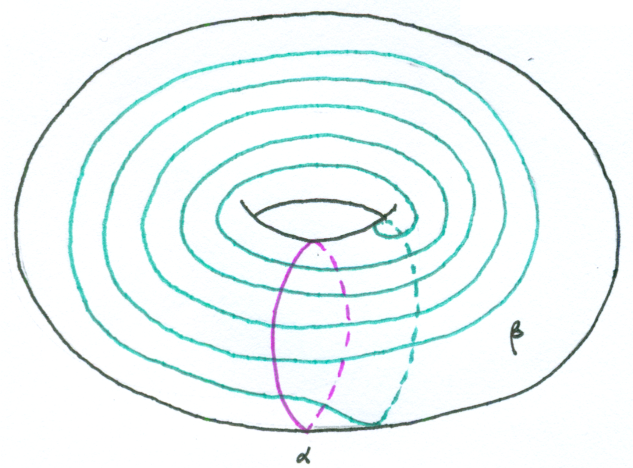


Figure 2.6: Same Heegaard diagram for $L(5,1)$ drawn on torus

Example.3: Let us describe a Heegaard decomposition of $S^1 \times S^2$. Consider a genus-1 surface with chosen pair of standard meridian and parallel. Let φ be a homeomorphism between two such tori such that φ send meridian to meridian and parallel to parallel. This map can be extended to solid tori and it gives a homeomorphism between two solid tori. Thus we attach two solid tori via this map. As meridian bounds a disk in the solid tori, when we attach two disks along their boundary circle we obtain a sphere. But meridian is homeomorphic to $\{p\} \times \partial D^2$ for any $p \in S^1$. Therefore, we get $S^1 \times S^2$ as a three-manifold.

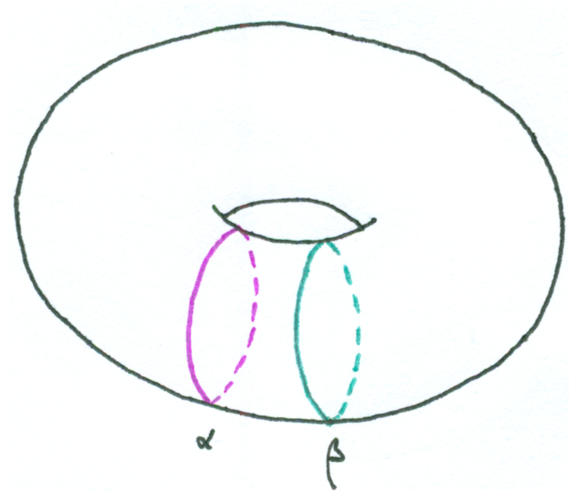


Figure 2.7: A Heegaard diagram for $S^1 \times S^2$

Different Heegaard diagrams for S^3:

Figure 2.8: A Heegaard diagram for S^3

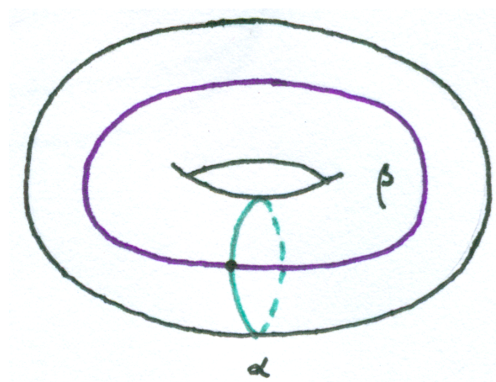


Figure 2.9: Same Heegaard diagram for S^3, drawn on torus

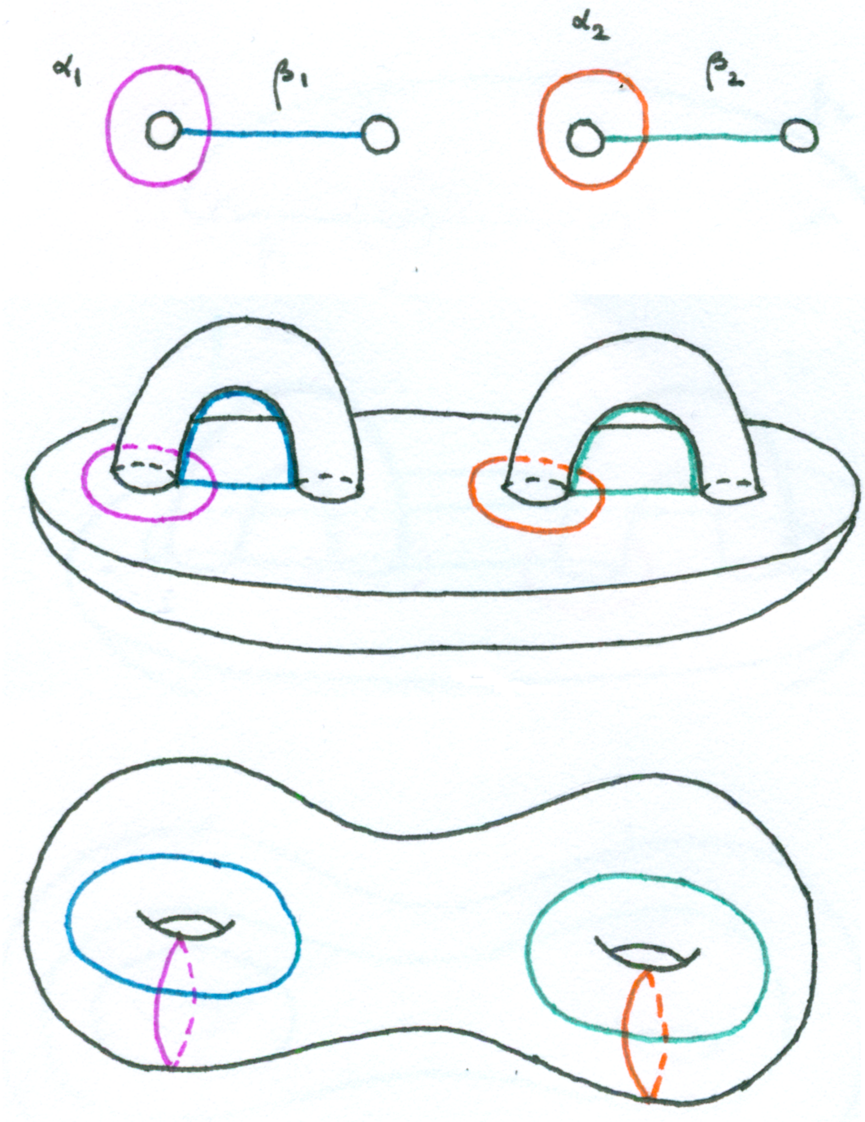


Figure 2.10: A Heegaard diagram for S^3 after stabilization move

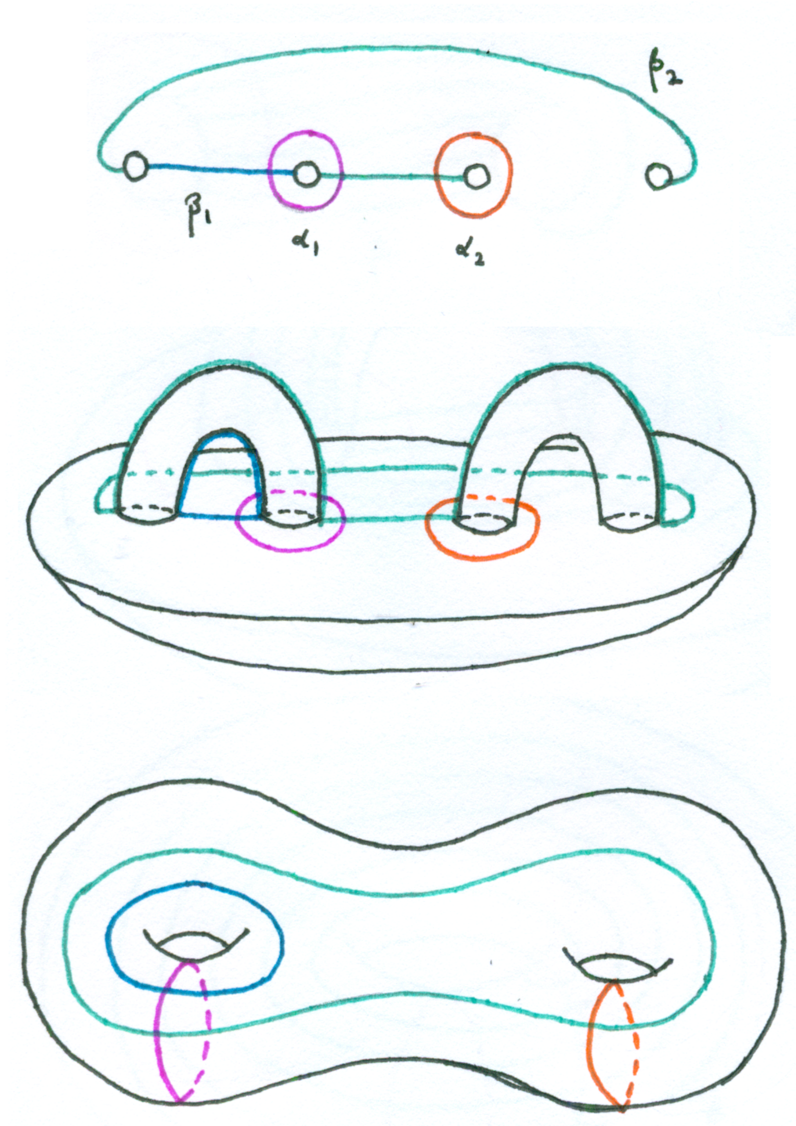


Figure 2.11: A Heegaard diagram for S^3 with stabilization and handle slide

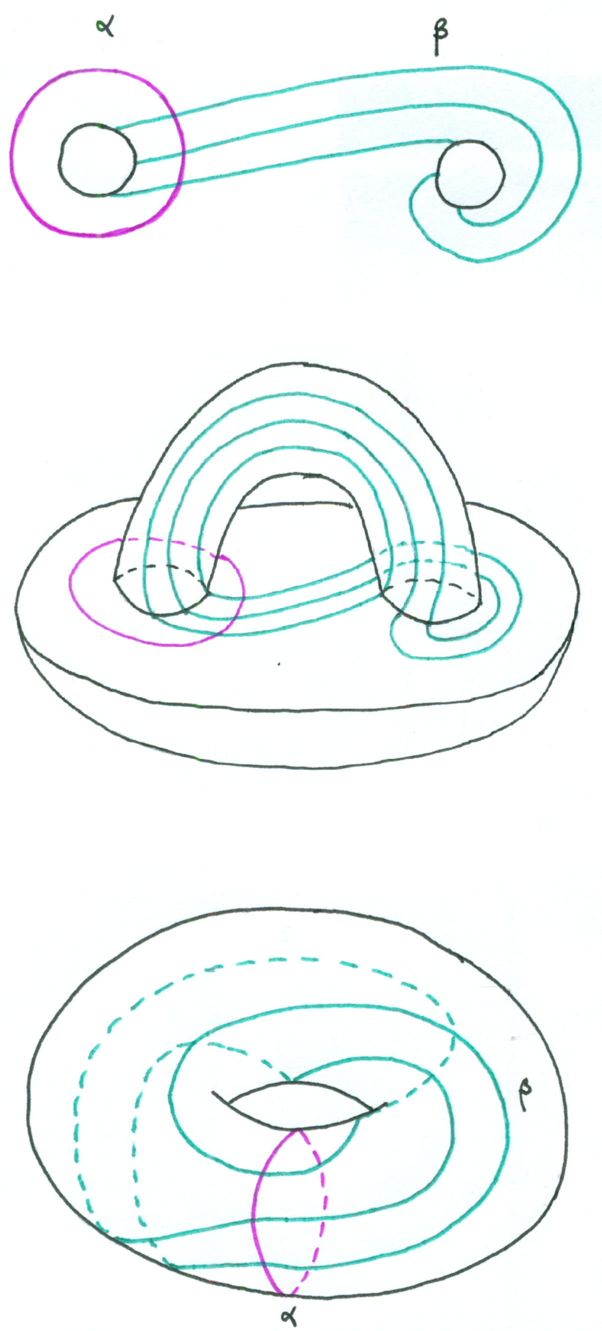


Figure 2.12: A Heegaard diagram for $L(3, 2)$

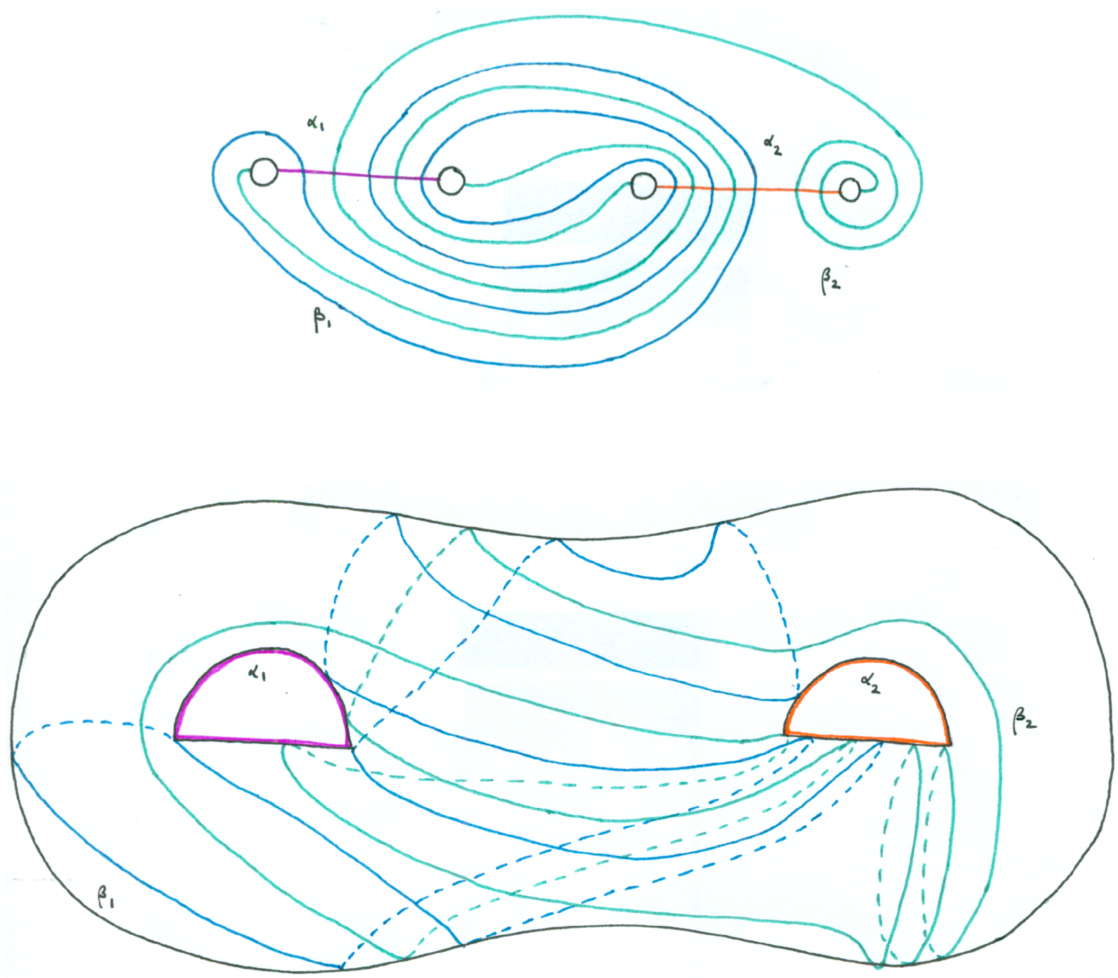


Figure 2.13: A genus-2 Heegaard diagram

2.3 Heegaard Decompositions Via Morse Theory

In this section we study Heegaard decompositions by using the techniques of Morse theory. We give some basics of Morse theory which we need and mostly we follow [22]. For a detailed description of the theory see [22], [24], and [14].

Morse theory studies the relation between functions on a space and the shape of the space by looking at critical points of a real-valued function and understand the shape of the space from the information about critical points.

Let M be an n-dimensional smooth manifold, i.e., at each point of M there is a smooth coordinate system $(x_1, .., x_n)$ and let $f : M \to \mathbb{R}$ be a smooth function, $p \in M$ is a *critical point* of f is for a chosen local coordinates $(x_1, .., x_n)$ for p we have

$\frac{\partial f}{\partial x_i}(p) = 0$ for all $1 \leq i \leq n$.

For a critical point p of f, consider the second partial derivatives and form the *Hessian* at p, $n \times n$ matrix $H(p)$ where $H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. Then a critical point is called *nondegenerate* if $detH(p) \neq 0$ i.e., $H(p)$ is nonsingular, and called *degenerate* if $detH(p) = 0$. Notice that $H(p)$ is a symmetric matrix as f is smooth implying that $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ for all i, j.

Definition 3. Let $f : M \to \mathbb{R}$ be a smooth function on an n-dimensional manifold M. If every critical point of f is nondegenerate then f is called a Morse function.

Example: Height function on S^2 is a Morse function $f : S^2 \to \mathbb{R}$, $f(x, y, z) = z$ with $x^2 + y^2 + z^2 = 1$. Here f has only 2 critical points $(0, 0, 1)$ and $(0, 0, -1)$ and both of them are nondegenerate.

Theorem 2.3.1. *Let p be a nondegenerate critical point of $f : M \to \mathbb{R}$. We can choose a local coordinate system $(x_1, .., x_n)$ centered at p such that the coordinate representation of f with respect to these coordinates has the following form*

$$f = -x_1^2 - x_2^2 - .. - x_\lambda^2 + x_{\lambda+1}^2 + .. + x_n^2 + c$$

where $f(p) = c$.

This theorem is called the *Morse Lemma* and the number of minus signs λ in the standard form is the number of negative diagonal entries of $H(p)$ after diagonalization and by Sylvester Law λ is independent of the diagonalization of H [22]. λ called the *index* of a critical point.

Corollary 2.3.2. *A Morse function defined on a compact manifold admits only finitely many critical points.*

The following theorem shows the existence of Morse functions.

Theorem 2.3.3. *Let M be a closed (compact without boundary) manifold and $g : M \to \mathbb{R}$ be a smooth function defined on M, there exists a Morse function $f : M \to \mathbb{R}$ arbitrarily close (C^2-close) to g.*

This says that Morse functions are dense in the set of smooth $\mathbb{R}$-valued functions on M and for detailed descriptions of above arguments, reader is referred to [22]and [24].

Definition 4. Let X be a vector field on M. A curve $c(t)$ is called an integral curve of X if

$$\frac{dc}{dt}(t) = X_{c(t)} \tag{2.1}$$

for every t where $c(t)$ is defined. Note that $\frac{dc}{dt}(t)$ is the velocity vector of the curve at time t and $X_{c(t)}$ at $c(t)$. So an integral curve of X is a flow line moving with X as its velocity vector.

If M is a compact manifold without boundary there is an integral curve $c_p(t)$ of X for $-\infty < t < \infty$ passing through p at $t = 0$, see [22]. By the equation (2.1) has a unique solution so that two distinct integral curves do not meet.

A Morse function on a manifold induces a handle decomposition, so that it is possible to rebuild the manifold by adding handles corresponding to each intersection points. This is basically done by studying how the level sets $f^{-1}((\infty, t)$ of a Morse function f changes when t passing thorough a critical point. Moreover, if M is connected, a Morse function on M can be chosen so that it has only one index-0 critical point and one index-n critical point, where n is the dimension of M.

Let us consider a closed, oriented three-manifold M and a Morse function f on it such that f has only one minimum and one maximum. The handle decomposition corresponding to the Morse function f consist of a one 0-handle and one 3-handle for maximum values, and same number of 1-handles and 2- handles, say g many. A three dimensional i-handle is $D^i \times D^{3-i}$, thus a 0-handle is a solid ball and we attach 1-handles along the attaching spheres to the belt sphere of the 0-handle. The handlebody consisting of only 0-handle and g many 1-handles is a genus-g handlebody. The 2-handles and a 3- handle are dual to 0-handle and 1-handles, as they correspond to 0-handle and 1- handles of the Morse function $-f$. Therefore, 2-handles and a 3-handle also represent a genus-g handlebody. This decomposition is a Heegaard splitting of the three-manifold M and as M is closed so there is no boundary component implies that after attaching 2-handles and 3-handles to the handlebody there is no remaining boundary. This briefly shows why the number of 1-handles and 2-handles should be same.

Take a Morse function f on closed, oriented three-manifold M such that it is self-indexing, which means $f(p) = ind(p)$ for every critical point. This Morse function induces a Heegaard decomposition via a handle decomposition. Consider the gradient vector field of this Morse function on M. Then for every point on the Heegaard surface Σ_g, consider the flow lines passing through these points. Let α_i denote the set of flow lines to index-1 critical points and let β_i denote the set of flow-lines to index-2 critical points. The integral curves flowing down to index-1 critical points and flowing up to index-2 critical points. As f is self-indexing, the curves α_i and β_i are closed curves on the Heegaard surface and they correspond to the attaching circles of the handlebodies in the Heegaard splitting obtained from the handle decomposition. We say that this Heegaard diagram is compatible with the Morse function f on M. The detailed description of the theory can be found in the references stated at the beginning of this section.

2.4 Relation Between Two Heegaard Diagrams for a Three-Manifold

In this section we study the relationship between two different Heegaard diagrams representing the same three-manifold. For more detail see [8], [29], and [37].

Definition 5. Two Heegaard diagrams (Σ, α, β) and $(\Sigma^{'}, \alpha^{'}, \beta^{'})$ are called diffeomorphic if there is a diffeomorphism $\phi : \Sigma \to \Sigma^{'}$ preserving orientation such that ϕ takes α to $\alpha^{'}$ and β to $\beta^{'}$.

We extend the set of attaching circles for the handlebody U. The stabilization move is equivalent to introducing canceling pair of index-1 and index-2 critical points, since in the handle decomposition it corresponds 2-handle intersecting transversally at a single point with the belt sphere of the 1-handle, [22]. Introducing canceling pair of index-0 and index-1 critical points is to add 1-handle to 0-handle such that one of the attaching spheres of 1-handle, as there are two points $\partial D^1 \times \{0\}$, intersects transversally at a single point with the belt sphere, $\{0\} \times \partial S^3$, of the 0-handle. Canceling pair index-3 and index-2 critical points is dual to canceling pair of index-0 and index-1 critical points. Moreover, canceling pair of index-1 and index-2 critical points increases the genus of the Heegaard surface by 1, therefore introducing canceling pair of index-0 and index-1 critical points does no change the genus of the Heegaard surface, but increases the number attaching circles for the handlebody U by 1. Canceling pair of critical points corresponds to deleting one of the attaching circles. For example in the below example the curve α_2 bounds a surface so it is null-homologous in $H_1(\Sigma)$.

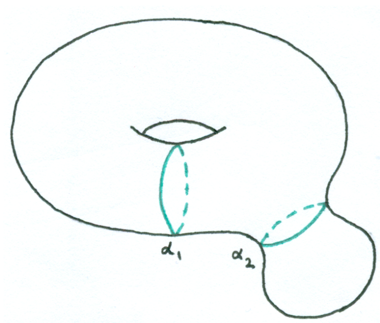


Figure 2.14: An illustration for canceling pair of index-1 and index-0 critical points

Remark 2.4.1 ([12])**.** Relative homology groups $H_n(X, A)$ for any pair (X, A) fit into a long exact sequence

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

Definition 6. A set $\{\alpha_1, .., \alpha_d\}$ containing pairwise disjoint embedded circles in Σ which bound embedded disks in U and span the image of the boundary homomorphism $\partial : H_2(U, \Sigma) \to H_1(\Sigma)$, is called an extended set of attaching circles for the handlebody U.

Lemma 2.4.2. *Let γ be a simple closed curve disjoint from a set of $\{\alpha_1, .., \alpha_g\}$ attaching circles in Σ for handlebody U. Then γ is either null-homologous or for some α_i, orγ is isotopic to a curve obtained by handlesliding α_i over some collection of α_j for $i \neq j$.*

Proof. Note that $\{\alpha_1, .., \alpha_g\}$ is linearly independent in $H_1(\Sigma;\mathbb{Z})$ and let $\{[\alpha_1], .., [\alpha_g], [b_1], .., [b_g]\}$ be a basis of $H_1(\Sigma)$ such that b_i's are simple closed curves on Σ around each genus. The curve γ homologically cannot contain b_i's in the linear combination of the basis because γ does not intersect with α curves. Let us surger out $\{\alpha_1, .., \alpha_g\}$ from Σ then we have S^2 with closed curve γ on it and g many pair of marked points (p_i, q_i) to remember the places of α_i's on Σ, attaching spheres of the 1-handles to be attached.

We claim that γ does not separate any (p_i, q_i) if and only if γ separates Σ. First assume that γ separates Σ which means it bounds a surface and γ is null-homologous in $H_1(\Sigma;\mathbb{Z})$. If $\alpha_1^{'}$ is obtained from handlesliding α_1 over α_2 then as they bound a pair of pants on Σ, homologically the sum of them is trivial. Thus $\alpha_1^{'}$ is homologically linear combination of α_1 and α_2. If γ separates p_i and q_i for some i the coefficient a_i of $[\gamma] = \sum a_i[\alpha_i]$ is nonzero. It contradicts with γ being null-homologous. Conversely, assume that γ does not separate Σ then $[\gamma] = \sum a_i[\alpha_i]$ is not zero, so at least one of $a_i \neq 0$ then γ separates (p_i, q_i), contradiction.

With this argument if γ does not separate any (p_i, q_i) then γ is null-homologous. If it does then the homology class of γ is linear combination of $[\alpha_i]$'s, thus γ is isotopic to a handleslide of α_i over some α_j's with $i \neq j$. □

Lemma 2.4.3. *Let $\{\alpha_1, .., \alpha_d\}$ be an extended set of attaching circles for handlebody U. Take two subsets containing g-many curves such that each subset forms a set of attaching circles for U. Then two sets can be related by isotopy and handleslide.*

Proof. We prove the statement by induction on the number of genus g. If $g = 1$, on torus as there is only one attaching circle for handlebody, thus any two different attaching curves are homologically same and they are isotopic. Let us assume that the statement is true for the genus $g - 1$ and prove for genus g. Let $\{\alpha_1, .., \alpha_d\}$ be extended set of attaching circles for U. Take any two subsets containing g-many circles as $\{\alpha_1, .., \alpha_g\}$ and $\{\alpha_1^{'}, .., \alpha_g^{'}\}$. Either two of these sets are disjoint or they contain common elements. Let us assume that, wlog, α_1 is a common element for both sets. If we surger out α_1 curve the the genus of the Heegaard surface is reduced by 1 with $(g-1)$ many attaching circles for handlebody with 2 marked points. Thus any isotopy on the Heegaard surface separating the two marked points is isotopic to handlesliding over α_1. After surgering out α_1 we have two sets of $g - 1$ many circles and by hypothesis they are related by isotopy and handleslide by the Lemma (2.4.2). If two sets $\{\alpha_1, .., \alpha_g\}$ and $\{\alpha_1^{'}, .., \alpha_g^{'}\}$ are disjoint then as $\alpha_1^{'}$ is not null-homologous by Lemma (2.4.2) it is isotopic to a handleslide of some α_i over some α_j's with $i \neq j$. Then we are in the first case concuding two sets can be related by isotopy and handslide. □

Now we are ready to state the most important theorem of this chapter.

Theorem 2.4.4. *If (Σ, α, β) and (Σ, α, β) are two Heegaard diagrams representing* *the same three-manifold Y then two diagrams are diffeomorphic after a finite sequence of Heegaard moves.*

Proof. Consider compatible Morse functions f and f' for these two Heegaard diagrams. Connect these Morse functions through generic family of functions f_t. Except for finitely many t, we get the induced Heegaard diagrams for Y such that the extended sets of attaching circles can be related via isotopy and handleslide. For the finitely many t, there is a stabilization move corresponding to the canceling pair of index-1 and index-2 critical points.

For a handlebody U, extend the two sets of attaching circles to $\{\alpha_1, .., \alpha_d\}$ and $\{\alpha_1^{'}, .., \alpha_d^{'}\}$ such that they are related by isotopy and handleslide. (Wlog) let $\alpha_1^{'}$ obtained from α_1 handlesliding over some α_j's. Consider the extended set $\{\alpha_1, .., \alpha_d, \alpha_1^{'}\}$, then any two subsets of g-many curves can be related by isotopy and handleslide by the Lemma(2.4.3). Continue this argument to conclude $\{\alpha_1, .., \alpha_g\}$ and $\{\alpha_1^{'}, .., \alpha_g^{'}\}$ are related by isotopy and handleslide. Same argument is true for β and $\beta^{'}$ curves proving the theorem. □

There is a special argument about stabilization as follows, its proof can be found in [37].

Theorem 2.4.5. *Let (Σ, α, β) and $(\Sigma^{'}, \alpha^{'}, \beta^{'})$ be two Heegaard diagrams, with genus g and g' Heegaard surfaces respectively, representing the same three-manifold Y. Then for some k large enough $(k - g')$-fold stabilization of the first decomposition is diffeomorphic to $(k - g)$-fold stabilization of the second decomposition.*

As an illustration think about genus-0 and genus-1 Heegaard decompositon of S^3.

Chapter 3

MORE TOPOLOGICAL BACKGROUND AND NECESSARY TOOLS

In this chapter we study some necessary topological tools to define Heegaard Floer homology. First we study symmetric product space and Whitney disks in this space then we define domains and intersection numbers, and lastly we discuss $Spin^c$ structures on three-manifolds.

3.1 Symmetric Product Space

In order to define Heegaard Floer homology, we study a configuration space rather than a Heegaard surface. For a three-manifold Y, let $(\Sigma_g, \alpha_1, .., \alpha_g, \beta_1, .., \beta_g, z)$ be a pointed Heegaard diagram. The following configuration space

$$Sym^g(\Sigma_g) = \Sigma_g \times .. \times \Sigma_g / S_g$$

where S_g is symmetric group with g elements is called *Symmetric product space*, denoted by $Sym^g(\Sigma_g)$. It is the set of unordered g-tuple of points in Σ_g where same points can appear more than once.

Proposition 3.1.1. *$Sym^g(\Sigma_g)$ is a smooth manifold of real dimension $2g$.*

Proof. Note that it is not a free action since there are elements other than identity fixing elements of $\Sigma_g \times .. \times \Sigma_g$. Any element of Σ_g which is of the form $\{x_1, .., x_g\}$ with $x_i \in \Sigma_g$ can be thought of as the roots of a monic polynomial $p(x) = (x - x_1)..(x - x_g)$ of degree g over $\mathbb{C}$. An open neighborhood of this point in $Sym^g(\Sigma_g)$ is homeomorphic to an open neighborhood of $p(x)$. As monic polynomials over $\mathbb{C}$ are homeomorphic locally to $\mathbb{C}^g$, $Sym^g(\Sigma_g)$ is a $2g$ dimensional smooth manifold. □

Definition 7. Let $(\Sigma_g, \alpha, \beta)$ be a Heegaard diagram. The set of attaching circles induce smoothly embedded g dimensional tori in $Sym^g(\Sigma_g)$:

$$\mathbb{T}_\alpha = \alpha_1 \times .. \times \alpha_g$$
$$\mathbb{T}_\beta = \beta_1 \times .. \times \beta_g$$

Note that the set of attaching circles $\{\alpha_1, .., \alpha_g\}$ are disjoint and different elements of $\alpha_1 \times .. \times \alpha_g$ are not in the same orbit. Therefore $\mathbb{T}_\alpha = \alpha_1 \times .. \times \alpha_g$ is homeomorphic to $S^1 \times .. \times S^1$, the g dimensional torus.

Definition 8. A complex structure on a vector space V is an automorphism $J : V \to V$ such that $J^2 = -\mathbb{1}$, and with such a structure V becomes a complex vector space. An almost complex structure on a real manifold is that its tangent bundle is equipped with a complex structure.

Definition 9. Let M be a complex manifold equipped with a complex structure J and let $S \subset M$. S is called totally real submanifold if $T_pS \cap JT_pS = (0)$ for every $p \in S$, i.e., any of its tangent spaces does not contain a J complex line.

Proposition 3.1.2. *The tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ as defined above are totally real submanifolds of $Sym^g(\Sigma_g)$.*

Proof. Consider the projection map $\pi : \Sigma_g \times .. \times \Sigma_g \to Sym^g(\Sigma_g)$. This is a holomorphic local diffeomorphism away from the diagonal $D \subset Sym^g(\Sigma_g)$, which consists of g-tuples of points in Σ_g such that at least two entries coincide. Note that $\mathbb{T}_\alpha \cap D = \emptyset$. Thus it suffices to prove that $\alpha_1 \times .. \times \alpha_g$ is totally real submanifold of $\Sigma_g \times .. \times \Sigma_g$. Any complex structure j on Σ_g induces a complex structure on $Sym^g(\Sigma_g)$ and a product complex structure J on $\Sigma_g \times .. \times \Sigma_g$. Take a vector v from a tangent space of $\Sigma_g \times .. \times \Sigma_g$ then $J(v) = (v_1, .., v_g) = (j(v_1), .., j(v_g))$. Now, take a vector w from a tangent space of $\mathbb{T}_\alpha$, $w = (w_1, .., w_g)$ where w_i is tangent to α_i but $j(w_i)$, if not zero, is not tangent to α_i, since tangent space of α_i has dimension 1 and $j^2 = -\mathbb{1}$ implies that it can not have a real eigenvector. So at least one of $w_i \neq 0$ and $J(w)$ is not in the tangent space of $\mathbb{T}_\alpha$ as none of its components is tangent to α_i. Therefore, $\alpha_1 \times .. \times \alpha_g \subset \Sigma_g \times .. \times \Sigma_g$ is a totally real submanifold. □

Definition 10. Let $(\Sigma_g, \alpha, \beta, z)$ be a pointed Heegaard diagram with the chosen basepoint z we define the following subspace

$$V_z = \{z\} \times Sym^{g-1}(\Sigma_g)$$

Note that z is disjoint from α and β curves, so it follows that $V_z \cap \mathbb{T}_\alpha = \emptyset$ and $V_z \cap \mathbb{T}_\beta = \emptyset$. Intersection with V_z will be more important as we proceed.

Theorem 3.1.3. *Let Σ be a genus g surface then*

$$\pi_1(Sym^g(\Sigma)) \simeq H_1(Sym^g(\Sigma)) \simeq H_1(\Sigma)$$

Proof. First let us prove the isomorphism between $H_1(Sym^g(\Sigma))$ and $H_1(\Sigma)$. The inclusion map $\Sigma \hookrightarrow Sym^g(\Sigma)$ induces an injective map on the level of the first homology

$$H_1(\Sigma) \to H_1(Sym^g(\Sigma))$$

sending basis elements $[\gamma]$ of $H_1(\Sigma)$ to $\{[\gamma], x, .., x\}$. Let us define the inverse map $H_1(Sym^g(\Sigma)) \to H_1(\Sigma)$. Diagonal D defined as before is codimension 1 in $Sym^g(\Sigma)$, because it is homeomorphic to the set of solutions of monic polynomials of degree g of codimension 1 in $\mathbb{C}^g$. Therefore, a generic circle intersects with the diagonal in finitely many points. Take a generic curve in $Sym^g(\Sigma)$, if it misses the diagonal then we have a continuous map $\gamma : S^1 \to \Sigma \times .. \times \Sigma$ where the image is a disjoint union of g circles corresponding to a g-fold cover of S^1 to Σ. If the curve intersects with the diagonal then $\gamma^{-1}(D) \neq \emptyset$ and $S^1 - \gamma^{-1}(D)$ is union of arcs then we have a continuous map $\gamma : S^1 - \gamma^{-1}(D) \to \Sigma \times .. \times \Sigma$. For every point in $\gamma^{-1}(D)$ we combine the corresponding coordinate functions and connect them; in other words we take a branch cover over a point. So we construct a continuous map where the circle are arranged to form a single circle. This is not necessarily a g-fold cover of S^1. But as long as we obtain a map to a disjoint union of circles then we have a homology class.

The map defined here is well-defined, which means homologous curves γ and γ' have homologous images. If γ and γ' are homologous they bound an oriented surface Z in $Sym^g(\Sigma)$. If Z does not intersect with the diagonal, the map $Z \to \Sigma \times .. \times \Sigma$ maps boundary to boundary which are the images of γ and γ', thus the image is homologous. Generically Z intersects with D in one dimensional subspaces like arcs and curves . By a similar argument above it will give a branched cover $\widetilde{Z}$ mapping to Σ. Its boundary curves are images of γ and γ'. Therefore the map is well-defined and these two maps are inverses of each other.

To prove the second isomorphism $\pi_1(Sym^g(\Sigma)) \simeq H_1(Sym^g(\Sigma))$, it suffices to show that $\pi_1(Sym^g(\Sigma))$ is Abelian. Indeed, the first homology group is the abelianization of the fundamental group. If it is Abelian already the result follows. In order to see this take a null-homologous curve $\gamma : S^1 \to Sym^g(\Sigma)$ such that it does not intersect with the diagonal. By the above argument, we can obtain a map $\widetilde{\gamma}$ of g-fold cover of S^1 to Σ which is also null-homologous, so it bounds a surface F in Σ. Let $\imath : F \hookrightarrow \Sigma$ be injection such that $\imath|_{\partial F} = \widetilde{\gamma}$. Extend this g-fold covering of circle to disk $\pi : F \to D$. For any $z \in D$, $z \mapsto \imath \circ \pi^{-1}(z)$ provides the nullhomotopy of γ. Therefore, $\pi_1(Sym^g(\Sigma))$ is Abelian. □

Proposition 3.1.4. *Let Y be a closed, oriented three-manifold with a Heegaard diagram (Σ, α, β). Then we have the following:*

$$\frac{H_1(Sym^g(\Sigma))}{H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)} \simeq \frac{H_1(\Sigma)}{[\alpha_1], .., [\alpha_g], [\beta_1], .., [\beta_g]} \simeq H_1(Y; \mathbb{Z}) \tag{3.1}$$

Proof. The isomorphism on the right can be obtained using Mayer-Vietoris sequence. By definition the attaching circles $\{[\alpha_1], .., [\alpha_g]\}$ and $\{[\beta_1], .., [\beta_g]\}$ are linearly independent in $H_1(\Sigma)$. The two set can also be linearly independent and in that case they generate $H_1(\Sigma)$ together. However they are not necessarily linearly independent and the map $H_1(\Sigma) \to H_1(Y, \mathbb{Z})$ has kernel $\{[\alpha_1], .., [\alpha_g], [\beta_1], .., [\beta_g]\}$. The isomorphism on the right can be obtained by the previous theorem where we showed the isomorphism between $H_1(Sym^g(\Sigma))$ and $H_1(\Sigma)$. Note that $\mathbb{T}_\alpha$ is smoothly embedded in $Sym^g(\Sigma)$ and $H_1(\mathbb{T}_\alpha)$ is generated by $\{[\alpha_i]\}$ with $1 \leq i \leq g$, and similar is true for $\mathbb{T}_\beta$. Therefore isomorphism follows. □

Let us continue with the topological properties of the symmetric product space, $Sym^g(\Sigma)$, and let us understand the holomorphic spheres in $Sym^g(\Sigma)$. Thus we study the second homotopy group. The fundamental group of a path-connected space X acts on higher homotopy groups of the same space. To see this take a path $\gamma : I \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$ then to each map $f : (I^n, \partial I^n) \to (X, x_1)$ associate a new map $\gamma f : (I^n, \partial I^n) \to (X, x_0)$ by shrinking the domain of f to a smaller concentric cube in I^n. This description formulates the action of π_1 on π_n as

$$\pi_1(X, x_0) \times \pi_n(X) \to \pi_n(X, x_0)$$

sending

$$([\gamma], [f]) \to [\gamma f]$$

The action is trivial if $[\gamma f] \simeq [f]$ for every $[\gamma] \in \pi_1(X)$. Let π'_n denote the quotient of π_n under the action of π_1. For more detailed description of this see [12].

Theorem 3.1.5. *Let Σ be a Riemann surface of genus $g \geq 2$ then*

$$\pi'_2(Sym^g(\Sigma)) \simeq \mathbb{Z}$$

If $g \geq 3$ action of $\pi_1(Sym^g(\Sigma))$ on $\pi_2(Sym^g(\Sigma))$ is trivial, in this case

$$\pi_2(Sym^g(\Sigma)) \simeq \mathbb{Z}$$

Proof. Let x be a generic point of Σ, by generecity we mean $x \in \Sigma - \alpha - \beta$. Let $V_x = \{x\} \times Sym^{g-1}(\Sigma)$ be subvariety and define the map

$$\phi : \pi'_2(Sym^g(\Sigma)) \to \mathbb{Z}$$

$$\phi(u) = \#(u \cap V_x)$$

ϕ counts the number of coordinates of u which are x. Take an orientation preserving hyper-elliptic involution $\tau : \Sigma \to \Sigma$ such that $\Sigma/\tau \simeq S^2$ call it $S_0 \subset Sym^g(\Sigma)$. With S_0 we can obtain sphere $S = S_0 \times x_3 \times .. \times x_g \subset Sym^g(\Sigma)$ as the set of elements $S = \{(y, \tau(y)), z, .., z\} \subset Sym^2(\Sigma)$. Note that $\#(S \cap V_x) = 1$. Indeed the first coordinate of S scans the points of Σ and it becomes x once so S and V_x intersects once in one coordinate. Therefore, we can take S as positive generator of $\pi'_2(Sym^g(\Sigma))$.

First let us show that this map is well-defined. We need to show that it is homotopy invariant. Let u and u' be different elements in the homotopy class of u. Let u_t be a homotopy from u to u'. By continuity of the homotopy, the map $\phi(u_t) = \#(u_t \cap V_x)$ changes continuously as $0 \leq t \leq 1$. But the intersection number is integer so this number must change continuously and it forces $\#(u \cap V_x) = \#(u' \cap V_x)$.

The map ϕ is a homomorphism. If we splice two spheres $u_1, u_2 \in \pi'_2(Sym^g(\Sigma))$, then intersection number is additive for $u_1 * u_2$ and

$$\#(u_1 * u_2 \cap V_x) = \#(u_1 \cap V_x) + \#(u_2 \cap V_x)$$

As we count the number of x appearing in coordinates attaching two spheres each other in one point result in adding the number of intersection points of each sphere. Intersection number will be discussed in more detail in next section.

The map is clearly onto as there exists an element assigned to a generator of $\mathbb{Z}$. Thus we can get all other numbers by splicing and changing orientations.

Now let us show that the map is injective. We will show by proving the kernel is trivial. Let $Z \in Ker(\phi)$ so $\phi(Z) = 0$ which means either Z does not intersect with V_x or its algebraic intersection adds up to 0. Generically Z meets with V_x in finitely many points as Z has real dimension 2 and V_x is codimension 2. Splice homotopy translates of S with appropriate signs, at intersection points then we obtain Z' a sphere whose geometric intersection with V_x is empty and homotopic to Z. As its algebraic number adds up to 0, the number of positive and negative intersections is same and by splicing even number of S we do not change the algebraic intersection number. So Z' is a sphere in $Sym^g(\Sigma - x)$. After the π_1 action on π_2 which provides freedom to change basepoint, the splicing operation is not related to the basepoint, thus this operation takes place in $\pi_2'(Sym^g(\Sigma))$. We claim that Z' is trivial in $Sym^g(\Sigma - x)$ for $g > 2$. Note that $\Sigma - x$ is homotopy equivalent to wedge sum of $2g$ circles. Consider cell structure of Σ. Taking out a point is homotopy equivalent to taking out a disk, therefore without a 2-cell the remaining is a 1-skeleton, which is a bouquet of $2g$-circles. $\Sigma - z$ is also homotopy equivalent to $\mathbb{C} - \{z_1, .., z_g\}$. $Sym^g(\mathbb{C} - \{z_1, .., z_g\})$ is the space of monic polynomials p of degree g such that $p(z_i) \neq 0$ with $1 \leq i \leq 2g$. Coefficients of p are in $\mathbb{C}^g$ minus $2g$ generic hyperplanes. By a theorem of Hattori [29] which says homology groups of the universal covering space of this complement are trivial except in dimension 0 or g. This shows that $\pi_2(Sym^g(\Sigma - x))$ is trivial. It shows that for any element in $Ker(\phi)$ we can find a sphere Z' homotopic to Z living in $Sym^g(\Sigma - x)$ and homotopically trivial. So ϕ is injective.

This proves that $\pi_2'(Sym^g(\Sigma)) \simeq \mathbb{Z}$ for $g > 2$. For $g = 2$, $Sym^g(\Sigma)$ is diffeomorphic to blowup of T^4, see [29], [8].

What remains is to show that for $g \geq 3$, the action of $\pi_1(Sym^g(\Sigma))$ is trivial, so that the map $\pi_1(Sym^g(\Sigma)) \times \pi_2(Sym^g(\Sigma)) \to \pi_2(Sym^g(\Sigma))$ sending $([\gamma], [\sigma])$ to $[\gamma\sigma]$, where $\gamma : S^1 \to Sym^g(\Sigma)$ and $\sigma : S^2 \to Sym^g(\Sigma)$ gives $[\gamma\sigma] \simeq [\sigma]$. By the previous theorem $\pi_1(Sym^g(\Sigma)) \simeq H_1(\Sigma)$. Thus $\gamma = \gamma' \times \{x, .., x\}$ for some $\gamma' : S^1 \to \Sigma$ and $\{x, .., x\} \in Sym^{g-1}(\Sigma)$ and replace σ with $\{x\} \times \sigma'$ where $\sigma' : S^2 \to Sym^{g-1}(\Sigma)$. The map $\gamma\sigma : S^1 \vee S^2 \to Sym^g(\Sigma)$ takes p to $\{x, .., x\}$. Denote $\gamma\sigma$ by $\gamma \vee \sigma$ on $S^1 \vee S^2$ and this can be extended to $\gamma' \times \sigma' : S^1 \times S^2 \to Sym^g(\Sigma)$ taking p to $\{x, .., x\}$. Action of $\pi_1(S^1 \times S^2)$ on $\pi_2(S^1 \times S^2)$ is trivial, therefore the map $\gamma \vee \sigma : S^1 \vee S^2 \to S^1 \times S^2$ sending (t, w) to (p, w) is homotopic to $\{p\} \times i$ and it maps to $Sym^g(\Sigma)$ after composing with $\gamma' \times \sigma'$, this gives

$$\gamma' \times \sigma'(\{p\} \times i) = \gamma'(\{p\}) \times \sigma'(i) = \{x\} \times \sigma' \simeq \sigma$$

Thus the map $\gamma \vee \sigma : S^1 \vee S^2 \to Sym^g(\Sigma)$ is homotopic to σ as desired. □

Remark 3.1.6. When $g = 1$ the map $\pi_2(x, y) \to \mathbb{Z} \oplus H^1(Y; \mathbb{Z})$ is injective.

