# On Milnor's Exotic Seven Spheres

by

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This is to certify that I have examined this copy of a master's thesis

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## ABSTRACT

In 1956, John Milnor constructed several examples of a compact smooth seven dimensional manifold without boundary that is homeomorphic but not diffeomorphic to the standard seven dimensional smooth sphere $\mathbb{S}^7$, which consequently earned him a Fields Medal in 1962. The aim of this work is to understand these smooth manifolds which are now known as the *exotic* seven spheres. To this end, after a brief introduction in the first section, we review some basic facts about fiber bundles and characteristic classes in section two. In the third section, we construct a smooth closed seven dimensional manifold and use Reeb's theorem to show that this manifold is homeomorphic to $\mathbb{S}^7$. The fact that its smooth structure is different from the standard smooth structure on $\mathbb{S}^7$ is explained in the last part of the third section.

All manifolds are assumed to be orientable, differentiable and compact, and all homology and cohomology groups are assumed to be with $\mathbb{Z}$ coefficients unless otherwise specified.

# ÖZET

1956'da, John Milnor standart, yedi boyutlu, pürüzsüz küre $\mathbb{S}^7$'ye homeomorfik olup difeomorfik olmayan, birçok tıkız, pürüzsüz, yedi boyutlu, kenarsız çokkatlı inşa etti. Bundan ötürü kendisine 1962'de Fields Madalyası verildi. Bu çalışmanın amacı, egzotik yedi boyutlu küreler olarak bilinen bu düzgün çokkatlıları anlamak. Bu amaçla, ilk bölümde kısa bir giriş yaptktan sonra, ikinci kısımda "lif demeti" ve "karakteristik sınıf" temel kavramlarıyla ilgili bazı temel gerçekleri tekrar ettik. Üçüncü bölümde pürüzsüz, kapalı, yedi boyutlu bir çokkatlı inşa edip, bu çokkatlının $\mathbb{S}^7$'ye homeomorfik olduğunu Reeb'in teoremini kullanarak gösterdik. Türevlenebilir yapısının $\mathbb{S}^7$ üzerindeki standart yapıdan farklı olduğunu ise üçüncü bölümün son kısımda açıkladık.

Tez boyunca, aksi belirtilmediği sürece, çokkatlıları yönlendirilebilir, türevlenebilir ve tıkız; homoloji ve kohomoloji gruplarını da $\mathbb{Z}$ katsayılı aldık.

*To my grandma, whom I miss so much...*

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# Contents

LIST OF SYMBOLS & ABBREVIATIONS

| | |
|---|---|
| $\cong$ | isomorphic to |
| $\approx$ | homeomorphic to |
| $\simeq$ | homotopy equivalent to |
| $\hookrightarrow$ | inclusion |
| $\mathbb{T}^n$ | $n$-dimensional torus |
| $\mathbb{D}^n$ | $n$-dimensional unit disk |
| $\mathbb{RP}^n$ | $n$-dimensional real projective space |
| $\mathbb{CP}^n$ | $n$-dimensional complex projective space |
| $\mathbb{HP}^n$ | $n$-dimensional quaternionic projective space |
| $\#$ | the connected sum |
| $\vee$ | the wedge sum |
| $\trianglelefteq$ | normal subgroup |

# 1 INTRODUCTION

The classification of 2-dimensional manifolds was well understood in the 19th century. The corresponding question in higher dimensions, however, turned out to be much more difficult.

## 1.1 Generalized Poincaré Conjecture

Henri Poincaré attempted to classify 3-dimensional manifolds and in 1904, he asked his famous question known as the ***Poincaré Conjecture***:

*If a closed 3-manifold $M^3$ is simply connected, does it follow that $M^3$ is homeomorphic to the* 3*-dimensional sphere $\mathbb{S}^3$?*

Since the hypothesis is equivalent to $M^3$ being homotopy equivalent to $\mathbb{S}^3$ ([13], by Whitehead's and Hurewicz's theorems), Poincaré's question can be rephrased as:

*If a closed 3-manifold $M^3$ is homotopy equivalent to $\mathbb{S}^3$, then is $M^3$ necessarily homeomorphic to $\mathbb{S}^3$?*

In fact, one can ask the same question in any dimension:

*If a closed n-manifold $M^n$ is homotopy equivalent to $\mathbb{S}^n$, does it follow that $M^n$ is homeomorphic to $\mathbb{S}^n$?*

This question, known as the ***Generalized Poincaré Conjecture*** has inspired topologists ever since, and attempts to prove it have led to many advances towards understanding the topology of manifolds [4].

In the late 1950's and early 1960's, it was discovered that higher dimensional manifolds are actually easier to work with than the 3- and 4-dimensional ones. One reason for this is the following: The relations between generators of the fundamental group correspond to 2-dimensional disks mapped into the manifold. In dimension 5 or greater, such disks can be put into a general position so that they are disjoint from each other, with no self-intersections. As a matter of fact, S. Smale was able to confirm the Generalizad Poincaré Conjecture in 1961 in dimensions five and higher.

In dimension 3 and 4 it may not be possible to avoid intersection of such disks, leading to serious difficulties. Nonetheless, M. Freedman proved the Generalizad Poincaré Conjecture in dimension four, using non-differentiable methods in 1982. In fact, he classified all closed, oriented, simply-connected topological 4-manifolds. The conjecture in dimension three turned out to be hardest and resisted many attempts until G. Perelman finally confirmed it in 2002 using R. Hamilton's program based on Ricci flow [3].

### 1.2 Milnor's Work

During the 1950's, while studying another problem, Milnor discovered the existence of some 7-dimensional smooth manifolds each of which is homeomorphic but not diffeomorphic to the standard 7-dimensional sphere $\mathbb{S}^7$. Surprisingly, the manifolds Milnor studied were well understood in the literature; these were some $\mathbb{S}^3$ bundles over $\mathbb{S}^4$. Such bundles are classified by $\pi_3(SO(4)) = \mathbb{Z} \oplus \mathbb{Z}$, i.e., just a pair of integers.

It is easy to see that the total spaces of some of these bundles, which are 7-dimensional smooth oriented manifolds, are homotopic to $\mathbb{S}^7$. On the other hand using differentiable methods Milnor showed that these manifolds can not possibly be the standard smooth $\mathbb{S}^7$. Initially, Milnor believed that he found a counterexample to the Generalized Poincaré Conjecture in dimension seven since he assumed that these manifolds could not even be homeomorphic to $\mathbb{S}^7$.

After a careful thought, Milnor proved that in fact any of the 7-manifolds at hand were homeomorphic to $\mathbb{S}^7$ by constructing a Morse function with only two critical points on each of them. To show that the differential structure on these manifolds are not standard, Milnor relied on a deep result of Thom and Hirzebruch which states that the signature $\sigma(W)$ of a *closed* smooth 8-dimensional manifold $W$ satisfies:

$$\sigma(W) = \frac{p_1^2[W] - 7p_2[W]}{45}$$

where $p_1^2[W]$ and $p_2[W]$ are the Pontrjagin numbers of $W$ [2].

In the following we provide further details of Milnor's construction of exotic 7-spheres which appeared in his ground-breaking article "On manifolds homeomorphic to the 7-sphere" [1].

# 2 PRELIMINARIES

## 2.1 Fibre Bundles

The exotic spheres are built up by gluing two trivial $\mathbb{S}^3$ *bundles* over $\mathbb{R}^4$, so we begin with defining bundles.

**Definition 1.** A *fiber bundle* $\beta = (E, \pi, B, F)$ consists of

**i.** a topological space $E$ called bundle/total space,

**ii.** a topological space $B$ called base space (we always take $B$ as paracompact),

**iii.** a continuous, onto map $\pi : E \to B$,

**iv.** a space $F$ called *the fiber* satisfying, for each point $b$ in $B$, $\pi^{-1}(b)$ is homeomorphic to $F$,

and lastly, for each $b \in B$ there should exist a neighborhood $U$ of $b$ and a homeomorphism $\phi : \pi^{-1}(U) \to U \times F$ such that $\pi \circ \phi^{-1}(b', f) = b'$ for all $b' \in U, f \in F$ (local trivialization).

Group of homeomorphisms of the fiber $F$ is called *the group of the bundle* which is topological group and acts effectively on $F$.

If we require the fiber to be not an arbitrary topological space but a vector space, then we get a special example of fiber bundles.

**Definition 2.** For a fiber bundle $\beta = (E, \pi, B, F)$, if the fiber $F$ is $\mathbb{R}^k$ then the bundle is called *rank $k$ vector bundle* or *$k$-plane bundle.*

Therefore in a rank $k$ vector bundle we have that for each point $b$ in the base, $\pi^{-1}(b)$ has a real $k$-vector space structure, and in the local triviality condition, for each $q \in U$, the restriction of $\phi$ to $\pi^{-1}(q)$ is a linear isomorphism between $\pi^{-1}(q)$ and $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

If in a vector bundle, all spaces are smooth manifolds, $\pi$ is smooth and the local trivialization maps are all diffeomorphisms then the bundle is called a *smooth vector bundle.*

As an example of vector bundles, the *tangent bundle* $\tau_M = (TM, \pi, M, \mathbb{R}^k)$ of a smooth $k$-manifold $M$ can be given. Here $TM = \coprod_{p \in M} T_pM$ is disjoint union of the tangent spaces at all points of $M$. The projection map sends each vector to its initial point, and the group $G$ is the full linear group operating on tangent spaces. Moreover, $TM$ is smooth $2k$-manifold [11].

**Definition 3.** For a rank $k$ vector bundle $\xi = (E, \pi, B, \mathbb{R}^k)$, if there exists a trivialization over all of the base space $B$, then $\xi$ is said to be the *trivial bundle* in which case we have $E \approx B \times \mathbb{R}^k$.

Thus any space $B$ admits a trivial bundle, just by taking $E$ equal to $B \times \mathbb{R}^k$ with appropriate maps. On the other hand, a well known example to nontrivial bundles is the *canonical line bundle* $\gamma_n^1$ over the real projective space $\mathbb{RP}^n$ which is constructed as follows:

We recall that $\mathbb{RP}^n$ is the quotient space of the standard $n$-sphere $\mathbb{S}^n$, where antipodal points are identified. The total space $E$ is

$$E = \{([x], k.x) \mid k \in \mathbb{R}\} \subset \mathbb{RP}^n \times \mathbb{R}^{n+1}$$

where $[x]$ is the image of the points $\{x, -x\} \in \mathbb{S}^n$ under the quotient map, and $\pi$ is defined as

$$\pi : E \to \mathbb{RP}^n,\ \pi([x], k.x) = [x].$$

Thus each fiber $\pi^{-1}([x])$ is a line through $\{x\}$ and $\{-x\}$ in $\mathbb{R}^{n+1}$ with the usual vector space structure. We can also check that the local triviality condition is satisfied, hence giving a vector bundle. To visualize, when $n = 1$, $\gamma_1^1$ is a bundle over $\mathbb{S}^1$, and the total space is the Möbius band [2]. To show that $\gamma_n^1$ is nontrivial we first define cross sections.

**Definition 4.** Let $\pi : E \to B$ be a rank $k$ vector bundle.

**i.** A *section* of this bundle is a continuous map $\sigma : B \to E$ satisfying $\pi \circ \sigma = Id_B$. If $\sigma$ is smooth then it is called *smooth section.*

**ii.** A *(smooth) frame* for $\pi : E \to B$ is an ordered $k$ tuple $(\sigma_1, .., \sigma_k)$ of (smooth) sections such that for each $b \in B$, $(\sigma_1(b), .., \sigma_k(b))$ is a basis for $\pi^{-1}(b)$.

A $k$-vector bundle is trivial if and only if it admits a frame. For a smooth manifold $M$, if its tangent bundle $\tau_M$ admits a smooth frame; i.e., if $\tau_M$ is trivial then $M$ is said to be *parallelizable.*

Now we can turn to the claim that $\gamma_n^1$ is nontrivial. We will show that it does not admit a frame (or equivalently does not admit a nowhere zero section as $\gamma_n^1$ is a rank 1 bundle). The proof follows from the intermediate value theorem. In fact, let $s : \mathbb{RP}^n \to E$ be any section taking each $[x]$ in $\mathbb{RP}^n$ to $([x], k.x) \in E$, where $k \in \mathbb{R}$ is determined by a continuous function

$t : \mathbb{S}^n \to \mathbb{R}$. Then we have $t(-x) = -t(x)$. Since $\mathbb{S}^n$ is connected there exists some $y \in \mathbb{S}^n$, $t(y) = 0$. Now, for $q : \mathbb{S}^n \to \mathbb{RP}^n$ the quotient map, $s \circ q$ takes $y \in \mathbb{S}^n$ to $([y], 0)$. In other words, the value of each section is zero for at least one point in the base. Hence $\gamma_n^1$ is a nontrivial vector bundle.

Before giving more examples to bundles, we need to introduce a couple of concepts.

**Definition 5.** Let $\eta = (E_\eta, \pi_\eta, B_\eta)$ and $\xi = (E_\xi, \pi_\xi, B_\xi)$ be two vector bundles. A *bundle map* from $\eta$ to $\xi$ is a pair of continuous maps $F : E_\eta \to E_\xi$, and $f : B_\eta \to B_\xi$, such that $\pi_\xi \circ F = f \circ \pi_\eta$ and restriction of $F$ to each fiber of $\eta$ is linear.

In other words, for each $b \in B_\eta$, $F$ carries the fiber of $b$ in $\eta$ to the fiber of $f(b)$ in $\xi$ isomorphically; i.e., $F$ is a vector space isomorphism when restricted to fibers. It is common to say that $F$ covers $f$ and call $F$ the bundle map.

