OPTIMAL PORTFOLIO INVESTMENT UNDER TRANSACTION COSTS

by

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
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To my family

ABSTRACT

In this thesis, we consider portfolio optimization problem in i.i.d. discrete-time markets under two different scenarios, where the market is modeled by a sequence of price relative vectors with log-normal distribution and with arbitrary discrete distributions. We provide novel approaches for both of these scenarios and introduce optimal portfolio selection algorithms that maximizes the expected cumulative wealth in i.i.d. markets with proportional transaction costs.

In the first part, we study optimal investment in a financial market having a finite number of assets from a signal processing perspective. We investigate how an investor should distribute capital over these assets and when he should reallocate the distribution of the funds over these assets to maximize the cumulative wealth over any investment period. In particular, we introduce a portfolio selection algorithm that maximizes the expected cumulative wealth in i.i.d. two-asset discrete-time markets where the market levies proportional transaction costs in buying and selling stocks. We achieve this using "threshold rebalanced portfolios", where trading occurs only if the portfolio breaches certain thresholds. Under the assumption that the price relative sequences have log-normal distribution from the Black-Scholes model, we evaluate the expected wealth under proportional transaction costs and find the threshold rebalanced portfolio that achieves the maximal expected cumulative wealth over any investment period. Our derivations can be readily extended to markets having more than two stocks, where these extensions are pointed out in the thesis. As predicted from our derivations, we significantly improve the achieved wealth over portfolio selection algorithms from the literature on historical data sets.

In the second part, we first construct portfolios that achieve the optimal expected growth in i.i.d. discrete-time two-asset markets under proportional transaction costs. We then extend our analysis to cover markets having more than two stocks. The market is modeled by a sequence of price relative vectors with arbitrary discrete distributions, which can also be used to approximate a wide class of continuous distributions. To achieve the optimal growth, we use threshold portfolios, where we introduce a recursive update to calculate the expected wealth. We then demonstrate that under the threshold rebalancing framework, the achievable set of portfolios elegantly form an irreducible Markov chain under mild technical conditions. We evaluate the corresponding stationary distribution of this Markov chain, which provides a natural and efficient method to calculate the cumulative expected wealth. Subsequently, the corresponding parameters are optimized yielding the growth optimal portfolio under proportional transaction costs in i.i.d. discrete-time two-asset markets. As a widely known financial problem, we next solve optimal portfolio selection in discrete-time markets constructed by sampling continuous-time Brownian markets. For the case that the underlying discrete distributions of the price relative vectors are unknown, we provide a maximum likelihood estimator that is also incorporated in the optimization framework in our simulations.

ÖZETÇE

Bu tez çalışmasında, bağımsız özdeşçe dağılmış zamanda ayrık piyasalarda portföy eniyileme problemi piyasanın logaritmik Gauss dağılımına sahip ve gelişigüzel ayrık dağılıma sahip göreli fiyat vektörleri ile modellendiği senaryolar olmak üzere iki farklı senaryo üzerinde incelenmektedir. Bu senaryoların her ikisi için yeni yaklaşımlara yer verilmekte ve orantılı hareket masrafı bulunan bağımsız özdeşçe dağılmış piyasalarda beklenen birikimli sermayeyi enbüyüten portföy seçim algoritmaları sunulmaktadır.

İlk olarak, sonlu sayıda aktife sahip mali piyasalarda en iyi yatırım, işaret işleme bakış açısından çalışılmaktadır. Yatırımcının birikimli sermayesini herhangi yatırım döneminde enbüyütmesi için elindeki sermayesini aktifler üzerinde nasıl dağıtması gerektiği ve ne zaman bu dağılımı yeniden tahsis etmesi gerektiği incelenmektedir. Özellikle aktif alım ve satımında orantılı hareket masrafı toplayan, bağımsız özdeşçe dağılmış iki aktifli, zamanda ayrık piyasalarda beklenen birikimli sermayeyi enbüyüten portföy seçim algoritması sunulmaktadır. Bu, alım-satımın yalnızca portföy belirli eşik değerlerini ihlal ettiğinde gerçekleştiği "Eşik Değerinde Yeniden Dengelenen Portföyler" kullanılarak gerçekleştirilmektedir. Göreli fiyat dizilerinin Black-Scholes modelindeki logaritmik Gauss dağılımına sahip olduğu varsayımı altında orantılı hareket masrafları göz önüne alındığında beklenen sermaye belirlenmekte ve herhangi yatırım döneminde en iyi beklenen birikimli sermayeyi elde eden eşik değerinde yeniden dengelenen portföy bulunmaktadır. Bu tezdeki türetmeler kolaylikla ikiden fazla aktifi bulunan piyasalara genellenebilmekte ve bu genellemelere tezdeki gereken yerlerde yer verilmektedir. Türetmelerden tahmin edildiği üzere geçmiş veri kümelerinde, literatürdeki portföy seçim algoritmalarına nazaran kazanılan sermayede önemli artış elde edilmektedir.

İkinci bölümde ise, öncelikle orantılı hareket masrafı toplayan, bağımsız özdeşçe dağılmış iki aktifli, zamanda ayrık piyasalarda beklenen birikimli sermayeyi enbüyüten portföy kurulmaktadır. Daha sonra bu analizler ikiden fazla aktifi bulunan piyasaları kapsayacak şekilde genişletilmektedir. Piyasa geniş bir sürekli dağılım sınıfını da yaklaşıklamakta kullanılabilen gelişigüzel ayrık dağılıma sahip göreli fiyat vektör dizisi ile modellenmektedir. En

iyi büyümeyi elde etmek için eşik değeri portföyleri kullanılmakta ve beklenen sermayeyi hesaplamak için bir özyineli güncelleme yöntemi sunulmaktadır. Daha sonra eşik değerinde yeniden dengeleme yöntemi çerçevesinde, portföylerin alabileceği değerler kümesinin ılımlı teknik koşullar altında bir indirgenemez Markov zinciri oluşturduğu gösterilmektedir. Bu Markov zincirine karşılık gelen durağan dağılım değerlendirilmekte ve beklenen birikimli sermayenin hesaplanması için doğal ve etkin bir yöntem sağlanmaktadır. Ardından karşılık gelen parametreler eniyilenmekte ve orantılı hareket masrafı toplayan, bağımsız özdeşçe dağılmış iki aktifli, zamanda ayrık piyasalarda en iyi büyümeli portföy sunulmaktadır. Daha sonra yaygın olarak bilinen bir finans problemi olan, zamanda sürekli Brown piyasalarının örneklenmesi ile oluşturulan zamanda ayrık piyasalarda en iyi portföy seçimi problemi çözülmektedir. Göreli fiyat vektörlerinin baz alınan ayrık dağılımının bilinmediği durumlar için en büyük olabilirlik kestirimi sunulmakta ve ayrıca benzetimlerdeki eniyileme çerçevesi ile ilintilendirilmektedir.

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Chapter 1

INTRODUCTION

Since the recent global crises have demonstrated the importance of sound financial modeling and reliable data processing, financial applications have attracted a growing interest from the signal processing community [30,36]. The financial markets produce vast amounts of temporal data ranging from stock prices to interest rates, which make them ideal media to apply signal processing methods. Furthermore, due to the integration of high performance, low-latency computing resourses and the financial data collection infrastructures, signal processing algorithms can be readily leveraged with full potential in financial stock markets. This thesis focuses particularly on the portfolio selection problem, which is one the most important financial applications and has already attracted substantial interest from the signal processing community [2–4, 43, 44].

Determining the optimum portfolio and the best portfolio rebalancing strategy that maximize the wealth in discrete-time markets with *no transaction fees* has been heavily investigated in information theory [13, 14], machine learning [20, 39, 47] and signal processing [25–28] fields. Assuming that the portfolio rebalancings, i.e., adjustments by buying and selling stocks, require no transaction fees and with some further mild assumptions on the stock prices, the portfolio that achieves maximum wealth is shown to be a constant rebalanced portfolio (CRP) [14,15]. A CRP is a portfolio strategy where the distribution of funds over the stocks are reallocated to a predetermined structure, also known as the target portfolio, at the start of each investment period. CRPs constitute a subclass of a more general portfolio rebalancing class, the calendar rebalancing portfolios, where the portfolio vector is rebalanced to a target vector on a periodic basis [29]. Numerous studies have been carried out to asymptotically achieve the performance of the best CRP tuned to the individual sequence of stock prices albeit either with different performance bounds or different performance results on historical data sets [14, 20, 47]. CRPs under transaction costs are further investigated in [8], where a sequential algorithm using a weighting similar to that introduced in [15], is also shown to be competitive under transaction costs; i.e., asymptotically achieving the performance of the best CRP under transaction costs. However, we emphasize that maintaining a CRP requires potentially significant trading due to possible rebalancings at each investment period [27]. As shown in [27], even the performance of the best CRP is severally affected by moderate transaction fees rendering CRPs ineffective in real life stock markets. Hence, it may not be enough to try to achieve the performance of the best CRP if the cost of rebalancing outweighs that which could be gained from rebalancing at every investment period. Clearly, one can potentially obtain significant gain in wealth by including unavoidable transactions fees in the market model and perform reallocation accordingly.

Along these lines, the optimal portfolio selection problem under transactions costs is extensively investigated for continuous-time markets [17, 31, 35, 41], where growth optimal policies that keep the portfolio in closed compact sets by trading only when the portfolio hits the compact set-boundaries are introduced. Naturally, the results for the continuous markets cannot be straightforwardly extended to the discrete-time markets, where continuous trading is not allowed. However, it has been shown in [21] that under certain mild assumptions on the sequence of stock prices, similar no trade zone portfolios achieve the optimal growth rate even for discrete-time markets under proportional transaction costs. For markets having two stocks; i.e., two-asset stock markets, these no trade zone portfolios correspond to threshold portfolios; i.e., the no trade zone is defined by thresholds around the target portfolio. As an example, for a market with two stocks, the portfolio is represented by a vector $\boldsymbol{b} = [b \;\; 1-b]^T$, $b \in [0, 1]$, assuming only long positions [29], where b is the ratio of the capital invested in the first stock. For this market, the no rebalancing region around a target portfolio $\boldsymbol{b} = [b \;\; 1-b]^T$, $b \in [0, 1]$, is given by a threshold ϵ, $\min\{b, 1-b\} \geq \epsilon \geq 0$, such that the corresponding portfolio at any investment period is rebalanced to a desired vector if the ratio of the wealth in the first stock breaches the interval $(b-\epsilon, b+\epsilon)$. In particular, unlike a calendar rebalancing portfolio, e.g., a CRP, a threshold rebalanced portfolio (TRP) rebalances by buying and selling stocks only when the portfolio breaches the preset boundaries, or "thresholds", and otherwise does not perform any rebalancing. Intuitively, by limiting the number of rebalancings due to this non rebalancing regions, threshold portfolios are able to avoid hefty transactions costs associated with excessive trading unlike calendar portfolios. Although TRPs are shown to be optimal in i.i.d. discrete-time two-asset markets (under certain technical conditions) [21], finding the TRP that maximizes the expected growth of wealth under proportional transaction costs has not been solved, except for basic scenarios [21], to the best of our knowledge.

This thesis is based on two papers [45,46]. In the first part of this thesis, we analyze i.i.d. discrete-time markets represented by the sequence of price relatives (defined as the ratio of the opening price to the closing price of stocks), where the sequence of price relatives follow log-normal distributions. In the second part, instead of using a continuous distribution, we implement discrete distributions. The sequence of price relative vectors are assumed to have "discrete" distributions; however, the discrete distributions on the vector of price relatives are arbitrary.

1.1 *Log-Normal Price Relatives*

In the first part of this thesis, we evaluate the expected wealth achieved by a TRP over any finite investment period given any target portfolio and threshold for two-asset discrete-time stock markets subject to proportional transaction fees. We emphasize that we study a two-asset market for notational simplicity and our derivations can be readily extended to markets having more than two assets as pointed out in the thesis where needed. We consider i.i.d. discrete-time markets represented by the sequence of price relatives (defined as the ratio of the opening price to the closing price of stocks), where the sequence of price relatives follow log-normal distributions. Note that the log-normal distribution is the assumed statistical model for price relative vectors in the well-known Black-Scholes model [29, 32]. This distribution accurately models real life stock prices which has been shown in many empirical studies [29]. Log-normal distribution which is extensively used in the financial literature is shown to accurately model empirical price relative vectors [9].

Under this setup, we provide an iterative relation that efficiently and recursively calculates the expected wealth in any i.i.d. market over any investment period. The terms in this recursion are evaluated by a certain multivariate Gaussian integral. We then use a randomized algorithm to calculate the given integral and obtain the expected growth. This expected growth is then optimized by a brute force method to yield the optimal target portfolio and the threshold to maximize the expected wealth over any investment period.

1.2 Discrete Price Relatives

In the second part, the sequence of price relative vectors are assumed to have “discrete” distributions; however, the discrete distributions on the vector of price relatives are arbitrary. The corresponding discrete distributions can also be used to approximate a wide class of continuous distributions on the price relatives that satisfy certain regularity conditions by appropriately increasing the size of the discrete sample space. The detailed market model is provided in Section IV. Under this general market model, we use “threshold rebalanced portfolios” (TRP)s, which are shown to yield optimal growth in general i.i.d. discrete-time two-asset markets.

We first recursively calculate the expected wealth achieved by a TRP over any investment period and then optimize the corresponding TRP to maximize expected wealth. We demonstrate that under certain technical conditions, the achievable portfolios in the TRP framework form an irreducible homogenous Markov chain with a finite number of states. This Markov chain can then be elegantly leveraged to calculate the expected growth. Subsequently, the parameters of the TRPs are optimized to achieve the maximum growth using a brute force search. Furthermore, we also solve the optimal portfolio selection problem in discrete-time markets produced by sampling continuous-time Brownian markets extensively studied in the financial literature [29].

1.3 Contributions

The contributions of this thesis are as follows. We first provide an iterative relation that efficiently and recursively calculates the expected wealth for the case where the sequence of price relatives follow log-normal distributions by evaluating a certain multivariate Gaussian integral. We then provide a randomized algorithm to calculate the given integral and obtain the expected growth. This expected growth is then optimized by a brute force method to yield the optimal target portfolio and threshold to maximize the expected wealth over any investment period. Furthermore, we also provide a maximum-likelihood estimator to estimate the parameters of the log-normal distribution from the sequence of price relative vectors, which is incorporated into the algorithmic framework in Simulations section since these parameters are naturally unknown in real life markets.

In the second part, we recursively evaluate the expected achieved wealth of a threshold portfolio for any b and ϵ over any investment period for the case where the sequence of price relative vectors have discrete distributions. We then demonstrate that under the threshold rebalancing framework, the achievable set of portfolios form an irreducible Markov chain under mild technical conditions. We evaluate the corresponding stationary distribution of this Markov chain, which provides a natural and efficient method to calculate the cumulative expected wealth. Subsequently, the corresponding parameters are optimized using a brute force approach yielding the growth optimal investment portfolio under proportional transaction costs in i.i.d. discrete-time two-asset markets. As a well studied problem, we also solve optimal portfolio selection in discrete-time markets constructed by sampling continuous-time Brownian markets. For the case that the underlying discrete distributions of the price relative vectors are unknown, we provide a maximum likelihood estimator to estimate the corresponding distributions that is incorporated in the optimization framework in the Simulations section.

1.4 Content

Chapter 2 begins with a detailed description of the market and the TRPs. We then calculate the expected wealth for the market where the sequence of price relatives follow log-normal distributions using a TRP in an i.i.d. two-asset discrete-time market under proportional transaction costs over any investment period in Section 2.1. First, we provide an iterative relation to recursively calculate the expected wealth growth. The terms in the iterative algorithm are calculated using a certain form of multivariate Gaussian integrals. We provide a randomized algorithm to calculate these integrals in Section 2.2. The maximum-likelihood estimation of the parameters of the log-normal distribution is given in Section 2.3. Section 4 presents the simulations of the iterative relation and the optimization of the expected wealth growth with respect to the TRP parameters using the ML estimator.

In Chapter 3, we investigate threshold rebalancing portfolios for the discrete market where the sequence of price relatives have discrete distributions. We first introduce a recursive update in Section 3.1. We then show that the TRP framework can be analyzed using finite state Markov chains in Section 3.2 and Section 3.3. The special Brownian market is analyzed in Section 3.4. The maximum likelihood estimator is derived in Section 3.5. We simulate the performance of our algorithms in Section 4.

Finally in Chapter 5, we summarize this work and give concluding remarks.

1.5 Notations

Throughout this document, boldface letters and regular letters with subscripts denote vectors and individual elements of vectors, respectively. Furthermore, capital letters and lowercase letters denote random variables and individual realizations of the corresponding random variable, respectively. The vector $[a_1, a_2, \ldots, a_n]^T$ is denoted by $\mathbf{a}^n$. The abbreviations "i.i.d.", "p.d.f.", and "w.l.o.g." are shorthands for the terms "independent identically distributed", and "probability distribution function", respectively. The time index is shown in the subscripts. The operator E[.] denotes the expectation operator. Here, $\mathcal{N}(\mu, \sigma^2)$ and $\ln \mathcal{N}(\mu, \sigma^2)$ denotes the Gaussian and Log-Normal distribution with mean μ and variance σ^2, respectively.

Chapter 2

OPTIMAL INVESTMENT UNDER TRANSACTION COSTS: A THRESHOLD REBALANCED PORTFOLIO APPROACH

In this chapter, we study the investment problem in a financial market having a finite number of assets. We concentrate on how an investor should distribute capital over these assets and when he should reallocate the distribution of the funds over those assets in time to maximize the overall cumulative wealth. In financial terms, distributing ones capital over various assets is known as the portfolio management problem and reallocation of this distribution by buying and selling stocks is referred as the rebalancing of the given portfolio [29]. Due to obvious reasons, the portfolio management problem has been investigated in various different fields from financial engineering [32], machine learning to information theory [13], with a significant room for improvement as the recent financial crises demonstrated. To this end, we investigate the portfolio management problem in discrete-time markets when the market levies proportional *transaction costs* in trading while buying and selling stocks, which accurately models a wide range of real life markets [29,32]. In discrete time markets, we have a finite number of assets and the reallocation of wealth (or rebalancing of the capital) over these assets is only allowed at discrete investment periods, where the investment period is arbitrary, e.g., each second, minute or each day [13,14]. Under this framework, we introduce algorithms that achieve the *maximal* expected cumulative wealth under proportional transaction costs in i.i.d. discrete-time markets extensively studied in the financial literature [29,32]. We further illustrate that our algorithms significantly improve the achieved wealth over the well-known algorithms in the literature on historical data sets under realistic transaction costs, as anticipated from our derivations. The precise problem description including the market and transaction cost models are provided in Section 2.2.

In this part, we first evaluate the expected wealth achieved by a TRP over any finite investment period given any target portfolio and threshold for two-asset discrete-time stock markets subject to proportional transaction fees. We emphasize that we study two-asset market for notational simplicity and our derivations can be readily extended to markets having more than two assets as pointed out in the chapter where needed. We consider i.i.d. discrete-time markets represented by the sequence of price relatives (defined as the ratio of the opening price to the closing price of stocks), where the sequence of price relatives follow log-normal distributions. Note that the log-normal distribution is the assumed statistical model for price relative vectors in the well-known Black-Scholes model [29, 32] and this distribution is shown to accurately model real life stock prices by many empirical studies [29]. Under this setup, we provide an iterative relation that efficiently and recursively calculates the expected wealth by evaluating a certain multivariate Gaussian integral. We then provide a randomized algorithm to calculate the given integral and obtain the expected growth. This expected growth is then optimized by a brute force method to yield the optimal target portfolio and threshold to maximize the expected wealth over any investment period. Furthermore, we also provide a maximum-likelihood estimator to estimate the parameters of the log-normal distribution from the sequence of price relative vectors, which is incorporated into the algorithmic framework in Simulations section since these parameters are naturally unknown in real life markets.

Portfolio management problem is studied with transaction costs in [23] on the horse race setting, which is a special discrete-time market where only one of the asset pays off and the others pay nothing on each period. This basic framework is then extended to general stock markets in [21], where threshold portfolios are shown to be growth optimal for two-asset markets. However, no algorithm, except for a special sampled Brownian market, is provided to find the optimal target portfolio or threshold in [21]. To achieve the performance of the best TRP, a sequential algorithm is introduced in [22] that is shown to asymptotically achieve the performance of the best TRP tuned to the underlying sequence of price relatives. This algorithm uses a similar weighting introduced in [15] to construct the universal portfolio. We emphasize that the universal investment strategies, e.g., [22], which are inspired by universal source coding ideas, based on Bayesian type weighting, are heavily utilized to construct sequential investment strategies [2, 3, 14, 25–28, 39, 47]. Although these methods are shown to "asymptotically" achieve the performance of the best portfolio in the competition class of portfolios, their non-asymptotic performance is acceptable only if a sufficient number of candidate algorithms in the competition class is overly successful [27] to circumvent the loss due to Bayesian type averaging. Since these algorithms are usually designed in a min-max (or universal) framework and hedge against (or should even work for) the worst case sequence, their average (or generic) performance may substantially suffer [10, 16, 20]. In our simulations, we show that our introduced algorithm readily outperforms a wide class of universal algorithms on the historical data sets, including [22]. Note that to reduce the negative effect of the transaction costs in discrete time markets, semiconstant rebalanced portfolio (SCRP) strategies have also been proposed and studied in [8,20,27]. Different than a CRP and similar to the TRPs, an SCRP rebalances the portfolio only at the determined periods instead of rebalancing at the start of each period. Since for an SCRP algorithm rebalancing occurs less frequently than a CRP, using an SCRP strategy may improve the performance over CRPs when transaction fees are present. However, no formulation exists to find the optimal rebalancing times for SCRPs to maximize the cumulative wealth. Although there exist universal methods [27, 39] that achieve asymptotically the performance of the best SCRP tuned to the underlying sequence of price relatives, these methods suffer in realistic markets since they are tuned to the worst case scenario [27] as demonstrated in the Simulations section.

2.1 Problem Description

In this chapter, all vectors are column vectors and represented by lower-case bold letters. Consider a market with m stocks and let $\{\mathbf{x}(t)\}_{t\geq 1}$ represent the sequence of price relative vectors in this market, where $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_m(t)]^T$ with $x_i(t) \in \mathbb{R}_p^+$ for $i \in \{1, 2, \ldots, m\}$. At each investment period, say period t, $\mathbf{b}(t)$ represents the vector of portfolios such that $b_i(t)$ is the fraction of money invested on the ith stock. We allow only long-trading such that $\sum_{i=1}^{m} b_i(t) = 1$ and $b_i(t) \geq 0$. After the price relative vector $\mathbf{x}(t)$ is revealed, we earn $\mathbf{b}^T(t)\mathbf{x}(t)$ at the period t. Assuming we started investing using 1 dollars, at the end of n periods, the wealth growth in a market with no transaction costs is given by

$$S(n) = \prod_{t=1}^{n} \mathbf{b}^T(t)\mathbf{x}(t). \tag{2.1}$$

If we use the well known the CRP [13], then we earn

$$\prod_{t=1}^{n} \mathbf{b}^T \mathbf{x}(t),$$

at the end of n periods ignoring the transaction costs. This method is called "constant rebalancing" since at the start of each investment period t, the portfolio vector $\mathbf{b}(t) = [b_1(t), b_2(t), \ldots, b_m(t)]$ is adjusted, or rebalanced, to a predetermined constant portfolio vector $\mathbf{b} = [b_1, b_2, \ldots, b_m]$ where $\sum_{i=1}^m b_i = 1$. As an example, at the start of each investment period t, since we invested using $\boldsymbol{b}$ at the investment period $t-1$ and observed $x(t-1)$, the current portfolio vector, say $\mathbf{b}_{\text{old}}(t)$,

$$\mathbf{b}_{\text{old}}(t) \triangleq \left[\frac{b_1 x_1(t-1)}{\sum_{i=1}^m b_i x_i(t-1)}, \ldots, \frac{b_m x_m(t-1)}{\sum_{i=1}^m b_i x_i(t-1)} \right]^T,$$

should be adjusted back to $\mathbf{b}$. If we assume a symmetric proportional transaction cost with cost ratio c for both selling and buying, then we need to spend approximately $\sum_{i=1}^m b_{i,\text{old}}(t) S(t) |b_{i,\text{old}}(t) - b_i| c$ dollars for rebalancing. Note that if the transaction costs are not symmetric, the analysis follows by assuming $c = c_{\text{sell}} + c_{\text{buy}}$ by [8], where c_{sell} and c_{buy} are the proportional transaction costs in selling and buying, respectively. Since a CRP should be rebalanced back to its initial value at the start of each investment period, a transaction fee proportional to the wealth growth up to the current period, i.e., $S(t)$, is required for each period t. Hence, constantly rebalancing at each time t may be unappealing for large c.

To avoid such frequent rebalancing, we use TRPs, where we denote a TRP with a target vector $\mathbf{b}$ and a threshold ϵ (with certain abuse of notation) as "TRP with $(\mathbf{b}, \epsilon)$". For a sequence of price relatives vectors $\mathbf{x}^n \triangleq [\mathbf{x}(1), \mathbf{x}(2), \ldots, \mathbf{x}(n)]$ with $\mathbf{x} \in \mathbb{R}_m^+$, a TRP with $(\mathbf{b}, \epsilon)$ rebalances the portfolio to $\mathbf{b}$ at the first time τ satisfying

$$\frac{b_j \prod_{t=1}^{\tau} x_j(t)}{\sum_{k=1}^m b_k \prod_{t=1}^{\tau} x_k(t)} \notin [b_j - \epsilon_j, b_j + \epsilon_j] \tag{2.2}$$

for any $j \in \{1, 2, \ldots, m\}$, thresholds ϵ_j, and does not rebalance otherwise, i.e., while the portfolio vector stays in the no rebalancing region. Starting from the first period of a no rebalancing region, i.e., where the portfolio is rebalanced to the target portfolio $\mathbf{b}$, say $t = 1$

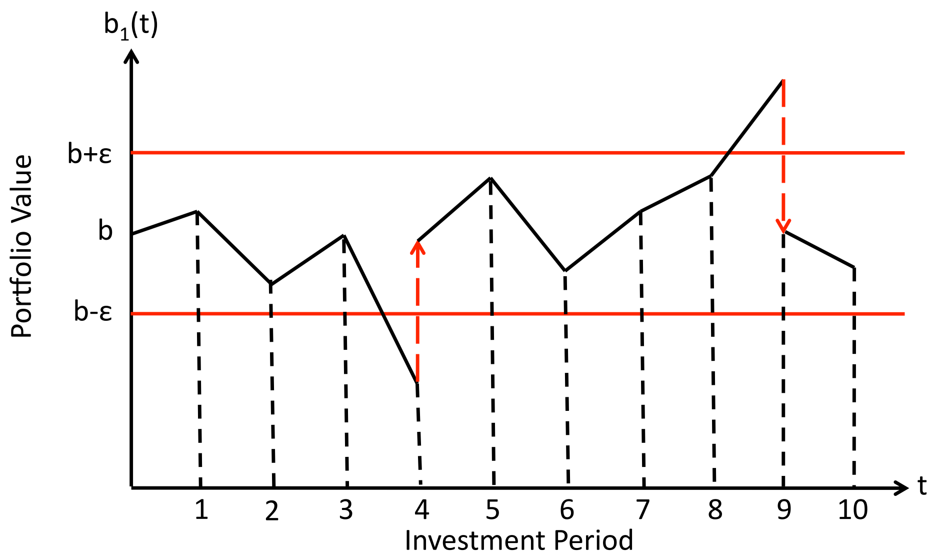


Figure 2.1: A sample scenario for threshold rebalanced portfolios.

for this example, the wealth gained during any no rebalancing region is given by

$$W(\mathbf{x}^n|\mathbf{b}^n \in \mathcal{E}_n^{\mathrm{nc}}) = \sum_{k=1}^{m} b_k \prod_{t=1}^{n} x_k(t), \tag{2.3}$$

where $\mathbf{b}^n = [\mathbf{b}(1), \mathbf{b}(2), \ldots, \mathbf{b}(n)]$ with $\mathbf{b}(t)$ is the portfolio at period t and $\mathcal{E}_n^{\mathrm{nc}}$ is the length n no rebalancing region defined as

$$\mathcal{E}_n^{\mathrm{nc}} = \{\mathbf{b}^n \mid \mathbf{b}(1) = \mathbf{b}, b_j(t) \in (b_j - \epsilon_j, b_j + \epsilon_j), j \in \{1, 2, \ldots, m\}, t \in \{1, 2, \ldots, n\}\}. \tag{2.4}$$

A TRP pays a transaction fee when the portfolio vector leaves the predefined no rebalancing region, i.e., goes out of the no rebalancing region $\mathcal{E}_n^{\mathrm{nc}}$, and rebalanced back to its target portfolio vector $\mathbf{b}$. Since the TRP may avoid constant rebalancing, it may avoid excessive transaction fees while securing the portfolio to stay close the target portfolio $\mathbf{b}$, when we have heavy transaction costs in the market.