3.2 Disks in Symmetric Products

Let $\mathbb{D} \subset \mathbb{C}$ be unit disk such that

$$e_1 = \{z \in \mathbb{D} | Re(z) \geq 0\}$$
$$e_2 = \{z \in \mathbb{D} | Re(z) \leq 0\}$$

an intersection point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is unordered g-tuple $\{x_1, .., x_g\}$ of intersection points of α_i's and β_j's such that they intersect exactly at one point x_k.

Definition 11. Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. A continuous map $u : \mathbb{D} \to Sym^g(\Sigma_g)$ with $u(-i) = x$ and $u(i) = y$ with $u(e_1) \subset \mathbb{T}_\alpha$ and $u(e_2) \subset \mathbb{T}_\beta$ is called a Whitney disk connecting x and y in $Sym^g(\Sigma)$.

Definition 12. $\pi_2(x, y)$ is the set of homotopy classes of Whitney disks connecting x and y.

Remark 3.2.1. Let $x, y, z \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Suppose we have a Whitney disk from x to y and a Whitney disk from y to z. We can glue them to get a disk from x to z. Moreover, there is a splicing action on $\pi_2(x, y)$. We can splice spheres from $\pi_2'(Sym^g(\Sigma))$ to a disk in $\pi_2(x, y)$ so that we attach a sphere to a disk in one point and as a result, it is still a Whitney disk from x to y satisfying the definition. We take spheres from $\pi_2'(Sym^g(\Sigma))$, therefore the basepoint does not matter.

Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Take two paths from x to y $a : [0, 1] \to \mathbb{T}_\alpha$ and $b : [0, 1] \to \mathbb{T}_\beta$ then $a - b$ is a loop in $Sym^g(\Sigma)$. By the isomorphism in Equation(3.1), let $\epsilon(x, y)$ be the image of $a - b$ in $H_1(Y, \mathbb{Z})$. The definition is well-defined which means $\epsilon(x, y)$ is independent of the paths chosen. Let $a' : [0, 1] \to \mathbb{T}_\alpha$ and $b' : [0, 1] \to \mathbb{T}_\beta$ be another pair of paths connecting x and y. Note that $a - a'$ is a loop based at x in $\mathbb{T}_\alpha$ and similarly $b - b'$ is a loop based at x in $\mathbb{T}_\beta$. We need to show that $(a - b) - (a' - b')$ is nullhomologous in $H_1(Y, \mathbb{Z})$. As $(a-b)-(a'-b') = (a-a')-(b-b')$ is nullhomologous in $H_1(Y, \mathbb{Z})$ under the map in Eqn. 3.1). By definition it follows that if $\epsilon(x, y) \neq 0$ then $\pi_2(x, y) = \emptyset$. Indeed if a chosen loop $a - b$ is essential in $H_1(Y, \mathbb{Z})$ then it is not in the kernel of the map in Eqn.(3.1). If there is a Whitney disk $u \in \pi_2(x, y)$ its boundary is a loop in $H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)$. Thus $\pi_2(x, y)$ must be empty.

We can calculate $\epsilon(x, y)$ on Σ with the help of the isomorphism

$$\pi_1(Sym^g(\Sigma)) \simeq H_1(\Sigma)$$

Intersection points x and y are unordered g-tuples $x = \{x_1, .., x_g\}$ and $y = \{y_1, .., y_g\}$, thus a path $a : [0, 1] \to \mathbb{T}_\alpha$ is a collection of arcs $\alpha_1 \cup .. \cup \alpha_g$ in Σ with $\partial a = \sum y_i - \sum x_i$, and a path $b : [0, 1] \to \mathbb{T}_\beta$ is also a collection of arcs $\beta_1 \cup .. \cup \beta_g$ in Σ with same boundary. This implies $a - b$ is closed 1-cycle in Σ and the image of $a - b$ in $H_1(Y, \mathbb{Z})$ is $\epsilon(x, y)$.

Proposition 3.2.2. *$\epsilon(x, y)$ is additive as $\epsilon(x, y) + \epsilon(y, z) = \epsilon(x, z)$.*

Proof. Let a and b be two paths as above from x to y. Take $a' : [0,1] \to \mathbb{T}_\alpha$ and $b' : [0,1] \to \mathbb{T}_\beta$ be two paths from y to z. Choose paths $c : [0,1] \to \mathbb{T}_\alpha$ and $d : [0,1] \to \mathbb{T}_\beta$ from x to z combining a with a' and b with b' respectively. Image of $a - b$ in $H_1(Y, \mathbb{Z})$ is $\epsilon(x, y)$ and image of $a' - b'$ is $\epsilon(y, z)$. Then $(a - b) - (a' - b') = (a + a') - (b + b')$ is loop in $Sym^g(\Sigma)$ whose image is $\epsilon(x, z)$ and result follows. □

Proposition 3.2.3. *For $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $x \sim y$ if $\epsilon(x, y) = 0$ is an equivalence relation.*

Proof. $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\epsilon(x, y) = 0$ implies $\pi_2(x, y) \neq \emptyset$ so there exists a Whitney disk connecting x and y. Let $u : \mathbb{D} \to Sym^g(\Sigma)$ with $u(i) = u(-i) = x$ and $u(e_1) \subset \mathbb{T}_\alpha$ and $u(e_2) \subset \mathbb{T}_\beta$. So $u \in \pi_2(x, x) \neq \emptyset$ implies $\epsilon(x, x) = 0$ and $x \sim x$. Suppose $x \sim y$ so $\epsilon(x, y) = 0$. Take two paths $a : [0,1] \to \mathbb{T}_\alpha$ and $b : [0,1] \to \mathbb{T}_\beta$ connecting x and y. Image of $a - b$ is 0 in $H_1(Y, \mathbb{Z})$. But $-a$ and $-b$ are paths from y to x and image of $b - a$ is also 0 in $H_1(Y, \mathbb{Z})$. This shows $\epsilon(y, x) = 0$ so $y \sim x$ and it is symmetric. Now, suppose $x \sim y$ and $y \sim z$ then $\epsilon(x, y) = 0$ and $\epsilon(y, z) = 0$. $\epsilon(x, y)$ is additive implies that $\epsilon(x, z) = \epsilon(x, y) + \epsilon(y, z) = 0$ so $x \sim z$. Thus, it is an equivalence relation. □

Remark 3.2.4. The set of intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ can be partitioned into equivalence classes by above relation.

Example: Consider a genus-1 decomposition of $L(p, q)$ and choose attaching curves α and β, so that they intersect at p points. All intersection points lie in different equivalence classes. Note that a Heegaard decomposition of $L(p, q)$ as described in Section 2.2 will look like below, which is for $L(5, 3)$:

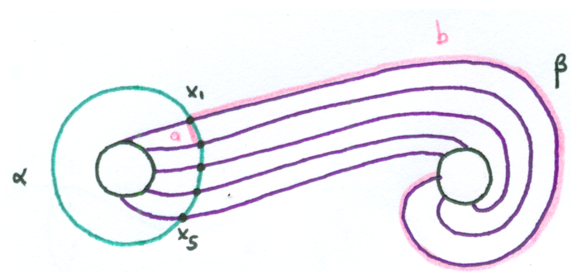


Figure 3.1: A Heegaard diagram of the Lens space $L(5, 3)$

Choose α and β curves as such to have p intersection points. Choose x_1 and x_2 which can be connected by a path a in $\mathbb{T}_\alpha$ and by a path b in $\mathbb{T}_\beta$. After attaching the 1-handle, b becomes a path from x_1 to x_2 in Figure(3.1). As $Sym^1(\Sigma_1) \simeq \Sigma/S^1 \simeq \Sigma_1$, the loop $a - b$ is homotopic to a standard parallel of the solid torus and by definition it is not $0 \in H_1(\Sigma_1)$, so $\epsilon(x_1, x_2) \neq 0$ implies $x_1 \nsim x_2$. In general, using a similar argument we see that loops obtained by connecting intersection points x_i and x_j with $i \neq j$ is homotopic to a standard parallel of solid torus and hence not equal to 0 in $H_1(\Sigma_1)$. Therefore, $x_i \nsim x_j$ for $i \neq j$.

3.3 Domains and Intersection Numbers

In this section, we learn domains in Σ_g which is helpful to understand disks in $Sym^g(\Sigma_g)$.

Definition 13. Let A be a set of maps, $A_{x,y} : \pi_2(x,y) \to \mathbb{Z}$ for every $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that $A_{x,y}(\phi) + A_{y,z}(\psi) = A_{x,z}(\phi * \psi)$, for every $\phi \in \pi_2(x,y)$ and $\psi \in \pi_2(y,z)$, is called an additive assignment.

Now, let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $w \in \Sigma_g - \alpha_1 - .. - \alpha_g - \beta_1 - .. - \beta_g$ be a basepoint. Define $n_w : \pi_2(x,y) \to \mathbb{Z}$ sending a homotopy class of a Whitney disk to its algebraic intersection number

$$n_w(\phi) = \#(\{w\} \cap Sym^{g-1}(\Sigma_g))$$

Remember that $V_w = \{w\} \cap Sym^{g-1}(\Sigma_g)$ and this map counts the number of coordinates of image of ϕ which is w considering orientation.

As a first step let us show that this algebraic intersection number is finite. Note that $V_w = \{w\} \cap Sym^{g-1}(\Sigma_g)$ is real codimension 2 in $Sym^g(\Sigma_g)$. Therefore, generically image of a 2 dimensional object $\mathbb{D}$ intersects with V_w at points. Image of $\mathbb{D}$ and v_w is compact, therefore they intersect at finitely many points.

This map is well-defined, so it is independent of the representative chosen in each homotopy class. Let ϕ_1, ϕ_2 be different representatives of $\phi \in \pi_2(x,y)$. As they are homotopic, let $\{\phi_t\}$ be a homotopy between ϕ_1 and ϕ_2. By definition ϕ_t is continuous with respect to t, as the algebraic intersection number V_w with image of ϕ_t is an integer and ϕ_t is continuous with respect to t imply $\#(\phi_t(\mathbb{D}) \cap V_w)$ is constant. Thus, $\#(\phi_0(\mathbb{D}) \cap V_w) = \#(\phi_1(\mathbb{D}) \cap V_w)$ implies $n_w(\phi_1) = n_w(\phi_2)$ and n_w is well-defined.

Now, let us show that n_w is additive. Take $x, y, z \in \mathbb{T}\alpha \cap \mathbb{T}\beta$ with $\phi \in \pi_2(x,y)$, $\psi \in \pi_2(y,z)$, and $w \in \Sigma_g - \alpha_1 - .. - \alpha_g - \beta_1 - .. - \beta_g$. $\phi * \psi$ is a Whitney disk connecting x and z so $\phi * \psi \in \pi_2(x,y)$. $n_w(\phi * \psi) = \#(\phi * \psi(\mathbb{D}) \cap V_w)$ counts the number of intersection points between V_w and $(\phi * \psi)(\mathbb{D})$. As $\phi * \psi$ is a disk obtained by gluing ϕ and ψ, the algebraic intersection number is the sum of algebraic intersection numbers of $\phi(\mathbb{D}) \cap V_w$ and $\psi(\mathbb{D}) \cap V_w$. Thus, $n_w(\phi * \psi) = n_w(\phi) + n_w(\psi)$ follows.

Definition 14. Let (Σ, α, β) be a Heegaard diagram and $D_1, .., D_m$ be closures of the components of $\Sigma_g - \alpha_1 - .. - \alpha_g - \beta_1 - .. - \beta_g$. For any $\phi \in \pi_2(x,y)$ we define domain for ϕ as formal linear combination of $D_1, .., D_m$ as:

$$\mathcal{D}(\phi) = \sum_{i=1}^{m} n_{z_i} D_i$$

where $z_i \in int(D_i)$ for $1 \leq i \leq m$. We denote $\mathcal{D}(\phi) \geq 0$ for all i.

Remark 3.3.1. Note that the definition of domain for ϕ is independent of the representative of the homotopy class of ϕ which follows by well-definedness of n_w map.

Example: Consider a genus-1 decomposition of S^3 where the Heegaard diagram has α and β curves intersecting as in Example.1 in Section 2.2. There is only one region D_1 as $\Sigma - \alpha - \beta$ is an open disk.

Proposition 3.3.2. *For $x, y, p \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\phi_1 \in \pi_2(x, y)$, $\phi_2 \in \pi_2(y, p)$. Then*

$$\mathcal{D}(\phi_1 * \phi_2) = \mathcal{D}(\phi_1) + \mathcal{D}(\phi_2)$$

and in particular

$$\mathcal{D}(S * \phi) = \mathcal{D}(\phi) + \sum_{i=1}^{m} D_i$$

where S is positive generator of $\pi_2'(Sym^g(\Sigma_g))$.

Proof. Above equalities follow easily from additivity of n_w and note that $\phi_1 * \phi_2 \in \pi_2(x, p)$.

$$\mathcal{D}(\phi_1 * \phi_2) = \sum_{i=1}^{m} n_{z_i}(\phi_1 * \phi_2) D_i = \sum_{i=1}^{m} [n_{z_i}(\phi_1) + n_{z_i}(\phi_2)] D_i = \mathcal{D}(\phi_1) + \mathcal{D}(\phi_2).$$

For the second equality we use the fact that $n_{z_i}(S) = 1$ and result follows. □

In the definition of domain, we have not mention a restriction for the interior point $z_i \in int(D_i)$ chosen. Let us show that this quantity is independent of the choice of interior point z_i and $\mathcal{D}(\phi)$ depends only on the homotopy class. As $n_w(\phi)$ counts algebraically the number of components of image of $\mathbb{D}$ under ϕ which are w. Consider the quotient of the surface Σ under identifying all α and β curves to points. This is equivalent to saying that collapse the boundary of D_i for all $1 \leq i \leq m$ into single point getting wedge of m spheres in the end. There is a similar quotient in the domain $\mathbb{D}$ as its boundary maps to $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. After composing the projection of ϕ to the surface with this quotient described above, $\phi : \mathbb{D} \to Sym^g(\Sigma)$ turns into a map from $S^2 \to \bigvee_{i=1}^{m} S^2$. If in the image there is a point in any one of the spheres in the wedge sum the other points of the same sphere must be in the image too. Therefore, $n_{z_i}(\phi) = n_{z_j}(\phi)$ for any $z_i, z_j \in int(D_k)$ proving that the definition of domain associated to $\phi \in \pi_2(x, y)$ $\mathcal{D}(\phi)$ is independent of the interior points chosen as basepoint and it depends only on the homotopy class of ϕ.

The region D_i without boundary are 2-cells, so we can see the domain $\mathcal{D}(\phi)$ associated to ϕ as a 2-chain, which is a formal linear combination of 2 simplices. Thus we can study its boundary. Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ be intersection points with $x = \{x_1, .., x_g\}$, $y = \{y_1, .., y_g\}$ where $x_i \in \alpha_i \cap \beta_i$ and $y_i \in \alpha_i \cap \beta_{\sigma^{-1}(i)}$ and σ is a permutation.

First, let us investigate the case $g = 2$, $x = \{x_1, x_2\}$, $y = \{y_1, y_2\}$ and σ can be identity or a transpose changing $(1, 2)$ to $(2, 1)$. If σ is the latter we have:

$$\begin{cases} x_i \in \alpha_i \cap \beta_i & \text{for } i = 1,2 \\ y_1 \in \alpha_1 \cap \beta_2 \\ y_2 \in \alpha_2 \cap \beta_1 \end{cases}$$

Place these 4 points on the plane, we can connect x_1 and y_1 by α_1 curve, x_2 and y_2 by α_2 curve, x_2 and y_1 by β_2 curve, and x_1 and y_2 by β_1 curve.

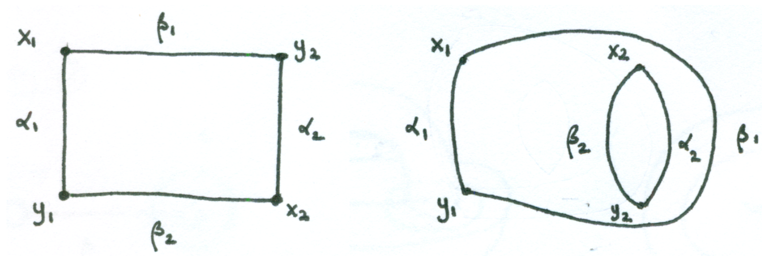


Figure 3.2: Domains of disks in $Sym^2(\Sigma)$

If σ is the identity permutation then we have $x_i, y_i \in \alpha_i \cap \beta_i$ for $i = 1,2$ and we have the following figure.

For arbitrary g and $\phi \in \pi_2(x, y)$, if we restrict boundary of $\mathcal{D}(\phi)$ to α_i it is a path between x_i and y_i for all $1 \leq i \leq m$. So this restriction is a 1-chain whose boundary is $y_i - x_i$. Similarly, if we restrict boundary of $\mathcal{D}(\phi)$ to β_i it is a path between x_i and $y_{\sigma(i)}$ as $y_{\sigma(i)} \in \alpha_{\sigma(i)} \cap \beta_{\sigma^{-1}(\sigma(i))} = \alpha_{\sigma(i)} \cap \beta_i$. It is also a 1-chain whose boundary is $x_i - y_{\sigma(i)}$. Thus we can say that $\partial(\mathcal{D}(\phi))$ connects x to y on α curves and y to x on β curves.

Let us study a specific domain type called a *periodic domain.*

Definition 15. Let $(\Sigma, \alpha, \beta, z)$ be a Heegaard diagram with basepoint z. A periodic domain is a 2-chain $\mathcal{P} = \sum_{i=1}^{m} a_i D_i$ such that its boundary is union of α and β curves with $n_z(\mathcal{P}) = 0$. For any $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, if for $\phi \in \pi_2(x, x)$, $n_z(\phi) = 0$ then ϕ is called a periodic class and the domain associated to a periodic class is called a periodic domain.

Remark 3.3.3. For $x, y, z \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, there is a generalized multiplication

$$* : \pi_2(x, y) \times \pi_2(y, z) \to \pi_2(x, z)$$

For $x = y$, $\pi_2(x, x)$ has a group structure, then the set of periodic classes, denoted as $\prod_x(z)$, is naturally a subgroup of $\pi_2(x, x)$.

Now let us state the main theorem of this section describing the topology of $\pi_2(x, y)$.

Theorem 3.3.4. *Let $g > 1$, then for every $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$,*

$$\pi_2(x, x) \simeq \mathbb{Z} \oplus H^1(Y, \mathbb{Z})$$

In general, for every $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, if $\epsilon(x, y) \neq 0$ then $\pi_2(x, y) = \emptyset$; otherwise

$$\pi_2(x, y) \simeq \mathbb{Z} \oplus H^1(Y, \mathbb{Z})$$

We will prove this theorem by using short exact sequence of spaces giving long exact sequences of homotopy groups. A short exact sequence of the form

$$A \xrightarrow{\ i\ } X \longrightarrow X/A$$

gives long exact sequence of homology groups but not long exact sequence of homotopy groups. However fiber bundle

$$F \longrightarrow E \longrightarrow B$$

is such a short exact sequence. Let us give some preliminary definitions and statements for the proof of the Theorem (3.3.4).

Definition 16. A map $p : E \to B$ has homotopy lifting property with respect to a space X if for a given homotopy $g_t : X \to B$ and a map $\widetilde{g_0} : X \to E$ lifting g_0 such that $p\widetilde{g_0} = g_0$ then there exists a homotopy $\widetilde{g_t} : X \to E$ lifting g_t.

Definition 17. A fibration is a map $p : E \to B$ having homotopy lifting property with respect to all spaces X.

Theorem 3.3.5 ([12]). *Suppose that the map $p : E \to B$ has the homotopy lifting property with respect to disks D^k for all $k \geq 0$. Choose basepoints $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. Then the map $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for all $n \geq 1$. If B is path-connected then there is a long exact sequence:*

$$\cdots \longrightarrow \pi_n(F, x_0) \longrightarrow \pi_n(E, x_0) \xrightarrow{\ p_*\ } \pi_n(B, b_0) \longrightarrow \pi_{n-1}(F, x_0) \longrightarrow \cdots$$
$$\longrightarrow \pi_0(E, x_0) \longrightarrow 0$$

Proof of this theorem uses relative form of homotopy lifting property and it will not be discussed here.

A map $p : E \to B$ satisfying homotopy lifting property for disks is called a *Serre fibration*. For detailed discussion of this topic and proofs, see fiber bundles in [12].

Proof of the Theorem 3.3.4. Let $\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ be the space of paths in $Sym^g(\Sigma)$ joining $\mathbb{T}_\alpha$ to $\mathbb{T}_\beta$. Consider the evaluation map $p : \Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta) \to \mathbb{T}_\alpha \times \mathbb{T}_\beta$ sending paths γ from a to b to its endpoints $(a, b) \in \mathbb{T}_\alpha \times \mathbb{T}_\beta$. Fiber $p^{-1}(a, b)$ is all paths in $\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ from a to b, and take fiber as the path space $\Omega Sym^g(\Sigma)$. Then we have a fiber bundle:

$$\Omega Sym^g(\Sigma) \longrightarrow \Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta) \longrightarrow \mathbb{T}_\alpha \times \mathbb{T}_\beta$$

First, let us show that this is a Serre fibration, so we need to prove the map $p : \Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta) \to \mathbb{T}_\alpha \times \mathbb{T}_\beta$ has homotopy lifting property with respect to D^k for all $k \geq 0$. Let $g_t : D^k \to \mathbb{T}_\alpha \times \mathbb{T}_\beta$ be a homotopy. The image of the map $g_0 : D^k \to \mathbb{T}_\alpha \times \mathbb{T}_\beta$ for every point can be seen as a path $\gamma = p^{-1}(g_0(x))$ in $\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ between endpoints $g_0(x) = (a, b)$. Thus we can extend it to $\widetilde{g_0} : D^k \to \Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ taking x to γ such that $p\widetilde{g_0} = g_0$, then by homotopy lifting property we have a homotopy $\widetilde{g_t} : D^k \to \Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$.

Serre fibration induces a long exact homotopy sequence by above theorem and using $\mathbb{T}_\alpha \times \mathbb{T}_\beta$ is path-connected then we have:

$$\cdots \longrightarrow \pi_n(\Omega Sym^g(\Sigma), x_0) \longrightarrow \pi_n(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), x_0) \xrightarrow{p_*} \pi_n(\mathbb{T}_\alpha \times \mathbb{T}_\beta, b_0)$$
$$\longrightarrow \pi_{n-1}(\Omega Sym^g(\Sigma), x_0) \longrightarrow \cdots \longrightarrow \pi_0(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), x_0) \longrightarrow 0$$

Note that the space $\pi_2(x, x)$ can be identified with the fundamental group of the space $\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ based at the constant path $x_0 = (x)$. Let $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ then a constant path (x) is a path joining $\mathbb{T}_\alpha$ to $\mathbb{T}_\beta$. Such a path correspond to image of a Whitney disk $u : \mathbb{D} \to Sym^g(\Sigma)$ such that $u(i) = u(-i) = x$ with $u(e_1) \subset \mathbb{T}_\alpha$ and $u(e_2) \subset \mathbb{T}_\beta$. Homotopy of loops in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ based at constant path (x) correspond to homotopy of Whitney disks joining x to x. Thus $\pi_2(x, x)$ can be identified with $\pi_1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta))$.

Consider the following part of the long exact homotopy sequence:

$$\cdots \longrightarrow \pi_2(\mathbb{T}_\alpha \times \mathbb{T}_\beta, b_0) \longrightarrow \pi_1(\Omega Sym^g(\Sigma), x_0) \longrightarrow \pi_1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), b_0)$$
$$\xrightarrow{p_*} \pi_1(\mathbb{T}_\alpha \times \mathbb{T}_\beta, b_0) \longrightarrow \pi_0(\Omega Sym^g(\Sigma), x_0) \longrightarrow \pi_0(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), x_0) \longrightarrow 0$$

We have the following isomorphisms:

- $\pi_2(\mathbb{T}_\alpha \times \mathbb{T}_\beta) = \pi_2(\mathbb{T}_\alpha) \times \pi(\mathbb{T}_\beta) \simeq 0$, (see [10])
- $\pi_i(\Omega Sym^g(\Sigma)) \simeq \pi_{i+1}(Sym^g(\Sigma))$ for $i = 0, 1$, (see [10])
- $\pi_1(Sym^g(\Sigma)) \simeq H_1(\Sigma) \simeq H^1(\Sigma)$, last isomorphism comes from Poincare Duality.
- $\pi_1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta, (x)) \simeq \pi_2(x, x)$
- $\pi_2(Sym^g(\Sigma)) \simeq \mathbb{Z}$ as $g \geq 2$ by Theorem 3.1.5

Consider the third isomorphism above, the images of $\pi_1(\mathbb{T}_\alpha)$ and $\pi_1(\mathbb{T}_\alpha)$ correspond to $H^1(U_0; \mathbb{Z})$ and $H^1(U_1; \mathbb{Z})$, so we have the following sequence combined with above isomorphisms:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_2(x, x) \longrightarrow H^1(U_0) \oplus H^1(U_1) \longrightarrow H^1(\Sigma)$$
$$\longrightarrow \pi_0(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), x_0) \longrightarrow 0$$

For the 3-manifold $Y = U_0 \cup U_1$ with $\Sigma = U_0 \cap U_1$, let us apply the Mayer-Vietoris sequence for cohomology with $\mathbb{Z}$ coefficients:

$$\cdots \longrightarrow H^n(Y) \longrightarrow H^n(U_0) \oplus H^n(U_1) \longrightarrow H^n(\Sigma) \longrightarrow H^{n+1}(Y) \longrightarrow \cdots$$

and consider the following piece:

$$\cdots \longrightarrow H^1(Y) \longrightarrow H^1(U_0) \oplus H^1(U_1) \longrightarrow H^1(\Sigma) \longrightarrow \cdots$$

As this is a chain complex composition of first two maps gives 0 and $H^1(Y)$ is subset of the kernel of the map $H^1(U_0) \oplus H^1(U_1) \to H^1(\Sigma)$ and we obtain the following short exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_2(x, x) \longrightarrow H^1(Y; \mathbb{Z}) \longrightarrow 0$$

The remaining part is to show that the sequence splits.

Lemma 3.3.6 (Splitting Lemma [12])**.** *For a short exact sequence of Abelian groups*

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ j\ } C \xrightarrow{\ j\ } 0$$

there is a homomorphism $p : B \to A$ such that $pi = \mathbb{1} : A \to A$ if and only if $B \simeq A \oplus C$.

The homomorphism $n_z : \pi_2(x, x) \to \mathbb{Z}$ which counts intersection points algebraically provides a splitting for the sequence and $\pi_2(x, x) \simeq \mathbb{Z} \oplus H^1(Y; \mathbb{Z})$ follows for $g > 2$. Moreover $\prod_x(z) \simeq H^1(Y; \mathbb{Z})$ follows easily as $\prod_x(z)$ set of $\phi \in \pi_2(x, x)$ such that $n_z(\phi) = 0$.

For $g > 2$ we use the fact that $\pi_2(Sym^g(\Sigma)) \simeq \mathbb{Z}$. Thus for $g = 2$ we need to divide by the action of $\pi_1(Sym^g(\Sigma))$.

For the case $x \neq y$ and $\epsilon(x, y) = 0$ which implies $\pi_2(x, y) \neq \emptyset$, we can apply the above reasoning to obtain the result. □

3.4 *Spin^c* Structures

In this section we will define *Spin^c* structures over Y and present its properties and its relation with the intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Proposition 3.4.1. *Let Y be an oriented, closed 3-manifold then it has trivial Euler characteristic.*

Proof. Consider a Heegaard decomposition of Y into genus-g handlebodies. This can be seen as a handle decomposition of Y with one 0-handle, g 1-handles and g 2-handles, and one 3-handle. By Corollary 4.19 in [22],

$$\chi(Y) = \sum_0^3 (-1)^i k_i$$

where k_i denotes the number of i-handles contained in Y. As the number of 1-handles is the same as the number of 2-handles, we have

$$\chi(Y) = \sum(-1)^0 1 + (-1)^1 g + (-1)^2 g + (-1)^3 1 = 0$$

as desired. □

We will need the following information for the following proofs and statements in this section.

Definition 18. [18] A vector field v is a continuous map $v : Y \to TY$ such that $\pi \circ v = \mathbb{1}_Y$ where $\pi : TY \to Y$ is projection map.

Definition 19. Let $f : S^n \to S^n$ be a continuous map and $f_* : \widetilde{H_n}(S^n) \to \widetilde{H_n}(S^n)$ be an induced homomorphism from infinite cyclic group to itself, so it must be of the form $f_*(\alpha) = d\alpha$ for some $d \in \mathbb{Z}$ depending only on f. This integer is called the degree of f and denoted by deg$f$.

Remark 3.4.2. If $f \simeq g$ then deg$f$ =deg$g$ as $f_* = g_*$. Converse of this statement is a theorem due to Hopf.

Theorem 3.4.3. *Maps between spheres of the same dimension are classified by their degrees up to homotopy.*

For a proof of this statement see [12].

Definition 20. [7] Let v be a vector field on an oriented surface S with isolated zero p, index of v, an integer associated to p, is defined as: Let $x : U \to S$ be an orthogonal parametrization at $p = x(0,0)$ compatible with the orientation of S and let $\alpha : [0,l] \to S$ be a simple, closed, positively oriented, piecewise regular parametrized curve such that $\alpha([0,l]) \subset x(U)$is boundary of a simple region R containing p as its only zero. Let $v = v(t)$, $t \in [0,l]$ be the restriction of v along α and let $\phi = \phi(t)$ be some determination of the angle from x_u to $v(t)$. As α is closed there is an integer I defined as:

$$2\pi I = \phi(l) - \phi(0) = \int_0^l \frac{d\phi}{dt} dt.$$

I is called the index of v at p.

This definition is independent of the choices made, see [7] for details.

Proposition 3.4.4. *Let Y be an oriented closed 3-manifold. As $\chi(Y) = 0$, Y admits a nowhere vanishing vector field.*

Proof. Choose a generic Morse function f and a generic metric g on Y. Consider the gradient vector field $\overrightarrow{\nabla} f$ of f, which vanishes only on the critical points of f. Take a path joining index-0 and index-3 critical points, and a path joining each index-1 critical point to a unique index-2 critical point. Tubular neighbourhood of each of these paths is homeomorphic to a 3-ball $B^2 \subset \mathbb{R}^3$. Note that we have nowhere vanishing vector field on the closure of B^3's and on each B^3 we have a vector field which is zero at exactly two points. Choose one of B^3's and restrict $\overrightarrow{\nabla} f$ on ∂B^3 and normalize this to obtain a map $S^2 \to S^2$ call it g. Note that g has no zeros, it is a degree-0 map. By theorem of Hopf [12, 25], g is homotopic to constant map from $S^2 \to S^2$. So we can extend $\overrightarrow{\nabla} f|_{\partial B^3}$ to B^3 as a nonvanishing vector field. Applying the same argument to each B^3 we obtain a nowhere vanishing vector field on Y as wanted. □

Definition 21. Let v_1 and v_2 be two nowhere vanishing vector fields on Y. We say v_1 homologous to v_2 is there is a ball B in Y such that $v_1|_{Y-B}$ is homotopic to $v_2|_{Y-B}$.

Proposition 3.4.5. *v_1 homologous to v_2 is an equivalence relation.*

Proof. Let v_1, v_2, v_3 be nowhere vanishing vector fields on Y. $v_1 \sim v_1$, indeed, for and ball B in Y, $v_1|_{Y-B}$ is homotopic to $v_1|_{Y-B}$ via identity map as family of maps. Suppose that $v_1 \sim v_2$ so there is a ball B in Y and a homotopy $\{f_t\}_{0 \leq t \leq 1}$ such that $f_0 = v_1|_{Y-B}$ and $f_1 = v_2|_{Y-B}$. Take $g_t = f_{1-t}$ for $0 \leq t \leq 1$ then we obtain a homotopy between $v_2|_{Y-B}$ and $v_1|_{Y-B}$, so $v_2 \sim v_1$. Now, suppose $v_1 \sim v_2$ and $v_2 \sim v_3$, then there exist balls B_1 and B_2 in Y such that $v_1|_{Y-B_1}$ homotopic to $v_2|_{Y-B_1}$ and $v_2|_{Y-B_2}$ homotopic to $v_3|_{Y-B_2}$. Let $B \supset (B_1 \cup B_2)$ be a ball in Y containing both B_1 and B_2, then $v_1|_{Y-B}$ homotopic to $v_2|_{Y-B}$ and $v_2|_{Y-B}$ homotopic to $v_3|_{Y-B}$, thus $v_1|_{Y-B}$ homotopic to $v_3|_{Y-B}$ follows. This is an equivalence relation. □

Definition 22. The space of *Spin*^{c} structures over Y, denoted as *Spin*$^c(Y)$, is the set of equivalence classes of above relation of nowhere vanishing vector fields on Y.

Let TY be the tangent bundle of Y then it satisfies the *local trivialization property* namely: For every $p \in Y$, there is a neighborhood $p \in U \subset Y$ and a homeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^3$ such that $\pi_1 \circ \phi = \pi$, where $\pi : TY \to Y$ is the projection map and $\pi_1 : U \times \mathbb{R}^3 \to U$ is the projection onto the first factor. If there is a trivialization over all of Y then TY is a trivial bundle homeomorphic to $Y \times \mathbb{R}^3$, see [18]. Moreover, any closed oriented 3-manifold has a trivial tangent bundle which can be shown by characteristic classes.

Proposition 3.4.6. *In the light of above theorem we can fix a trivialization τ of the tangent bundle TY of Y. Then there is a one-to-one correspondence between vector fields v over Y and maps $f_v : S^2 \to S^2$.*

Proof. $\tau : TY \to Y \times \mathbb{R}^3$ is a homeomorphism and $v : Y \to TY$ a vector field. Consider the following composition of maps. First, compose τ and v to get a map $\tau \circ v : Y \to Y \times \mathbb{R}^3$ and compose it by π to project to $\mathbb{R}^3$. As v is nonvanishing we can compose with the map η taking the norm. In the end we obtain a map $f_v = \eta \circ \pi \circ \tau \circ v$ from $Y \to S^2$. Note that f_v depends on the vector field v so it is one-to-one. A map $f : Y \to S^2 \subset \mathbb{R}^3$ sending every point $p \in Y$ to $v \in S^2$ living in $\mathbb{R}^3$ can be seen as a vector at p. Therefore f correspond to a vector field on Y which is nonvanishing. □

Remark 3.4.7. We cam also define an equivalence relation on maps $f : Y \to S^2$ as: $f_0, f_1 : Y \to S^2$ are homologous if they are homotopic in the complement of a 3-ball B (or finitely many balls) in Y.

For a fixed generator μ of $H^2(S^2;\mathbb{Z}) \simeq \mathbb{Z}$ and a trivialization τ of the tangent bundle TY then there is a $1-1$ correspondence depending on the trivialization as follows

$$\delta^\tau : Spin^c(Y) \to H^2(Y;\mathbb{Z})$$

$$v \mapsto f_v^*(\mu)$$

and $Spin^c(Y)$ becomes an affine space, see [8, 29]

Now we will define a natural map from intersection points of totally real tori in $Sym^g(\Sigma)$ to $Spin^c$ structures, which is one of the main ingredients for the definition of Heegaard Floer homology. Let $(\Sigma, \alpha, \beta, z)$ be a pointed Heegaard diagram. The map

$$s_z : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to Spin^c(Y)$$

is defined as follows: Take a Morse function f on Y compatible with the α and β-curves, and a Riemannian metric g on Y which gives and inner product on TY and allows us to define the gradient vector field $\overrightarrow{\nabla} f$ on Y. Take $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $x = \{x_1, .., x_g\}$ g-tuple elements and each one of x_i determines a path joining an index-1 critical point to an index-2 critical point by uniqueness of the Equation (2.1). We know via Morse theory that, we can choose f so that it has only one index-0 critical point and one index-3 critical point, so the basepoint z determines a path joining index-0 critical point to index-3 critical point. Now we have $g+1$ paths and each connects a pair of critical points of f. Tubular neighborhood of each of those paths is homeomorphic to a 3-ball, B^3. On the complement of a tubular neighborhood we have nonvanishing vector field $\overrightarrow{\nabla} f$, as it vanishes only on the ciritical points of f which are in these 3-balls. By the same argument in the proof of the Theorem(3.4.4) we can extend this nonvanishing vector field on the boundary sphere as a nonvanishing vector field on B^3 for each 3-ball. In the end, this gives a nonvanishing vector field over Y which can be normalized to get a unit vector field over Y. Homology class of nowhere vanishing vector field obtained in this manner is called a $Spin^c$ structure $s_x(x)$. Thus s_z is a map sending an intersection point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ to the corresponding $Spin^c$ structure $s_z(x)$.

Remark 3.4.8. This map can be seen as a refinement of the equivalence classes given by $\epsilon(x, y) \in H_1(Y;\mathbb{Z})$.

Next we will show that the map $s_z : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to Spin^c(Y)$ is well-defined. It is independent of the choice of the compatible Morse function f and the extension of the gradient vector field $\vec{\nabla} f$ to 3-balls.

Let us take a different Morse function f_1 compatible with α and β-curves. We use the Morse function to determine the paths joining critical points of index-1 and 2 and critical points of index-0 and 3. Changing f to f_1 can give different trajectories but they will differ by an isotopy move on the Heegaard surface, as both compatible Morse funcitons corresponding to the same Heegaard diagram. Thus, their gradient vector fields $\vec{\nabla} f$ and $\vec{\nabla} f_1$ will be homotopic on the complement of finitely many ($g+1$ precisely) 3-balls so the map is independent of the compatible Morse function.

$s_z(x)$ is also independent of the extension of $\vec{\nabla} f$ to 3-balls. A different extension will give a different nonvanishing vector field v over Y, but the extension of $\vec{\nabla} f$ into balls and v are same in the complement of finitely many 3-balls. So they are homologous and they correspond to the same $Spin^c$ structure.

Next let us see how $s_z(x)$ depends on x and the basepoint z.

Theorem 3.4.9. *Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ then*

1. $s_z(y) - s_z(x) = PD[\epsilon(x, y)]$

2. *If $z_1, z_2 \in \Sigma - \alpha_1 - .. - \alpha_g - \beta_1 - .. - \beta_g$ can be connected by an arc z_t in Σ from z_1 to z_2 staying disjoint from β such that intersection number $\#(\alpha_i \cap z_t) = 1$ and $\#(\alpha_j \cap z_t) = 0$ for $i \neq j$ then for all $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$,*

$$s_{z_2}(x) - s_{z_1}(x) = \alpha_i^*$$

where $\alpha_i^* \in H^2(Y; \mathbb{Z})$ Poincare dual to homology class in Y induced from a curve γ in Σ with $\alpha_i \cdot \gamma = 1$ and $\alpha_j \cdot \gamma = 0$ for $i \neq j$.

Proof. Take a Morse function $f : M \to \mathbb{R}$ compatible with α and β curves. For $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $x = \{x_1, .., x_g\}$ is a g-tuple of points and each x_i determines a path for $\vec{\nabla} f$ connecting index-1 to index-2 critical points passing through x_i. The basepoint z also determines a path for $\vec{\nabla} f$ joining index-0 to index-3 critical points. Thus we have $g+1$ trajectories γ_x and γ_z. For $y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, y also determines g trajectories γ_y and $\gamma_x - \gamma_y$ is a closed loop in Y. From the intersection points x and y we obtain $Spin^c$ structures $s_z(x)$ and $s_z(y)$ which can obtained from $\vec{\nabla} f$ modifying it in a neighborhood of $\gamma_x \cup \gamma_z$ for x and $\gamma_y \cup \gamma_z$ for y.

Remember that $Spin^c(Y)$ is an affine space, $s_z(x) - s_z(y) \in Spin^3(Y)$ and there is a $1-1$ correspondance between $Spin^c(Y)$ and $H^2(Y; \mathbb{Z})$, so the difference $s_z(x) - s_z(y)$ can be represented by a cohomology class. Note that

$$H^2(Y; \mathbb{Z}) \simeq H_1(Y; \mathbb{Z}) \simeq \frac{H_1(\Sigma)}{[\alpha_1], .., [\alpha_g], [\beta_1], .., [\beta_g]}$$

where the first isomorphism comes from the Poincare duality. For the intersection points $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ we can choose $Spin^c$ structure $s_z(x)$ and $s_z(y)$ such that they agree on the complement of $\gamma_x \cup \gamma_y$. Therefore, the difference $s_z(x) - s_z(y)$ can be represented by a compactly supported cohomology class in a neighborhood of $\gamma_x - \gamma_y$.

If we assume that the curve is connected then $s_z(x) - s_z(y)$ can be taken as a multiple of $PD(\gamma_x - \gamma_y)$, Poincare dual of $\gamma_x - \gamma_y$. Let us find the coefficient. Take a disk D_0 transversally intersecting with the closed loop $\gamma_x - \gamma_y$ in Y. For $x_i \in x$ such that $x_i \notin y$, D_0 can be taken as a small neighborhood of x_i on Σ. If there is no such $x_i \notin y$ then $x = y$ and $\gamma_x = \gamma_y$ so $s_z(x) - s_z(y) = 0$ follows easily. Now take a representative v_x of $s_z(x)$ such that $v_x = \overrightarrow{\nabla} f$ near D_0, similarly take representative v_y of $s_z(y)$ such that $v_y = \overrightarrow{\nabla} f$ near D_0. Fix a trivialization $\tau : TY \to Y \times \mathbb{R}^3$ to obtain f_x and $f_y : Y \to S^2$ corresponding to v_x and v_y respectively such that $f_x|_{\partial D_0} = f_y|_{\partial D_0}$. In order to see the difference, let us compare their degrees.

$$s_z(x) - s_z(y) = [deg_{D_0}(v_x) - deg_{D_0}(v_y)]PD(\gamma_x - \gamma_y)$$

Take another disk D_1 with the same boundary of D_0 such that $D_1 \cup D_0$ bounds a 3-ball in Y. Index-1 critical point corresponding to, say x_1 is in $D_1 \cup D_0$, but there is no critical point inside. We can take $v_x = \overrightarrow{\nabla} f$ over D_1 so v_x does not vanish inside this 3-ball, therefore it has index-0. We have the following:

$$0 = deg_{D_0}(v_x) + deg_{D_1}(v_x) = deg_{D_0}(v_x) + deg_{D_1}(\overrightarrow{\nabla} f)$$

So $deg_{D_0}(v_x) = -deg_{D_1}(v_x)$. As $v_y = \overrightarrow{\nabla} f$ over D_0 we have:

$$deg_{D_0}(v_x) - deg_{D_0}(v_y) = -deg_{D_1}(\overrightarrow{\nabla} f) - deg_{D_0}(\overrightarrow{\nabla} f) = 1$$

Last equality follows from that the index of $\overrightarrow{\nabla} f$ around index-1 critical point is -1. Thus we have:

$$s_z(x) - s_z(y) = PD(\gamma_x - \gamma_y)$$

Let a be a path in $Sym^g(\Sigma)$ from x to y in $\mathbb{T}_\alpha$, similarly b be a path in $Sym^g(\Sigma)$ from x to y in $\mathbb{T}_\beta$. So $a \subset \mathbb{T}_\alpha$ is a collection of arcs $a \subset \alpha_1 \cup .. \cup \alpha_g$ with the boundary $y - x$, and $b \subset \mathbb{T}_\beta$ is a collection of arcs $b \subset \beta_1 \cup .. \cup \beta_g$ with the boundary $y - x$. Note that $\epsilon(x, y)$ represents the image of $a - b$ in $H_1(Y; \mathbb{Z})$. Let us see the relation between $\epsilon(x, y)$ and $\gamma_x - \gamma_y$. Take an arc $a_i \subset a$ connecting x_i to y_i, then a_i is homotopic to relative to endpoints to a piece of a loop in U_1 connecting index-1 to index-2 critical points. Similarly take an arc $b_i \subset b$ connecting x_i to y_i which is also homotopic relative to endpoints to the rest of the loop in U_0 connecting index-1 to index-2 critical points in Y. Therefore the loop $a - b$ in Y homologous to $\gamma_y - \gamma_x$ and $\epsilon(x, y)$ is the image of $\gamma_y - \gamma_x$ in $H_1(Y; \mathbb{Z})$. Thus $s_z(y) - s_z(x) = PD[\epsilon(x, y)]$ follows proving the first equality.

In order to understand how the map s_z depends on the basepoint z we will proceed as in the first case. Take two basepoints z_1 and z_2 on Σ away from α and β curves such that z_1 is connected to z_2 on Σ by an arc z_t as described in the hypothesis.