If $\xi, \eta$ are over the same base space $B$, then for $f$ we simply take the identity map of $B$ and say that $F$ is the bundle map.

**Definition 6.** Two vector bundles $\eta$, $\xi$ over the same base space are said to be *isomorphic*, if there exists a bijective bundle map $F : E_\eta \to E_\xi$ whose inverse is also a bundle map.

This means that isomorphic bundles have homeomorphic total spaces.

**Definition 7.** Given a vector bundle $\xi = (E, \pi, B)$, an arbitrary topological space $B_1$ and a map $f : B_1 \to B$, the *induced/pullback bundle* $f^*\xi = (E_1, \pi_1, B_1)$ of $\xi$ is a vector bundle over $B_1$, where

$$E_1 = \{(b, e) \in B_1 \times E \mid f(b) = \pi(e)\}, \text{ and } \pi_1(b, e) = b.$$

From this we see that for each $b \in B_1$ the fiber $\pi_1^{-1}(b)$ is $b \times \pi^{-1}(f(b))$. In addition, $f$ and the map $F : E_1 \to E$ defined as $F(b, e) = e$, give a bundle map between $f^*\xi$ and $\xi$. If we restrict $F$ to the fiber of $b$, then $F : b \times \pi^{-1}(f(b)) \to \pi^{-1}(f(b))$ is a linear isomorphism. In short, the fiber over each point $b$ in $f^*\xi$ is identified with the fiber over its image $f(b)$ in $\xi$. In fact, $f^*\xi$ admits a unique vector bundle structure [9].

To put it another way, when we are given a vector bundle $\xi = (E, \pi, B)$ and a map $f : B_1 \to B$, the induced bundle exists. Moreover, it is unique up to isomorphism.

To exemplify induced bundles, we consider the constant map $f : B \to b$ where $b \in B$. Let $\xi$ be a (trivial) rank $k$ vector bundle over $b$. Then total space of $f^*\xi$ is $B \times \mathbb{R}^k$ giving a trivial bundle over $B$. The converse is also true. That is to say,

**Proposition 2.1.** *A trivial bundle is the pullback of a vector bundle over a point.*

*Proof.* Indeed, given a trivial rank $n$ bundle $\xi = (B_1 \times \mathbb{R}^n, \pi_1, B_1)$, we choose a point $b$ in the base. Then for $f : B_1 \to b$ the constant map and $\beta = (\mathbb{R}^n, \pi, b)$ the bundle over $b$, we have $f^*\beta = \xi$. □

Moreover,

**Theorem 2.2.** *([14], Corollary 1.8) Every vector bundle over a contractible base space is trivial.*

General statement for fiber bundles can be found in [8], Corollary 11.6.

To give an another example of induced bundles, let $\xi = (E, \pi, B)$ be a vector bundle. We take a subspace $A$ of $B$ and the inclusion map $\iota : A \hookrightarrow B$. Then the bundle $(\pi^{-1}(A), \pi|_{\pi^{-1}(A)}, A)$ restricted to $A$ is $\iota^*\xi$. This bundle is said to be the *portion of* $\xi$ *over* $A$.

Instead of restricting the base space, we can also 'restrict the fibers' of a given bundle.

**Definition 8.** Let $\xi = (E_\xi, \pi_\xi, B)$ and $\eta = (E_\eta, \pi_\eta, B)$ be two vector bundles over the same space $B$, with $E_\xi \subset E_\eta$. If for each $b \in B$, $\pi_\xi^{-1}(b)$ is a subspace of $\pi_\eta^{-1}(b)$, then $\xi$ is a *subbundle* of $\eta$. We denote this as $\xi \subset \eta$.

More generally, let $\beta = (E, \pi, B, F)$ be a fiber bundle with group $G$. If a subspace $F'$ of the fiber $F$ is invariant under $G$, then it gives us a new bundle $\beta'$ over the same base, with fiber $F'$. We say that $\beta'$ is the subbundle of $\beta$ determined by $F'$.

Let $\xi$ and $\eta$ be vector bundles and $\xi \subset \eta$. To define the *orthogonal complement* of $\xi$ in $\eta$, we need put a metric on these vector bundles. A *Euclidean vector bundle* $\eta = (E_\eta, \pi_\eta, B_\eta)$ is a real vector bundle with a continuous map $\mu : E_\eta \to \mathbb{R}$ such that the restriction of $\mu$ to each fiber is a Euclidean metric; i.e., positive definite and quadratic function.

**Definition 9.** Let $\xi_1 = (E_1, \pi_1, B)$ and $\xi_2 = (E_2, \pi_2, B)$ be two vector bundles over the same space $B$. The bundle $\xi_1 \oplus \xi_2 = (E, \pi, B)$ is called the *Whitney Sum* of $\xi_1$ and $\xi_2$, where $E = \{(v_1, v_2) \in E_1 \times E_2 \mid \pi_1(v_1) = \pi_2(v_2)\}$, and $\pi(v_1, v_2) = \pi_1(v_1) = \pi_2(v_2)$.

Hence for each $b \in B$, we have $\pi^{-1}(b) = \pi_1^{-1}(b) \oplus \pi_2^{-1}(b)$.

Now, let $\xi$ and $\eta$ be Euclidean vector bundles and $\xi \subset \eta$. The *orthogonal complement* $\xi^\perp = (E_\perp, \pi_\perp, B)$ of $\xi = (E, \pi, B)$ *in* $\eta$ is a bundle whose fibers are orthogonal complements of the corresponding fibers of $\xi$ and whose total space is the union of these fibers. Then we have $\eta \cong \xi \oplus \xi^\perp$, as the fiber of each point in $\eta$ is the direct sum of its fibers in $\xi$ and $\xi^\perp$.

To illustrate, let $M \subset N$ be smooth manifolds with Riemannian metric. Then the tangent bundle $\tau_M$ of $M$ is a subbundle of $\tau_N|_M$ and its orthogonal complement $\nu$ in $\tau_N|_M$ is called the *normal bundle* of $M$ in $N$. Thus we have $\tau_M \oplus \nu \cong \tau_N|_M$.

### 2.1.1 Sphere Bundles

In the sequel, we are mainly concerned with *sphere bundles* which are other examples of fiber bundles. Before that, we recall the orthogonal group.

**Definition 10.** Let $GL_n(\mathbb{R})$ is the general linear group; i.e., the group of $n \times n$ invertible matrices with real entries. The group $\{A \in GL_n(\mathbb{R}) \mid AA^T = AA^T = I\}$ is called the *orthogonal group* and denoted by $O(n)$.

Equivalently, $O(n)$ is the group of isometries of $\mathbb{R}^n$ fixing the origin and transitively acts on $\mathbb{S}^{n-1}$. It can be topologized as a subspace of $\mathbb{R}^{n^2}$, where each element $A$ of $O(n)$ corresponds to a point $p$ in $\mathbb{R}^{n^2}$ such that the coordinates of $p$ are the entries of $A$. Besides, as the columns of a matrix in $O(n)$ are unit vectors, $O(n)$ is a subspace of $\prod_{i=1}^{n} \mathbb{S}^{n-1}$.

**Definition 11.** The kernel of the determinant map $d : O(n) \to \{\pm 1\}$ is called the *special orthogonal group*, denoted by $SO(n)$. Being the rotation group of $\mathbb{S}^{n-1}$, it is also called the *rotation group.*

In fact, $SO(n)$ is a closed orientable manifold of dimension $n(n-1)/2$ [7]. A property of $SO(n)$ to be used in the sequel is that it is path connected. Indeed, each element of $SO(n)$ can be joined to the identity by a path of rotations. More precisely, let us take $A = [a_1, ..., a_n] \in SO(n)$ where $a_i$'s denote the columns. We can rotate $A$ until $a_n$ coincides with $(0, ..., 0, 1)$. Then $a_1, ..., a_{n-1}$ lie in $\mathbb{R}^{n-1} \times \{0\}$ and by a rotation in $\mathbb{R}^{n-1}$ we can bring $a_{n-1}$ in coincidence with $(0, ..., 0, 1, 0)$. Continuing this procedure we continuously deform $[a_1, ..., a_n]$ through a family of orthonormal basis until $a_2, ..., a_n$ coincide with the last $n-1$ elements of the standard basis of $\mathbb{R}^n$. At the last step we must either have $a_1 = (1, 0, ..., 0)$ or $a_1 = (-1, 0, ..., 0)$. But the latter is impossible because of the positivity of the determinant of $A$, and continuous deformations of it through a family of basis preserve the sign of the determinant. Consequently $a_1 = (1, 0, ..., 0)$ and we are done.

Now we will define sphere bundles.

**Definition 12.** Any bundle in which the fiber is an $n$-sphere and the group is the orthogonal group $O(n+1)$ is called an *$n$-sphere bundle.*

Such a bundle is obtained from a vector bundle by taking only vectors of unit length in the total space. In other words, an $n$-sphere bundle $\epsilon$ is a subbundle of a rank $n+1$ vector bundle $\xi$, where fibers of $\epsilon$ are vectors of unit length in the total space of $\xi$.

If the group of a sphere bundle is the rotation group, then it is called an *orientable sphere bundle.*

#### 2.1.2 Quaternions

In this part we briefly remember the set $\mathbb{H}$ called quaternions and $\mathbb{HP}^n$, the projective space. $\mathbb{H}$ is a four-dimensional vector space over $\mathbb{R}$ with the basis $\{1, i, j, k\}$ and with operations; addition, scalar multiplication, and noncommutative quaternion multiplication (with the "$i, j, k$ rule"). As a set, $\mathbb{H}$ is equal to $\mathbb{R}^4$. A quaternion is a number of the form $q = a_1 + a_2 i + a_3 j + a_4 k$ where $a_i \in \mathbb{R}$, $1 \leq i \leq 4$ and its conjugate equals $\overline{q} = a_1 - a_2 i - a_3 j - a_4 k$. The norm $|q|$ of $q$ is given by $\sqrt{\sum_{i=1}^{4} a_i^2}$. For any two quaternions $|q_1|$ and $|q_2|$, the norm satisfies $|q_1 q_2| = |q_1||q_2|$. Lastly, the inverse $q^{-1}$ of $q$ is defined as $\overline{q}/|q|^2$, therefore for a unit quaternion, its inverse equals its conjugate.

The quaternionic projective space $\mathbb{HP}^n$ is the set of all quaternionic lines through the origin in $\mathbb{H}^{n+1}$, which is a smooth $4n$-manifold. The cell structure of $\mathbb{HP}^n$ is

$$\mathbb{HP}^n = e^0 \cup e^4 \cup ... \cup e^{4n} \text{ [13]}.$$

In fact, $\mathbb{HP}^n$ is obtained from $\mathbb{HP}^{n-1}$ inductively, by attaching a cell $e^{4n}$ via the quotient map $\mathbb{S}^{4n-1} \to \mathbb{HP}^{n-1}$ (which is one of the Hopf maps defined below when $n = 2$).

### 2.1.3 The Hopf Fibration

Hopf fibration is a kind of fibering of spheres by spheres. We will describe the fibration of $\mathbb{S}^3$ over $\mathbb{S}^2$ with fiber $\mathbb{S}^1$, denoted by $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \to \mathbb{S}^2$. We have $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ and $\mathbb{S}^2 \approx \mathbb{CP}^1$ is the set of equivalence classes $[z_1, z_2]$ of points $(z_1, z_2)$, with $z_1, z_2 \in \mathbb{C}$ not both zero, where the equivalence relation is defined as $[z_1, z_2] = [w_1, w_2]$ if and only if $(w_1, w_2) = (\lambda z_1, \lambda z_2)$ for $\lambda \neq 0$.

The Hopf map $p : \mathbb{S}^3 \to \mathbb{S}^2$ is defined as $p((z_1, z_2)) = [z_1, z_2]$. This map is onto since any $[z_1, z_2] \in \mathbb{S}^2 \approx \mathbb{CP}^1$ equals $[z_1/\lambda, z_2/\lambda]$, where $\lambda = \sqrt{|z_1|^2 + |z_2|^2}$. Therefore $p(z_1/\lambda, z_2/\lambda) = [z_1, z_2]$. In addition, the inverse image of a point of $\mathbb{S}^2$ is $\mathbb{S}^1$. For if $|\lambda| = 1$ and $(z_1, z_2)$ is in $\mathbb{S}^3$ then so is $(\lambda z_1, \lambda z_2)$, hence they go to the same point in $\mathbb{S}^2$. Conversely, if $p((z_1, z_2)) = p((t_1, t_2))$ then $(t_1, t_2) = (\lambda z_1, \lambda z_2)$ for $|\lambda| = 1$. Hence the fibers are simply $\mathbb{S}^1$. This decomposition of $\mathbb{S}^3$ into great circles $\mathbb{S}^1$ with $\mathbb{S}^2$ the decomposition space is one example of the Hopf fibrations.

By considering quaternionic case, we similarly get the fibration $\mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \to \mathbb{S}^4 \approx \mathbb{HP}^1$. We have indeed two more instances $\mathbb{S}^7 \hookrightarrow \mathbb{S}^{15} \to \mathbb{S}^8$ and $\mathbb{S}^0 \hookrightarrow \mathbb{S}^1 \to \mathbb{S}^1$, and these are all the Hopf fibrations [13].

## 2.2 Characteristic Classes

Characteristic classes are cohomology classes associated to the vector bundles. Briefly speaking, they are diffeomorphism invariants and measure how a vector bundle is twisted or nontrivial. We follow [2] and [14] to give a brief description of special types of them, namely Euler, Chern and Pontrjagin classes.