For notational clarity, in the remaining of the chapter, we assume that the number of stocks in the market is equal to 2, i.e., $m = 2$. Note that our results can be readily extended to the case when $m > 2$. We point out the necessary modifications to extend our derivations to the case $m > 2$. Then, the threshold rebalanced portfolios are described as follows.

Given a TRP with target portfolio $\mathbf{b} = [b, 1-b]^T$ with $b \in [0,1]$ and a threshold ϵ, the no rebalancing region of a TRP with $(\mathbf{b}, \epsilon)$ is represented by $(b-\epsilon, b+\epsilon)$. Given a TRP with $(b-\epsilon, b+\epsilon)$, we only rebalance if the portfolio leaves this region, which can be found using only the first entry of the portfolio (since there are two stocks), i.e., if $b_{1,\text{old}}(t) \notin (b-\epsilon, b+\epsilon)$. In this case, we rebalance $b_{1,\text{old}}(t)$ to b. Fig. 2.1 represents a sample TRP in a discrete-time two-asset market and when the portfolio is rebalanced back to its initial value if it leaves the no rebalancing interval.

Before our derivations, we emphasize that the performance of a TRP is clearly effected by the threshold and the target portfolio. As an example, choosing a small threshold ϵ, i.e., a low threshold, may cause frequent rebalancing, hence one can expect to pay more transaction fees as a result. However, choosing a small ϵ secures the TRP to stay close to the target portfolio $\boldsymbol{b}$. Choosing a larger threshold ϵ, i.e., a high threshold, avoids frequent rebalancing and degrades the excessive transaction fees. Nevertheless, the portfolio may drift to risky values that are distant from the target portfolio $\boldsymbol{b}$ under large threshold. Furthermore, we emphasize that proportional transaction cost c is a key factor in determining the ϵ. Under mild stochastic assumptions it has been shown in [14, 15] that in a market with no transaction costs, CRPs achieve the maximum possible wealth. Therefore in a market with no transaction costs, i.e., $c = 0$, the maximum wealth can be achieved when we choose a zero threshold, i.e., $\epsilon = 0$ and a target portfolio $b^* = \arg\max_b E[\log(bx_1 + (1-b)x_2)]$, where x_1 and x_2 represent the price relatives of two-asset market [15]. On the other hand, in a market with high transaction costs, choosing a high threshold, i.e., a large ϵ, eliminates the unappealing effect of transaction costs. For instance, for the extreme case where the transaction cost is infinite, i.e., $c = \infty$, the best TRP should either have $\epsilon = 1$ or $b \in \{0, 1\}$ to ensure that no rebalancing occurs.

In this chapter, we assume that the price relative vectors have a log-normal distribution following the well-known Black-Scholes model [29]. This distribution that is extensively used in financial literature is shown to accurately model empirical price relative vectors [9]. Hence, we assume that $\mathbf{x}(t) = [x_1(t), x_2(t)]^T$ has an i.i.d. log-normal distribution with mean $\boldsymbol{\mu} = [\mu_1, \mu_2]$ and standard deviation $\boldsymbol{\sigma} = [\sigma_1, \sigma_2]$, respectively, i.e., $\mathbf{x}(t) \sim \ln \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\sigma}^2)$. In this chapter, we first optimize the wealth achieved by a TRP for the discrete-time market, where the distributions of the price relatives are known. We then provide a ML estimator for these parameters to cover the case where the means and variances are unknown. The ML estimator is incorporated in the algorithmic framework in the Simulations section since the corresponding parameters are unknown in real life markets. The details of the maximum-likelihood estimation are given in Section 2.3.

2.2 Threshold Rebalanced Portfolios

In this section, we analyze the TRPs in a discrete-time market with proportional transaction costs as defined in Section 2.1. We first introduce an iterative relation, as a theorem, to recursively evaluate the expected achieved wealth of a TRP over any investment period. The terms in this iterative equation are calculated using a certain form of multivariate Gaussian integrals. We provide a randomized algorithm to calculate these integrals. We then use the given iterative equation to find the optimal ϵ and b that maximize the expected wealth over any investment period.

2.2.1 An Iterative Equation to Calculate the Expected Wealth

In this section, we introduce an iterative equation to evaluate the expected cumulative wealth of a TRP with $(b-\epsilon, b+\epsilon)$ over any period n, i.e., $E[S(n)]$. As seen in Fig. 2.2, for a TRP with $(b-\epsilon, b+\epsilon)$, any investment scenario can be decomposed as the union of consecutive no-crossing blocks such that each rebalancing instant, to the initial $\mathbf{b}$, signifies the end of a block. Hence, based on this observation, the expected gain of a TRP between any consecutive crossings, i.e., the gain during the no rebalancing regions, directly determines the overall expected wealth growth. Hence, we first calculate the conditional expected gain of a TRP over no rebalancing regions and then introduce the iterative relation based on these derivations.

For a TRP with $(b-\epsilon, b+\epsilon)$, we call a no rebalancing region of length n as "period n with no-crossing" such that the TRP with the initial and target portfolio $\mathbf{b} = [b, 1-b]$ stays in the $(b-\epsilon, b+\epsilon)$ interval for $n-1$ consecutive investment periods and crosses one of the thresholds at the nth period. We next calculate the expected gain of a TRP over any no-crossing period as follows.

The wealth growth of a TRP with $(b-\epsilon, b+\epsilon)$ for a period τ with no-crossing can be written as [1]

$$S_{\rm nc}(\tau) \stackrel{\triangle}{=} b\prod_{t=1}^{\tau}[x_1(t)] + (1-b)\prod_{t=1}^{\tau}[x_2(t)], \tag{2.5}$$

without the transaction cost that arises at the last period. To find the total achieved wealth for a period τ with no-crossing, we need to subtract the transaction fees from (2.5). If portfolio $b_1(t)$ crosses the threshold at the investment period $t = \tau$, then we need to rebalance it back to b, i.e., $b_1(t) = b$ and approximately pay

$$S_{\rm nc}(\tau)c\left|\frac{b\prod_{t=1}^{\tau}(x_1(t))}{b\prod_{t=1}^{\tau}(x_1(t)) + (1-b)\prod_{t=1}^{\tau}(x_2(t))} - b\right|, \tag{2.6}$$

where c represents the symmetrical commission cost, to rebalance two stocks, i.e., $b_{1,\rm old}(\tau{+}1)$ to b, and $b_{2,\rm old}(\tau+1) = 1 - b_{1,\rm old}(\tau+1)$ to $1-b$. Hence, the net overall gain for a period τ with no-crossing becomes

$$\begin{aligned} S(\tau) &= S_{\rm nc}(\tau) - S_{\rm nc}(\tau)c\left|\frac{b\prod_{t=1}^{\tau}(x_1(t))}{b\prod_{t=1}^{\tau}(x_1(t)) + (1-b)\prod_{t=1}^{\tau}(x_2(t))} - b\right| \\ &= b\prod_{t=1}^{\tau}[x_1(t)] + (1-b)\prod_{t=1}^{\tau}[x_2(t)] - c(b-b^2)\left|\prod_{t=1}^{\tau}[x_1(t)] - \prod_{t=1}^{\tau}[x_2(t)]\right| \\ &= \zeta_1\prod_{t=1}^{\tau}[x_1(t)] + \zeta_2\prod_{t=1}^{\tau}[x_2(t)], \end{aligned} \tag{2.7}$$

where $\zeta_1 \stackrel{\triangle}{=} b - 2c(b-b^2)$ and $\zeta_2 \stackrel{\triangle}{=} 1 - b + 2c(b-b^2)$ for $b+\epsilon$ hitting and $\zeta_1 \stackrel{\triangle}{=} b + 2c(b-b^2)$ and $\zeta_2 \stackrel{\triangle}{=} 1 - b - 2c(b-b^2)$ for $b-\epsilon$ hitting. Thus, the conditional expected gain of a TRP conditioned on that the portfolio stays in a no rebalancing region until the last period of the region can be found by calculating the expected value of (2.7). Since, we now have the conditional expected gains, we next introduce an iterative relation to find the expected wealth growth of a TRP with $(b-\epsilon, b+\epsilon)$ for period n, $E[S(n)]$, by using the expected gains of no-crossing periods as shown in Fig. 2.2.

In order to calculate the expected wealth $E[S(n)]$ iteratively, let us first define the variable $R(\tau)$, which is the expected cumulative gain of all possible portfolios that hit any

[1] This is the special case of (2.3) for $m = 2$.

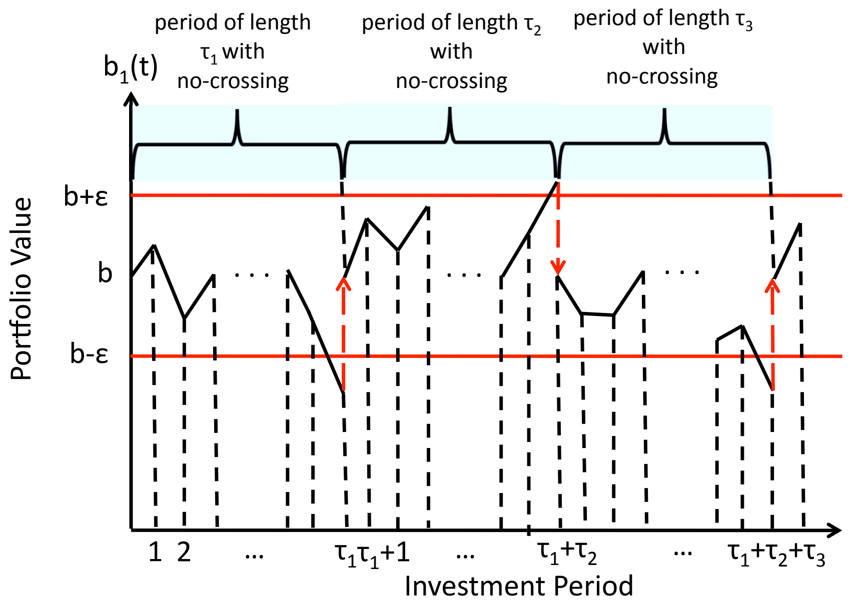


Figure 2.2: No-crossing intervals of threshold rebalanced portfolios.

of the thresholds first time at the τth period, i.e.,

$$R(\tau) = E\left[S(\tau)\,\middle|\,\mathbf{b}^{\tau} \in \mathcal{E}_{\tau}^{\text{fc}}\right], \tag{2.8}$$

where $\mathcal{E}_{\tau}^{\text{fc}}$ denotes the set of all possible portfolios with initial portfolio b and that stay in the no rebalancing region for $\tau - 1$ consecutive periods and hits one of the $b - \epsilon$ or $b + \epsilon$ boundary at the τth period, i.e.,

$$\mathcal{E}_{\tau}^{\text{fc}} \triangleq \{\mathbf{b}^{\tau} \in \mathcal{B}_{\tau}(b,\epsilon)\,|\,b(1) = b, b(i) \in (b-\epsilon, b+\epsilon)\,\forall i \in \{2,\ldots,\tau-1\}, b(\tau) \notin (b-\epsilon, b+\epsilon)\}. \tag{2.9}$$

Here, $\mathcal{B}_{\tau}(b,\epsilon)$ is defined as the set of all possible threshold rebalanced portfolios with initial and target portfolio b and a no rebalancing interval $(b-\epsilon, b+\epsilon)$. Similarly we define the variable $T(\tau)$, which is the expected growth of all possible portfolios of length τ with no threshold crossings, i.e.,

$$T(\tau) = E\left[S(\tau)\,\middle|\,\mathbf{b}^{\tau} \in \mathcal{E}_{\tau}^{\text{nc}}\right], \tag{2.10}$$

where $\mathcal{E}_\tau^{\rm nc}$ denotes the set of portfolios with initial portfolio b and that stay in the no rebalancing region for τ consecutive periods, i.e.,[2]

$$\mathcal{E}_\tau^{\rm nc} \triangleq \{\mathbf{b}^\tau \in \mathcal{B}_\tau(b,\epsilon) \,|\, b(1) = b, b(i) \in [b-\epsilon, b+\epsilon] \, \forall i \in \{2,\ldots,\tau\}\}. \tag{2.11}$$

Given the variables $R(\tau)$ and $T(\tau)$, we next introduce a theorem that iteratively calculates the expected wealth growth of a TRP over any period n. Hence, to calculate the expected achieved wealth, it is sufficient to calculate $R(\tau)$, $T(\tau)$, threshold crossing probabilities $P\left(\mathbf{b}^n \in \mathcal{E}_n^{\rm fc}\right)$ and $P\left(\mathbf{b}^n \in \mathcal{E}_n^{\rm nc}\right)$, which are explicitly evaluated in the next section.

Theorem 2.2.1 *The expected wealth growth of a TRP $(b-\epsilon, b+\epsilon)$, i.e. $E[S(n)]$, over any i.i.d. sequence of price relative vectors $\mathbf{x}^n = [\mathbf{x}(1), \mathbf{x}(2), \ldots, \mathbf{x}(n)]$, satisfies*

$$E[S(n)] = \sum_{i=1}^{n} P(\mathcal{E}_i^{\rm fc}) R(i) E[S(n-i)] + P(\mathcal{E}_n^{\rm nc}) T(n), \tag{2.12}$$

where we define $S_0 = 1$, $R(n)$ in (2.8), $T(n)$ in (2.10), $\mathcal{E}_i^{\rm fc}$ in (2.11) and $\mathcal{E}_n^{\rm nc}$ in (2.9).

We emphasize that by Theorem 2.2.1, we can recursively calculate the expected growth of any TRP over any i.i.d. discrete-time market under proportional transaction costs. Theorem 2.2.1 holds for i.i.d. markets having either $m = 2$ or $m > 2$ provided that the corresponding terms in (2.12) can be calculated.

Proof: By using the law of total expectation [40], $E[S(n)]$ can be written as

$$E[S(n)] = \int_{\mathbf{b}^n \in \mathcal{B}_n(b,\epsilon)} E[S(n)|\mathbf{b}^n] P(\mathbf{b}^n) \mathrm{d}\mathbf{b}^n, \tag{2.13}$$

where $\mathcal{B}_n(b,\epsilon)$ is defined as the set of all possible TRPs with the initial and target portfolio b and threshold ϵ. To obtain (2.12), we consider all possible portfolios as a union of $n+1$ disjoint sets: (1) the portfolios which cross one of the thresholds first time at the 1st

[2] This is the special case of the definition in (2.4) for $m = 2$.

period; (2) the portfolios which cross one of the thresholds first time at the 2nd period; and continuing in this manner, (3) the portfolios which cross one of the thresholds first time at the nth period; and finally (4) the portfolios which do not cross the thresholds for n consecutive periods. Clearly these market portfolio sets are disjoint and their union provides all possible portfolio paths. Hence (2.13) can also be written as

$$\begin{aligned} E[S(n)] = &\sum_{i=1}^{n} \int_{\mathbf{b}_1^i \in \mathcal{E}_i^{\mathrm{fc}}, \mathbf{b}_{i+1}^n \in \mathcal{B}n-i(b,\epsilon)} E[S(n)|\mathbf{b}_1^i \in \mathcal{E}_i^{\mathrm{fc}}, \mathbf{b}_{i+1}^n \in \mathcal{B}_{n-i}(b,\epsilon)] P(\mathbf{b}_1^i \in \mathcal{E}_i^{\mathrm{fc}}, \mathbf{b}_{i+1}^n \in \mathcal{B}_{n-i}(b,\epsilon)) \mathrm{d}\mathbf{b}^n \\ &+ \int_{\mathbf{b}^n \in \mathcal{E}_n^{\mathrm{nc}}} E[S(n)|\mathbf{b}^n \in \mathcal{E}_n^{\mathrm{nc}}] P(\mathbf{b}^n \in \mathcal{E}_n^{\mathrm{nc}}) \mathrm{d}\mathbf{b}^n, \end{aligned} \tag{2.14}$$

where $\mathbf{b}_i^j \triangleq [b(i), b(i+1), \ldots, b(j)]$. To continue with our derivations, we define $S_{i \to j}$ as the wealth growth from the period i to period j, i.e., $S_{i \to j} \triangleq \frac{S(j)}{S(i)}$. Assume that in the period τ, the portfolio crosses one of the thresholds and a rebalancing occurs. In that case, regardless of the portfolios before the period τ, the portfolio is rebalanced back to its initial value in the τth period, i.e., to $[b, 1-b]^T$. Since the price relative vectors are independent over time, we can conclude that the portfolios before the period τ are independent from the portfolios after the period τ, i.e., $b(\tau) = b$ and every portfolio $b(i)$ for $i \in \{1, 2, \ldots, \tau - 1\}$ are independent from the portfolios $b(j)$ for $j \in \{\tau + 1, \tau + 2, \ldots, n\}$. Hence, the investment period where the portfolio path crosses one of the thresholds, i.e., τ, divides the whole investment block into uncorrelated blocks in terms of price relative vectors and portfolios. Thus, the wealth growth acquired up to the period τ, $S_{1 \to \tau}$, is uncorrelated to the wealth growth acquired after that period, i.e., $S_{\tau+1 \to n}$. Hence, if we assume that a threshold crossing occurs at the period τ, then we have

$$\begin{aligned} E\left[S(n)|\mathbf{b}_1^\tau \in \mathcal{E}_\tau^{\mathrm{fc}}, \mathbf{b}_{\tau+1}^n \in \mathcal{B}_{n-\tau}(b,\epsilon)\right] &= E\left[S_{1\to\tau} S_{\tau+1\to n}|\mathbf{b}_1^\tau \in \mathcal{E}_\tau^{\mathrm{fc}}, \mathbf{b}_{\tau+1}^n \in \mathcal{B}_{n-\tau}(b,\epsilon)\right] \\ &= E\left[S_{1\to\tau}|\mathbf{b}_1^i \in \mathcal{E}_i^{\mathrm{fc}}\right] E\left[S_{\tau+1\to n}|\mathbf{b}_{i+1}^n \in \mathcal{B}_{n-i}(b,\epsilon)\right]. \end{aligned} \tag{2.15}$$

Applying (2.15) to (2.14), we get

$$E\left[S(n)\right] = \sum_{i=1}^{n} \int_{\mathbf{b}_1^i \in \mathcal{E}_i^{\text{fc}}, \mathbf{b}_{i+1}^n \in \mathcal{B}_{n-i}(b,\epsilon)} E\left[S_{1\to i}|\mathbf{b}_1^i \in \mathcal{E}_i^{\text{fc}}\right] E\left[S_{i+1\to n}|b(i)=b, \mathbf{b}_{i+1}^n \in \mathcal{B}_{n-i}(b,\epsilon)\right]$$
$$\times P\left(\mathbf{b}_1^i \in \mathcal{E}_i^{\text{fc}}\right) P\left(\mathbf{b}_{i+1}^n \in \mathcal{B}_{n-i}(b,\epsilon)\right) \mathrm{d}\mathbf{b}^n + \int_{\mathbf{b}^n \in \mathcal{E}_n^{\text{nc}}} E\left[S(n)|\mathbf{b}^n \in \mathcal{E}_n^{\text{nc}}\right]$$
$$\times P\left(\mathbf{b}^n \in \mathcal{E}_n^{\text{nc}}\right) \mathrm{d}\mathbf{b}^n. \tag{2.16}$$

Since the integral in (2.16) can be decomposed into two disjoint integrals, (2.14) yields

$$E[S(n)] = \sum_{i=1}^{n} \int_{\mathbf{b}_1^i \in \mathcal{E}_i^{\text{fc}}} E[S_{1\to i}|\mathbf{b}_1^i \in \mathcal{E}_i^{\text{fc}}] P(\mathbf{b}_1^i \in \mathcal{E}_i^{\text{fc}}) \mathrm{d}\mathbf{b}_1^i$$
$$\times \int_{\mathbf{b}_{i+1}^n \in \mathcal{B}_{n-i}(b,\epsilon)} E[S_{i+1\to n}|b(i)=b, \mathbf{b}_{i+1}^n \in \mathcal{B}_{n-i}(b,\epsilon)] P(\mathbf{b}_{i+1}^n \in \mathcal{B}_{n-i}(b,\epsilon)) \mathrm{d}\mathbf{b}_{i+1}^n$$
$$+ \int_{\mathbf{b}^n \in \mathcal{E}_n^{\text{nc}}} E[S(n)|\mathbf{b}^n \in \mathcal{E}_n^{\text{nc}}] P(\mathbf{b}^n \in \mathcal{E}_n^{\text{nc}}) \mathrm{d}\mathbf{b}^n. \tag{2.17}$$

We next write (2.17) as a recursive equation.

To accomplish this, we first note that

(i) $R(i)$ is defined as the expected gain of TRPs with length i, which crosses one of the thresholds first time at the i-th period, it follows that

$$R(i) = E\left[S(\tau) \,\middle|\, \mathbf{b}^i \in \mathcal{E}_i^{\text{fc}}\right] \tag{2.18}$$
$$= \frac{1}{P(\mathcal{E}_i^{\text{fc}})} \int_{\mathbf{b}_1^i \in \mathcal{E}_i^{\text{fc}}} E[S_{1\to i}|\mathbf{b}_1^i \in \mathcal{E}_i^{\text{fc}}] P(\mathbf{b}_1^i \in \mathcal{E}_i^{\text{fc}}) \mathrm{d}\mathbf{b}_1^i, \tag{2.19}$$

where we write $P(\mathcal{E}_i^{\text{fc}})$ instead of $P(\mathbf{b}_1^i \in \mathcal{E}_i^{\text{fc}})$.

(ii) Then, as the second term, $T(n)$ is defined as the expected gain of TRPs of length n, which does not cross one of the thresholds for n consecutive periods. This yields

$$T(n) = E\left[S(n) \,\middle|\, \mathbf{b}^n \in \mathcal{E}_n^{\text{nc}}\right] \tag{2.20}$$
$$= \frac{1}{P(\mathcal{E}_n^{\text{nc}})} \int_{\mathbf{b}^n \in \mathcal{E}_n^{\text{nc}}} E[S(n)|\mathbf{b}^n \in \mathcal{E}_n^{\text{nc}}] p(\mathbf{b}^n \in \mathcal{E}_n^{\text{nc}}) \mathrm{d}\mathbf{b}^n. \tag{2.21}$$

(iii) Finally, observe that the second integral in (2.17) is the expected wealth growth of a

TRP of length $n-i$, i.e.,

$$E[S(n-i)] = \int_{\mathbf{b}_{i+1}^n \in \mathcal{B}_{n-i}(b,\epsilon)} E[S_{i+1\to n}|b(i)=b, \mathbf{b}_{i+1}^n \in \mathcal{B}_{n-i}(b,\epsilon)] p(\mathbf{b}_{i+1}^n \in \mathcal{B}_{n-i}(b,\epsilon)) \mathrm{d}\mathbf{b}_{i+1}^n, \tag{2.22}$$

where $p(\mathbf{b}_{i+1}^n \in \mathcal{B}_{n-i}(b,\epsilon)) = 1$ by the definition of the set $\mathcal{B}_{n-i}(b,\epsilon)$.

Hence, if we apply (2.19), (2.21) and (2.22) to (2.17), we can write (2.12) as

$$E[S(n)] = \sum_{i=1}^{n} P(\mathcal{E}_i^{\text{fc}}) R(i) E[S(n-i)] + P(\mathcal{E}_n^{\text{nc}}) T(n), \tag{2.23}$$

hence the proof follows. □

Theorem 2.2.1 provides a recursion to iteratively calculate the expected wealth growth $E[S(n)]$, when $R(\tau)$ and $T(\tau)$ are explicitly calculated for a TRP with $(b-\epsilon, b+\epsilon)$. Hence, if we can obtain $P\left(\mathcal{E}_\tau^{\text{fc}}\right) R(\tau)$ and $P\left(\mathcal{E}_\tau^{\text{nc}}\right) T(\tau)$ for any τ, then (2.12) yields a simple iteration that provides the expected wealth growth for any period n. We next give the explicit definitions of the events $\mathcal{E}_\tau^{\text{fc}}$ and $\mathcal{E}_\tau^{\text{nc}}$ in order to calculate the conditional expectations $R(\tau)$ and $T(\tau)$. Following these definitions, we calculate $P\left(\mathcal{E}_\tau^{\text{fc}}\right) R(\tau)$ and $P\left(\mathcal{E}_\tau^{\text{nc}}\right) T(\tau)$ to evaluate the expected wealth growth $E[S(\tau)]$, iteratively from Theorem 2.2.1 and find the the optimal TRP, i.e., optimal b and ϵ, by using a brute force search.

In the next section, we provide the explicit definitions for $\mathcal{E}_\tau^{\text{fc}}$ and $\mathcal{E}_\tau^{\text{nc}}$, and define the conditions for staying in the no rebalancing region or hitting one of the boundaries to find the corresponding probabilities of these events.

2.2.2 Explicit Calculations of $R(n)$ and $T(n)$

In this section, we first define the conditions for the market portfolios to cross the corresponding thresholds and calculate the probabilities for the events $\mathcal{E}_\tau^{\text{fc}}$ and $\mathcal{E}_\tau^{\text{nc}}$. We then calculate the conditional expectations $R(n)$ and $T(n)$ as certain multivariate Gaussian integrals. The explicit calculation of multivariate Gaussian integrals are given in Section 2.2.3.