Take $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, x will give g trajectories connecting index-1 critical points to index-2 critical points. For z_1 and z_2, we have $g+1$ trajectories γ_x and γ_{z_1} and $g+1$ trajectories γ_x and γ_{z_2}. Corresponding $Spin^c$ structures $s_{z_1}(x)$ and $s_{z_2}(x)$ agree on the complement of $\gamma_{z_1} \cup \gamma_{z_2}$. The difference $s_{z_1}(x) - s_{z_2}(x)$ is represented by a cohomology class which is nonzero around $\gamma_{z_1} - \gamma_{z_2}$ then it follows $s_{z_1}(x) - s_{z_2}(x)$ is some multiple of $PD(\gamma_{z_1} - \gamma_{z_2})$. To find the coefficient, let us take a disk D_0 transversally intersecting with γ_{z_1} positively such that D_0 is disjoint from γ_{z_2}, and take another disk D_1 such that $D_0 \cup D_1$ bounds a 3-ball in Y containing the index-0 critical point. Take a representative v_{z_1} of $s_{z_1}(x)$ and v_{z_2} of $s_{z_2}(x)$ such that $v_{z_1} = \overrightarrow{\nabla} f$ over D_1 and $v_{z_2} = \overrightarrow{\nabla} f$ over D_0. Note that v_{z_1} does not vanish inside the 3-ball that $D_0 \cup D_1$ bounds so it has index 0.

$$0 = deg_{D_0}(v_{z_1}) + deg_{D_1}(v_{z_1})$$

so $deg_{D_0}(v_{z_1}) = -deg_{D_1}(v_{z_1})$ then

$$\begin{aligned} deg_{D_0}(v_{z_1}) - deg_{D_0}(v_{z_2}) &= -deg_{D_1}(v_{z_1}) - deg_{D_0}(v_{z_2}) \\ &= -deg_{D_1}(\overrightarrow{\nabla} f) - deg_{D_0}(\overrightarrow{\nabla} f) \\ &= -1 \end{aligned}$$

As $\overrightarrow{\nabla} f$ has index 1 around index-0 critical points so $s_{z_1}(x) - s_{z_2}(x) = -PD(\gamma_{z_1} - \gamma_{z_2})$.

We need to understand the relation between $\gamma_{z_1} - \gamma_{z_2}$ and α_i^*. We know that z_t is an arc from z_1 to z_2. Take another arc z' connecting z_2 to z_1 such that it lies on Σ and does not intersect with and α curves. We have a curve $\gamma = z_t \cup z'$ on Σ with $\#(\alpha_i \cap \gamma) = 1$ and $\#(\alpha_j \cap \gamma) = 0$ for $i \neq j$. z_t is homotopic to a piece of $\gamma_{z_1} - \gamma_{z_2}$ lying in U_1 relative to endpoints z_1 and z_2, and z' is homotopic to the rest of $\gamma_{z_1} - \gamma_{z_2}$ lying in U_0 relative to endpoints. It follows then $\gamma_{z_1} - \gamma_{z_2}$ homologous to γ, so they are same in $H_1(Y;\mathbb{Z})$. We call the Poincare dual of γ in $H^2(Y;\mathbb{Z})$ as α_i^* and γ satisfies the desired properties stated in the hypothesis. Hence

$$s_{z_2}(x) - s_{z_1}(x) = \alpha_i^*$$

finishes the proof of the second statement. As a result we have the full understanding of the map s_z, its well-definedness and how it depends on the basepoints and intersection points. □

We can define a natural map on $Spin^c(Y)$ which is an involution map sending $s \in Spin^c(Y)$ to $\overline{s}$, called *conjugate* $Spin^c$ structure of s with $\overline{s} = -s$. By using $Spin^c(Y)$ is an affine space and the $1-1$ correspondence between $Spin^c(Y)$ and $H^2(Y;\mathbb{Z})$ we can define a map $c_1 : Spin^c(Y) \to H^2(Y;\mathbb{Z})$ sending $s \mapsto s - \overline{s}$, this map is called the *first Chern class* which is one of the characteristic classes. Note that $c_1(\overline{s}) = \overline{s} - \overline{(\overline{s})} = -c_1(s)$. For more detailed description of the Chern classes the reader is referred to [25].

Chapter 4

ANALYTICAL BACKGROUND

This chapter includes necessary analytic background which we need to build up the theory. We mention the required statements and give references for the proofs and detailed examination of the subject of the subject. The most important parts are the theorems of transversality and compactness. Moreover we study the moduli space of holomorphic disks, the Maslov index, and nondegenerate disks.

Definition 23. An almost complex structure (on M^{2n}) is a complex structure on the tangent bundle, $J : TM \to TM$ a differentiable map such that J preserves each fiber and it is linear on each fiber with $J^2 = -\mathbb{1}$

Definition 24.

1. A symplectic form on a smooth manifold M is a closed, non-degenerate 2-form.

2. Let M be a complex manifold with a complex structure J and a compatible Riemannian metric g, the alternating 2-form $\eta(X, Y) = g(JX, Y)$ is called the associated Kähler form. [2]

3. (M, w) is a symplectic manifold with a symplectic form w. An almost complex structure J tame w if $w(\epsilon, J\epsilon) > 0$ for every nonzero tangent vector $\epsilon \in TM$.

A *Kähler* form η over Σ induces a *Kähler* form $\pi_*(w_0)$ over $Sym^g(\Sigma) - D$, where $w_0 = n^{\times g}$, $\pi : \Sigma^{\times g} \to Sym^g(\Sigma)$ the quotient map, and $D \subset Sym^g(\Sigma)$ the diagonal. [29]

Definition 25. An almost complex structure J on $Sym^g(\Sigma)$ is called (j, η, V)-nearly symmetric for a fixed triple (j, η, V), where j is an almost complex structure on Σ, η is a *Kähler* form on Σ, $\{z_i\}_{i=1}^m$ the set of points on Σ which elements are in the connected components disjoint from α and β curves, and V is an open set such that

$$(\{z_i\}_{i=1}^m \times Sym^{g-1}(\Sigma) \cup D) \subset V \subset Sym^g$$

with

$$\overline{V} \cap (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) = \emptyset$$

if J tames $\pi_*(w_0)$ over $Sym^{g-1}(\Sigma) - V$ and $J = Sym^g(j)$ over V.

Let us assume α_i and β_j intersect transversally, then we say that $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ intersect transversally. For an infinite strip $[0, 1] \times i\mathbb{R}$ in the complex plane, which is clearly nonempty and not all of $\mathbb{C}$ can be changed to the unit disk $\mathbb{D}$ in $\mathbb{C}$ by Riemann Mapping theorem. Fix a path of almost complex structures J_s for $s \in [0, 1]$

over $Sym^g(\Sigma)$. For any $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ the *moduli space of homolorhic (pseudo-holomorphic) strips* connecting x and y is defined as follows:

$$\mathcal{M}_{J_s}(x,y) = \left\{ u : \mathbb{D} \to Sym^g(\Sigma) \;\middle|\; \begin{array}{c} u(\{1\} \times \mathbb{R}) \subset \mathbb{T}_\alpha \\ u(\{0\} \times \mathbb{R}) \subset \mathbb{T}_\beta \\ \lim_{t \to -\infty} u(s+it) = x \\ \lim_{t \to +\infty} u(s+it) = y \\ \frac{du}{ds} + J(s)\frac{du}{dt} = 0 \end{array} \right\}$$

For $\phi \in \pi_2(x,y)$, $M_{J_s}(\phi) \subset M_{J_s}(x,y)$ is the space of representatives which are holomorphic. Unit disk can be seen as an infinite strip where $\{1\} \times i\mathbb{R}$ corresponds to e_1 and $\{0\} \times i\mathbb{R}$ corresponds to e_2. Vertical translations preserve e_1 and e_2. So for any holomorphic disk $u \in M_{J_s}(\phi)$, if we precompose u with a vertical translation, it also gives a holomorphic disk. These translations stand for the different parametrizations, then under the $\mathbb{R}$ action, we get the unparametrized moduli space

$$\widehat{\mathcal{M}}_{J_s}(\phi) = \frac{\mathcal{M}_{J_s}(\phi)}{\mathbb{R}}$$

This $\mathbb{R}$ action is a free action, because other than the identity element there is no element of $\mathbb{R}$ fixing any point. However, if $\phi \in \pi_2(x,y)$ connects x to x and $D(\phi) = 0$ then ϕ is a constant map and the action is not free.

Proposition 4.0.10. *For any $u \in \mathcal{M}_{J_s}(\phi)$, $D(u) \geq 0$.*

Proof. We will give a sketch of the proof. $D(u) = \sum_{i=1}^{m} n_{z_i}(\phi) D_i$, we will show that $n_{z_i}(\phi) \geq 0$ for every i which implies $D(u) \geq 0$. Any $u \in \mathcal{M}_{J_s}(\phi)$ is a pseudo-holomorphic disk, so we have an almost complex structure and a canonical orientation. The subvariety V_{z_i} has an almost complex structure being a subspace of $Sym^g(\Sigma)$, so it also has a canonical orientation. The intersection sign will be $+1$ for every intersection point if the frame obtained from u and V_{z_i} matches with the frame of $Sym^g(\Sigma)$, as $dimSym^g(\Sigma) = dimu + dimV_{z_i}$ [22]. By the canonical orientation if u and V_{z_i} intersect they intersect non-negatively. Thus $D(u) \geq 0$ finishing the proof. □

Definition 26. The dimension of the moduli space $\mathcal{M}_{J_s}(\phi)$ is called the Maslov index and denoted by $\mu(\phi)$.

Remark 4.0.11. The dimension of the unparametrized moduli space $\widehat{\mathcal{M}}_{J_s}(\phi)$ is $\mu(\phi) - 1$.

Proposition 4.0.12. *The Maslov index has the following useful properties:*

1. *Let $S \in \pi_2'(Sym^g(\Sigma))$ be the positive generator. Then for any $\phi \in \pi_2(x,y)$,*

$$\mu(\phi + k[S]) = \mu(\phi) + 2k$$

2. *The Maslov index additive:* $\mu(\phi_1 * \phi_2) = \mu(\phi_1) + \mu(\phi_2)$.

The proof of the first statement is in [29] and of the second one is in [8].

Remark 4.0.13. If we add a topological sphere to a disk it changes the Maslov index by at least two.

$$\mu([Z]) = 2\langle c_1, [Z]\rangle$$

for the generator S, $\mu([S]) = 2$ as $\langle c_1, [S]\rangle = 1$, [8]. Moreover, let $\phi \in \pi_2(x, x)$ be the homotopy class of the constant map then the moduli space $\mathcal{M}_{J_s}(\phi)$ consists of single element and $\mu(\phi) = 0$ then $\mu(\phi + k[S]) = 2k$.

Let us now state the first important statement for the moduli spaces $\mathcal{M}_{J_s}(x, y)$, the transversality theorem, a proof can be found in [29].

Theorem 4.0.14. *Let (Σ, α, β) be a Heegaard diagram such that α_i meets with β_j transversally and fix (j, η, V). Then for a dense set of paths J_s of (j, η, V)-nearly symmetric almost-complex structures the moduli space $\mathcal{M}_{J_s}(x, y)$ becomes a smooth manifold.*

Remark 4.0.15. In this theorem we need a path J_s of nearly symmetric almost-complex structures rather than one J. However, in some cases we can take $J = Sym^g(j)$ induced from the complex structure j over Σ, so by definition J is nearly symmetric. In that case we can reach the tranversality by putting α and β curves in general position so that the dimensions of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ add up to the dimension of $Sym^g(\Sigma)$. The existence of isotopy of $Sym^g(\Sigma)$ will guarantee that $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ intersect at finitely many points, [22](Theorem 4.25). But we do not take an arbitrary almost-complex structure J on $Sym^g(\Sigma)$ which is (j, η, V)-nearly symmetric.

Definition 27. For $\phi \in \pi_2(x, y)$, we say that the domain $D(\phi)$ is α-injective if all of its multiplicities (i.e., $n_{z_i}(\phi)$) are 0 or 1, if its interior is disjoint from each α_i for all i and the boundary contains intervals from each α_i.

We can reach the transversality theorem for a constant nearly symmetric-almost complex structure as follows.

Theorem 4.0.16. *For an α-injective homotopy class $\phi \in \pi_2(x, y)$, fix a complex structure j on Σ inducing a complex structure $Sym^g(j)$ on $Sym^g(\Sigma)$. For a generic perturbations of the α curves, the moduli space $\mathcal{M}(\phi)$ is a smooth manifold.*

A proof of the statement can be found in [29].

For each path J_s the moduli space $\mathcal{M}_{\mathcal{J}_f}(\phi)$ for $\phi \in \pi_2(x, y)$ is a manifold and it has an orientation. However the orientation of each moduli space should be consistent, thus we define coherent orientation systems on moduli spaces.

Definition 28 ([29]). Fix a $Spin^c$ structure $s \in Spin^c(Y)$, then a coherent system of orientations is a choice of nonvanishing sections $o(\phi)$ of the determinant line bundle for every $\phi \in \pi_2(x, y)$ and for each $x, y \in \mathcal{S} = \{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta | s_z(x) = s\}$ such that the sections are compatible with gluing as $o(\phi_1) \wedge o(\phi_2) = o(\phi_1 * \phi_2)$ under the identification coming from splicing, and $o(u * S) = o(u)$ under the identification coming from the canonical orientation for the moduli space of holomorphic spheres.

The more detailed description of the coherent system of orientations and the reason why we take sections of the determinant line bundle can be found in [29, 36]. Note that when we define Whitney disks as a map from the unit disk $\mathbb{D} \subset \mathbb{C}$ to $Sym^g(\Sigma)$, we fix an orientation on the disk and the convention is how we described it.

Definition 29. Take a nearly symmetric almost-complex structure J over $Sym^g(\Sigma)$ then for every intersection point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, the moduli space of α- degenerate disks is defined as:

$$\mathcal{N}_J(x) = \left\{ u : [0, \infty) \times \mathbb{R} \to Sym^g(\Sigma) \;\middle|\; \begin{array}{c} u(\{0\} \times \mathbb{R}) \subset \mathbb{T}_\alpha \\ \lim_{z \to \infty} u(z) = x \\ \frac{du}{ds} + J\frac{du}{dt} = 0 \end{array} \right\}$$

We can interpret $\mathcal{N}_J(x)$ as the moduli space of J-holomorphic disks such that the boundary $\partial\mathbb{D}$ is mapped into $\mathbb{T}_\alpha$ and i is mapped to x. Similar to the Whitney disks, for any $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, let $\pi_2(x)$ denote the homotopy classes of maps. As $\mathbb{T}_\alpha$ is embedded in $Sym^g(\Sigma)$, $\pi_1(\mathbb{T}_\alpha) \to \pi_1(Sym^g(\Sigma))$ is injective. The map $\varphi : \pi_2(x) \to \mathbb{Z}$ sending $u \mapsto n_z(u)$, the algebraic intersection number of u with V_z turns into an isomorphism [29]. Let $\mathcal{O}_x \in \pi_2(x)$ denote the generator of the set then any other $u \in \pi_2(x)$ is of the form $u = \mathcal{O}_x + k[S]$, similar to $\pi_2(x, y)$ and isomorphic to $\mathbb{Z}$.

For any $u \in \mathcal{N}_J(x)$, real dilation gives a reparametrization of u, similarly pure imaginary translation also gives a different reparametrization. Then the unparametrized moduli space $\widehat{\mathcal{N}}_J(x)$ is obtained by taking the quotient of $\mathcal{N}_J(x)$ with this 2-dimensional action.

For the moduli space of α-degenerate disks we have the smoothness as follows.

Theorem 4.0.17. *Let $x \in \mathbb{T}_\alpha$ be such that it is not in any $Sym^g(j)$-holomorphic spheres in $Sym^g(\Sigma)$. Then there exists a contractible neighborhood U of $Sym^g(j)$ in the space of (j, η, V)-nearly symmetric almost-complex structures $\mathcal{J}(j, \eta, V)$ such that for a generic $J \in U$, the moduli space $\widehat{\mathcal{N}}_J(\mathcal{O}_x + k[S])$ is compact and 0-dimensional smooth manifold.*

Theorem 4.0.18. *Take a finite set of points $\{x_i\} \subset Sym^g(\Sigma)$ and a complex structure j over Σ such that $Sym^g(j)$-holomorphic spheres misses the set $\{x_i\}$. Then there exists a contractible neighborhood of $Sym^g(j)$, $U \subset \mathcal{J}(j, \eta, V)$ such that for a generic $J \in U$, the total signed number of points in $\widehat{\mathcal{N}}_J(\mathcal{O}_x + k[S])$ is zero.*

The proofs of the last two important theorems can be found in the main article [29]. In the last statement such a complex structure j over Σ for which there exists a $Sym^g(j)$-holomorphic sphere containing at least one of the x_i is real codimension 2, see [29], thus we have sufficient complex structures over Σ which does not contain any of these points. These last two theorems are important for boundary degenerations while proving the map we define becomes a boundary map in the Heegaard Floer chain complex.

The second most important theorem of this chapter is compactness of the unparametrized moduli space $\widehat{\mathcal{M}}(\phi)$.

Theorem 4.0.19. *Take a Heegaard diagram (Σ, α, β) such that the α and β curves are in general position, then for any generic path J_s of nearly symmetric almost-complex structures there is no nonconstant J_s-holomorphic disk u such that $\mu(u) \leq 0$ and for every $\phi \in \pi_2(x, y)$ with $\mu(\phi) = 1$, the unparametrized moduli space $\widehat{\mathcal{M}}(\phi) = \frac{\mathcal{M}(\phi)}{\mathbb{R}}$ is compact and zero dimensional manifold.*

Proof. We will give a sketch of the proof. First part of the statement that $\widehat{\mathcal{M}}(\phi)$ is a manifold follows from the transversality theorem (4.0.14). With the energy bound in [29] we obtain the compactness via the Gromov Compactness theorem, see [11]. By [29], it says that a sequence of J-holomorphic curves with bounded energy has a convergent subsequence whose limit is a union of J-holomorphic curves α and β_j where α is a holomorphic curve and β_j's are finite number of bubbles attached to the curve at a point. By a bubble we mean a J-holomorphic sphere having a transverse intersection with the rest of the curve α. Then by the compactness theorem, a sequence of holomorphic disks in the moduli space $\mathcal{M}(\phi)$ converges to broken flow-lines, boundary bubbling, or sphere bubbling, as the bubbles can occur in the interior of the disk or on the boundary. Remember that the Maslov index is additive by the Proposition (4.0.12). Consider the limit of the sequence of disks $\lim_{i \to +\infty} u_i$, and the Maslov index of each u_i is 1. If the limit can be expressed by a broken flow line namely $\phi = \phi_1 * \phi_2$ then $\mu(\phi)$ becomes 2 which can not be the case. Similarly sphere bubbling off and the boundary bubbling increase the Maslov index by at least two, this also can not occur. Therefore the sequence converges to a disk implying $\widehat{\mathcal{M}}(\phi)$ is compact. □

Chapter 5

HEEGAARD FLOER HOMOLOGY GROUPS

So far we have defined all the necessary tools that we need and now we are ready to give the definition of the Heegaard Floer homology groups. The techniques come from the Lagrangian Floer homology but in a little bit different way. The totally real submanifolds $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ become Lagrangian submanifolds in the symmetric product space $Sym^g(\Sigma)$ by fixing some auxiliary data, so it can be thought as we are studying the Lagrangian Floer Homology on $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$.

We will proceed as follows. We divide the construction into two steps with respect to the first betti numberof the three-manifold we study: first the case $b_1(Y) = 0$ and then the case $b_1(Y) > 0$. Then we give definitions of $\widehat{CF}$, CF^∞ with the subcomplex CF^- and the quotient complex CF^+ with the corresponding boundary maps, and we prove that with these maps they become chain complexes whose homology groups are $\widehat{HF}, HF^\infty, HF^-, HF^+$ respectively. We continue with some immediate properties and give some trivial examples.

5.1 *The Definition of* $\widehat{HF}$ *and* HF^∞ *when* $b_1(Y) = 0$

First we define the chain complex when $b_1(Y) = 0$ and in the next section we define the chain complex when $b_1(Y) > 0$. By using the Universal Coefficient theorem if $b_1(Y) = 0$ then Y is a rational homology three-sphere. In this case $\pi_2(x, y) \simeq \mathbb{Z}$ and there is no periodic domains. However when $b_1(Y) > 0$, by the Theorem (3.3.4) it is $\pi_2(x, y) \simeq \mathbb{Z} \oplus H^1(Y; \mathbb{Z})$. By the presence of periodic domains we consider some special Heegaard diagrams for compactness of the moduli spaces and relative $\mathbb{Z}$ grading is well-defined modulo an indeterminacy. Therefore we separate into two cases based on the first betti number of the manifold.

We need to fix some auxiliary data as follows:

- A pointed Heegaard diagram $(\Sigma, \alpha, \beta, z)$ for Y with genus at least 1, and α and β curves are in general position so that $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ intersect at finitely many points.

- Choose a $Spin^c$ structure $s \in Spin^c(Y)$ and let $\mathcal{S} = \{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta | s_z(x) = s\}$

- Choose a coherent orientation system, o.

- A generic complex structure j over Σ so that every intersection point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is disjoint from $Sym^g(j)$-holomorphic spheres in $Sym^g(\Sigma)$. Note that the j's for which $Sym^g(j)$-holomorphic spheres containing at least one such point has real codimension 2.

- A generic path J_s of nearly symmetric almost-complex structures over $Sym^g(\Sigma)$ contained in an contractible neighborhood U of $Sym^g(j)$.

Let us define $\widehat{CF}(\alpha, \beta, s)$ as a free Abelian group which is generated by the intersection points $x \in \mathcal{S} \subset \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. We give a relative grading on $\widehat{CF}(\alpha, \beta, s)$.

Definition 30. An Abelian group is called relatively graded if it is generated by the elements partitioned into equivalence classes $\mathcal{S}$ with a relative grading function $gr : \mathcal{S} \times \mathcal{S} \to \mathbb{Z}$ such that for every $x, y, w \in \mathcal{S}$

$$gr(x, y) + gr(y, w) = gr(x, w)$$

The relative grading on $\widehat{CF}(\alpha, \beta, s)$ is given as

$$gr(x, y) = \mu(\phi) - 2n_z(\phi) \tag{5.1}$$

for any $\phi \in \pi_2(x, y)$.

First let us verify that it is a relative grading and then show that this grading is well-defined.

Take $x, y, w \in \mathcal{S} \subset \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\phi \in \pi_2(x, y)$ and $\psi \in \pi_2(y, w)$ then $\phi * \psi$ is a disk connecting x and w so that $\phi * \psi \in \pi_2(x, w)$.

$$\begin{aligned} gr(x, y) + gr(y, w) &= \mu(\phi) - 2n_z(\phi) + \mu(\psi) - 2n_z(\psi) \\ &= \mu(\phi * \psi) - 2n_z(\phi * \psi) \\ &= gr(x, w) \end{aligned}$$

Remember that the Maslov index is additive and we showed that the intersection number is also additive, therefore the grading defined is a relative grading.

Proposition 5.1.1. *The relative grading defined above on $\widehat{CF}(\alpha, \beta, s)$ is independent of the choice of the homotopy class of the Whitney disk $\phi \in \pi_2(x, y)$ and the representative of ϕ.*

Proof. By the definition of $\mu(\phi)$ it is independent of the representative of ϕ and in Section 3.3 we showed that the intersection number $n_z(\phi)$ is also independent of the representative of ϕ. Thus, the relative grading is independent of the representative of the homotopy class.

As $\pi_2(x,y) \simeq \mathbb{Z} \oplus H^1(Y;\mathbb{Z})$ by the theorem (3.3.4) and Y is a rational homology 3-sphere, we have $\pi_2(x,y) \simeq \mathbb{Z}$. Thus for $\phi \in \pi_2(x,y)$, any other homotopy class ϕ' is of the form $\phi * k[S]$ where S is positive generator of ${\pi_2}'(Sym^g(\Sigma))$, then

$$\begin{aligned}\mu(\phi') - 2n_z(\phi') &= \mu(\phi * k[S]) - 2n_z(\phi * k[S]) \\ &= \mu(\phi) + 2k\mu([S]) - 2n_z(\phi) - 2kn_z([S]) \\ &= \mu(\phi) - 2n_z(\phi) + 2k - 2k \\ &= \mu(\phi) - 2n_z(\phi)\end{aligned}$$

Therefore the relative grading is independent of the chosen $\phi \in \pi_2(x,y)$. $\square$

Let us define the differential map on the generators when the intersection number is zero as follows:

$$\partial : \widehat{CF}(\alpha,\beta,s) \to \widehat{CF}(\alpha,\beta,s)$$

sending

$$\partial x = \sum_{\{y\in\mathcal{S}|gr(x,y)=1\}} \#(\widehat{\mathcal{M}_0}(x,y)).y$$

where $\widehat{\mathcal{M}_0}(x,y) = \widehat{\mathcal{M}}(\phi)$ for $\phi \in \pi_2(x,y)$ with $n_z(\phi) = 0$ and $\mu(\phi) = 1$.

This is actually a double sum as

$$\partial x = \sum_{y\in\mathcal{S}} \sum_{\{\phi\in\pi_2(x,y)|n_z(\phi)=0,\mu(\phi)=1\}} \#(\widehat{\mathcal{M}}(\phi)).y$$

Y is a rational 3-sphere, therefore $\pi_2(x,y) \simeq \mathbb{Z}\oplus H^1(Y;\mathbb{Z}) \simeq \mathbb{Z}$. Any other homotopy class ϕ' is of the form $\phi * k[S]$ where S is positive generator of ${\pi_2}'(Sym^g(\Sigma))$. Thus $\mu(\phi') = \mu(\phi * k[S]) = \mu(\phi) + 2k = 1$ implies $k = 0$ so we can not have a different homotopy class than ϕ. There is at most one homotopy class with $n_z(\phi) = 0$ and $\mu(\phi) = 1$. Actually there is exactly one such homotopy class. If we connect x to y by a path $a \subset \mathbb{T}_\alpha$ and $b \subset \mathbb{T}_\beta$ then $a - b$ gives a loop in $Sym^g(\Sigma)$. Image of $a - b$ in $H_1(Y;\mathbb{Z})$ is defined by $\epsilon(x,y)$. However, $H_1(Y;\mathbb{Z}) = 0$ implies that $\epsilon(x,y) = 0$, so $\pi_2(x,y) \neq \emptyset$, proving there is exactly one such homotopy class.

Note that $\widehat{\mathcal{M}}(\phi)$ is 0-dimensional compact manifold so the boundary map is well-defined and it counts the number of pseudo-holomorphic disks connecting x to y in $Sym^g(\Sigma)$. This counting is a signed count, so it depends on the orientation of the moduli space that we fixed a coherent orientation system in the beginning. We will study how the homology groups depend on the orientation later.

$n_z(\phi) = 0$ is a geometric intersection number because if two holomorphic disks intersect the intersection number is positive, so it can not be zero. Rather than counting holomorphic disks in $Sym^g(\Sigma)$ connecting intersection points, equivalently we can count them in $Sym^g(\Sigma - z)$. As we showed in the proposition (4.0.10) that any $u \in \mathcal{M}_{J_s}(x, y)$ either does not intersect with the subvariety V_z or intersect non-negatively.

Let us show that we actually obtain a chain complex.

Theorem 5.1.2. *If $b_1(Y) = 0$ then $(\widehat{CF}(\alpha, \beta, s), \partial)$ is a chain complex, meaning $\partial^2 = 0$.*

Proof. We need to show that $\partial^2 = 0$. This will follow from the compactification of the 1-dimensional moduli space $\widehat{\mathcal{M}}(\phi)$. From Floer's theory, we need to consider the ends of the moduli space $\widehat{\mathcal{M}}(\phi)$ for $\phi \in \pi_2(x, w)$ with $\mu(\phi) = 2$. By the Gromov compactification, there are three kinds of ends:

1. Ends corresponding to the "broken flow-lines": we have a pair of disks $u \in \mathcal{M}(x, y)$ with $\mu(u) = 1$ and $v \in \mathcal{M}(y, w)$ with $\mu(v) = 1$.

2. Ends corresponding to the "sphere bubbling off": a holomorphic sphere $S \in Sym^g(\Sigma)$ meets with a holomorphic disk $u \in \mathcal{M}(x, w)$.

3. Ends corresponding to the "boundary bubbling": for a $v \in \mathcal{M}(x, w)$ there exists a holomorphic map $u : \mathbb{D} \to Sym^g(\Sigma)$ mapping the boundary into $\mathbb{T}_\alpha$ or $\mathbb{T}_\beta$ meeting with the boundary at a point.

These point out that a sequence of disks in $\mathcal{M}(x, w)$, if it converges, converges to a disk in the same moduli space, or to a broken flow-line, or to a disk with deformation. By a deformation we mean in the limit a "bubble" which is a holomorphic-sphere attached to a disk at one point can occur. This bubble can be in the interior of the disk or on the boundary.

Let us see that multiple degenerations can not occur at the same time. Without loss of generality, assume that we have a broken flow-line and a sphere bubbling off at the same time. As we want to understand $\partial^2 = 0$, grading can differ by 2. However, after a broken flow it decreases by 1 and by a sphere bubbling off it decreases also by 2. Thus grading differs by 3 rather than 2.

Moreover, if $x' \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta - \mathcal{S}$ there is no Whitney disk connecting x to x'. Because $s_z(x') \neq s$ so the difference $s_z(x') - s_z(x) = PD[\epsilon(x, y)] \neq 0$ is nontrivial where the latter is a cohomology class in $H^2(Y; \mathbb{Z})$ so $\epsilon(x, x') \neq 0$ in $H_1(Y; \mathbb{Z})$ implying $\pi_2(x, x') = \emptyset$.

Now let us show that there is no sphere bubbling off and boundary bubbling. Note that we consider the case $n_z(\phi) = 0$, rather than taking spheres from $Sym^g(\Sigma)$ we can take from $Sym^g(\Sigma - z)$. But we showed in the proof of the theorem (3.1.5) that $\pi_2(Sym^g(\Sigma - z))$ is trivial, so there is no nontrivial spheres in $\pi_2(Sym^g(\Sigma - z))$. We have no sphere bubbling off and boundary bubbling which is a degenerate disk whose boundary lies entirely inside $\mathbb{T}_\alpha$ or $\mathbb{T}_\beta$. Therefore only boundary components in the compactification are broken flow-lines. The sum of the ends of the moduli space of $\widehat{\mathcal{M}}(\phi)$ is the following:

$$\sum_{\{\varphi\in\pi_(x,y)|\mu(\varphi)=1\}} \#\widehat{\mathcal{M}}(\varphi) \sum_{\{\psi\in\pi_(y,w)|\mu(\psi)=1\}} \#\widehat{\mathcal{M}}(\psi) = 0 \tag{5.2}$$

where $\phi = \varphi * \psi$. As $\widehat{\mathcal{M}}(\psi)$ and $\widehat{\mathcal{M}}(\varphi)$ are 0-dimensional and compact spaces, they consists of discrete points. Therefore the signed number of those points gives us 0.

Now let us see that how it is related to the boundary operator and show that $\partial^2 = 0$. Apply ∂ on the generators of $\widehat{CF}(\alpha, \beta, s)$

$$\partial^2 x = \partial(\sum_{\{y\in\mathcal{S}|gr(x,y)=1\}} \#(\widehat{\mathcal{M}}_0(x,y)).y)$$

$$= \sum_{\{y\in\mathcal{S}|gr(x,y)=1\}} \#(\widehat{\mathcal{M}}_0(x,y)).y \sum_{\{w\in\mathcal{S}|gr(y,w)=1\}} \#(\widehat{\mathcal{M}}_0(y,w)).w$$

where

$$\widehat{\mathcal{M}}_0(x,y) = \widehat{\mathcal{M}}(\varphi) \text{ for } \varphi \in \pi_2(x,y) \text{ with } n_z(\varphi) = 0 \text{ and } \mu(\varphi) = 1$$
$$\widehat{\mathcal{M}}_0(y,w) = \widehat{\mathcal{M}}(\psi) \text{ for } \psi \in \pi_2(x,y) \text{ with } n_z(\psi) = 0 \text{ and } \mu(\psi) = 1$$

as we mentioned there is exactly one homotopy class with the Maslov index is 1, and

$$\begin{aligned} gr(x,w) &= gr(x,y) + gr(y,w) \\ &= \mu(\varphi) - 2n_z(\varphi) + \mu(\psi) - 2n_z(\psi) \\ &= \mu(\varphi * \psi) - 2n_z(\varphi * \psi) \\ &= \mu(\phi) - 2n_z(\phi) \\ &= 2 \end{aligned}$$

The coefficient in the composition corresponds to the signed count of the ends of the moduli space of $\widehat{\mathcal{M}}(\phi)$ with $\mu(\phi) = 2$, which only have broken flow lines and by the above calculation (5.2)and we showed that it is 0. Thus for every w and y, $\partial^2 x = 0$ for each generator $x \in \mathcal{S}$ implying $\partial^2 = 0$. Hence $(\widehat{CF}(\alpha, \beta, s), \partial)$ is a chain complex. □

Definition 31. The Floer homology groups $\widehat{HF}(\alpha, \beta, s)$ are the homology groups of the chain complex $(\widehat{CF}(\alpha, \beta, s), \partial)$.

Remark 5.1.3. Note that there is a relative grading on the complex $\widehat{CF}(\alpha, \beta, s)$ thus we can not see the homology grading explicitly as r-th chain group $\widehat{CF}^r(\alpha, \beta, s)$ and the r-th chain map ∂^r.

As a first step we consider the case where the intersection number is 0 and the grading difference is 1. Now let us generalize and enrich the subject. Let us define $CF^\infty(\alpha,\beta,s)$ as a free Abelian group generated by $[x,i]$ where $x \in \mathcal{S}$ and $i \in \mathbb{Z}$. Similarly, we give a relative grading to $CF^\infty(\alpha,\beta,s)$ defined on the generators as:

$$gr([x,i],[y,j]) = gr(x,y) + 2i - 2j$$

Let us verify that this is a relative grading. Let $[x,i],[y,j],[w,k]$ be generators of $CF^\infty(\alpha,\beta,s)$. Then

$$\begin{aligned} gr([x,i],[y,j]) + gr([y,j],[w,k]) &= gr(x,y) + 2i - 2j + gr(y,w) + 2j - 2k \\ &= gr(x,y) + gr(y,w) + 2i - 2k \\ &= gr(x,w) + 2i - 2k \\ &= gr([x,i],[w,k]) \end{aligned}$$

Now define the boundary map $\partial^\infty : CF^\infty(\alpha,\beta,s) \to CF^\infty(\alpha,\beta,s)$ as:

$$\partial^\infty[x,i] = \sum_{y\in\mathcal{S}} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} \#(\widehat{\mathcal{M}}(\phi)).[y, i - n_z(\phi)] \tag{5.3}$$

Remember that we consider the case $b_1(Y) = 0$, therefore $\pi_2(x,y) \simeq \mathbb{Z}$. Even though we write the boundary map as a double sum, there is at most one homotopy class in $\pi_2(x,y)$ with $\mu(\phi) = 1$.

Theorem 5.1.4. *For $b_1(Y) = 0$, the pair $(CF^\infty(\alpha,\beta,s),\partial^\infty)$ is a chain complex.*

Proof. We will prove that $(\partial^\infty)^2 = 0$. We will proceed similarly as in the hat homology case. The proof is mainly based on the compactification of 1-dimensional moduli space $\widehat{\mathcal{M}}(\phi)$ with $\mu(\phi) = 2$. By the Gromov compactification "ends" or equivalently the limit of a sequence of disks in $\mathcal{M}(\phi)$ can converge to a broken flow-line or to a disk with a bubble which is either attached to the interiror or to the boundary of the disk. By the same reasoning in the previous proof, we can not have multiple degenerations at once.

A disk $u \in \mathcal{M}(x,w)$ whose boundary lies entirely in $\mathbb{T}_\alpha$ or $\mathbb{T}_\beta$ has a corresponding domain $D(u)$ which is a multiple of the Heegaard surface Σ. u is homologically a sphere in $Sym^g(\Sigma)$ and $\pi_2{}'(Sym^g(\Sigma)) \simeq \mathbb{Z}$ generated by $[S]$ where S is coming from the hyperelliptic involution on Σ such that for a generic basepoint z on Σ disjoint from α and β curves, $[S]$ intersects with the subvariety V_z once so $n_z([S]) = 1$. As $\pi_2{}'(Sym^g(\Sigma))$ is cyclic so u must be a multiple of $[S]$, say l then $n_z(u) = l$.

$$D(u) = \sum_{i=1}^{m} n_{z_i}(u) D_i$$

$n_{z_i}(u)$ is same for each i, so $D(u) = l[\Sigma]$ for some $l \in \mathbb{Z}$.

If u is pseudo-holomorphic then by the proposition (4.0.10) $D(u) \geq 0$ so $l \geq 0$. If $D(u) = 0$ then u must be constant. Because let us take a sequence of disks $u_i \in \pi_2(x, y)$ such that $\mu(u_i) = 2$ for each i, then the limit $\lim_{i\to+\infty} u_i = u_\infty + v$ converges to a disk u_∞ with a possible bubble v. Note that $\mu(\lim_{i\to+\infty} u_i) = 2$ and a holomorphic sphere increases the Maslov index by at least two as $\mu(v) = 2\langle c_1, [v]\rangle$. If we have bubble in the limit then $\lim_{i\to+\infty} u_i = v$ is only a bubble with no disk u_∞, in that case u must be a constant and $x = w$ should be the same points.

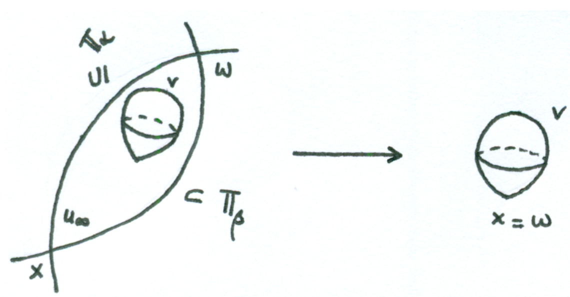


Figure 5.1: Convergence of holomorphic disks to a bubble in the limit

If there is a boundary bubbling it can occur only in the case $x = w$. For $x \neq w$ boundary bubbling is excluded. For $x = w$ there is no sphere bubbling off because in the limit we can only have a bubble not a bubble attached to a disk. Moreover, for generic complex structure j on Σ, $Sym^g(j)$-holomorphic spheres miss the intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ so there is no sphere bubbling off.

For $x \neq w$ there is only broken flow-lines and the signed count of the ends of the moduli space $\mathcal{M}(\phi)$:

$$\sum_{y\in\mathcal{S}} \sum_{\{\varphi\in\pi_2(x,y),\psi\in\pi_2(y,w)|\varphi*\psi=\phi\}} \#(\widehat{\mathcal{M}}(\varphi)).\#(\widehat{\mathcal{M}}(\psi)) = 0$$

with $\varphi * \psi = \phi$ for each homotopy class $\varphi \in \pi_2(x, y)$ and $\psi \in \pi_2(y, w)$. $\widehat{\mathcal{M}}(\varphi)$ and $\widehat{\mathcal{M}}(\psi)$ are both 0-dimensional and compact manifolds, there is only finitely many points therefore the signed count gives us 0.

When $x = w$ in addition to broken flow-lines, there is also boundary bubbling. Therefore the signed count of the ends of the moduli space $\mathcal{M}(\phi)$:

$$\#\widehat{\mathcal{N}}^\alpha(x) + \#\widehat{\mathcal{N}}^\beta(x) + \sum_{y\in\mathcal{S}} \sum_{\{\varphi\in\pi_2(x,y),\psi\in\pi_2(y,w)|\varphi*\psi=\phi\}} \#(\widehat{\mathcal{M}}(\varphi)).\#(\widehat{\mathcal{M}}(\psi)) = 0$$

Via the theorem (4.0.18) the signed count for $\widehat{\mathcal{N}}^{\alpha}(x)$ and $\widehat{\mathcal{N}}^{\beta}(x)$ are both zero. This implies the signed count for the broken flow-lines is also 0.

Let us compute the boundary map on the generators.

$$(\partial^{\infty})^2[x,i] = \partial^{\infty}(\sum_{y\in\mathcal{S}} \sum_{\{\varphi\in\pi_2(x,y)|\mu(\varphi)=1\}} \#(\widehat{\mathcal{M}}(\varphi)).[y, i-n_z(\varphi)])$$

$$= \sum_{y\in\mathcal{S}} \sum_{\{\varphi\in\pi_2(x,y)|\mu(\varphi)=1\}} \sum_{w\in\mathcal{S}} \sum_{\{\psi\in\pi_2(y,w)|\mu(\psi)=1\}} \#(\widehat{\mathcal{M}}(\varphi))\#(\widehat{\mathcal{M}}(\psi)).[w, i-n_z(\varphi)-n_z(\psi)]$$

such that for the homotopy classes $\varphi \in \pi_2(x,y)$ and $\psi \in \pi_2(y,w)$ we have $\varphi * \psi \in \pi_2(x,w)$.

In the discussion of the ends of the moduli space we can only have boundary bubbling which occurs in the case $x = w$. However the signed count of $\widehat{\mathcal{N}}^{\alpha}(x)$ and $\widehat{\mathcal{N}}^{\beta}(x)$ are both zero. Therefore in either of the case $x = w$ or $x \neq w$:

$$\sum_{y\in\mathcal{S}} \sum_{\{\varphi\in\pi_2(x,y),\psi\in\pi_2(y,w)|\varphi*\psi=\phi\}} \#(\widehat{\mathcal{M}}(\varphi))\#(\widehat{\mathcal{M}}(\psi)) = 0$$

implying $(\partial^{\infty})^2[x,i] = 0$ for every generator $[x,i]$. Hence $(\partial^{\infty}) = 0$ proving the pair $(CF^{\infty}(\alpha,\beta,s),\partial^{\infty})$ is a chain complex. □

Definition 32. The Floer Homology groups are the homology groups of the chain complex $(CF^{\infty}(\alpha,\beta,s),\partial^{\infty})$.

There exists a chain map $U : (CF^{\infty}(\alpha,\beta,s) \to (CF^{\infty}(\alpha,\beta,s)$ lowering the grading by two. It is defined on the generators as

$$U[x,i] = [x,i-1]$$

and

$$gr([x,i],[x,i-1]) = gr(x,x) + 2i - 2(i-1) = gr(x,x) + 2 = 2$$

Proposition 5.1.5. *The map U defined above is a chain map. i.e.,* $\partial^{\infty} \circ U = U \circ \partial^{\infty}$ *and the corresponding diagram is commutative.*

Proof. It easily follows as:

$$U \circ \partial^{\infty}[x,i] = U(\sum_{y\in\mathcal{S}} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} \#(\widehat{\mathcal{M}}(\phi)).[y, i-n_z(\phi)]))$$

$$= \sum_{y\in\mathcal{S}} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} \#(\widehat{\mathcal{M}}(\phi)).[y, i-n_z(\phi)-1]))$$

and similarly

$$\partial^\infty \circ U[x,i] = \partial^\infty[x,i-1]$$

$$= \sum_{y\in\mathcal{S}} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} \#(\widehat{\mathcal{M}}(\phi)).[y,i-1-n_z(\phi)])$$

proving the diagram commutes and the map U is a chain map. □

If we restrict the generators $[x,i]$ such that $i<0$ then we obtain the subgroup $CF^-(\alpha,\beta,s)$ of $CF^\infty(\alpha,\beta,s)$ which is also a freely generated Abelian group. As $CF^-(\alpha,\beta,s) \lhd CF^\infty(\alpha,\beta,s)$ we can define the quotient

$$CF^+(\alpha,\beta,s) = CF^\infty(\alpha,\beta,s)/CF^-(\alpha,\beta,s)$$

Proposition 5.1.6. *$CF^-(\alpha,\beta,s)$ is a subcomplex of $CF^\infty(\alpha,\beta,s)$ and there is a short exact sequence of chain complexes as:*

$$0 \longrightarrow CF^-(\alpha,\beta,s) \xrightarrow{i} CF^\infty(\alpha,\beta,s) \xrightarrow{\pi} CF^+(\alpha,\beta,s) \longrightarrow 0$$

where the first map is injection and second map is the projection.

Proof. By definition $CF^-(\alpha,\beta,s)$ is a freely generated Abelian group already. Restrict the differential map ∂^∞ of $CF^\infty(\alpha,\beta,s)$ on $CF^-(\alpha,\beta,s)$ as $\partial^- = \partial^\infty|_{CF^-(\alpha,\beta,s)}$ Then we need to show that $(\partial^-)^2=0$. The proof follows easily from the same arguments used in ∂^∞ is a boundary map. Let us check that the map is well-defined.

$$\partial^-[x,i] = \sum_{y\in\mathcal{S}} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} \#(\widehat{\mathcal{M}}(\phi)).[y,i-n_z(\phi)]$$

By the proposition (4.0.10) $n_z(\phi)\geq 0$ then $i-n_z(\phi)<0$ holds so $[y,i-n_z(\phi)]$ is a generator for $CF^-(\alpha,\beta,s)$ and ∂^- is well-defined map on $CF^-(\alpha,\beta,s)$. Applying the boundary map twice on the generators:

$$(\partial^-)^2[x,i] = \partial^-(\sum_{y\in\mathcal{S}} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} \#(\widehat{\mathcal{M}}(\phi)).[y,i-n_z(\phi)])$$

$$= \sum_{y\in\mathcal{S}} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} \sum_{w\in\mathcal{S}} \sum_{\{\psi\in\pi_2(y,w)|\mu(\psi)=1\}} \#(\widehat{\mathcal{M}}(\phi))\#(\widehat{\mathcal{M}}(\psi)).[w,i-n_z(\phi)-n_z(\psi)]$$

By the nonnegativity of $n_z(\phi)$ and $n_z(\psi)$, $i-n_z(\phi)-n_z(\psi)<0$ follows. By the ends of the 1-dimensional moduli space argument as in the proof of the theorem (5.1.4) we obtain $(\partial^-)^2=0$. Therefore, $(CF^-(\alpha,\beta,s),\partial^-)$ is a subcomplex of $(CF^\infty(\alpha,\beta,s),\partial^\infty)$.