### 2.2.1 Orientation and Euler Classes

**Definition 13.** An *orientation of a real vector space* $V$ of dimension $n > 0$ is an equivalence class of bases, where the equivalence relation is defined as follows. Let $\{v_1, ..., v_n\}$ and $\{v'_1, ..., v'_n\}$ be two ordered bases. They are said to be equivalent if the transformation matrix $[a_{ij}]$, defined as $v'_i = \sum a_{ij} v_j$ has positive determinant.

Thus to every vector space, we can assign precisely two distinct orientations (as there are two possibilities for the sign of the determinant). On $\mathbb{R}^n$ we take canonical orientation, corresponding to its canonical ordered basis.

**Definition 14.** An *orientation for a rank n vector bundle* $\xi = (E, \pi, B, \mathbb{R}^n)$ is a function which assigns an orientation to each fiber of $\xi$, satisfying the following local compatibility condition. For every point $b \in B$, there should exist a neighborhood $U$ of $b$ and a homeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$, so that for each $b' \in U$ the restricted map $\phi : \pi^{-1}(b') \to b' \times \mathbb{R}^n \cong \mathbb{R}^n$ is orientation preserving [2].

In terms of cohomology this may be stated as follows. To each fiber $F$ there is assigned a preferred generator $u_F \in H^n(F, F - \{0\})$, and for every $b \in B$ there exist a neighborhood $U$ of $b$ and a cohomology class $u \in H^n(\pi^{-1}(U), \pi^{-1}(U) - \{0\})$ so that

$$u|_{(F,F-\{0\})} = u_F \in H^n(F, F - \{0\}) \text{ for each fiber } F \text{ over } U.$$

If an orientation exists for a vector bundle $\xi$, then $\xi$ is called orientable. Furthermore, if an orientation is specified then $\xi$ is said to be oriented.

**Theorem 2.3.** *(Thom Isomorphism)([2], Theorem 10.4) If a rank n vector bundle $\xi = (E, \pi, B, F)$ is oriented, then there is a unique class $u \in H^n(E, E - \{0\})$ such that $u|_{(F,F-\{0\})}$ is equal to the preferred generator $u_F \in H^n(F, F - \{0\})$ for each fiber $F$.*

We call $u$ the *fundamental class.* In addition, for an oriented rank n vector bundle $\xi = (E, \pi, B, F)$, the inclusion $(E, \emptyset) \hookrightarrow (E, E - \{0\})$ induces a restriction homomorphism

$$\Gamma : H^n(E, E - \{0\}) \to H^n(E),$$

where $E - \{0\}$ is the set of nonzero vectors in $E$. Under $\Gamma$, $u$ gives a new class $u|_E$ in $H^n(E)$. On the other hand, the base $B$ is the deformation retract of $E$ by the zero section. Indeed, let $g : B \to E$ be the zero section. Then $\pi \circ g = Id_B$ and $g \circ \pi \simeq Id_E$ via the homotopy $F : E \times [0, 1] \to E$, $F(v, t) = t \cdot v$. Therefore $B$ is homotopy equivalent to $E$ implying that $H^n(E) \cong H^n(B)$ canonically. Hence

**Definition 15.** The *Euler class* of an oriented rank $n$ vector bundle is the cohomology class

$$e(\xi) \in H^n(B)$$

corresponding to the restriction of the fundamental class $u|_E$ under the canonical isomorphism between $H^n(E)$ and $H^n(B)$.

Clearly, if the orientation of $\xi$ is reversed, then $e(\xi)$ changes sign.

Orientation for manifolds is defined similarly.

**Definition 16.** An *orientation for an $n$ dimensional manifold $M$* is a function which assigns to each $x \in M$ a local orientation $\mu_x$ (i.e. a choice of one of the two possible generators of $H_n(M, M - \{x\}) \cong \mathbb{Z}$) subject to the following condition. For each $a \in M$ there exists a compact neighborhood $N$ of $a$ and a class $\mu_N \in H_n(M, M - N)$ so that $\rho_x(\mu_N) = \mu_x$ for each $x \in N$ where $\rho_x$ is an isomorphism from $H_n(M, M - N)$ to $H_n(M, M - \{x\})$.

The pair consisting of a manifold and an orientation is called an *oriented manifold.* Similar to the bundle case, we know that for a closed, oriented $n$-manifold $M$, there is a unique class $\mu \in H_n(M)$ which satisfies $\rho_x(\mu) = \mu_x$ for each $x \in M$ where $\rho_x : H_n(M) \to H_n(M, M - \{x\})$ is an isomorphism. This class $\mu$ is said to be the *fundamental homology class* of $M$. ([2], Theorem A.8, [13], Theorem 3.26)

An orientation for $M$ gives rise to an orientation for its tangent bundle $\tau_M$, and conversely ([2], Lemma 11.6). Thus the terms "the orientation of a manifold" and "the orientation of its tangent bundle" are interchangeable.

### 2.2.2 Complex Vector Bundles and Chern Classes

The Chern class is another type of characteristic classes which is defined for *complex vector bundles*. Complex vector bundles are defined in the same way as real rank $k$ vector bundles, except we replace $\mathbb{R}^k$ with $\mathbb{C}^k$ in Definition 2. One method to construct such a bundle is to start with a real rank $2k$ bundle and to give each fiber a *complex vector space structure*.

**Definition 17.** A *complex structure* on a real rank $2k$ bundle $\xi = (E_\xi, \pi, B)$ is a continuous map $J : E_\xi \to E_\xi$ which maps every fiber linearly onto itself and $J(J(v)) = -v$, $\forall v \in E_\xi$.

Now suppose that a complex structure $J$ is given for a real rank $2k$ bundle $\xi = (E_\xi, \pi, B)$. For every complex number $x + yi$ and $v \in E_\xi$, we set $(x + yi)v = xv + J(yv)$ which makes each fiber a complex vector space. As a result, we get a complex vector bundle.

Conversely, given any complex rank $k$ bundle $w$, we can ignore the complex structure and see each fiber as a real $2k$-vector space. This gives us the *underlying real bundle* $w_\mathbb{R}$ of $w$. Furthermore,

**Lemma 2.4.** *The underlying real bundle $w_\mathbb{R}$ of a complex $n$-plane bundle $w$ has a canonical preferred orientation.*

*Proof.* We take one of the fibers $F$ (a vector space over $\mathbb{C}$) of $w$ and choose a basis $\{v_1, ..., v_n\}$ of $F$. Then $\{v_1, iv_1, ..., v_n, iv_n\}$ form an ordered basis for $F_\mathbb{R}$, hence determines an orientation for $F_\mathbb{R}$. Moreover this orientation does not depend on the choice of the complex basis $\{v_1, ..., v_n\}$. In fact, $GL(n, \mathbb{C})$ is connected (being the preimage of $\mathbb{R}^2 - \{0\}$ under the determinant function) so that we can pass from one basis to another continuously without changing the orientation.

By applying this construction to every fiber of $w$, we get the orientation for $w_\mathbb{R}$. ☐

As a consequence, given a complex bundle $w = (E, \pi, B)$ the Euler class $e(w_\mathbb{R})$ is defined and $e(w_\mathbb{R}) \in H^{2n}(B)$.

To define Chern classes of $w$, we recall the *Gysin exact sequence*.

**Theorem 2.5.** *([2], Theorem 12.2) To any real, oriented $n$-plane bundle $\xi = (E, \pi, B)$ there is an exact sequence:*

$$.. \to H^i(B) \overset{\smile e}{\to} H^{i+n}(B) \overset{\pi_0^*}{\to} H^{i+n}(E-\{0\}) \to H^{i+1}(B) \overset{\smile e}{\to} ..$$

*so called the Gysin exact sequence.*

*Here the symbol* $\smile e$ *stands for the homomorphism* $a \mapsto a \smile e(\xi)$ *where* $e(\xi)$ *is the Euler class of* $\xi$*, and* $\pi_0^*$ *is the map induced from* $\pi_0 : E - \{0\} \to B$ *which is the restriction of* $\pi$ *to the set of nonzero vectors* $E - \{0\}$ *in* $E$*.*

We will now give an inductive definition of the Chern classes for a complex $n$-plane bundle $w = (E, \pi, B)$. First, we construct a complex $(n-1)$-plane bundle $w_0$ over $E - \{0\}$. A point in $E - \{0\}$ is specified by taking a fiber $F$ of $w$ and then a nonzero vector $v$ in that fiber. Now, the fiber of $w_0$ over $v$ is defined as the quotient space $F/(\mathbb{C}v)$, where $\mathbb{C}v$ is the one dimensional subspace spanned by $v$.

For this complex $n$-(therefore a real $2n$-) plane bundle $w = (E, \pi, B)$, the Gysin exact sequence reads:

$$.. \to H^{2i-2n}(B) \to H^{2i}(B) \overset{\pi_0^*}{\to} H^{2i}(E-\{0\}) \to H^{2i-2n+1}(B) \to ..,$$

so that when $i < n$, $\pi_0^* : H^{2i}(B) \to H^{2i}(E - \{0\})$ is isomorphism. Hence we have,

**Definition 18.** Given a complex $n$-plane bundle $w = (E, \pi, B)$, the *Chern class* $c_i(w) \in H^{2i}(B)$ is defined by induction on $n$ as follows.

$$c_i(w) = \begin{cases} e(w_{\mathbb{R}}), & \text{if i = n,} \\ \pi_0^{*-1} c_i(w_0), & \text{if i < n,} \\ 0, & \text{otherwise} \end{cases}$$

where $w_0$ is a complex $n-1$ bundle over $E - \{0\}$ and $\pi_0^*$ is the isomorphism above.

**Definition 19.** Let $B$ be a topological space. $H^\Pi(B)$ will denote the ring consisting of all formal infinite series $a = a_0 + a_1 + ..$ with $a_i \in H^i(B)$. The product in $H^\Pi(B)$ is defined as

$$\begin{aligned}(a_0 + a_1 + ..)(b_0 + b_1 + ..) &= (a_0 \smile b_0) + (a_1 \smile b_0 + a_0 \smile b_1) \\ &+ (a_2 \smile b_0 + a_1 \smile b_1 + a_0 \smile b_2) + ...\end{aligned}$$

and it is associative. (When working with $\mathbb{Z}/2\mathbb{Z}$ coefficients, also commutative as $a_i \smile b_j = (-1)^{i+j} b_j \smile a_i$.)

In fact $H^{\Pi}(B)$ the Cartesian product of the groups $H^i(B)$'s.

For a complex $n$-plane bundle $w$ over $B$, the formal sum $c(w) = 1 + c_1(w) + ... + c_n(w)$ in the ring $H^{\Pi}(B)$ is said to be the *total Chern class* of $w$, where $c_0(w)$ is defined to be 1.

Chern classes are *natural* in the sense that for two complex $n$-plane bundles $w$ and $w'$ over $B$ and $B'$ respectively, where there is a map $f : B \to B'$ covered by a bundle map, we have $c_i(w) = f^* c_i(w')$ for all $i$. We write this as $c(w) = f^* c(w')$.

Furthermore, for two complex vector bundles $w$ and $\phi$ over a common base space, their Chern classes satisfy: $c(w \oplus \phi) = c(w)c(\phi)$ [2], [14].

If we change the complex structure of a complex bundle then we get another type of bundle.

**Definition 20.** Let $w = (E, \pi, B)$ be a complex vector bundle. The *conjugate bundle* $\overline{w} = (\overline{E}, \pi, B)$ is defined to be the complex bundle whose underlying real vector bundle is the same as that of $w$, but complex structure is the opposite.

In other words, for a complex number $\lambda$ and an element $v$ of a fiber of $\overline{E}$, the scalar multiplication $\lambda v$ in $\overline{E}$ is defined to be equal to the multiplication $\overline{\lambda} v$ in $E$. Then the local trivializations for $\overline{E}$ are obtained from the local trivializations for E by composing with the coordinatewise conjugation map $\mathbb{C}^n \to \mathbb{C}^n$ in each fiber. The Chern classes of $\overline{w}$ can be computed via below lemma.

**Lemma 2.6.** *Let $w$ be a rank $n$ complex vector bundle and $\overline{w}$ be its conjugate bundle. Then their Chern classes $c_k(w)$ and $c_k(\overline{w})$ satisfies*

$$c_k(\overline{w}) = (-1)^k c_k(w),\ k = 0, ..., n.$$

*Proof.* Let us choose a basis $\{v_1, .., v_n\}$ for a fiber $F$ of $w$, where $F$ is $n$-vector space over $\mathbb{C}$. Then the bases $\{v_1, iv_1, .., v_n, iv_n\}$ and $\{v_1, -iv_1, .., v_n, -iv_n\}$ determine the orientations for $F_{\mathbb{R}}$ and $(\overline{F})_{\mathbb{R}}$ respectively. By checking the determinant of the transformation matrix, $w_{\mathbb{R}}$ and $(\overline{w})_{\mathbb{R}}$ have the same orientation if $n$ is even, the opposite orientation if $n$ is odd. Therefore $e((\overline{w})_{\mathbb{R}}) = (-1)^n e(w_{\mathbb{R}})$, since the Euler class changes sign when the orientation is reversed. Hence by definition of the top Chern class, we get

$c_n(\overline{w}) = e((\overline{w})_{\mathbb{R}}) = (-1)^n e(w_{\mathbb{R}}) = (-1)^n c_n(w)$. The rest of the proof for the other classes $c_k(\overline{w})$ is by induction on $k$. □

Before passing to next section, we will also state the axiomatic definition of Chern classes ([2], Section 4):

**Definition 21.** To each complex $n$-plane vector bundle $w = (E, \pi, B)$ there corresponds a unique sequence of cohomology classes

$$c_i(w) \in H^{2i}(B),\ i = 0, 1, ...$$

called the *Chern classes* of $w$ which satisfy the followings.