To get the explicit definitions of the events $\mathcal{E}_\tau^{\text{fc}}$ and $\mathcal{E}_\tau^{\text{nc}}$, we note that we have two different boundary hitting scenarios for a TRP, i.e., starting from the initial portfolio b, the portfolio can hit $b-\epsilon$ or $b+\epsilon$. From b, the portfolio crosses $b-\epsilon$ boundary if

$$\frac{b\prod_{t=1}^{\tau}(x_1(t))}{b\prod_{t=1}^{\tau}(x_1(t))+(1-b)\prod_{t=1}^{\tau}(x_2(t))} \leq b-\epsilon, \tag{2.24}$$

where τ is the first time the crossing happens without ever hitting any of the boundaries before. Since $x_1(i), x_2(i) > 0$ for all i, (2.24) happens if

$$\prod_{t=1}^{\tau}\frac{x_2(t)}{x_1(t)} \geq \frac{b(1-b+\epsilon)}{(1-b)(b-\epsilon)}, \tag{2.25}$$

which is equivalent to

$$\Pi_2(\tau) \geq \gamma_1 \Pi_1(\tau),$$

where $\Pi_1(i) \triangleq \prod_{t=1}^{i} x_1(t)$, $\Pi_2(i) \triangleq \prod_{t=1}^{i} x_2(t)$ and $\gamma_1 \triangleq \frac{b(1-b+\epsilon)}{(1-b)(b-\epsilon)}$. Since $\mathbf{x}(i)$'s have log-normal distributions, i.e., $\mathbf{x}(t) \sim \ln\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\sigma}^2)$, $\Pi_1(i)$ and $\Pi_2(i)$ are log-normal, too [40]. Furthermore, to calculate the required probabilities, we note that

$$\begin{aligned} p\left(\Pi_1(i), \Pi_1(k-1), \Pi_1(k)\right) &= p\left(\Pi_1(i), \Pi_1(k-1)\right) p\left(\Pi_1(k)|\Pi_1(k-1), \Pi_1(i)\right) \\ &= p\left(\Pi_1(i)\right) p\left(\Pi_1(k-1)|\Pi_1(i)\right) p\left(\Pi_1(k-1)x_1(k)|\Pi_1(k-1), \Pi_1(i)\right) \\ &= p\left(\Pi_1(i)\right) p\left(\Pi_1(k-1)|\Pi_1(i)\right) p\left(\Pi_1(k)|\Pi_1(k-1)\right), \end{aligned} \tag{2.26}$$

$\forall i \in \{0, 1, \ldots, k-2\}$, where (2.26) follows since $x(k)$ is independent of $\Pi_1(i)$ for $k > i$. Hence $\Pi_1(i)$'s form a Markov chain such that $\Pi_1(i) \leftrightarrow \Pi_1(k-1) \leftrightarrow \Pi_1(k)\ \ \forall i \in \{0, 1, \ldots, k-2\}$. Following the similar, steps we also obtain that $\Pi_2(i) \leftrightarrow \Pi_2(k-1) \leftrightarrow \Pi_2(k)\ ,\forall i \in \{0, 1, \ldots, k-2\}$. Note that, by extending the definitions Π_1 and Π_2 one can obtain $\Pi_1, \Pi_2, \ldots, \Pi_m$ for the case $m > 2$. Furthermore, taking the logarithm of both sides of (2.25) we have

$$\Sigma_1^{\tau} \triangleq \sum_{t=1}^{\tau} z(t) \geq \theta_1,$$

where $z(t) \triangleq \ln\left(\frac{x_2(t)}{x_1(t)}\right)$ and $\theta_1 \triangleq \ln\frac{b(1-b+\epsilon)}{(1-b)(b-\epsilon)} = \ln\gamma_1$. The partial sums of $z(t)$'s are defined as $\Sigma_i^k = \sum_{t=i}^{k} z(t)$ for notational simplicity. Since $\mathbf{x}(t) \sim \ln\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\sigma}^2)$, $z(t)$'s are Gaussian, i.e. $z(t) \sim \mathcal{N}(\mu, \sigma^2)$, where $\mu = \mu_2 - \mu_1$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2$, their sums, Σ_i^k's, are Gaussian too. Furthermore note that, $\Sigma_1^k = \sum_{t=1}^{k} z(t) = \sum_{t=1}^{k}\ln\left(\frac{x_2(t)}{x_1(t)}\right) = \ln\left(\prod_{t=1}^{k}\frac{x_2(t)}{x_1(t)}\right) = \ln\frac{\Pi_2(k)}{\Pi_1(k)}$.

Similarly with an initial value b, market portfolio crosses $b+\epsilon$ boundary if

$$\frac{b\prod_{t=1}^{\tau}(x_1(t))}{b\prod_{t=1}^{\tau}(x_1(t))+(1-b)\prod_{t=1}^{\tau}(x_2(t))} \geq b+\epsilon, \tag{2.27}$$

where τ is the first crossing time without ever hitting any of the boundaries before. Again, since $x_1(i), x_2(i) > 0$ for all i, (2.27) happens if

$$\prod_{t=1}^{\tau}\frac{x_2(t)}{x_1(t)} \leq \frac{b(1-b-\epsilon)}{(1-b)(b+\epsilon)}, \tag{2.28}$$

which can be written of the form

$$\Pi_2(t) \leq \gamma_2\Pi_1(t).$$

Equation (2.28) yields

$$\Sigma_1^{\tau} = \sum_{t=1}^{\tau} z(t) \leq \theta_2,$$

where $\theta_2 \triangleq \ln\frac{b(1-b-\epsilon)}{(1-b)(b+\epsilon)} = \ln\gamma_2$.

Hence, we can explicitly describe the event that the market threshold portfolio $(b-\epsilon, b+\epsilon)$ does not hit any of the thresholds for τ consecutive periods, $\mathcal{E}_{\tau}^{\mathrm{nc}}$, as the intersection of the events as

$$\mathcal{E}_{\tau}^{\mathrm{nc}} \triangleq \bigcap_{i=1}^{\tau}\{\Sigma_1^i \in [\theta_2, \theta_1]\} = \bigcap_{i=1}^{\tau}\{\gamma_2\Pi_1(i) \leq \Pi_2(i) \leq \gamma_1\Pi_1(i)\}. \tag{2.29}$$

Similarly, the event of the market threshold portfolio $(b-\epsilon, b+\epsilon)$ hitting any of the thresholds first time at the τ-th period, $\mathcal{E}_{\tau}^{\mathrm{fc}}$, can be defined as the intersections of the events

$$\begin{aligned}\mathcal{E}_{\tau}^{\mathrm{fc}} &\triangleq \bigcap_{i=1}^{\tau-1}\{\Sigma_1^i \in [\theta_2, \theta_1]\} \bigcap \Big[\{\Sigma_{\tau} \in [-\infty, \theta_2)\} \bigcup \{\Sigma_{\tau} \in (\theta_1, \infty]\}\Big] \\ &= \bigcap_{i=1}^{\tau-1}\{\gamma_2\Pi_1(i) \leq \Pi_2(i) \leq \gamma_1\Pi_1(i)\} \bigcap \Big[\{\Pi_2(\tau) \geq \gamma_1\Pi_1(\tau)\} \bigcup \{\Pi_2(\tau) \leq \gamma_2\Pi_1(\tau)\}\Big],\end{aligned} \tag{2.30}$$

yielding the explicit definitions of the events $\mathcal{E}_{\tau}^{\mathrm{fc}}$ in (2.30) and $\mathcal{E}_{\tau}^{\mathrm{nc}}$ in (2.29). Note that the definitions of $\mathcal{E}_{\tau}^{\mathrm{nc}}$ and $\mathcal{E}_{\tau}^{\mathrm{fc}}$ can be extended for the case $m > 2$ by employing the updated definitions of $\Pi_1, \Pi_2, \ldots, \Pi_m$.

Since we have the quantitive definitions of the events $\mathcal{E}_\tau^{\text{fc}}$ and $\mathcal{E}_\tau^{\text{nc}}$, we can express the expected overall gain of τ-period no hitting portfolios, $T(\tau)$, as

$$\begin{aligned} T(\tau) &= E\Big[S(\tau) \;\Big|\; \mathcal{E}_\tau^{\text{nc}}\Big] \\ &= E\Big[b\prod_{t=1}^{\tau}[x_1(t)] + (1-b)\prod_{t=1}^{\tau}[x_2(t)] \;\Big|\; \mathcal{E}_\tau^{\text{nc}}\Big] \\ &= E\Big[b\Pi_1(\tau) + (1-b)\Pi_2(\tau) \;\Big|\; \mathcal{E}_\tau^{\text{nc}}\Big]. \end{aligned} \tag{2.31}$$

The expectation $E\Big[b\Pi_1(\tau) + (1-b)\Pi_2(\tau) \;\Big|\; \mathcal{E}_\tau^{\text{nc}}\Big]$ can be expressed in an integral form as

$$\begin{aligned} E\Big[b\Pi_1(\tau) + (1-b)\Pi_2(\tau) \;\Big|\; \mathcal{E}_\tau^{\text{nc}}\Big] = \int_0^\infty \int_0^\infty & (b\pi_1 + (1-b)\pi_2) \\ & \times P\Big(\Pi_1(\tau) = \pi_1, \Pi_2(\tau) = \pi_2 \;\Big|\; \mathcal{E}_\tau^{\text{nc}}\Big)\, \mathrm{d}\pi_2 \mathrm{d}\pi_1 \end{aligned} \tag{2.32}$$

by definition of conditional expectation. Note that for the case $m > 2$ the double integral in the definition of T_τ (2.32) is replaced by an m-dimensional integral over updated random variables $\Pi_1, \Pi_2, \ldots, \Pi_m$. Combining (2.32) and (2.31) yields

$$\begin{aligned} T(\tau) &= \int_0^\infty \int_0^\infty (b\pi_1 + (1-b)\pi_2)\;\, P\Big(\Pi_1(\tau) = \pi_1, \Pi_2(\tau) = \pi_2 \;\Big|\; \mathcal{E}_\tau^{\text{nc}}\Big)\, \mathrm{d}\pi_2 \mathrm{d}\pi_1 \\ &= \frac{1}{P\left(\mathcal{E}_\tau^{\text{nc}}\right)} \int_0^\infty \int_0^\infty (b\pi_1 + (1-b)\pi_2)\;\, P\left(\Pi_1(\tau) = \pi_1, \Pi_2(\tau) = \pi_2\right) \\ &\times P\Big(\mathcal{E}_\tau^{\text{nc}} \;\Big|\; \Pi_1(\tau) = \pi_1, \Pi_2(\tau) = \pi_2\Big)\, \mathrm{d}\pi_2 \mathrm{d}\pi_1 \end{aligned} \tag{2.33}$$

by Bayes' theorem that $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$. If we write the explicit definition of $\mathcal{E}_\tau^{\text{nc}}$

given in (2.29), then we obtain

$$P\left(\mathcal{E}_\tau^{\mathrm{nc}}\right)T(\tau)=\int_0^\infty\int_0^\infty(b\pi_1+(1-b)\pi_2)\;P\left(\Pi_1(\tau)=\pi_1,\Pi_2(\tau)=\pi_2\right)P\Big[\gamma_2\Pi_1(1)\le\Pi_2(1)\le\gamma_1\Pi_1(1)$$
$$,\ldots,\gamma_2\Pi_1(\tau)\le\Pi_2(\tau)\le\gamma_1\Pi_1(\tau)\Big|\;\Pi_1(\tau)=\pi_1,\Pi_2(\tau)=\pi_2\Big]\,\mathrm{d}\pi_2\mathrm{d}\pi_1$$
$$=\int_0^\infty\int_{\gamma_2\pi_1}^{\gamma_1\pi_1}(b\pi_1+(1-b)\pi_2)\;P(\Pi_1(\tau)=\pi_1,\Pi_2(\tau)=\pi_2)$$
$$\times P\Big[\gamma_2\frac{\pi_1}{\prod_{t=2}^\tau x_1(t)}\le\frac{\pi_2}{\prod_{t=2}^\tau x_2(t)}\le\gamma_1\frac{\pi_1}{\prod_{t=2}^\tau x_1(t)},\gamma_2\frac{\pi_1}{\prod_{t=3}^\tau x_1(t)}\le\frac{\pi_2}{\prod_{t=3}^\tau x_2(t)}\le\gamma_1\frac{\pi_1}{\prod_{t=3}^\tau x_1(t)}$$
$$,\ldots,\gamma_2\frac{\pi_1}{x_1(\tau)}\le\frac{\pi_2}{x_2(\tau)}\le\gamma_1\frac{\pi_1}{x_1(\tau)}\Big]\,\mathrm{d}\pi_2\mathrm{d}\pi_1 \tag{2.34}$$

where (2.34) follows by the definitions of $\Pi_1(i)$ and $\Pi_2(i)$, i.e., $\Pi_1(i)=\prod_{t=1}^i x_1(t)=\frac{\Pi_1(\tau)}{\prod_{t=i+1}^\tau x_1(t)}$ and $\Pi_2(i)=\prod_{t=1}^i x_2(t)=\frac{\Pi_2(\tau)}{\prod_{t=i+1}^\tau x_2(t)}$. If we rearrange the inequalities in (2.34) to put the product terms together, which does not affect the direction of the inequality since all terms are positive, then we obtain

$$P\left(\mathcal{E}_\tau^{\mathrm{nc}}\right)T(\tau)=\int_0^\infty\int_{\gamma_2\pi_1}^{\gamma_1\pi_1}(b\pi_1+(1-b)\pi_2)\;P(\Pi_1(\tau)=\pi_1,\Pi_2(\tau)=\pi_2)P\Big[\frac{\pi_2}{\pi_1\gamma_1}\le\prod_{t=2}^\tau\frac{x_2(t)}{x_1(t)}\le\frac{\pi_2}{\pi_1\gamma_2},$$
$$\frac{\pi_2}{\pi_1\gamma_1}\le\prod_{t=3}^\tau\frac{x_2(t)}{x_1(t)}\le\frac{\pi_2}{\pi_1\gamma_2},\ldots,\frac{\pi_2}{\pi_1\gamma_1}\le\frac{x_2(\tau)}{x_1(\tau)}\le\frac{\pi_2}{\pi_1\gamma_2}\Big]\,\mathrm{d}\pi_2\mathrm{d}\pi_1$$
$$=\int_0^\infty\int_{\gamma_2\pi_1}^{\gamma_1\pi_1}(b\pi_1+(1-b)\pi_2)\;P(\Pi_1(\tau)=\pi_1,\Pi_2(\tau)=\pi_2)P\Big(\Sigma_2^\tau\in[\kappa-\theta_1,\kappa-\theta_2],\Sigma_3^\tau\in[\kappa-\theta_1,\kappa-\theta_2],$$
$$\ldots,\Sigma_\tau^\tau\in[\kappa-\theta_1,\kappa-\theta_2]\Big)\,\mathrm{d}\pi_2\mathrm{d}\pi_1, \tag{2.35}$$

which follows from the definition of Σ_i^k where $\kappa\stackrel{\triangle}{=}\ln\frac{\pi_2}{\pi_1}$. The first probability in (2.35) can be calculated as

$$P\left(\Pi_1(\tau)=\pi_1,\Pi_2(\tau)=\pi_2\right)=P\left(\Pi_1(\tau)=\pi_1\right)P\left(\Pi_2(\tau)=\pi_2\right)$$
$$=\frac{1}{\pi_1\sqrt{2\pi\,\tau\sigma_1^2}}e^{-\frac{(\ln\pi_1-\tau\mu_1)^2}{2\,\tau\sigma_1^2}}+\frac{1}{\pi_1\sqrt{2\pi\,\tau\sigma_2^2}}e^{-\frac{(\ln\pi_2-\tau\mu_2)^2}{2\,\tau\sigma_2^2}} \tag{2.36}$$

which follows since $\Pi_1(\tau)\stackrel{\triangle}{=}\prod_{t=1}^\tau x_1(t)$ and $\Pi_2(\tau)\stackrel{\triangle}{=}\prod_{t=1}^\tau x_2(t)$, we have $\Pi_1(\tau)\sim\ln\mathcal{N}(\tau\mu_1,\tau\sigma_1^2)$ and $\Pi_2(\tau)\sim\ln\mathcal{N}(\tau\mu_2,\tau\sigma_2^2)$. Note that the corresponding terms in (2.35) is written as a multi variable integral calculated in Section 2.2.3.

Following similar steps, we can obtain the expected overall gain $R(\tau)$ as

$$\begin{aligned} R(\tau) &= E\left[S(\tau) \;\middle|\; \mathcal{E}_\tau^{\text{fc}}\right] \\ &= E\left[b\prod_{t=1}^{\tau}[x_1(t)] + (1-b)\prod_{t=1}^{\tau}[x_2(t)] - 2c(b-b^2)|\prod_{t=1}^{\tau}[x_1(t)] - \prod_{t=1}^{\tau}[x_2(t)]| \;\middle|\; \mathcal{E}_\tau^{\text{fc}}\right]. \end{aligned} \tag{2.37}$$

The conditional expectation $E\left[S(\tau) \;\middle|\; \mathcal{E}_\tau^{\text{fc}}\right]$ can also be expressed in an integral form as

$$E\left[S(\tau) \;\middle|\; \mathcal{E}_\tau^{\text{fc}}\right] = \int_0^\infty \int_0^\infty S(\tau)\; P\left(\Pi_1(\tau) = \pi_1, \Pi_2(\tau) = \pi_2 \;\middle|\; \mathcal{E}_\tau^{\text{fc}}\right)\, \mathrm{d}\pi_2 \mathrm{d}\pi_1, \tag{2.38}$$

which follows from the definition of conditional expectation. Combining (2.38) and (2.37) yields

$$\begin{aligned} R(\tau) &= \int_0^\infty \int_0^\infty S(\tau)\; P\left(\Pi_1(\tau) = \pi_1, \Pi_2(\tau) = \pi_2 \;\middle|\; \mathcal{E}_\tau^{\text{fc}}\right) \mathrm{d}\pi_2 \mathrm{d}\pi_1 \\ &= \frac{1}{P\left(\mathcal{E}_\tau^{\text{fc}}\right)} \int_0^\infty \int_0^\infty S(\tau)\; P\left(\Pi_1(\tau) = \pi_1, \Pi_2(\tau) = \pi_2\right) \\ &\times P\left(\mathcal{E}_\tau^{\text{fc}} \;\middle|\; \Pi_1(\tau) = \pi_1, \Pi_2(\tau) = \pi_2\right) \mathrm{d}\pi_2 \mathrm{d}\pi_1, \end{aligned} \tag{2.39}$$

where (2.39) follows from the Bayes' theorem. Note that the definition of $R(\tau)$ (2.39) can be extended for the case $m > 2$ by replacing the double integral with an m-dimensional integral over the updated random variables $\Pi_1, \Pi_2, \ldots, \Pi_m$. If we replace the event $\mathcal{E}_\tau^{\text{fc}}$ with its explicit definition in (2.30), then we get

$$\begin{aligned} P\left(\mathcal{E}_\tau^{\text{fc}}\right) R(\tau) &= \int_0^\infty \int_0^\infty (\zeta_1\pi_1 + \zeta_2\pi_2)\; P\left(\Pi_1(\tau) = \pi_1, \Pi_2(\tau) = \pi_2\right) P\Big[\gamma_2\Pi_1(1) \le \Pi_2(1) \le \gamma_1\Pi_1(1), \ldots, \\ &\gamma_2\Pi_1(\tau-1) \le \Pi_2(\tau-1) \le \gamma_1\Pi_1(\tau-1), \gamma_1\Pi_1(\tau) \le \Pi_2(\tau) \;\Big|\; \Pi_1(\tau) = \pi_1, \Pi_2(\tau) = \pi_2\Big] \mathrm{d}\pi_2\mathrm{d}\pi_1 \\ &+ \int_0^\infty \int_0^\infty (\zeta_3\pi_1 + \zeta_4\pi_2)\, P\left(\Pi_1(\tau) = \pi_1, \Pi_2(\tau) = \pi_2\right) P\Big[\gamma_2\Pi_1(1) \le \Pi_2(1) \le \gamma_1\Pi_1(1), \ldots, \\ &\gamma_2\Pi_1(\tau-1) \le \Pi_2(\tau-1) \le \gamma_1\Pi_1(\tau-1), \gamma_2\Pi_1(\tau) \ge \Pi_2(\tau) \;\Big|\; \Pi_1(\tau) = \pi_1, \Pi_2(\tau) = \pi_2\Big] \mathrm{d}\pi_2\mathrm{d}\pi_1, \end{aligned} \tag{2.40}$$

where $\zeta_1 \stackrel{\triangle}{=} b - 2c(b-b^2)$, $\zeta_2 = 1-b+2c(b-b^2)$, $\zeta_3 = b+2c(b-b^2)$ and $\zeta_4 = 1-b-2c(b-b^2)$. We next calculate the first integral in (2.40) and the second integral follows similarly.

By the definitions of $\Pi_1(i)$ and $\Pi_2(i)$, we have $\Pi_1(i) = \prod_{t=1}^{i} x_1(t) = \frac{\Pi_1(\tau)}{\prod_{t=i+1}^{\tau} x_1(t)}$ and $\Pi_2(i) = \prod_{t=1}^{i} x_2(t) = \frac{\Pi_2(\tau)}{\prod_{t=i+1}^{\tau} x_2(t)}$, hence the first integral in (2.40) can be written as

$$\int_0^\infty \int_{\gamma_1\pi_1}^\infty (\zeta_1\pi_1 + \zeta_2\pi_2)\, P(\Pi_1(\tau)=\pi_1, \Pi_2(\tau)=\pi_2) P\Big[\gamma_2 \frac{\pi_1}{\prod_{t=2}^{\tau} x_1(t)} \leq \frac{\pi_2}{\prod_{t=2}^{\tau} x_2(t)} \leq \gamma_1 \frac{\pi_1}{\prod_{t=2}^{\tau} x_1(t)},$$
$$\gamma_2 \frac{\pi_1}{\prod_{t=3}^{\tau} x_1(t)} \leq \frac{\pi_2}{\prod_{t=3}^{\tau} x_2(t)} \leq \gamma_1 \frac{\pi_1}{\prod_{t=3}^{\tau} x_1(t)}, \ldots, \gamma_2 \frac{\pi_1}{x_1(\tau)} \leq \frac{\pi_2}{x_2(\tau)} \leq \gamma_1 \frac{\pi_1}{x_1(\tau)}\Big]\, \mathrm{d}\pi_2 \mathrm{d}\pi_1. \tag{2.41}$$

If we gather the product terms in (2.41) into the same fraction, then we obtain

$$\int_0^\infty \int_{\gamma_1\pi_1}^\infty (\zeta_1\pi_1 + \zeta_2\pi_2)\;\; P(\Pi_1(\tau)=\pi_1, \Pi_2(\tau)=\pi_2) P\Big[\frac{\pi_2}{\pi_1\gamma_1} \leq \prod_{t=2}^{\tau} \frac{x_2(t)}{x_1(t)} \leq \frac{\pi_2}{\pi_1\gamma_2},$$
$$\frac{\pi_2}{\pi_1\gamma_1} \leq \prod_{t=3}^{\tau} \frac{x_2(t)}{x_1(t)} \leq \frac{\pi_2}{\pi_1\gamma_2}, \ldots, \frac{\pi_2}{\pi_1\gamma_1} \leq \frac{x_2(\tau)}{x_1(\tau)} \leq \frac{\pi_2}{\pi_1\gamma_2}\Big]\, \mathrm{d}\pi_2 \mathrm{d}\pi_1 \tag{2.42}$$
$$= \int_0^\infty \int_{\gamma_1\pi_1}^\infty (\zeta_1\pi_1 + \zeta_2\pi_2)\;\; P(\Pi_1(\tau)=\pi_1, \Pi_2(\tau)=\pi_2) P\Big(\Sigma_2^\tau \in [\kappa-\theta_1, \kappa-\theta_2], \Sigma_3^\tau \in [\kappa-\theta_1, \kappa-\theta_2],$$
$$\ldots, \Sigma_\tau^\tau \in [\kappa-\theta_1, \kappa-\theta_2]\Big)\, \mathrm{d}\pi_2 \mathrm{d}\pi_1, \tag{2.43}$$

which follows from the definition of Σ_i^k where $\kappa \stackrel{\triangle}{=} \ln \frac{\pi_2}{\pi_1}$. Following similar steps that yields (2.43), we can calculate (2.40) as

$$\begin{aligned} P\left(\mathcal{E}_\tau^{\mathrm{fc}}\right) R(\tau) = & \int_0^\infty \int_{\gamma_1\pi_1}^\infty (\zeta_1\pi_1 + \zeta_2\pi_2)\;\; P(\Pi_1(\tau)=\pi_1, \Pi_2(\tau)=\pi_2) P\Big(\Sigma_2^\tau \in [\kappa-\theta_1, \kappa-\theta_2], \\ & \Sigma_3^\tau \in [\kappa-\theta_1, \kappa-\theta_2], \ldots, \Sigma_\tau^\tau \in [\kappa-\theta_1, \kappa-\theta_2]\Big)\, \mathrm{d}\pi_2 \mathrm{d}\pi_1 \\ & + \int_0^\infty \int_0^{\gamma_2\pi_1} (\zeta_3\pi_1 + \zeta_4\pi_2)\;\; P(\Pi_1(\tau)=\pi_1, \Pi_2(\tau)=\pi_2) P\Big(\Sigma_2^\tau \in [\kappa-\theta_1, \kappa-\theta_2], \\ & \Sigma_3^\tau \in [\kappa-\theta_1, \kappa-\theta_2], \ldots, \Sigma_\tau^\tau \in [\kappa-\theta_1, \kappa-\theta_2]\Big)\, \mathrm{d}\pi_2 \mathrm{d}\pi_1, \end{aligned} \tag{2.44}$$

where the probability $P(\Pi_1(\tau)=\pi_1, \Pi_2(\tau)=\pi_2)$ can be obtained via (2.36). Hence to calculate $P\left(\mathcal{E}_\tau^{\mathrm{nc}}\right) T(\tau)$ and $P\left(\mathcal{E}_\tau^{\mathrm{fc}}\right) R(\tau)$, we need to calculate the probability $P\Big(\Sigma_2^\tau \in [\kappa-\theta_1, \kappa-\theta_2], \Sigma_3^\tau \in [\kappa-\theta_1, \kappa-\theta_2], \ldots, \Sigma_\tau^\tau \in [\kappa-\theta_1, \kappa-\theta_2]\Big)$ in (2.35) and (2.44).