The second part of the statement follow from that the first map is injective and second map is projection. $Im(i) = CF^-(\alpha, \beta, s) = Ker(\pi : CF^\infty(\alpha, \beta, s) \to CF^\infty(\alpha, \beta, s)/CF^-(\alpha, \beta, s)$. □

Remark 5.1.7. If we restrict the chain map $U : CF^\infty(\alpha, \beta, s) \to CF^\infty(\alpha, \beta, s)$ to $CF^-(\alpha, \beta, s)$ it will give a morphism from $CF^-(\alpha, \beta, s)$ to itself lowering the grading by 2. Similarly it also induces a self morphism on the quotient complex $CF^+(\alpha, \beta, s)$. Let us denote this action U on $CF^-(\alpha, \beta, s)$ (respectively $CF^+(\alpha, \beta, s)$) by U^- (respectively U^+).

Proposition 5.1.8. *Let us define a map* $\iota : \widehat{CF}(\alpha, \beta, s) \to CF^+(\alpha, \beta, s)$ *on the generators by:*

$$\iota(x) = [x, 0].$$

Then there exists a short exact sequence of chain complexes:

$$0 \longrightarrow \widehat{CF}(\alpha, \beta, s) \xrightarrow{\iota} CF^+(\alpha, \beta, s) \xrightarrow{U^+} CF^+(\alpha, \beta, s) \longrightarrow 0$$

Proof. It suffices to show that $Im(\iota) = Ker(U^+)$.

$$Im(\iota) = \{[x, 0] | x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, s_z(x) = s\}$$

Under the map U, $U[x, 0] = [x, -1]$, but in the quotient it is 0 so $U^+[x, 0] = 0$. Thus $Im(\iota) \subset Ker(U^+)$. Conversely, let $[x, i] \in Ker(U^+)$, so $[x, i] \in CF^+(\alpha, \beta, s)$ and i can not be negative, as the map U^+ maps $[x, i]$ to 0 this forces $i = 0$. $Ker(U^+)$ contains elements of the form $[x, 0]$ where $x \in \mathcal{S}$, then $Ker(U^+) \subset Im(\iota)$ implying $Im(\iota) = Ker(U^+)$. Hence we have short exact sequence of chain complexes and we can imbed $\widehat{CF}(\alpha, \beta, s)$ into $CF^+(\alpha, \beta, s)$ and the U action on $\widehat{CF}(\alpha, \beta, s)$ is taken to be trivial as a convention. □

Definition 33. The Floer homology groups $HF^\infty(\alpha, \beta, s), HF^-(\alpha, \beta, s), HF^+(\alpha, \beta, s)$ are the homology groups of the chain complexes $CF^\infty(\alpha, \beta, s), CF^-(\alpha, \beta, s)$, and $CF^+(\alpha, \beta, s)$ respectively. By the U-action the homology groups become $\mathbb{Z}[U]$-modules.

Remark 5.1.9. The short exact sequence of chain complexes in the proposition (5.1.6) induces a long exact homology sequence.

$$\cdots \longrightarrow HF^-(\alpha, \beta, s) \xrightarrow{i_*} HF^\infty(\alpha, \beta, s) \xrightarrow{\pi_*} HF^+(\alpha, \beta, s) \longrightarrow \cdots$$

5.2 *The Definitions of Heegaard Floer Homology Groups When* $b_1(Y) > 0$

The first betti number, $b_1(Y)$ is the rank of the Abelian group $H_1(Y)$ or equivalently the vector space dimension of $H_1(Y; \mathbb{Q})$. In Sections 5.1 we defined the Heegaard Floer Homology groups when $H_1(Y; \mathbb{Q}) = 0$. Now we will study the case $b_1(Y) > 0$, i.e., when the rank of $H_1(Y; \mathbb{Q})$ is nontrivial. We give relative grading on the complex which will differ slightly from Equation (5.1), we consider special kind of Heegaard diagrams to obtain the compactness of the moduli spaces $\mathcal{M}(\phi)$ with $\mu(\phi) = 1$, and then we define the chain complex.

Let $s \in Spin^c(Y)$ be a $Spin^c$ structure and $\mathcal{S} = \{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta | s_z(x) = s\}$. The relative grading we defined in Equation (5.1) is true modulo some indeterminacy which is given as

$$\varrho(s) = \underset{\epsilon \in H_2(Y;\mathbb{Z})}{gcd} \langle c_1(s), \epsilon \rangle \tag{5.4}$$

This definition makes sense, as $c_1(s) \in H^2(Y;\mathbb{Z})$ then we can evaluate a second cohomology class on second homology class.

Remark 5.2.1. When $b_1(Y) = 0$, Y is a rational homology 3-sphere. $H_2(Y;\mathbb{Z})$ is trivial so there is no indeterminacy and in this case there exists $\mathbb{Z}$ grading rather that $\mathbb{Z}/\varrho(s)$.

We use the compactness of moduli spaces $\mathcal{M}(\phi)$ with $\mu(\phi) = 1$ in order to have this property, we use special Heegaard diagrams.

Definition 34. Take $s \in Spin^c(Y)$ then

1. A pointed Heegaard diagram $(\Sigma, \alpha, \beta, z)$ is called strongly admissible for s if every nontrivial periodic domain D with $\langle c_1(s), H(D) \rangle = 2n \geq 0$ has some coefficient $> n$.

2. A pointed Heegaard diagram $(\Sigma, \alpha, \beta, z)$ is called weakly admissible for s if every nontrivial periodic domain D with $\langle c_1(s), H(D) \rangle = 0$ has both positive and negative coefficients.

Proposition 5.2.2. *We have the following two properties:*

1. *If $s \in Spin^c(Y)$ is torsion $Spin^c$ structure then the strong and weak admissibility coincide.*

2. *If a Heegaard diagram is strongly admissible for any torsion $Spin^c$ structure, then it is weakly admissible for every $Spin^c$ structure.*

Proof. If $s \in Spin^c(Y)$ is torsion then for any nontrivial periodic domain D then $\langle c_1(s), H(D) \rangle = 0$ where $H(D)$ is the corresponding homology class in $H_2(Y;\mathbb{Z})$ for D. Then:

1. If s is torsion, then by definition a pointed Heegaard diagram is strongly admissible and weakly admissible at the same time. Thus for torsion $Spin^c$ structures weak and strong admissibility conditions coincide.

2. Suppose that a Heegaard diagram is strongly admissible for any torsion $Spin^c$ structure, say for s. Then for any nontrivial periodic domain D, $\langle c_1(s), H(D)\rangle = 0$ then D has both positive and negative coefficients. This says every nontrivial periodic domain D has both positive and negative coefficients implying that the Heegaard diagram is weakly s-admissible for any $Spin^c$ structures.

□

Theorem 5.2.3. *If a pointed Heegaard diagram $(\Sigma, \alpha, \beta, z)$ is weakly admissible for a fixed $Spin^c$ structure s, then for fixed $j, k \in \mathbb{Z}$ and for each $x, y \in \mathcal{S}$ there are finitely many homotopy classes $\phi \in \pi_2(x, y)$ with $\mu(\phi) = j$, $n_z(\phi) = k$, and $D(\phi) \geq 0$.*

Proof. Take some $\varphi \in \pi_2(x, y)$ with $\mu(\phi) = j$. Any other $\phi \in \pi_2(x, y)$ with $\mu(\phi) = j$ is of the form

$$\phi = \varphi + P_x - \frac{\langle c_1(s), H(P)\rangle}{2}[S] \tag{5.5}$$

where $P_x \in \prod_x$ is a periodic class, P is the domain associated to P_x, and S is the positive generator of $\pi_2{}'(Sym^g(\Sigma))$. This is because $\pi_2(x, y) \simeq \mathbb{Z} \oplus H^1(Y; \mathbb{Z})$ by the theorem (3.3.4). This isomorphism is not canonical and two elements $\varphi, \phi \in \pi_2(x, y)$ differ by a constant in $\mathbb{Z}$ and a cohomology class as

$$\phi = \varphi + P + k[S] \text{ for some } k \in \mathbb{Z}$$

where S is the positive generator of $\pi_2{}'(Sym^g(\Sigma))$. As we require $\mu(\phi) = \mu(\varphi)$ this forces k in the equation to be as in (5.5).

Let us verify that $\mu(\phi) = \mu(\varphi)$:

$$\begin{aligned}
\mu(\phi) &= \mu(\varphi) + P_x - \frac{\langle c_1(s), H(P)\rangle}{2}[S]) \\
&= \mu(\varphi) + \mu(P_x) - \mu(\frac{\langle c_1(s), H(P)\rangle}{2}[S]) \\
&= \mu(\varphi) + \mu(P_x) - \frac{\langle c_1(s), H(P)\rangle}{2} \\
&= \mu(\varphi)
\end{aligned}$$

the last equality follows from the Theorem 4.9 of [29] which says that the Maslov index of each periodic class $\phi \in \prod_x$, $\mu(\phi) = \langle c_1(s), H(\phi)\rangle$, where $H(\phi)$ is the homology class in $H_2(Y; \mathbb{Z})$ corresponding to ϕ (remember the $1-1$ correspondence between periodic classes and $H^1(Y; \mathbb{Z})$ via Theorem (3.3.4)).

If in addition $n_z(\phi) = n_z(\varphi)$ observe that:

$$\begin{aligned} n_z(\varphi) &= n_z(\phi) \\ &= n_z(\varphi + P_x - \frac{\langle c_1(s), H(P)\rangle}{2}[S]) \\ &= n_z(\varphi) + n_z(P_x) - \frac{\langle c_1(s), H(P)\rangle}{2} \end{aligned}$$

as $n_z(P_x) = 0$ this forces $\langle c_1(s), H(P)\rangle = 0$. Then for the associated domain of ϕ and φ we have

$$D(\phi) = D(\varphi) + P$$

where P is a periodic domain such that $\langle c_1(s), H(P)\rangle = 0$.

We also want $D(\phi) \geq 0$, so $D(\phi) = D(\varphi) + P \geq 0$ then $P \geq -d(\varphi)$. It turn out now we need to show that for a fixed $\varphi \in \pi_2(x, y)$ there is only finitely many periodic domains P in

$$\mathcal{Q} = \{P \in \textstyle\prod_x |\langle c_1(s), H(P)\rangle = 0, P \geq -D(\varphi)\}$$

Assume that there are infinitely many periodic domains in $\mathcal{Q}$. Every periodic domain P is of the form $P = \sum_{i=1}^{m} p_i D_i$ where $\{D_i\}_{i=1}^m$ components of $\Sigma - \alpha - \beta$. As every periodic domain can be expressed as a linear combination of D_i, each elements can be thought as an element in an m-dimensional vector space with basis $\{D_i\}_{i=1}^m$. Then $\mathcal{Q}$ becomes the set of lattice points in this space. $\mathcal{Q}$ has infinitely many points implies that it is unbounded. We can define a Euclidean norm for each periodic domain $P = \sum_{i=1}^{m} p_i D_i$ as

$$\parallel P \parallel = (\sum_{i=1}^{m} |p_i|^2)^{1/2}$$

$\mathcal{Q}$ is unbounded so there exists a sequence $\{P_i\}$ in $\mathcal{Q}$ such that $\parallel P_i \parallel \to \infty$, otherwise $\mathcal{Q}$ can not have infinitely many points. However the sequence $\frac{P_i}{\|P_i\|}$ converges to a unit vector in the space of periodic domains. This sequence has a subsequence whose limit is a unit vector $\widetilde{P} = \sum_{i=1}^{m} p_j D_j$ such that $p_j \in \mathbb{R}$ and $\langle c_1(s), H(P)\rangle = 0$. Note that for every periodic domain in $\mathcal{Q}$ as $P \geq -D(\varphi)$, there is a lower bound for the coefficients and $\{P_i\}$ is divergent implies that each coefficients of $\widetilde{P}$ is nonnegative. The subspace of periodic domains in $H_2(Y; \mathbb{Z})$ for which $\langle c_1(s), H(P)\rangle = 0$ with nonnegative multiplicities has a nontrivial real vector, then it should have a rational vector also. We can obtain a periodic domain $\overline{P}$ with nonnegative integer coefficients if we cancel the denominators such that $\langle c_1(s), H(\overline{P})\rangle = 0$. This result contradicts with the hypothesis that the Heegaard diagram is weakly admissible. Therefore, $\mathcal{Q}$ has finitely many lattice points. As a result for a given $j, k \in \mathbb{Z}$ there is only finitely many homotopy classes $\phi \in \pi_2(x, y)$ with $\mu(\phi) = j$ and $n_z(\phi) = k$ satisfying $D(\phi) \geq 0$. □

We also have a similar statement for strongly admissible pointed Heegaard diagrams.

Theorem 5.2.4. *If a pointed Heegaard diagram $(\Sigma, \alpha, \beta, z)$ is strongly admissible for a fixed $s \in Spin^c(Y)$, then for a fixed $j \in \mathbb{Z}$ there exists finitely many homotopy classes $\phi \in \pi_2(x, y)$ with $\mu(\phi) = j$ and $D(\phi) \geq 0$.*

Proof. Take $\varphi \in \pi_2(x, y)$ with $\mu(\phi) = j$ then any other $\phi \in \pi_2(x, y)$ with $\mu(\phi) = j$ is of the form

$$\phi = \varphi - P_x + \tfrac{\langle c_1(s), H(P)\rangle}{2}[S]$$

as in (5.5) in the previous theorem. In the statement there is no restriction about the intersection numbers $n_z(\phi)$ and $n_z(\varphi)$.

$$\begin{aligned} n_z(\phi) &= n_z(\varphi - P_x + \frac{\langle c_1(s), H(P)\rangle}{2}[S]) \\ &= n_z(\varphi) + \frac{\langle c_1(s), H(P)\rangle}{2} \end{aligned}$$

then the corresponding domain of ϕ:

$$\begin{aligned} D(\phi) &= \sum_{i=1}^{m} n_{z_i}(\phi) D_i \\ &= \sum_{i=1}^{m} [n_{z_i}(\varphi) + \frac{\langle c_1(s), H(P)\rangle}{2}] D_i \\ &= D(\varphi) + \frac{\langle c_1(s), H(P)\rangle}{2} D_i[\Sigma] - P \end{aligned}$$

for some periodic domain P. We also require $D(\phi) \geq 0$ this forces

$$-P + \frac{\langle c_1(s), H(P)\rangle}{2}[\Sigma] \geq -D(\varphi) \tag{5.6}$$

Now as in the weak admissibility case we need to show that for each $\varphi \in \pi_2(x, y)$ with $\mu(\phi) = j$

$$\mathcal{Q} = \{P \in \textstyle\prod_x \mid -P + \frac{\langle c_1(s), H(P)\rangle}{2}[\Sigma] \geq -D(\varphi)\}$$

has finitely many elements. By the same reasoning as in the previous theorem if there exists finitely many elements in $\mathcal{Q}$, we can obtain a periodic domain P with real coefficients such that $\frac{\langle c_1(s), H(P)\rangle}{2}[\Sigma] \geq 0$. This will imply there is also a periodic domain satisfying the equation (5.6) with rational coefficients. From this it is easy to obtain a periodic domain P with integral coefficients satisfying (5.6). But then for $\langle c_1(s), H(P)\rangle = 2n \geq 0$, P can not have a coefficient grater than n. This contradicts with the stong admissibility of the pointed Heegaard diagram. Therefore there are finitely many elements in $\mathcal{Q}$ and for fixed $j \in \mathbb{Z}$, and there exists finitely many homotopy classes $\phi \in \pi_2(x, y)$ with $\mu(\phi) = j$ and $D(\phi) \geq 0$. □

These two theorems are the main reasons for defining the admissibility condition. We will use admissible diagrams when $b_1(Y) > 0$ to prove the boundary map is a finite sum so there is no convergence issue, and it will give a chain map.

Remark 5.2.5. We have introduced the admissibility criteria for the pointed Heegaard diagrams and derived some important results. But the question in mind should be is there admissible Heegaard diagrams. We will turn to this question later in Section 6.4.

Let us define the chain complexes for $b_1(Y) > 0$. For a 3-manifold Y with $b_1(Y) > 0$, fix a pointed Heegaard diagram $(\Sigma, \alpha, \beta, z)$, a $Spin^c$ structure $s \in Spin^c(Y)$, a coherent orientation system o, and assume that the Heegaard diagram is strongly s-admissible.

We will define $\widehat{CF}(\alpha, \beta, s, o)$ and $CF^\infty(\alpha, \beta, s, o)$ as in Section 5.1 with the same boundary maps.

$\widehat{CF}(\alpha, \beta, s, o)$ is a free Abelian group generated by the elements $x \in \mathcal{S} = \{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta | s_z(x) = s\}$ and $CF^\infty(\alpha, \beta, s, o)$ is a free Abelian group generated by the elements $[x, i]$ with $x \in \mathcal{S}$ and $i \in \mathbb{Z}$. It has a subgroup $CF^-(\alpha, \beta, s, o)$ generated by $[x, i]$ with $i < 0$ and the quotient group $CF^+(\alpha, \beta, s, o)$. There is a relative grading on $\widehat{CF}(\alpha, \beta, s, o)$ given as in the equation (5.1) as $gr(x, y) = \mu(\phi) - 2n_z(\phi)$ for any $\phi \in \pi_2(x, y)$. When $b_1(Y) > 0$ this turns out to a $\mathbb{Z}/\varrho(s)$-grading with the indeterminacy $\varrho(s)$ given by the equation(5.4).

Differential map on $CF^\infty(\alpha, \beta, s, o)$ is same as (5.3)

$$\partial^\infty[x, i] = \sum_{y \in \mathcal{S}} \sum_{\{\phi \in \pi_2(x,y) | \mu(\phi)=1\}} \#(\widehat{\mathcal{M}}(\phi)) \cdot [y, i - n_z(\phi)]$$

an it induces a differential map on the quotient complex as:

$$\partial^+[x, i] = \sum_{\{y \in \mathcal{S}} \sum_{\phi \in \pi_2(x,y) | \mu(\phi)=1, i \geq n_z(\phi)\}} \#(\widehat{\mathcal{M}}(\phi)) \cdot [y, i - n_z(\phi)] \tag{5.7}$$

This map is well-defined. Note that for $i \geq n_z(\phi)$, $i - n_z(\phi) \geq 0$. Otherwise if $i - n_Z(\phi) < 0$ then as $[y, i - n_Z(\phi)]$ is a generator of $CF^-(\alpha, \beta, s, o)$, so the map makes sense. Moreover, the Heegaard diagram is strongly s-admissible then for fixed $\mu(\phi) = 1$ there exists finitely many homotopy classes $\phi \in \pi_2(x, y)$ via the theorem (5.2.4), then both maps ∂^∞ and ∂^+ are finite sums.

Let us state the main theorem in this section and see how the admissibility criteria is involved.

Theorem 5.2.6. *Let Y be a 3-manifold with $b_1(Y) > 0$ and fix a $Spin^c$ structure $s \in Spin^c(Y)$. Then,*

1. *If the pointed Heegaard diagram $(\Sigma, \alpha, \beta, z)$ for Y is strongly s-admissible then the pair $(CF^\infty(\alpha, \beta, s, o), \partial^\infty)$ is a chain complex with a subcomplex $CF^-(\alpha, \beta, s, o)$ and the quotient complex $CF^+(\alpha, \beta, s, o)$.*

2. *If the pointed Heegaard diagram $(\Sigma, \alpha, \beta, z)$ for Y is weakly s-admissible then the pair $(CF^+(\alpha, \beta, s, o), \partial^+)$ is a chain complex with a subcomplex $\widehat{CF}(\alpha, \beta, s, o)$.*

Proof. We use the admissibility criterion to show that the boundary maps ∂ for $\widehat{CF}$, ∂^∞ for CF^∞, ∂^- for CF^-, and ∂^+ for CF^+ are finite sums.

1. If the pointed Heegaard diagram is strongly s-admissible then the boundary map ∂^∞ is a finite sum. Because for $\mu(\phi) = 1$ and as we count the number of pseudo-holomorphic disks meeting with V_z, $n_z(\phi) \geq 0$ implying $D(\phi) \geq 0$. Thus there are finitely many homotopy classes $\phi \in \pi_2(x, y)$ with such conditions via the Theorem (5.2.4). Then it is a chain map $(\partial^\infty)^2 = 0$ follows from the same arguments used in Theorem (5.1.4) so $(CF^\infty(\alpha, \beta, s, o), \partial^\infty)$ is a chain complex. The boundary maps ∂^- for $CF^-(\alpha, \beta, s, o)$, which is the restriction of ∂^∞ to $CF^-(\alpha, \beta, s, o)$ and is well-defined, and ∂^+ for $CF^+(\alpha, \beta, s, o)$ are also finite sums and $(\partial^-)^2 = 0$ and $(\partial^+)^2 = 0$ follows similarly as in the case $b_1(Y) = 0$.

2. Suppose that the pointed Heegaard diagram is weakly s-admissible. The differential map for $CF^+(\alpha, \beta, s, o)$ in (5.7), take a generator $[x, i]$, for this i we restrict $n_z(\phi)$ where $\phi \in \pi_2(x, y)$ to $i \geq n_z(\phi) \geq 0$ (the 2nd inequality is true via the proposition (4.0.10)) and $\mu(\phi) = 1$. Then by the Theorem (5.2.3) there exist finitely many $\phi \in \pi_2(x, y)$, so the map ∂^+ is a finite sum. There is an embedding of $\widehat{CF}(\alpha, \beta, s, o)$ into $CF^+(\alpha, \beta, s, o)$ via the map $\iota(x) = [x, 0]$ in the proposition (5.1.8). If we restrict ∂^+ onto $\widehat{CF}(\alpha, \beta, s, o)$ with $n_z(\phi) = i = 0$, it is also a finite sum. The argument $CF^+(\alpha, \beta, s, o)$ and $\widehat{CF}(\alpha, \beta, s, o)$ are chain complex proved similarly as in the Theorem (5.1.4) and the Theorem (5.1.2).

□

5.3 Examples and Properties

In this section we give some basic properties of the Heegaard Floer homology groups and some examples. Properties include finiteness of $Spin^c$ structures for nontrivial homology, the difference between homology groups with the $Spin^c$ structure s and its conjugate s'. In the examples part we focus on S^3, and the general version $L(p, q)$, and $S^2 \times S^1$. In this section we basically follow the paper [28] and for more detail and calculations the reader is referred to this paper.

Let us begin with basic properties.

Theorem 5.3.1. *Let Y be a closed, oriented three-manifold and let $(\Sigma, \alpha, \beta, z)$ be a pointed Heegaard diagram for Y, fix a $Spin^c$ structure $s \in Spin^c(Y)$ and a coherent orientation system o then $\widehat{HF}(\alpha, \beta, s)$ is nontrivial if and only if $HF^+(\alpha, \beta, s)$ is nontrivial.*

Proof. By the proposition (5.1.8) there is a short exact sequence

$$0 \longrightarrow \widehat{CF}(\alpha, \beta, s) \xrightarrow{\iota} CF^+(\alpha, \beta, s) \xrightarrow{U} HF^+(\alpha, \beta, s) \longrightarrow 0$$

and it induces a long exact homology sequence:

$$\cdots \longrightarrow \widehat{HF}(\alpha, \beta, s) \xrightarrow{\iota_*} HF^+(\alpha, \beta, s) \xrightarrow{U_*} HF^+(\alpha, \beta, s) \longrightarrow \cdots$$

Suppose that $\widehat{HF}(\alpha, \beta, s)$ is nontrivial then U is not an isomorphism as it can not be $1{-}1$, so $Ker(U)$ is not trivial implies that $HF^+(\alpha, \beta, s)$ can not be trivial. Conversely, if $\widehat{HF}(\alpha, \beta, s)$ is trivial then U is an isomorphism so $Ker(HF^+(\alpha, \beta, s)) = 0$, but U is a chain map lowering the grading by 1 so $HF^+(\alpha, \beta, s)$ is not nontrivial. Moreover under for sufficiently power of U, $HF^+(\alpha, \beta, s)$ turns out to be trivial. In particular the rank of $\widehat{HF}(\alpha, \beta, s)$ is nonzero if and only if the rank of $HF^+(\alpha, \beta, s)$ is nonzero. □

The next theorem is about the finiteness of $Spin^c$ structures with nontrivial Heegaard Floer homology.

Theorem 5.3.2. *For a closed, oriented three-manifold Y there exist finitely many $Spin^c$ structures $s \in Spin^c(Y)$ such that the homology groups $HF^+(\alpha, \beta, s)$ is nontrivial.*

Proof. If $\widetilde{s} \in Spin^c(Y)$ is a torsion $Spin^c$ structure then the pointed Heegaard diagram $(\Sigma, \alpha, \beta, z)$ for Y is both weakly and strongly admissible for $\widetilde{s}$. Moreover if a pointed Heegaard diagram is strongly admissible for any torsion $Spin^c$ structure then it is weakly admissible for every $Spin^c$ structures by the proposition (5.2.2). If a Heegaard diagram $(\Sigma, \alpha, \beta, z)$ is weakly admissible then $CF^+(\alpha, \beta, s)$ is a chain complex with subcomplex $\widehat{CF}(\alpha, \beta, s)$ by the theorem (5.2.6). Then the corresponding homology groups $HF^+(\alpha, \beta, s)$ and $\widehat{HF}(\alpha, \beta, s)$ can be computed via this diagram $(\Sigma, \alpha, \beta, z)$ for every $Spin^c$ structures. The α and β curves have finitely many intersections, thus $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ has finitely many intersection points. From each intersection point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ we can obtain a $Spin^c$ structures via the map $s_z : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to Spin^c(Y)$. In the chain map we choose the intersection points such that $s_z(x) = s$, for the chosen $Spin^c$ structure $s \in Spin^c(Y)$. There is only finitely many $Spin^c$ structures such that the chain complexes $CF^\infty(\alpha, \beta, s)$ and $\widehat{CF}(\alpha, \beta, s)$ are nontrivial. Therefore there exist finitely many $Spin^c$ structures such that $HF^+(\alpha, \beta, s)$ and thus $\widehat{HF}(\alpha, \beta, s)$ are nonzero. □

Next we study how the Heegaard Floer homology groups change if we change the $Spin^c$ structure s to $\overline{s}$, the conjugate of s which can be thought of as $-s$.

Theorem 5.3.3. *Let Y be a closed, oriented three-manifold with a fixed $s \in Spin^c(Y)$. If we replace s with $\overline{s}$, then the Heegaard Floer homology groups do not change under this involution, i.e., there are $\mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^*(H_1(Y;\mathbb{Z})/Tors)$-module isomorphisms between the homology groups:*

- $HF^\infty(\alpha, \beta, s) \simeq HF^\infty(\alpha, \beta, \overline{s})$
- $HF^+(\alpha, \beta, s) \simeq HF^+(\alpha, \beta, \overline{s})$
- $HF^-(\alpha, \beta, s) \simeq HF^-(\alpha, \beta, \overline{s})$
- $\widehat{HF}(\alpha, \beta, s) \simeq \widehat{HF}(\alpha, \beta, \overline{s})$

Proof. For a fixed $s \in Spin^c(Y)$ assume that the pointed Heegaard diagram $(\Sigma, \alpha, \beta, s)$ for Y is strongly s-admissible. Thus $CF^\infty(\alpha, \beta, s)$ is a chain complex with subcomplex $CF^-(\alpha, \beta, s)$ and quotient complex $CF^+(\alpha, \beta, s)$. The α and β curves in the statement is from the Heegaard diagram $(\Sigma, \alpha, \beta, s)$. We did not prove that the Heegaard Floer homology groups are invariant under the Heegaard diagrams representing the same three-manifold Y, instead of using (Y, s) we insist on the notation as (α, β, s).

On the set of $Spin^c$ structures there is an involution, for any $s \in Spin^c(Y)$ the conjugate of s, $\overline{s}$. Note that s corresponds to a nonvanishing vector field over Y, so $-s$ also corresponds to a nonvanishing vector field. If s is obtained from a Morse function f compatible with the α and β curves, then $-s$ can be obtained from $-f$, the negative Morse function. On the Heegaard surface Σ the effect of this change can be seen by changing the roles of α and β curves and reverse the orientation of Σ. Let U_0 be the handlebody with α curves and U_1 be the handlebody with β curves. The orientation on the boundary of handlebodies are opposite. When we change the roles of α and β curves, the roles of handlebodies U_0 and U_1 are changed. However to make the underlying manifold Y unchanged, as the orientations of U_0 and U_1 are reversed, we also need to change the orientation of Σ.

α and β curves are same in both cases, therefore the set of intersection points does not change. For every $\phi \in \pi_2(x, y)$, $\mathcal{M}(\phi)$ is also the same for both. After the roles of the handlebodies are reversed, the Morse function $-f$ is compatible with the new Heegaard diagram. For the intersection point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ for which $s_z(x) = s$, with the new diagram and the Morse function $-f$, under the map $s_z(x) = \overline{s}$, we obtain the conjugate of $Spin^c$ structure. The new Heegaard diagram is also strongly $\overline{s}$-admissible.

There are two chain complexes $CF^\infty(\alpha, \beta, s)$ and $CF^\infty(\alpha, \beta, \overline{s})$ with the same generators $[x, i]$, where $i \in \mathbb{Z}$. The corresponding boundary map of the chain complexes are also the same. Therefore the homology groups are isomorphic

$$HF^\infty(\alpha, \beta, s) \simeq HF^\infty(\alpha, \beta, \overline{s})$$

The same argument is true also for the homology groups of the subcomplex, quotient complex, and $\widehat{CF}(\alpha, \beta, s)$ proving the desired isomorphisms. $\square$

Let us turn to some examples and computations starting with the homology 3-sphere.

Example: Let $Y = S^3$ be a 3-sphere, so $b_1(Y) = 0$. Consider the genus-1 Heegaard decomposition of S^3 described in Section 2.2 with the corresponding Heegaard diagram.

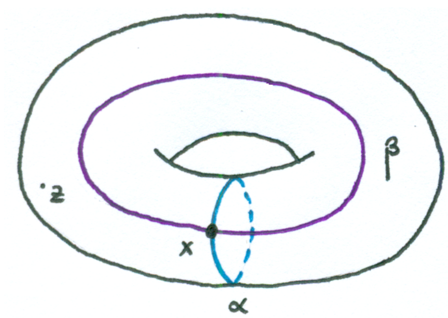


Figure 5.2: Heegaard diagram for S^3

$\mathbb{T}_\alpha = \alpha$ and $\mathbb{T}_\beta = \beta$ in the Symmetric product space $Sym^1(\Sigma) \simeq \Sigma$, and there is only one intersection point corresponding to intersection of α and β curves. There is a $1-1$ correspondence between $Spin^c(S^3)$ and $H^2(S^3; \mathbb{Z})$, but the latter one is trivial implies that there is only one $Spin^c$ structure on S^3, call it s. Take a basepoint $z \in \Sigma - \alpha - \beta$ then as $\widehat{CF}(\alpha, \beta, s)$ has only one generator, $\widehat{CF}(\alpha, \beta, s) \simeq \mathbb{Z}$. The boundary map is as follows.

$$\partial : \widehat{CF}(\alpha, \beta, s) \to \widehat{CF}(\alpha, \beta, s)$$

$$\partial(x) = \sum_{\{\phi \in \pi_2(x,x) | \mu(\phi)=1, n_z(\phi)=0\}} \#(\widehat{\mathcal{M}}(\phi)) \cdot x$$

There is only one generator, so we consider homotopy classes $\phi \in \pi_2(x, x)$ with $n_z(\phi) = 0$, which means ϕ is a periodic class. $\prod_z(x) \simeq H^1(S^3; \mathbb{Z}) \simeq \{0\}$ implies that the boundary map is trivial. $Im\partial = 0$ and $Ker\partial = \mathbb{Z}$, then we have

$$\widehat{HF}(\alpha, \beta, s) \simeq \mathbb{Z}$$

$CF^\infty(\alpha, \beta, s)$ is generated by $[x, i]$ for $i \in \mathbb{Z}$ and is a $\mathbb{Z}[U]$-module with the boundary map ∂^∞ is defined as

$$\partial^\infty : CF^\infty(\alpha, \beta, s) \to CF^\infty(\alpha, \beta, s)$$

$$\partial^\infty[x,i] = \sum_{\{\phi\in\pi_2(x,x)|\mu(\phi)=1\}} \#(\widehat{\mathcal{M}}(\phi))\cdot[x,i-n_z(\phi)]$$

The relative grading of two generators $[x,i]$ and $[x,i-1]$ in the free Abelian group $CF^\infty(\alpha,\beta,s)$ is 2. It is a $\mathbb{Z}[U]$-module and these generators differ by a U-action which lowers the grading by 2. However the boundary map lowers the grading by 1. There are no elements differ by 1 in $CF^\infty(\alpha,\beta,s)$. Therefore the boundary map ∂^∞ is trivial implying the homology group $HF^\infty(\alpha,\beta,s)$ is a graded $\mathbb{Z}[U]$-module denoted as $\mathbb{Z}[U,U^{-1}]$, where U is the chain map lowering the grading by 2 (arbitrary power U^k lowers the grading by $2k$.)

$CF^-(\alpha,\beta,s)$ is generated by $[x,i]$ with $i<0$ thus it is a submodule of $\mathbb{Z}[U]$-module generated by elements of grading less than or equal to -2. As the grading difference between $[x,i]$ and $[x,i-1]$ is 2, the boundary map is trivial implying that the homology groups $HF^-(\alpha,\beta,s)$ is a free $\mathbb{Z}[U]$-module.The quotient complex $CF^+(\alpha,\beta,s)$ has the induced grading with a trivial boundary map by the same reasoning. As a $\mathbb{Z}[U]$-module $HF^+(\alpha,\beta,s)$ isomorphic to $\mathbb{Z}[U,U^{-1}]/\mathbb{Z}[U]$.

For $Y=S^3$ we have

- $\widehat{HF}(\alpha,\beta,s)\simeq\mathbb{Z}$
- $HF^\infty(\alpha,\beta,s)\simeq\mathbb{Z}[U,U^{-1}]$
- $HF^-(\alpha,\beta,s)\simeq\mathbb{Z}[U]$
- $HF^+(\alpha,\beta,s)\simeq\mathbb{Z}[U,U^{-1}]/\mathbb{Z}[U]$

Let us consider the stabiliazation of the Heegaard diagram in the above figure for S^3 as:

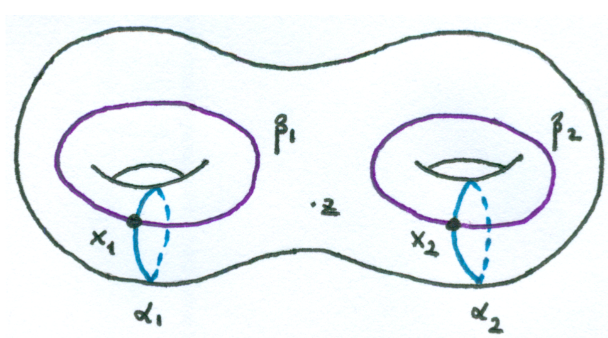


Figure 5.3: Heegaard diagram for S^3 with stabilization move

where $\mathbb{T}_\alpha=\alpha_1\times\alpha_2$, $\mathbb{T}_\beta=\beta_1\times\beta_2$ with a unique intersection number $\mathbb{T}_\alpha\cap\mathbb{T}_\beta=x=\{x_1,x_2\}$ in the Symmetric product space. Then the chain complex $CF^\infty(\alpha,\beta,s)\simeq\mathbb{Z}$

with $HF^{\infty}(\alpha, \beta, s) \simeq \mathbb{Z}$, and $CF^{\infty}(\alpha, \beta, s)$ generated by $[x, i]$ as $\mathbb{Z}[U]$-module with $HF^{\infty}(\alpha, \beta, s) \simeq \mathbb{Z}[U, U^{-1}]$. The same results follow. If we continue stabilize the same diagram we still have the same results.

Example: Let $Y = L(p, q)$ be the Lens space which is a closed, oriented three-manifold with $p \geq 3$ and $(p, q) = 1$. In Section 2.2 we described a genus-1 Heegaard decomposition of $L(p, q)$ in detail, and see below the figure for $L(3, 1)$.

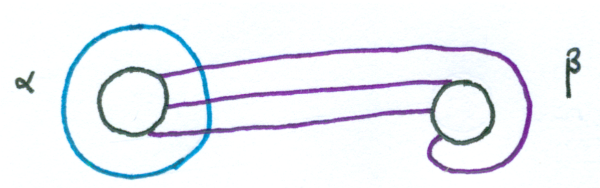


Figure 5.4: A Heegaard diagram for $L(3, 1)$

Note that $H^2(L(p, q); \mathbb{Z}) \simeq \mathbb{Z}_p$ and there is $1-1$ correspondence with $Spin^c(L(p, q))$. In the diagram there are three intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, in general the diagram of $L(p, q)$ can be arranged so that there are p intersection points. From each intersection points we obtain different $Spin^c$ structures. Let x and y be two different intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ then $s_z(x) - s_z(y) = PD[\epsilon(x, y)]$. We can connect x and y on Σ by an arc on the α curve and by an arc on the β curves so that the curve γ connecting intersection points x and y is essential in $H_1(L(p, q); \mathbb{Z})$, so $\epsilon(x, y) \neq 0$ implies that $s_z(x) \neq s_z(y)$. Thus for each $Spin^c$ structure $s \in Spin^c(L(p, q))$ there is only one intersection point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that $s_z(x) = s$. $CF^{\infty}(\alpha, \beta, s)$ is generated by $[x_j, i]$ where $i \in \mathbb{Z}$ and $j \in \{1, .., p\}$ with the boundary map

$$\partial^{\infty}[x, i] = \sum_{\{\phi \in \pi_2(x,x) | \mu(\phi)=1\}} \#(\widehat{\mathcal{M}}(\phi)) \cdot [x, i - n_z(\phi)]$$

The boundary map is trivial because of the relative grading between the generator differ by an even number which is at least 2. Therefore we obtain:

- $\widehat{HF}(\alpha, \beta, s) \simeq \mathbb{Z}$
- $HF^{\infty}(\alpha, \beta, s) \simeq \mathbb{Z}[U, U^{-1}]$ a graded $\mathbb{Z}[U]$-module.
- $HF^{-}(\alpha, \beta, s) \simeq \mathbb{Z}[U]$
- $HF^{+}(\alpha, \beta, s) \simeq \mathbb{Z}[U, U^{-1}]/\mathbb{Z}[U]$

Example: Let $Y = S^1 \times S^2$ be the next example. A genus-1 Heegaard decomposition of $S^1 \times S^2$ with the attaching curves is the following diagram (Σ, α, β), where α is an embedded essential curve on Σ and β is isotopic translate of α.

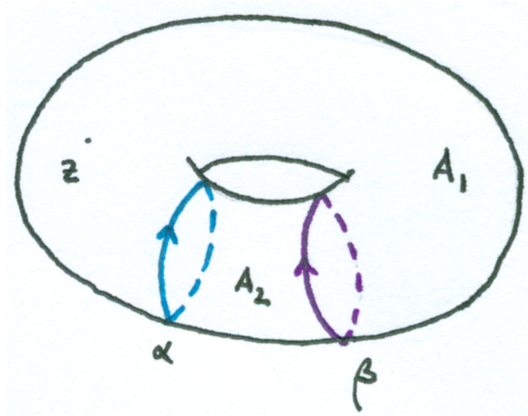


Figure 5.5: A Heegaard diagram for $S^1 \times S^2$

According to this recipe we obtain three-manifold $S^1 \times S^2$. $\mathbb{T}_\alpha = \alpha$ and $\mathbb{T}_\beta = \beta$ and as $g = 1$ then $Sym^g(\Sigma) \simeq \Sigma$ with $\mathbb{T}_\alpha \cap \mathbb{T}_\beta = \emptyset$. There is no intersection point in Σ. $\Sigma - \alpha - \beta$ has 2 components which are both annuli. If we choose the basepoint $z \in A_1$ then as the boundary of A_2 is sum of α and β curves with the algebraic intersection number 0 implies that A_2 is a periodic domain. If the basepoint $z \in A_2$ is chosen to be in A_2 then A_1 is a periodic domain.

There is a $1-1$ correspondence between $Spin^c(S^1 \times S^1)$ and $H^2(S^1 \times S^2; \mathbb{Z}) \simeq H_1(S^1 \times S^2; \mathbb{Z}) \simeq Ab\pi_1(S^1 \times S^1) \simeq \mathbb{Z}$. Thus there is only one torsion $Spin^c$ structure, call it s_0. The Heegaard diagram is not weakly admissible for s_0 because periodic domains does not have both positive and negative coefficients. Moreover, for any $Spin^c$ structure it is not strongly admissible. Since for the torsion $Spin^c$ structure the Heegaard diagram is not weakly admissible implies it is not strongly admissible. For any $Spin^c$ structure different than the torsion one, as periodic domain is nontrivial here, the evaluation $\langle c_1(s), H(P)\rangle = 2n > 0$ must imply P has coefficient greater than 1, but it does not have. With these arguments we can not study with the given Heegaard diagram. Thus we introduce canceling pair of intersection points between α and β curves to make the diagram weakly admissible for the torsion $Spin^c$ structure s_0. This can be done by fixing one of the curves say α and move β in a one parameter family so that they have two transverse intersection points, x^+ and x^- with a pair of nonhomotopic homolorphic disks connecting them.

Let us denote the disks by D_1 and D_2, and let $\phi_1, \phi_2 \in \pi_2(x^+, x^-)$ be homotopy classes such that $D(\phi_1) = D_1$ and $D(\phi_2) = D_2$. We begin with genus-1 Heegaard diagram so that $Sym^g(\Sigma) \simeq T^2$ and the disks D_1 and D_2 are convex polygons which are simply-connected then it follows $\mu(\phi_1) = \mu(\phi_2) = 1$, see the appendix of [35]. Note that $\phi_1 - \phi_2 \in \pi_2(x^4, x^-)$ with $n_z(\phi_1 - \phi_2) = 0$, so $\phi_1 - \phi_2$ is a periodic class which generates the periodic classes in $Sym^g(\Sigma) \simeq T^2$. It follows two intersection points represent the trivial *spin*^{c} structure, and the periodic domains have both positive and negative coefficients. Therefore, the new Heegaard diagram is weakly admissible for the torsion *Spin*^{c} structure s_0.

The moduli spaces $\widehat{\mathcal{M}}(\phi_1)$ and $\widehat{\mathcal{M}}(\phi_2)$ have unique solutions and if there is $\phi \in \pi_2(x^+, x^-)$ with $n_z(\phi) = 0$ different from ϕ_1 and ϕ_2, the moduli space $\widehat{\mathcal{M}}(\phi) = \emptyset$, because $D(\phi)$ has negative coefficients. We can choose the orientation such that the signed count gives 0 and it implies that the boundary map is trivial. Therefore, with 2 generators and trivial boundary map $\widehat{HF}(\alpha', \beta', s_0) \simeq H_*(S^1; \mathbb{Z})$.

Remark 5.3.4. [29] Consider the three-manifold $\#^g(S^1 \times S^2)$ then by similar arguments we have $\widehat{HF}(\alpha', \beta', s_0) \simeq H_*(T^g; \mathbb{Z})$

Chapter 6

INVARIANCE, ACTION, AND ADMISSIBILITY

In Chapter 5 we gave the definition of Heegaard Floer homology groups which is the main part of this thesis. In this chapter we deal with more specific and further topics about the Heegaard Floer homology groups and we mainly focus on two major topics in this chapter.

The first one is the dependence of the Heegaard Floer homology groups on the coherent orientation system and, for a chosen complex structure j over the Heegaard surface Σ, the path J_s of nearly symmetric almost-complex structures and then the complex structure j. These are very important two steps to see how Heegaard Floer homology groups eventually become invariants for 3-manifolds. Moreover, these two steps are the easiest and the shortest ones as compared to understand the dependence of the homology groups on the Heegaard surface.

The second part is that we give some further information about the theory. There is an action on the Heegaard Floer homology groups. When $b_1(Y) = 0$ we defined the U-action and for $b_1(Y) > 0$ we define a new action on the Heegaard Floer homology groups. Then we study the admissibility in more detail as promised in chapter 4.

Section 6.1 is about the dependence of the Heegaard Floer homology groups on the coherent orientation systems. In Section 6.2, we define of a new action on the homology groups. In Section 6.3 we study the dependence of the Heegaard Floer homology groups on the path J_s and the chosen complex structure j over the Heegaard surface Σ. In Section 6.4 we show the existence of admissible Hegaard diagrams. As we defined the Heegaard Floer homology groups for admissible Heegaard diagrams in Chapter 5 when $b_1(Y) > 0$.

6.1 Dependence on the Coherent Orientation System

Before giving the definition of chain complexes we fixed a coherent orientation system o, so that the orientations of the moduli spaces $\mathcal{M}(\phi)$ are compatible with each other. In this section we will study how the Heegaard Floer homology groups depend on the coherent orientation system.

Take two coherent orientation system o and o', the question is to understand the difference between them. This difference is defined by

$$\delta = \delta(o, o') = \in Hom(H^1(Y; \mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$$

as a map from $H^1(Y;\mathbb{Z})$ to $\mathbb{Z}/2\mathbb{Z}$. Intuitively, if they are similar, they differ by 0, if different, they differ by 1. Let $\phi \in \pi_2(x,x)$ be a periodic class, using the $1-1$ correspondance between the periodic classes and cohomology classes in $H^1(Y;\mathbb{Z})$, ϕ corresponds to a cohomology class $H \in H^1(Y;\mathbb{Z})$. A coherent orientation system is a choice of nonvanishing section $o(\phi)$ of the determinant line bundle over each $\phi \in \pi_2(x,y)$ which is compatible with the gluing. Then the nonvanishing section o of the determinant line bundle over the component corresponding to ϕ is a positive multiple of o' or a negative multiple of o'. If positive we define $\delta(H)=0$, if negative we define $\delta(H)=1$. We will see that it suffices to understand the difference between two coherent orientation systems on periodic classes in $\pi_2(x,x)$. By the theorem (3.3.4) we know that $\prod_z(x) \simeq H^1(Y;\mathbb{Z})$. For each periodic class we obtain a coherent orientation system.

Definition 35. Two coherent orientation systems o and o' are said to be equivalent if $\delta(H)=0$ for every $H \in H^1(Y;\mathbb{Z})$, which means their difference vanishes on periodic domains.