1. The class $c_0(w) = 1 \in H^0(B)$ and $c_i(w) = 0$, for $i > n$.
2. Let $w' = (E', \pi', B')$ be another complex $n$-plane vector bundle. If there is a map $f : B \to B'$ covered by a bundle map, then

$$c_i(w) = f^* c_i(w') \text{ for all } i.$$

3. (Whitney Product) Let $\phi$ be a complex $n$-plane bundle over the same base space $B$. Then

$$c_k(w \oplus \phi) = \Sigma_{i=0}^{k} c_i(w) \smile c_{k-i}(\phi) \text{ for all } k.$$

   Equivalently,

$$c(w \oplus \phi) = c(w)c(\phi).$$

4. For the canonical complex line bundle $\gamma_1^1$ over $\mathbb{CP}^1$, the Chern class $c_1(\gamma_1^1)$ is nonzero.

### 2.2.3 Pontrjagin Classes

In this section we will ignore the odd Chern classes and define Pontrjagin Classes in terms of (even) Chern Classes. First we need the following.

**Definition 22.** The *complexification* $\xi \otimes \mathbb{C}$ of a real $n$-plane bundle $\xi$ is complexification of each of its fiber $\mathbb{R}^n$. That is to say, we take the direct sum of $\mathbb{R}^n$ with itself, and on $\mathbb{R}^n \oplus \mathbb{R}^n$ we define scalar multiplication by $i$ as $i(x, y) = (-y, x)$.

Thus each fiber of $\xi \otimes \mathbb{C}$ is $\mathbb{C}^n$; i.e., $\xi \otimes \mathbb{C}$ is a complex $n$-plane bundle. From this definition we see that the underlying real bundle $(\xi \otimes \mathbb{C})_{\mathbb{R}}$ is canonically isomorphic to $\xi \oplus \xi$.

**Definition 23.** Let $\xi$ be a real rank $n$ vector bundle with base $B$. The $i^{th}$ *Pontrjagin class* $p_i(\xi)$ is defined as

$$p_i(\xi) = (-1)^i c_{2i}(\xi \otimes \mathbb{C}) \in H^{4i}(B).$$

For $i > n/2$, $c_{2i}(\xi \otimes \mathbb{C})$ so $p_i(\xi)$ are zero. Therefore the total Pontrjagin class is

$$p(\xi) = 1 + p_1(\xi) + .. + p_{[n/2]}(\xi) \in H^{\Pi}(B),$$

where [.] is the greatest integer function.

***Notational convention.*** For a smooth manifold $M$, it is common to write $p(M)$ for the total Pontrjagin class $p(\tau_M)$ of its tangent bundle $\tau_M$, and to call the class $p(\tau_M) = p(M)$ the Pontrjagin class of $M$.

Pontrjagin classes satisfy naturality, too: For two real $n$-plane bundles $w$ and $w'$ over $B$ and $B'$ respectively, suppose that a map $f : B \to B'$ is covered by a bundle map. Then

$$p(w) = f^* p(w').$$

This axiom and Proposition (2.1) imply that if $\varepsilon$ is a trivial bundle then $p_i(\varepsilon) = 0$, for $i > 0$. Another consequence is that, if $w$ and $w'$ are isomorphic bundles then ($f$ is the identity map and therefore) $p_i(\xi) = p_i(\eta)$.

Pontrjagin classes also satisfy the Whitney product rule.

**Theorem 2.7.** *([2], Theorem 15.3) If $\xi$ and $\eta$ are real $n$-plane bundles over the same base space, then*

$$p_k(\xi \oplus \eta) = \Sigma_{i=0}^{k} p_i(\xi) \smile p_{k-i}(\eta), \; 0 \leq k \leq n. \quad (1)$$

Combining the above results, given two vector bundles $\varepsilon$ and $\xi$ over the same base, where $\varepsilon$ is trivial, we have

$$p(\xi \oplus \varepsilon) = p(\xi).$$

**Example 1.** *As an application, we consider the tangent bundle* $\tau_{\mathbb{S}^n}$ *and the normal bundle* $\nu$ *of the* $n$*-sphere* $\mathbb{S}^n \in \mathbb{R}^{n+1}$*. Then* $\nu \cong \mathbb{S}^n \times \mathbb{R}$ *is trivial hence* $p(\tau_{\mathbb{S}^n}) = p(\tau_{\mathbb{S}^n} \oplus \nu)$*. Moreover,* $\tau_{\mathbb{S}^n} \oplus \nu \cong \mathbb{S}^n \times \mathbb{R}^{n+1}$ *is also trivial implying that* $p_i(\tau_{\mathbb{S}^n} \oplus \nu) = 0$ *for* $i > 0$*. As a result,* $p(\tau_{\mathbb{S}^n} \oplus \nu) = 1 = p(\tau_{\mathbb{S}^n})$ *and thus all the Pontrjagin classes of tangent bundle of a sphere, except* $p_0$*, are zero.*

For any complex vector bundle $w$, $w_{\mathbb{R}} \otimes \mathbb{C}$ is canonically isomorphic to $w \oplus \overline{w}$ ([2], Lemma 15.4). That gives an identity to be used in the sequel.

**Theorem 2.8.** *The Chern classes* $c_i(w)$ *of a complex* $n$*-plane bundle* $w$ *and the Pontrjagin classes* $p_i(w_{\mathbb{R}})$ *of the underlying real bundle* $w_{\mathbb{R}}$ *satisfies:*

$$1 - p_1(w_{\mathbb{R}}) + p_2(w_{\mathbb{R}}) - ... \pm p_n(w_{\mathbb{R}}) = (1 + c_1(w) + c_2(w) + ... + c_n(w))(1 - c_1(w) + c_2(w) - ... \pm c_n(w)).$$

*Proof.* We know that $p_i(w_{\mathbb{R}}) = (-1)^i c_{2i}(w_{\mathbb{R}} \otimes \mathbb{C})$. Thus the left hand side equals to $c(w_{\mathbb{R}} \otimes \mathbb{C})$ as we ignore the odd Chern classes. On the other hand, $w_{\mathbb{R}} \otimes \mathbb{C}$ is canonically isomorphic to $w \oplus \overline{w}$. Thus $c(w_{\mathbb{R}} \otimes \mathbb{C}) = c(w \oplus \overline{w}) = c(w)c(\overline{w})$, which equals to the right hand side, since $c_k(\overline{w}) = (-1)^k c_k(w)$. □

**Lemma 2.9.** *The Pontrjagin classes of a real rank* $n$ *vector bundle* $\xi = (E, \pi, B)$ *are independent from the orientation of* $\xi$*.*

*Proof.* Let $\xi$ and $\xi^-$ denote the bundles over the same base space $B$ with opposite orientations. We consider the underlying bundles $(\xi \otimes \mathbb{C})_{\mathbb{R}}$ and $(\xi^- \otimes \mathbb{C})_{\mathbb{R}}$ which are isomorphic to $\xi \oplus \xi$ and $\xi^- \oplus \xi^-$ respectively. Then for any fiber $\mathbb{R}^k \oplus \mathbb{R}^k$ of $\xi \oplus \xi$ and the corresponding fiber (with opposite orientation) $\mathbb{R}^{k^-} \oplus \mathbb{R}^{k^-}$ of $\xi^- \oplus \xi^-$, the determinant of the transformation matrix is positive. Hence the Euler classes of $(\xi \otimes \mathbb{C})_{\mathbb{R}}$ and $(\xi^- \otimes \mathbb{C})_{\mathbb{R}}$ are the same. Therefore by induction, other Chern classes of $(\xi \otimes \mathbb{C})$ and $(\xi^- \otimes \mathbb{C})$ are also the same, which implies that $\xi$ and $\xi^-$ have equal Pontrjagin classes. □

Hence we will define a diffeomorphism invariant for smooth manifolds; the Pontrjagin number.

### 2.2.4 Pontrjagin Numbers

We begin with *Kronecker index notation*, $<,>$. Let $X$ be a topological space and $\Lambda$ be the coefficient ring. The value of a cochain $f \in C^n(X, \Lambda)$ on a chain $a \in C_n(X,\Lambda)$ is denoted by $< f, a > \in \Lambda$. Now let $\beta \in H^n(X,\Lambda)$, $w \in H_n(X,\Lambda)$ be given. To define $< \beta, w >$, we choose representatives $\gamma \in Z^n(X,\Lambda)$ and $c \in Z_n(X,\Lambda)$ such that $[\gamma] = \beta$, $[c] = w$ and set $< \beta, w > = < \gamma, c >$. We need to show that this equality is independent of the choice of representatives. We do it in two steps.

First, take $c, c' \in Z_n(X,\Lambda)$ such that they represent the same class $w$ in $H_n(X,\Lambda)$; i.e., $c - c' = \partial b$, for some $b \in C_{n+1}(X,\Lambda)$. Now for $\gamma \in Z^n(X,\Lambda)$, we have $< \gamma, c > = < \gamma, c' >$; i.e., $\gamma(c - c') = \gamma(\partial b) = 0$. Indeed $\gamma(\partial b) = \delta\gamma(b) = 0$ as $\gamma$ is in the kernel of the operator $\delta : C^n(X,\Lambda) \to C^{n+1}(X,\Lambda)$.

Similarly, we take $\gamma, \gamma' \in Z^n(X,\Lambda)$, $[\gamma] = [\gamma'] = \beta$. Then $\exists \psi \in C^{n-1}(X,\Lambda)$ such that $\gamma - \gamma' = \delta\psi$. Hence $(\gamma - \gamma')(c) = \delta\psi(c) = \psi(\partial c) = 0$ as $c \in Z_n(X,\Lambda)$, so the Kronecker index is well defined for cohomology and homology classes.

**Definition 24.** Let $M^{4n}$ be a smooth, compact, oriented $4n$-manifold. For each partition $I = i_1, ..., i_r$ of $n$ (i.e. an unordered sequence of positive integers with $\Sigma_{j=1}^r i_j = n$), the $I^{th}$ *Pontrjagin number* $p_I[M^{4n}] = p_{i_1}...p_{i_r}[M^{4n}]$ is defined to be the integer

$$< p_{i_1}(\tau_M) \smile ... \smile p_{i_r}(\tau_M), \mu >$$

where $\tau_M$ is the tangent bundle and $\mu$ is the fundamental homology class.

Note that each $p_{i_j}$ is in $H^{4i_j}(M)$ and $\Sigma_{j=1}^r 4i_j = 4n$. Then by definition of the cup product, $p_{i_1}(\tau_M) \smile ... \smile p_{i_r}(\tau_M)$ is in $H^{4n}(M)$ and since $\mu \in H_{4n}(M)$, this definition makes sense.

***Remark.*** Let $M^{4n}$ be a smooth, closed, oriented $4n$-manifold. When the orientation of $M^{4n}$ is reversed, although the Pontrjagin classes remain unchanged (Lemma 2.9), $\mu$ changes sign. Hence each Pontrjagin number also changes sign. Therefore, if some Pontrjagin number of $M^{4n}$ is nonzero then it cannot have an orientation reversing diffeomorphism.

### 2.2.5 Grassmann Manifolds and Universal Bundles

In this section we define the Grassmann Manifolds which determine the characteristic classes of vector bundles.

**Definition 25.** The *Grassmann manifold* $G_n(\mathbb{R}^{n+k})$ is the set of all $n$ dimensional planes through the origin of $\mathbb{R}^{n+k}$.

It is topologized as a quotient space of the *Stiefel manifold* $V_n(\mathbb{R}^{n+k})$ which is the set of all $n$ frames in $\mathbb{R}^{n+k}$. Indeed the quotient map $q : V_n(\mathbb{R}^{n+k}) \to G_n(\mathbb{R}^{n+k})$ sends each $n$ frame to the $n$-plane which it spans. Therefore $G_n(\mathbb{R}^{n+k})$ has the quotient topology. $G_n(\mathbb{R}^{n+k})$ is a topological manifold of dimension $nk$ and $G_n(\mathbb{R}^{n+k})$ is homeomorphic to $G_k(\mathbb{R}^{n+k})$, where an $n$-plane in $G_n(\mathbb{R}^{n+k})$ is sent to its orthogonal $k$-plane in $G_k(\mathbb{R}^{n+k})$ [2].

We construct a canonical vector bundle $\gamma^n(\mathbb{R}^{n+k})$ over $G_n(\mathbb{R}^{n+k})$ as follows. Let $E$ be the set of all pairs

$$(n\text{-plane in } \mathbb{R}^{n+k}, \text{ vector in that } n\text{-plane}).$$

Then $E$ is topologized as a subset of $G_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}$. For the total space of $\gamma^n(\mathbb{R}^{n+k})$, we take $E$ and we define the projection map $\pi : E \to G_n(\mathbb{R}^{n+k})$ by $\pi(X, x) = X$.