Following from the definition of Σ_i^ks, we have

$$\begin{aligned} p(\Sigma_i^k, \Sigma_{i+1}^k, \Sigma_j^k) &= p(\Sigma_{i+1}^k, \Sigma_j^k) p(\Sigma_i^k | \Sigma_{i+1}^k, \Sigma_j^k) \\ &= p(\Sigma_j^k) p(\Sigma_{i+1}^k | \Sigma_j^k) p(\Sigma_{i+1}^k + z(i) | \Sigma_{i+1}^k, \Sigma_j^k) \\ &= p(\Sigma_j^k) p(\Sigma_{i+1}^k | \Sigma_j^k) p(\Sigma_i^k | \Sigma_{i+1}^k) \end{aligned} \tag{2.45}$$

$\forall i \in \{0, 1, \dots, k-2\}$, where (2.45) follows since $z(i)$ is independent of Σ_j^k for $j > i$. Then, Σ_i^k's form a Markov chain such that $\Sigma_j^k \leftrightarrow \Sigma_{i+1}^k \leftrightarrow \Sigma_i^k$ $\forall i \in \{0, 1, \dots, k-2\}$ and $j > i$. Hence, we can write the probability

$$\begin{aligned} & P\Big(\Sigma_2^\tau \in [\kappa - \theta_1, \kappa - \theta_2], \Sigma_3^\tau \in [\kappa - \theta_1, \kappa - \theta_2], \dots, \Sigma_\tau^\tau \in [\kappa - \theta_1, \kappa - \theta_2]\Big) \\ &= \int_{\kappa-\theta_1}^{\kappa-\theta_2} \int_{\kappa-\theta_1}^{\kappa-\theta_2} \dots \int_{\kappa-\theta_1}^{\kappa-\theta_2} P(\Sigma_\tau^\tau = s_1, \Sigma_{\tau-1}^\tau = s_2, \dots, \Sigma_2^\tau = s_{\tau-1}) \, \mathrm{d}s_{\tau-1} \mathrm{d}s_{\tau-2} \dots \mathrm{d}s_1 \\ &= \int_{\kappa-\theta_1}^{\kappa-\theta_2} \int_{\kappa-\theta_1}^{\kappa-\theta_2} \dots \int_{\kappa-\theta_1}^{\kappa-\theta_2} P(\Sigma_2^\tau = s_{\tau-1} | \Sigma_3^\tau = s_{\tau-2}) P(\Sigma_3^\tau = s_{\tau-2} | \Sigma_4^\tau = s_{\tau-3}) \dots \\ & P(\Sigma_{\tau-1}^\tau = s_2 | \Sigma_\tau^\tau = s_1) P(\Sigma_\tau^\tau = s_1) \, \mathrm{d}s_{\tau-1} \mathrm{d}s_{\tau-3} \dots \mathrm{d}s_2 \mathrm{d}s_1, \end{aligned} \tag{2.46}$$

where (2.46) follows by the chain rule and Σ_i's form a Markov chain. We can express the conditional probabilities in (2.46), which are of the form $P(\Sigma_i^\tau = s_{\tau-i} | \Sigma_{i+1}^\tau = s_{\tau-i-1})$, as

$$\begin{aligned} P(\Sigma_i^\tau = s_{\tau-i+1} | \Sigma_{i+1}^\tau = s_{\tau-i}) &= P(\Sigma_{i+1}^\tau + z(i) = s_{\tau-i+1} | \Sigma_{i+1}^\tau = s_{\tau-i}) \\ &= P(s_{\tau-i} + z(i) = s_{\tau-i+1} | \Sigma_{i+1}^\tau = s_{\tau-i}) \\ &= P(z(i) = s_{\tau-i+1} - s_{\tau-i} | \Sigma_{i+1}^\tau = s_{\tau-i}) \\ &= P(z(i) = s_{\tau-i+1} - s_{\tau-i}) \end{aligned} \tag{2.47}$$

where (2.47) follows from the independence of $z(i)$ and $z(k)$'s for $i < k \le \tau$ or the independence of $z(i)$ and $\Sigma_{i+1}^\tau = \sum_{k=i+1}^{\tau} z(k)$. If we replace (2.47) with the conditional probabilities in (2.46) and use $P(\Sigma_\tau^\tau = s_1) = P(z(\tau) = s_1)$, then we obtain

$$P\Big(\Sigma_2^\tau \in [\kappa-\theta_1, \kappa-\theta_2], \Sigma_3^\tau \in [\kappa-\theta_1, \kappa-\theta_2], \ldots, \Sigma_\tau^\tau \in [\kappa-\theta_1, \kappa-\theta_2]\Big)$$
$$= \int_{\kappa-\theta_1}^{\kappa-\theta_2} \int_{\kappa-\theta_1}^{\kappa-\theta_2} \cdots \int_{\kappa-\theta_1}^{\kappa-\theta_2} f_z(s_{\tau-1} - s_{\tau-2}) f_z(s_{\tau-2} - s_{\tau-3}) \ldots f_z(s_2 - s_1) f_z(s_1)\, \mathrm{d}s_{\tau-1}\mathrm{d}s_{\tau-2} \ldots \mathrm{d}s_2\mathrm{d}s_1$$
$$= \int_{\kappa-\theta_1}^{\kappa-\theta_2} \int_{\kappa-\theta_1}^{\kappa-\theta_2} \cdots \int_{\kappa-\theta_1}^{\kappa-\theta_2} (\frac{1}{2\pi\sigma^2})^{\frac{\tau-1}{2}} e^{\frac{-1}{2\sigma^2}\sum_{i=2}^{\tau-1}(s_i - s_{i-1}-\mu)^2 + (s_1-\mu)^2}\, \mathrm{d}s_{\tau-1}\mathrm{d}s_{\tau-2} \ldots \mathrm{d}s_2\mathrm{d}s_1, \tag{2.48}$$

where (2.48) follows since $z(i)$'s are Gaussian, $z \sim \mathcal{N}(\mu, \sigma^2)$, i.e., $f_z(.)$ is the normal distribution. Hence in order to iteratively calculate the expected wealth growth of a TRP, we need to calculate the multivariate Gaussian integral given in (2.48), which is investigated in the next section.

2.2.3 Multivariate Gaussian Integrals

In order to complete calculation of the iterative equation in (2.12), we next evaluate the definite multivariate Gaussian integral given in (2.48) on the multidimensional $[\kappa-\theta_1, \kappa-\theta_2]^n$ space. We emphasize that the corresponding multivariate integral cannot be calculated using common diagonalizing methods [42]. Although, in (2.48), the coefficient matrix of the multivariate integral is symmetric positive-definite, common diagonalizing methods cannot be directly applied since the integral bounds after a straightforward change of variables dependent on y_i. However, (2.48) can be represented as certain error functions of Gaussian distributions.

We note that the multivariate Gaussian integral given in (2.48) is the "non-central multivariate normal integral" or non-central MVN integral [18] and general MVN integrals are in the form [18]

$$\Phi_k(\mathbf{a}, \mathbf{b}, \mathbf{\Sigma}) = \frac{1}{\sqrt{|\mathbf{\Sigma}|(2\pi)^k}} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_k}^{b_k} e^{\frac{-1}{2}\mathbf{x}^T\mathbf{\Sigma}^{-1}\mathbf{x}}\, \mathrm{d}x_k \ldots \mathrm{d}x_2\mathrm{d}x_1, \tag{2.49}$$

where $\mathbf{\Sigma}$ is a symmetric, positive definite covariance matrix. In our case, (2.48) is a non-central MVN integral which can be written of the form (2.49) where $k = \tau - 1$ and the inverse of the covariance matrix is given by

A Pseudo-code of QMC Algorithm for MVN Integrals:
1. get $\mathbf{\Sigma}$, $\mathbf{a}$, $\mathbf{b}$, N, M and α
2. compute lower triangular Cholesky factor L for $\mathbf{\Sigma}$, permuting $\mathbf{a}$ and $\mathbf{b}$, and rows and columns of $\mathbf{\Sigma}$ for variable prioritization.
3. initialize $P = 0$, $N = 0$, $V = 0$, and $q = \sqrt{p}$ with $\mathbf{p} = (2, 3, 5, \dots, p_k)$ where p_j is the j-th prime.
4. for $i = 1, 2, \dots, M$ do
$I_i = 0$ and generate uniform random $\mathbf{\Delta} \in [0, 1]^k$ shift vector.
for $j = 1, 2, \dots, N$ do
$\mathbf{w} = \lvert 2(j\mathbf{q} + \mathbf{\Delta}) - \mathbf{1} \rvert$,
$d_1 = \Phi\left(\frac{a_1}{l_{1,1}}\right)$, $e_1 = \Phi\left(\frac{b_1}{l_{1,1}}\right)$ and $f_1 = e_1 - d_1$.
for $m = 2, 3, \dots, k$ do
$y_{m-1} = \Phi^{-1}(d_{m-1} + w_{m-1}(e_{m-1} - d_{m-1}))$,
$d_m = \Phi\left(\frac{a_m - \sum_{n=1}^{m-1} l_{m,n} y_j}{l_{m,m}}\right)$,
$e_m = \Phi\left(\frac{b_m - \sum_{n=1}^{m-1} l_{m,n} y_j}{l_{m,m}}\right)$,
$f_m = (e_m - d_m) f_{m-1}$.
endfor
$I_i = I_i + (f_m - I_i)/j$.
endfor
$\sigma = (I_i - t)/i$, $P = P + \sigma$, $V = (i-2)V/i + \sigma^2$ and $E = \alpha\sqrt{V}$
endfor
5. output $P \approx \Phi_k(\mathbf{a}, \mathbf{b}, \mathbf{\Sigma})$ with error estimate E.

Figure 2.3: A randomized QMC algorithm proposed in [18] to compute MVN probabilities for hyper-rectangular regions.

$$\mathbf{\Sigma}^{-1} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

which is a symmetric positive definite matrix with $|\mathbf{\Sigma}| = 1$, the lower bound vector is of the form

$$\mathbf{a} = \begin{bmatrix} \kappa - \theta_1 - \mu \\ \kappa - \theta_1 - 2\mu \\ \vdots \\ \kappa - \theta_1 - (\tau - 1)\mu \end{bmatrix}$$

and the upper bound vector is given by

$$\mathbf{b} = \begin{bmatrix} \kappa - \theta_2 - \mu \\ \kappa - \theta_2 - 2\mu \\ \vdots \\ \kappa - \theta_2 - (\tau - 1)\mu \end{bmatrix}$$

where $-k\mu$ terms in the lower and the upper bounds follow from the non-central property of (2.48). We emphasize that the MVN integral in (2.49) cannot be calculated in a closed form [18] and most of the results on this integral correspond to either special cases or coarse approximations [7,18]. Hence, in this part, we use the randomized QMC algorithm, provided in Fig. 2.3 [18] for completeness, to compute MVN probabilities over hyper rectangular regions. Here, the algorithm uses a periodized and randomized QMC rule [37] where the output error estimate E in Fig. 2.3 is the usual Monte Carlo standard error based on N samples of the randomly shifted QMC rule, and scaled by the confidence factor α. We observe in our simulations that the algorithm in Fig. 2.3 produce satisfactory results on the historical data [27]. We emphasize that different algorithms can be used instead of the Quasi-Monte Carlo (QMC) algorithm to calculate the multivariable integrals in (2.48), however, the derivations still hold.

2.3 Maximum-Likelihood Estimation of Parameters of the Log-Normal Distribution

In this section, we give the MLEs for the mean and variance of the log-normal distribution using the sequence of price relative vectors, which are used sequentially in the Simulations section to evaluate the optimal TRPs. Since the investor observes the sequence of price relatives sequentially, he or she needs to estimate $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ at each investment period to find the maximizing b and ϵ. Without loss of generality we provide the MLE for $x_1(t)$, where the MLE for $x_2(t)$ directly follows.

For these derivations, we assume that we observed a sequence of price relative vectors of length N, i.e., $(x_1(1), x_1(2), \ldots, x_1(N))$. Note that the sample data need not to belong to N consecutive periods such that the sequential representation is chosen for ease of presentation.

Then, we find the parameters μ_1 and σ_1^2 that maximize the log-likelihood function

$$\ln \mathcal{L}(\mu_1, \sigma_1^2 \,|\, x_1(1), x_1(2), \ldots, x_1(N)) = \ln f(x_1(1), x_1(2), \ldots, x_1(N) \,|\, \mu_1, \sigma_1^2) = \sum_{i=1}^{N} \ln f(x_1(i) \,|\, \mu_1, \sigma_1),$$

where $f(x|\mu_1, \sigma_1^2) = \frac{1}{x\sqrt{2\pi{\sigma_1}^2}} e^{-\frac{(\ln x - \mu_1)^2}{2{\sigma_1}^2}}$. The log-likelihood function in (2.50) can also be written as

$$\begin{aligned} \ln \mathcal{L}(\mu_1, \sigma_1^2 \,|\, x_1(1), x_1(2), \ldots, x_1(N)) &= \sum_{i=1}^{N} \ln \frac{1}{x_1(i)\sqrt{2\pi{\sigma_1}^2}} e^{-\frac{(\ln x_1(i) - \mu_1)^2}{2{\sigma_1}^2}} \\ &= \sum_{i=1}^{N} \ln \frac{1}{x_1(i)\sqrt{2\pi{\sigma_1}^2}} - \sum_{i=1}^{N} \frac{(\ln x_1(i) - \mu_1)^2}{2{\sigma_1}^2}. \end{aligned} \quad (2.50)$$

We start with maximizing the log-likelihood function $\ln \mathcal{L}$ with respect to μ_1, i.e., find the estimator $\hat{\mu_1}$ that satisfies $\frac{\partial \ln \mathcal{L}}{\partial \mu_1} = 0$. If we take the partial derivative of the expression in (2.50) with respect to μ_1, then we obtain

$$\frac{\partial \ln \mathcal{L}}{\partial \mu_1} = \sum_{i=1}^{N} \frac{\ln x_1(i) - \mu_1}{{\sigma_1}^2}.$$

Hence μ_1, which satisfies $\frac{\partial \mathcal{L}}{\partial \mu_1} = 0$, or the ML estimator $\hat{\mu_1}$ of μ_1, can be found as

$$\hat{\mu_1} = \frac{1}{N} \sum_{i=1}^{N} \ln x_1(i). \quad (2.51)$$

To find the ML estimator of the variance σ_1^2, we find $\hat{\sigma_1^2}$ that satisfies $\frac{\partial \ln \mathcal{L}}{\partial \sigma_1^2} = 0$. Since μ_1 that satisfies $\frac{\partial \hat{l}}{\partial \mu_1} = 0$ in (2.51) does not depend on σ_1^2, we can use it in (2.50). Let us define $\bar{x_1} = \sum_{i=1}^{N} \frac{\ln x_1(i)}{N}$ for notational clarity. By replacing $\bar{x_1}$ with μ_1 in (2.51) and taking the partial derivative of the expression with respect to σ_1^2, we obtain

$$\frac{\partial \ln \mathcal{L}}{\partial \sigma_1^2} = -\frac{N}{2\sigma_1^2} + \frac{1}{2(\sigma_1^2)^2} \sum_{i=1}^{N} (\ln x_1(i) - \bar{x_1})^2.$$

Hence

$$\hat{\sigma_1^2} = \frac{1}{N}\sum_{i=1}^{N}(\ln x_1(i) - \bar{x_1})^2. \tag{2.52}$$

Following similar steps, the ML estimators for $x_2(t)$ yield

$$\hat{\mu_2} = \frac{1}{N}\sum_{i=1}^{N}\ln x_2(i), \tag{2.53}$$

and

$$\hat{\sigma_2^2} = \frac{1}{N}\sum_{i=1}^{N}(\ln x_2(i) - \bar{x_2})^2, \tag{2.54}$$

where $\bar{x_2} \triangleq \sum_{i=1}^{N}\frac{\ln x_2(i)}{N}$. Note that the ML estimators $\hat{\mu_1}$, $\hat{\sigma_1^2}$, $\hat{\mu_2}$ and $\hat{\sigma_2^2}$ are consistent [38], i.e., they converge to the true values as the size of the data set goes to infinity, i.e., $N \to \infty$ [40].

2.4 Simulations

In this section, we illustrate the performance our algorithm under different scenarios. We first use TRPs over simulated data of two stocks, where each stock is generated from a log-normal distribution. We then continue to test the performance over the historical "Ford - MEI Corporation" stock pair chosen for its volatility [12] from the New York Stock Exchange. As the final set of experiments, we use our algorithm over the historical data set from [10] and illustrate the average performance. In all these trials, we compare the performance of our algorithm with portfolio selection strategies from [13, 22, 27].

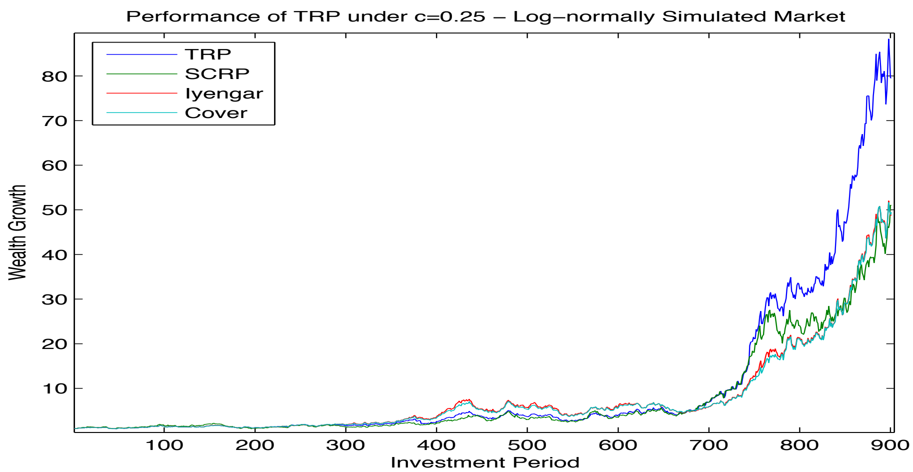
In the first example, each stock is generated from a log-normal distribution such that $x_1(t) \sim \ln\mathcal{N}(0.006, 0.05)$ and $x_2(t) \sim \ln\mathcal{N}(0.003, 0.05)$, where the mean and variance values are arbitrarily selected. We observe that the results do not depend on a particular choice of model parameters as long as they resemble real life markets. We simulate the performance over 1100 investment periods. Since the mean and variance parameters are not known by the investor, we use the ML estimators from Section 2.3, which are then used to determine the target portfolio b and the threshold value ϵ. We start by calculating the ML estimators using the initial 200 samples and find the target portfolio $\mathbf{b} = [b1 - b]$ and the threshold ϵ that maximize the expected wealth growth by a brute-force search. Then, we use the corresponding bb and ϵ during the following 200 samples. In similar lines, we calculate and use the optimal TRP for a total of 900 days, where b and are estimated over every window of 200 samples and used in the following window of 200 samples. We choose a window of size 200 samples to get reliable estimates for the means and variances based on the size of the overall data. In Fig. 2.4, we show the performances of: this sequential TRP algorithm "TRP", the Covers universal portfolio selection algorithm [13] "Cover", the Iyengars universal portfolio algorithm [22] "Iyengar" and a semiconstant rebalanced portfolio (SCRP) algorithm [27] "SCRP", where the parameters are chosen as suggested in [15]. As seen in Fig. 2.4, the TRP with the parameters sequentially calculated using the ML estimators is the best rebalancing strategy among the others as expected from our derivations. In Fig. 2.4b and Fig. 2.4a, we present results for a mild transaction cost $c = 0.01$ and a hefty transaction cost $c = 0.025$, respectively, where c is the fraction paid in commission for each transaction, i.e., $c = 0.01$ is a 1% commission. We observe that the performance of the TRP algorithm is better than the other algorithms for these transaction costs. However, the relative gain is larger for the large transaction cost since the TRP approach, with the optimal parameters chosen as in this part, can hedge more effectively against the transaction costs.

As the next example, we apply our algorithm to historical data from [13] from the New York Stock Exchange collected over a 22-year period. We first apply algorithms on the "Ford - MEI Corporation" pair as shown in Fig. 2.5, which are chosen because of their volatility [12]. In Fig. 2.5, we plot the wealth growth of: the sequential TRP algorithm with the optimal parameters sequentially calculated, the Covers universal portfolio, the Iyengars universal portfolio and the SCRP algorithm with the suggested parameters in [27]. We use the ML estimators to choose the optimal TRP as in the first set of experiments, however, since the historical data contains 5651 days we use a window of size 1000 days. Hence, the performance results are shown over 4651 days. As seen from Fig. 2.5, the proposed TRP algorithm significantly outperforms other algorithms for this data set. Similar to the simulated data case, we investigate the performance of the TRP algorithm under different transaction costs, i.e., a moderate transaction cost $c = 0.01$ in Fig. 2.5b and a hefty transaction cost $c = 0.025$ in Fig. 2.5a. Comparing the results from the Fig. 2.5a and Fig. 2.5b, we conclude that the TRP with the optimal sequential parameter selection can better handle the transaction costs when the stocks are volatile for this experiment.

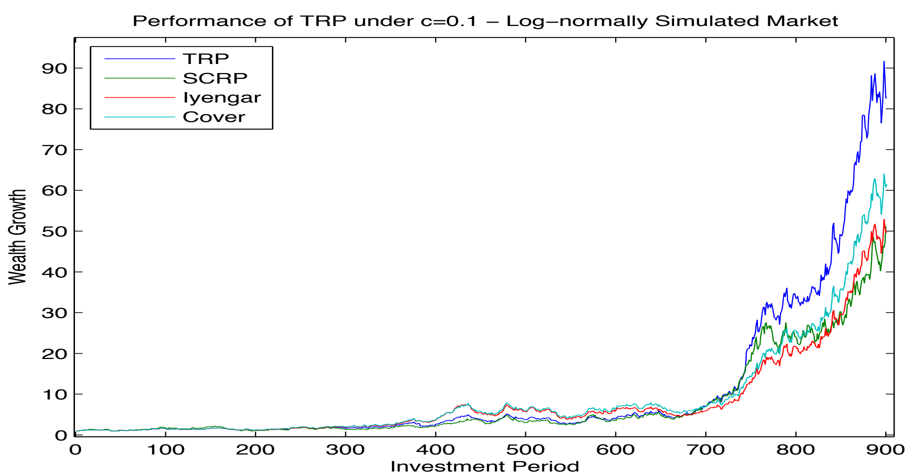
Finally, to remove any bias on a particular stock pair, we show the average performance of the TRP algorithm over randomly selected stock pairs from the historical data set from [13]. The total set includes 34 different stocks, where the Iroquois stock is removed due to its peculiar behavior. We first randomly select pairs of stocks and invest using: the sequential TRP algorithm with the sequential ML estimators, the Covers universal portfolio algorithm, the Iyengars universal portfolio algorithm and the SCRP algorithm. The sequential selection of the optimal TRP parameters are performed similar to the previous case, i.e., we use ML estimators on an investment block of 1000 days and use the calculated optimal TRP in the next block of 1000 days. For each stock pair, we simulate the performance of the algorithms over 4651 days. In Fig. 2.6, we present the wealth achieved by these algorithms, where the results are averaged over 10 independent trials. We present the achieved wealth over random sets of stock pairs under a moderate transaction cost $c = 0.01$ in Fig. 2.6b and a hefty transaction cost $c = 0.025$ in Fig. 2.6a. As seen from the figures, the TRP algorithm with the ML estimators readily outperforms the other strategies under different transaction costs on this historical data set.

2.5 Conclusions

In this chapter, we studied an important financial application, the portfolio selection problem, from a signal processing perspective. We investigated the portfolio selection problem in i.i.d. discrete time markets having a finite number of assets, when the market levies proportional transaction fees for both buying and selling stocks. We introduced algorithms based on threshold rebalanced portfolios that achieve the maximal growth rate when the sequence of price relatives have the log-normal distribution from the well-known Black-Scholes model [29]. Under this setup, we provide an iterative relation that efficiently and recursively calculates the expected wealth in any i.i.d. market over any investment period. The terms in this recursion are evaluated by a certain multivariate Gaussian integral. We then use a randomized algorithm to calculate the given integral and obtain the expected growth. This expected growth is then optimized by a brute force method to yield the optimal target portfolio and the threshold to maximize the expected wealth over any investment period. We also provide a maximum-likelihood estimator to estimate the parameters of the log-normal distribution from the sequence of price relative vectors. As predicted from our derivations, we significantly improve the achieved wealth over portfolio selection algorithms from the literature on the historical data set from [13].

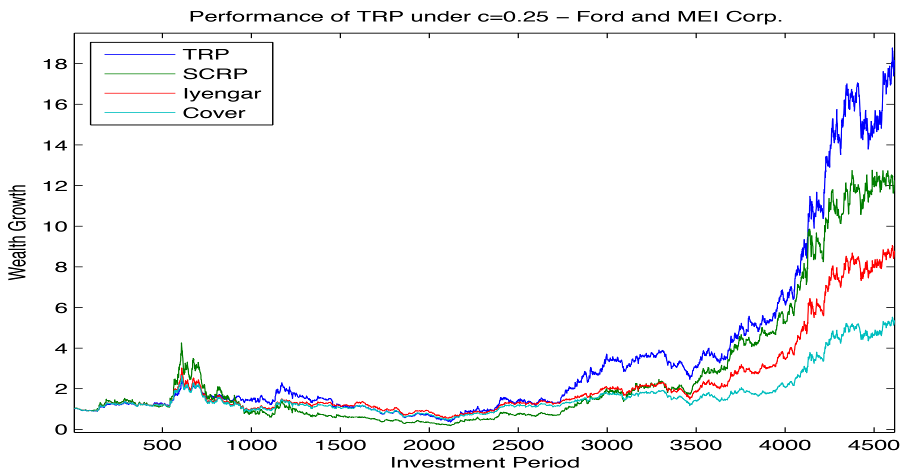


(a)

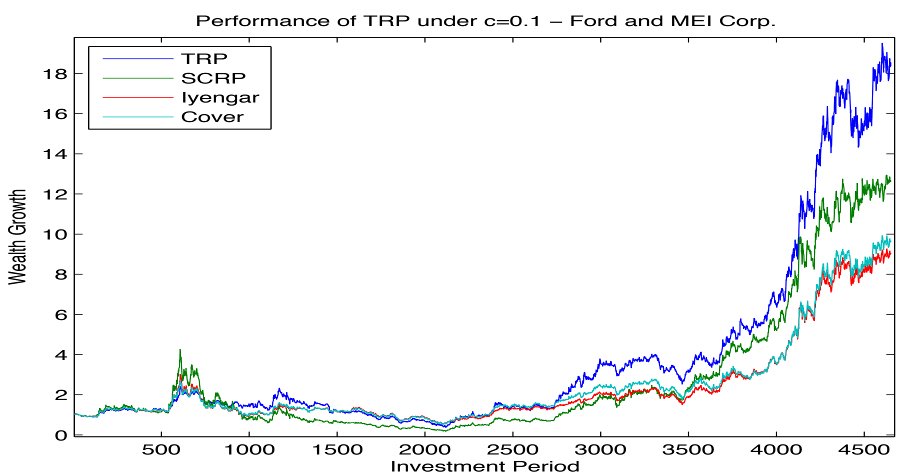


(b)

Figure 2.4: Performance of various portfolio investment algorithms on a Log-normally simulated two-stock market. (a) Wealth growth under hefty transaction cost (c=0.025). (b) Wealth growth under moderate transaction cost (c=0.01).

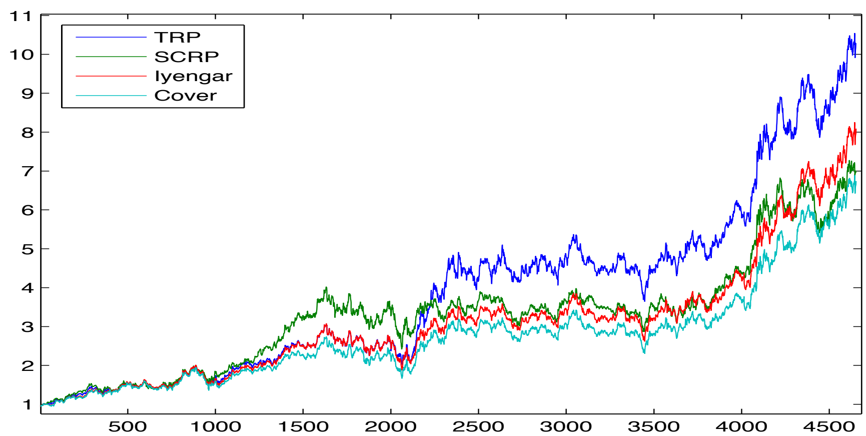


(a)

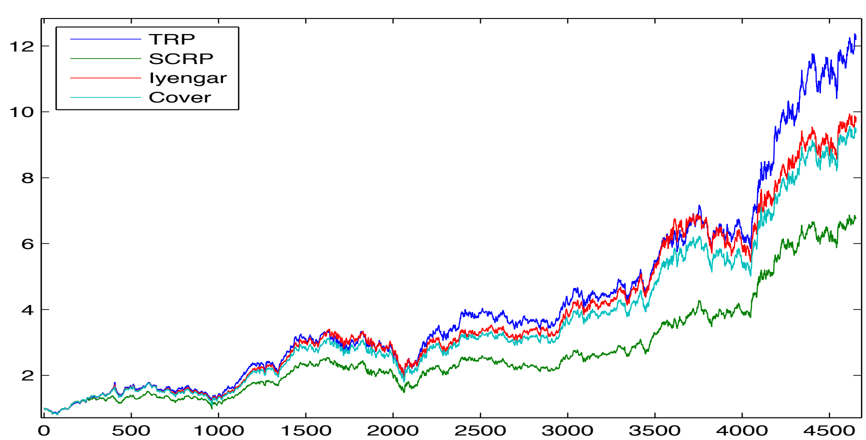


(b)

Figure 2.5: Performance of various portfolio investment algorithms on Ford - MEI Corporation pair. (a) Wealth growth under hefty transaction cost (c=0.025). (b) Wealth growth under moderate transaction cost (c=0.01).