Remark 6.1.1. If the difference of two coherent orientation system is zero, this gives an equivalence relation on the set of coherent orientation systems. Thus we obtain equivalence classes of orientations.

Remark 6.1.2. There are $2^{b_1(Y)}$ inequivalent choices of coherent orientation systems. Note that as $H_0(Y;\mathbb{Z})$ is free, by the Universal Coefficient Theorem

$$\begin{aligned} H^1(Y;\mathbb{Z}) &\simeq Hom(H_1(Y;\mathbb{Z}),\mathbb{Z}) \\ &\simeq Hom(\mathbb{Z}^{b_1(Y)}) \oplus Hom(T,\mathbb{Z}) \\ &\simeq \mathbb{Z}^{b_1(Y)} \end{aligned}$$

where T is the torsion part of $H_1(Y;\mathbb{Z})$. Therefore, there are $2^{b_1(Y)}$ inequivalent coherent orientation systems corresponding to each generator and choice.

Remark 6.1.3. When $b_1(Y)=0$ there is only one class of coherent orientation system. That is why we did not say much about orientation when defining the Heegaard Floer homology groups in this case.

We will also see that there are $2^{b_1(Y)}$ different chain complexes corresponding to the variations of the equivalence classes of coherent orientation systems via the next theorem where we will see the dependence of the Heegaard Floer homology groups on the coherent orientation systems.

Theorem 6.1.4. *If there are two equivalent coherent orientation systems o and o' then the corresponding chain complexes $CF^\infty(\alpha,\beta,s,o)$ and $CF^\infty(\alpha,\beta,s,o')$ are isomorphic. Similarly the corresponding chain complexes CF^-, CF^+, and $\widehat{CF}$.*

Proof. We need to show that there is a chain isomorphism $\mathcal{F} : CF^\infty(\alpha,\beta,s,o) \to CF^\infty(\alpha,\beta,s,o')$. As o and o' are equivalent, they differ by $\delta(o,o')=0$. Fix an intersection point $\widetilde{x} \in \mathcal{S}$, for any other $x \in \mathcal{S}$, a homotopy class of a Whitney disk $\phi \in \pi_2(\widetilde{x}, x)$ can be thought as a path from $\widetilde{x}$ to x. There is a sign $\sigma(x) \in \{+1, -1\}$ such that

$$o(\phi) = \sigma(x) \cdot o'(\phi)$$

Note that as o and o' are isomorphic coherent orientation systems, the sign $\sigma(x)$ does not depend on the chosen $\phi \in \pi_2(\widetilde{x}, x)$. First, $\sigma(x)$ is independent of the representative of ϕ. A representative u of ϕ is homotopic to ϕ and $\sigma(x)$ is an integer either $+1$ or -1, thus by the continuity $\sigma(x)$ is same for both. Moreover, for any other $\psi \in \pi_2(\widetilde{x}, x)$ their difference $\phi * \psi$ is a periodic class and some multiple of $[S]$, where S is positive generator of $\pi_2{}'(Sym^g(\Sigma))$. By the definition of the coherent system of orientation, Definition (28), $[S]$ does not affect the sign as $o(\phi * [S]) = o(\phi)$, and as o and o' are equivalent their difference vanishes on the periodic classes. Therefore, the sign $\sigma(x)$ must be same for both ψ and ϕ.

Define a map $\mathcal{F} : CF^\infty(\alpha, \beta, s, o) \to CF^\infty(\alpha, \beta, s, o')$ as $\mathcal{F}[x, i] = \sigma(x) \cdot [x, i]$. First let us see $\mathcal{F}$ is a chain map, so it is compatible with the boundary maps:

$$\begin{array}{ccc} CF^\infty(\alpha, \beta, s, o) & \xrightarrow{\mathcal{F}} & CF^\infty(\alpha, \beta, s, o') \\ \Big\downarrow \partial^\infty & & \Big\downarrow \partial^{\infty\prime} \\ CF^\infty(\alpha, \beta, s, o) & \xrightarrow{\mathcal{F}} & CF^\infty(\alpha, \beta, s, o') \end{array}$$

For a fixed $\widetilde{x} \in \mathcal{S}$, say $\sigma(x)$ is the sign.

$$\partial^{\infty\prime} \circ \mathcal{F}[x, i] = \partial^{\infty\prime}(\sigma(x) \cdot [x, i])$$

$$= \sum_{y \in \mathcal{S}} \sum_{\{\phi \in \pi_2(x,y) | \mu(\phi) = 1\}} \sigma(x) \cdot \#(\widehat{\mathcal{M}}(\phi)) \cdot [y, i - n_z(\phi)]$$

$$\mathcal{F} \circ \partial^{\infty\prime}[x, i] = \mathcal{F}(\sum_{y \in \mathcal{S}} \sum_{\{\phi \in \pi_2(x,y)\}} \#(\widehat{\mathcal{M}}(\phi)) \cdot [y, i - n_z(\phi)])$$

$$= \sum_{y \in \mathcal{S}} \sum_{\{\phi \in \pi_2(x,y)\}} \sigma(y) \cdot \#(\widehat{\mathcal{M}}(\phi)) \cdot [y, i - n_z(\phi)])$$

In the first composition we use the coherent orientation system o' on the moduli spaces and in the second composition we use the coherent orientation system o. On periodic domains their difference vanishes. However on arbitrary homotopy classes in $\pi_2(x, y)$ they differ by a sign. Thus, if we take into account this orientation argument it follows that $\mathcal{F}$ is compatible with the boundary maps. It is already defined on the generators, so $\mathcal{F}$ is $1-1$ and onto. Therefore, $\mathcal{F}$ is a chain isomorphism and the Heegaard Floer homology groups depend on the equivalence classes of coherent orientation system. □

6.2 Actions on the Heegaard Floer Homology Groups

For $b_1(Y) = 0$ we defined a chain map $U : CF^\infty(\alpha, \beta, s) \to CF^\infty(\alpha, \beta, s)$ which lowers the grading by 2. Then U^+ and U^- are the corresponding (chain maps) actions on the quotient complex $CF^+(\alpha, \beta, s)$ and the subcomplex $CF^-(\alpha, \beta, s)$. With this action the Heegaard Floer homology groups becomes $\mathbb{Z}[U]$-modules. For $b_1(Y) > 0$, the map $U : CF^\infty(\alpha, \beta, s, o) \to CF^\infty(\alpha, \beta, s, o)$ is still a chain map lowering the grading by 2 and induces also action on the subcomplex $CF^-(\alpha, \beta, s, o)$ and the quotient complex $CF^+(\alpha, \beta, s, o)$. Thus, the Heegaard Floer homology groups become $\mathbb{Z}[U]$-modules.

Now for $b_1(Y) > 0$ we define a new action on the Heegaard Floer homology groups. Let $(\Sigma, \alpha, \beta, z)$ be a pointed Heegaard diagram. Let $\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ the space of paths connecting $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. By the proof of the theorem (3.3.4) the space $\pi_2(x, x)$ can be identified with the fundamental group of the path space $\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ based at the constant path. By the Universal Coefficient Theorem for cohomology (see [12]) we have:

$$H^1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta); \mathbb{Z}) \simeq Ext(H_0(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta); \mathbb{Z}), \mathbb{Z}) \oplus Hom(H_1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta); \mathbb{Z}), \mathbb{Z})$$

By using the properties of the Ext functor, $H_0(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta); \mathbb{Z})$ is free and $\pi_1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta))$ is Abelian we have:

$$H^1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta); \mathbb{Z}) \simeq Hom(\pi_1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)), \mathbb{Z}) \tag{6.1}$$

Remember the identification between $\pi_1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta))$ based at constant path x and $\pi_2(x, x)$ then

$$Hom(\pi_1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)), \mathbb{Z}) \simeq Hom(\pi_2(x, x), \mathbb{Z})$$

When $g \geq 3$ we have $\pi_2(x, x) \simeq \mathbb{Z} \oplus H^1(Y; \mathbb{Z})$. Here $\mathbb{Z}$ corresponds to $\pi_2{}'(Sym^g(\Sigma)) \simeq \mathbb{Z}$, thus in general we have:

$$\pi_2(x, x) \simeq \pi_2{}'(Sym^g(\Sigma)) \oplus H^1(Y; \mathbb{Z})$$

Combining these with the equation (6.1) we have:

$$\begin{aligned} H^1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta); \mathbb{Z}) &\simeq Hom(\pi_2{}'(Sym^g(\Sigma)) \oplus H^1(Y; \mathbb{Z}), \mathbb{Z}) \qquad (6.2) \\ &\simeq Hom(\pi_2{}'(Sym^g(\Sigma)), \mathbb{Z}) \oplus Hom(H^1(Y; \mathbb{Z}), \mathbb{Z}) \\ &\simeq \pi_2{}'(Sym^g(\Sigma)) \oplus Hom(H^1(Y; \mathbb{Z}), \mathbb{Z}) \end{aligned}$$

Here we used the properties of the *Hom* functor which can be found in [13]. For $g > 2$ we can take $\pi_2{}'(Sym^g(\Sigma))$ as $\pi_2(Sym^g(\Sigma))$ as π_1 action becomes trivial as in the theorem (3.1.5). In the end we have the following isomorphisms:

$$H^1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta); \mathbb{Z}) \simeq Hom(\pi_1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)), \mathbb{Z})$$

$$\simeq \pi_2(Sym^g(\Sigma)) \oplus Hom(H^1(Y; \mathbb{Z}), \mathbb{Z})$$

The aim of this section is to prove the following theorem. Let $\Sigma(\alpha, \beta, s, z)$ be a Heegaard diagram for Y when $b_1(Y) > 0$.

Theorem 6.2.1. *$H^1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta); \mathbb{Z})$ acts on the Heegaard Floer homology groups $HF^\infty(\alpha, \beta, s)$, $HF^+(\alpha, \beta, s)$, $HF^-(\alpha, \beta, s)$, and $\widehat{HF}(\alpha, \beta, s)$ as lowering grading by 1. Moreover this action induces action of the exterior algebra $\Lambda^*(H_1(Y; \mathbb{Z})/Tors) \subset \Lambda^*(H^1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta); \mathbb{Z}); \mathbb{Z})$ on each Heegaard Floer homology groups.*

Let us study some preliminaries before attempting to prove the theorem. Recall from the basic algebraic topology, see [12], the qth chain group of X is given by $C^q(X) = \{f | f : C_q \to \mathbb{Z}, f \text{ is a homomorphism}\}$ with the coboundary map $\delta^q : C^q(X) \to C^{q+1}(X)$ taking $f \mapsto f \circ \partial_{q+1}$ where $\partial_{q+1} : C_{q+1} \to C_q$ is boundary map between chain groups in homology. Then $Ker\delta^q = Z^q(X) \subset C^q(X)$ is the *cocycle group* and $Im\delta^{q-1} = B^q(X) \subset C^q(X)$ is the *coboundary group.*

Fix a pointed Heegaard diagram $(\Sigma, \alpha, \beta, s)$ for Y and let $\xi \in Z^1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), \mathbb{Z}) \subset C^1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), \mathbb{Z})$ be one-cocycle in the path space $\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$. The map between chain complexes

$$\mathcal{A}_\xi : CF^\infty(\alpha, \beta, s) \to CF^\infty(\alpha, \beta, s)$$

is defined as

$$\mathcal{A}_\xi[x, i] = \sum_{y \in \mathcal{S}} \sum_{\{\phi \in \pi_2(x,y) | \mu(\phi)=1\}} \xi(\phi) \cdot \#(\widehat{\mathcal{M}}(\phi)) \cdot [y, i - n_z(\phi)] \tag{6.3}$$

lowering the grading by 1. Let us verify this first:

$$\begin{aligned} gr([x, i], [y, i - n_z(\phi)]) &= gr(x, y) + 2i - 2(i - n_z(\phi)) \\ &= \mu(\phi) - 2n_z(\phi) + 2i - 2i + 2n_z(\phi) \\ &= 1 \end{aligned}$$

The map in (6.3) is well-defined, meaning the evaluation $\xi(\phi)$ is independent of the representative of ϕ. Any representative of ϕ say ϕ' is a path in $\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ connecting the constant paths x and y. As ϕ and ϕ' are homotopic in $Sym^g(\Sigma)$, the paths ϕ and ϕ' in $\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ are homotopic too. As ξ is a cocycle the evaluation makes sense and it is independent of the representative of ϕ. Note that the map $\mathcal{A}_\xi$ is similar to the boundary map ∂^∞ of the chain complex $CF^\infty(\alpha, \beta, s)$.

Proposition 6.2.2. *$\mathcal{A}_\xi : CF^\infty(\alpha, \beta, s) \to CF^\infty(\alpha, \beta, s)$ is a chain map.*

Proof. Let us compute the maps $\mathcal{A}_\xi \circ \partial^\infty$ and $\partial^\infty \circ \mathcal{A}_\xi$ on the generators. At some point the proof turns out to be similar to the proof of $(\partial^\infty)^2 = 0$. Suppose for $\phi \in \pi_2(x, w)$ with $\mu(\phi) = 2$, $n_z(\phi) = k$ then

$$(\mathcal{A}_\xi \circ \partial^\infty)[x, i] = \mathcal{A}_\xi(\sum_{y \in \mathcal{S}} \sum_{\{\varphi \in \pi_2(x,y) | \mu(\varphi)=1\}} \#(\widehat{\mathcal{M}}(\varphi)) \cdot [y, i - n_z(\phi)])$$

$$= \sum_{y,w \in \mathcal{S}} \sum_{\{\varphi \in \pi_2(x,y) | \mu(\varphi)=1\}} \sum_{\{\psi \in \pi_2(y,w) | \mu(\psi)=1\}} \xi(\psi) \cdot \#(\widehat{\mathcal{M}}(\varphi)) \cdot \#(\widehat{\mathcal{M}}(\psi)) \cdot [w, i - n_z(\phi) - n_z(\psi)]$$

$$(\partial^\infty \circ \mathcal{A}_\xi)[x, i] = \partial^\infty(\sum_{y \in \mathcal{S}} \sum_{\{\varphi \in \pi_2(x,y) | \mu(\varphi)=1\}} \xi(\varphi) \#(\widehat{\mathcal{M}}(\varphi)) \cdot [y, i - n_z(\varphi)])$$

$$= \sum_{y,w \in \mathcal{S}} \sum_{\{\varphi \in \pi_2(x,y) | \mu(\varphi)=1\}} \sum_{\{\psi \in \pi_2(y,w) | \mu(\psi)=1\}} \xi(\varphi) \cdot \#(\widehat{\mathcal{M}}(\varphi)) \cdot \#(\widehat{\mathcal{M}}(\psi)) \cdot [w, i - n_z(\varphi) - n_z(\psi)]$$

Now for $\phi \pi_2(x, y)$ with $\mu(\phi) = 2$ and $n_z(\phi) = k$, for the ends of the unparametrized moduli space $\widehat{\mathcal{M}}(\phi)$ we have:

$$\#(\text{ends of } \widehat{\mathcal{M}}(\phi)) = 0 \tag{6.4}$$

Note also that ξ is a cocycle so it is a homomorphism and it respects to the group structure, therefore we have

$$\xi(\varphi * \psi) = \xi(\varphi) + \xi(\psi)$$

Then we have

$$0 = \xi(\phi) \cdot [\#\text{ends of } \widehat{\mathcal{M}}(\phi)]$$

$$= \xi(\psi * \varphi) \cdot \sum_{\{\varphi * \psi = \phi | \mu(\varphi) = \mu(\psi) = 1\}} \#(\widehat{\mathcal{M}}(\varphi)) \cdot \#(\widehat{\mathcal{M}}(\psi))$$

$$= \sum_{\{\varphi * \psi = \phi | \mu(\varphi) = \mu(\psi) = 1\}} [\xi(\varphi) + \xi(\psi)] \#(\widehat{\mathcal{M}}(\varphi)) \cdot \#(\widehat{\mathcal{M}}(\psi))$$

Then

$$(\mathcal{A}_\xi \circ \partial^\infty + \partial^\infty \circ \mathcal{A}_\xi)[x,i]$$

$$= \sum_{y,w\in\mathcal{S}} \sum_{\{\varphi\in\pi_2(x,y),\psi\in\pi_2(y,w)|\mu(\varphi)=\mu(\psi)=1\}} [\xi(\varphi)+\xi(\psi)]\#(\widehat{\mathcal{M}}(\varphi))\cdot\#(\widehat{\mathcal{M}}(\psi))[w,i-k]$$

the right hand side gives 0. Thus $\mathcal{A}_\xi$ is a chain map lowering the grading by 1. (Note that the diagram here is anti-commutative.) □

Proposition 6.2.3. *If ξ is a coboundary then $\mathcal{A}_\xi$ is chain homotopic to zero.*

Proof. Suppose that ξ is a coboundary then $\xi \in Im\delta^0$ and in $H^1(\Omega(\mathbb{T}_\alpha,\mathbb{T}_\beta))$ ξ represents trivial element. There is a zero cochain $B \in C^0(\Omega(\mathbb{T}_\alpha,\mathbb{T}_\beta),\mathbb{Z})$ such that if γ is an arc in $\Omega(\mathbb{T}_\alpha,\mathbb{T}_\beta)$, which can also be seen as a disk connecting two intersection points as

$$\xi(\gamma) = B(\gamma(0)) - B(\gamma(1))$$

Then we define a map $H : CF^\infty(\alpha,\beta,s) \to CF^\infty(\alpha,\beta,s)$

$$H[x,i] = B(x)\cdot[x,i]$$

where B is a zero cochain and $B(x)$ is calculated by taking x as a constant path from $\mathbb{T}_\alpha$ to $\mathbb{T}_\beta$. Then by definition it follows

$$\mathcal{A}_\xi = \partial^\infty \circ H - H \circ \partial^\infty$$

$$(\partial^\infty \circ H) = \partial^\infty(B(x)\cdot[x,i])$$

$$= \sum_{y\in\mathcal{S}} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} B(x)\#(\widehat{\mathcal{M}}(\phi))[y,i-n_z(\phi)]$$

$$(H\circ\partial^\infty)[x,i] = H(\sum_{y\in\mathcal{S}} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} \#(\widehat{\mathcal{M}}(\phi))[y,i-n_z(\phi)])$$

$$= \sum_{y\in\mathcal{S}} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} B(y)\#(\widehat{\mathcal{M}}(\phi))[y,i-n_z(\phi)]$$

$$(\partial^\infty \circ H - H \circ \partial^\infty)[x,i] = \sum_{y\in\mathcal{S}} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} [B(x)-B(y)]\#(\widehat{\mathcal{M}}(\phi))[y, i-n_z(\phi)]$$

$$= \sum_{y\in\mathcal{S}} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} \xi(\phi)\#(\widehat{\mathcal{M}}(\phi))[y, i-n_z(\phi)]$$

$$= \mathcal{A}_\xi[x,i]$$

We can think of $\phi \in \pi_2(x,y)$ as an arc in $\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ and $\xi(\phi) = [B(x)-B(y)]$. We verified on the generators and $\mathcal{A}_\xi = \partial^\infty \circ H - H \circ \partial^\infty$ follows. Let 0 denote the zero chain map then

$$(\mathcal{A}_\xi - 0) = \partial^\infty \circ H - H \circ \partial^\infty$$

and by definition of chain homotopy this implies the desired result. □

Now we can prove the theorem (6.2.1) stated at the beginning of the section.

Proof of the Theorem 6.2.1. We showed that $\mathcal{A}_\xi$ is a chain map lowering the grading by 1. Then $\mathcal{A}_\xi$ descends to an action of $H^1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta))$ on HF^∞. Because for each cocycle $\mathcal{A}_\xi$ gives a chain map lowering the grading by 1 and for each coboundary it is chain homotopic to zero. This shows that $\mathcal{A}_\xi$ is an action on the cohomology level. By the Universal Coefficient Theorem $H^1(Y;\mathbb{Z}) \simeq Hom(H_1(Y;\mathbb{Z}),\mathbb{Z})$ and the Equation (6.2), this action can be considered as the action of $H_1(Y;\mathbb{Z})/Tors$ on the homology groups. Note that rather than taking Hom we can take the quotient by the torsion to consider the free part.

Moreover, $\mathcal{A}_\xi$ descends to an action of the exterior algebra $\Lambda^*(H_1(Y;\mathbb{Z})/Tors)$. In order to see this we need to show that $\mathcal{A}_\xi \circ \mathcal{A}_\xi = 0$. We will prove this with an alternate description of $\mathcal{A}_\xi$. Take a one-cocycle $\xi \in Z^1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), \mathbb{Z})$ and let $f : \Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta) \to S^1$ be a map that represents ξ, by the correspondance between cohomology classes and the homotopy classes of maps from $Y \to S^1$, see [12]. For a generic point $p \in S^1$, let $V = f^{-1}(p)$ similar to the definition of the subvariety V_z. We define the action

$$\mathcal{A}_\xi[x,i] = \sum_{y\in\mathcal{S}} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} a(\xi,\phi)\cdot[y, i-n_z(\phi)]$$

where $a(\xi,\phi) = \{u \in \mathcal{M}(\phi) | u([0,1]\times\{0\}) \in V\}$.

Note that u is in the unparametrized moduli space. We can think of u as an infinite strip $[0,1]\times\mathbb{R}$. A parametrization of $[0,1]\times\{0\}$ corresponds to translation of the line segment in $\mathbb{R}$ direction as $[0,1]\times\{s\}$ for $s\in\mathbb{R}$. As it is fixed at 0 we do not need to consider the unparametrized moduli space $\widehat{\mathcal{M}}(\phi)$. Because the segment at $[0,1]\times\{s\}$ of another parametrization of u stays in V not the 0th level. V is codimension 1 so the elements in $a(\xi,\phi)$ is finitely many. Here we count the number of homolomorphic disks such that image of $u([0,1]\times\{0\})$ under u stays in $f^{-1}(p)$.

If $\mathcal{A}_\xi$ is applied twice we need to consider the ends of $\mathcal{M}(\phi)$ where $\phi\in\pi_2(x,w)$ with $\mu(\phi)=2$. For generic points p and p', let $V=f^{-1}(p)$ and $V'=f^{-1}(p')$ and consider

$$\mathbb{M}=\{s\in[0,\infty),u\in\mathcal{M}(\phi)|u([0,1]\times\{s\})\in V,u([0,1]\times\{-s\})\in V'\}$$

If $s=0$ then $\mathbb{M}$ has no ends. Because take a sequence of disks u_i in $\mathbb{M}$. For each u_i we have $u([0,1]\times\{0\})\in V,V'$ so if the limits exists it will be of this form too. The ends occur as $s\to\infty$. If boundary degeneration occur by the theorem (5.1.4) the algebraic contribution of them vanishes and we have broken flow-lines

$$\{u\in\mathcal{M}(\varphi)|u([0,1]\times\{0\})\in V\}\times\{u\in\mathcal{M}(\psi)|u([0,1]\times\{0\})\in V'\} \tag{6.5}$$

such that $\varphi*\psi=\phi$. In the proof of the theorem (5.1.4) the oriented count of the points of these spaces gives 0. Let us see this by applying the map $\mathcal{A}_\xi$ twice:

$$\mathcal{A}_\xi\circ\mathcal{A}_\xi[x,i]=\mathcal{A}_\xi(\sum_{y\in\mathcal{S}}\sum_{\{\varphi\in\pi_2(x,y)|\mu(\varphi)=1\}}a(\xi,\varphi)[y,i-n_z(\varphi)])$$

$$=\sum_{y,w\in\mathcal{S}}\sum_{\{\varphi\in\pi_2(x,y)|\mu(\varphi)=1\}}\sum_{\{\psi\in\pi_2(x,y)|\mu(\psi)=1\}}a(\xi,\varphi)a(\xi,\psi)[w,i-n_z(\varphi)-n_z(\psi)]$$

the coefficients $a(\xi,\varphi)a(\xi,\psi)$ corresponds to the equation (6.5) and the sum gives 0. Therefore, $\mathcal{A}_\xi\circ\mathcal{A}_\xi=0$ follows and the action of $H_1(Y;\mathbb{Z})/Tors$ on $HF^\infty(\alpha,\beta,s)$ descends to an action of the exterior algebra $\Lambda^*(H_1(Y;\mathbb{Z})/Tors)$. □

Remark 6.2.4. $\mathcal{A}_\xi$ is a chain map on $CF^\infty(\alpha,\beta,s)$ lowering the grading by 1. Therefore for each generator $[x,i]$ of $CF^-(\alpha,\beta,s)$ is preserved under the map $\mathcal{A}_\xi$ by the nonnegativity of the intersection number. Thus $\mathcal{A}_\xi$ can be restricted to $CF^-(\alpha,\beta,s)$. This implies there is an induced action of $\mathcal{A}_\xi$ on the homology groups $\widehat{HF}(\alpha,\beta,s)$, $HF^+(\alpha,\beta,s$, and $HF^-(\alpha,\beta,s)$.

Remark 6.2.5. By the isomorphism in the equation (6.2), for $g>2$ as π_1 acts on π_2 trivially $\pi_2(Sym^g(\Sigma))\simeq\mathbb{Z}$. The action of $H^1(\Omega(\mathbb{T}_\alpha,\mathbb{T}_\beta))$ can be seen as the action of $\mathbb{Z}\oplus Hom(H_1(Y;\mathbb{Z}),\mathbb{Z})$. The action of $\mathbb{Z}$ is trivial on the Heegaard Floer homology groups. In order to see this, one need to show the coefficients in $\xi(\phi)$ in the chain map $\mathcal{A}_\xi$ is zero.

The action of $H_1(Y;\mathbb{Z})/Tors$ on $HF^\infty(\alpha,\beta,s)$ can be observed geometrically as follows. This is the interesting part of the action. Let (Σ,α,β,s) be a pointed Heegaard diagram for Y. Take a curve γ on the Heegaard surface such that it does not meet with the intersection points of the α and β curves. There is an isomorphism

$$\frac{H_1(\Sigma)}{[\alpha_1],..,[\alpha_g],[\beta_1],..,[\beta_g]} \simeq H_1(Y;\mathbb{Z})$$

So γ can be realized as a homology class $[\gamma] \in H_1(Y;\mathbb{Z})$. Then the action

$$\mathcal{A}_{[\gamma]}[x,i] = \sum_{y\in\mathcal{S}} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} a([\gamma],\phi)[y, i-n_z(\phi)]$$

where $a([\gamma],\phi)$ is defined similarly as before is a finite set:

$$a([\gamma],\phi) = \#\{u \in \mathcal{M}()\phi | u([0,1]\times\{0\}) \in (\gamma \times Sym^{g-1}(\Sigma)) \cap \mathbb{T}_\alpha\}$$

$V = f^{-1}(p)$ is codimension 1 subspace of $\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ which means the space of certain paths connecting $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ for a chosen point $p \in S^1$. Thus here V corresponds to $\gamma \times Sym^{g-1}(\Sigma)$ intersected with $\mathbb{T}_\alpha$, note that $u(\{1\}\times\mathbb{R})$ maps into $\mathbb{T}_\alpha$. The coefficient $a([\gamma],\phi)$ can be interpreted by using $\#(\widehat{\mathcal{M}}(\phi))$. We multiply $\#(\widehat{\mathcal{M}}(\phi))$ by the the algebraic intersection number of the subvariety V_z with $(\gamma \times Sym^{g-1}(\Sigma)) \cap \mathbb{T}_\alpha$.

6.3 Dependence on The Complex Structure and The Path

The aim of this section is to understand how the Heegaard Floer homology groups depend on the choice of the complex structure j over Σ and the path J_s of nearly symmetric almost-complex structures. Remember that before giving the definition of the chain complexes in the beginning of the Section 5.1, we fixed some auxiliary data which contain a fixed coherent orientation system and a generic path J_s of nearly symmetric almost complex structure. In the first section we discussed the orientation and now let us turn to j and the path J_s.

The answer of the above question and the main part of this section is the following theorem.

Theorem 6.3.1. *For a closed, oriented 3-manifold Y, fix a $Spin^c$ structure $s \in Spin^c(Y)$ and let $(\Sigma, \alpha, \beta, z)$ be a pointed strongly s-admissible Heegaard diagram for Y with an equivalence class of coherent orientation system o. Then the Heegaard Floer homology groups $HF^\infty(\alpha,\beta,s,o)$, $HF^-(\alpha,\beta,s,o)$, $HF^+(\alpha,\beta,s,o)$, and $\widehat{HF}(\alpha,\beta,s,o)$ are independent(up to isomorphism) of the chosen complex structure j over Σ and the path J_s of nearly symmetric almost-complex structures.*

Proof. First we will prove the statement for the case $b_1(Y) = 0$ and then for $b_1(Y) > 0$. In the first case when Y is a rational homology three sphere, for a fixed complex structure j over Σ we will show that the Heegaard Floer homology groups are independent of the path J_s.

Let $(\Sigma, \alpha, \beta, z)$ be a pointed Heegaard diagram for Y and let j be a generic complex structure over Σ such that the intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ are disjoint from $Sym^g(j)$-holomorphic spheres in the Symmetric product. Then by the definition of nearly symmetric almost complex structure on the Symmetric product space in the

Definition (25) obtained from the complex structure j, we obtain a path J_s of (j, η, V)-nearly symmetric almost complex structures such that the path J_s is contained in a contractible open neighborhood U of $Sym^g(j)$ as in the Theorem (4.0.18) to handle the boundary degenerations.

Take two paths $J_{s(0)}$ and $J_{s(1)}$ of nearly symmetric almost-complex structures contained in the contractible open neighborhood U of $Sym^g(j)$. U is contractible implies U is simply-connected, thus $J_{s(0)}$ and $J_{s(1)}$ can be connected by a path in U. Take a path $J_s : [0,1] \to U$ in U in parameter t such that $J_s(t)$ is a path for each $t \in [0,1]$. This can be interpreted as a two parameter family of function $J : [0,1] \times [0,1] \to U$, which is a path of paths. It is possible to arrange J_s around the boundary points $t \in [0,1]$ so that we can extend t to all real line $\mathbb{R}$, see [29]. Corresponding to these two paths there are two chain complexes $(CF^\infty(\alpha, \beta, s), \partial^\infty_{J_{s(0)}})$ and $(CF^\infty(\alpha, \beta, s), \partial^\infty_{J_{s(1)}})$. We define a map between two chain complexes and show that this induces a chain homotopy to the identity map, so that the corresponding homology groups become isomorphic. This proves the independence of the homology groups from the choice of the path of nearly symmetric almost complex structures.

The chain map described above is

$$\Phi^\infty_{J_{s,t}} : (CF^\infty(\alpha, \beta, s), \partial^\infty_{J_{s(0)}}) \to (CF^\infty(\alpha, \beta, s), \partial^\infty_{J_{s(1)}})$$

defined as

$$\Phi^\infty_{J_{s,t}}[x, i] = \sum_y \sum_{\{\phi \in \pi_2(x,y) | \mu(\phi) = 0\}} \#(\mathcal{M}_{J_{s,t}}(\phi)) \cdot [y, i - n_z(\phi)]$$

for each $t \in [0,1]$, $\mathcal{M}_{J_{s,t}}(\phi)$ is the corresponding moduli space of dimension 0 is describes as

$$\mathcal{M}_{J_{s,t}}(\phi) = \left\{ u : \mathbb{D} \to Sym^g(\Sigma) \;\middle|\; \begin{array}{l} \frac{du}{ds} + J(s,t)\frac{du}{dt} = 0 \\ u(\{1\} \times \mathbb{R}) \subset \mathbb{T}_\alpha, u(\{0\} \times \mathbb{R}) \subset \mathbb{T}_\beta \\ \lim_{t \to -\infty} u(s + it) = x, \lim_{t \to +\infty} u(s + it) = y \end{array} \right\}$$

for each $t \in [0,1]$ the moduli space $\mathcal{M}_{J_{s,t}}(\phi)$ is 0 dimensional compact manifold by the theorem (4.0.19) it has finitely many points, therefore the map $\Phi^\infty_{J_{s,t}}$ is a finite sum and well-defined.

First observe that

$$\begin{aligned} gr([x, i], [y, i - n_z(\phi)]) &= gr(x, y) + 2i - 2n_z(\phi) \\ &= \mu(\phi) - 2n_z(\phi) + 2i - 2i + 2n_z(\phi) \\ &= \mu(\phi) \\ &= 0 \end{aligned}$$

so the map $\Phi^{\infty}_{J_{s,t}}$ does not change the grading. Let us show that $\Phi^{\infty}_{J_{s,t}}$ is a chain map so it must commute with the boundary maps. We need the following diagram is commutative

$$\begin{array}{ccc} CF^{\infty}(\alpha,\beta,s) & \xrightarrow{\Phi^{\infty}_{J_{s,t}}} & CF^{\infty}(\alpha,\beta,s) \\ \big\downarrow \partial^{\infty}_{J_{s(0)}} & & \big\downarrow \partial^{\infty}_{J_{s(1)}} \\ CF^{\infty}(\alpha,\beta,s) & \xrightarrow{\Phi^{\infty}_{J_{s,t}}} & CF^{\infty}(\alpha,\beta,s) \end{array}$$

or equivalently $\Phi^{\infty}_{J_{s,t}} \circ \partial^{\infty}_{J_{s(0)}} = \partial^{\infty}_{J_{s(1)}} \circ \Phi^{\infty}_{J_{s,t}}$.

The boundary maps change the grading by 1 therefore we consider the Whitney disk of Maslov index 1.

$$\Phi^{\infty}_{J_{s,t}} \circ \partial^{\infty}_{J_{s(0)}} = \Phi^{\infty}_{J_{s,t}}(\sum_{y} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} \#(\widehat{\mathcal{M}}(\phi))[y, i-n_z(\phi)])$$

$$= \sum_{y,w} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} \sum_{\{\psi\in\pi_2(y,w)|\mu(\psi)=0\}} \#(\widehat{\mathcal{M}}_{J_{s(0)}}(\phi))\#(\mathcal{M}_{J_{s,t}}(\psi))[y, i-n_z(\phi)-n_z(\psi)]$$

$$\partial^{\infty}_{J_{s(1)}} \circ \Phi^{\infty}_{J_{s,t}} = \partial^{\infty}_{J_{s(1)}}(\sum_{y} \sum_{\{\psi\in\pi_2(x,y)|\mu(\psi)=0\}} \#(\mathcal{M}_{J_{s,t}}(\psi))[y, i-n_z(\psi)])$$

$$= \sum_{y,w} \sum_{\{\psi\in\pi_2(x,y)|\mu(\psi)=0\}} \sum_{\{\phi\in\pi_2(y,w)|\mu(\phi)=1\}} \#(\mathcal{M}_{J_{s,t}}(\psi))\#(\widehat{\mathcal{M}}_{J_{s(1)}}(\phi))[y, i-n_z(\psi)-n_z(\phi)]$$

Now let us understand the coefficient of the difference below, which is complicated to write.

$$\Phi^{\infty}_{J_{s,t}} \circ \partial^{\infty}_{J_{s(0)}} - \partial^{\infty}_{J_{s(1)}} \circ \Phi^{\infty}_{J_{s,t}} \tag{6.6}$$

The ends of the moduli space $\mathcal{M}_{J_{s,t}}(\varphi)$ with $\mu(\phi)=1$ contains only broken flow lines. If there is a bubble in the limit which is in the interior of a disk or on the boundary, it changes the Maslov index by at least 2. This situation can not happen as $\mu(\phi)=1$. Thus we can only have broken flow-lines, with $\mu(\phi)=1$ we can have $\varphi=\phi*\psi$ with $\mu(\phi)=1$ and $\mu(\psi)=0$ or vice versa. The coefficients in the above map (6.6) corresponds to the ends which has the following relation

$$(\prod_{\phi*\psi=\varphi} \mathcal{M}_{J_{s,t}}(\psi) \times \widehat{\mathcal{M}}_{J_{s(0)}}(\phi)) \prod (\prod_{\phi*\psi=\varphi} \widehat{\mathcal{M}}_{J_{s(1)}}(\phi) \times \mathcal{M}_{J_{s,t}}(\psi)) = 0$$

With the orientation on the moduli spaces we count with sign and obtain 0 which corresponds to the coefficients of the desired relation (6.6). Thus $\Phi^{\infty}_{J_{s,t}}$ is a chain map.

Next we need to show that the corresponding homology groups of the chain complexes are isomorphic. We claim that the map $\Phi^{\infty}_{J_{s,t}} \circ \Phi^{\infty}_{J_{s,1-t}}$ is chain homotopic to the identity map. the combination of $J_{s,t}$ with $J_{s,1-t}$ is a complex structure. Let $J_{s,t,\tau}$ be a family of maps connecting the juxtaposition of $J_{s,t}$ with $J_{s,1-t}$ at $\tau = 0$ and $J_{s,t}(1)$ at $\tau = 1$. If we take $J_{s,t}(1) = J_{s(0)}$ as the path corresponding to the identity, $J_{s,t}(1)$ becomes independent of the parameter t. For each $J_{s,t,\tau}$ there is a corresponding moduli space, let for $\tau \in [0, 1]$

$$\mathcal{M}_{J_{s,t,\tau}}(\phi) = \bigcup_{\tau\in[0,1]} \mathcal{M}_{J_{s,t(\tau)}}(\phi)$$

be the moduli space. Note that $\mathcal{M}_{J_{s,t,\tau}}(\phi)$ is of dimension $\mu(\phi) + 1$. By using the transversality theorem, Theorem(4.0.14) and for generic $J_{s,t,\tau}$, tha space $\mathcal{M}_{J_{s,t,\tau}}(\phi)$ is a manifold of dimension $\mu(\phi) + 1$.

First we connect two paths $J_{s(0)}$ and $J_{s(1)}$ by one parameter family $J_s(t)$ in U. Then we defined two parameter family $J_{s,t}(\tau)$ for a fixed $\tau \in [0, 1]$ connecting the paths $J_{s(0)}$ and the complex structures obtained by combining $J_{s,t}$ and $J_{s,1-t}$. With this new family we define the homotopy between the corresponding chain complexes.

$$H^{\infty}_{J_{s,t,\tau}}[x, i] = \sum_{y} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=-1\}} \#(\mathcal{M}_{J_{s,t,\tau}}(\phi)) \cdot [y, i - n_z(\phi)]$$

Let us compare the grading difference between the generators:

$$\begin{aligned} gr([x, i], [y, i - n_z(\phi)]) &= gr(x, y) + 2i - 2(i - n_z(\phi)) \\ &= \mu(\phi) - 2n_z(\phi) + 2i - 2i + 2n_z(\phi) \\ &= \mu(\phi) \\ &= -1 \end{aligned}$$

Thus $H^{\infty}_{J_{s,t,\tau}}$ is a map changing the grading by -1. We want $H^{\infty}_{J_{s,t,\tau}}$ is a chain homotopy between $\Phi^{\infty}_{J_{s,t}} \circ \Phi^{\infty}_{J_{s,1-t}}$ and the identity map $\mathbb{1}_{J_{s(0)}}$, thus we need

$$\Phi^{\infty}_{J_{s,t}} \circ \Phi^{\infty}_{J_{s,1-t}} + \mathbb{1}_{J_{s(0)}} + \partial^{\infty}_{J_{s(0)}} \circ H^{\infty}_{J_{s,t,\tau}} + H^{\infty}_{J_{s,t,\tau}} \circ \partial^{\infty}_{J_{s(0)}} = 0 \tag{6.7}$$

Note that first two maps above does not change the grading whereas the boundary map changes the grading by 1 and $H^{\infty}_{J_{s,t,\tau}}$ changes by -1, as a composition they do not change the grading too. This is the reason why in the definition of $H^{\infty}_{J_{s,t,\tau}}$ we take Whitney disks of Maslov index -1.

The coefficients of the relation in (6.7) corresponds to counting the ends of the moduli space $\mathcal{M}_{J_{s,t,\tau}}(\phi)$ with $\mu(\phi) = 1$. The map $H^{\infty}_{J_{s,t,\tau}}$ is a family of maps indexed by τ from the composition $\Phi^{\infty}_{J_{s,t}} \circ \Phi^{\infty}_{J_{s,1-t}}$ to $\mathbb{1}_{J_{s(0)}}$. Therefore for $\tau = 0$ the ends correspond to the coefficients of the composition $\Phi^{\infty}_{J_{s,t}} \circ \Phi^{\infty}_{J_{s,1-t}}$:

$$\sum_{y,w} \sum_{\{\varphi\in\pi_2(x,y)|\mu(\varphi)=0\}} \sum_{\{\psi\in\pi_2(y,w)|\mu(\psi)=0\}} \#(\mathcal{M}_{J_{s,1}}(\varphi))\#(\mathcal{M}_{J_{s,0}}(\psi))[w, i - n_z(\varphi) - n_z(\psi)]$$

and for $\tau = 1$ it corresponds to the map $\mathbb{1}_{J_{s(0)}}$. However for arbitrary $\tau \in (0,1)$ the ends correspond to the coefficient of the map

$$\partial^{\infty}_{J_{s(0)}} \circ H^{\infty}_{J_{s,t,\tau}} + H^{\infty}_{J_{s,t,\tau}} \circ \partial^{\infty}_{J_{s(0)}}$$

which is the sum of the coefficients of the below maps.

$$\partial^{\infty}_{J_{s(0)}} \circ H^{\infty}_{J_{s,t,\tau}} =$$

$$\sum_{y,w} \sum_{\{\phi_1 \in \pi_2(x,y) | \mu(\phi_1) = -1\}} \sum_{\{\phi_2 \in \pi_2(y,w) | \mu(\phi_2) = 1\}} \#(\mathcal{M}_{J_{s,t,\tau}}(\phi_1)) \#(\widehat{\mathcal{M}}_{J_{s,t,\tau}}(\phi_2))[w, i - n_z(\phi_1) - n_z(\phi_2)]$$

$$H^{\infty}_{J_{s,t,\tau}} \circ \partial^{\infty}_{J_{s(0)}} =$$

$$\sum_{y,w} \sum_{\{\phi_2 \in \pi_2(x,y) | \mu(\phi_2) = 1\}} \sum_{\{\phi_1 \in \pi_2(y,w) | \mu(\phi_1) = -1\}} \#(\widehat{\mathcal{M}}_{J_{s,t,\tau}}(\phi_2)) \#(\mathcal{M}_{J_{s,t,\tau}}(\phi_1))[w, i - n_z(\phi_1) - n_z(\phi_2)]$$

The ends of the moduli space $\mathcal{M}_{J_{s,t,\tau}}(\phi)$ with $\mu(\phi) = 0$ contains only broken flow-lines. Because as an end there is no bubble attached to the interior or to the boundary of a disk. A bubble increases the Maslov index by at least 2 but $\mu(\phi) = 0$ prevents the boundary degenerations. Therefore $\Phi^{\infty}_{J_{s,t}} \circ \Phi^{\infty}_{J_{s,1-t}}$ is chain homotopic to the identity map $\mathbb{1}_{J_{s(0)}}$. The corresponding homology groups are isomorphic so $(HF^{\infty}(\alpha, \beta, s), J_{s(0)}) \simeq (HF^{\infty}(\alpha, \beta, s), J_{s(1)})$.

The chain map $\Phi^{\infty}_{J_{s,t}}$ defined on the complex $CF^{\infty}(\alpha, \beta, s)$ induces a map on the subcomplex $CF^{-}(\alpha, \beta, s)$ and thus on the quotient complex $CF^{+}(\alpha, \beta, s)$. For the chain complex $\widehat{CF}(\alpha, \beta, s)$ we can consider the same map $\Phi^{\infty}_{J_{s,t}}$ with $n_z(\phi) = 0$ and it will give the same result. This shows that the Heegaard Floer homology groups, for a fixed complex structure j over Σ, do not depend on the choice of the path of nearly symmetric almost complex structures.

Moreover the Heegaard Floer homology groups have $\mathbb{Z}[U]$-module structure. Thus we need to chow that the map $\Phi^{\infty}_{J_{s,t}}$ commutes with the U action. (Similarly the induced maps of $\Phi^{\infty}_{J_{s,t}}$ on the subcomplex and the quotient complex commute with

U.) This follows easily from the definition

$$\begin{aligned}\Phi^{\infty}_{J_{s,t}} \circ U[x,i] &= \Phi^{\infty}_{J_{s,t}}[x,i-1] \\ &= \sum_{y} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=0\}} \#\mathcal{M}_{J_{s,t}}(\phi)[y,i-1-n_z(\phi)] \\ &= U(\sum_{y} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=0\}} \#\mathcal{M}_{J_{s,t}}(\phi)[y,i-n_z(\phi)]) \\ &= U \circ \Phi^{\infty}_{J_{s,t}}[x,i]\end{aligned}$$

Thus $\Phi^{\infty}_{J_{s,t}}$ respects the $\mathbb{Z}[U]$-module structure of the Heegaard Floer homology groups.

So far we showed that the chain complex is invariant under small perturbation of the path J_s. If a complex structure j is close to other complex structure j' over Σ, we can approximate the path J_s by the path $J_s{}'$ and the corresponding Heegaard Floer homology groups are isomorphic. We choose special complex structures j over Σ as mentioned in the auxiliary data in Section 5.1 and the set of allowed complex structures are dimension 2 and it is connected, see [29]. Therefore for two complex structures j_1 and j_2 over Σ there is a path j connecting j_1 to j_2. At each time $t \in [0,1]$ J_s^t is the corresponding path. By invariance of the Heegaard Floer homology groups (up to isomorphism) under small perturbations of the path, we obtain the corresponding homology groups for j_1 and j_2 are isomorphic. This finishes the proof of the first part of the statement. If Y is a rational homology three-sphere the Heegaard Floer homology groups are independent of the path J_s and te complex structure j over Σ.

Next, let us attempt to the proof of the case $b_1(Y) > 0$. In addition let us assume that the pointed Heegaard diagram $(\Sigma, \alpha, \beta, z)$ representing Y is strongly s-admissible, so that $CF^{\infty}(\alpha,\beta,s)$ is chain complex with the subcomplex $CF^{-}(\alpha,\beta,s)$ and the quotient complex $CF^{+}(\alpha,\beta,s)$.