Let $M$ be a smooth $n$ manifold in $\mathbb{R}^{n+k}$ and $\tau_M = (TM, \pi, M, \mathbb{R}^k)$ be the tangent bundle. Then $\tau_M$ can be mapped into $\gamma^n(\mathbb{R}^{n+k})$. In fact, the generalized Gauss map

$$g : M \to G_n(\mathbb{R}^{n+k}),$$

which sends each $x \in M$ to its tangent plane $T_xM \in G_n(\mathbb{R}^{n+k})$, is covered by a bundle map

$$G : TM \to E,$$

where $G(x, v) = (T_xM, v)$. More generally,

**Lemma 2.10.** *([2], Lemma 5.3) For any $n$-plane bundle $\xi$, there exists a bundle map sending $\xi$ into $\gamma^n(\mathbb{R}^{n+k})$ provided that $k$ is sufficiently large.*

For this reason $\gamma^n(\mathbb{R}^{n+k})$ is called a *universal bundle.*

If we let the dimension of $\mathbb{R}^{n+k}$ tend to infinity, then we get an *infinite Grassmann manifold.* For this, let $\mathbb{R}^k$ denote the vector space consisting of infinite sequences $(x^1, x^2, ..., x^k, 0, 0, ...)$ of real numbers for which all but a finite number $(\leq k)$ of the $x^i$ are zero. Then we define

$$\mathbb{R}^\infty = \bigcup_{k=1}^\infty \mathbb{R}^k.$$

**Definition 26.** The *infinite Grassmann manifold* $G_n = G_n(\mathbb{R}^\infty)$ is the set of all $n$ dimensional linear subspaces of $\mathbb{R}^\infty$ with the weak topology, which means that

$$G_n = \bigcup_{k=0}^\infty G_n(\mathbb{R}^{n+k}).$$

**Definition 27.** The *universal bundle* $\gamma^n$ is a bundle over $G_n$, whose total space $E(\gamma^n)$ is the set of all pairs

$$(n\text{-plane in } \mathbb{R}^\infty, \text{ vector in that } n\text{-plane})$$

and the projection $\pi : E(\gamma^n) \to G_n$ is defined as in the finite dimensional case. The following theorem asserts that $\gamma^n$ is universal.

**Theorem 2.11.** *([2], Theorem 5.6-5.7) For any n-plane bundle $\xi$, there exists a bundle map sending $\xi$ into $\gamma^n$ and the map between total spaces is unique up to homotopy.*

In addition, the characteristic classes of $\mathbb{R}^n$ bundles are determined by the cohomology classes of $G_n$. That is to say, let $\xi$ be an $n$-plane bundle over a topological space $B$ and $c(\xi) \in H^i(B, \Lambda)$ be a characteristic class of $\xi$, where $\Lambda$ is any coefficient ring. We know that there is a bundle map $(F, f)$ sending $\xi$ into $\gamma^n$. Then there exists $c(\gamma^n) \in H^i(G_n, \Lambda)$ such that

$$c(\xi) = f^*(c(\gamma^n)).$$

# 3 MILNOR'S EXOTIC SEVEN SPHERES

In this section, we give the construction of the Milnor's spheres $M^7_{\mathrm{k}}$ and show that these are homeomorphic to standard seven sphere $\mathbb{S}^7$ but not diffeomorphic to it for some values of $k$. Indeed, each $M^7_{\mathrm{k}}$ is the total space of a 3-sphere bundle over $\mathbb{S}^4$ where the group of the bundle is $SO(4)$, so we begin with classifying such bundles. But before that let us define the higher homotopy groups.

For $n \geq 2$, the definition of the $n^{th}$ *homotopy group* $\pi_n(X, x_0)$ of a space $X$ is analogous to that of $\pi_1(X, x_0)$. We replace the interval $I = [0, 1]$ with the $n$-cube $I^n = \{(t_1, ..., t_n) \in \mathbb{R}^n \mid 0 \leq t_i \leq 1, i = 1, ..., n\}$. The union of the $n-1$ faces forms the boundary $\partial I^n$ of $I^n$. We then consider the maps from $I^n$ into $X$ which carry $\partial I^n$ to $x_0$. The set of homotopy classes of such maps is $\pi_n(X, x_0)$. If we pinch $\partial I^n$ to a point $y_0$, then $\pi_n(X, x_0)$ can be described as the set of homotopy classes of maps from $\mathbb{S}^n$ into $X$, sending $y_0$ to $x_0$. The sum operation in $\pi_n(X, x_0)$ is as follows

$$(f+g)(t_1, ..., t_n) = \begin{cases} f(2t_1, t_2, ..., t_n), & t_1 \in [0, 1/2], \\ g(2t_1 - 1, t_2, ..., t_n), & t_1 \in [1/2, 1]. \end{cases}$$

For $n = 0$, $\pi_0(X, x_0)$ is defined as the set of path components of $X$.

We now compute $\pi_3(SO(4))$ to be used in the next section.

**Theorem 3.1.** *$\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$.*

*Proof.* We give a sketch of the proof in five steps.

***i.*** $SO(4) \approx \mathbb{S}^3 \times SO(3)$**.**

To see this, we identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$ and $\mathbb{S}^3$ with unit quaternions. Let us denote the elements of $O(4)$ by $[v_1, ..., v_4]$, $v_i$'s being orthonormal vectors in $\mathbb{R}^4$. $O(4)$ is the group of isometries of $\mathbb{R}^4$ and acts transitively on $\mathbb{S}^3$. We take $1 = (1, 0, 0, 0) \in \mathbb{S}^3$. The subgroup leaving 1 fixed, acts on $\mathbb{S}^3 - \{1\} = \mathbb{R}^3$ and it is $O(3)$. Therefore we denote the elements of $O(3)$ as $[1, v_2, v_3, v_4]$.

On the other hand, to each $q \in \mathbb{S}^3$ there corresponds a transformation $\sigma(q) \in O(4)$ as follows:

$$\sigma : \mathbb{S}^3 \to O(4), \text{ such that } \sigma(q)q' = q'q.$$

Quaternionic multiplication is understood on the right hand side. This clearly is an isometry.

Hence we define a homeomorphism $\Psi : \mathbb{S}^3 \times O(3) \to O(4)$ by

$$\Psi(q, [1, v_2, v_3, v_4]) = \sigma(q)[1, v_2, v_3, v_4] = [q, v_2 q, .., v_4 q].$$

Then

$$\Psi^{-1}[v_1, ..., v_4] = (v_1, [1, v_2 v_1^{-1}, ..., v_4 v_1^{-1}])$$

is continuous. By restricting $\Psi$ we get that $\mathbb{S}^3 \times SO(3) \approx SO(4)$.

***ii.*** $\pi_3(SO(4)) \cong \pi_3(\mathbb{S}^3) \oplus \pi_3(SO(3))$***.***

In fact, for spaces $X, Y$ we have the relation $\pi_n(X \times Y) \cong \pi_n(X) \oplus \pi_n(Y)$, $n \geq 1$ ([8],17.8). Thus from ***i***, $\pi_3(SO(4)) \cong \pi_3(\mathbb{S}^3) \oplus \pi_3(SO(3))$.

***iii.*** $\pi_3(\mathbb{S}^3) \cong \mathbb{Z}$***.***

This follows from the following theorem.

**Theorem 3.2.** *([8], 15.9) The first nonzero homotopy group of* $\mathbb{S}^n$ *is* $\pi_n(\mathbb{S}^n)$ *and this group is infinite cyclic.*

***iv.*** $SO(3) \approx \mathbb{RP}^3$***.***

We consider the map $\varphi : \mathbb{D}^3 \to SO(3)$ sending a nonzero vector $x$ to the rotation through the angle $|x|\pi$ about the axis formed by the line through the origin in the direction of $x$, where an orientation convention is made. Since antipodal points of $\partial\mathbb{D}^3$ are sent to the same rotation, $\varphi$ induces $\overline{\varphi} : \mathbb{RP}^3 \to SO(3)$ where $\mathbb{RP}^3$ is regarded as $\mathbb{D}^3$ with antipodal points are identified. It can be shown that this map is a homeomorphism.

***v.*** $\pi_3(SO(3)) \cong \mathbb{Z}$***.***

We have

**Theorem 3.3.** *(Covering Space)([8], 17.6) If* $p : B \to X$ *is a covering,* $b \in B$ *then*

$$p_* : \pi_n(B, b) \to \pi_n(X, p(b)),\ n \geq 2$$

*is an isomorphism.*

Since $SO(3) \approx \mathbb{RP}^3$, $\mathbb{S}^3$ is a covering space of $SO(3)$, $\pi_3(SO(3)) \cong \pi_3(\mathbb{S}^3) \cong \mathbb{Z}$.

To conclude we have, $\pi_3(SO(4)) \cong \pi_3(\mathbb{S}^3) \oplus \pi_3(SO(3)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

□

## 3.1 Classification of Bundles Over the $n$-Sphere

Vector bundle isomorphism defined above is indeed a specialization of more general notion, *fiber bundle isomorphism.* Two fiber bundles $\beta, \beta'$ with the same base, fiber and group, are called *isomorphic/equivalent* if there exists a bundle map from $\beta$ to $\beta'$ inducing the identity map of the common base space. This implies that equivalent fiber bundles have homeomorphic total spaces. In fact, this relation is an equivalence relation and our aim now is to classify the equivalence classes of bundles over $\mathbb{S}^n$.

**Definition 28.** (Characteristic map of a bundle) Let $\beta = (E, \pi, \mathbb{S}^n, F)$ be a fiber bundle over $\mathbb{S}^n$ and $\mathbb{S}^{n-1}$ be the great sphere on $\mathbb{S}^n$, determining two closed hemispheres $E_1, E_2$. We take $V_1, V_2$ open $n$-cells containing $E_1, E_2$ and bounded by $n-1$ spheres parallel to $\mathbb{S}^{n-1}$. Then $V_1$ and $V_2$ cover $\mathbb{S}^n$ and $V_1 \cap V_2$ is an equatorial band around $\mathbb{S}^{n-1}$.

As $V_1$ and $V_2$ are contractible, by Theorem 2.2, a bundle over any one of them is a product bundle. Thus, for the portions $\beta_i = (\pi^{-1}(V_i), \pi|_{\pi^{-1}(V_i)}, V_i)$, there exist trivializations

$$\phi_i : \pi^{-1}(V_i) \to V_i \times F, \tag{2}$$

over the base spaces $V_i$ where $i = 1, 2$. In addition, for each $q \in V_i$ if we restrict $\phi_i$ to the fiber $\pi^{-1}(q)$, we get $\pi^{-1}(q) \approx \{q\} \times F$ which in turn homeomorphic to $F$ via, say $\psi$. Let us call the composition $\psi \circ \phi_i|_{\pi^{-1}(q)}$, $\phi_{i_q}$. Therefore for each $q \in V_1 \cap V_2$, $\phi_{1_q} \circ \phi_{2_q}^{-1}$ is an element of $G$, as a homeomorphism from $F$ to itself. Thus we define a map

$$g : V_1 \cap V_2 \to G, \quad g(q) = \phi_{1_q} \circ \phi_{2_q}^{-1}. \tag{3}$$

Then we restrict $g$ to the great sphere $\mathbb{S}^{n-1}$ and denote this restricted map by $T$. We call $T : \mathbb{S}^{n-1} \to G$ the *characteristic map of the bundle.* We note that the homotopy class $[T]$ of $T$ is an element of the group $\pi_{n-1}(G)$.

The homotopy class $[T]$ determines precisely the equivalence class of the bundle. In fact, any map $T : \mathbb{S}^{n-1} \to G$ is the characteristic map of some bundle over $\mathbb{S}^n$, and two bundles over a sphere with the same fiber and group are equivalent if and only if their characteristic maps are homotopic, provided that the group is path connected [8]. In addition,

**Theorem 3.4.** *(Bundle Classification)([8], Section 18) The equivalence classes of bundles over $\mathbb{S}^n$ with group $G$, are in one to one correspondence with the elements of $\pi_{n-1}(G)$, if $G$ is path connected.*

## 3.2 Construction of a Seven Dimensional Manifold $M^7_{\mathrm{k}}$

Recall that each $M^7_{\mathrm{k}}$ is the total space of a 3-sphere bundle over the 4-sphere with rotation group $SO(4)$ as structural group. Since $SO(4)$ is path connected (see Section 2.1.1), the Bundle Classification Theorem reads as follows. The equivalence classes of $\mathbb{R}^4$ bundles over $\mathbb{S}^4$ with group $SO(4)$, are in one to one correspondence with the elements of the group $\pi_3(SO(4))$ which is $\mathbb{Z} \oplus \mathbb{Z}$ by the previous section. Therefore for each element of $\mathbb{Z} \oplus \mathbb{Z}$ we construct an $\mathbb{R}^4$ bundle over $\mathbb{S}^4$ with $SO(4)$ as the group.

As above, we take two charts $\mathbb{S}^4 - \{N\} \approx \mathbb{R}^4$ and $\mathbb{S}^4 - \{S\} \approx \mathbb{R}^4$, where $N, S$ are the north and the south poles respectively. Being contractible, these charts determine trivial bundles $\mathbb{R}^4 \times \mathbb{R}^4$. We will "glue" these two trivial bundles. For the base spaces, note that $(\mathbb{S}^4 - \{N\}) \cap (\mathbb{S}^4 - \{S\}) \approx \mathbb{R}^4 - \{0\}$. We will identify the points in this intersection via stereographic projections:

$$\sigma : \mathbb{S}^4 - \{N\} \to \mathbb{R}^4,\ \sigma(x^1, .., x^5) = (x^1, .., x^4)/1 - x^5$$

and

$$\sigma' : \mathbb{S}^4 - \{S\} \to \mathbb{R}^4,\ \sigma(x^1, .., x^5) = (x^1, .., x^4)/1 + x^5.$$

For $u = (u^1, .., u^4) \in \mathbb{R}^4$, the inverse of $\sigma$ is $\sigma^{-1}(u) = (2u^1, .., 2u^4, |u|^2 - 1)/|u|^2 + 1$ and the transition map is $\sigma' \circ \sigma^{-1}(u) = u/|u|^2$ which will be the identification map for the points in the intersection of two charts of the base.