(a)



(b)

Figure 2.6: Average performance of various portfolio investment algorithms on random stock pairs. (a) Wealth growth under hefty transaction cost (c=0.025). (b) Wealth growth under moderate transaction cost (c=0.01).

Chapter 3

GROWTH OPTIMAL PORTFOLIOS IN DISCRETE-TIME MARKETS UNDER TRANSACTION COSTS

In this chapter, we study investment problem in markets that allow trading at discrete periods, where the discrete period is arbitrary, e.g., it can be seconds, minutes or days [29]. Furthermore the market levies transaction fees for both selling and buying an asset proportional to the volume of trading at each transaction, which accurately models a broad range of financial markets [8, 29]. In our discussions, we first consider markets with two assets, i.e., two-asset markets. We emphasize that the two-stock markets are extensively studied in financial literature and are shown to accurately model a wide range of financial applications [29], e.g., the practically significant "Stock and Bond Market", where an investor holds a portfolio between a set of stocks and U.S. treasury bonds [12]. We then extend our analysis to markets having more than two assets, i.e., m-stock markets, where m is arbitrary but determined by the investor. Following the extensive literature [14, 27, 29, 32, 33, 44], the market is modelled by a sequence of price relative vectors, say $\{\boldsymbol{X}(n)\}_{n\geq 1}$, $\boldsymbol{X}(n) \in [0,\infty)^m$, where each entry of $\boldsymbol{X}(n)$, i.e., $X_i(n) \in [0,\infty)$, is the ratio of the closing price to the opening price of the ith stock per investment period. In this sense, each entry of $\boldsymbol{X}(n)$ quantifies the gain (or the loss) of that asset at each investment period. The sequence of price relative vectors is assumed to have an i.i.d. "discrete" distribution [29, 32, 33, 44], however, the discrete distributions on the vector of price relatives are arbitrary. In this sense, the corresponding discrete distributions can approximate a wide class of continuous distributions on the price relatives that satisfy certain regularity conditions by appropriately increasing the size of the discrete sample space. We first assume that we know the discrete distributions on the price relative vectors and then extend our analysis to cover when the underlying distributions are unknown. We emphasize that the i.i.d. assumption on the sequence of price relative vectors is shown to hold in most realistic markets [21, 29]. The detailed market model is provided in Section IV. At each investment period, the diversification of the capital over these assets is represented by a portfolio vector $\boldsymbol{b}(n)$, where $1 \geq b_i(n) \geq 0$, $\sum_{i=1}^{m} b_i(n) = 1$, and $b_i(n)$ is the ratio of the capital invested in the ith asset at investment period n. Note that if we invest using $\boldsymbol{b}(n)$, we earn (or loose) $\boldsymbol{b}^T(n)\boldsymbol{X}(n)$ at the investment period n after $\boldsymbol{X}(n)$ is revealed. Given that we start with one dollars, after an investment period of N days, we have a growth of wealth $\prod_{n=1}^{N} \boldsymbol{b}^T(n)\boldsymbol{X}(n)$. Under this general market model, we provide algorithms that maximize the expected growth over any period N by using "threshold rebalanced portfolios" (TRP)s, which are extensively used in Stock and Bond Markets [29] and are shown to yield optimal growth in general i.i.d. discrete-time markets [21].

Under mild assumptions on the sequence of price relatives and without any transaction costs, Cover et. al [14] showed that the portfolio that achieves the maximal growth is a constant rebalanced portfolio (CRP) in i.i.d. discrete-time markets. A CRP is a portfolio investment strategy where the fraction of wealth invested in each stock is kept constant at each investment period. A problem extensively studied in this framework is to find sequential portfolios that asymptotically achieve the wealth of the best CRP tuned to the underlying sequence of price relatives. This amounts to finding a daily trading strategy that has the ability to perform as well as the best asset diversified, constantly rebalanced portfolio. Several sequential algorithms are introduced that achieve the performance of the best CRP either with different convergence rates or performance on historical data sets [1, 14, 20, 24, 47]. Even under transaction costs, sequential algorithms are introduced that achieve the performance of the best CRP [8]. Nevertheless, we emphasize that keeping a CRP may require extensive trading due to possible rebalancing at each investment period deeming CRPs, or even the best CRP, ineffective in realistic markets even under mild transaction costs [27].

In continuous-time markets, however, it has been shown that under transaction costs, the optimal portfolios that achieve the maximal wealth are certain class of "no-trade zone" portfolios [11, 17, 41]. In simple terms, a no-trade zone portfolio has a compact closed set such that the rebalancing occurs if the current portfolio breaches this set, otherwise no rebalancing occurs. Clearly, such a no-trade zone portfolio may avoid hefty transaction costs since it can limit excessive rebalancing by defining appropriate no-trade zones. Analogous to continuous time markets, it has been shown in [21] that in two-asset i.i.d. markets under proportional transaction costs, compact no-trade zone portfolios are optimal such that they achieve the maximal growth under mild assumptions on the sequence of price relatives. In two-asset markets, the compact no trade zone is represented by thresholds, e.g., if at investment period n, the portfolio is given by $\boldsymbol{b}(n) = [b(n) \;\; (1 - b(n))]^T$, where $1 \geq b(n) \geq 0$, then rebalancing occurs if $b(n) \notin (\alpha, \beta)$, given the thresholds α, β, where $1 \geq \beta \geq \alpha \geq 0$. Similarly, the interval (α, β) can be represented using a target portfolio b and a region around it, i.e., $(b - \epsilon, b + \epsilon)$, where $\min\{b, 1 - b\} \geq \epsilon \geq 0$ such that $\alpha = b - \epsilon$ and $\beta = b + \epsilon$. Extension of TRPs to markets having more than two stocks is straightforward and explained in Section 3.2.2.

However, how to construct the no-trade zone portfolio, i.e., selecting the thresholds that achieve the maximal growth, has not yet been solved except in elementary scenarios [21]. We emphasize that a sequential universal algorithm that asymptotically achieves the performance of the best TRP specifically tuned to the underlying sequence of price relatives is introduced in [22]. This algorithm leverages Bayesian type weighting from [14] inspired from universal source coding and requires no statistical assumptions on the sequence of price relatives. In similar lines, various different universal sequential algorithms are introduced that achieve the performance of the best algorithm in different competition classes in [2, 5, 16, 25–28, 39]. However, we emphasize that the performance guarantees in [22] (and in [2, 5, 16, 25–27, 39]) on the performance, although without any stochastic assumptions, is given for the worst case sequence and only optimal in the asymptotic. For any finite investment period, the corresponding order terms in the upper bounds may not be negligible in financial markets, although they may be neglected in source coding applications (where these algorithms are inspired from). We demonstrate that our algorithm readily outperforms these universal algorithms over historical data [14], where similar observations are reported in [10, 28].

Our main contributions are as follows. We first consider two-asset markets and recursively evaluate the expected achieved wealth of a threshold portfolio for any b and ϵ over any investment period. We then extend this analysis to markets having more than two-stocks. We next demonstrate that under the threshold rebalancing framework, the achievable set of portfolios form an irreducible Markov chain under mild technical conditions. We evaluate the corresponding stationary distribution of this Markov chain, which provides a natural and efficient method to calculate the cumulative expected wealth. Subsequently, the corresponding parameters are optimized using a brute force approach yielding *the growth optimal investment portfolio under proportional transaction costs in i.i.d. discrete-time* two-asset markets. We note that for the case with irreducible Markov chain, which covers practically all scenarios, the optimization of the parameters is offline and carried out only once. However, for the case with recursive calculations, the algorithm requires an exponential computational complexity in terms of number of states. However, in our simulations, we observe that a reduced complexity form of the recursive algorithm that keeps only a constant number of states by appropriately pruning provides nearly identical results to the "optimal" algorithm. Furthermore, as a well studied problem, we also solve optimal portfolio selection in discrete-time markets constructed by sampling continuous-time Brownian markets. When the underlying discrete distributions of the price relative vectors are unknown, we provide a maximum likelihood estimator to estimate the corresponding distributions that is incorporated in the optimization framework in the Simulations section. For all these approaches, we also provide the corresponding complexity bounds.

3.1 Problem Description

We consider discrete-time stock markets under transaction costs. We first consider a market with two stocks and then extend the analysis to markets having more than two stock. We model the market using a sequence of price relative vectors $\mathbf{X}(n)$. A vector of price relatives $\mathbf{X}(n) = [X_1(n), \ldots, X_m(n)]^T$ represents the change in the prices of the assets over investment period n in a market with m assets, i.e., $X_i(n)$ is the ratio of the closing to the opening price of the ith stock over period n. For a market having two assets $\mathbf{X}(n) = [X_1(n)\, X_2(n)]^T$. We assume that the price relative sequences $X_1(n)$ and $X_2(n)$ are independent and identically distributed (i.i.d.) over with possibly different discrete sample spaces $\mathcal{X}_1$ and $\mathcal{X}_2$, i.e., $X_1(n) \in \mathcal{X}_1$ and $X_2(n) \in \mathcal{X}_2$, respectively [21]. For technical reasons, in our derivations, we assume that the sample space is $\mathcal{X} \triangleq \mathcal{X}_1 \cup \mathcal{X}_2 = \{x_1, x_2, \ldots, x_K\}$ for both $X_1(n)$ and $X_2(n)$ where $|\mathcal{X}| = K$ is the cardinality of the set $\mathcal{X}$. The probability mass function (pmf) of $X_1(n)$ is $p_1(x) \triangleq \Pr(X_1 = x)$ and the probability mass function of $X_2(n)$ is $p_2(x) \triangleq \Pr(X_2 = x)$. We define $p_{i,1} = p_1(x_i)$ and $p_{i,2} = p_2(x_i)$ for $x_i \in \mathcal{X}$ and the probability mass vectors $\mathbf{p}_1 = [p_{1,1}\ p_{2,1}\ \ldots\ p_{K,1}]^T$ and $\mathbf{p}_2 = [p_{1,2}\ p_{2,2}\ \ldots\ p_{K,2}]^T$, respectively. Here, we first assume that the corresponding probability mass vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ are known. We then extend our analysis where $\mathbf{p}_1$ and $\mathbf{p}_2$ are unknown and sequentially estimated using a maximum likelihood estimator in Section 3.3.

An allocation of wealth over two stocks is represented by the portfolio vector $\mathbf{b}(n) = [b(n)\ 1-b(n)]$, where $b(n)$ and $1-b(n)$ represents the proportion of wealth invested in the first and second stocks, respectively, for each investment period n. In two stock markets, the portfolio vector $\mathbf{b} = [b\ 1-b]$ is completely characterized by the proportion b of the total wealth invested in the first stock. For notational clarity, we use $b(n)$ to represent $b_1(n)$ throughout the chapter.

We denote a threshold rebalancing portfolio with an initial and target portfolio b and a threshold ϵ by TRP(b,ϵ). At each market period n, an investor rebalances the asset allocation only if the portfolio leaves the interval $(b-\epsilon, b+\epsilon)$. When $b(n) \notin (b-\epsilon, b+\epsilon)$, the investor buys and sells stocks so that the asset allocation is rebalanced to the initial allocation, i.e., $b(n) = b$, and he/she has to pay transaction fees. We emphasize that the rebalancing can be made directly to the closest boundary instead of to b as suggested in [21], however, we rebalance to b for notational simplicity and our derivations hold for that case also. We model transaction cost paid when rebalancing the asset allocation by a fixed proportional cost $c \in (0,1)$ [8, 21, 27]. For instance, if the investor buys or sells S dollars of stocks, then he/she pays cS dollars of transaction fees. Although we assume a symmetric transaction cost ratio, all the results can be carried over to markets with asymmetric costs [21, 27]. Let $S(N)$ denote the achieved wealth at investment period N and assume, without loss of generality, that the initial wealth of the investor is 1 dollars. For example, if the portfolio $b(n)$ does not leave the interval $(b-\epsilon, b+\epsilon)$ and the allocation of wealth is not rebalanced for N investment periods, then the current proportion of wealth invested in the first stock is given by

$$b(N) = \frac{b\prod_{n=1}^{N} X_1(n)}{b\prod_{n=1}^{N} X_1(n) + (1-b)\prod_{n=1}^{N} X_2(n)}$$

and achieved wealth is given by

$$S(N) = b\prod_{n=1}^{N} X_1(n) + (1-b)\prod_{n=1}^{N} X_2(n).$$

If the portfolio leaves the interval $(b-\epsilon, b+\epsilon)$ at period N, i.e., $b(N) \notin (b-\epsilon, b+\epsilon)$, then the investor rebalances the asset distribution to the initial distribution and pays approximately $S(N)|b(N) - b|c$ dollars for transaction costs [8].

In the next section, we first evaluate the expected achieved wealth $E[S(N)]$ so that we can optimize b and ϵ. We also analyze the number of calculations required to evaluate $E[S(N)]$, i.e., the complexity of the algorithm. We extend our results on expected achieved wealth to markets having more than two assets, i.e., m-asset markets. We then present conditions under which the set of all achievable portfolios has finite elements and derive the expected achieved wealth under these conditions. Finally, we consider the well-known Brownian market with two stocks and find the expected wealth growth [17,21] which is then optimized.

3.2 Threshold Rebalanced Portfolios

In this section, we investigate threshold rebalancing portfolios in discrete-time two-asset markets under proportional transaction costs. We first calculate the expected achieved wealth at a given investment period by an iterative algorithm. Then, we present an upper bound on the complexity of the algorithm. We also extend the expected achieved wealth calculations to markets having more than two assets, i.e., m-asset markets for an arbitrary m. We next give the necessary and sufficient conditions such that the achievable portfolios are finite at any investment period. This result is important when we calculate the expected achieved wealth since the complexity of the algorithm does not grow when the set of achievable portfolios is finite at any period. We also show that the portfolio sequence converges to a stationary distribution and derive the expected achieved wealth. Based on the calculation of the expected achieved wealth, we optimize b and ϵ using a brute-force search. Finally, with these derivations, we consider the well-known discrete-time two-asset Brownian market with proportional transaction costs and investigate the asymptotic expected achieved wealth to optimize b and ϵ.

3.2.1 An Iterative Algorithm

In this section, we calculate the expected wealth growth of a TRP with an iterative algorithm and find an upper bound on the complexity of the algorithm. To accomplish this, we first define the set of achievable portfolios at each investment period since the iterative

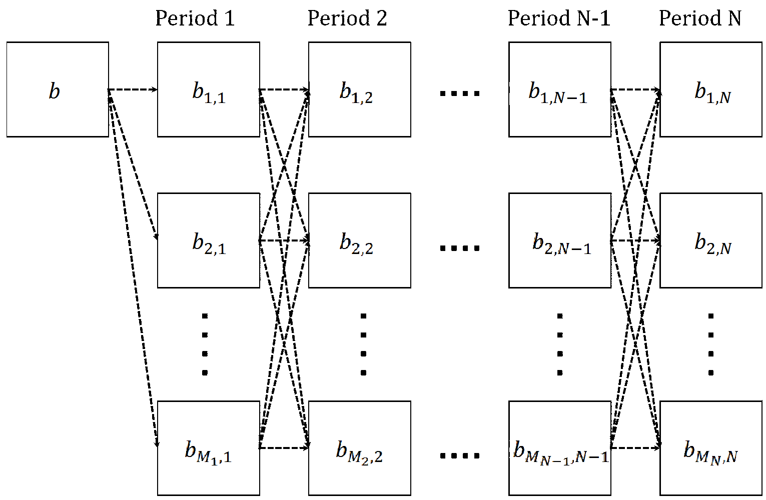


Figure 3.1: Block diagram representation of N period investment.

calculation of the expected achieved wealth is based on the achievable portfolio set. We next introduce the portfolio transition sets and the transition probabilities of achievable portfolios at successive investment periods in order to find the probability of each portfolio state iteratively. We evaluate the expected achieved wealth $E[S(N)]$ at a given investment period N based on the set of achievable portfolios, the transition probabilities and the set of price relative vectors connecting the portfolio states. We then optimize b and ϵ using a brute-force search.

We define the set of achievable portfolios at each investment period as follows. Since the sample space of the price relative sequences $X_1(n)$ and $X_2(n)$ is finite, i.e., $|\mathcal{X}| = K$, the set of achievable portfolios at period N can only have finitely many elements. We define the set of achievable portfolios at period N as $\mathcal{B}_N = \{b_{1,N}, \ldots, b_{M_N,N}\}$, where $M_N \triangleq |B_N|$ is the size of the set $\mathcal{B}_N$ for $N \geq 1$. As an example, we have

$$\mathcal{B}_1 = \left\{ b_{1,1}, \ldots, b_{M_1,1} \mid b_{l,1} = \frac{bu}{bu + (1-b)v} \in (b-\epsilon, b+\epsilon) \text{ or } b_{l,1} = b,\ u, v \in \mathcal{X} \right\}.$$

As illustrated in Fig. 3.1, for each achievable portfolio $b_{l,N} \in \mathcal{B}_N$, there is a certain set of portfolios in $\mathcal{B}_{N-1}$ that are connected to $b_{l,n}$, by definition of $b_{l,n}$. At a given investment period N, the set of achievable portfolios $\mathcal{B}_N$ is given by

$$\mathcal{B}_N = \left\{ b_{1,N}, \ldots, b_{M_N,N} \mid b_{l,N} = \frac{b_{k,N-1}u}{b_{k,N-1}u + (1-b_{k,N-1})v} \in (b-\epsilon, b+\epsilon) \text{ or } b_{l,N} = b,\ u,v \in \mathcal{X} \right\}.$$

We let, without loss of generality, $b_{1,N} = b$ for each $N \in \mathbb{N}$. Note that in Fig. 3.1, the size of the set of achievable portfolios at each period may grow in the next period depending on the set of price relative vectors. We next define the transition probabilities as $q_{k,l,N} = \Pr(b(N) = b_{l,N} | b(N-1) = b_{k,N-1})$ for $k = 1, \ldots, M_{N-1}$ and $l = 1, \ldots, M_N$ and the set of achievable portfolios that are connected to $b_{l,N}$, i.e., the portfolio transition set, as $\mathcal{N}_{l,N} = \{b_{k,N-1} \in \mathcal{B}_{N-1} \mid q_{k,l,N} > 0,\ k = 1, \ldots, M_{N-1}\}$ for $l = 1, \ldots, M_N$. Hence, the probability of each portfolio state is given by

$$\begin{aligned}
\Pr(b(N) = b_{l,N}) &= \sum_{b_{k,N-1} \in \mathcal{B}_{N-1}} \Pr(b(N) = b_{l,N} | b(N-1) = b_{k,N-1}) \Pr(b(N-1) = b_{k,N-1}) \\
&= \sum_{b_{k,N-1} \in \mathcal{N}_{l,N}} q_{k,l,N} \Pr(b(N-1) = b_{k,N-1}) \qquad (3.1)
\end{aligned}$$

for $l = 1, \ldots, M_N$. Therefore, we can calculate the probability of achievable portfolios iteratively. Using these iterative equations, we next iteratively calculate the expected achieved wealth $E[S(N)]$ at each period as follows.

By definition of $\mathcal{B}_N$ and using the law of total expectation [6], the expected achieved wealth at investment period N can be written as

$$E[S(N)] = \sum_{l=1}^{M_N} \Pr(b(N) = b_{l,N}) E[S(N) | b(N) = b_{l,N}]. \qquad (3.2)$$

To get $E[S(N)]$ in (3.2) iteratively, we evaluate $\Pr(b(N) = b_{l,N}) E[S(N) | b(N) = b_{l,N}]$ for each $l = 1, \ldots, M_N$ from $\Pr(b(N-1) = b_{k,N-1}) E[S(N-1) | b(N-1) = b_{k,N-1}]$ for $k = 1, \ldots, M_{N-1}$. To achieve this, we first find the transition probabilities (not the state probabilities) between the achievable portfolios.

We define the set of price relative vectors that connect $b_{k,N-1}$ to $b_{l,N}$ as $\mathcal{U}_{k,l,N}$ where

$$\mathcal{U}_{k,l,N} = \left\{ \mathbf{w} = [w_1\ w_2]^T \in \mathcal{X}^2 \mid b_{l,N} = \frac{w_1 b_{k,N-1}}{w_1 b_{k,N-1} + w_2(1-b_{k,N-1})} \right\}$$

for $k = 1, \ldots, M_{N-1}$ and $l = 2, \ldots, M_N$. We consider the price relative vectors that connect $b_{k,N-1}$ to $b_{1,N} = b$ separately since, in this case, there are two cases depending on whether the portfolio leaves the interval $(b - \epsilon, b + \epsilon)$ or not. We define $\mathcal{U}_{k,1,N}$ as

$$\mathcal{U}_{k,1,N} = \mathcal{V}_{k,1,N} \cup \mathcal{R}_{k,1,N},$$

where $\mathcal{V}_{k,1,N}$ is the set of price relative vectors that connect $b_{k,N-1}$ to $b_{1,N} = b$ such that the portfolio does not leave the interval $(b - \epsilon, b + \epsilon)$ at period N, i.e.,

$$\mathcal{V}_{k,1,N} = \left\{ \mathbf{w} = [w_1 \ w_2]^T \in \mathcal{X}^2 \mid \frac{w_1 b_{k,N-1}}{w_1 b_{k,N-1} + w_2(1 - b_{k,N-1})} = b \right\},$$

and $\mathcal{R}_{k,1,N}$ is the set of price relative vectors that connect $b_{k,N-1}$ to $b_{1,N}$ such that the portfolio leaves the interval $(b - \epsilon, b + \epsilon)$ at period N and is rebalanced to $b_{1,N} = b$, i.e.,

$$\mathcal{R}_{k,1,N} = \left\{ \mathbf{w} = [w_1 \ w_2]^T \in \mathcal{X}^2 \mid \frac{w_1 b_{k,N-1}}{w_1 b_{k,N-1} + w_2(1 - b_{k,N-1})} \notin (b - \epsilon, b + \epsilon) \right\}.$$

Then, the transition probabilities are given by

$$\begin{aligned} q_{k,l,N} = \Pr\left(b(N) = b_{l,N} | b(N-1) = b_{k,N-1}\right) &= \Pr\left(\mathbf{X}(N) \in \mathcal{U}_{k,l,N}\right) \\ &= \sum_{\mathbf{w}=[w_1 \ w_2]^T \in \mathcal{U}_{k,l,N}} p_1(w_1) p_2(w_2) \end{aligned} \tag{3.3}$$

for $k = 1, \ldots, M_{N-1}$ and $l = 1, \ldots, M_N$ so that we can calculate $\Pr\left(b(N)\right) = b_{l,N})$ iteratively for each $l = 1, \ldots, M_N$ by (3.1). Since we have recursive equations for the state probabilities, we next perform the iterative calculation of the expected achieved wealth based on the achievable portfolio sets and the transition probabilities.

Given the recursive formulation for the state probabilities, we can evaluate the term $\Pr\left(b(N) = b_{l,N}\right) E[S(N) | b(N) = b_{l,N}]$ for $l = 1, \ldots, M_N$ from $\Pr\left(b(N-1) = b_{k,N-1}\right) E[S(N-1) | b(N-1) = b_{k,N-1}]$ for $k = 1, \ldots, M_{N-1}$ iteratively to calculate $E[S(N)]$ by (3.2) as follows. To evaluate $\Pr\left(b(N) = b_{l,N}\right) E[S(N) | b(N) = b_{l,N}]$, we need to consider two cases separately based on the value of $b_{l,N}$.

In the first case, we see that if the portfolio $b(N) = b_{l,N}$, where $l = 2, \ldots, N$, then the portfolio does not leave the interval $(b - \epsilon, b + \epsilon)$ at period N. Hence, no transaction cost is paid so that we can express $\Pr\left(b(N)=b_{l,N}\right)E[S(N)|b(N)=b_{l,N}]$ as a summation of the conditional expectations for all $b_{k,N-1}\in\mathcal{N}_{l,N}$ by the law of total expectation [6] as

$$\begin{aligned}
&\Pr\left(b(N)=b_{l,N}\right)E[S(N)|b(N)=b_{l,N}]\\
&=\sum_{b_{k,N-1}\in\mathcal{N}_{l,N}} E\left[S(N)|b(N)=b_{l,N},b(N-1)=b_{k,N-1}\right]\Pr\left(b(N-1)=b_{k,N-1}|b(N)=b_{l,N}\right)\Pr\left(b(N)=b_{l,N}\right)\\
&=\sum_{b_{k,N-1}\in\mathcal{N}_{l,N}} E\left[S(N)|b(N)=b_{l,N},b(N-1)=b_{k,N-1}\right]\Pr\left(b(N-1)=b_{k,N-1}\right)q_{k,l,N},
\end{aligned} \tag{3.4}$$

where (3.4) follows from Bayes' theorem [40]. We note that given $b(N-1)=b_{k,N-1}$ and $b(N)=b_{l,N}$, the price relative vector $\mathbf{X}(N)$ can take values from $\mathcal{U}_{k,l,N}$ and $q_{k,l,N}=\Pr\left(\mathbf{X}(N)\in\mathcal{U}_{k,l,N}\right)$ so that (3.4) can be written as a summation of the conditional expectations for all $\mathbf{X}(N)=\mathbf{w}\in\mathcal{U}_{k,l,N}$ [6] after replacing $q_{k,l,N}$

$$\begin{aligned}
&\Pr\left(b(N)=b_{l,N}\right)E[S(N)|b(N)=b_{l,N}]\\
&=\sum_{b_{k,N-1}\in\mathcal{N}_{l,N}}\ \sum_{\mathbf{w}=[w_1\ w_2]^T\in\mathcal{U}_{k,l,N}} E\left[S_N|b(N)=b_{l,N},b(N-1)=b_{k,N-1},\mathbf{X}(N)=\mathbf{w}\right]\\
&\times\Pr\left(b(N-1)=b_{k,N-1}\right)\Pr\left(\mathbf{X}(N)=\mathbf{w}|\mathbf{X}(N)\in\mathcal{U}_{k,l,N}\right)\Pr\left(\mathbf{X}(N)\in\mathcal{U}_{k,l,N}\right).
\end{aligned} \tag{3.5}$$

Now, given that $b(N-1)=b_{k,N-1}$, $b(N)=b_{l,N}$ and $\mathbf{X}(N)=\mathbf{w}=[w_1\ w_2]^T$, we observe that $\Pr\left(\mathbf{X}(N)=\mathbf{w}|\mathbf{X}(N)\in\mathcal{U}_{k,l,N}\right)\Pr\left(\mathbf{X}(N)\in\mathcal{U}_{k,l,N}\right)=\Pr\left(\mathbf{X}(N)=\mathbf{w}\right)$ and

$$\begin{aligned}
&E\left[S_N|b(N)=b_{l,N},b(N-1)=b_{k,N-1},\mathbf{X}(N)=\mathbf{w}\right]\\
&=E\left[S(N-1)(b_{k,N-1}w_1+(1-b_{k,N-1})w_2)|b(N-1)=b_{k,N-1}\right],
\end{aligned} \tag{3.6}$$

and by using (3.6) in (3.5), we have

$$\begin{aligned}
&\Pr\left(b(N)=b_{l,N}\right)E[S(N)|b(N)=b_{l,N}]\\
&=\sum_{b_{k,N-1}\in\mathcal{N}_{l,N}}\ \sum_{\mathbf{w}=[w_1\ w_2]^T\in\mathcal{U}_{k,l,N}} E\left[S(N-1)(b_{k,N-1}w_1+(1-b_{k,N-1})w_2)|b(N-1)=b_{k,N-1}\right]\\
&\times\Pr\left(b(N-1)=b_{k,N-1}\right)\Pr\left(\mathbf{X}(N)=\mathbf{w}\right).
\end{aligned}$$

Therefore, we can write $\Pr(b(N) = b_{l,N}) E[S(N)|b(N) = b_{l,N}]$ from $\Pr(b(N-1) = b_{k,N-1}) E[S(N-1)|b(N-1) = b_{k,N-1}]$ as

$$\begin{aligned}
&\Pr(b(N) = b_{l,N}) E[S(N)|b(N) = b_{l,N}] \\
&= \sum_{b_{k,N-1} \in \mathcal{N}_{l,N}} \Pr(b(N-1) = b_{k,N-1}) E[S(N-1)|b(N-1) = b_{k,N-1}] \\
&\times \sum_{\mathbf{w}=[w_1\ w_2]^T \in \mathcal{U}_{k,l,N}} (b_{k,N-1}w_1 + (1-b_{k,N-1})w_2) p_1(w_1) p_2(w_2)
\end{aligned} \tag{3.7}$$

for $l = 2, \ldots, M_N$, where we use $\Pr(\mathbf{X}(N) = \mathbf{w}) = p_1(w_1)p_2(w_2)$.