Fix a complex structure j over the Heegaard surface Σ and take two paths $J_{s(0)}$ and $J_{s(1)}$ of nearly symmetric almost-complex structures contained in a contractible open neighborhood U of $Sym^g(j)$. Let us use the same map $\Phi^{\infty}_{J_{s,t}} : (CF^{\infty}(\alpha,\beta,s), \partial^{\infty}_{J_{s(0)}}) \to (CF^{\infty}(\alpha,\beta,s), \partial^{\infty}_{J_{s(1)}})$

$$\Phi^{\infty}_{J_{s,t}}[x,i] = \sum_{y} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=0\}} \#(\mathcal{M}_{J_{s,t}}(\phi)) \cdot [y,i-n_z(\phi)]$$

Note that the Heegaard diagram is strongly s-admissible and in the definition of the map $\Phi^{\infty}_{J_{s,t}}$ we fix the Maslov index and by the Theorem (5.2.4) there exist finitely many homotopy classes $\phi \in \pi_2(x,y)$, thus $\Phi^{\infty}_{J_{s,t}}$ is a finite sum and it is well-defined.

This map induces isomorphism on the Heegaard Floer homology groups as in the same way in the first case $b_1(Y) = 0$. Independence of the complex structure j over Σ of the homology groups is the same as in the first case too. We just need to show that the map $\Phi^{\infty}_{J_{s,t}}$ respects the module structure when $b_1(Y) > 0$, Heegaard Floer homology groups are modules over $\mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^* H_1(Y;\mathbb{Z})/Tors$. Thus we claim that for $\xi \in H_1(Y;\mathbb{Z})/Tors$,

$$\Phi^{\infty}_{J_{s,t}} \circ \mathcal{A}_{\xi} = \mathcal{A}_{\xi} \circ \Phi^{\infty}_{J_{s,t}}$$

Let $\xi \in H^1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta);\mathbb{Z})$ be an element such that it does not meet with any constant path. An element of this space is a path connecting $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, or equivalently corresponds to the intersection points of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. Let $f : \Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta) \to S^1$ be a map representing ξ and for a generic point $p \in S^1$, let $V = f^{-1}(p)$ be codimension 1 subspace. Let us define a map $h : CF^{\infty}(\alpha, \beta, s) \to CF^{\infty}(\alpha, \beta, s)$ as

$$h[x,i] = \sum_{y} \sum_{\{\phi \in \pi_2(x,y) | \mu(\phi)=0\}} \#\{(r,u) \in \mathcal{M}_{J_{s,t}} \mid u([0,1] \times \{r\}) \in V\} \cdot [y, i - n_z(\phi)]$$

where $r \in \mathbb{R}$ and h is map which does not change the grading between the generators. The space defined as

$$\mathcal{M} = \{(r,u) \in \mathbb{R} \times \mathcal{M}_{J_{s,t}}(\psi) \mid u([0,1] \times \{r\}) \in V\}$$

is a moduli space of dimension 2. The ends of this moduli space includes the broken flow-lines and the boundary degenerations occur as $r \to \infty$ or $r \to -\infty$. A sequence of disks u_i in $\mathcal{M}$ where $\psi \in \pi_2(x,w)$ can have limits corresponding to the following figures. The second figure corresponds to the composition $\partial^{\infty}_{J_{s(1)}} \circ h$ and the third one corresponds to the composition $h \circ \partial^{\infty}_{J_{s(0)}}$.

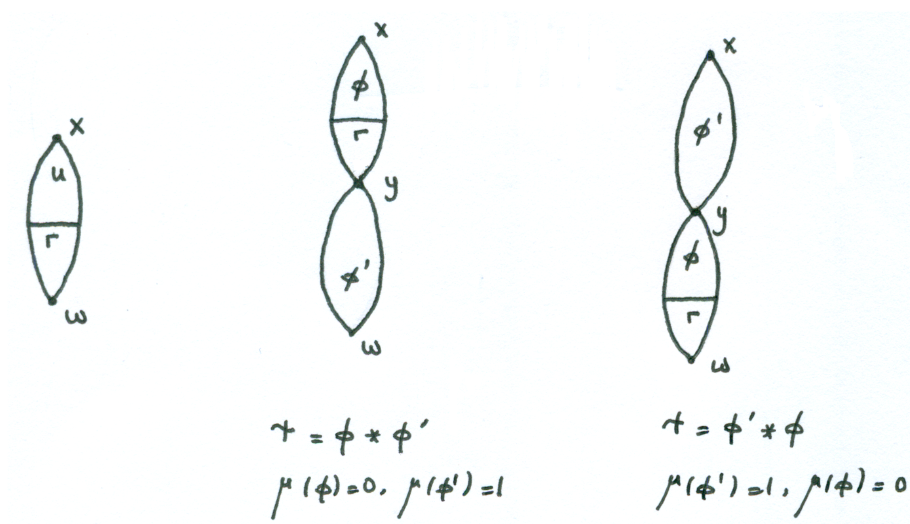


Figure 6.1: Illustration of the limit

If $r \to \infty$ or $r \to -\infty$ this corresponds to a bubbling where the bubble is attached to the disk at x or w. In that case the ends of the moduli space $\mathcal{M}$ correspond to the commutator of $\mathcal{A}_\xi$ and $\Phi^\infty_{J_{s,t}}$, $[\mathcal{A}_\xi, \Phi^\infty_{J_{s,t}}] = \mathcal{A}_\xi^{-1} \circ {\Phi^\infty_{J_{s,t}}}^{-1} \circ \mathcal{A}_\xi \circ \Phi^\infty_{J_{s,t}}$, see [29] for more detail. The ends where the bubble occurs correspond to the commutator of h and the boundary maps ∂^∞. We obtain then

$$\mathcal{A}_\xi \circ \Phi^\infty_{J_{s,t}} - \Phi^\infty_{J_{s,t}} \circ \mathcal{A}_\xi = \partial^\infty_{J_{s(1)}} \circ h - h \circ \partial^\infty_{J_{s(0)}}$$

The explanation of the equality is more like a sketch and the reader is referred to [29]. In the homology level the right hand side gives 0 implies that $\Phi^\infty_{J_{s,t}}$ commutes with the action $\mathcal{A}_\xi$, so it respects to the module structure of the Heegaard Floer homology groups when $b_1(Y) > 0$.

Hence in the end we proved that the Heegaard Floer homology groups are independent of the path J_s and the complex structure j over Σ. □

6.4 Admissibility

For a three-manifold Y with $b_1(Y) > 0$ we defined the Heegaard Floer homology groups by using admissible Heegaard diagrams for a given *Spin*^{c} structure s. In this section, we prove the existence of such admissible Heegaard diagrams. This construction is a result of a simple trick which is called special Heegaard moves.

Definition 36. Let Σ be a genus-g surface and γ be a simple closed and oriented curve on Σ. Winding along γ is diffeomorphism of Σ obtained by integrating a vector field X supported in a tubular neighbourhood of γ such that $d(\theta) > 0$ for the chosen coordinate system $(t, \theta) \in (-\epsilon, \epsilon) \times S^1$ as a tubular neighbourhood of $\gamma = \{0\} \times S^1$.

We can visualize this move as follows. Let (Σ, α, β) be a Heegaard diagram and take a curve γ on Σ such that γ meets α_1 transversally and γ is disjoint from other α curves. Let ϕ be a diffeomorphism of Σ such that ϕ winds along γ. If the image of α_1 meets with α_1 at $2k$ points in a tubular neighbourhood of γ then ϕ winds α_1 along γ k-times. Image of $\alpha_1$under ϕ is isotopic to α_1. More intuitively this is pushing α_1 in the direction of γ from the transverse intersection point of γ and α_1 such that the image of α_1 winds around γ k-times resulting $2k$ intersection points. $\phi(\alpha_1)$ is homologically same as α_1 and via this winding move we increase the intersection points by $2k$, where k many intersection points occur on the left of the intersection point and k many intersection points occur on the right of the intersection point as illustrated below.

Definition 37. Let (Σ, α, β) be a pointed Heegaard diagram representing a three-manifold Y. Take $s \in$ *Spin*$^c(Y)$ then the Heegaard diagram is called s-realized if there is an intersection point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that $s_z(x) = s$

Theorem 6.4.1. *Let Y be a closed, oriented three-manifold with a fixed Spin^c structure s, then there is an s-realized pointed Heegaard diagram representing Y.*

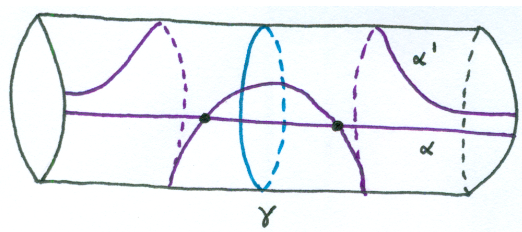


Figure 6.2: Winding α along γ

Proof. Let (Σ, α, β) be a Heegaard diagram for Y, choose a collection of closed, oriented and pairwise disjoint curves γ on Σ such that

$$\#(\alpha_i \cap \gamma_j) = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

We can assume that $\mathbb{T}_\beta \cap \mathbb{T}_\gamma$ is nonempty, if not by isotopy translates of β curves we can arrange that $\mathbb{T}_\beta \cap \mathbb{T}_\gamma$ is nonempty. Then choose a basepoint on Σ such that $z \in \Sigma - \alpha - \beta - \gamma$ and it is also disjoint from the tubular neighbourhood of γ, the winding region. As $\mathbb{T}_\beta \cap \mathbb{T}_\gamma \neq \emptyset$ take $x \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$ such that $x = \{x_1, .., x_g\}$ with $x_i = \beta_i \cap \gamma_i$. When α_i winds along γ_i a pair of intersection points are created, one of them is on the right of α_i the other one is on the left. We name them as ${x_i}^+(1)$ and ${x_i}^-(1)$. If we wind the new α_i along γ_i then another pair of intersection points are created ${x_i}^+(2)$ and ${x_i}^-(2)$.

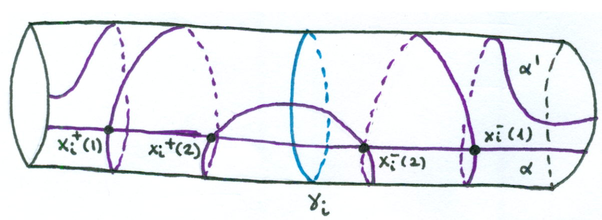


Figure 6.3: Newly created intersection points after winding

Note that $\{x_1^+(i_1), .., x_g^+(i_g)\} \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_\beta$ call it as $x(i_1, .., i_g)$. If we connect x_i to $x_i(k+1)$ by an arc in α_i and by an arc in β_i relative to endpoints we obtain a curve homologically same as γ_i on the Heegaard surface Σ.

Take two intersection points $x(i_1, .., i_g)$ and $x(j_i, .., j_g)$ and consider the difference of the corresponding $Spin^c$ structures. The theorem (3.4.9) and the image of the curve joining each components of the intersection points in $Sym^g(\Sigma)$ is defined by ϵ imply

$$s_z(x(i_1, .., i_g)) - s_z(x(j_1, .., j_g)) = ((i_1 - j_1)PD[\gamma_1], .., (i_g - j_g)PD[\gamma_g])$$

Note that the number i_k's correspond to the number of winding around γ_i, to compare the difference we take the winding along each γ_i which corresponds to the difference $(i_k - j - k)$ for each k.

For the intersection point $x(1,..,1)$, which is farthest from the intersection point, consider the corresponding *Spin*^{c} structure, it is independent of the number of times α_i winds along γ_i. Because winding α_i along γ_i produces an isotopic copy of α_i, and under this isotopy nonvanishing vector field changes but the homology class remains fixed under this isotopy. Call the *Spin*^{c} structure corresponding to the intersection number $x(1,..,1)$ as s_0. Then for any *Spin*^{c} structure corresponding to an intersection number differ from s_0 by a nonnegative multiple of $[\gamma_1],..,[\gamma_g]$. The coefficient is nonnegative since the intersection point $x(i_1,..,i_g)$ gives coefficients $(1-i_1),..,(1-i_g)$ depending on the place of $x(i_1,..,i_g)$ to the reference intersection point, so $1 - i_k \leq 0$ for each k.

If we choose $\{\gamma_1^-,..,\gamma_g^-\}$ parallel copies of γ_i with opposite orientation and wind along those γ_i^-'s also we can obtain *Spin*^{c} structures which differ from s_0 by some multiple of $[\gamma_1],..,[\gamma_g]$. As the group $H^2(Y;\mathbb{Z})$ is generated by Poincare duals of $[\gamma]$, it is possible to obtain all *Spin*^{c} structures in this way. Thus we can have intersection points for each *Spin*^{c} structures. □

With this winding argument we can reach the strong admissibility of the Heegaard diagram.

Definition 38. Let $s \in Spin^c(Y)$ be fixed. An s-renormalized periodic domain is a two chain $Q = \sum a_i D_i$ on Σ such that the boundary of Q is sum of α and β curves with intersection number $n_z(Q) = -\frac{\langle c_1(s), H(Q)\rangle}{2}$

Remark 6.4.2. The difference between s-renormalized periodic domains and periodic domains is the intersection number. Thus there is a $1-1$ correspondence between them via taking a periodic domain P to $P - \frac{\langle c_1(s), H(Q)\rangle}{2}[\Sigma]$ which is an s-renormalized periodic domain.

Theorem 6.4.3. *Given a closed, oriented three-manifold Y and $s \in Spin^c(Y)$, there exists a strongly s-admissible Heegaard diagram representing Y.*

Proof. For this *Spin*^{c} structure s there is a Heegaard diagram $(\Sigma, \alpha, \beta, s)$ such that for some $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $s_z(x) = s$. We claim that each s-renormalized periodic domain obtained from this pointed Heegaard diagram via using winding argument has both positive and negative coefficients. A periodic domain P gives rise to an s-renormalized periodic domain $Q = \sum(a_i - \frac{\langle c_1(s), H(Q)\rangle}{2})D_i$. Thus every nontrivial periodic domain P if $\langle c_1(s), H(Q)\rangle \geq 2n$, P has a coefficient $> n$ follows. Then by definition, the Heegaard diagram is strongly s-admissible.

If we choose to use rational coefficients for the homology and as $\mathbb{Q}$ is a field then the first betti number for Y, $b_1(Y)$ represents the vector space dimension of $H_1(Y;\mathbb{Q})$.

Let $b = b_1(Y)$ and there is a $1-1$ correspondence between periodic domains and $H^1(Y) \simeq Hom(H_1(Y), \mathbb{Z}) \simeq H_1(Y)$ by the Universal Coefficient theorem. Also by the above remark there is a $1-1$ correspondence between periodic domains and s-renormalized periodic domains. Thus we can choose $\{Q_1, .., Q_b\}$ as a basis for the vector space of s-renormalized periodic domains. An s-renormalized periodic domain Q can be determined by α coefficients of its boundary uniquely, since by definition $Q = \sum a_i D_i$ if we fix α parts of the boundary and it is a two-chain it follows that the β coefficients of the boundary can be determined to give Q. For each basis element Q_i, the map sending each basis element to its α coefficients of the boundary induce an injective map between vector spaces. For each basis element (if necessary reorder) the boundary can be expressed as $\partial Q_i = \alpha_i + \sum\limits_{j=b+1}^{g} a_{i,j}\alpha_i + b_{i,j}\beta_j$ with rational coefficients. Choose closed, oriented γ curves which are pairwise disjoint on Σ and choose basepoints on γ curves as $w_i \in \gamma_i$ for $1 \le i \le b$ such that w_i is disjoint from each α and β curves. Then let $c_i = \max_{j=1,..b} |n_{w_i}(Q_j)|$ and choose sufficiently large integer N such that $N > b(\max_{i=1,..,b} c_i)$

We obtain new periodic domains $\{Q_1^{'}, .., Q_b^{'}\}$ if we wind α curves $\{\alpha_1, .., \alpha_b\}$ N times around $\{\gamma_1, .., \gamma_b\}$ and N times in reverse direction along $\{\gamma_1^-, .., \gamma_b^-\}$, the parallel copies of $\{\gamma_1, .., \gamma_b\}$. The basepoints w_i are on γ_i not on the parallel copies, thus winding α_i along γ_i N times increases the intersection number of Q_i with the subvariety V_{w_i} by N. The boundary of Q_i contains α_i and after winding we increase the intersection number giving $n_{w_i}(Q_i^{'}) = n_{w_i}(Q_i) + N$. By the definition of N, $N > b \cdot c_i$. Now the sum

$$n_{w_i}(Q_i) + b \cdot c_i = n_{w_i}(Q_i) + b \cdot \max_{j=1,..,b} |n_{w_i}(Q_j)| \ge (b-1) \cdot \max_{j=1,..,i-1,i+1,..,b} |n_{w_i}(Q_j)|$$

Note that $n_{w_i}(Q_j)$ and $n_{w_i}(Q_j^{'})$ differ if $i = j$ otherwise the basepoint $w_i \in \gamma_i$ and boundary of Q_j does not contain α_i for $i \neq j$. Thus $\max_{j=1,..,i-1,i+1,..,b} |n_{w_i}(Q_j)| = \max_{j=1,..,i-1,i+1,..,b} |n_{w_i}(Q_j^{'})|$ If we combine the arguments above we have

$$\begin{aligned} n_{w_i}(Q_j^{'}) &= n_{w_i}(Q_i) + N \\ &= n_{w_i}(Q_i) + b \cdot c_i \\ &= (b-1) \cdot \max_{j=1,..,i-1,i+1,..,b} |n_{w_i}(Q_j)| \\ &= (b-1) \cdot \max_{j=1,..,i-1,i+1,..,b} |n_{w_i}(Q_j^{'})| \end{aligned}$$

If we choose the basepoints w_i^- on the parallel copies γ_i^- of γ curves each argument follows similarly but changes sign due to the orientation. As we wind along γ_i^- in reverse direction $n_{w_i}(Q_i^{'}) = n_{w_i}(Q_i) - N$ then we have:

$$\begin{aligned} n_{w_i}(Q_j^{'}) &= n_{w_i}(Q_i) - N \\ &= n_{w_i}(Q_i) - b \cdot c_i \\ &= -(b-1) \cdot \max_{j=1,.,i-1,i+1,.,b} |n_{w_i}(Q_j)| \\ &= -(b-1) \cdot \max_{j=1,.,i-1,i+1,.,b} |n_{w_i}(Q_j^{'})| \end{aligned}$$

For each s-renormalized periodic domain Q express it using the basis elements $\{Q_i\}_{i=1}^{b}$ then using the winding argument we obtain linear combination of $\{Q_i^{'}\}_{i=1}^{b}$ such that we can find basepoints w with n_w is positive and a basepoint w' with $n_{w'}$ is negative. Thus each s-renormalized periodic domain has both positive and negative coefficients and the new Heegaard diagram is strongly s-admissible. □

If two strongly s- admissible Heegaard diagrams are isotopic via an isotopy such that all the intermediate Heegaard diagrams are also strongly s-admissible are called *strongly s-isotopic*. If two strongly s-admissible Heegaard diagrams are isotopic by an isotopy which does not cross the basepoint z then they are strongly s-isotopic. The similar argument for weakly s-admissible Heegaard diagrams is defined as follows. If two weakly admissible Heegaard diagrams are isotopic such that the isotopy does not cross the basepoint then they are called weakly s-isotopic. The important result is the following.

Proposition 6.4.4. *A weakly s-admissible Heegaard diagram is weakly s-isotopic to a strongly s-admissible Heegaard diagram.*

The detailed description and proofs of the statements about the admissibility condition can be found in [29].

So far we only studied the dependence of Floer homology groups on the choice of coherent orientation system and the choice of the complex structure over the Heegaard surface and the path of nearly symmetric almost-complex structure over the symmetric product space. Moreover Floer homology groups are also independent of the chosen Heegaard diagrams. To prove the latter we need to study the dependence of the Floer homology groups to the Heegaard moves, thus we can understand the difference between the homology groups corresponding to given two different Heegaard diagrams representing the same three-manifold. This main result is in [29].

Chapter 7

KNOT FLOER HOMOLOGY

So far we gave the definition of Heegaard Floer homology with some examples and properties. In this chapter we study knot Floer homology whose definition is very similar to Heegaard Floer homology. It is defined in general for oriented links L in a closed, oriented three-manifold Y, but we continue on the subject with oriented nullhomologous knots K in a closed, oriented three-manifold Y. First we define knot Floer complex for oriented knots K in S^3, then we study knot filtration. We mention some important properties of knot Floer homology. Moreover, we introduce another knot invariant Khovanov homology. First we briefly describe the the chain complex and prove the invariance of the homology of the chain complex, which we is defined as Khovanov homology. Then we present some basic similarities of these two knot invariants. In this part we basically follow [32], [33], [8], and [3].

7.1 Preliminaries

Proposition 7.1.1. *If we have an oriented, n-component link in Y then it represents a knot in $Y\#^{n-1}(S^1 \times S^2)$*

Proof. We focus on knots in closed, oriented three-manifolds because by a simple trick it is possible to pass from links to knots. We connect n-component link L by 1-handles on Y as follows. First take pairs of points $\{p_i, q_i\}$ on L for $i \leq i \leq n-1$ which are pairwise grouped. If the pair (p_i, q_i) is identified for each i then L becomes connected. Let these points (p_i, q_i) represent the attaching spheres of a 1-handle to be attached to Y. Thus if we attach 1-handle to Y for each pair of points then it connects the components of the link which contains p_i and the one contains q_i by a band such that the orientation of the band is compatible with the orientation of the components of L. After attaching the 1-handles to Y this corresponds to taking connected sum as $Y\#^{g-1}(S^1 \times S^2)$. □

Therefore, this proposition justifies why it suffices to study with knots.

Let (Σ, α, β) be a Heegaard diagram for a closed, oriented three-manifold Y with two base points $z, w \in \Sigma - \alpha - \beta$. Let us connect z and w via an arc γ_1 in $\Sigma - \alpha$ and γ_2 in $\Sigma - \beta$. If we push these two arcs into handlebodies U_0 and U_1, staying disjoint from α and β curves, we have a closed embedded circle $\gamma_1 - \gamma_2$, a knot K in Y. This tells that choosing a second base point automatically generates a knot in Y and $(\Sigma, \alpha, \beta, w, z)$ is called a doubly pointed Heegaard diagram for the manifold Y compatible with the knot K.

We can have the above analogy using Morse theory. First choose a Morse-Smale pair (f, g) for Y where f is a self-indexing Morse function and g is a Riemannian metric on Y. Let $(\Sigma_g, \alpha, \beta)$ be a Heegaard diagram for Y with two base points z and w. We can choose Morse function f on Y so that it has only one index-3 and index-0 critical points, then take two trajectories η_1 and η_2 connecting index-3 and index-0 critical points passing through the base points w and z respectively. Then we obtain a closed, embedded circle $\eta_1 - \eta_2$ in Y which is a knot. Note that we can choose η_1 and η_2 disjoint from α and β curves. for each point we have unique flow line passing through it and two distinct flow lines do no meet, for detailed description see [22]

Note that these two descriptions represent the same knot in three manifold. In the second description push two flow lines η_1 and η_2 into the handlebodies remaining disjoint from α and β curves gives us the knot in the first description.

Proposition 7.1.2. *Every knot K in Y can be represented by a doubly pointed Heegaard diagram.*

Proof. Let us consider a height function from the knot K into $\mathbb{R}$ which is a Morse function. We can choose it to have two critical points with image of maximum is 3 and minimum is 0. Then if we add 1 and 2 handles to tubular neighborhood of K which is a solid torus, we obtain Heegaard decomposition for Y. Equivalently, we can extend height function on K to a self-indexing Morse function on Y. K is disjoint from α and β curves and it intersects with the Heegaard surface. Call these points as w and z which are disjoint also from α and β curves so they really are base points. □

For an oriented, embedded, null-homologous knot K in a closed, oriented three-manifold Y, we associate to the pair (Y, K) a Heegaard diagram $(\Sigma_g, \alpha, \beta_0, \mu)$ where Σ_g is Heegaard surface, α is set of attaching circles $\{\alpha_1, .., \alpha_g\}$, β_0 is set of $g-1$ attaching circles $\{\beta_2, .., \beta_g\}$, and μ is closed, embedded circle on Σ_g disjoint from β_0 curves which is meridian of knot K on the Heegaard surface. Note that $(\Sigma, \alpha, \beta_0)$ represents the knot complement $Y - nd(K)$. Because if we attach 1-handles on the 0-handle along α-curves then 2-handles along β_0-curves, in order to obtain three manifold Y we need to attach a 2-handle along μ and a 3-handle to cap off. We can think of a 3-handle attached to a 2-handle as a solid torus, as they correspond to 0-handle and 1-handle respectively and this is neighborhood of K. Thus for $\beta = \beta_0 \cup \{\mu\}$, we have a Heegaard diagram $(\Sigma_g, \alpha, \beta)$ for Y.

Definition 39. A marked Heegaard diagram for a knot (Y, K) is $(\Sigma, \alpha, \beta_0, \mu, m)$ where $m \in \mu \cap (\Sigma - \alpha_1 - .. - \alpha_g)$ is on μ.

The difference between marked Heegaard diagram and pointed Heegaard diagram is for the pointed Heegaard diagram, the base point is chosen on Σ disjoint from all α and β curves, whereas for marked Heegaard diagram we insist m to be on μ and disjoint from α and β_0 curves.

Remark 7.1.3. We can obtain doubly pointed Heegaard diagram for Y from a marked Heegaard diagram for (Y, K). Choose an arc γ intersecting transversally with μ at the marked point m, then call initial and terminal points as z and w respectively. K is an oriented knot and longitude λ of K has the same orientation with K. Choose an orientation of γ to agree with orientation of λ so that γ is an arc from z to w.

There is no much difference studying with a Heegaard diagram for Y or a Heegaard diagram for Y compatible with a null-homologous knot in it. In the latter, we have a second base point and it specifies a knot in the three-manifold. We will use doubly pointed Heegaard diagram for Y having in mind it comes from a marked Heegaard diagram for (Y, K). In order to define Heegaard Floer homology for (Y, K) we have the same ingredients as before. Let $(\Sigma, \alpha, \beta, w, z)$ be doubly pointed Heegaard diagram for Y then $Sym^g(\Sigma)$ is symmetric product space, $\mathbb{T}_\alpha = \alpha_1 \times .. \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times .. \times \beta_g$ are totally real tori in $Sym^g(\Sigma)$. For $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\pi_2(x, y)$ is the set of homotopy classes of disks connecting x and y. For a base point $v \in \Sigma - \alpha - \beta$ the local multiplication number is $n_v(\phi) = \#\phi^{-1}(\{v\} \times Sym^g(\Sigma))$. A complex structure j on Σ induces a complex structure $Sym^g(j)$ on $Sym^g(\Sigma)$ such that holomorphic representatives of Whitney disks ϕ in the moduli space $\mathcal{M}(\phi)$ has $n_v(\phi) \geq 0$ by the Theorem (4.0.10).

Defining Heegaard Floer homology for a fixed $s \in Spin^c(Y)$ we considered intersection points $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that $s_w(x) = s$ and we count the number of pseudo-holomorphic disks intersecting with the basepoint w. However, in knot Floer homology with a second base point we have filtration by the following arguments.

Let $(\Sigma, \alpha, \beta, w)$ be a Heegaard diagram for Y. We defined the map $s_w : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to Spin^c(Y)$ in Section 3.4 where we obtain $Spin^c$ structures from intersection points. Now we will try to understand the similar map for (Y, K) using surgery on (Y, K).

Definition 40. Consider ϵ-neighbourhood of null-homologous knot K in Y with meridian μ. Take this neighbourhood of K which is a solid torus and repaste it by identifying μ with the curve $p\mu + q\lambda$ with $(p, q) = 1$ and λ is longitude of K.

Now consider zero surgery $Y_0(K)$. We identify meridian μ of the repasted torus with λ and it will bound a disk in Y now.

Let $(\Sigma, \alpha, \beta_0, m)$ be Heegaard diagram for (Y, K). Note that meridian μ of K is on the Heegaard surface and we can view longitude λ of K lying on the Heegaard surface as follows. Consider handle decomposition of Y. First we add 1-handles along α-curves on the 0-handle, then we add 2-handles along β_0-curves. The resulting manifold represents the knot complement with boundary a torus. There is a longitude λ of the knot on this boundary and it is disjoint from β_0-curves. λ can lie on 2-handles which we attach on β_0-curves, so we can move λ down to the Heegaard surface remaining disjoint from β_0-curves. Thus longitude is on the Heegaard surface, disjoint from β_0-curves intersecting transversally at a single point with μ.

Let us study the map $\underline{s}_m : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to Spin^c(Y_0(K))$ where the set $\underline{Spin}^c(Y, K) := Spin^c(Y_0(K))$ denotes the *relative* $Spin^c$ *structures* for (Y, K).

Proposition 7.1.4. $\underline{Spin}^c(Y, K) \simeq Spin^c(Y) \times \mathbb{Z}$

Proof. Let $\varphi : \underline{Spin}^c(Y, K) \to Spin^c(Y) \times \mathbb{Z}$ be a map. For $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\underline{s}_m(x) = \underline{t} \in \underline{Spin}^c(Y, K)$, the restriction $\underline{t}|_{Y-K}$ is same as restricting $s_w(x)$ to $Y - K$. Note that nonvanishing vector field on $Y - K$ can be extending uniquely to solid torus. Thus $\underline{t}|_{Y-K}$ can be extended uniquely to Y. Let φ maps $\underline{t} \in \underline{Spin}^c(Y, K)$ to the unique extension of $\underline{t}|_{Y-K}$ to Y, so φ is clearly $1 - 1$. A surface F whose boundary is a knot is called a *Seifert surface*. After a zero surgery, let $\widehat{F}$ be a surface in $Y_0(K)$ corresponding to F capped off by a disk, and let $[\widehat{F}]$ denote 2nd homology class in Y. Then we define φ as sending $\underline{t}$ to $\frac{1}{2}\langle c_1(\underline{t}), [\widehat{F}]\rangle$ on the 2nd factor. The question is if $\langle c_1(\underline{t}), [\widehat{F}]\rangle$ is an even number.

Let $s'_w : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \underline{Spin}^c(Y, K)$ be a usual map associating $Spin^c$ structures to intersection points. As $[\widehat{F}] \in H_2(Y_0(K), \mathbb{Z})$ let P be a periodic domain representing $[\widehat{F}]$ with a basepoint w. Then by [28] we have $\langle c_1(s'_w(y)), [\widehat{F}]\rangle = \chi(P) + 2\overline{n}_y(P)$ where $\overline{n}_y$ is the generalization of local multiplicity $n_y(P)$ in [28] such that as y is in the interior of D_i, we have $\overline{n}_y(P) = 1$. By Lemma 7.3 in [28], $\chi(P)$ is even integer finishing the discussion. Thus we have the desired isomorphism in the statement.

Let $(\Sigma, \alpha, \beta_0, \mu, m)$ be a marked Heegaard diagram of (Y, K) and let us define the map $\underline{s}_m(x) : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to Spin^c(Y_0(K))$. After a zero surgery on Y, meridian μ is sent to longitude λ via $\mu \to 0 \cdot \mu + \lambda$, so we obtain $Y_0(K)$. Let us replace meridian with a longitude λ such that λ winds along μ once without crossing the marked point. With this isotopy, we increase the number of intersection points. A pair of intersection points (x', x'') are created closest to x. Let $\gamma = \beta_0 \cup \{\lambda\}$ then $(\Sigma, \alpha, \gamma, w)$ is a Heegaard diagram for $Y_0(K)$ after the zero surgery. Let $\underline{s}'_w : \mathbb{T}_\alpha \cap \mathbb{T}_\gamma \to Spin^c(Y_0(K))$ be the usual map from intersection points to $Spin^c$ structures and by the Theorem 3.4.9, we have for any $x', x'' \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$, the difference is $s'_w(x') - s'_w(x'') = PD[\epsilon(x', x'')]$. Let a be an arc connecting x' and x'' on α curves and b be an arc connecting x' and x'' on β curves, then the curve $a - b$ already bounds a disk on Σ_g. Thus $a - b$ being null homologous in $H_1(Y_0(K) : \mathbb{Z})$ implies $\epsilon(x', x'') = 0$ and $s'_w(x') = s'_w(x'')$. therefore we define the map $\underline{s}_m : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to Spin^c(Y_0(K))$ as sending an intersection point to $\underline{s}_m(x) = s'_w(x') = s'_w(x'')$ giving a $Spin^c$ structure $s'_w(x')$ over $Y_0(K)$.

We could have taken the point z rather than w. Note that in Figure (7.1) basepoints w and z are in the same component of $\Sigma - \alpha - \gamma$. To obtain a $Spin^c$ structure from intersection points, we use the basepoint to determine a trajectory connecting index-3 and index-0 critical points. Instead of using such a trajectory passing from w we can use the trajectory passing through z. The corresponding nonvanishing vector

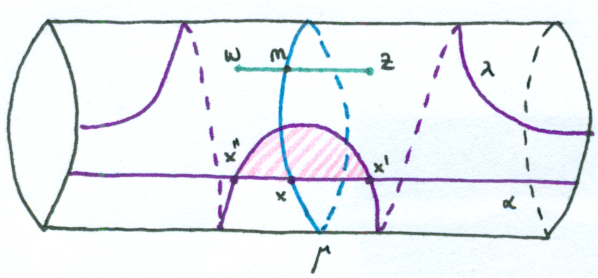


Figure 7.1: Marked point and creation of new intersection point by winding

fields are the same except finitely many 3-balls thus $s'_w(x') = s'_z(x')$ and the map $\underline{s}_m : \mathbb{T}_\alpha \cap \mathbb{T}_\gamma \to \underline{Spin}^c(Y, K)$ is well-defined.

□

Let us understand how the map $\underline{s}_w : \mathbb{T}_\alpha \cap \mathbb{T}_\gamma \to \underline{Spin}^c(Y, K)$ depends on the variable.

Theorem 7.1.5. *Let K be an oriented knot in closed, oriented 3-manifold Y with a marked Heegaard diagram $(\Sigma, \alpha, \beta_0, \mu, m)$. Let $\beta = \beta_0 \cup \{\mu\}$, then given any $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and any $\phi \in \pi_2(x, y)$ we have*

$$\underline{s}_m(x) - \underline{s}_m(y) = [n_z(\phi) - n_w(\phi)] \cdot PD[\mu]$$

where $[\mu]$ is homology class of μ in $H_1(Y_0(K) : \mathbb{Z})$ such that $\#(\mu \cap F) = 1$, where F is a Seifert surface whose boundary is K.

Before the proof of the statement, we give preliminaries for Heegaard triple diagrams and holomorphic triangles. We basically follow Section 8 of [29].

Definition 41. A Heegaard triple diagram of genus g is an oriented two manifold and 3 g-tuples α, β and γ complete sets of attaching circles for handlebodies U_α, U_β and U_γ respectively

Let

$$Y_{\alpha,\beta} = U_\alpha \cup U_\beta$$
$$Y_{\beta,\gamma} = U_\beta \cup U_\gamma$$
$$Y_{\alpha,\gamma} = U_\alpha \cup U_\gamma$$

be the three-manifolds obtained gluing those handlebodies. A Heegaard triple diagram naturally specifies a cobordism $X_{\alpha,\beta,\gamma}$ between these 3 manifolds which can be described as a pair of pants connecting $Y_{\alpha,\beta}$, $Y_{\beta,\gamma}$, and $Y_{\alpha,\gamma}$ with the induced orientation on boundary

$$\partial X_{\alpha,\beta,\gamma} = -Y_{\alpha,\beta} - Y_{\beta,\gamma} + Y_{\alpha,\gamma}$$

Let Δ be a 2 simplex with vertices v_α, v_β and v_γ labeled clockwise. Let e_γ be an edge between v_α and v_β, e_α be an edge between v_β and v_γ, and e_β be an edge between v_α and v_γ.

Definition 42. Let $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $y \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, and $w \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ then a continuous map $u : \Delta \to Sym^g(\Sigma)$ with the following conditions

$$u(v_\gamma) = x \ u(e_\alpha) \subseteq \mathbb{T}_\alpha$$
$$u(v_\alpha) = y \ u(e_\beta) \subseteq \mathbb{T}_\beta$$
$$u(v_\beta) = w \ u(e_\gamma) \subseteq \mathbb{T}_\gamma$$

is called a Whitney triangle connecting x, y and w.

We say that two Whitney triangles are homotopic if the maps are homotopic through maps which are all Whitney triangles.

Proof of the Theorem 7.1.5. Let $(\Sigma, \alpha, \beta, \gamma, w)$ be a Heegaard triple diagram of the cobordism $X_{\alpha,\beta,\gamma}$ from $Y_{\alpha,\beta}$ to $Y_{\alpha,\gamma}$ which are Y and $Y_0(K)$ respectively. Note that we begin with a marked Heegaard diagram for (Y, K) and from this we obtain doubly pointed Heegaad diagram with two basepoints z and w, which are in the same components of $\Sigma - \alpha - \gamma$.

The manifold obtained from gluing $U_\beta \cup U_\gamma$ represents the 3 manifold $S^3 \#^{g-1}(S^1 \times S^2)$ with $\beta = \beta_0 \cup \{\mu\}$. Note that $\beta = \beta_0 \cup \{\mu\}$ and $\gamma = \beta_0 \cup \{\lambda\}$ differ only by one attaching circle. Thus the Heegaard diagram corresponds to the 3 manifold $S^3 \#^{g-1}(S^1 \times S^2)$ with one intersection point, call it θ.

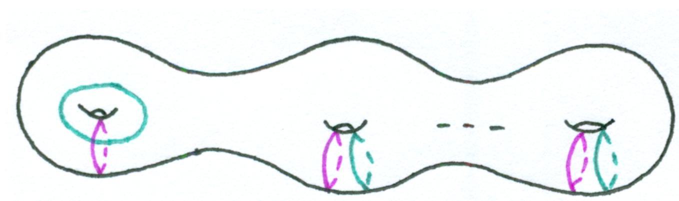


Figure 7.2: A Heegaard diagram for $\#^g S^1 \times S^2$

Let $\psi \in \pi_2(x, \theta, y)$ be a Whitney triangle, where $\theta \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ and $s'_w(y) : \mathbb{T}_\alpha \cap \mathbb{T}_\gamma \to Spin^c(Y_0(K))$ is the natural map. Thus we have:

$$s'_w(y) = \underline{s}_m(x) + [n_w(\psi) - n_z(\psi)]PD[\mu] \tag{7.1}$$

Take $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $k \in \mathbb{Z}$, and let

$$S(x,k) = \{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma | \psi \in \pi_2(x,\theta,y) \text{ Whitney triangle, } n_w(\psi) - n_z(\psi) = k\}$$

The set $S(x,k)$ contains only one $Spin^c$ equivalence class of intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ for $Y_0(K)$. If $y_1, y_2 \in S(x,k)$ thus

$$\begin{aligned} s'_w(y_1) - s'_w(y_2) &= \underline{s}_m(x) + [n_w(\psi_1) - n_z(\psi_1)]PD[\mu] \\ &- \underline{s}_m(x) - [n_w(\psi_2) - n_z(\psi_2)]PD[\mu] \\ &= 0 \end{aligned}$$

Thus they correspond to the same (up to homology) $Spin^c$ structure over $Y_0(K)$. Moreover, if $k = 0$ then $s'_w(y) = \underline{s}_m(x)$. As $n_w(\psi) = n_z(\psi)$, y corresponds to x' or x'' and $\underline{s}_m(x) = s'_w(y)$ follows. To show the Equation(7.1), it suffices to show for holomorphic triangles staying inside the winding region, i.e. the region where λ winds along μ sufficiently many times. If $n_w(\psi_k) - n_z(\psi_k) = k \leq 0$, then wind λ along μ sufficiently many times on one side of μ such that $n_w(\psi_k) = 0$ and $n_z(\psi_k) = k$. Then $\psi_k \in \pi_2(x, \theta, x'_k)$, where x'_k is newly created intersection points on $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ closest to x. By [28] the difference is:

$$\begin{aligned} \langle c_1(s'_w(x_i), [\widehat{F}])\rangle - \langle c_1(s'_w(x_j), [\widehat{F}])\rangle &= \chi(P) + 2\overline{n}_{x_i}(P) - \chi(P) - 2\overline{n}_{x_j}(P) \\ &= 2(\overline{n}_{x_i}(P) - \overline{n}_{x_j}(P)) = 2(i-j) \end{aligned}$$

This verifies the equation (7.1).

In order to obtain the desired equation in the statement, let $\psi \in \pi_2(y, \theta, y')$ be a Whitney triangle where $y' \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $n_w(\psi) = n_z(\psi) = 0$, and $\phi \in \pi_2(x,y)$ is any Whitney disk. Then by juxtaposing the triangle ψ by ϕ, we obtain a Whitney triangle $\psi' = \phi * \psi \in \pi_2(x, \theta, y')$. Then,

$$\begin{aligned} n_w(\psi') - n_z(\psi') &= n_w(\phi * \psi) - n_z(\phi * \psi) \\ &= n_w(\phi) + n_w(\psi) - n_z(\phi) - n_z(\psi) \\ &= n_w(\phi) - n_z(\phi) \end{aligned}$$

and

$$\begin{aligned} s'_w(y') &= \underline{s}_m(x) + \underline{s}_m(x) + [n_w(\psi) - n_z(\psi)]PD[\mu] \\ &= \underline{s}_m(x) + [n_w(\phi) - n_z(\phi)]PD[\mu] \end{aligned}$$

By definition $s'_w(y') = \underline{s}_m(y)$, thus $\underline{s}_m(y) - \underline{s}_m(x) = [n_w(\phi) - n_z(\phi)]PD[\mu]$ follows. □

7.2 Filtration and Bigrading

In this section we give the definition of a filtered complex and bigrading on $C(K)$, for an oriented nullhomologous knot K in three-manifold Y.

Definition 43. Let S be a partially ordered set, i.e., reflexive, antisymmetric, and transitive, then an S-filtered group C is a free Abelian group generated by a set of generators $\mathcal{G}$ admitting a map $\mathcal{F} : \mathcal{G} \to S$.

Elements of S-filtered group is of the form as a formal sum $\sum_{\sigma \in \mathcal{G}} a_\sigma \cdot \sigma$ with integer coefficients.

Let $(C, \mathcal{F}, \mathcal{G})$ and $(C', \mathcal{F}', \mathcal{G}')$ be two S-filtered groups with two elements $a = \sum_{\sigma \in \mathcal{G}} a_\sigma \cdot \sigma \in C$ and $b = \sum_{\sigma \in \mathcal{G}'} b_\sigma \cdot \sigma \in C'$, we can compare these two elements as follows. We say that $a \leq b$ if

$$\max_{\{\sigma \in \mathcal{G} | a_\sigma \neq 0\}} \mathcal{F}(\sigma) \leq \max_{\{\sigma \in \mathcal{G}' | b_\sigma \neq 0\}} \mathcal{F}'(\sigma)$$

A morphism between S-filtered groups $\phi : (C, \mathcal{F}, \mathcal{G}) \to (C', \mathcal{F}', \mathcal{G}')$ is a group homomorphism and for every $a \in C$ we have $\phi(a) \leq a$.

Definition 44. An S-filtered chain complex is an S-filtered group and the boundary map is an S-filtered morphism.

Definition 45. A chain map between S-filtered complexes is an S-filtered morphism.

Definition 46. If $T \subset S$ is subset of a partially ordered set such that for $b \in T$, for every $a \in S$ with $a \leq b$ is also an element of T. Then if $(C_*, \partial, \mathcal{F})$ is an S-filtered chain complex, the subset T of S gives subcomplex of$(C_*, \partial, \mathcal{F})$.

With given definitions consider the following. Let Y be a closed, oriented three-manifold with a pointed Heegaard diagram $(\Sigma, \alpha, \beta, z)$ representing it. For a fixed $s \in Spin^c(Y)$, let $CF^\infty(\alpha, \beta, s)$ be a free Abelian group generated by elements of the form $[x, i]$ with $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that $s_z(x) = s$ and $i \in \mathbb{Z}$, so $[x, i] \in (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \mathbb{Z}$. Then we have the following:

Proposition 7.2.1. *The map $\mathcal{F}[x, i] = i$ induce a filtration on the chain complex $CF^\infty(\alpha, \beta, s)$, therefore on the subcomplex $CF^-(\alpha, \beta, s)$, and on the quotient complex $CF^+(\alpha, \beta, s)$.*

Proof. First $CF^\infty(\alpha,\beta,s)$ is a $\mathbb{Z}$-filtered group. Note that $\mathbb{Z}$ is partially ordered set and by definition $CF^\infty(\alpha,\beta,s)$ is a free Abelian group generated by elements in $(\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \mathbb{Z}$ and $\mathcal{F}[x,i] = i$ is a well-defined map on the set of generators. Every element of $CF^\infty(\alpha,\beta,s)$ is of the form $\sum\limits_{[x,i]} a_{[x,i]}[x,i]$ with $a_{[x,i]} \in \mathbb{Z}$. Thus $CF^\infty(\alpha,\beta,s)$ is a $\mathbb{Z}$-filtered group.

Next $(CF^\infty(\alpha,\beta,s), \partial^\infty)$ is a $\mathbb{Z}$-filtered chain complex. We need to show that ∂^∞ is a $\mathbb{Z}$-filtered morphism, so for every $a \in CF^\infty(\alpha,\beta,s)$, $\partial^\infty(a) \leq a$. It is sufficient to show the latter on the generators. For any $[x,i]$,

$$\partial^\infty[x,i] = \sum_y \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} \#(\widehat{\mathcal{M}}(\phi)) \cdot [y, i - n_z(\phi)]$$

$$\begin{aligned}\max_{\{y|\#(\widehat{\mathcal{M}}(\phi))\neq 0\}} \mathcal{F}[y, i-n_z(\phi)] &= \max_{\{y|\#(\widehat{\mathcal{M}}(\phi))\neq 0\}} (i - n_z(\phi)) \\ &< i \\ &= \mathcal{F}[x,i]\end{aligned}$$

which follows by the nonnegativity of the intersection number of pseudo-holomorphic disks. Thus $\partial^\infty[x,i] \leq [x,i]$ and ∂^∞ is a $\mathbb{Z}$-filtered morphism. Therefore, the map $\mathcal{F} : (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \mathbb{Z} \to \mathbb{Z}$ induces a filtration on the chain complex $(CF^\infty(\alpha,\beta,s), \partial^\infty)$.