For the fibers, on the other hand, we will first define the map $g$ as in (3). Let

$$g : (\mathbb{S}^4 - \{N\}) \cap (\mathbb{S}^4 - \{S\}) \approx \mathbb{R}^4 - \{0\} \to SO(4),$$

so that $g(u)$ is a map from the fiber $\mathbb{R}^4$ to itself. To describe how to identify the fibers, we define the action of $g(u)$ on $\mathbb{R}^4$ as follows.

$$g(u)v = u^h v u^j/|u|, \text{ for } u \in \mathbb{R}^4 - \{0\},\ v \in \mathbb{R}^4$$

(with quaternionic multiplication on the right). If we take $(h, j) \in \mathbb{Z} \oplus \mathbb{Z}$ satisfying $h + j = 1$ then the norm of any element $v \in \mathbb{S}^3$ is preserved under this diffeomorphism. Hence we can restrict this action to $\mathbb{S}^3$.

Therefore, we identify the subsets $(\mathbb{R}^4 - \{0\}) \times \mathbb{S}^3$ under the diffeomorphism

$$(u, v) \mapsto (u', v') = (u/|u|^2, u^h v u^j/|u|) \tag{4}$$

where $v \in \mathbb{S}^3$ and $h + j = 1$ this time. Hence we have constructed an $\mathbb{S}^3$ bundle over $\mathbb{S}^4$. The map above makes the differentiable structure of the total space precise.

Indeed, we know that the equivalence class of this $\mathbb{R}^4$ bundle (and in turn $\mathbb{S}^3$ bundle) is determined by the homotopy class of its characteristic map. The characteristic map $f_{hj}$ is obtained by restricting $g$ to the great sphere $\mathbb{S}^3$:

$$f_{hj} : \mathbb{S}^3 \to SO(4), \quad f_{hj}(u) \cdot v = u^h v u^j \tag{5}$$

where $v \in \mathbb{R}^4$ (again with the quaternionic multiplication). Hence, we now know that for each $(h, j) \in \mathbb{Z} \oplus \mathbb{Z}$, there corresponds a vector bundle with the characteristic map $f_{hj}$. We take

$$h + j = 1 \text{ and let } h - j = k \text{ so that } k \text{ is an odd integer.}$$

For such $i$ and $j$, we denote the sphere bundle determined by $f_{hj}$ as $\xi_{hj}$ and call the total space $M^7_{\mathrm{k}}$ as the conditions on $h$ and $j$ uniquely determine an odd number $k$.

We consider only the bundles $\xi_{hj}$ with $h + j = 1$ and $h - j = k$ and will show that the total spaces $M^7_{\mathrm{k}}$ of these bundles are exotic spheres, for some values of $k$.

### 3.3 $M^7_{\mathrm{k}}$ is a Topological Seven Sphere

In this part we will show that, for each odd number $k$, $M^7_{\mathrm{k}}$ constructed above is homeomorphic to $\mathbb{S}^7$ by defining a Morse function $f$ (i.e., a real valued, smooth function with all critical points are non-degenerate) on it.

**Theorem 3.5.** *(Reeb)([6], Theorem 4.1) If $M$ is a closed $n$-manifold and $f$ is a Morse function on $M$ with only two critical points, then $M$ is homeomorphic to the $n$-sphere.*

For the proof, we need two lemmas.

**Lemma 3.6.** *(Morse)([6], Lemma 2.2) Let $p$ be a non-degenerate critical point for a smooth function $f : M \to \mathbb{R}$ defined on an $n$-manifold $M$ and $\lambda$ is the* index *of $f$ at $p$. Then there is a local coordinate system $(y^1, ..., y^n)$ in a neighborhood $U$ of $p$ with $y^i(p) = 0$ for all $i$ and the identity*

$$f = f(p) - (y^1)^2 - ... - (y^\lambda)^2 + (y^{\lambda+1})^2 + ... + (y^n)^2$$

*holds throughout $U$.*

**Lemma 3.7.** *([6], Theorem 3.1) Let $f$ be a smooth real valued function on a manifold $M$. Let $a < b \in \mathbb{R}$ and suppose that $f^{-1}[a, b] = \{p \in M | \ a \leq f(p) \leq b\}$ is compact and contains no critical points of $f$. Then $M^a = f^{-1}(-\infty, a] = \{p \in M | \ f(p) \leq a\}$ is diffeomorphic to $M^b = f^{-1}(-\infty, b]$.*

*Proof.* (of Theorem 3.5) The two critical points must be minimum and maximum points as $M$ is compact. Let us normalize the image of $f$, and take $f(p) = 0$ as the minimum and $f(q) = 1$ as the maximum. By the Morse Lemma, for $0 < \varepsilon < 1/2$, the sets $f^{-1}[1-\varepsilon, 1]$ and $M^\varepsilon = f^{-1}[0, \varepsilon]$ are closed $n$-cells. In fact, since $f(q) = 1$ is the maximum value, $f$ cannot take values larger than 1. Thus, $f$ has the form $f = f(q) - (y^1)^2 - ... - (y^n)^2$ near $q$. Then the set $f^{-1}[1 - \varepsilon, 1] = \{y \in M | \ 1 - \varepsilon \leq f(y) \leq 1\}$ equals

$$\{y = (y^1, .., y^n) \in M | \ 1 - \varepsilon \leq 1 - (y^1)^2 - ... - (y^n)^2 \leq 1\}$$

which is diffeomorphic to the $n$-disk $\mathbb{D}^n$. Similarly, $f^{-1}[0, \varepsilon]$ is diffeomorphic to the $n$-disk $\mathbb{D}^n$, too.

In addition, there is no critical points in $f^{-1}[\varepsilon, 1 - \varepsilon]$ so $M^\varepsilon$ is homeomorphic to $M^{1-\varepsilon}$ by Lemma 3.7. Thus $M$ is the union of two $n$-disks $M^{1-\varepsilon}$ and $f^{-1}[1 - \varepsilon, 1]$ matched along their boundaries $\mathbb{S}^{n-1}$. This implies that $M$ is homeomorphic to $\mathbb{S}^n$, because the attaching homeomorphism $h : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ can be extended to a homeomorphism $\overline{h} : \mathbb{D}^n \to \mathbb{D}^n$ [7]. □

Now, we will define a function satisfying the hypotheses of the theorem to show that the manifold $M^7_k$ is homeomorphic to $\mathbb{S}^7$. Let $f_1$ be a real valued function defined by

$$f_1(u,v) = Re(v)/\sqrt{1+|u|^2}$$

on one of the sets $V_1 := \mathbb{R}^4 \times \mathbb{S}^3$'s where $(\mathbb{R}^4 - \{0\}) \times \mathbb{S}^3$'s were identified. We replace the coordinates $(u', v')$ on the other subset given by the equation (4), to $(u'', v')$ such that $u'' = u'(v')^{-1}$. Let $f_2$ be a real valued function defined on $V_2 := \mathbb{R}^4 \times \mathbb{S}^3$ by

$$f_2(u'',v') = Re(u'')/\sqrt{1+|u''|^2}.$$

Now we define a function $f : M^7_k \to \mathbb{R}$ as

$$f|_{V_i} = f_i \text{ for } i = 1, 2.$$

Then $f$ is well defined and smooth on $M^7_k$. Indeed, for any point $x \in V_1 \cap V_2$ we have

$$f(x) = Re(v)/\sqrt{1+|u|^2} = Re(u'')/\sqrt{1+|u''|^2}.$$

To see this equality, first observe that $Re(u'')/\sqrt{1+|u''|^2} = Re(u'')|u|/\sqrt{1+|u|^2}$ since $|u''| = 1/|u|$. Thus we check whether $Re(u'')|u|$ equals $Re(v)$. We have $(v')^{-1} = (u^{-j}v^{-1}u^{-h})|u|$, so

$$\begin{aligned} Re(u'')|u| &= Re(u'(v')^{-1})|u| = Re[(u/|u|^2)(u^{-j}v^{-1}u^{-h})|u|]|u| \ \ (h+j=1) \\ &= Re(u^h v^{-1} u^{-h}) = Re((u^h/|u|^h)v^{-1}(u^{-h}/|u|^{-h})) \end{aligned}$$

which is in turn equal to $Re(v^{-1})$, as conjugation by a unit quaternion does not change the real part; and it is equal to $Re(v)$, as $v$ is a unit quaternion and $v^{-1} = \overline{v}$. Hence

$$f(x) = Re(v)/\sqrt{1+|u|^2} = Re(u'')/\sqrt{1+|u''|^2}, \tag{6}$$

and since $f_1$ and $f_2$ are smooth, $f$ is a smooth function on $M^7_k$.

Now we show that this function has only two non-degenerate critical points, to conclude that $M^7_k \approx \mathbb{S}^7$. For $f(u,v) = Re(v)/\sqrt{1+|u|^2}$ where $|v| = 1$, we write $u = (u^1, .., u^4)$, $v = (v^1, .., v^4)$. Then

$$f(u,v) = \sqrt{1-\textstyle\sum_{i=2}^4 (v^i)^2}/\sqrt{1+\textstyle\sum_{i=1}^4 (u^i)^2}.$$

We compute that

$$\partial f/\partial u^i = (-u^i)(1-\sum_{i=2}^4 (v^i)^2)^{1/2}/(1+\sum_{i=1}^4 (u^i)^2)^{3/2},$$

$$\partial f/\partial v^i = (-v^i)/[(1-\sum_{i=2}^4 (v^i)^2)(1+\sum_{i=1}^4 (u^i)^2)]^{1/2},\ i=2,3,4.$$

Therefore $df = 0$ if and only if $v^i = 0,\ i = 2,3,4$ and $u^i = 0,\ i = 1,2,3,4$ giving two critical points $(u,v) = (0,\pm 1)$. On the other chart $f(u'',v') = Re(u'')/\sqrt{1+|u''|^2} = x^1/\sqrt{1+\sum_{i=1}^4 (x^i)^2}$, for $u'' = (x^1,..,x^4)$. Then

$$\partial f/\partial x^i = (1+\sum_{i=2}^4 (x^i)^2)/(1+\sum_{i=1}^4 (x^i)^2)^{3/2}$$

Therefore there is no critical points on this chart.

Lastly, it is also easy to check that the points $(0,\pm 1)$ are nondegenerate, thus by Reeb's theorem we have

**Corollary 3.8.** *For each $(h,j) \in \mathbb{Z}\oplus\mathbb{Z}$ such that $h+j=1$, the space $M^7_{\mathrm{k}}$ is homeomorphic to $\mathbb{S}^7$.*

### 3.4 $M_k^7$ is Exotic

In this section we show that the spheres $M_k^7$ constructed above are not diffeomorphic to $\mathbb{S}^7$ for $k^2 \not\equiv 1 \, mod \, 7$. For that, we need $\lambda(M^7)$ which is an invariant for oriented, differentiable 7-manifolds $M^7$ satisfying $H^3(M^7) = H^4(M^7) = 0$. It is known that every closed 7-manifold $M^7$ is the boundary of an 8-manifold $B^8$ [1]. Thus, $\lambda(M^7)$ will be defined as a function of the Pontrjagin class $p_1$ of (the tangent bundle of) $B^8$ and the *signature* $\sigma(B^8)$ defined below.

**Definition 29.** The *signature/index* $\sigma(M)$ of a compact and oriented $n$ manifold $M$ is defined as follows. If $n = 4k$ for some $k$, we choose a basis $\{a_1, ..., a_r\}$ for $H^{2k}(M^{4k}, \mathbb{Q})$ so that the *symmetric* matrix $[< a_i \smile a_j, \mu >]$ is diagonal where $\mu$ is the fundamental class. Then $\sigma(M^{4k})$ is the number of positive diagonal entries minus the number of negative ones. Otherwise (if $n$ is not a multiple of 4) $\sigma(M)$ is defined to be zero [2]. (The symmetricity of $[< a_i \smile a_j, \mu >]$ follows from the rule: $b \smile a = (-1)^{mn} a \smile b$, for cohomology classes $a \in H^m(X)$ and $b \in H^n(X)$.)

To put it another way, after choosing a basis $\{a_1, ..., a_r\}$ making $[< a_i \smile a_j, \mu >]$ diagonal, $\sigma(M^{4k})$ is the signature of the rational quadratic form $a \mapsto < a \smile a, \mu >$ for any $a \in H^{2k}(M^{4k}, \mathbb{Q})$. It is a homotopy invariant of closed and oriented manifolds and has four properties:

1. $\sigma(M \# M') = \sigma(M) + \sigma(M')$.
2. $\sigma(M \times M') = \sigma(M)\sigma(M')$.
3. If $M^{4k}$ is a boundary of some oriented $(4k+1)$-manifold, then $\sigma(M) = 0$ [2].
4. Changing the orientation of $M$ changes the sign of the signature [18].

Now we take an oriented, differentiable 7-manifold $M^7$ satisfying $H^3(M^7) = H^4(M^7) = 0$ with the fundamental class $\mu \in H_7(M^7)$. Let $M^7$ bound $B^8$ whose signature is $\sigma(B^8)$ and the first Pontrjagin class is $p_1 \in H^4(B^8)$. We take the class $\nu \in H_8(B^8, M^7)$ satisfying $\partial\nu = \mu$ where $\partial : H_8(B^8, M^7) \to H_7(M^7)$. On the other hand, since $H^3(M^7) = H^4(M^7) = 0$, from the long exact sequence of relative cohomology groups we get the isomorphism

$$i : H^4(B^8, M^7) \to H^4(B^8). \tag{7}$$

Therefore it is legitimate to define a number

$$q(B^8) =< i^{-1}(p_1) \smile i^{-1}(p_1), \nu >=:< (i^{-1}p_1)^2, \nu >$$

and the number $\lambda(M^7)$ is defined as:

$$\lambda(M^7) \equiv 2q(B^8) - \sigma(B^8) \quad mod\, 7. \tag{8}$$

**Theorem 3.9.** *$\lambda(M^7)$ is well defined.*

*Proof.* We will show that $\lambda(M^7)$ does not depend on the choice of the manifold $B^8$. To that end let us take two disjoint manifolds $B_1^8$, $B_2^8$ with boundary $M^7$. By gluing them along their boundaries we get a closed 8-manifold $C^8 = B_1^8 \cup B_2^8$ whose differential structure is compatible with that of $B_1^8$ and $B_2^8$. We choose an orientation $\mu$ for $C^8$ consistent with the orientation $\mu_1$ of $B_1^8$ (hence with $-\mu_2$). Let $q(C^8)$ be the Pontrjagin number $< p_1^2(C^8), \mu >$.