In the second case, if the portfolio $b(N) = b_{1,N}$, then there are two sets of price relative vectors that connect $b_{k,N-1}$ to $b_{1,N}$, i.e., $\mathcal{V}_{k,1,N}$ and $\mathcal{R}_{k,1,N}$. Depending on the value of the price vector, the portfolio may be rebalanced to $b_{1,N} = b$. If $\mathbf{X}(N) \in \mathcal{V}_{k,1,N}$, then the portfolio is not rebalanced and no transaction fee is paid. If $\mathbf{X}(N) \in \mathcal{R}_{k,1,N}$, then the portfolio is rebalanced and transaction cost is paid. We can find $\Pr(b(N) = b_{1,N}) E[S(N)|b(N) = b_{1,N}]$ from $\Pr(b(N-1) = b_{k,N-1}) E[S(N-1)|b(N-1) = b_{k,N-1}]$ as a summation of the conditional expectations for all $b_{k,N-1} \in \mathcal{N}_{1,N}$ [6] as

$$\begin{aligned}
&\Pr(b(N) = b_{1,N}) E[S(N)|b(N) = b_{1,N}] \\
&= \sum_{b_{k,N-1} \in \mathcal{N}_{1,N}} E[S(N)|b(N) = b_{1,N}, b(N-1) = b_{k,N-1}] \Pr(b(N-1) = b_{k,N-1}|b(N) = b_{1,N}) \\
&\times \Pr(b(N) = b_{1,N}) \\
&= \sum_{b_{k,N-1} \in \mathcal{N}_{1,N}} E[S(N)|b(N) = b_{1,N}, b(N-1) = b_{k,N-1}] \Pr(b(N-1) = b_{k,N-1}) q_{k,l,N}.
\end{aligned} \tag{3.8}$$

We note that given $b(N-1) = b_{k,N-1}$ and $b(N) = b_{1,N}$, the price relative vector $\mathbf{X}(N)$ can take values from $\mathcal{V}_{k,1,N}$ or $\mathcal{R}_{k,1,N}$, $q_{k,l,N} = \Pr(\mathbf{X}(N) \in \mathcal{U}_{k,l,N})$ and $\Pr(\mathbf{X}(N) = \mathbf{w}|\mathbf{X}(N) \in \mathcal{U}_{k,l,N})$

$\times \Pr(\mathbf{X}(N) \in \mathcal{U}_{k,l,N}) = \Pr(\mathbf{X}(N) = \mathbf{w})$ which yields in (3.8) that

$$\begin{aligned}
&\Pr(b(N) = b_{1,N})\, E[S(N)|b(N) = b_{1,N}] \\
&= \sum_{b_{k,N-1} \in \mathcal{N}_{l,N}} \Bigg\{ \sum_{\mathbf{w}=[w_1\ w_2]^T \in \mathcal{V}_{k,1,N}} E\left[S_N | b(N) = b_{l,N}, b(N-1) = b_{k,N-1}, \mathbf{X}(N) = \mathbf{w}\right] \\
&\times \Pr(b(N-1) = b_{k,N-1}) \Pr(\mathbf{X}(N) = \mathbf{w}) \\
&+ \sum_{\mathbf{w}=[w_1\ w_2]^T \in \mathcal{R}_{k,1,N}} E\left[S_N | b(N) = b_{l,N}, b(N-1) = b_{k,N-1}, \mathbf{X}(N) = \mathbf{w}\right] \\
&\times \Pr(b(N-1) = b_{k,N-1}) \Pr(\mathbf{X}(N) = \mathbf{w}) \Bigg\}.
\end{aligned}$$

If $\mathbf{X}(N) = \mathbf{w} \in \mathcal{V}_{k,1,N}$, then it follows that

$$\begin{aligned}
&E\left[S_N | b(N) = b_{1,N}, b(N-1) = b_{k,N-1}, \mathbf{X}(N) = \mathbf{w}\right] \\
&= E\left[S(N-1)(b_{k,N-1} w_1 + (1 - b_{k,N-1}) w_2) | b(N-1) = b_{k,N-1}\right].
\end{aligned} \tag{3.9}$$

If $\mathbf{X}(N) = \mathbf{w} \in \mathcal{R}_{k,1,N}$, then transaction cost is paid which results

$$\begin{aligned}
&E\left[S_N | b(N) = b_{1,N}, b(N-1) = b_{k,N-1}, \mathbf{X}(N) = \mathbf{w}\right] \\
&= E\left[S(N-1)(b_{k,N-1} w_1 + (1 - b_{k,N-1}))\left(1 - c\left|\frac{b_{k,N-1} w_1}{b_{k,N-1} w_1 + (1 - b_{k,N-1}) w_2} - b\right|\right) \Big| b(N-1) = b_{k,N-1}\right].
\end{aligned} \tag{3.10}$$

Hence, we can write (3.8) after using (3.9) and (3.10) as

$$
\begin{aligned}
&\Pr\left(b(N)=b_{1,N}\right)E[S(N)|b(N)=b_{1,N}]\\
&=\sum_{b_{k,N-1}\in\mathcal{N}_{1,N}}\Pr\left(b(N-1)=b_{k,N-1}\right)\\
&\times\Bigg\{\sum_{\mathbf{w}=[w_1\ w_2]^T\in\mathcal{V}_{k,1,N}}\Pr\left(\mathbf{X}(N)=\mathbf{w}\right)E\left[S(N-1)(b_{k,N-1}w_1+(1-b_{k,N-1})w_2)|b(N-1)=b_{k,N-1}\right]\\
&+\sum_{\mathbf{w}=[w_1\ w_2]^T\in\mathcal{R}_{k,1,N}}\Pr\left(\mathbf{X}(N)=\mathbf{w}\right) \qquad (3.11)\\
&\times E\left[S(N-1)(b_{k,N-1}w_1+(1-b_{k,N-1}))\left(1-c\left|\frac{b_{k,N-1}w_1}{b_{k,N-1}w_1+(1-b_{k,N-1})w_2}-b\right|\right)|b(N-1)=b_{k,N-1}\right]\Bigg\}.
\end{aligned}
$$

Thus, we can write $\Pr\left(b(N)=b_{1,N}\right)E[S(N)|b(N)=b_{1,N}]$ from $\Pr\left(b(N-1)=b_{k,N-1}\right)E[S(N-1)|b(N-1)=b_{k,N-1}]$ as

$$
\begin{aligned}
&\Pr\left(b(N)=b_{1,N}\right)E[S(N)|b(N)=b_{1,N}]\\
&=\sum_{b_{k,N-1}\in\mathcal{N}_{1,N}}\Pr\left(b(N-1)=b_{k,N-1}\right)E\left[S(N-1)|b(N-1)=b_{k,N-1}\right]\\
&\times\Bigg\{\sum_{\mathbf{w}=[w_1\ w_2]^T\in\mathcal{V}_{k,1,N}}(b_{k,N-1}w_1+(1-b_{k,N-1})w_2)p_1(w_1)p_2(w_2) \qquad (3.12)\\
&+\sum_{\mathbf{w}=[w_1\ w_2]^T\in\mathcal{R}_{k,1,N}}(b_{k,N-1}w_1+(1-b_{k,N-1}))\left(1-c\left|\frac{b_{k,N-1}w_1}{b_{k,N-1}w_1+(1-b_{k,N-1})w_2}-b\right|\right)p_1(w_1)p_2(w_2)\Bigg\},
\end{aligned}
$$

which yields the recursive expressions for $\Pr\left(b(N)=b_{l,N}\right)E[S(N)|b(N)=b_{l,N}]$ iteratively for each $l=1,\ldots,M_N$ with (3.7) and (3.12).

Hence, in the first case where the portfolio $b(N)=b_{l,N}$ for $l=2,\ldots M_N$, we can calculate $E\left[S(N)|b(N)=b_{l,N}\right]\Pr\left(b(N)=b_{l,N}\right)$ from $E\left[S(N-1)|b(N-1)=b_{k,N-1}\right]\Pr\left(b(N-1)=b_{k,N-1}\right)$ for $b_{k,N-1}\in\mathcal{N}_{l,N}$ by (3.7). In the second case where the portfolio $b(N)=b_{1,N}=b$, we can calculate $E\left[S_N|b(N)=b_{1,N}\right]\Pr\left(b(N)=b_{1,N}\right)$ from $E\left[S(N-1)|b(N-1)=b_{k,N-1}\right]$ $\times\Pr\left(b(N-1)=b_{k,N-1}\right)$ for $b_{k,N-1}\in\mathcal{N}_{1,N}$ by (3.12). Therefore, we can evaluate $E[S(N)]$ iteratively by (3.2). Since, we have the recursive formulation, we can optimize b and ϵ by a brute force search as shown in the Simulations section. For this recursive evaluation, we have to find the set of achievable portfolios at each investment period to compute $E[S(N)]$ by (3.2). Hence, we next analyze the number of calculations required to evaluate the expected achieved wealth $E[S(N)]$.

Complexity Analysis of the Iterative Algorithm

We next investigate the number of achievable portfolios at a given market period to determine the complexity of the iterative algorithm. We show that the set of achievable portfolios at period N is equivalent to the set of achievable portfolios when the portfolio $b(n)$ does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods. We first demonstrate that if the portfolio never leaves the interval $(b-\epsilon, b+\epsilon)$ for N periods, then $b(N)$ is given by

$$b(N) = \frac{1}{1 + \frac{1-b}{b} e^{\sum_{n=1}^{N} Z(n)}},$$

where $Z(n) \stackrel{\triangle}{=} \ln \frac{X_2(n)}{X_1(n)}$ with a sample space $\mathcal{Z} = \left\{ z = \ln \frac{u}{v} \mid u, v \in \mathcal{X} \right\}$ where $|\mathcal{Z}| = M$. Then, we argue that the number of achievable portfolios at period N, M_N, is equal to the number of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take when the portfolio does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods. We point out that $M \leq K^2 - K + 1$ since the price relative sequences $X_1(n)$ and $X_2(n)$ are elements of the same sample space $\mathcal{X}$ with $|\mathcal{X}| = K$ and by using this, we find an upper bound on the number of achievable portfolios.

Lemma 3.2.1 *The number of achievable portfolios at period N, M_N, is equal to the number of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take when the portfolio $b(n)$ does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods and is bounded by $\binom{N+K^2-K}{N}$, i.e., $M_N = |\mathcal{B}_N| \leq \binom{N+K^2-K}{N}$.*

Proof: The proof is in the Appendix A. □

Remark 3.2.1 *Note that the complexity of calculating $E[S(N)]$ is bounded by $\mathcal{O}\left(\sum_{n=1}^{N} \binom{n+K^2-K}{n} / N\right)$ since at each period $n = 1, \ldots, N$, we calculate $E[S(n)]$ as a summation of M_n terms, i.e., $E[S(n)] = \sum_{l=1}^{M_n} E[S(n) | b(n) = b_{l,n}] \Pr(b(n) = b_{l,n})$ and $M_n \leq \binom{n+K^2-K}{n}$.*

In the next section, we extend the given iterative algorithm to calculate the expected achieved wealth in a market with m-assets, where m is an arbitrary number determined by the investor. This result implies that the given optimal threshold rebalanced portfolio method can be employed not only in a two asset market like "Stock and Bond market", but a general stock market where an investor diversify capital into several assets.

3.2.2 Generalization of the Iterative Algorithm to the m-asset Market Case

In this section, we generalize the iterative method introduced in Section 3.2.1 to a market with m assets where $m \in \mathbb{Z}^+$. We model the market as a sequence of i.i.d. price relative vectors $\mathbf{X}(n) = [X_1(n)\ X_2(n) \ldots X_m(n)]$, where $X_i(n) \in \mathcal{X}$ and the p.m.f. of $X_i(n)$ is $p_i(x) \triangleq \Pr(X_i(n) = x)$. For m-asset case, the portfolio vector is given by $\mathbf{b}(n) = [b_1(n)\ b_2(n) \ldots b_m(n)]$, target portfolio vector is defined as $\mathbf{b} = [b_1\ b_2 \ldots b_m]$ and the threshold vector is given by $\epsilon = [\epsilon_1\ \epsilon_2 \ldots \epsilon_m]$. Along these lines, TRP$(\mathbf{b}, \epsilon)$ rebalances the wealth allocation $\mathbf{b}(n)$ to $\mathbf{b}$ only when $\mathbf{b}(n) \notin \mathbf{b}^\epsilon \triangleq [b_1 - \epsilon_1, b_1 + \epsilon_1] \times [b_2 - \epsilon_2, b_2 + \epsilon_2] \times \ldots \times [b_m - \epsilon_m, b_m + \epsilon_m]$. In this case, if the wealth allocation is not rebalanced for N investment periods, then the proportion of wealth invested in the ith asset becomes

$$b_i(N) = \frac{b_i \prod_{n=1}^{N} X_i(N)}{\sum_{k=1}^{m} b_k \prod_{n=1}^{N} X_k(N)}$$

and achieved wealth is given by

$$S(N) = \sum_{k=1}^{m} b_k \prod_{n=1}^{N} X_k(N).$$

We define the set of achievable portfolios at period N as

$$\mathcal{B}_N = \left\{ \mathbf{b}_{1,N}, \mathbf{b}_{2,N}, \ldots, \mathbf{b}_{M_N,N} \mid \mathbf{b}_{k,N} = \frac{\mathbf{b}_{l,N-1} \circ \mathbf{x}}{\mathbf{x}^T \mathbf{b}_{l,N-1}} \in \mathbf{b}^\epsilon \text{or} \mathbf{b}_{k,N} = \mathbf{b},\ \mathbf{x} \in \mathcal{X}^m \right\}$$

where $M_N = |\mathcal{B}_N|$. In accordance with the definitions given in 2-asset market case, the definitions of the portfolio transition sets and the transition probabilities of achievable portfolios follows. Then similar to the iterative algorithm introduced in Section 3.2.1 and the equations (3.7) and (3.12), we can evaluate the term $\Pr(\mathbf{b}(N) = \mathbf{b}_{l,N}) E[S(N)|\mathbf{b}(N) = \mathbf{b}_{l,N}]$ for $l = 1, \ldots, M_N$ from $\Pr(\mathbf{b}(N-1) = \mathbf{b}_{k,N-1}) E[S(N-1)|\mathbf{b}(N-1) = \mathbf{b}_{k,N-1}]$ for

$k = 1, \ldots, M_{N-1}$ iteratively to calculate $E[S(N)]$. Therefore for m-asset market case, by using

$$E[S(N)] = \sum_{l=1}^{M_N} \Pr\left(\mathbf{b}(N) = \mathbf{b}_{l,N}\right) E\left[S(N) | \mathbf{b}(N) = \mathbf{b}_{l,N}\right],$$

the expected achieved wealth $E[S(N)]$ can be evaluated iteratively.

In the next section, we show that the set of all achievable portfolios, $\mathcal{B} \triangleq \cup_{n=1}^{\infty} \mathcal{B}_n$, is finite under mild technical conditions. This result is important when we analyze the asymptotic behavior of the expected achieved wealth since the the complexity of the algorithm that evaluates $E[S(n)]$ is constant when the set of achievable portfolios is finite. We demonstrate that the portfolio sequence forms a Markov chain with a finite state space and converges to a stationary distribution. Finally, we analyze the limiting behavior of the expected achieved wealth and then optimize b and ϵ with a brute-force algorithm.

3.2.3 Finitely Many Achievable Portfolios

In this section, we investigate the cardinality of the set of achievable portfolios $\mathcal{B}$ and demonstrate that $\mathcal{B}$ is finite under certain conditions in the following theorem, Theorem 3.2.1. This result is significant since when $\mathcal{B}$ is finite, we can derive a recursive update with a constant complexity, i.e., the number of states does not grow, to calculate the expected achieved wealth at any investment period. Then, we can investigate the limiting behavior of the expected achieved wealth using this update to optimize b and ϵ. Before providing the main theorem, we first state a couple of lemmas that are used in the derivation of the main result of this section.

We first point out that in Lemma 3.2.1, we showed that the number of achievable portfolios at period N is equal to the number of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take when the portfolio $b(n)$ does not leave the interval $(b - \epsilon, b + \epsilon)$ for N investment periods. Then, we observed that the cardinality of the set $\mathcal{B}$ is equal to the number of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take for any $N \in \mathbb{N}$ when the portfolio $b(n)$ never leaves the interval $(b - \epsilon, b + \epsilon)$. We next show that the portfolio $b(n)$ does not leave the interval $(b - \epsilon, b + \epsilon)$ for N periods if and only if the sum $\sum_{n=1}^{k} Z(n) \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$, where $\alpha_1 \triangleq \ln \frac{b(1-b+\epsilon)}{(1-b)(b-\epsilon)} > 0$ and $\alpha_2 \triangleq \ln \frac{b(1-b-\epsilon)}{(1-b)(b+\epsilon)} < 0$. Moreover, we also prove that the number of achievable portfolios is equal to the cardinality of the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$ where we define the set $\mathcal{M}$ as

$$\mathcal{M} = \{m_1 z_1 + m_2 z_2 + \ldots + m_{M^+} z_{M^+} \mid m_i \in \mathbb{Z},\ z_i \in \mathcal{Z}^+ \text{ for } i = 1, \ldots, M^+\}, \tag{3.13}$$

$\mathcal{Z}^+ \triangleq \{z \in \mathcal{Z} \mid z \geq 0\}$, $M^+ \triangleq |\mathcal{Z}^+|$. Note that $\mathcal{Z}^+$ is the set of positive elements of the set $\mathcal{Z}$ and any value that the sum $\sum_{n=1}^{N} Z(n)$ can take is an element of $\mathcal{M}$. Hence, if we can demonstrate that the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$ is finite under certain conditions, then it yields the cardinality of the set $\mathcal{B}$ since $\mathcal{B}$ is finite if and only if $M \cap (\alpha_2, \alpha_1)$ is finite.

In the following lemma, we prove that the portfolio $b(n)$ does not leave the interval $(b - \epsilon, b + \epsilon)$ for N periods if and only if the sum $\sum_{n=1}^{k} Z(n) \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$.

Lemma 3.2.2 *The portfolio $b(n)$ does not leave the interval $(b - \epsilon, b + \epsilon)$ for N investment periods if and only if the sum $\sum_{n=1}^{k} Z(n) \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$.*

Proof: The proof is in the Appendix B. □

In the following lemma, we demonstrate that if the condition $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ is satisfied for each $z \in \mathcal{Z}^+$, then for any element $m \in \mathcal{M} \cap (\alpha_2, \alpha_1)$, there exists an N-period market scenario where the portfolio does not leave the interval $(b - \epsilon, b + \epsilon)$ for N investment periods and $\{Z(n) = Z^{(n)}\}_{n=1}^{N}$ such that $m = \sum_{n=1}^{N} Z^{(n)}$ for some $\{Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z}$ and $N \in \mathbb{N}$. It follows that the set of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take for any $N \in \mathbb{N}$ when the portfolio never leaves the interval $(b - \epsilon, b + \epsilon)$ for N investment periods is equivalent to the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$. Hence, we show that the cardinality of the set of achievable portfolios is equal to the cardinality of the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$. After this lemma, we present conditions under which the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$ is finite so that the set of achievable portfolios is also finite.

Lemma 3.2.3 *If $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ for $z \in \mathcal{Z}^+$, then any element of $\mathcal{M} \cap (\alpha_2, \alpha_1)$ can be written as a sum $\sum_{n=1}^{N} Z^{(n)}$ for some $N \in \mathbb{N}$ where $\{Z(n) = Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z}$ and $\sum_{n=1}^{k} Z^{(n)} \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$.*

Proof: In Lemma 3.2.1, we showed that for any investment period N, the number of different portfolio values that $b(N)$ can take is equal to the number of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take where $\sum_{n=1}^{k} Z(n) \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$. Since this is true for any investment period N, it follows that the number of all achievable portfolios is equal to the number of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take for any $N \in \mathbb{N}$ such that $\sum_{n=1}^{N} Z(n) \in (\alpha_2, \alpha_1)$.

Here, we show that if $m \in \mathcal{M} \cap (\alpha_2, \alpha_1)$, then there exists a sequence $\{Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z}$ for some $N \in \mathbb{N}$ such that $m = \sum_{n=1}^{N} Z^{(n)}$ and $\sum_{n=1}^{k} Z^{(n)} \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$. Let $m \in \mathcal{M} \cap (\alpha_2, \alpha_1)$. Then, it can be written as $m = m_1 z_1 + \ldots + m_{M^+} z_{M^+}$ for some $m_i \in \mathbb{Z}$ and $z_i \in \mathcal{Z}^+$, $i = 1, \ldots, M^+$. We define $S(k) = \sum_{n=1}^{k} Z^{(n)}$ for $k \geq 1$ and construct a sequence $\{Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z}$ for some $N \in \mathbb{N}$ such that $m = \sum_{n=1}^{N} Z^{(n)}$ and $S(k) \in (\alpha_2, \alpha_1)$ for each $k = 1, \ldots, N$ as follows. We choose $z_i \in \mathcal{Z}^+$ such that $m_i > 0$, let $Z^{(1)} = z_i$ and decrease m_i by 1. We see that $S(1) = Z^{(1)} \in (\alpha_2, \alpha_1)$ since $z_i < \min\{|\alpha_1|, |\alpha_2|\}$. Next, we choose $z_j \in \mathcal{Z}^+$ such that $m_j < 0$, let $Z^{(2)} = -z_j$ and increase m_j by 1. Then, it follows that $S(2) = Z^{(1)} + Z^{(2)} = z_i - z_j \in (\alpha_2, \alpha_1)$ since $z_i, z_j < \min\{|\alpha_1|, |\alpha_2|\}$. At any time $k \geq 3$, if

- $S(k) \geq 0$, we choose $z_l \in \mathcal{Z}^+$ such that $m_l < 0$, let $Z^{(k+1)} = -z_l$ and increase m_l by 1. Note that $S(k+1) \in (\alpha_2, \alpha_1)$ since $S(k) \in (\alpha_2, \alpha_1)$, $S(k) \geq 0$ and $Z^{(k+1)} < 0$. Now assume that there exists no $z_l \in \mathcal{Z}^+$ such that $m_l < 0$, i.e., $m_j \geq 0$ for $j = 1, \ldots, M$. If we let $I \triangleq \{j \in \{1, \ldots, M\} \mid m_j \geq 0\} = \{k_1, \ldots, k_T\}$ where $T \triangleq |I|$ and

$$Z^{(l)} = z_{k_j}, \quad l = k + 1 + \sum_{i=1}^{j-1} k_i, \ldots, k + \sum_{i=1}^{j} k_i$$

for $j = 1, \ldots, T$, then we get that $m = S(N) = \sum_{n=1}^{N} Z^{(n)}$ where $N = k + \sum_{i=1}^{T} k_i$. We observe that $S_i \in (\alpha_2, \alpha_1)$ for $i = k+1, \ldots, N$ since $m \in (\alpha_2, \alpha_1)$, $\sum_{j=1}^{T} m_{k_j} x_{k_j} \geq 0$ and $S(k) > 0$.

- $S(k) < 0$, we choose $z_l \in \mathcal{Z}^+$ such that $m_l > 0$, let $Z^{(k+1)} = z_l$ and decrease m_l by 1. Note that $S(k+1) \in (\alpha_2, \alpha_1)$ since $S(k) \in (\alpha_2, \alpha_1)$, $S(k) < 0$ and $Z^{(k+1)} \geq 0$. Assume that there exists no $z_l \in \mathcal{Z}^+$ such that $m_l \geq 0$, i.e., $m_j < 0$ for $j = 1, \ldots, M$. If we let $J \triangleq \{j \in \{1, \ldots, M\} \mid m_j \leq 0\} = \{k_1, \ldots, k_W\}$ where $W \triangleq |J|$ and

$$Z^{(l)} = z_{k_j}, \quad l = k + 1 + \sum_{i=1}^{j-1} k_i, \ldots, k + \sum_{i=1}^{j} k_i$$

for $j = 1, \ldots, W$, then we get that $m = S(N) = \sum_{n=1}^{N} Z^{(n)}$ where $N = k + \sum_{i=1}^{W} k_i$. We see that $S_i \in (\alpha_2, \alpha_1)$ for $i = k+1, \ldots, N$ since $m \in (\alpha_2, \alpha_1)$, $\sum_{j=1}^{W} m_{k_j} x_{k_j} \leq 0$ and $S(k) < 0$.

Therefore, we can write $m = \sum_{n=1}^{N} Z^{(n)}$ for some $N \geq 1$ where $\{Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z}$ and $\sum_{n=1}^{k} Z^{(n)} \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$. □

Hence, we showed that if the condition $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ is satisfied for each $z \in \mathcal{Z}^+$, then any element of the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$ can be written as a sum $\sum_{n=1}^{N} Z(n)$ for some $N \in \mathbb{N}$ when the portfolio does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods. It follows that the set of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take for any $N \in \mathbb{N}$ when the portfolio does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods is equivalent to the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$. Thus, the number of achievable portfolios is equal to the cardinality of the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$. In the following theorem, we demonstrate that if $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ for $z \in \mathcal{Z}^+$ and the set $\mathcal{M}$ has a minimum positive element, then $\mathcal{M} \cap (\alpha_2, \alpha_1)$ is finite. Hence, the set of achievable portfolios is also finite under these conditions. Otherwise, we show that the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$ contains infinitely many elements so that the set of achievable portfolios is also infinite. Thus, we show that the set of achievable portfolios is finite if and only if the minimum positive element of the set $\mathcal{M}$ exists.

Theorem 3.2.1 *If $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ for $z \in \mathcal{Z}^+$ and the set $\mathcal{M}$ has a minimum positive element, i.e., if*

$$\delta = \min\{m \in \mathcal{M} \mid m > 0\}$$

exists, then the set of achievable portfolio $\mathcal{B} = \cup_{n=1}^{\infty} \mathcal{B}_n$ is finite. If such a minimum positive element does not exist, then $\mathcal{B}$ is countably infinite.

In Theorem 3.2.1 we present a necessary and sufficient condition for the achievable portfolios to be finite. We emphasize that the required condition, i.e., $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ for $z \in \mathcal{Z}^+$, is a necessary required technical condition which assures that the TRP thresholds are large enough to prohibit constant rebalancings at each investment period. In this sense, this condition does not limit the generality of the TRP framework.

By Theorem 3.2.1, we establish the conditions for a unique stationary distribution of the achievable portfolios. With the existence of a unique stationary distribution, in the next section, we provide the asymptotic behavior of the expected wealth growth by presenting the growth rate.