Let $T \subset \mathbb{Z}$ be the set of negative integers. Thus for any $b \in T$, for any $a \in \mathbb{Z}$ with $a \leq b$ is also in T. The by definition T gives rise to a filtration on the subcomplex of $(CF^-(\alpha,\beta,s), \partial^-)$ and on the quotient complex $CF^+(\alpha,\beta,s)$. □

Let K be a knot in 3-sphere and let $(\Sigma,\alpha,\beta,z,w)$ be a doubly pointed Heegaard diagram for (S^3,K) and let $C(K)$ be free Abelian group generated by $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. As there is only one $Spin^c$ structure s over S^3, thus every intersection point gives rise to the same equivalence class $Spin^c$ structure. On $C(K)$ let us define two gradings corresponding to the function $F, G : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{Z}$.

Definition 47. Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ be any two intersection points, then define

$$f(x,y) = n_z(\phi) - n_w(\phi)$$

for any $\phi \in \pi_2(x,y)$.

Proposition 7.2.2. *The map f defined above is well-defined. i.e., f is independent of representative of ϕ and the chosen $\phi \in \pi_2(x,y)$.*

Proof. We discussed that the algebraic intersection number is independent of the representative of the homotopy class ϕ. Let $\psi \in \pi_2(x,y)$ be another homotopy class. As $\pi_2(x,y) \simeq \mathbb{Z} \oplus H^1(S^3\mathbb{Z}) \simeq \mathbb{Z}$. Then $\psi = \phi * k[S]$ for some $k \in \mathbb{Z}$ where S is positive generator of $\pi_2'(Sym^g(\Sigma))$, then

$$n_z(\psi) = n_z(\phi * k[S]) = n_Z(\phi) + kn_z([S]) = n_z(\phi) + k$$

and

$$f(x, y) = n_z(\psi) - n_w(\psi) = n_z(\phi) + k - n_w(\phi) - k = n_z(\phi) - n_w(\phi)$$

Thus f is independent of the chosen $\phi \in \pi_2(x, y)$ and it is well-defined. □

Proposition 7.2.3. *For any $x, y, p \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $f(x, y) + f(y, p) = f(x, p)$*

Proof. Let $\phi \in \pi_2(x, y)$ and $\psi \in \pi_2(y, p)$ then as $\psi * \psi \in \pi_2(x, p)$ the result follows easily from the definitions. □

Proposition 7.2.4 ([8]). *The map f can be lifted to a function $F : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{Z}$ in a unique way such that F satisfies two properties:*

1. $F(x) - F(y) = f(x, y)$
2. $\#\{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \mid F(x) = i\} = \#\{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \mid F(x) = -i\}(mod 2)$ *for every* $i \in \mathbb{Z}$

where the second property is called the additional symmetry.

Another map H satisfying the same properties with F differ from F by a constant. The description can be found in [8] and the second property of F is follows from the symmetry of the coefficients of the Alexander polynomial and the number of intersection points at each level i corresponds to a_ith coefficient of the Alexander polynomial which follows from the Fox calculus, see [35]. F is called the Alexander grading on $C(K)$ which correspond to the filtration. The map $G : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{Z}$ is called the Maslov grading and it corresponds to the homolgoy grading on the chain complex.

Definition 48. Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ then define $g(x, y) = \mu(\phi) - 2n_w(\phi)$

The map g is relative grading definition which is also called the *Maslov grading* and it is well-defined, i.e., it is independent of representative and the chosen $\phi \in \pi_2(x, y)$. For any $\psi \in \pi_2(x, y) \simeq \mathbb{Z}$ is of the form $\psi = \phi * k[S]$ for some $k \in \mathbb{Z}$, and by definition it follows that $g(x, y) = \mu(\psi) - n_w(\psi) = \mu(\phi) - n_w(\phi)$.

Proposition 7.2.5 ([8]). *Let $(\Sigma, \alpha, \beta, w)$ be a pointed Heegaard diagram for S^3, then the chain complex is $\widehat{CF}(\alpha, \beta, z) \simeq \mathbb{Z}$ with homology group $\widehat{HF}(\alpha, \beta, z) \simeq \mathbb{Z}$. The homology group is supported in grading 0 then the map g can be lifted to a map $G : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{Z}$ in a unique way.*

We give the definition of knot Floer chain complex for an oriented, nullhomologous knot K in S^3. Let $(\Sigma, \alpha, \beta, z, w)$ be a doubly pointed Heegaard diagram for (S^3, K). Then define $C(K)$ as a free Abelian group generated by $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. From each intersection point we obtain the same class of $Spin^c$ structure as there is only one. Choose a generic complex structure j over Σ and a path J_s of nearly symmetric almost-complex structure over $Sym^g(\Sigma)$. Then define the boundary map

$$\partial_K : C(K) \to C(K)$$
$$\partial_K(x) = \sum_y \sum_{\{\psi \in \pi_2(x,y) | \mu(\phi)=1, n_z(\phi)=n_w(\phi)=0\}} \#(\widehat{\mathcal{M}}(\phi)) \cdot y$$

Theorem 7.2.6. $(C(K), \partial_K)$ *is a chain complex. i.e.,* ${\partial_K}^2 = 0$

Proof. The difference between knot Floer complex and Heegaard Floer complex is the choice of a second basepoint w and we take into account the second algebraic intersection number $n_w(\phi)$. The proof of ${\partial_K}^2 = 0$ follows from counting the ends of the moduli space $\mathcal{M}(\phi)$ with $\mu(\phi) = 2$. The proof is same as the proof of the Theorem (5.1.2). □

For example let U be an unknot in S^3, where its regular neighbourhood is a solid torus. Let $(\Sigma, \alpha, \beta, z, w)$ be a Heegaard diagram for S^3 as in the figure (2.2) with the second basepoint w. Then there exists only one generator giving $C(K) \simeq \mathbb{Z}$ and as the boundary map is trivial ∂_K is trivial. Thus $H(U) \simeq \mathbb{Z}$.

Definition 49. With two grading F, G on $C(K)$, let $C_{i,j}$ be the Abelian group generated freely by the intersection points $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that $F(x) = i$ and $G(x) = j$.

Theorem 7.2.7. *Let K be a knot in S^3 with a doubly pointed Heegaard diagram for (S^3, K). Then the free Abelian group $C(K)$ can be decomposed as*

$$C(K) = \bigoplus_{i,j} C_{i,j}$$

giving $\partial_K(C_{i,j}) \subset C_{i,j-1}$.

Proof. Let $x \in C_{i,j}$ be a generator so $F(x) = i$ and $G(x) = j$. Consider the image under the boundary map

$$\partial_K(x) = \sum_y \sum_{\{\phi \in \pi_2(x,y) | \mu(\phi)=1, n_w(\phi)=n_z(\phi)=0\}} \#(\widehat{\mathcal{M}}(\phi)) \cdot y$$

Then

$$F(x) - F(y) = f(x,y) = n_z(\phi) - n_w(\phi) = 0$$
$$G(x) - G(y) = g(x,y) = \mu(\phi) - n_w(\phi) = 1$$

giving

$$F(x) = F(y) = i$$
$$G(y) = G(x) - 1 = j - 1$$

implies $y \in C_{i,j-1}$ showing $C_{i,j} \subset C_{i,j-1}$ □

Remark 7.2.8. The decomposition of $C(K) = \bigoplus_{i,j} C_{i,j}$ implies that we can decompose $H(K)$ as $H(K) = \bigoplus_{i,j} H_{i,j}(K)$. At each i-th level G corresponds to the homology grading and we have a chain complex $C_i(K)$ implies the decomposition on the homology groups.

7.3 Knot Filtration

In previous section we study knots in S^3 and give bigrading on $C(K)$. In this section we generalize this to oriented, nullhomologous knots in an oriented, closed three-manifold and define knot filtration.

Let K be an oriented null homologous knot in Y and let $(\Sigma, \alpha, \beta, w, z)$ be a doubly pointed Heegaard diagram for (Y, K). Fix an auxiliary data described in Section 5.1. A generic allowed complex structure j over Σ and a generic path J_s of nearly symmetric almost complex structure over $Sym^g(\Sigma)$. Fix a coherent orientation system o. Then we give a $\mathbb{Z} \times \mathbb{Z}$ filtration on the chain complex.

Let $CF^\infty(\alpha, \beta, w, z)$ be an Abelian group generated freely by $[x, i, j]$ where $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $i, j \in \mathbb{Z}$ with a differential map

$$\partial^\infty[x, i, j] = \sum_y \sum_{\{\phi \in \pi_2(x,y) | \mu(\phi)=1\}} \#(\widehat{\mathcal{M}}(\phi))[y, i - n_w(\phi), j - n_z(\phi)]$$

Theorem 7.3.1. *$(CF^\infty(\alpha, \beta, w, z), \partial^\infty)$ is a chain complex i.e. $(\partial^\infty)^2 = 0$*

Proof. The proof is similar to the proof of the Theorem 5.1.4 based on counting the ends of the moduli space $M(\phi)$ with $\mu(\phi) = 2$. □

There exists a U action on the chain complex defined as

$$U : CF^\infty(\alpha, \beta, w, z) \to CF^\infty(\alpha, \beta, w, z)$$

$$U[x, i, j] = [x, i - 1, j - 1]$$

The map U is a chain map so it commutes with the boundary maps ∂^∞ which can be verified easily and it gives a $\mathbb{Z}[U]$-module structure on the chain complex $CF^\infty(\alpha, \beta, w, z)$.

Definition 50. We define the filtration on $CF^\infty(\alpha, \beta, w, z)$ as

$$F : (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times (\mathbb{Z} \times \mathbb{Z}) \to \mathbb{Z} \times \mathbb{Z}$$

$$F[x, i, j] = (i, j)$$

The partial ordering on $(\mathbb{Z} \times \mathbb{Z})$ is given by $(i, j) \leq (i', j')$ if $i \leq i'$ and $j \leq j'$. Then it follows by definition, $CF^\infty(\alpha, \beta, w, z)$ is a $\mathbb{Z} \times \mathbb{Z}$-filtered group and the boundary map ∂^∞ is a $\mathbb{Z} \times \mathbb{Z}$-filtered morphism. It suffices to show on the generators of $CF^\infty(\alpha, \beta, w, z)$ that $\partial^\infty[x, i, j] \leq [x, i, j]$ for every $[x, i, j]$. This is true because

$$\begin{aligned} \max_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} F[y, i - n_w(\phi), j - n_z(\phi)] &= \max_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} (i - n_w(\phi), j - n_z(\phi)) \\ &\leq (i, j) \\ &= F[x, i, j] \end{aligned}$$

by the nonnegativity of $n_w(\phi)$ and $n_z(\phi)$ which follows from the Proposition 4.0.10. Thus the map $F : (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \mathbb{Z}^2 \to \mathbb{Z}^2$ induces a filtration on the chain complex $(CF^\infty(\alpha, \beta, w, z), \partial^\infty)$.

Let CFK^∞ denote the chain complex $CF^\infty(\alpha, \beta, w, z)$. Then the complex CFK^∞ can be also decomposed into a sum of complexes. For example, the generators $[x, i, j]$ and $[y, l, m]$ are contained in the same summand if there is a homotopy class of a Whitney disk $\phi \in \pi_2(x, y)$ such that $n_w(\phi) = i - l$ and $n_z(\phi) = j - m$. Because $[x, i, j]$ maps to $[y, l, m]$ under the differential map ∂^∞ as

$$\begin{aligned}\partial^\infty[x, i, j] &= \sum_y \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} \#(\widehat{\mathcal{M}}(\phi))[y, i - n_w(\phi), j - n_z(\phi)] \\ &= \sum_y \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} \#(\widehat{\mathcal{M}}(\phi))[y, l, m]\end{aligned}$$

The decomposition of the complex CFK^∞ into a sum of complexes can be interpreted via $Spin^c$ structures. We begin with a doubly pointed Heegaard diagram, however we know that from a marked Heegaard diagram for (Y, K) we can obtain a doubly pointed Heegaard diagram for a null homologous knot K in Y. Let $s \in Spin^c(Y)$ be a $Spin^c$ structure then by the isomorphism

$$\underline{Spin}^c(Y, K) \simeq Spin^c(Y) \times \mathbb{Z}$$

in the Proposition (7.1.4). Let $\underline{t} \in \underline{Spin}^c(Y, K)$ be the $Spin^c$ structure such that $\underline{t}$ projects to s. Then for the $Spin^c$ structure $\underline{t} \in \underline{Spin}^c(Y, K)$, the subset $CFK^\infty(\alpha, \beta, \underline{t}) \subset CF^\infty(\alpha, \beta, w, z)$ is generated by $[x, i, j]$ such that

$$s_w(x) = s \text{ and } \underline{s}_m(x) + (i - j)PD[\mu] = \underline{t}$$

where μ is meridian of K a closed curve in $Y_0(K)$ and $[\mu] \in H_1(Y_0(K))$ is the corresponding homology class.

Proposition 7.3.2. *$CFK^\infty(\alpha, \beta, \underline{t})$ is a subcomplex of $CF^\infty(\alpha, \beta, w, z)$*

Proof. We need to show that, image of each generator of $CFK^\infty(\alpha, \beta, \underline{t})$ under the boundary map is contained in $CFK^\infty(\alpha, \beta, \underline{t})$ and $(\partial^\infty)^2 = 0$ implies that restriction of ∂^∞ on $CFK^\infty(\alpha, \beta, \underline{t})$ also gives a chain complex, thus $CFK^\infty(\alpha, \beta, \underline{t})$ is a subcomplex of $CF^\infty(\alpha, \beta, w, z)$.

Let $[x, i, j]$ be a generator of CFK^∞ then $s_w(x) = s$ where s is the projection of $\underline{t}$ of the map in Proposition 7.1.4 and $\underline{s}_m(x) - (i - j)PD[\mu] = \underline{t}$

$$\partial^\infty[x, i, j] = \sum_{\{y\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta|s_w(y)=s\}} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} \#(\widehat{\mathcal{M}}(\phi))[y, i - n_w(\phi), j - n_z(\phi)]$$

Apply the Theorem 7.1.5 then for $\phi \in \pi_2(x,y)$

$$\underline{s}_m(x) - \underline{s}_m(y) = [n_z(\phi) - n_w(\phi)]PD[\mu]$$
$$\underline{t} - (i-j)PD[\mu] - \underline{s}_m(y) = [n_z(\phi) - n_w(\phi)]PD[\mu]$$

then

$$\underline{s}_m(y) + [i - n_w(\phi) - j + n_z(\phi)]PD[\mu] = \underline{t}$$

as desired. Thus $[y, i - n_w(\phi), j - n_z(\phi)] \in CFK^\infty(Y,K,\underline{t})$ and $CFK^\infty(Y,K,\underline{t})$ is a subcomplex. □

Theorem 7.3.3. *Let $\underline{t}_1$ and $\underline{t}_2$ be $Spin^c$ structures over $Y_0(K)$ extending the same $Spin^c$ structure then the corresponding chain complexes*

$$CFK^\infty(Y,K,\underline{t}_1) \text{ and } CFK^\infty(Y,K,\underline{t}_2)$$

are isomorphic as chain complexes.

Proof. Let $s \in Spin^c(Y)$ be the $Spin^c$ structure such that it is the restriction of both $\underline{t}_1$ and $\underline{t}_2$. The complex $CFK^\infty(Y,K,\underline{t}_1)$ is generated by $[x,i,j]$ such that

$$s_w(x) = s \text{ and } \underline{s}_m(x) + (i-j)PD[\mu] = \underline{t}_1$$

and $CFK^\infty(Y,K,\underline{t}_2)$ is generated by $[x,l,m]$ such that

$$s_w(x) = s \text{ and } \underline{s}_m(x) + (l-m)PD[\mu] = \underline{t}_2$$

Note that the intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ giving the same $Spin^c$ structure s is same for both $CFK^\infty(Y,K,\underline{t}_1)$ and $CFK^\infty(Y,K,\underline{t}_2)$.

Define a map $\varphi : CFK^\infty(Y,K,\underline{t}_1) \to CFK^\infty(Y,K,\underline{t}_2)$ sending

$$[x,i,j] \to [x,l,m]$$

such that $n_w(\phi) = i - l$ and $n_w(\phi) = j - m$. Then it easily follows that φ commutes with the boundary maps and H gives the desired chain isomorphism. □

Remark 7.3.4. We can split CFK^∞ using $Spin^c$ structures over $Y_0(K)$ then two isomorphic chain complexes $CFK^\infty(Y,K,\underline{t}_1)$ and $CFK^\infty(Y,K,\underline{t}_2)$ differ only by a shift in the $\mathbb{Z}\times\mathbb{Z}$ filtration.

Fix $\underline{t}_0 \in \underline{Spin^c}(Y,K)$ and let $s \in Spin^c(Y)$ such that $\underline{t}_0$ projects to s under the isomorphism in Theorem 7.1.5. The complex $CFK^\infty(Y,K,\underline{t}_0)$ gives a $\mathbb{Z}$-filtration on $CF^\infty(\alpha,\beta,s)$ via the map

$$\prod_1 : CFK^\infty(Y,K,\underline{t}_0) \to CF^\infty(\alpha,\beta,s)$$

$$\prod_1[x,i,j] = [x,i]$$

The map $\prod_1$ is an isomorphism which follows easily as it is defined on the generators. The map

$$F[x,i,j] = j$$

gives a $\mathbb{Z}$-filtration on $CFK^{\infty}(Y,K,\underline{t}_0)$.

The subset $CFK^{-,*}(Y,K,\underline{t}_0) \subset CFK^{\infty}(Y,K,\underline{t}_0)$ generated by elements $[x,i,j]$ with $i<0$ gives a subcomplex, as the boundary map ∂^{∞} restricted to $CFK^{-,*}(Y,K,\underline{t}_0)$ is well defined and it has a quotient complex $CFK^{+,*}(Y,K,\underline{t}_0)$. $CFK^{\infty}(Y,K,\underline{t}_0)$ induce a $\mathbb{Z}$-filtration on $CF^{\infty}(Y,s)$. Therefore, the subcomplex $CFK^{-,*}(Y,K,\underline{t}_0)$ and the quotient complex $CFK^{+,*}(Y,K,\underline{t}_0)$ induce filtration on $CF^{-}(\alpha,\beta,s)$ and $CF^{+}(\alpha,\beta,s)$ respectively.

Consiter the U-action on $CFK^{+,*}(Y,K,\underline{t}_0)$ then

$$Ker(U) : CFK^{+,*}(Y,K,\underline{t}_0) \to CFK^{+,*}(Y,K,\underline{t}_0)$$

is the set of elements $[x,0,j]$ with all $i=0$ gives the subcomplex $CFK^{0,+}(Y,K,\underline{t}_0)$ of $CFK^{+,*}(Y,K,\underline{t}_0)$ which induces filtration on $\widehat{CF}(\alpha,\beta,s)$. Fix a $Spin^c$ structure $\underline{t}_0 \in Spin^c(Y,K)$ then the map,

$$CFK^{0,*}(Y,K,\underline{t}_0) \to \widehat{CF}(Y,s)$$
$$\text{sending } [x,0,j] \to [x,0]$$

gives $\mathbb{Z}$-grading.

Similar to the chain complex defined in Section 7.2 for a fixed $\underline{t} \in Spin^c(Y_0(K))$ and (Σ,α,β,m) be a marked Heegaard diagram for (Y,K) define $\widehat{CFK}(\alpha,\beta,\underline{t})$ be a free Abelian group generated by $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\underline{s}_m(x) = \underline{t}$ with the boundary map $\widehat{\partial} : \widehat{CFK}(\alpha,\beta,\underline{t}) \to \widehat{CFK}(\alpha,\beta,\underline{t})$ defined as

$$\widehat{\partial}x = \sum_{y} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1, n_w(\phi)=n_z(\phi)=0\}} \#(\widehat{\mathcal{M}}(\phi)) \cdot y$$

gives a chain complex. Then the complex $CFK^{0,*}(Y,K,\underline{t}_0)$ which is graded by $Spin^c$ structures over $Y_0(K)$ can be expressed as,

$$CFK^{0,*}(Y,K,\underline{t}_0) = \bigoplus_{\{\underline{t}\in Spin^c(Y_0(K))|\underline{t} \text{ extends } s\}} \widehat{CFK}(\alpha,\beta,\underline{t})$$

A generator $[x,0,j] \in CFK^{0,*}(Y,K,\underline{t}_0)$ corresponds to an element of $\widehat{CFK}(Y,K,\underline{t})$ for some $\underline{t}$ extending s. The generator has the properties

$$s_w(x) = s \text{ and } \underline{s}_m(x) = s \text{ and}$$
$$\underline{s}_m(x) + (i,j)PD[\mu] = \underline{t} \text{ as } i=0 \text{ and } \underline{t}_0$$

extends s thus $\underline{s}_m(x) = \underline{t}_0$. Thus $[x,0,j]$ corresponds to an element x in $\widehat{CFK}(Y,K,\underline{t})$ such that $\underline{t}_0 - jPD[\mu] = \underline{t}$

7.4 Properties of Knot Floer Homology

In this section for an oriented n-component link L in a closed, oriented three-manifold Y, or in particular an oriented knot K in Y, we briefly review the basic properties of knot Floer homology, which highlights its importance. The proofs and more details with calculations can be found in [32], [31], [33],and [35]. In knot theory to distinguish two knots is very important and there are many tools like polynomials and invariants such as Jones polynomial, Alexander polynomial, Kauffman polynomial, HOMFLY polynomial, and Vassiliev invariants, see [34] and [19].

The one of the most important results of the knot Floer holomogy groups is the following theorem whose proof can be found in [32] and [8].

Theorem 7.4.1. *For an oriented knot K in a closed, oriented three-manifold Y, fix $\underline{t} \in \underline{Spin}^c(Y, K)$. The filtered chain homotopy type of the chain complex $CFK^\infty(Y, K, \underline{t})$ is a topological invariant for the oriented knot K in Y and the $Spin^c$ structure $\underline{t}$. Therefore, it is independent of the chosen admissible marked Heegaard diagram.*

Thus the knot Floer homology groups $\widehat{HFK}(Y, K, \underline{t}) = H_*(\widehat{CFK}(Y, K, \underline{t}))$ are topological invariants of the oriented knot K in Y and $\underline{t} \in \underline{Spin}^c(Y, K)$. The homology groups are independent of the chosen complex structure j and the path J_s of nearly symmetric almost-complex structures, [8].

We mentioned in Section 7.1 that if we have an oriented n-component link L in Y. Then by adding 1-handles to Y, it corresponds to an oriented knot $\widetilde{K}$ in $Y\#(S^1 \times S^2)$ call it $\widetilde{Y}$. For a fixed $\underline{t} \in \underline{Spin}^c(\widetilde{Y}, \widetilde{K})$ the corresponding homology groups is defined as

$$CFK^\infty(Y, L, \underline{t}) = CFK^\infty(\widetilde{Y}, \widetilde{K}, \underline{t})$$

Then for an oriented n-component link L in closed, oriented three-manifold Y and fixed $\underline{t} \in \underline{Spin}^c(Y, K)$, $CFK^\infty(Y, L, \underline{t})$ is a link invariant.

Let us assume that $Y = S^3$ is a three-sphere. Then we have the following properties. The proofs can be found in the major papers about the subject as [32], [31], [33],and [35]. We give these properties for the completeness of the subject, to emphasize the importance of knot Floer homology groups.

Let L be an oriented link in $Y = S^3$ then the graded Abelian groups $\widehat{HFK}(L, i)$ where $i \in \mathbb{Z}$ can be given rational coefficients with $\widehat{HFK}(L, i, \mathbb{Q}) \simeq \widehat{HFK}(L, i) \otimes_{\mathbb{Z}} \mathbb{Q}$ following from the Universal Coefficient theorem.

For an oriented link L, consider the skein moves for each crossing with L_+, L_-, L_0 which are link diagrams obtained after resolving a crossing. Then the *Alexander-Conway polynomial* in one variable is defined by the skein relation as:

$$\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\Delta(L_0)$$

Then the Euler characteristic is related to the Alexander-Conway polynomial of L, $\Delta_L(T)$ by

$$\sum \chi(\widehat{HFK}(L, i, \mathbb{Q})) \cdot T^i = (T^{-1/2} - T^{1/2})^{n-1} \cdot \Delta_L(T)$$

where n is the number of components of the link diagram L.

If we take the mirror image of the link projection L we change under crossing (resp. over crossing) with over crossing (resp. under crossing) then the difference between the corresponding knot Floer homology groups of L and its mirror image $\overline{L}$ is

$$\widehat{HFK}_d(L, i, \mathbb{Q}) \simeq \widehat{HFK}_{-d}(\overline{L}, -i, \mathbb{Q})$$

Alexander polynomial is symmetric which is $\Delta_K(T) = \Delta_K(T^{-1})$ for every knot K, see [26], then the knot Floer homology groups have the conjugation symmetry as

$$\widehat{HFK}_d(L, i, \mathbb{Q}) \simeq \widehat{HFK}_{d-2i}(L, -i, \mathbb{Q})$$

Let L_1 and L_2 be two disjoint oriented link diagrams. Consider the connected sum $L_1\#L_2$ of two links such that they can be separated by S^2 the the knot Floer homology groups of $L_1\#L_2$ respects the KÃ¼nneth principle giving

$$\widehat{HFK}(L_1\#L_2, i, \mathbb{Q}) \simeq \bigoplus_{i_1+i_2=i} \widehat{HFK}(L_1, i_1, \mathbb{Q}) \otimes_{\mathbb{Q}} \widehat{HFK}(L_2, i_2, \mathbb{Q})$$

Definition 51. [9] Let L be an oriented link in S^3. L is called a fibered link if the complement $S^3 - L$ is a surface bundle over the circle such that its fiber F over $1 \in S^3$ is the interior of a compact oriented surface F whose boundary is L.

Let L be an oriented n-component link in S^3 and let the degree of the Alexander-Conway polynomial of L is of degree d then

$$\widehat{HFK}(L, d + \tfrac{n-1}{2}) \simeq \mathbb{Z}$$

Definition 52. Let K be an oriented knot in S^3 then an embedded, oriented surface S in S^3 such that $\partial S = K$ is called a Seifert surface for K. The minimal genus of any Seifert surface for K is called the Seifert genus of K, denoted as $g(K)$.

Note that if K is an unknot the surface admitting the unknot as boundary is of genus 0, conversely if a Seifert surface is of genus 0 then its boundary corresponds to the unknot. Moreover we have the following whose proof can be found in [27]

Theorem 7.4.2. *Let K be an oriented knot in S^3 then let $degH_{i,j} = max\{i \in \mathbb{Z} \mid \bigoplus_j H_{i,j}(K) \neq 0\}$ be the degree of the knot Floer homology then the Seifert genus*

$$g(K) = degH_{i,j}(K)$$

Remark 7.4.3. The above theorem shows that knot Floer homology groups detects whether or knot a given knot is different from the unknot.

Moreover, by [32] we have that Knot Floer homology groups of left handed trefoil (considered in S^3) and right handed trefoil are different. As left handed trefoil and right handed trefoil are not isotopic to the right handed trefoil whose Jones polynomial, even Kauffman polynomial are different, see [34]. Thus $\widehat{HFK}$ distinguishes right handed trefoil from the left handed trefoil. In addition $\widehat{HFK}$ distinguishes Pretzel knots, whose definition is given in next section, of the form $P(2a+1, 2b+1, 2c+1)$ from the unknot.

There is a move on the link diagrams called *mutation move*, see [19] , which is described in next section. The famous example is Kinoshita-Terasaka knot and Conway knot pair such that one is obtained from the other by a mutation move. $\widehat{HFK}$ is sensitive to Conway mutation. However, the knot invariants like Alexander polynomial, Jones polynomial, and Kauffman polynomial are mutation invariant, [34], [32], [33], so they can not detect for example the difference between the Kinoshita-Terasaka knot and Conway knot which are nonisotopic knots, see [19]. For more details for the sensitivity of $\widehat{HFK}$ for mutation move and detailed calculation see [33].

7.5 Khovanov Homology

7.5.1 Introduction

Classification of knots lies in the heart of Knot Theory. We want to determine when given two knots present the same knot, and in particular to determine a given knot to represent the unknot. To achieve this goal we use invariants which are algebraic objects assigned to a knot diagram depending only on the isotopy class of the knot. We discussed knot Floer homology as such an invariant for oriented, nullhomologous knots in three-manifolds, and in this section we introduce another knot invariant called Khovanov homology.

We begin by discussing Jones polynomial. We define it using Kauffman bracket and state diagrams then show that it is a invariant and provide examples to demonstrate how it detects nonisotopic knots. Next, we discuss its weakness and study some examples of nonisotopic links having the same Jones polynomial which alerts the need for more powerful invariant.

In the next section we provide some background information for Khovanov Homology including graded vector spaces, height and degree shifts, and graded Euler characteristic. In Subsection 7.5.4 we describe the construction of Khovanov Homology and prove that it is a link invariant generalizing the Jones polynomial. Therefore Khovanov Homology is a stronger link invariant than Jones polynomial. Then by definition and some properties of Khovanov homology we see some similarities between knot Floer homology and Khovanov homology.

7.5.2 Jones Polynomial

Jones Polynomial, which was discovered by V. F. R Jones in 1984, is a Laurent polynomial in integer coefficients associated with an oriented link diagram L. It is a link invariant therefore isotopic knots have the same Jones polynomial and it does not change under three Reidemeister moves and plane isotopies.

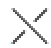
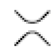
Jones polynomial, even though inadequate, is related to statistical mechanics and is useful to see the difference between a link diagram and its mirror image [16]. Moreover, Jones polynomial is generalized developing into a theory and it gives invariants for three dimensional manifolds via the help of quantum theory [19].

We will construct Jones polynomial from the Kauffman bracket using states of a link diagram. Then in a similar way we will construct Khovanov Homology.

7.5.2.1 Kauffman Bracket

The Kauffman bracket is a polynomial for nonoriented links and it is described by the following relation

(7.2)

where we resolve every crossing  into  A-smoothing (or 0-smoothing) or  A^{-1}-smoothing (or 1-smoothing). If we resolve every crossing of a link diagram then we obtain a disjoint union of closed cycles. Label every crossing with respect to how we smooth them into A or A^{-1}, then we have a sequence s of A's or A^{-1}'s. s is called a state or a smoothing of a link diagram L. Note that we have $2^{c(L)}$ many states for L, where $c(L)$ denotes the number of crossing of the link diagram. Then we define the state expansion of bracket polynomial as

$$\langle L\rangle = \sum_s \langle L|s\rangle(-A^2 - A^{-2})^{\|s\|}$$

where $\|s\|$ is the number of loops obtained after smoothing every crossing of L accordingly to the state s and $\langle L|s\rangle$ is the crossing weight as we label every crossing of a L as A or A^{-1}.

Remark 7.5.1. Kauffman bracket polynomial is invariant under $R2$ and $R3$ but not $R1$, for details see [34].

Remark 7.5.2. Instead of using relation (7.2) we will make a change of variable [16] for practical purposes. Let $c(L)$ be the number of crossing in a links diagram L, then multiply $\langle L\rangle$ by $A^{-c(L)}$ and replace A^2 by $-q^{-1}$ in resulting polynomial, so (7.2) becomes:

(7.3)

Now assign every smoothing α of L to a vertex of the n-cube $\{0,1\}^n$ with $\alpha = (\alpha_1, .., \alpha_n)$ where α_i is the smoothing type for the i-th crossing. The *Height* of a smoothing α is the number of 1's in the sequence. To every smoothing associate a polynomial $V_\alpha(L) = (q+q^{-1})^k(-q)^r$, where k is the number of cycles obtained after smoothing every crossing and r is the height of α. Then we define Kuffman bracket as follows:

$$\langle L\rangle = \sum_{\alpha\in\{0,1\}^n} V_\alpha(L)$$

Kauffman bracket satisfies the following defining relations:

1. $\langle \emptyset \rangle = 1$
2. $\langle \bigcirc \sqcup L\rangle = (q+q^{-1})\langle L\rangle$

3. 

We can easily deduce that Kauffman bracket of unknot is $(q + q^{-1})$.

As we mentioned before Kauffman bracket is not a link invariant, but a slight modification of it gives a link invariant.

Definition 53. For oriented link diagram L, let n_+ (respectively n_-) be the number of positive (respectively negative) crossing according to the right hand rule.

Definition 54. Unnormalized Jones polynomial of L is defined as

$$\widehat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$$

and the normalized Jones polynomial of L is

$$J(L) = \widehat{J}(L) \big/ (q + q^{-1})$$

Theorem 7.5.3. *Jones polynomial is isotopy invariant for links, i.e., it is invariant under Reidemeister moves and plane isotopies.*

For the proof of the above statement, reader is referred to [34] or [1]

Examples: Consider Hopf link with the orientation in the [34] and left trefoil then the normalized Jones polynomial of those two knots are:

$$J(H) = -q - q^{-5} \text{ and } J(T) = q^{-2} + q^{-6} - q^{-8} \text{ , see [34]}$$

consider the change of variable as described above. They are nonisotopic links as one of them is link the other one is knot.

7.5.2.2 Weakness of the Jones Polynomial

We will study two examples to demonstrate why Jones polynomial is not adequate. One of the problems arise from connected sum of links with more than one components. The connected sum operation is not well defined for links with more than one component, because depending on which components we connect two links, we can have non-isotopic links as a result of different choices see the examples in [34].

Example: The next problem is related to knots. Consider the following nonisotopic knots whose Jones polynomials match. In the figure below, the one on the left is Kinoshita-Terasaka knot and the one on the right is the Conway knot.

These are nonisotopic knots which can be determined by their knot groups [19] and they also have different knot genus, knot genus of the Conway Knot is 3 but knot genus of the Kinoshita-Terasaka is 2. However, these two knots can be obtained from each other by a *mutation move*. Consider a solid ball whose boundary intersects one of the

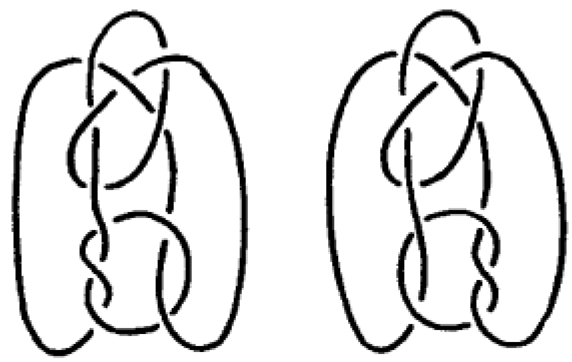


Figure 7.3: Kinoshita-Terasaka and Conway Knots ([34])

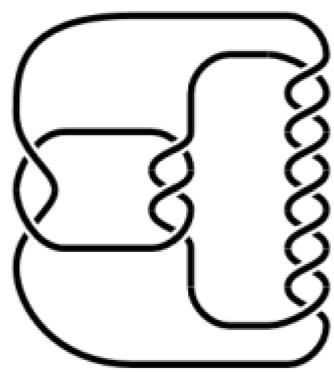


Figure 7.4: Pretzel link $P(-2,3,7)$

knots in four points. Remove this part and rotate it by an angle π and then replace it. We then obtain the other knot and they have the same Jones polynomial [19, 34].

A simple mutation move can be seen on Pretzel links as in the figure above, $P(p_1,..,p_n)$ where there are p_i left handed crossings in the i-th tangle. A mutation on Pretzel link changes p_i and p_{i+1}, but Jones polynomial does not change even though we obtain nonisotopic links. Moreover, as the number of crossings of a link diagram increases, it gets more difficult and impractical to calculate Jones polynomial.

7.5.3 Background Information

In this section we briefly review some necessary materials for the construction of Khovanov Homology.

Definition 55. A graded vector space $W = \bigoplus_m W_m$ with homogenous components W_m is a vector space with a grading on it, so that W can be expressed as direct sum of its subspaces. Graded dimension of W is a power series:

$$qdimW = \sum_m q^m dimW_m$$

If we choose a basis for each component W_m of W then we end up with a basis for W. If we have vector space $W = V^{\otimes k}$ then choosing a basis for W induces a grading on it as follows. First assign degrees to basis elements of V and then define the degree of a basis element of $V^{\otimes k}$ to be sum of degrees of tensor factors. Then the span of the basis elements of degree m is a subspace W_m of W and we have: $W = \bigoplus_m W_m$.

Proposition 7.5.4. *For graded vector spaces W_1 and W_2 we have the following:*

1. $qdim(W_1 \oplus W_2) = qdimW_1 + qdimW_2$
2. $qdim(W_1 \otimes W_2) = (qdimW_1)(qdimW_2)$

Proof. Let $W_1 = \bigoplus_i W_{1i}$ and $W_2 = \bigoplus_j W_{2j}$. Let us prove the first statement, $W = W_1 \oplus W_2$ is also a graded vector space where the m-th graded component is direct sum of m-th graded components of W_1 and W_2. As

$$dim(W_{1m} \otimes W_{2m}) = dimW_{1m} + dimW_{2m}$$

then it follows

$$qdimW_1 \oplus W_2 = \sum_m q^m dimW_m = \sum_m q^m (dimW_{1m} + dimW_{2m}) = qdimW_1 + qdimW_2$$

Now prove the second statement, $W = W_1 \otimes W_2$ is a graded vector space whose m-th graded component is $W_m = \bigoplus_{i+j=m} W_{1i} \otimes W_{2j}$. As

$$dimW_m = \sum_{i+j=m} dim(W_{1i} \otimes W_{2j}) = \sum_{i+j=m} (dimW_{1i})(dimW_{2j})$$

then it follows

$$qdimW = \sum_m g^m dimW_m = \sum_m \sum_{i+j=m} (dimW_{1i})(dimW_{2j}) = (qdimW_1)(qdimW_2)$$

□

Definition 56. Degree shift by $.\{l\}$ of a graded vector space $W = \bigoplus_m W_m$ is a graded vector space $W\{l\}$ where $W\{l\}_m = W_{m-l}$.

Remark 7.5.5. We have the following:

$$\begin{aligned} qdimW\{l\} &= \sum q^m dimW\{l\}_m \\ &= \sum q^m dimW_{m-l} \\ &= q^l \sum q^{m-l} dimW_{m-l} \\ &= q^l qdimW. \end{aligned}$$

Definition 57. Let (C, d) be a chain complex

$$\cdots \longrightarrow C^r \xrightarrow{d^r} C^{r+1} \xrightarrow{d^{r+1}} \cdots$$

then a height shift by $.[s]$ of (C, d) is a chain complex $(C[s], d[s])$ where $C[s]^r = C^{r-s}$ and shifting differential maps accordingly.

Definition 58. Let (C, d) be a chain complex of vector spaces where each chain group C^r is also a graded vector space $C^r = \bigoplus_i C_i^r$, then differential map is of degree k if $d^r(C_i^r) \subset C_{i+k}^{r+1}$.

Remark 7.5.6. If such a chain complex (C, d) has degree 0 differential map then for fixed i, C_i is a subcomplex

$$\cdots \longrightarrow C_i^{r-1} \longrightarrow C_i^r \longrightarrow C_i^{r+1} \longrightarrow \cdots$$

where differential maps d^r are restricted to C_i^r as a map from C_i^r to C_i^{r+1}.

Definition 59. For a chain complex (C, d), Euler characteristic of C is

$$\chi(C) = \sum_n (-1)^n dim\mathcal{H}^n(C)$$

Proposition 7.5.7. *If only finitely many chain groups are nonzero in a chain complex and they are finite dimensional then Euler characteristic of C can be expressed as:*

$$\chi(C) = \sum_n (-1)^n dimC^n$$

Definition 60. Graded Euler characteristic of a chain complex (C, d) is

$$\chi_q(C) = \sum_n (-1)^n qdim\mathcal{H}^n(C)$$

Proposition 7.5.8. *If the differential map is of degree zero of a chain complex (C, d) and C has finitely many nonzero chain groups which also are finite dimensional then*

$$\chi_q(C) = \sum_n (-1)^n qdim\mathcal{C}^n$$

We skip the proofs of the last two propositions which can be seen easily.

7.5.4 Khovanov Homology

Khovanov suggested associating a chain complex of graded vector spaces to a link diagram called *Khovanov* bracket $[\![L]\!]$, where grading is chosen appropriately so that the Jones polynomial transforms into a homological object.

7.5.4.1 Construction

Let L be an oriented link diagram and let χ be a set of crossings with $n = |\chi|$, n_+ (respectively n_-) is the number of positive (respectively negative) crossings. Let V be a vector space over $\mathbb{Z}$, spanned by two basis elements v_+ and v_- whose degrees are chosen to be $+1$ and -1 respectively. Thus V is a graded vector space as $V = V_{+1} \oplus V_{-1}$ where $V_{+1} = \langle v_+ \rangle$ and $V_{-1} = \langle v_- \rangle$, then $qdimV = qdimV_{+1} + q^{-1}dimV_{-1} = q + q^{-1}$. Note that two generators of V correspond to two labelings of a crossing.

Let us express every smoothing α of L as a sequence of $\{0, 1\}$. Then for every smoothing there is a corresponding vertex in the n-cube $\{0, 1\}^\chi$ organized such that vertices at the same height stand in the same column. To every vertex α of the n-cube $\{0, 1\}^\chi$ we associate a graded vector space $V_\alpha(L) = V^{\otimes k}\{r\}$ where k is the number of cycles obtained after the smoothing α and r is height of it. Then define r-th chain group

$$[\![L]\!]^r = \bigoplus_{|\alpha|=r} V_\alpha(L)$$

for $0 \leq r \leq n$ which is also a graded vector space. then $[\![L]\!]$ with a differential map which we will define soon will be a chain complex. Let us define another chain complex as $C(L) = [\![L]\!][-n_-]\{n_+ - 2n_-\}$. The figure below demonstrates what we have defined so far on a right trefoil.

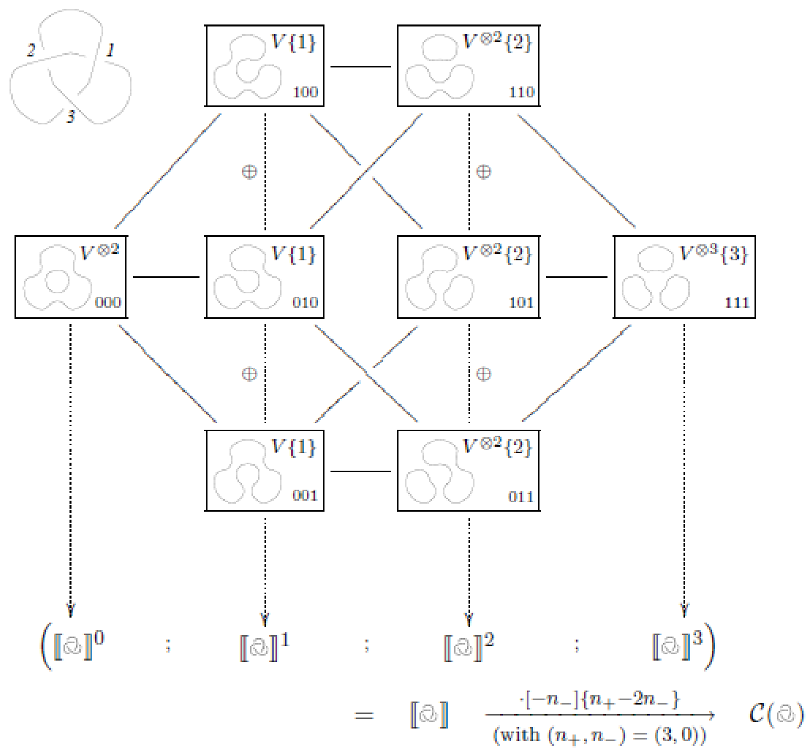


Figure 7.5: Right Trefoil and cube Diagram

Next let us see how Jones polynomial is getting involved:

Theorem 7.5.9. *The graded Euler characteristic of* $C(L)$ *is equal to the unnormalized Jones polynomial of* L.

Proof. $C(L)$ is a finitely supported graded chain complex with finite dimensional chain groups. We will show that the differential map is of degree zero, then $\chi_q(C(L))$ is equal to the alternating sum of the graded dimensions of chain groups of $C(L)$.

As $C(L) = [\![L]\!][-n_-]\{n_+ - 2n_-\}$ we have:

$$\begin{aligned}
\chi_q(C(L)) &= \sum_{r=o}^{n} (-1)^r qdim C(L)^r \\
&= \sum_{r=o}^{n} (-1)^r qdim([\![L]\!]^{r+n_-}\{n_+ - 2n_-\}) \\
&= \sum_{r=o}^{n} (-1)^r q^{n_+ - 2n_-} qdim[\![L]\!]^{r+n_-} \\
&= \sum_{r=o}^{n} (-1)^r q^{n_+ - 2n_-} qdim(\bigoplus_{|\alpha|=r+n_-} V_\alpha(L)) \\
&= \sum_{r=o}^{n} (-1)^r q^{n_+ - 2n_-} \sum_{|\alpha|=r+n_-} qdim V_\alpha(L) \\
&= \sum_{r=o}^{n} (-1)^r q^{n_+ - 2n_-} \sum_{|\alpha|=r+n_-} dim V^{\otimes k}\{r + n_-\} \\
&= \sum_{r=o}^{n} (-1)^r q^{n_+ - 2n_-} \sum_{|\alpha|=r+n_-} (q + q^{-1})^k q^{r+n_-} \\
&= q^{n_+ - 2n_-} \sum_{r=o}^{n} \sum_{|\alpha|=r+n_-} (-1)^r q^{r+n_-} (q + q^{-1})^k \\
&= q^{n_+ - 2n_-} (-1)^{-n_-} \sum_{r=o}^{n} \sum_{|\alpha|=r+n_-} (-1)^{r+n_-} q^{r+n_-} (q + q^{-1})^k \\
&= q^{n_+ - 2n_-} (-1)^{n_-} \sum_{\alpha \in \{0,1\}^n} (q + q^{-1})^k (-q)^r \\
&= \widehat{J}(L)
\end{aligned}$$

Note that height shift $[-n_-]$ corresponds to $(-1)^{n_-}$ factor in the Jones polynomial. □

7.5.4.2 Khovanov and Cube Categories

Before defining the differential map for $[\![L]\!]$ let us pause and think about how to define the differential map such that homology becomes a link invariant. In this part we will basically follow [16].