Hirzebruch formula (given for closed manifolds) [2] implies that

$$\sigma(C^8) =< (7p_2(C^8) - p_1^2(C^8))/45, \mu >$$

and so by bilinearity

$$45\sigma(C^8) + q(C^8) = 7 < p_2(C^8), \mu >\equiv 0 \quad mod\, 7.$$

Equivalently,

$$2q(C^8) \equiv \sigma(C^8) \quad mod\, 7. \tag{9}$$

For now we assume the lemma below.

**Lemma 3.10.** *Under the conditions above,*

$$\sigma(C^8) = \sigma(B_1^8) - \sigma(B_2^8) \quad \& \quad q(C^8) = q(B_1^8) - q(B_2^8). \tag{10}$$

Hence by inserting (10) in (9), we get $2q(B_1^8) - \sigma(B_1^8) \equiv 2q(B_2^8) - \sigma(B_2^8) \quad mod\, 7$, proving the theorem. □

We note that this theorem also shows that $\lambda(M^7)$ is a diffeomorphism invariant for the seven manifolds with the stated properties.

*Proof.* (of Lemma 3.10) We consider the diagram

$$\begin{array}{ccc} H^4(C^8, M^7) & \xrightarrow{\ h\ } & H^4(B_1^8, M^7) \oplus H^4(B_2^8, M^7) \\ \Big\downarrow j & & \Big\downarrow i_1 \oplus i_2 \\ H^4(C^8) & \xrightarrow{\ k\ } & H^4(B_1^8) \oplus H^4(B_2^8) \end{array}$$

The maps in the columns are isomorphisms from the exact cohomology sequences, and the one at the bottom is also an isomorphism from the Mayer-Vietoris sequence. Hence $h$ is an isomorphism too. Let $\alpha_1 \in H^4(B_1^8, M^7)$ and $\alpha_2 \in H^4(B_2^8, M^7)$. If $\alpha = jh^{-1}(\alpha_1, \alpha_2) \in H^4(C)$, then

$$\begin{aligned} < \alpha^2, \mu > &= < jh^{-1}(\alpha_1^2, \alpha_2^2), \mu > = < (\alpha_1^2, \alpha_2^2), (\nu_1, (-\nu_2)) > \\ &= < \alpha_1^2, \nu_1 > - < \alpha_2^2, \nu_2 > \end{aligned} \tag{11}$$

where $\nu_i$ is the image of $\mu_i$ under the map $H_8(B_i^8) \to H_8(B_i^8, M^7)$. Therefore the result that the quadratic form of $C^8$ is the direct sum of the form of $B_1^8$ and minus the form of $B_2^8$ gives us $\sigma(C^8) = \sigma(B_1^8) - \sigma(B_2^8)$.

To obtain the second equality in (10), we consider the inclusion $\iota : B_1^8 \hookrightarrow C^8$ which is an embedding. Thus $TB_1^8 = \iota^*(TC^8)$ and $p_1(B_1^8) = \iota^* p_1(C^8)$, and similarly for $B_2^8$. Hence the restriction map $H^4(C) \to H^4(B_1^8) \oplus H^4(B_2^8)$ sends $p_1(C^8)$ to $(p_1(B_1^8), p_1(B_2^8))$. So that the computation in (11) with $\alpha = p_1(C^8)$, $\alpha_1 = i_1^{-1} p_1(B_1^8)$, $\alpha_2 = i_2^{-1} p_1(B_2^8)$ shows that

$$\begin{aligned} < p_1^2(C^8), \mu > &= < (i_1^{-1} p_1(B_1^8))^2, \nu_1 > - < (i_2^{-1}(p_1)(B_2^8))^2, \nu_2 > \\ &= q(B_1^8) - q(B_2^8). \end{aligned}$$

Hence the proof of the lemma is complete. □

***Remark.*** From the definition of $\lambda(M^7)$ we see that $\lambda(\mathbb{S}^7) = 0$. In fact, $\mathbb{S}^7 = \partial \mathbb{D}^8$ and $p_1 \in H^4(D^8) = 0$ and $\sigma(D^8)$, therefore $\lambda(S^7)$ are equal to zero. For the sphere $M_{\mathrm{k}}^7$, we will show that $\lambda(M_{\mathrm{k}}^7) \neq 0$ and conclude that $M_{\mathrm{k}}^7$ is not diffeomorphic to $\mathbb{S}^7$ for all $k$'s such that $k^2 \not\equiv 1 \ mod\ 7$. Indeed, we will compute $\lambda(M_{\mathrm{k}}^7)$.

**Theorem 3.11.** *The invariant* $\lambda(M_{\mathrm{k}}^7)$ *satisfies* $\lambda(M_{\mathrm{k}}^7) \equiv k^2 - 1$ *in mod 7.*

Before starting the proof, we recall that $M_{\mathrm{k}}^7$ is the total space of a 3-sphere bundle $\xi_{hj}$ over $\mathbb{S}^4$ with $h + j = 1$, $h - j = k$. $M_{\mathrm{k}}^7$ was constructed by taking the two trivial $\mathbb{S}^3$ bundles over $\mathbb{R}^4$ and identifying the subsets $(\mathbb{R}^4 - \{0\} \times \mathbb{S}^3)$ under the diffeomorphism given in (4):

$$(u, v) \mapsto (u', v') = (u/|u|^2, u^h v u^j/|u|).$$

*Proof.* To begin with, let us denote the projection map of $\xi_{hj}$ as $\varepsilon_k$. Then, since

$$\xi_{hj} = (M^7_{\mathrm{k}}, \varepsilon_k, \mathbb{S}^4, \mathbb{S}^3)$$

is a sphere bundle, by definition, it is obtained from a vector bundle

$$\eta_{hj} = (E_k, \pi, \mathbb{S}^4, \mathbb{R}^4)$$

with group $SO(4)$, by taking only vectors of unit length in $E_k$. The smooth structure of the total space $M^7_{\mathrm{k}}$ of $\xi_{hj}$ is induced from $E_k$ and the action of $SO(4)$ on $\mathbb{S}^3$ is given by the restriction of its action on $\mathbb{R}^4$. In the same way, by replacing the fiber $\mathbb{R}^4$ with the unit disk $\mathbb{D}^4$, we get a 4-cell bundle

$$\beta_{hj} = (B^8_k, \rho_k, \mathbb{S}^4, \mathbb{D}^4).$$

Here $B^8_k$ is a smooth manifold whose boundary is $M^7_{\mathrm{k}}$ and $\rho_k$ is the restriction of $\pi$. We now divide the proof in six steps.

Step 1. We will find the index $\sigma(B^8_k)$ at first. We note that since $\pi_1(\mathbb{S}^4) = 0$, $\eta_{hj}$ is orientable. In fact, we have

**Proposition 3.12.** *([20], p.116) Every vector bundle over a simply connected base is orientable.*

Therefore when an orientation for $\eta_{hj}$ is specified, this induces an orientation on $B^8_k$ (and thus on the boundary $M^7_{\mathrm{k}}$). Let $\iota$ be the standard generator for the cohomology group $H^4(\mathbb{S}^4)$. Then the element $\alpha = \rho^*_k(\iota)$ generates $H^4(B^8_k) \cong \mathbb{Z}$. Now we choose the orientation $\nu \in H_8(B^8_k, M^7_{\mathrm{k}})$ satisfying

$$< (i^{-1}\alpha)^2, \nu >= +1 \qquad (12)$$

where $i$ is the isomorphism as in (7). Consequently, the index $\sigma(B^8_k)$ equals $+1$.

Step 2. To calculate the invariant $\lambda(M^7_{\mathrm{k}}) \equiv 2q(B^8_k) - \sigma(B^8_k) \quad mod\ 7$, we need to find $p_1(B^8_k)$. But we will first compute $p_1(\xi_{hj})$. Since $\xi_{hj}$ is induced from the vector bundle $\eta_{hj}$, what is meant by the Pontrjagin class $p_1(\xi_{hj})$ of $\xi_{hj}$ is $p_1(\eta_{hj}) \in H^4(\mathbb{S}^4)$ as Pontrjagin classes are defined for vector bundles. Clearly $p_1(\eta_{hj})$ depends on $\iota$, and also on $h$ and $j$ as the bundle is determined by (5). Further, it is a linear function [8]. Let $p_1(\eta_{hj}) = (ch + dj)\iota$. If the orientation of the fiber $\mathbb{R}^4$ is reversed, $p_1(\eta_{hj})$ does not change by Lemma 2.9, but $f_{hj}$ becomes $f_{-j-h}$. Indeed, let $v \in \mathbb{R}^4$ be an element of the fiber whose image under $f_{hj}$ is $u^h v u^j$. Then under the orientation reversal, $v$ goes to $\overline{v}$ and $u^h v u^j$ goes to $\overline{u^h v u^j} = u^{-j}\overline{v}u^{-h}$. Hence the map sending $\overline{v}$ to $u^{-j}\overline{v}u^{-h}$ is $f_{-j-h}$ and the resulting bundle is $\eta_{-j-h}$ whose first Pontrjagin class is $p_1(\eta_{-j-h}) = (-cj - dh)\iota$.

Nevertheless, as we noted above $p_1(\eta_{hj}) = (ch + dj)\iota = (-cj - dh)\iota$, so $p_1(\eta_{hj})$ is given as $c(h - j)\iota = ck\iota$, for some constant $c$.

Step 3. Now we will find $p_1(B_k^8)$. The tangent bundle $\tau_{B_k^8}$ of $B_k^8$ decomposes as Whitney sum of the bundle of vectors tangent to fiber and the bundle of vectors normal to fiber. The former is induced from the vector bundle $\eta_{hj}$ and the latter from $\tau_{\mathbb{S}^4}$ under $\rho_k$. Indeed $\rho$ induces

$$\begin{array}{ccccccc}
E_{\|} & \xrightarrow{\rho_k'} & E_k & \quad & E_{\perp} & \xrightarrow{\rho_k''} & T\mathbb{S}^4 \\
\big\downarrow & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow & & \big\downarrow \\
B_k^8 & \xrightarrow{\rho_k} & \mathbb{S}^4 & & B_k^8 & \xrightarrow{\rho_k} & \mathbb{S}^4.
\end{array}$$

Then we have

$$\begin{aligned}
p_1(B_k^8) = p_1(\tau_{B_k^8}) &= p_1(\rho_k^*(\eta_{hj}) \oplus \rho_k^*(\tau_{\mathbb{S}^4})) \\
&= p_1(\rho_k^*(\eta_{hj})) + p_1(\rho_k^*(\tau_{\mathbb{S}^4})) && \text{(Whitney Product Theorem)} \\
&= \rho_k^*(p_1(\eta_{hj})) + \rho_k^*(p_1(\tau_{\mathbb{S}^4})) && \text{(by naturality)} \\
&= \rho_k^*(ck\iota) + 0 && \text{(from Step 2 and Example 1)} \\
&= ck\alpha.
\end{aligned}$$

In the next two steps, we will find the value of $c$.

Step 4. We consider the special case where $k = 1$. We will show that in this case $\xi_{10}$ is the *usual Hopf fibration* and that $B_1^8$ (the total space of the associated disk bundle $\beta_{hj}$) is the quaternionic projective plane $\mathbb{HP}^2$ minus an open 8-cell [16].

Firstly, we note that when $k = 1$ the map in (4) becomes:

$$(u, v) \mapsto (u', v') = (u/|u|^2, uv/|u|). \tag{13}$$

We will show that this map coincides with the transition map of the *standard Hopf construction of* $\mathbb{S}^7$ which is given as follows ([19], p.207):

We take the stereographic projections $\sigma$ and $\sigma'$ as above. Call the charts $\mathbb{S}^4 - \{N\}$ and $\mathbb{S}^4 - \{S\}$, $V_1$ and $V_2$ respectively and the local trivializations are taken as

$$\begin{aligned} \phi_1 \quad &: \pi^{-1}(V_1) \to V_1 \times \mathbb{S}^3 \\ \tfrac{(v,uv)}{\sqrt{1+|u|^2}} &\mapsto (\sigma^{-1}(u), v), \end{aligned}$$

$$\begin{aligned} \phi_2 \quad &: \pi^{-1}(V_2) \to V_2 \times \mathbb{S}^3 \\ \tfrac{(u'^{-1}v',v')}{\sqrt{1+|u'|^2}} &\mapsto (\sigma'^{-1}(u'), v'). \end{aligned}$$

Now we find the transition map $g(q) = \phi_{1_q} \circ \phi_{2_q}^{-1}$ given in (3), where $q$ is a point in $V_1 \cap V_2$. Because $q$ is in the intersection of the charts $V_1, V_2$ we have $q = \sigma^{-1}(u) = \sigma'^{-1}(u')$, $u, u' \in \mathbb{R}^4$, therefore $\sigma' \circ \sigma^{-1}(u) = u' = u/|u|^2$.