Proof: For any investment period N, we showed in Lemma 3.2.1 that the number of different portfolio values that $b(N)$ can take is equal to the number of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take where the sum $\sum_{n=1}^{k} Z(n) \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$. In the Lemma 3.2.3, we showed that the set of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take where the sum $\sum_{n=1}^{k} Z(n) \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$ is equivalent to the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$. We let $\mathcal{H}$ be the set of values that the sum $\sum_{n=1}^{N} Z(n) \in (\alpha_2, \alpha_1)$ can take for any $N \in \mathbb{N}$, i.e., $\mathcal{H} = \{\sum_{n=1}^{N} Z^{(n)} \mid \{Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z},\ \sum_{n=1}^{k} Z^{(n)} \in (\alpha_2, \alpha_1) \text{for} k = 1, \ldots, N,\ N \in \mathbb{N}\}$. Now, assume that the minimum positive element δ exists. We next illustrate that the sum $\sum_{n=1}^{N} Z^{(n)}$ for any sequence $\{Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z}$ can be written as $k\delta$ for some $k \in \mathbb{Z}$, i.e., $\sum_{n=1}^{N} Z^{(n)} = k\delta$.

Assume that there exists a sequence $\{Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z}$ such that the sum $Z = \sum_{n=1}^{N} Z^{(n)} \neq k\delta$ for any $k \in \mathbb{Z}$. If we divide the real line into intervals of length δ, then Z should lie in one of the intervals, i.e., there exists $k_0 \in \mathbb{Z}$ such that $k_0\delta < Z < (k_0 + 1)\delta$ so that we can write $Z = k_0\delta + \eta$ where $0 < \eta < \delta$. By definition of $\mathcal{M}$, an integer multiple of any element of $\mathcal{M}$ is also an element of $\mathcal{M}$ so that $k_0\delta \in \mathcal{M}$ since $\delta \in \mathcal{M}$. Moreover, for any two elements of $\mathcal{M}$, their difference is also an element of $\mathcal{M}$ so that $\eta = Z - k_0\delta \in \mathcal{M}$ since $Z \in \mathcal{M}$ and $k_0\delta \in \mathcal{M}$. However, this contradicts to the fact that δ is the minimum positive element of $\mathcal{M}$ since $0 < \eta < \delta$ and $\eta \in \mathcal{M}$. Hence, it follows that any element of $\mathcal{H}$ can be written as $k\delta$ for some $k \in \mathbb{Z}$. Note that there are finitely many elements in $\mathcal{H}$ since any element $h \in \mathcal{H}$ can be written as $h = k\delta$ for some $k \in \mathbb{Z}$ and $\alpha_2 < h < \alpha_1$. Since $|\mathcal{B}| = |\mathcal{H}|$, it follows that the set of achievable portfolios $\mathcal{B}$ is finite.

To show that if δ does not exist then $\mathcal{B}$ contains infinitely many elements, we assume that δ does not exist. Since every finite set of real numbers has a minimum, there are either countably infinitely many positive elements in the set $\mathcal{M}$ or none. We know that there exists $z_i \neq 0$ so that there are positive numbers in $\mathcal{M}$. Therefore, there are infinitely many elements in $\mathcal{M}$. Now assume that there exists $\gamma_1 > 0$ that can be written as a sum $\sum_{n=1}^{N} Z^{(n)}$ for some $N \in \mathbb{N}$ where $\{Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z}$ and $\sum_{n=1}^{k} Z^{(n)} \in (\alpha_2, \alpha_1)$. Then, by Lemma 3.2.3, it follows that $\gamma_1 \in \mathcal{M} \cap (0, \alpha_1)$ and since there exists no positive minimum element of $\mathcal{M}$, there exists $\gamma_2 > 0$ such that $\gamma_2 < \gamma_1$ so that $\gamma_2 \in \mathcal{M} \cap (0, \alpha_1)$. In this way, we can construct a decreasing sequence $\{\gamma_n\}$ such that $\gamma_n \in \mathcal{M} \cap (0, \alpha_1)$ for each $n \in \mathbb{N}$. Note that for any $n \in \mathbb{N}$, γ_n is also element of $\mathcal{H}$ by Lemma 3.2.3 so that there are countably infinite elements in $\mathcal{H}$. Hence, it follows that $\mathcal{B}$ has countably infinitely many elements. □

We showed that if $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ for $z \in \mathcal{Z}^+$ and the minimum positive element of the set $\mathcal{M}$ exists, then the set of achievable portfolios, $\mathcal{B}$, is finite. If the minimum positive element of the set $\mathcal{M}$ does not exist, then the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$ is countably infinite so that the number of achievable portfolios is also countably infinite. Hence, the set of achievable portfolios is finite if and only if the minimum positive element of the set $\mathcal{M}$ exists. However, Theorem 3.2.1 does not specify the exact number of achievable portfolios. In the following corollary, we demonstrate that the number of achievable portfolios is $\lfloor \frac{\alpha_1 - \alpha_2}{\delta} \rfloor$ if the set of achievable portfolios is finite.

Corollary 3.2.1 *If* $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ *for* $z \in \mathcal{Z}^+$ *and* $\delta = \min\{m | m > 0\, m \in \mathcal{M}\}$ *exists, then the number of achievable portfolios is* $\lfloor \frac{\alpha_1 - \alpha_2}{\delta} \rfloor$[1].

Proof: Assume that δ exists and there exists $\theta > 0$ such that θ can be written as a sum $\sum_{n=1}^{N} Z^{(n)}$ for some $N \in \mathbb{N}$ and $\{Z(n) = Z^{(n)}\}_{n=1}^{N} \in \mathcal{Z}$ such that $\sum_{n=1}^{k} Z^{(n)} \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N$. Note that such a θ exists, e.g., $\theta = z > 0$ where $z \in \mathcal{Z}^+$ since $z \in (\alpha_2, \alpha_1)$. Then, by Lemma 3.2.3, it follows that $\theta \in \mathcal{M} \cap (0, \alpha_1)$. Since δ is the minimum positive element of $\mathcal{M}$, it follows that $0 < \delta \leq \theta$ and $\delta \in \mathcal{M} \cap (0, \alpha_1)$. Hence, by Lemma 3.2.3, we get that δ can be written as a sum $\sum_{n=1}^{N'} Z^{(n)}$ for some $N' \in \mathbb{N}$ and $\{Z^{(n)}\}_{n=1}^{N'} \in \mathcal{Z}$ where $\sum_{n=1}^{k} Z^{(n)} \in (\alpha_2, \alpha_1)$ for $k = 1, \ldots, N'$. We note that δ is an element of the set of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take for any $N \in \mathbb{N}$ and $Z(n) \in \mathcal{Z}$ for $n = 1, \ldots, N$ such that the portfolio does not leave the interval $(b - \epsilon, b + \epsilon)$. We showed in Theorem 3.2.1 that any element of $\mathcal{M}$ can be written as $k\delta$ for some $k \in \mathbb{Z}$ so that the number of elements in $\mathcal{M} \cap (\alpha_2, \alpha_1)$ is $\lfloor \frac{\alpha_1 - \alpha_2}{\delta} \rfloor$. Hence, it follows that there are exactly $\lfloor \frac{\alpha_1 - \alpha_2}{\delta} \rfloor$ achievable portfolios since Lemma 3.2.3 implies that the set $\mathcal{M} \cap (\alpha_2, \alpha_1)$ is equivalent to the set of different values that the sum $\sum_{n=1}^{N} Z(n)$ can take for any $N \in \mathbb{N}$ and $Z(n) \in \mathcal{Z}$ for $n = 1, \ldots, N$ such that the sum $\sum_{n=1}^{k} Z(n) \in (\alpha_2, \alpha_1)$ for each $k = 1, \ldots, N$ and the cardinality of the latter set is equal to the number of achievable portfolios. □

[1]Here, $\lfloor x/y \rfloor$ is the largest integer less than or equal to x/y

In Theorem 3.2.1, we introduce conditions on the cardinality of the set of all achievable portfolio states, $\mathcal{B}$, and showed that if $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ for all $z \in \mathcal{Z}^+$ and the minimum positive element of the set $\mathcal{M}$ exists, then $\mathcal{B}$ is finite. This result is significant when we analyze the asymptotic behavior of the expected achieved wealth, i.e., in the following, we demonstrate that when $\mathcal{B}$ is finite, the portfolio sequence converges to a stationary distribution. Hence, we can determine the limiting behavior of the expected achieved wealth so that we can optimize b and ϵ. To accomplish this, specifically, we first present a recursive update to evaluate $E[S(n)]$. We then maximize $g(b,\epsilon) \stackrel{\triangle}{=} \lim_{n\to\infty} \frac{1}{n} \log E[S(n)]$ over b and ϵ with a brute-force search, i.e., we calculate $g(b,\epsilon)$ for different (b,ϵ) pairs and find the one that yields the maximum.

3.2.4 Finite State Markov Chain for Threshold Portfolios

If we assume that $|z| < \min\{|\alpha_1|, |\alpha_2|\}$ for all $z \in \mathcal{Z}^+$ and $\delta = \min\{m \in \mathcal{M} \mid m > 0\}$ exists, then the set of all achievable portfolios $\mathcal{B}$ is finite. By Corollary 3.2.1, it follows that there are exactly $L = \lfloor \frac{\alpha_1 - \alpha_2}{\delta} \rfloor$ achievable portfolios. We let $\mathcal{B} = \{b_1, \ldots, b_L\}$ and, without loss of generality, $b_1 = b$. We define the probability mass vector of the portfolio sequence as $\boldsymbol{\pi}(n) = [\pi_1(n) \ \ldots \ \pi_L(n)]^T$ where $\pi_i(n) \stackrel{\triangle}{=} \Pr(b(n) = b_i)$. The portfolio sequence $b(n)$ forms a homogeneous Markov chain with a finite state space $\mathcal{B}$ since the transition probabilities between states are independent of period n. We see that $b(n)$ is irreducible since each state communicates with other states so that all states are null-persistent since $\mathcal{B}$ is finite [19]. Then, it follows that there exists a unique stationary distribution vector $\boldsymbol{\pi}$, i.e., $\boldsymbol{\pi} = \lim_{n\to\infty} \boldsymbol{\pi}(n)$. To calculate $\boldsymbol{\pi}$, we first observe that the set of portfolios that are connected to b_l, $\mathcal{N}_{l,n}$, and the set of price relative vectors that connect b_k to b_l, $\mathcal{U}_{k,l,n}$, are independent of investment period since the price relative sequences are i.i.d. for $k = 1, \ldots, L$ and $l = 1, \ldots, L$. Hence, we write $\mathcal{U}_{k,l,n} = \mathcal{U}_{k,l}$ and $\mathcal{N}_{l,n} = \mathcal{N}_l$ for $n \in \mathbb{N}$. We next note that the state transition probabilities are also independent of investment period and write $q_{k,l,n} = \Pr(b(n) = b_l | b(n-1) = b_k) = q_{k,l}$ for $n \in \mathbb{N}$, $k = 1, \ldots, L$ and $l = 1, \ldots, L$.

Therefore, we can write $\Pr(b(n) = b_l)$ as

$$\Pr(b(n) = b_l) = \sum_{b_k \in \mathcal{N}_l} q_{k,l} \Pr(b(n-1) = b_k) = \sum_{k=1}^{L} q_{k,l} \Pr(b(n-1) = b_k), \tag{3.14}$$

where $q_{k,l} = 0$ if $b_k \notin \mathcal{N}_l$. Now, by using the definition of $\boldsymbol{\pi}(n)$ and (3.14), we get $\boldsymbol{\pi}(n+1) = \mathbf{P}\boldsymbol{\pi}(n)$ for each n, where $\mathbf{P}$ is the state transition matrix, i.e., $\mathbf{P}_{ij} = q_{i,j}$.

We next determine the limiting behavior of the expected achieved wealth $E[S(n)]$ to optimize b and ϵ as follows. In Section 3.2.1, we showed that $E[S(n)]$ can be calculated iteratively by (3.2), (3.7) and (3.12). If we define the vector $\mathbf{e}(n) = [e_1(n) \ \ldots \ e_L(n)]^T$ where $e_i(n) \stackrel{\triangle}{=} \Pr(b(n) = b_i) \, E[S(n)|b(n) = b_i]$, then we can calculate $E[S(n)]$ as the sum of the entries of $\mathbf{e}(n)$ by (3.2), i.e.,

$$E[S(n)] = \sum_{i=1}^{L} \Pr(b(n) = b_i) \, E[S(n)|b(n) = b_i] = \sum_{i=1}^{L} e_i(n) = \mathbf{1}^T \mathbf{e}(n), \tag{3.15}$$

where $\mathbf{1}$ is the vector of ones. Hence, by definition of $\mathbf{e}(n)$, we can write

$$\mathbf{e}(n+1) = \mathbf{Q}\mathbf{e}(n), \tag{3.16}$$

where the matrix $\mathbf{Q}$ is given by

$$\mathbf{Q} = \tag{3.17}$$

$$\begin{bmatrix} \sum\limits_{\mathbf{w}=[w_1 \ w_2]^T \in \mathcal{U}_{1,1}} (b_1 w_1 + (1-b_1) w_2) \, p_1(w_1) p_2(w_2) & \cdots & \sum\limits_{\mathbf{w}=[w_1 \ w_2]^T \in \mathcal{U}_{L,1}} (b_L w_1 + (1-b_L) w_2) \, p_1(w_1) p_2(w_2) \\ \vdots & \ddots & \vdots \\ \sum\limits_{\mathbf{w}=[w_1 \ w_2]^T \in \mathcal{U}_{1,L}} (b_1 w_1 + (1-b_1) w_2) \, p_1(w_1) p_2(w_2) & \cdots & \sum\limits_{\mathbf{w}=[w_1 \ w_2]^T \in \mathcal{U}_{L,L}} (b_L w_1 + (1-b_L) w_2) \, p_1(w_1) p_2(w_2) \end{bmatrix}$$

where we ignore rebalancing for presentation purposes. From (3.7) and (3.12), $\mathbf{Q}$ does not depend on period n since there are finitely many portfolio states, i.e., $\mathbf{Q}$ is constant. If we take rebalancing into account, then only the first row of the matrix $\mathbf{Q}$ changes and the other rows remain the same where

$$\mathbf{Q}_{1,j} = \sum_{\mathbf{w}=[w_1\ w_2]^T \in \mathcal{V}_{j,1}} (b_1 w_1 + (1-b_1)w_2)\, p_1(w_1)p_2(w_2)$$
$$+ \sum_{\mathbf{w}=[w_1\ w_2]^T \in \mathcal{R}_{j,1}} (b_1 w_1 + (1-b_1)w_2) \left(1 - c\left|\frac{b_1 w_1}{b_1 w_1 + (1-b_1)w_2} - b\right|\right) p_1(w_1)p_2(w_2),$$

$\mathcal{V}_{j,1}$ is the set of price relative vectors that connect b_j to $b_1 = b$ without crossing the threshold boundaries and $\mathcal{R}_{j,1}$ is the set of price relative vectors that connect b_j to $b_1 = b$ by crossing the threshold boundaries for $i = j, \ldots, L$. Note that we can find the matrix $\mathbf{Q}$ by using the set of achievable portfolios $\mathcal{B}$ and the probability mass vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ of the price relative sequences.

Here, we analyze $E[S(n)]$ as $n \to \infty$ as follows. We assume that the matrix $\mathbf{Q}$ is diagonalizable with the eigenvalues $\lambda_1, \ldots, \lambda_L$ and, without loss of generality, $\lambda_1 \geq \ldots \geq \lambda_L$, which is the case for a wide range of transaction costs [40]. Then, there exists a nonsingular matrix $\mathbf{B}$ such that $\mathbf{Q} = \mathbf{B}\boldsymbol{\Lambda}\mathbf{B}^{-1}$ where $\boldsymbol{\Lambda}$ is the diagonal matrix with entries $\lambda_1, \ldots, \lambda_L$. We observe that the matrix $\mathbf{Q}$ has nonnegative entries. Therefore, it follows from Perron-Frobenius Theorem [34] that the matrix $\mathbf{Q}$ has a unique largest eigenvalue $\lambda_1 > 0$ and any other eigenvalue is strictly smaller than λ_1 in absolute value, i.e., $\lambda_1 > |\lambda_j|$ for $j = 2, \ldots, L$. Then, the recursion (3.16) yields

$$\mathbf{e}(n) = \mathbf{Q}^n \mathbf{e}(0) = \mathbf{B}\boldsymbol{\Lambda}^n \mathbf{B}^{-1}\mathbf{e}(0) = \mathbf{B} \begin{bmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_L^n \end{bmatrix} \mathbf{B}^{-1}\mathbf{e}(0).$$

Hence, the expected achieved wealth $E[S(n)]$ is given by

$$E[S(n)] = \mathbf{1}^T \mathbf{e}(n) = \mathbf{1}^T \mathbf{B} \begin{bmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_L^n \end{bmatrix} \mathbf{B}^{-1}\mathbf{e}(0) = \mathbf{u}^T \begin{bmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_L^n \end{bmatrix} \mathbf{v}$$

$$= \sum_{i=1}^{L} u_i v_i \lambda_i^n,$$

where $\mathbf{u} \triangleq [u_1 \ \dots \ u_L]^T = \mathbf{B}^T\mathbf{1}$ and $\mathbf{v} \triangleq [v_1 \ \dots \ v_L] = \mathbf{B}^{-1}\mathbf{e}(0)$. Then, it follows that

$$g(b,\epsilon) = \lim_{n\to\infty} \frac{1}{n} \log E[S(n)] = \lim_{n\to\infty} \frac{1}{n} \log \left\{ \sum_{i=1}^{L} u_i v_i \lambda_i^n \right\} = \lim_{n\to\infty} \frac{1}{n} \log \left\{ \lambda_1^n \left[\sum_{i=1}^{L} u_i v_i \left(\frac{\lambda_i}{\lambda_1} \right)^n \right] \right\}$$

$$= \lim_{n\to\infty} \log \lambda_1 + \lim_{n\to\infty} \frac{1}{n} \log \left\{ \sum_{i=1}^{L} u_i v_i \left(\frac{\lambda_i}{\lambda_1} \right)^n \right\}$$

$$= \log \lambda_1$$

since $\lim_{n\to\infty} \left(\frac{\lambda_i}{\lambda_1}\right)^n = 0$ for $i = 2, \dots, L$. Hence, we can optimize b and ϵ as

$$[b^*, \epsilon^*] = \underset{b\in[0,1], 0<\epsilon}{\arg\max}\ g(b,\epsilon) = \underset{b\in[0,1], 0<\epsilon}{\arg\max}\ \log \lambda_1.$$

To maximize $g(b,\epsilon)$, we evaluate it for different values of (b,ϵ) pairs and find the pair that maximizes $g(b,\epsilon)$, i.e., by a brute-force search in the Simulations section.

In this section, we first demonstrated that the set of achievable portfolios is finite under certain conditions. We then showed that the portfolio forms a Markov chain with a finite state space and find the corresponding transition matrix and the stationary state probabilities. When $\mathcal{B}$ is finite, we derived a recursive update with a constant complexity, i.e., the number of states does not grow, to calculate the expected achieved wealth. Finally, we investigated the asymptotic behavior of the expected achieved wealth using this update to optimize b and ϵ with a brute-force search.

In the next section, we investigate the well-studied two-asset Brownian market model with transaction costs. We first show that the set of achievable portfolios is finite and calculate the state transition probabilities. Then, we calculate the asymptotic behavior of the expected achieved wealth to optimize b and ϵ.

3.2.5 Two Stock Brownian Markets

In this section, we consider the well-known two-asset Brownian market, where stock price signals are generated from a standard Brownian motion [17, 21, 41]. Portfolio selection problem in continuous time two-asset Brownian markets with proportional transaction costs was investigated in [41], where the growth optimal investment strategy is shown to be a threshold portfolio. Here, as usually done in the financial literature [17], we first convert the continuous time Brownian market by sampling to a discrete-time market [21]. Then, we calculate the expected achieved wealth and optimize b and ϵ to find the best portfolio rebalancing strategy for a discrete-time Brownian market with transaction costs. Note that although, the growth optimal investment in discrete-time two-asset Brownian markets with proportional transaction costs was investigated in [21], the expected achieved wealth and the optimal threshold interval $(b-\epsilon, b+\epsilon)$ has not been calculated yet.

To model the Brownian two-asset market, we use the price relative vector $\mathbf{X} = [X_1\ X_2]^T$ with $X_1 = 1$ and $X_2 = e^{kZ}$ where k is constant and Z is a random variable with $\Pr(Z = \pm 1) = \frac{1}{2}$. This price relative vector is obtained by sampling the stock price processes of the continuous time two-asset Brownian market [21, 41]. We emphasize that this sampling results a discrete-time market identical to the binomial model popular in asset pricing [21]. We first present the set of achievable portfolios and the transition probabilities between portfolio states. We then investigate the asymptotic behavior of the expected achieved wealth to optimize b and ϵ.

Since the price of the first stock is the same over investment periods, the portfolio leaves the interval $(b-\epsilon, b+\epsilon)$ if either the money in the second stock grows over a certain limit or falls below a certain limit. If the portfolio $b(n)$ does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods, then the money in the first stock is b dollars and the money in the second stock is $(1-b)e^{ki}$ for some $-N \le i \le N$ so that the portfolio is $b(N) = \frac{b}{b+(1-b)e^{ki}}$. Note that $\frac{b}{b+(1-b)e^{ki}} \in (b-\epsilon, b+\epsilon)$ if and only if $i_{\min} \le i \le i_{\max}$, where $i_{\min} \triangleq \left\lceil \frac{1}{k} \ln \frac{b(1-b-\epsilon)}{(1-b)(b+\epsilon)} \right\rceil$[2]

[2]Here, $\lceil x/y \rceil$ is the largest integer greater or equal to the x/y.

and $i_{\max} \stackrel{\triangle}{=} \left\lfloor \frac{1}{k} \ln \frac{b(1-b+\epsilon)}{(1-b)(b-\epsilon)} \right\rfloor$. Hence, the set of achievable portfolios is given by

$$\mathcal{S} = \left\{ b_i = \frac{b}{b + (1-b)e^{(i+i_{\min}-1)k}} \mid i = 1, \ldots, i_{\max} - i_{\min} + 1 \right\} = \{b_1, \ldots, b_S\},$$

where $|\mathcal{S}| = S$ and $S \stackrel{\triangle}{=} i_{\max} - i_{\min} + 1$ and $b_{1-i_{\min}} = b$. We see that the portfolio is rebalanced to $b_{1-i_{\min}} = b$ only if it is in the state b_1 and $X_2 = e^{-k}$ or if it is in the state b_S and $X_2 = e^k$. Therefore, the transition probabilities are given by

$$\Pr(b_i|b_j) = \begin{cases} \frac{1}{2} : i = 2, \ldots, S-1 \text{ and } j = i \pm 1 \text{ ,or } i = 1 \text{ and } j \in \{2, 1-i_{\min}\}, \text{ or } i = S \\ \quad \text{and } j \in \{S-1, 1-i_{\min}\} \\ 0 : \text{otherwise,} \end{cases}$$

where $P(b_i|b_j)$ is the probability that the portfolio $b(n) = b_i$ given that $b(n-1) = b_j$ for any period n. We now calculate $E[S(n)]$ using (3.15) and (3.16) as follows. The sets of price relative vectors that connect portfolio states are given by

$$\mathcal{U}_{i,j} = \begin{cases} \{[1 \; e^k]^T\} & : i = 1, \ldots, S-1 \text{ and } j = i+1, \text{ or } i = S \text{ and } j = 1 - i_{\min} \\ \{[1 \; e^{-k}]^T\} & : i = 2, \ldots, S-1 \text{ and } j = i-1, \text{ or } i = 1 \text{ and } j = 1 - i_{\min} \\ \varnothing & : \text{otherwise.} \end{cases}$$

Hence, we can calculate the matrix $\mathbf{Q}$ defined in (3.17) as

$$\mathbf{Q}_{i,j} = \begin{cases} \frac{1}{2}(b_j + (1-b_j)e^k) & : i = 2, \ldots, S \text{ and } j = i-1 \\ \frac{1}{2}(b_j + (1-b_j)e^{-k}) & : i = 1, \ldots, S-1 \text{ and } j = i+1 \\ 0 & : \text{otherwise,} \end{cases}$$

where we ignore rebalancing. If we take rebalancing into account, then

$$\mathbf{Q}_{1-i_{\min},1} = \frac{1}{2}(b_1 + (1-b_1)e^{-k}) \left(1 - c \left| \frac{b_1}{b_1 + (1-b_1)e^{-k}} - b \right| \right)$$

and

$$\mathbf{Q}_{1-i_{\min},S} = \frac{1}{2}(b_S + (1-b_S)e^k)\left(1 - c\left|\frac{b_S}{b_S + (1-b_S)e^k} - b\right|\right).$$

Then, by (3.15) and (3.16), $E[S_n]$ is given by $\mathbf{Q}^n\mathbf{e}(0)$. Moreover, we maximize

$$g(b,\epsilon) = \lim_{n\to\infty}\frac{1}{n}\log E[S_n] = \log\lambda_1,$$

where λ_1 is the largest eigenvalue of the matrix $\mathbf{Q}$. Here, we optimize b and ϵ with a brute-force search, i.e., we find λ_1 for different (b,ϵ) pairs and find the one that achieves the maximum.

In the next section, we sequentially estimate the probability mass vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ of the price relative sequences $X_1(n)$ and $X_2(n)$ using a maximum likelihood estimator.