For a link diagram L, the set of states form a category in the shape of a cube. Then a functor from a cube category to a module category induces a homology theory in a natural way as follows. First let $S(L)$ be the category of states for L, where the objects are states and a morphism is an arrow from a state with a given number of

0's to a state with fewer number of 0's. Next let $D^n = \{0,1\}^n$ be n-cube category with objects are n-element sequences of $\{0,1\}$ and a morphism is an arrow from a sequence with given number of 0's to a sequence with fewer number of 0's. Let us see the correspondence between $S(L)$ and D^n.

Let $\mathcal{F}_1 : D^n \to S(L)$ be a functor where the link diagram has n crossings and each of them is labeled 1 through n. Then $\mathcal{F}_1$ maps sequences to states where i-th term in the sequence matches with the smoothing of i-th crossing. It is clearly one-to-one, so we can define $\mathcal{F}_2 : S(L) \to D^n$ as $\mathcal{F}_2$ takes each state to a sequence whose terms match with smoothings in corresponding crossing. Therefore the composition of two morphisms are identity maps on their category.

Let $\mathcal{M}$ be a category of modules of finite sums containing 0 element. Let $\mathcal{F} : D^n \to \mathcal{M}$ be a functor taking sequences with n terms to some tensor powers corresponding to a state α of L. For any $\alpha \in D^n$, $\alpha = (\alpha_1, .., \alpha_n)$ where $\alpha_i = 0$ or 1 and a morphism

$$d_i : (\alpha_1, .., \alpha_i, .., \alpha_n) \to (\alpha_1, .., \overline{\alpha}_i, .., \alpha_n)$$

with $\overline{\alpha}_i = 1$ if $\alpha_i = 0$. Let us define

$$\partial_i : C(\alpha_1, .., \alpha_i, .., \alpha_n) \to C(\alpha_1, .., \overline{\alpha}_i, .., \alpha_n)$$

if d_i is defined. Then r-th chain group of C is $C^r = \bigoplus_{\alpha} C(\alpha_1, .., \alpha_n)$ where every sequence α has r 1's. We can define the differential map as

$$\partial(v) = \sum_{r=0}^{n} (-1)^{c(\alpha,i)} \partial_i(v) \text{ for } v \in C(\alpha_1, .., \alpha_n)$$

and $c(\alpha, i)$ is the number of 0's in the sequence α preceding α_i. As we need $\partial \circ \partial = 0$ to turn C into a chain complex, by the construction this is equivalent to $\partial_i \circ \partial_j = \partial_j \circ \partial_i$ as long as the composition and maps are defined. If we can set this relation in cube category, which is same as state category, then the functor $\mathcal{F}$ will induce a chain complex and homology.

7.5.4.3 Differential Maps

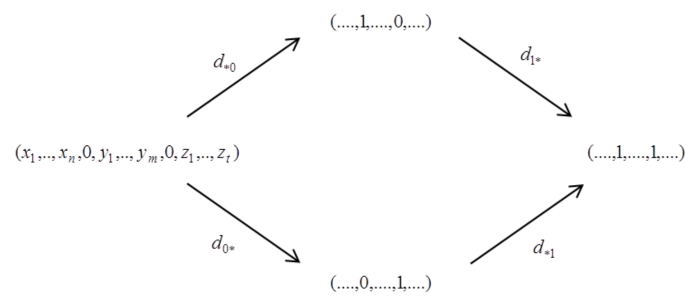
We need to define the differential map to make $[\![L]\!]$ and $C(L)$ into chain complexes. First let us define edge maps for edges of the state diagram cube $\{0,1\}^{\chi}$. Label each edge of the cube via elements of $\{0, 1, \star\}$ where $\star = 0$ corresponds to tail of the edge and $\star = 1$ corresponds to head of the edge. So each edge map is from a vertex at height r to a vertex at height $r + 1$. *Height of an edge* ϵ, denoted as $|\epsilon|$, is height of the vertex corresponding to the tail of ϵ. Then r-th differential map

$$d^r = \sum_{|\epsilon|=r} (-1)^{\epsilon} d_{\epsilon}$$

is the sum of all maps from graded vector spaces at height r to graded vector spaces at height $r+1$. Now we need $d \circ d = 0$ and we will see that it is sufficient if all square faces of the state diagram cube are anti-commutative. To obtain anti-commutativity of faces, we first establish d_ϵ's so that faces will be commutative and multiply some of the edges by (-1) for anti-commutativity.

Proposition 7.5.10. *If we multiply each edge map* d_ϵ *by* $(-1)^\epsilon = (-1)^{\sum_{i<j} \epsilon_i}$ *then all faces will be anti-commutative. Here* ϵ_i *is the i-th term of* ϵ *and* j *is the location of* $\star$ *in* ϵ.

Proof. Remember that each edge map d_ϵ is a map from a space at height r to a space at height $r+1$, for every face we have the following picture:



So the following hold:

- $d_{\star 0}(-1)^\epsilon = d_{\star 0}(-1)^{i_1+...+i_n}$
- $d_{0\star}(-1)^\epsilon = d_{0\star}(-1)^{i_1+...+i_n+j_1+...+j_m}$
- $d_{1\star}(-1)^\epsilon = d_{1\star}(-1)^{i_1+...+i_n+1+j_1+...+j_m}$
- $d_{\star 1}(-1)^\epsilon = d_{\star 1}(-1)^{i_1+...+i_n+j_1+...+j_m}$

If the sum $j_1 + ... + j_m$ is even (respectively odd) the first two edge maps have the same (respectively different) sign. Therefore, the last two edge maps have the opposite (respectively same) sign. Only one of those four maps has a different sign from the other three which implies for each face we have an odd number of negative signed edge maps which gives $d_{\star 1} \circ d_{1\star} \circ d_{\star 0} = 0$. This proves the statement. □

Two vertices corresponding to the head and tail of an edge map d_ϵ have the following difference: either the number of cycles increases by one from tail to head or decreases by one. We then have two linear maps m and Δ corresponding to merging of two cycles into one cycle and splitting one cycle into two cycles respectively. We assigned graded vector spaces $V_\alpha(L)$ to each vertex α and we will assign certain tensor factors to each cycle to define m and Δ.

Let us define $m : V \otimes V \to V$ and $\Delta : V \to V \otimes V$ to be identity on cycles which does not contribute to merging or splitting. As there is no order for cycles in the vertex α, m should be commutative and Δ to be co-commutative. By those arguments we define m and Δ as follows:

$$m : V \otimes V \to V,\ m : \begin{cases} v_+ \otimes v_+ \mapsto v_+ \\ v_+ \otimes v_- \mapsto v_- \\ v_- \otimes v_+ \mapsto v_- \\ v_- \otimes v_- \mapsto 0 \end{cases}$$

$$\Delta : V \to V \otimes V,\ \Delta : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$$

Note that the maps m and Δ are degree -1 by their definition. With these maps all faces commute. Indeed, we can merge first or split and vice versa, merge and merge, or split and split. By considering matrix representation of those maps the faces commute.

Theorem 7.5.11. *Sequences $[\![L]\!]$ and $C(L)$ are chain complexes.*

Proof. Let us first show that $([\![L]\!], d)$ is a chain complex. By definition $[\![L]\!]$ is a graded, free abelian group. we need to show that $d^2 = 0$. We have the following sequence:

$$\cdots \xrightarrow{d^{r-1}} [\![L]\!]^r \xrightarrow{d^r} [\![L]\!]^{r+1} \xrightarrow{d^{r+1}} [\![L]\!]^{r+2} \xrightarrow{d^{r+2}} \cdots$$

Now take any $v \in [\![L]\!]^r = \bigoplus_{|\alpha|=r} V_\alpha(L)$, $v = (v_1, .., v_n)$ such that $v_i \in V_\alpha(L)$ for some α at height r. Let us first understand $d^{r+1} \circ d^r(0, .., v_i, .., 0)$.

$d^{r+1} \circ d^r$ takes α to each states where 2 0-smoothing of α changes into 1-smoothing. Let β be one of such states. By the above diagram [diagram-3] there are two ways to go from α to β. All faces are commutative via edge maps d_ϵ, but we multiplied them by $(-1)^\epsilon$ such that they become then anti-commutative, then sum of two maps are 0. Then sum of two maps from α at height r to any β at height $r + 2$ is 0. It follows then sum of all maps from smoothing at height r to smoothing at height $r+2$ is 0.

This shows that $d^{r+1} \circ d^r(0, .., v_i, .., 0) = 0$ for all i, but the map is linear so we can extend it to v then we have $d^{r+1} \circ d^r(v) = 0$ for all $v \in [\![L]\!]^r$. Thus $d^2 = 0$ follows proving that $([\![L]\!], d)$ is a chain complex.

The sequence $C(L) = [\![L]\!][-n_-]\{n_+ - 2n_-\}$ has shifts in degree and height then by definition it also becomes a chain complex, by changing the boundary maps accordingly. □

Let us denote by $(H)^r(L)$ the rth cohomology group of $C(L)$. It is a graded vector space and depends on the link projection. We define $Kh(L)$ as the graded *Poincare polynomial* of the chain complex $C(L)$ in variable t as:

$$Kh(L) = \sum_r t^r qdim \mathcal{H}^r(L)$$

where by a *Poincare polynomial* of an n dimensional M we mean a polynomial in variable t such that $P(M,t) = \sum_r b_q(M)t^q$ where the coefficients are the Betti numbers of the manifold so they contain information about homological and/or cohomological properties of the manifold. For detailed information of Poincare polynomial the reader is referred to the appendix of [6].

The following theorem is the main statement and purpose of this term paper.

Theorem 7.5.12. *The graded dimension of the homology groups $\mathcal{H}^r(L)$ are link invariants. Therefore, $Kh(L)$, a polynomial in variable t and q, becomes a link invariant.*

Before the proof of the statement let us remark the followings.

Remark 7.5.13. Note that at $t = -1$, $Kh(L)$ is simply the unnormalized Jones polynomial

$$Kh(L) = \sum_r (-1)^r qdim \mathcal{H}^r(L)$$

Remark 7.5.14. Even though this subject is called Khovanov Homology, by definitions above and by the information given so far, it is actually a cohomology theory.

7.5.5 Invariance

In this section we will prove the main theorem stated at the end of the previous section. Before that we will give some homological algebra background that we need in the steps of the proof.

Khovanov himself uses cobordism in an elegant way for the invariance part where his techniques are beyond the scope of this thesis and his construction is more general than we studied here. As he uses a polynomial ring of degree 2 variable as the coefficient ring. For the original proof we refer the reader to [17].

We will proceed as follows for the proof. We will show that rth cohomology group of $C(L)$ is invariant under Reidemeister moves. Then as diagrams of isotopic links can be connected via Reidemeister moves and plane isotopies, this will show that isotopic links have isomorphic homology groups $\mathcal{H}^r(L)$. But let us introduce some necessary tools.

7.5.5.1 Some Homological Algebra

Proposition 7.5.15. *Let C be a chain complex and let $C' \subset C$ be a subcomplex. Then*

1. *If C' is acyclic (all homology groups are trivial) then $H(C) \simeq H(C/C')$.*

2. *If C/C' is acyclic then $H(C) \simeq H(C')$.*

Proof. Consider the following short exact sequence:

$$0 \longrightarrow C' \xrightarrow{i} C \xrightarrow{\pi} C/C' \longrightarrow 0$$

where the 2nd map is injection and 3rd map is projection. Then we have the following long exact homology sequence:

$$\cdots \longrightarrow H^r(C') \longrightarrow H^r(C) \longrightarrow H^r(C/C') \longrightarrow H^{r+1}(C') \longrightarrow \cdots$$

First suppose that $H^r(C')$ is trivial for every r then we have:

$$\cdots \longrightarrow 0 \longrightarrow H^r(C) \xrightarrow{f} H^r(C/C') \longrightarrow 0 \longrightarrow H^{r+1}(C) \longrightarrow \cdots$$

Note that f is injective and $Im(f)$ is the kernel of the next map which is all of $H^r(C/C')$ thus f is onto. It follows then f is an isomorphism and $H(C) \simeq H(C/C')$ proving the first statement.

Similarly, if $H^r(C/C')$ is trivial for every r then we have:

$$\cdots \longrightarrow 0 \longrightarrow H^r(C') \xrightarrow{f} H^r(C) \longrightarrow 0 \longrightarrow H^{r+1}(C') \longrightarrow \cdots$$

$Ker(f)$ is trivial so f is $1-1$ and $Im(f)$ is all of $H^r(C)$ being the kernel of the next map, so f is surjective. Thus, $H(C) \simeq H(C')$ follows proving the second statement. □

Let (B, d_B), (C, d_C) be chain complexes and $f : B \to C$ be a chain map. We can obtain a new chain complex called *mapping cone* by using this chain map f. We will basically follow the material in [38]. The mapping cone of f is a chain complex $Cone(f)$ such that its nth chain group is:

$$Cone(f)^n = B_{n-1} \oplus C_n$$

with the differential map $d(b, c) = (-d(b), d(c) - f(b))$ which can be given via the matrix

$$\begin{bmatrix} -d_B & 0 \\ -f & d_C \end{bmatrix}$$

With this differential map $Cone(f)$ becomes a chain complex. Indeed $d^2 = 0$ as:

$$\begin{bmatrix} -d_B & 0 \\ -f & d_c \end{bmatrix} \begin{bmatrix} -d_B & 0 \\ -f & d_c \end{bmatrix} = \begin{bmatrix} d_B^2 & 0 \\ f d_B - d_c f & d_C^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We can define $Cone(f)$ for cochain complexes also. For a cochain map $f : B \to C$ of cochain complexes, the mapping cone $Cone(f)$ is a cochain complex whose nth cochain group is:

$$Cone(f)^n = B^{n+1} \oplus C^n$$

The coboundary map can be given by the same matrix as in the homology case.

Remark 7.5.16. Here the diagrams are commutative, but the diagrams(faces) we consider in Khovanov Homology are anti-commutative. Thus in our case, to have a chain complex the differential map has the following matrix representation. the matrix

$$\begin{bmatrix} d_B & 0 \\ f & d_C \end{bmatrix}$$

Similarly, with this differential map $Cone(f)$ is a chain complex i.e., $d^2 = 0$:

$$\begin{bmatrix} d_B & 0 \\ f & d_c \end{bmatrix} \begin{bmatrix} d_B & 0 \\ f & d_c \end{bmatrix} = \begin{bmatrix} d_B^2 & 0 \\ f d_B + d_c f & d_C^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

where the 0 in lower left corner comes from anticommutativity.

Consider the following short exact sequence of chain complexes:

$$0 \longrightarrow C[1] \longrightarrow Cone(f) \longrightarrow B \longrightarrow 0$$

We use the height shift in the first complex to match the grading, as without height shift we have:

$$0 \longrightarrow C^{r-1} \longrightarrow Cone(f)^r \longrightarrow B^r \longrightarrow 0$$

but with the shift we have:

$$0 \longrightarrow C[1]^r \longrightarrow Cone(f)^r \longrightarrow B^r \longrightarrow 0$$

where the second map from $C[1] \to Cone(f)$ sends $c \mapsto (0, c)$ and the third map from $Cone(f) \to B[1]$ sends $(b, c) \mapsto b$ so that the first one is injective and the second one is surjective at each grading r showing this is really a short exact sequence of chain complexes.

Note that by definition of the height shift $H^n(B[1]) \simeq H^{n-1}(B)$ then we have the long exact sequence:

$$\cdots \longrightarrow H^n(C[1]) \longrightarrow H^n(Cone(f)) \longrightarrow H^n(B) \longrightarrow \cdots$$

which equals to the following by the above isomorphism:

$$\cdots \longrightarrow H^{n-1}(C) \longrightarrow H^n(Cone(f)) \longrightarrow H^n(B) \longrightarrow \cdots$$

Definition 61. A cochain (respectively chain) map $f : B \to C$ is called a quasi-isomorphism if the induced homomorphisms $H^n(B^*) \to H^n(C^*)$ (respectively $H_n(B^*) \to H_n(C^*)$) are isomorphisms for all n.

Proposition 7.5.17. *$Cone(f)$ is acyclic if and only if $f : B \to C$ is quasi-isomorphism.*

Proof. If $Cone(f)$ is acyclic then $f : B \to C$ is quasi-isomorphism follows easily :

$$\cdots \longrightarrow H^{n-1}(C) \longrightarrow 0 \longrightarrow H^n(B) \longrightarrow H^n(C) \longrightarrow 0 \cdots$$

from the long exact sequence. Conversely if $f : B\ C$ is quasi-isomorphism then by the isomorphism on the homology levels we have $H^n(Cone(f)) \subset Ker(H^n(B) \to H^n(C)) = \{0\}$ implies $H^n(Cone(f))$ is trivial for all n. □

7.5.5.2 Relations for the Khovanov Bracket

In section 2.1 we gave defining relations for the Kauffman bracket. Likewise, we can give some relations also for the Khovanov bracket $[\![L]\!]$ for a link diagram L which is a chain complex of graded vector spaces whose graded dimension is $\langle L \rangle$.

If we have nothing as a link projection but an empty set then the chain complex associated with the empty set is not a complex of graded vector spaces, as there is no crossing in the diagram which is empty set. In this case we only have the coefficient ring which is $\mathbb{Z}$ in our case. So we associate the complex

$$0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

to the Khovanov bracket of the empty set.

Let L be a link projection. suppose that we take disjoint union of L with an unknot $\bigcirc$. Let us see how the Khovanov bracket of $[\![L]\!]$ of L changes. We have

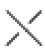
$$[\![L]\!]^r = \bigoplus_{|\alpha|=r} V_\alpha(L)$$

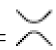
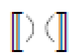
where $V_\alpha(L) = V^{\times k}\{r\}$ and k is the number of cycles. With the disjoint union $L \sqcup \bigcirc$

we increase the number of cycles by 1. Thus

$$\begin{aligned} [\![L\bigcirc]\!]^r &= \bigoplus_{|\alpha|=r} V_{\alpha(L\sqcup\bigcirc)} = \bigoplus_{|\alpha|=r} (V^{\otimes(k+1)}\{r\}) \\ &= \bigoplus_{|\alpha|=r} (V \otimes V^{\otimes k}\{r\}) = V \otimes (\bigoplus_{|\alpha|=r} (V^{\otimes k}\{r\})) \\ &= V \otimes [\![L]\!]^r \end{aligned}$$

for every r.

Now let us consider resolving the complex  analogous to 7.3. Let us resolve  into  0 and  1 smoothing, and then consider the chain complexes , , and .

Let us rename $A =$ , $B =$ , and $C =$  for the sake of simplicity.

Note that rth chain group of A includes all smoothings of B^r and C^{r-1}. The last one has grading $r-1$ because it corresponds to 1-smoothing which increases the grading by 1. Therefore as vector spaces we have the following:

(7.4)

Let  be a chain map (here it is taken as differential map). Then by using the cone construction discussed in previous section we obtain a chain complex, $Cone(d)$ whose rth chain group is:

We need to show that the differential map $\widetilde{d}$ of  is compatible with the differential map $d_{Cone(d)}$ of $Cone(d)$. It is sufficient to compare two maps on the basis elements. Take a basis element $v \in A^r$ then if it is in B^r or C^{r-1} then under the differential map $\widetilde{d^r}$ it is mapped to a smoothing at height $r+1$ by merging m or splitting map Δ. It will correspond to the restriction of the boundary map $\widetilde{d^r}$ to boundary maps of B^r or C^{r-1}. Moreover,it has a compatible image under the differential map $d_{Cone(d)}$ of $Cone(d)$ namely either

$$d_{Cone(d)}(v) = (d_B(v), d(v)) \text{ or } d_{Cone(d)}(v) = (0, d(v) + d_C(v))$$

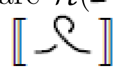
We need to consider the case if $v \in B^r$ then after 1-smoothing it may be an element of $C^{r-1}\{1\}$ which can be understood via the map $d(v)$. Note that the degree shift $\{1\}$ is necessary as differential maps are sums(considering sign) of merging m and/or splitting maps Δ which are of degree 1. If we consider these complexes A, B, and

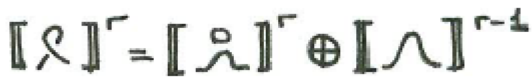
C as cube diagrams. We described here how to turn double complexes into a single complex by taking direct sums of each complexes and then flatten these cubes to have the rth chain group from each column. As a result we obtain:

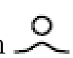
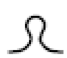
therefore the relation for the Khovanov bracket can be summarized as:

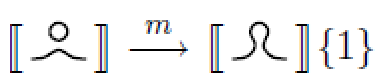
1. $[\![\emptyset]\!] = 0 \longrightarrow \mathbb{Z} \longrightarrow 0$
2. $[\![L\bigcirc]\!] = V \otimes [\![L]\!]$
3. 

7.5.5.3 Invariance Under the R1

We want to compare $\mathcal{H}($$)$ and $\mathcal{H}($$)$ under the first Reidemeister move. Consider the complex  and resolve  into  and . For the second one 1-smoothing increases the height by 1. By previous discussions, as vector spaces we have:

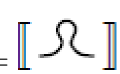
 (7.5)

From  to  the number of cycles increases which corrresponds to the merging map m,



between complexes.

First let us show that m is a chain map. In order to express in an easy way let us rename

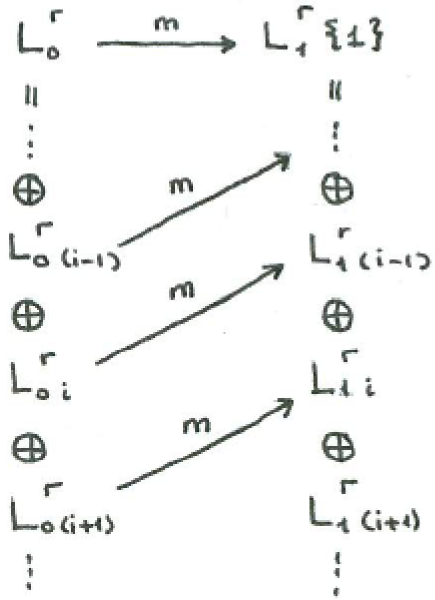
$L_0 =$  and $L_1 =$ 

Then at rth grade we have

$$L_0^r \xrightarrow{\ m\ } L_1^r\{1\}$$

We need to consider the degree shift by 1 because when we define edge maps, we mentioned m is a map of degree -1, so to have a degree 0 map we shift by 1.

L_0^r is itself a graded vector space $L_0^r = \bigoplus_i {L_0}_i^r$ and similarly $L_1^r = \bigoplus_j {L_1}_j^r$. Note also that $L_1^r\{1\}_i = L_{1(i-1)}^r$. And we have the following diagram:

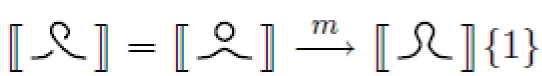


Then at the rth level diagram is anti-commutative:

$$\begin{array}{ccc} L_0^r & \xrightarrow{m} & L_1^r\{1\} \\ \downarrow{\scriptstyle d_0^r} & & \downarrow{\scriptstyle d_1^r} \\ L_0^{r+1} & \xrightarrow{m} & L_1^{r+1}\{1\} \end{array}$$

To see this take an element $v \in L_0^r$, for simplicity take $v \in L_{0i}^r$ then by diagram chasing we see that the diagram is anti-commutative similar to anti-commutativity of faces in the cube diagram. Therefore, m is a chain map of degree 0.

We claim that



is a cone construction $Cone(m)$ with the differential map $d_{Cone(m)}$ and also as vector spaces we have the equality (7.5). The differential map $d_{Cone(m)}$ sends

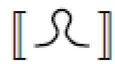
$$(v, w) \mapsto (d_{L_0}(v), m(v) + d_{L_1}(w))$$

This is a chain map as we discussed in section 5.1 but let us verify that $d^2_{Cone(m)} = 0$.

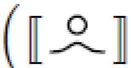
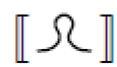
$$\begin{aligned} d(d(v,w)) &= d(d_{L_0}(v), m(v) + d_{L_1}(w)) \\ &= (d_{L_0}d_{L_0}(v), md_{L_0}(v) + d_{L_1}(m(v) + d_{L_1}(w))) \\ &= (d_{L_0}d_{L_0}(v), md_{l_0}(v) + d_{L_1}m(v) + d_{L_1}d_{L_1}(w)) \\ &= 0 \end{aligned}$$

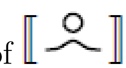
the 0's comes from L_0 and L_1 are chain complexes and the diagram above is anti-commutative.

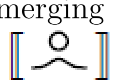
Boundary maps of the cone $Cone(m)$ and the complex  are comptatible. A basis element v in the rth chain group of  goes to $d(v) = \sum_{|\epsilon|=r} (-1)^{\epsilon} d_{\epsilon}(v)$. If it is already in L_0^r then $d(v)$ is restriction to $d_{L_0}(v)$ but we also need to consider $m(v)$. Because by a 1-smoothing we may end up a vertex belonging to the cube diagram of L_1. Similarly if $v \in L_1^r$ we need to consider $m(v)$ too. Therefore compatible boundary maps result isomorphic homology groups. So rather than working with the chain complex  we will work with the cone of m:

 $\xrightarrow{m}$  $\{1\}$

The complex $Cone(m)$ has a subcomplex C' which is also a cone defined naturally as

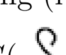
$C' = \Big($  $_{v_+} \xrightarrow{m}$  $\{1\}\Big)$

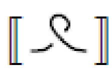
Remember that the complex $[\![L]\!]$ is direct sum of $[\![L]\!]^r$ such that $[\![L]\!]^r = \bigoplus_{|\alpha|=r} (V^{\otimes k}\{r\})$ where each tensor factor corresponds to a cycle in the smoothing. Mark each of these cycles by an element of V and remember that V is a vector space generated by $\{v_+, v_-\}$. Then  $_{v_+}$ denotes the subspace of  with the cycle on the top in the bracket is labeled by v_+. Thus C' is a subcomplex of $Cone(m)$. Note also that by definition of the merging map m, v_+ is a unit for m.

Let us denote $L_{0+} =$  $_{v_+}$ for simplicity. Now as v_+ is unit for m, it then follows $m : L_{0+} \to L_1\{1\}$ is a quasi-isomorphism, so by the proposition 7.5.17 C' is acyclic and by the proposition 7.5.15 we have $\mathcal{H}(Cone(m)) \simeq \mathcal{H}(Cone(m)/C')$.

Define a map $\varphi : Cone(m) \to (L_0/L_{0+} \to 0)$. As $Cone(m)^r = L_0^r \oplus L_1^{r-1}$ so that at the rth level it is $\varphi_r : L_0^r \oplus L_1^{r-1} \to L_0^r/L_{0+}^r \oplus 0$ as target space of φ is also a cone. $\varphi_r(v, w) = (\overline{v}, 0)$, where $\overline{v}$ is the image in the quotient. this map is a homomorphism and surjective at each r which can be seen easily. Take $(v, w) \in Ker(\varphi_r)$ so $\varphi_r(v, w) = (0, 0)$. For every $w \in L_1^{r-1}$ maps to 0 and v maps to 0 if $v \in L_{0+}^r$. So $Ker\varphi_r = L_{0+}^r \oplus L_1^{r-1}$. Then it follows:

$$Cone(m)/C' \simeq (L_0/L_{0+} \to 0) \simeq L_0/L_{0+} \simeq L_1\{1\}$$

Note that V/v_+ is generated only by v_- and  has + crossing (i.e., writhe number of the crossing is +1). Thus when we change  into $C($$)$ we make height and degree shifts as

 $\xrightarrow{[-n_-]\{n_4 - 2n_-\}}$ $C($$)$

And for $[\![\ ]\!]$ we have: $[\![\ ]\!][-n_-]\{n_+ - 2n_- - 1\}\{1\} = [\![\ ]\!][-n_-]\{n_+ - 2n_-\}$.

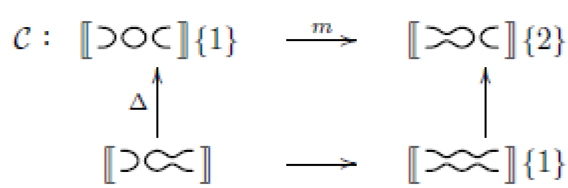
$$[\![\ ]\!] \xrightarrow{[-n_-]\{n_+ - 2n_-\}} C(\ )$$

Thus we will get isomorphic homologies from complexes $Cone(m)/C'$ and $L_1\{1\}$. Moreover, we also have the isomorphism $\mathcal{H}(Cone(m)) \simeq \mathcal{H}(Cone(m)/C')$. Then under the first Reidemeister move we proved that $\mathcal{H}(\ ) \simeq \mathcal{H}(\ )$.

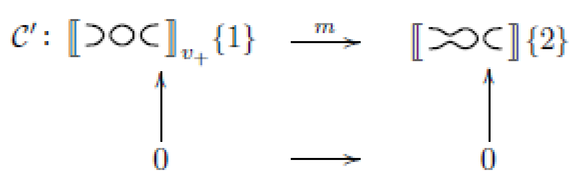
7.5.5.4 Invariance Under the R2

We want to see the difference in the homology $\mathcal{H}(\ )$ under the 2nd Reidemeister move. All diagrams from now on from the paper of Bar-Natan [3].

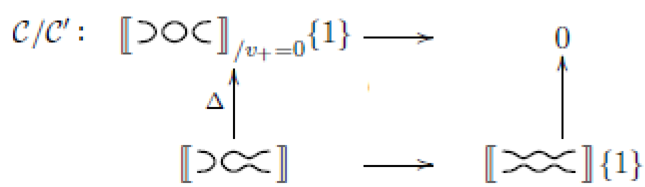
Let us compare the complexes $[\![\ ]\!]$ and $[\![\ ]\!]$ If we resolve every crossing of $\mathcal{C} = [\![\ ]\!]$ we have:



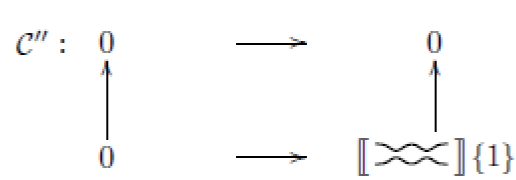
If we rotate the diagram clockwise by an angle $\pi/2$ we will obtain a cube diagram. For example direct sum of lower right and upper left corners correspond to the first chain group. We can take a subcomplex of $\mathcal{C}'$ of $\mathcal{C}$ as



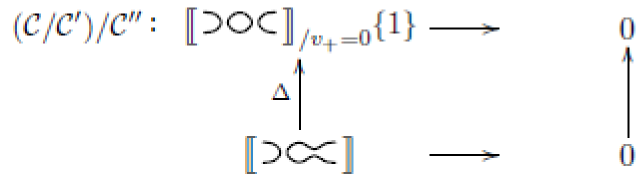
Note that m induce an isomorphism on the homology levels as v_+ is unit for m. Therefore $\mathcal{C}'$ is acyclic implying $H(C) \simeq H(C/C')$. Let us consider the quotient complex $\mathcal{C}/\mathcal{C}'$



with subcomplex



The quotient complex $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$ is acyclic.

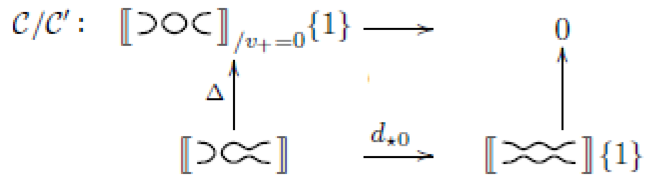


where the map Δ is as isomorphism as $v_+ = 0$ in the quotient. Δ induces an isomorphism on homology then it follows $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$ is acyclic.

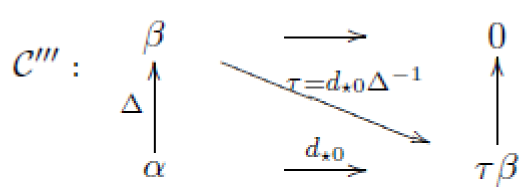
By the proposition 7.5.15 we have $H(\mathcal{C}'') \simeq H(\mathcal{C}/\mathcal{C}')$. If we combine what we have so far we get $H(\mathcal{C}) \simeq H(\mathcal{C}/\mathcal{C}') \simeq H(\mathcal{C}'')$ where $H(\mathcal{C}'')$ corresponds to the homology of the complex $[\![\;]\!]\{1\}$ if we flatten the cube diagram of $\mathcal{C}''$. By shifting degrees and height accordingly as in the case of $R1$ we will have $C(\;) \simeq C(\;)$ and they will induce isomorphic homologies proving that $\mathcal{H}(\;) \simeq \mathcal{H}(\;)$.

Second Proof: We will prove the invariance of $\mathcal{H}$ under the $R2$ in a different way. This proof will be the key ingredient for the invariance of $\mathcal{H}$ under the $R3$. Rather than using the edge maps of the cube diagram we will define a new map on the diagonal of the cube diagram to conclude the proof. Let us use the same notation.

Consider the complex $\mathcal{C}/\mathcal{C}'$:



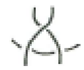
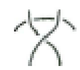
The map Δ on the left column is an isomorphism as $v_+ = 0$ in the quotient. We define a new map $\tau = d_{*0} \circ \Delta^{-1}$ composing the edge map with Δ^{-1}. consider then the subcomplex $\mathcal{C}'''$ of $\mathcal{C}/\mathcal{C}'$ containing all $\alpha \in [\![\;]\!]$ and all pairs $(\beta, \tau\beta) \in [\![\;]\!]_{/v_+=0}\{1\} \oplus [\![\;]\!]\{1\}$

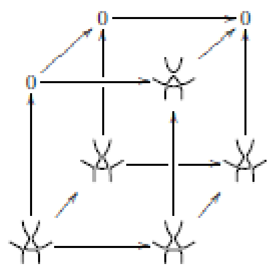


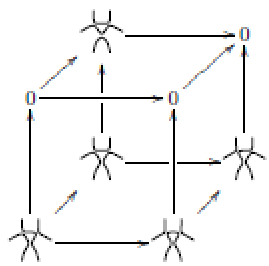
The map Δ becomes bijective in the subcomplex $\mathcal{C}'''$ implying $\mathcal{C}'''$ is acyclic and $H(\mathcal{C}/\mathcal{C}') \simeq H((\mathcal{C}/\mathcal{C}')/\mathcal{C}''')$ follows. If we show that $(\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$ is isomorphic to $\mathcal{C}''$ it will finishes the proof. In the quotient $(\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$ lower left corner becomes trivial.

Note that for every β in the upper left corner is identified with $\tau\beta$ in the lower right corner. Then $\mathcal{C}''$ is isomorphic to $(\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$ follows and the rest of the proof is same in the first one.

7.5.5.5 Invariance Under the $R3$

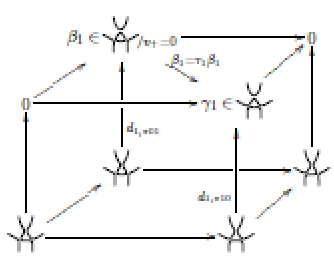
We will prove the isomorphism of $\mathcal{H}$ under the $R3$ as a last part. It will be a sketch basically, the result will follows by combining the techniques in 5.3 and 5.4. Let us first consider the cube diagrams of  and  obtained by resolving every crossing.

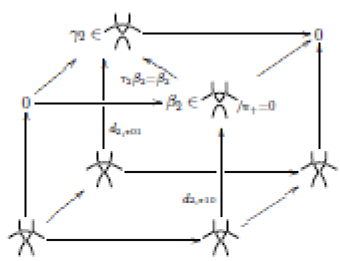


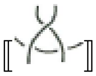
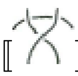


The bottom layers are isomorphic corresponding to the plane isotopy and the top layers are also isomorphic by using the invariance under the $R2$. Let us reduce the top layers to $\mathcal{C}''$ subcomplex as in section 5.4. But we can not continue from here. Even though the bottom and the top layers are isomorphic the maps connecting the bottom and the top parts are not. So we can not conclude that the cube diagrams are isomorphic. Let us use the second proof for the invariance under the $R2$.

We use the same notation as in the previous section. Reduce the top layers of both cubes to the subcomplexes $\mathcal{C}'$ and $\mathcal{C}'''$ and then consider the quotient complex $(\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$. WE know the isomorphism on the homologies $H(C) \simeq H((\mathcal{C}/\mathcal{C}')/\mathcal{C}''')$. Then we have the following cubes

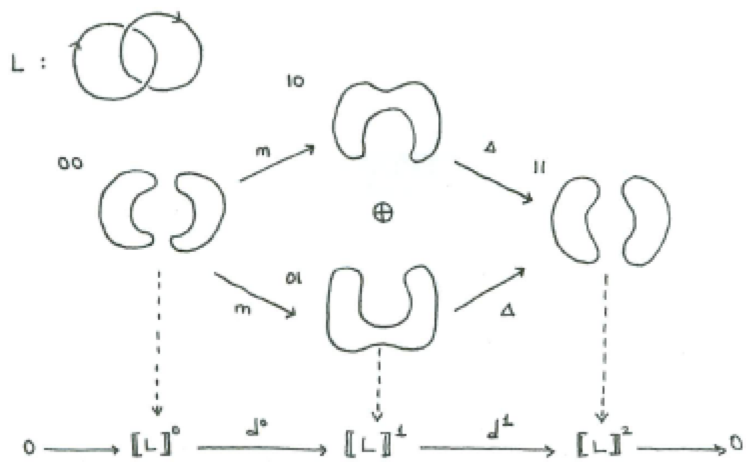




We claim that the two cubes are isomorphic. Define a map ϕ between two cubes such that as the bottom layers are isomorphic ϕ does not change anything on the bottom layers but let it transpose the top layers such that $\phi(\beta_1, \gamma_1) = (\beta_2, \gamma_2)$. This will induce an isomorphism on complexes because the maps between the bottom and the top levels are compatible such that the composition $\tau_1 \circ d_{1*01}$ in the first cube is same as the edge map d_{2*10} in the second cube. By degree and height shifts in the complexes ⟦⟧ and ⟦⟧ they will be isomorphic. therefore under the $R3$ $\mathcal{H}$ does not change and the proof of the main theorem is complete now. Note that other versions of the Reidemeister moves like left twist version of $R1$, other resolving of $R2$, and moving over for $R3$ can be proved similarly.

7.5.6 Example

So far we have developed the theory part of the Khovanov homology, now let us see how all these machinery works on an example. We will compute the Khovanov homology for the Hopf link for the chosen orientation as in the figure below. Let L denote the diagram of the Hopf link. After resolving every crossing we have the following cube diagram



and the following chain complex:

$$0 \longrightarrow V^{\otimes 2} \xrightarrow{d^0} V\{1\} \oplus V\{1\} \xrightarrow{d^1} V^{\otimes 2}\{2\} \longrightarrow 0$$

d_0 is the map as a sum of merging maps m: $d^0 = \sum_{|\epsilon|=0} (-1)^{\epsilon} d_{\epsilon} = d_{*0} + d_{0*}$. Let us understand the map on the basis elements.

$$0 \longrightarrow V \otimes V \xrightarrow{d^0} V\{1\} \oplus V\{1\}$$

$$d^0 : \begin{cases} v_+ \otimes v_+ \mapsto (v_+, v_+) \\ v_+ \otimes v_- \mapsto (v_-, v_-) \\ v_- \otimes v_+ \mapsto (v_-, v_-) \\ v_- \otimes v_- \mapsto (0, 0) \end{cases}$$

At first compute homology without degree and height shifts. Now let us find the kernel of d^0. Note that it is generated by $v_- \otimes v_-$ and $v_+ \otimes v_- - v_- \otimes v_+$. Thus

$$Ker d^0 = \langle v_- \otimes v_-, v_+ \otimes v_- - v_- \otimes v_+ \rangle$$

Image of the first map is 0 so

$$\mathcal{H}^0([\![L]\!]) = Kerd^0/\{0\} = \langle v_- \otimes v_-, v_+ \otimes v_- - v_- \otimes v_+ \rangle$$

$$\mathcal{H}^0([\![L]\!]) = W_0 \oplus W_{-2}$$

where $W_0 = \langle v_+ \otimes v_- - v_- \otimes v_+ \rangle$ and $W_{-2} = \langle v_- \otimes v_- \rangle$ then

$$qdim\mathcal{H}^0([\![L]\!]) = q^0 dimW_0 + q^{-2} dimW_{-2} = q^0 + q^{-2}$$

Let us now understand the first boundary map d^1.

$$d^1 = \sum_{|\epsilon|=1} (-1)^\epsilon d_\epsilon = (-1)^\epsilon d_{1*} + (-1)^\epsilon d_{*1} = (-1)^{0+1} d_{1*} + (-1)^0 d_{*1} = d_{*1} - d_{1*}$$

on the basis elements we have

$$V\{1\} \oplus V\{1\} \xrightarrow{d^1} V \otimes V\{2\}$$

$$d^1 : \begin{cases} (0, v_+) \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ (0, v_-) \mapsto v_- \otimes v_- \\ (v_+, 0) \mapsto -(v_+ \otimes v_- + v_- \otimes v_+) \\ (v_-, 0) \mapsto -(v_- \otimes v_-) \end{cases}$$

$$Im(d^0) = \langle (v_-, v_-), (v_+, v_+) \rangle$$

$$\begin{aligned} Ker(d^1) &= \langle (v_+, 0) + (0, v_+), (v_-, 0) + (0, v_-) \rangle \\ &= \langle (v_+, v_+), (v_-, v_-) \rangle \end{aligned}$$

$$\mathcal{H}^1([\![L]\!]) = Ker(d^1)/Im(d^0) = \langle (v_+, v_+), (v_-, v_-) \rangle / \langle (v_+, v_+), (v_-, v_-) \rangle \simeq \{0\}$$

$$qdim\mathcal{H}^1([\![L]\!]) = 0$$

$Im(d^1) = \langle v_+ \otimes v_- + v_- \otimes v_+, v_- \otimes v_- \rangle$ and $d^2 : V \otimes V\{2\} \to 0$ maps everything to 0 then $Ker(d^2) = \langle v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_- \rangle$.

$$\mathcal{H}^2([\![L]\!]) = Ker(d^2)/Im(d^1) \simeq \langle v_+ \otimes v_+, v_+ \otimes v_- - v_- \otimes v_+ \rangle$$

Say $W_2 = \langle v_+, v_+ \rangle$ and $W_0 = \langle v_+ \otimes v_- - v_- \otimes v_+ \rangle$ then

$$qdim\mathcal{H}^2([\![L]\!]) = (q^2 dimW_2 + q^0 dimW_0)q^2$$

We should multiply via q^2 as $Ker(d^2) = V^{\otimes k}\{2\}$ has degree shift by 2 then $qdim\mathcal{H}^2([\![L]\!]) = (q^2 + q^0)q^2 = q^4 + q^2$.

Note that $C(L) = [\![L]\!][-n_-]\{n_+ - 2n_-\}$ then observe:

$$\begin{aligned}
Kh(L) &= \sum_r t^r qdim\mathcal{H}^r(C(L)) \\
&= \sum_r t^r qdimC(L)^r \\
&= \sum_r t^r qdim([\![L]\!][-n_-]\{n_+ - 2n_-\}) \\
&= \sum_r t^r q^{n_+ - 2n_-} qdim[\![L]\!]^{r+n_-} \\
&= q^{n_+ - 2n_-} \sum_r t^{r+n_-} t^{-n_-} qdim[\![L]\!]^{r+n_-} \\
&= q^{n_+ - 2n_-} t^{-n_-} \sum_r t^{r+n_-} qdim[\![L]\!]^{r+n_-} \\
&= q^{n_+ - 2n_-} t^{-n_-} \sum_r t^r qdim\mathcal{H}^r([\![L]\!])
\end{aligned}$$

Therefore for L is the Hopf link after degree and height shifts and the chosen orientation in the figure we have $n_- = 2$ and $n_+ = 0$.

$$\begin{aligned}
Kh(L) &= q^{-4}t^{-2}[t^0(q^0 + q^{-2}) + t(0) + t^2(q^2 + q^4)] \\
&= q^{-4}t^{-2}[(1 + q^{-2}) + t^2q^2 + t^2q^4] \\
&= q^{-4}t^{-2} + q^{-6}t^{-2} + q^{-2} + 1
\end{aligned}$$

at $t = -1$ the Poincare polynomial becomes $q^{-6} + q^{-4} + q^{-2} + 1$ which is the unnormalized Jones polynomial of the Hopf link.

Remember that $\widehat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$ is the unnormalized Jones polynomial where $\langle L \rangle = \sum_\alpha V_\alpha(L) = \sum_\alpha (q + q^{-1})^k (-q)^r$. If L is the diagram of the Hopf link

$$\langle L \rangle = q^4 + q^2 + 1 + q^{-2}$$

$$\widehat{J}(L) = 1 + q^{-2} + q^{-4} + q^{-6}$$

mathcing with our result above.

7.5.7 Further Remarks

There should be a question in mind about Khovanov homology. Is it a sufficient link invariant? Can two nonisotopic links have the same homology groups? We mentioned that Jones polynomial is not sensitive to mutation move, but Khovanov distinguishes mutations that swap arcs between link components [5]. However, odd Khovanov homology and Khovanov homology over $\mathbb{Z}/2\mathbb{Z}$ are mutation invariant [5]. On the other hand, knot Floer homology is sensitive to Conway mutation where Khovanov homology fails in some cases. Euler characteristic of knot Floer is related to Alexander polynomial and Euler characteristic of Khovanov homology is related to Jones polynomial.

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