Let $v, v' \in \mathbb{S}^3$ be vectors in the fibers of $q$ above the charts $V_1, V_2$ respectively. Then

$$\phi_1(\pi^{-1}(q)) = (q, v) = (\sigma^{-1}(u), v)$$

and

$$\phi_2(\pi^{-1}(q)) = (q, v') = (\sigma'^{-1}(u'), v').$$

Therefore

$$\phi_2 \circ \phi_1^{-1}(q, v) = (q, v') = \phi_2(\frac{u'^{-1}v', v'}{\sqrt{1+|u'|^2}}).$$

On the other hand,

$$\phi_2(\phi_1^{-1}(q, v)) = \phi_2(\frac{v, uv}{\sqrt{1+|u|^2}}).$$

By equating the second components we get

$$uv/\sqrt{1+|u|^2} = v'/\sqrt{1+|u'|^2} = v'/\sqrt{1+(|u/|u|^2|)^2} = v'|u|/\sqrt{1+|u|^2}.$$

Hence $v' = uv/|u|$, and we already have $u' = u/|u|^2$. Therefore in the standard Hopf construction of $\mathbb{S}^7$, the map is defined as

$$(u,v) \mapsto (u',v') = (u/|u|^2, uv/|u|) \tag{14}$$

which is the diffeomorphism in (13). Thus when $k=1$, we have that $\xi_{10}$ is the usual Hopf fibration $\mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \to \mathbb{S}^4$ and $M_1^7$ is $\mathbb{S}^7$.

Secondly, we will show that the total space $B_1^8$ of the $\mathbb{D}^4$ bundle $\beta_{10}$ associated to $\xi_{10} = \mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \to \mathbb{S}^4$ is the space $\mathbb{HP}^2$ - {an open 8-cell}. Indeed, from the cell decomposition $\mathbb{HP}^2 = e^0 \cup e^4 \cup e^8$ we know that $\mathbb{HP}^2$ is obtained from $\mathbb{HP}^1 \approx \mathbb{S}^4$ by attaching an 8-cell $e^8$ via the Hopf map $\mathbb{S}^7 \to \mathbb{S}^4$.

Now we consider the *Thom space* of $\eta_{10}$. Before that, for a $k$-plane bundle $\xi$ with Euclidean metric, the Thom space is defined as follows. Let $A$ be the set $\{v \in E \mid |v| \geq 1\}$ in the total space $E$ of $\xi$. The space $T = E/A$ where $A$ is pinched to a point is called the Thom space of $\xi$. Thus $T$ has a base point $t$ and $T - \{t\}$ is the set $\{v \in E \mid |v| < 1\}$.

**Lemma 3.13.** *([2], Lemma 18.1) Let $\xi$ be a $k$-plane bundle with Euclidean metric. If the base space $B$ is a CW-complex then the Thom space is a $(k-1)$-connected CW complex, having (in addition to the base point $t$) one $(n+k)$-cell corresponding to each $n$-cell of $B$.*

Therefore, for the vector bundle $\eta_{10}$, the cell decomposition of its Thom space $T$ consists of one 0-cell, one 4-cell and one 8-cell and we showed that $\xi_{10}$ is the quaternionic Hopf bundle implying that the Thom space $T$ is $\mathbb{HP}^2$. On the other side, $T$ can be obtained by adjoining a cone $(\mathbb{S}^7 \times I)/(\mathbb{S}^7 \times \{1\})$ over the boundary of $B_1^8$. Thus, $B_1^8$ is $\mathbb{HP}^2$ - {an open 8-cell}.

Alternatively, one can see also [8], 26.3, 26.7 or [14], Section 3.2.

Step 5. Recall that our aim is to find $c$ for which we have $p_1(B_1^8) = c\alpha$, where $\alpha$ is a generator of $H^4(B_1^8)$. We know from Section 2.2.5 that the tangent bundle $\tau_{B_1^8}$ is sent to a universal bundle and its characteristic classes are pullbacks of the characteristic classes of this universal bundle.

Because $B_1^8 = \mathbb{HP}^2$ - {an open 8-cell}, we consider the inclusion $i : B_1^8 \hookrightarrow \mathbb{HP}^2$ which induces an isomorphism

$$i^* : H^4(\mathbb{HP}^2) \to H^4(B_1^8)$$

by the exact cohomology sequence and excision applied to $\mathbb{HP}^2$ and $B_1^8$. Since $\mathbb{HP}^2$ is the Grassmann manifold $G_4(\mathbb{R}^{12})$, we can send $\tau_{B_1^8}$ to the universal bundle $\gamma^4(\mathbb{R}^{4+8}) = (E, \pi, G_4(\mathbb{R}^{12}))$ where the map between base spaces is $i$.

Therefore we will compute the first Pontrjagin class of $\gamma^4(\mathbb{R}^{4+8})$ and show that

$$p_1(\gamma^4(\mathbb{R}^{4+8})) = 2 \cdot (a\ generator\ of\ H^4(\mathbb{HP}^2)).$$

Then we will pullback this class via $i^*$ and get

$$p_1(\tau_{B_1^8}) = 2 \cdot (a\ generator\ of\ H^4(B_1^8)) = \pm 2\alpha$$

which shows that $c = \pm 2$.

Indeed, $\gamma^4(\mathbb{R}^{4+8})$ is the underlying real 4-plane bundle of the canonical line bundle $\gamma$ over the quaternionic projective space $\mathbb{HP}^2$ which is constructed as follows. The total space of $\gamma$ is $E_\gamma = \{(L, v) \mid v \in L\}$, where $L$ is a quaternionic one dimensional subspace of $\mathbb{H}^3$, $v$ is a vector in it and the projection map sends $(L, v)$ to $L$. Therefore we denote $\gamma^4(\mathbb{R}^{12})$ by $\gamma_\mathbb{R}$. In addition, we denote the underlying complex 2-plane bundle of $\gamma$ by $\gamma_\mathbb{C}$.

To compute the first Pontrjagin class of $\gamma^4(\mathbb{R}^{4+8}) = \gamma_\mathbb{R}$, we consider its Gysin exact sequence. As the set of nonzero vectors in the total space $E - \{0\}$ is homotopy equivalent to $\mathbb{S}^{11}$ in $\mathbb{H}^3$, the sequence reads

$$.. \to H^{i+3}(\mathbb{S}^{11}) \to H^i(\mathbb{HP}^2) \overset{\smile e}{\to} H^{i+4}(\mathbb{HP}^2) \overset{\pi_0^*}{\to} H^{i+4}(\mathbb{S}^{11}) \to ..$$

By considering the cases $i = 0, -2$ we get

$$H^4(\mathbb{HP}^2) = \mathbb{Z},\ H^2(\mathbb{HP}^2) = 0$$

and the Euler class of $\gamma_\mathbb{R}$ generates $H^4(\mathbb{HP}^2)$.

Then, since $H^2(\mathbb{HP}^2) = 0$, $c_1(\gamma_\mathbb{C}) = 0$ and as $\gamma_\mathbb{C}$ is complex 2-plane bundle, $c_i(\gamma_\mathbb{C}) = 0$ for $i > 2$. Then the total Chern class $c(\gamma_\mathbb{C})$ is $1 + c_2(\gamma_\mathbb{C}) = 1 + e$ which in turn equals $c(\overline{\gamma_\mathbb{C}})$ by Lemma 2.6. Now from Theorem 2.8 we have

$$1 - p_1(\gamma_\mathbb{R}) + p_2(\gamma_\mathbb{R}) = c(\gamma_\mathbb{C})c(\overline{\gamma}_\mathbb{C}) = (1 + e)(1 + e).$$

Therefore

$$p(\gamma_{\mathbb{R}}) = 1 - 2e + e^2, \text{ so } p_1(\gamma_{\mathbb{R}}) = -2e$$

which shows that

$$p_1(B_1^8) = \pm 2\alpha. \tag{15}$$

In conclusion, the constant $c$ in Step 3 must be $\pm 2$, implying that

$$p_1(B_k^8) = \pm 2k\alpha. \tag{16}$$

Step 6. Hence we can compute $\lambda(M_k^7)$. First,

$$q(B_k^8) =< (i^{-1}(p_1(B_k^8)))^2, \nu >=< (i^{-1}(\pm 2k\alpha))^2, \nu >= 4k^2 < (i^{-1}\alpha)^2, \nu >= 4k^2$$

by equation (12). That gives us

$$\lambda(M_k^7) \equiv 2q(B_1^8) - \sigma(B_1^8) = 8k^2 - 1 \equiv k^2 - 1 \quad mod\ 7$$

finishing the proof of the theorem.

□

Consequently, we have the equivalence $\lambda(M_k^7) \equiv k^2 - 1$ in mod 7. This implies that when $k^2 \not\equiv 1\ mod\ 7$ , $\lambda(M_{\mathrm{k}}^7) \neq 0$. On the other hand, by the remark following Corollary 4.3, $\lambda(\mathbb{S}^7) = 0$. Hence

**Corollary 3.14.** *The topological 7-sphere $M_{\mathrm{k}}^7$ and the standard 7-sphere $\mathbb{S}^7$ have different differentiable structures.* □

To conclude, by combining the above result with the corollary of the previous section we get

**Theorem 3.15.** *For $k^2 \not\equiv 1\ mod\ 7$, the manifold $M_{\mathrm{k}}^7$ is homeomorphic, but not diffeomorphic to $\mathbb{S}^7$.* ■

Therefore $M_{\mathrm{k}}^7$ is an exotic seven sphere for each odd number $k \not\equiv \pm 1\ mod\ 7$.

***Remark.*** We note that this result does not guarantee that the differentiable structures of any two exotic seven spheres are different. Indeed, there exist only finitely many exotic seven spheres which is the result of Milnor and Kervaire.

Milnor and Kervaire studied the commutative, associative semigroup $\mathcal{M}_n$. It is the group of oriented diffeomorphism classes of smooth $n$-manifolds with the connected sum operation and the class of $\mathbb{S}^n$ is the identity element. The sub-semigroup $\mathcal{S}_n$ of $\mathcal{M}_n$ consists of the oriented diffeomorphism classes of smooth manifolds homeomorphic to $\mathbb{S}^n$. They computed that $\mathcal{S}_n = 0$ for $n = 1, 2, 3, 5, 6$ hence concluding that there are no exotic spheres in these dimensions. They also came up with the result that $\mathcal{S}_7 = \mathbb{Z}/28\mathbb{Z}$, so there are 28 exotic 7-spheres. Indeed, $\mathcal{S}_n$ is a finite abelian group for $n > 4$ and it is now completely computed for $n \leq 64$, except for the case $n = 4$.

Today it is known that for $n \neq 4$, the topological space $\mathbb{R}^n$ has a unique differentiable structure up to diffeomorphism, but $\mathbb{R}^4$ has uncountably many ones. Thus for a smooth, orientable 4-manifold $M$ which is homeomorphic to $\mathbb{S}^4$, there are infinitely many candidates for the possible differentiable structure of $M$. Indeed it is known that, *there are uncountably many distinct diffeomorphism classes (i.e. differentiable structures) of smooth manifolds homeomorphic to* $\mathbb{R}^4$. Therefore $\mathcal{S}_4$ is completely unknown and the question whether there is an exotic 4-sphere still remains open.

## References


[1] J. Milnor, *On Manifolds Homeomorphic to the 7-Sphere*, The Annals of Mathematics, Second Series, Vol. 64, No. 2, USA, Sep., 1956, p.399-405.

[2] J. W. Milnor, J. D. Stasheff, *Characteristic Classes*, Princeton University Press and University of Tokyo Press, Princeton, New Jersey, 1974.

[3] J. W. Milnor, *Spheres-Abel Lecture*, Stony Brook University, Oslo, May 25, 2011.

[4] J. W. Milnor, *The Poincaré Conjecture One Hundred Years Later: A Progress Report*, http://www.math.sunysb.edu/ jack/PREPRINTS/poi-04a.pdf, 2004.

[5] J. W. Milnor, *Collected Paper III, Differential Topology Introduction to Part 1: Exotic Spheres*, http://www.math.sunysb.edu/~jack/intro1-alone.pdf.

[6] J. Milnor, *Morse Theory*, Princeton University Press, Princeton, New Jersey, USA, second printing, 1969.

[7] Y. Matsumoto, *An Introduction to Morse Theory*, American Mathematical Society, USA, 2002.

[8] N. Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, Princeton, New Jersey, seventh printing, 1999.

[9] D. Husemöller, *Fiber Bundles*, Springer-Verlag, New York, third edition, 1994.

[10] W. S. Massey, *A Basic Course in Algebraic Topology*, Springer-Verlag, New York, 1991.

[11] J. M. Lee, *Introduction to Smooth Manifolds*, Springer-Verlag, New York, 2003.

[12] J. M. Lee, *Introduction to Topological Manifolds*, Springer, USA, second edition, 2011.

[13] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[14] A. Hatcher, *Vector Bundles and K-Theory*, 2009.

[15] G.L. Naber, *Topology, Geometry and Gauge Fields*, Springer, 1997.

[16] J. Francis, *Math 465- Topology of Manifolds Lecture Notes*, http://math.northwestern.edu/∼jnkf/classes/mflds/, 2010.

[17] B. Conrad, *Math 396- Differential Geometry Handouts*, http://math.stanford.edu/∼conrad/diffgeomPage/handouts/, 2006.

[18] A.Scorpan, *The Wild World of 4-Manifolds*, American Mathematical Society, USA, 2005.

[19] T. Asselmeyer-Maluga & C. H. Brans, *Exotic Smoothness and Physics-Differential Topology and Spacetime Models*, World Scientific, USA, 2007.

[20] R. Bott, L. W. Tu, *Differential Forms in Algebraic Topology*, Springer, 1923.