3.3 Maximum Likelihood Estimators of The Probability Mass Vectors

In this section, we sequentially estimate the probability mass vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ corresponding to $X_1(n)$ and $X_2(n)$, respectively, using a maximum likelihood estimator (MLE). In general, these vectors may not be known or change in time, hence, could be estimated at each investment period prior to calculation of $E[S(n)]$. The maximum likelihood estimator for a pmf on a finite set is well-known [40], but we provide the corresponding derivations here for completeness. We consider, without loss of generality, the price relative sequence $X_1(n)$ and assume that its realizations are given by $X_1(n) = w_n \in \mathcal{X}$ for $n = 1,\ldots,N$ and estimate $\mathbf{p}_1$. Similar derivations follow for the price relative sequence $X_2(n)$ and $\mathbf{p}_2$. Note that as demonstrated in the Simulations section, the corresponding estimation can be carried out over a finite length window to emphasize the most recent data. We define the realization vector $\mathbf{w} = [w_1,\ldots,w_N]$ and the probability mass function as $p_\theta(x_i) = p_1(x_i|\theta) = \theta_{x_i}$ for $i = 1,\ldots,K$ and the parameter vector $\theta \stackrel{\triangle}{=} [\theta_{x_1},\ldots,\theta_{x_K}]$. Then, the MLE of the probability mass vector $\mathbf{p}_1$ is given by

$$\theta_{\mathrm{MLE}} = \mathop{\arg\max}_{\theta:\sum_{i=1}^K \theta_{x_i}=1} p_1(\mathbf{w}|\theta) = \mathop{\arg\max}_{\theta:\sum_{i=1}^K \theta_{x_i}=1} \Pr\left(X_1(1)=w_1,\ldots,X_1(N)=w_N|\theta\right). \quad (3.18)$$

Since the price relative sequence $X_1(n)$ is i.i.d., it follows that

$$p_1(\mathbf{w}|\theta) = \prod_{i=1}^{N} p_1(w_i|\theta) = \prod_{i=1}^{N} \theta_{w_i} = \prod_{i=1}^{N} \prod_{j=1}^{K} \theta_{x_j}^{\mathrm{I}(w_i=x_j)}, \tag{3.19}$$

where (3.19) follows since I(.) is the indicator function, i.e., $\mathrm{I}(w_i = x_j) = 1$ if $w_i = x_j$ and $\mathrm{I}(w_i = x_j) = 0$ if $w_i \neq x_j$. If we change the order of the product operators in (3.19), then we obtain

$$p_1(\mathbf{w}|\theta) = \prod_{i=1}^{N} \prod_{j=1}^{K} \theta_{x_j}^{\mathrm{I}(w_i=x_j)} = \prod_{j=1}^{K} \prod_{i=1}^{N} \theta_{x_j}^{\mathrm{I}(w_i=x_j)} = \prod_{j=1}^{K} \theta_{x_j}^{\sum_{i=1}^{N} \mathrm{I}(w_i=x_j)} = \prod_{j=1}^{K} \theta_{x_j}^{N_j},$$

where $N_j \triangleq \sum_{i=1}^{N} \mathrm{I}(w_i = x_j)$, i.e., the number of realizations that are equal to $x_j \in \mathcal{X}$ for $j = 1, \ldots, K$. Note that $\sum_{j=1}^{K} N_j = N$. Hence, we can write (3.18) as

$$\begin{aligned} \theta_{\mathrm{MLE}} = \underset{\theta:\sum_{i=1}^{K} \theta_{x_i}=1}{\arg\max}\; p_1(\mathbf{w}|\theta) = \underset{\theta:\sum_{i=1}^{K} \theta_{x_i}=1}{\arg\max} \prod_{j=1}^{K} \theta_{x_j}^{N_j} &= \underset{\theta:\sum_{i=1}^{K} \theta_{x_i}=1}{\arg\max} \frac{1}{N} \log\left(\prod_{j=1}^{K} \theta_{x_j}^{N_j}\right) \\ &= \underset{\theta:\sum_{i=1}^{K} \theta_{x_i}=1}{\arg\max} \sum_{j=1}^{K} \frac{N_j}{N} \log \theta_{x_j}, \end{aligned} \tag{3.20}$$

where (3.20) follows that log(.) is a monotone increasing function. If we define the vector $\mathbf{h} = [h_{x_1}, \ldots, h_{x_K}]$, where $h_{x_j} \triangleq \frac{N_j}{N}$ for $j = 1, \ldots, K$, then we see that $h_{x_j} \geq 0$ for $j = 1, \ldots, K$ and $\sum_{j=1}^{K} h_{x_j} = 1$. Since $\mathbf{h}$ and θ are probability vectors, i.e., their entries are nonnegative and sum to one, it follows that $\mathrm{D}(\mathbf{h}\|\theta) \triangleq \sum_{i=1}^{K} h_{x_j} \log\left(\frac{h_{x_j}}{\theta_{x_j}}\right) \geq 0$ and $\mathrm{D}(\mathbf{h}\|\theta) = 0$ if and only if $\theta = \mathbf{h}$, i.e., their relative entropy is nonnegative [15]. Therefore, we get that

$$\begin{aligned} \sum_{j=1}^{K} \frac{N_j}{N} \log \theta_{x_j} = \sum_{j=1}^{K} h_{x_j} \log \theta_{x_j} &= \sum_{j=1}^{K} h_{x_j} \log\left(\frac{\theta_{x_j}}{h_{x_j}}\right) + \sum_{j=1}^{K} h_{x_j} \log h_{x_j} \\ &= -\mathrm{D}(\mathbf{h}\|\theta) + \sum_{j=1}^{K} h_{x_j} \log h_{x_j} \leq \sum_{j=1}^{K} h_{x_j} \log h_{x_j}, \end{aligned}$$

where the equality is reached if and only if $\theta = \mathbf{h}$. Hence, it follows that

$$\theta_{\mathrm{MLE}} = \underset{\theta:\sum_{i=1}^{K} \theta_{x_i}=1}{\arg\max}\; p_1(\mathbf{w}|\theta) = \underset{\theta:\sum_{i=1}^{K} \theta_{x_i}=1}{\arg\max} \sum_{j=1}^{K} \frac{N_j}{N} \log \theta_{x_j} = \mathbf{h}$$

so that we estimate the probability mass vector $\mathbf{p}_1$ with $\mathbf{h} = \left[\frac{N_1}{N}, \ldots, \frac{N_K}{N}\right]$ at each investment period N where $\frac{N_j}{N}$ is the proportion of realizations up to period N that are equal to x_j for $x_j \in \mathcal{X}$.

3.4 Simulations

In this section, we demonstrate the performance of TRPs with several different examples. We first analyze the performance of TRPs in a discrete-time two-asset Brownian market introduced in Section 3.2.5. As the next example, we apply TRPs to historical data from [13, 27] collected from the New York Stock Exchange over a 22-year period and compare the results to those obtained from other investment strategies [13, 22, 26, 27]. Using the historical data set, we first simulate the performance of TRPs, the semiconstant rebalanced portfolio (SCRP) [27], the Iyengar's algorithm [22], the Cover's algorithm [13] and the switching portfolio from [26] on a randomly selected stock pair. Finally, we then present the average performance of TRPs on randomly selected pairs of stocks and show that the performance of the TRP algorithm is significantly better than the portfolio investment strategies from [13, 22, 26, 27] in historical data sets as expected from Section III.

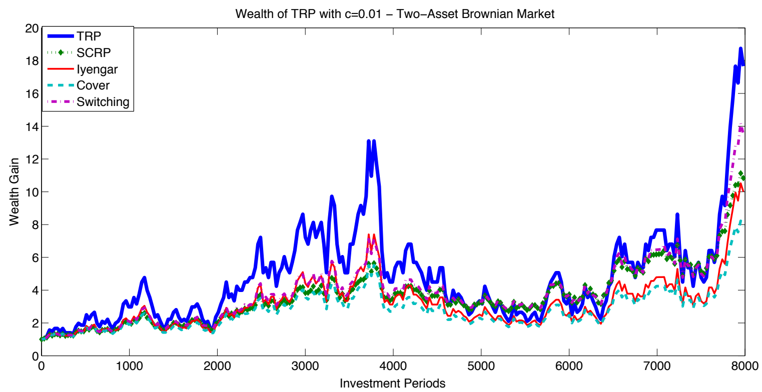
As the first scenario, we apply TRPs to a discrete-time two-asset Brownian market. Under this well studied market in the financial literature [29], the price relative vector is given by $\boldsymbol{X} = [X_1 \; X_2]^T$, where $X_1 = 1$, $X_2 = e^{kZ}$ and $Z = \pm 1$ with equal probabilities and we set $k = 0.03$ [21]. Here, the sample spaces of the price relative sequences X_1 and X_2 are $\mathcal{X}_1 = \{1\}$ and $\mathcal{X}_2 = \{0.97, 1.03\}$, respectively, and $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 = \{x_1, x_2, x_3\}$, where $x_1 = 1$, $x_2 = 0.97$, $x_3 = 1.03$. Hence, the probability mass vectors of the price relative sequences X_1 and X_2 are given by $\mathbf{p}_1 = [1\;0\;0]^T$ and $\mathbf{p}_2 = [0\;0.5\;0.5]^T$, respectively. Based on this data, we evaluate the growth rate for different (b, ϵ) pairs to find the best TRP that maximizes the growth rate using the approach introduced in Section 3.2.5, i.e., we form the matrix $\mathbf{Q}$ and evaluate the corresponding maximum eigenvalues to find the pair that achieves the largest maximum eigenvalue since this pair also maximizes the growth rate. Then, we invest 1 dollars in a randomly generated two-asset Brownian market using: the TRP, labeled as, "TRP", i.e., TRP(b,ϵ) with calculated (b, ϵ) pair, the SCRP algorithm with the target portfolio vector $\boldsymbol{b} = [0.5\;0.5]$, labeled as "SCRP", as suggested in [27], the Iyengar's algorithm, labeled as "Iyengar", the Cover's algorithm, labeled as "Cover", and the switching portfolio, labeled as "Switching", with parameters suggested in [26]. In Fig. 3.2, we plot the wealth achieved by each algorithm for transaction costs $c = 0.01$ and $c = 0.03$, where c is the proportion paid when rebalancing, i.e, $c = 0.03$ is a 3% commission. As expected from the derivations in Section III, we observe that, in both cases, the performance of the TRP algorithm is significantly better than the other algorithms under transaction costs.

We next present results that illustrate the performance of TRPs on historical data sets [13]. As for the first example, we present results on the stock pair Morris and Commercial Metals (randomly selected) from the historical data sets [13, 27] for a mild transaction cost $c = 0.015$ and a hefty transaction cost $c = 0.03$ to better illustrate the effect of transaction costs. The data includes the price relative sequences of the stock pair for 5651 investment periods (days). Since the brute force algorithm introduced in Section 3.2.1 requires the sample spaces of the price relative sequences, we proceed as follows. We first calculate the sample spaces and the probability mass vectors of the price relative sequences from the first 1000-day realizations of X_1 and X_2, where the sample spaces are simply constructed by quantizing the observed realizations into bins. We observed that the performance of the TRP is not effected by the number of bins provided that there are an adequate number of bins to approximate the continuous valued price relatives. Then, we optimize b and ϵ using the MLE introduced in Section IV and the brute force algorithm from Section III, and invest using this TRP for the next 1000 periods, i.e., from period 1001 to period 2000. We then update (b, ϵ) pair using the first 2000-day realizations of the price relative vectors and invest using the best TRP for the next 1000 periods. We repeat this process through all available data. Hence, we invest on the two stocks using TRP for 4651 periods where we update (b, ϵ) pair at each 1000 periods. In Fig. 3.3, we present the performances of the TRP algorithm, the SCRP algorithm, the Iyengar's algorithm [22], the Cover's algorithm and the switching portfolio algorithm [26]. We observe that although the performance of the algorithms other than the TRP degrade with increasing transaction cost, the performance of the TRP, using the MLE, is not significantly effected since it can avoid excessive rebalancings. In both cases, the TRP readily outperforms the other simulated algorithms for these simulations.

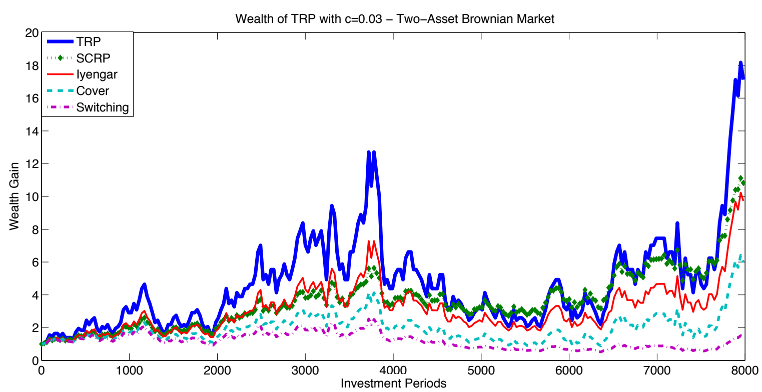
Finally, we illustrate the average performance of the threshold rebalancing strategy on a number of stock pairs to avoid any bias to particular stock pairs. In this set of simulations, we first randomly select pairs of stocks from the historical data that includes 34 stocks (where the Kin Ark stock is excluded) and invest using: the TRP algorithm, the SCRP algorithm, the Cover's algorithm, the Iyengar's algorithm and the switching portfolio, under a mild transaction cost $c = 0.015$ and a hefty transaction cost $c = 0.03$. In Fig. 3.4, we present the wealth gain for each algorithm, where the results are averaged over randomly selected 10 independent stock pairs. We observe from these simulations that the average performance of the TRP is better than the average performance of the other portfolio investment strategies commonly used in the literature.

3.5 Conclusions

We studied growth optimal investment in i.i.d. discrete-time markets under proportional transaction costs. Under this market model, we studied threshold portfolios that are shown to yield the optimal growth. We first introduced a recursive update to calculate the expected growth for a two-asset market and then extend our results to markets having more than two assets. We next demonstrated that under the threshold rebalancing framework, the achievable set of portfolios form an irreducible Markov chain under mild technical conditions. We evaluated the corresponding stationary distribution of this Markov chain, which provides a natural and efficient method to calculate the cumulative expected wealth. Subsequently, the corresponding parameters are optimized using a brute force approach yielding the growth optimal investment portfolio under proportional transaction costs in i.i.d. discrete-time two-asset markets. We also solved the optimal portfolio selection in discrete-time markets constructed by sampling continuous-time Brownian markets. For the case that the underlying discrete distributions of the price relative vectors are unknown, we provide a maximum likelihood estimator. We observed in our simulations, which include simulations using the historical data sets from [13], that the introduced TRP algorithm significantly improves the achieved wealth under both mild and hefty transaction costs as predicted from our derivations.

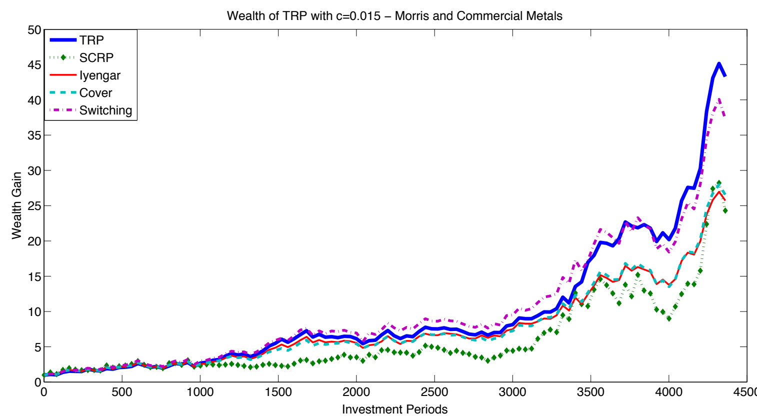


(a)

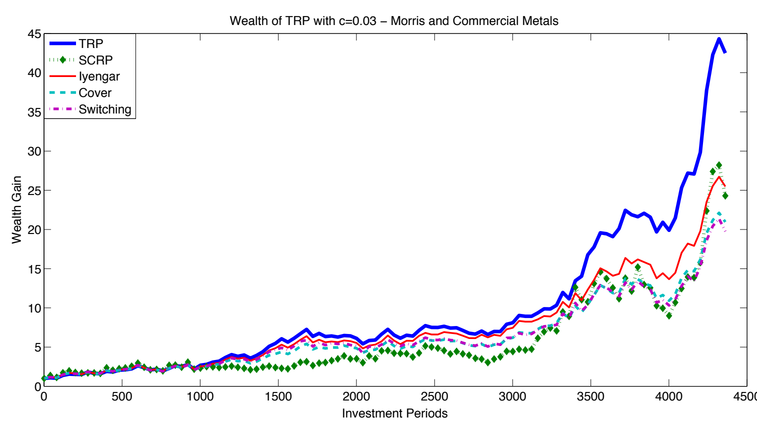


(b)

Figure 3.2: Performance of portfolio investment strategies in the two-asset Brownian market. (a) Wealth gain with the cost ratio $c = 0.01$. (b) Wealth gain with the cost ratio $c = 0.03$.

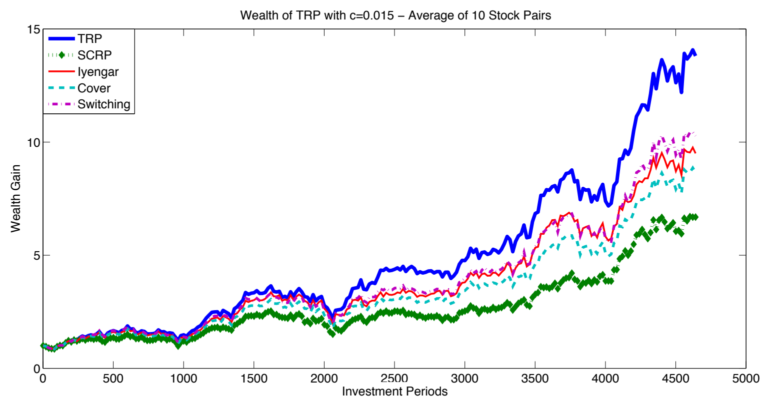


(a)

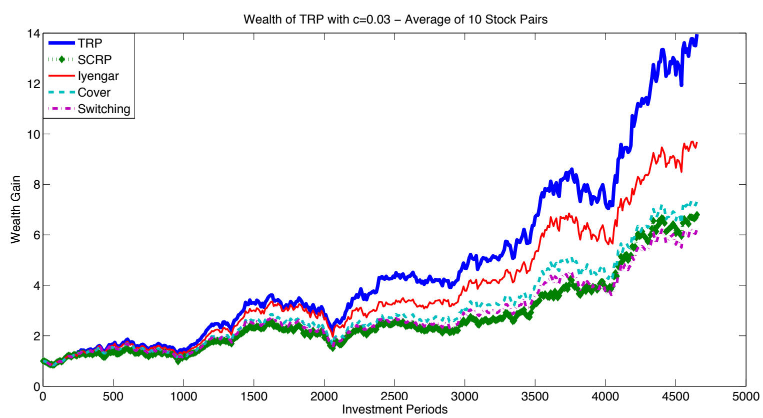


(b)

Figure 3.3: Performance of portfolio investment strategies on the Morris-Commercial Metals stock pair. (a) Wealth gains with the cost ratio $c = 0.015$. (b) Wealth gains with the cost ratio $c = 0.03$.



(a)



(b)

Figure 3.4: Average performance of portfolio investment strategies on independent stock pairs. (a) Wealth gain with the cost ratio $c = 0.015$. (b) Wealth gain with the cost ratio $c = 0.03$.

Chapter 4

CONCLUSIONS

In this thesis, we considered portfolio optimization problem in i.i.d. discrete-time markets under two different scenarios, where the market is modeled by a sequence of price relative vectors with log-normal distribution and with arbitrary discrete distributions. Chapter 2 deals with maximizing the expected cumulative wealth in i.i.d. discrete-time markets where the market levies proportional transaction costs under the assumption that the price relative sequences have log-normal distribution and Chapter 3 is dedicated to construct portfolios that achieve the optimal expected growth in i.i.d. discrete-time markets modeled by a sequence of price relative vectors with arbitrary discrete distributions under proportional transaction costs.

In Chapter 2, we investigated the portfolio selection problem in i.i.d. discrete time markets having a finite number of assets, when the market levies proportional transaction fees for both buying and selling stocks. We introduced algorithms based on threshold rebalanced portfolios that achieve the maximal growth rate when the sequence of price relatives have the log-normal distribution from the well-known Black-Scholes model. Under this setup, we provide an iterative relation that efficiently and recursively calculates the expected wealth in any i.i.d. market over any investment period. The terms in this recursion are evaluated by a certain multivariate Gaussian integral. We then use a randomized algorithm to calculate the given integral and obtain the expected growth. This expected growth is then optimized by a brute force method to yield the optimal target portfolio and the threshold to maximize the expected wealth over any investment period. We also provide a maximum-likelihood estimator to estimate the parameters of the log-normal distribution from the sequence of price relative vectors. As predicted from our derivations, we significantly improve the achieved wealth over portfolio selection algorithms from the literature on the historical data set from.

In Chapter 3, we first introduced a recursive update to calculate the expected growth for a two-asset market modeled by a sequence of price relative vectors with arbitrary discrete distributions and then extend our results to markets having more than two assets. We next demonstrated that under the threshold rebalancing framework, the achievable set of portfolios form an irreducible Markov chain under mild technical conditions. We evaluated the corresponding stationary distribution of this Markov chain, which provides a natural and efficient method to calculate the cumulative expected wealth. Subsequently, the corresponding parameters are optimized using a brute force approach yielding the growth optimal investment portfolio under proportional transaction costs in i.i.d. discrete-time two-asset markets. We also solved the optimal portfolio selection in discrete-time markets constructed by sampling continuous-time Brownian markets. For the case that the underlying discrete distributions of the price relative vectors are unknown, we provide a maximum likelihood estimator. We observed in our simulations, which include simulations using the historical data sets from [13], that the introduced TRP algorithm significantly improves the achieved wealth under both mild and hefty transaction costs as predicted from our derivations.

Chapter 5

APPENDIX A

Proof of Lemma 3.2.1:

We analyze the cardinality of the set $\mathcal{B}_N$ of achievable portfolios at period N, M_N, as follows. If we assume that an investor invests with a TRP(b,ϵ) for N investment periods and the sequence of price relative vectors are given by $\left\{[X_1(n)\ X_2(n)] = \left[X_1^{(n)}\ X_2^{(n)}\right]\right\}_{n=1}^{N}$ and the portfolio sequence is given by $\{b(n) = b_n\}_{n=1}^{N}$, then we see that the portfolio could leave the interval at any period depending on the realizations of the price relative vector. We define an N-period market scenario as a sequence of portfolios $\{b(n)\}_{n=1}^{N}$. We can find the number of achievable portfolios at period N as the number of different values that the last element of N-period market scenarios can take. Here, we partition the set of N-period market scenarios according to the last time the portfolio leaves the interval $(b-\epsilon, b+\epsilon)$ and show that any achievable portfolio at period N can be achieved by an N-period market scenario where the portfolio does no leave the interval $(b-\epsilon, b+\epsilon)$ for N periods as follows. If we define the set $\mathcal{P}$ as the set of N-period market scenarios, i.e.,

$$\mathcal{P} = \left\{\{b_n\}_{n=1}^{N} \mid b_n \in \mathcal{B}_n\ , n = 1, \ldots, N\right\} = \bigcup_{i=1}^{N+1} \mathcal{P}_i,$$

where $\mathcal{P}_i$ is the set of N-period market scenarios where the portfolio leaves the interval $(b-\epsilon, b+\epsilon)$ last time at period i, i.e.,

$$\mathcal{P}_i = \left\{\{b_n\}_{n=1}^{N} \mid b_n \in \mathcal{B}_n\ , n = 1, \ldots, N,\ b(n)\,\mathrm{leavestheinterval}\,(b-\epsilon, b+\epsilon)\,\mathrm{lasttimeatperiod}\,i\right\}$$

for $i = 1, \ldots, N$ and $\mathcal{P}_{N+1}$ is the set of N-period market scenarios where the portfolio does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods, i.e.,

$$\mathcal{P}_{N+1} = \left\{\{b_n\}_{n=1}^{N} \mid b_n \in \mathcal{B}_n\ , n = 1, \ldots, N,\ b(n)\,\mathrm{neverleavestheinterval}\,(b-\epsilon, b+\epsilon)\,\mathrm{forNperiods}\right\}.$$

We point out that $\mathcal{P}_i$'s are disjoint, i.e., $\mathcal{P}_i \cap \mathcal{P}_j = \varnothing$ for $i \neq j$ and their union gives the set of all N-period market scenarios, i.e., $\bigcup_{i=1}^{N+1} \mathcal{P}_i = \mathcal{P}$ so that they form a partition for $\mathcal{P}$. We see that the set $\mathcal{B}_N$ of achievable portfolios at period N is the set of last elements of N-period market scenarios, i.e., $\mathcal{B}_N = \{b_N \mid \{b_n\}_{n=1}^N \in \mathcal{P}\}$. We next show that the last element of any N-period market scenario from $\mathcal{P}_i$ for $i = 1, \ldots, N$ is also a last element of an N-period market scenario from $\mathcal{P}_{N+1}$. Therefore, we demonstrate that any element of the set $\mathcal{B}_N$ is achievable by a market scenario from $\mathcal{P}_{N+1}$ and $\mathcal{B}_N = \{b_N \mid \{b_n\}_{n=1}^N \in \mathcal{P}_{N+1}\}$.

Assume that $\{b_n\}_{n=1}^N \in \mathcal{P}_i$ for some $i \in \{1, \ldots, N\}$ so that $b_i = b$, i.e., the portfolio is rebalanced to b last time at period i. Note that b_N can also be achieved by an N-period market scenario $\left\{b_n^{'}\right\}_{n=1}^N$ where the portfolio never leaves the interval $(b-\epsilon, b+\epsilon)$, $b_j^{'} = b_{i+j}$ for $j = 1, \ldots, N-i$ and $X_1^{(j)} = X_2^{(j)}$ for $j = N-i+1, \ldots, N$ so that $b_N^{'} = b_{N-i}^{'} = b_N$. Hence, it follows that the set of achievable portfolios at period N is the set of achievable portfolios by N-period market scenarios from $\mathcal{P}_{N+1}$. We next find the number of different values that $b(N)$ can take where the portfolio does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods.

When the portfolio never leaves the interval $(b-\epsilon, b+\epsilon)$ for N investment periods, $b(N)$ is given by

$$b(N) = \frac{b \prod_{i=1}^N X_1(n)}{b \prod_{i=1}^N X_1(n) + (1-b) \prod_{i=1}^N X_2(n)}.$$

If we write the reciprocal of $b(N)$ as

$$\frac{1}{b(N)} = 1 + \frac{1-b}{b} \prod_{n=1}^{N} \frac{X_2(n)}{X_1(n)} = 1 + \frac{1-b}{b} e^{\sum_{n=1}^N Z(n)},$$

then we observe that the number of different values that the portfolio $b(N)$ can take is the same as the number of different values that the sum $\sum_{n=1}^N Z(n)$ can take. Since the price relative sequences $X_1(n)$ and $X_2(n)$ are elements of the same sample space $\mathcal{X}$ with $|\mathcal{X}| = K$, it follows that $|\mathcal{Z}| = M \leq K^2 - K + 1$. Since the number of different values that the sum $\sum_{n=1}^N Z(n)$ can take is equal to $\binom{N+M-1}{M-1}$ and $M \leq K^2 - K + 1$, it follows that the number of achievable portfolios at period N is bounded by $\binom{N+K^2-K}{K^2-K}$, i.e., $|\mathcal{B}_N| = M_N \leq \binom{N+K^2-K}{K^2-K}$ and the proof follows.

Chapter 6

APPENDIX B

Proof of Lemma 3.2.2:

If the portfolio does not leave the interval $(b-\epsilon, b+\epsilon)$ for N investment periods, then $b(n) \in (b-\epsilon, b+\epsilon)$ for $n = 1, \ldots, N$ and it is not adjusted to b at these periods so that

$$b(n) = \frac{b \prod_{i=1}^{n} X_1(i)}{b \prod_{i=1}^{n} X_1(i) + (1-b) \prod_{i=1}^{n} X_2(i)} \in (b-\epsilon, b+\epsilon)$$

for each $n = 1, \ldots, N$. Taking the reciprocal of $b(n)$, we get that

$$\frac{b(1-b-\epsilon)}{(1-b)(b+\epsilon)} < \prod_{i=1}^{n} \frac{X_2(i)}{X_1(i)} < \frac{b(1-b+\epsilon)}{(1-b)(b-\epsilon)}.$$

Noting that $\frac{X_2(i)}{X_1(i)} = e^{Z(i)}$ and taking the logarithm of each side, it follows that

$$\ln \frac{b(1-b-\epsilon)}{(1-b)(b+\epsilon)} = \alpha_2 < \sum_{i=1}^{n} Z(i) < \ln \frac{b(1-b+\epsilon)}{(1-b)(b-\epsilon)} = \alpha_1,$$

i.e., $\sum_{i=1}^{n} Z(i) \in (\alpha_2, \alpha_1)$ for $n = 1, \ldots, N$. Now, if the portfolio leaves the interval $(b-\epsilon, b+\epsilon)$ first time at period k for some $k \in \{1, \ldots, N\}$, then we get that $b(k) \geq b+\epsilon$ or $b(k) \leq b-\epsilon$ so that we get

$$\sum_{i=1}^{k} Z(i) \geq \alpha_1 \text{or} \sum_{i=1}^{k} Z(i) \leq \alpha_2,$$

i.e., $\sum_{i=1}^{k} Z(i) \notin (\alpha_2, \alpha_1)$.

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