Revenue Management with Markov Modulated Demand

by

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and have found that it is complete and satisfactory in all respects,
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ABSTRACT

The admission decision is one of the fundamental categories of demand-management decisions. In the dynamic model of the single-resource capacity control problem, distribution of demand at each stage is known, but it is either fixed or time-dependent. However, in reality demand may depend on the current external environment which represents the prevailing economic, financial, social or other factors that affect customer behavior. In this thesis, we formulate a Markov decision process (MDP) model in which the state of the process is described by the remaining inventory level and the current environment. We derive some structural results for this MDP model, including the existence of an environment-dependent threshold policy and a comparison of threshold levels in different environments. Then we extend our study to a dynamic pricing problem in which there is only one type of product and the objective is to find the optimal pricing policy to maximize the revenue of the firm. Another important research topic of revenue management is modeling consumer behavior. In such models, the change in consumer behavior towards the set of products offered is also considered. We formulate a discrete choice model of consumer behavior in fluctuating demand environment with a Markovian structure. We derive some structural results on the optimal policy for revenue management. Even though there may be an environment process, we may not observe it directly. We also model such a problem in which we just observe an external process that gives a probability distribution for the true state of the current environment. This model can be classified as Hidden Markov Decision Process and we derive structural results for it. We also present some computational results for all these three models in order to illustrate our structural properties.

ÖZETÇE

Kabul kararları talep yönetim kararlarının temel kategorilerindendir. Tek kaynaklı kapasite kontrol problemlerinin dinamik versiyonunda, talebin olasılık dağılımının bilindiği ve başka dış etkenlerden bağımsız olduğu varsayılmaktadır. Fakat gerçekte talep ekonomik, finansal, sosyal ya da müşterinin kararlarını etkileyen diğer etkenleri içinde barındıran anlık dış çevreye bağımlı olarak değişebilir. Bu tezde, karar değişkeni kalan envanter seviyesi ve anlık çevre olan bir Markov Karar Süreci modeli oluşturulup bu modelin yapısal sonuçları incelenecektir. Bu yapısal sonuçlar arasında anlık çevreye bağımlı eşik değeri politikasının varlığı ve bu eşik değerlerinin farklı çevrelerdeki kıyaslaması yer almaktadır. Bu çalışmanın sonuçları tek ürünün fiyatlandırıldığı ve amacın beklenen hastılatını ençoklayacak fiyatlandırma politikasını bulmak olduğu dinamik fiyatlandırma problemi için de genişletilmiştir. Hasılat yönetimindeki bir başka genel araştırma konusu da müşteri davranışlarına göre modellemedir. Bu modellerde işletme tarafından müşteriye küme olarak sunulan ürünlere karşı müşterinin davranışları da dikkate alınır. Bu tezde, yukarıda bahsedilen çevre bazlı modellere ek olarak, Markov yapıda değişen çevredeki müşteri davranış karar modeli oluşturulmuş ve yapısal sonuçlar incelenmiştir. Tüm bunlara ek olarak, çevre sürecinin direk gözlenemediği koşullar da dikkate alınmıştır. Böyle bir durum için de çevrenin o andaki durum değişkeni hakkında olasılık dağılımı bilinen bir dış etkenler sürecinin de içinde olduğu bir model oluşturulmuştur. Saklı Markov Karar Süreci sınıfına giren bu model için de yapısal sonuçlar elde edilmiştir. Her üç modelde de bulunan yapısal sonuçları örneklendirmek ve zenginleştirmek adına hesaplamalı sonuçlar kullanılmıştır.

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NOMENCLATURE

X_t	:	Random environment at time t
P	:	Transition matrix of environment process
x	:	Inventory level
r_{ja}	:	Probability that fare class a arrives in environment j
q_{jab}	:	Probability that b units of product is requested when a arrives in environment j
c	:	Reward function
F_j	:	Distribution of price that customer is willing to pay
$P_a^j(S)$	:	Probability that product a is chosen when S is offered in environment j
λ_j	:	Probability of an arrival in environment j
Y_t	:	Observation at time t
E	:	Emission matrix
π_t	:	belief vector at time t

Chapter 1

INTRODUCTION

Revenue management is a field that originates in the Airline Deregulation Act of 1978 (Talluri and van Ryzin [30]). There have been many studies since 1978 on different topics of revenue management. One of these topics is single resource capacity control. It is common in airline companies to sell identical seats at different fares. The major issue is the decision process of accepting or rejecting a booking request of a certain class.

Despite the improvements on models in the revenue management literature, most of the models assume that the arrival process of fare classes is either time independent or dependent with a known rate. Van Ryzin [33] emphasizes the needs for better demand modeling for revenue management. In particular, he mentions that standard demand models in revenue management treat causal variations based on external factors as noise. The environment-based framework addresses this issue. Van Ryzin [33] points out short term market conditions as a significant factor. These include competitors' availabilities and prices. Airline industry is also a very competitive field, customers make their decisions according to the available options for the flight which they want to use and companies need to consider the strategies of other competitors. In addition, there is evidence that the aggregate demand is affected by external market forces such as currency exchange rates and energy prices. Finally, weather conditions, such as forecasted snow storms or heat waves are important short term external factors that are known to impact demand in hotel and airline revenue management. Although these external factors seem to be very different from each other, they all influence the demand. This motivates the need for modeling the effects of such factors through an environment-dependent demand model. This kind of models are investigated in inventory replenishment problems (See Song and Zipkin [28] Özekici and Parlar [26]) Finally, there is reason to believe that environment-based demand may have a bigger impact in revenue management problems than in inventory replenishment problems. In inventory replenishment, the ordering decision helps absorbing some of the variability in demand. But in revenue management there is typically no replenishment opportunity and demand variability has to be addressed only by admission or pricing decisions.

Although these external factors seem to be very different from each other, such a model has not been discussed widely in a revenue management context. To our knowledge, the only study that seems to be related to our work is done by Barz [5]. She considers an infinite horizon problem that guarantees the termination of the selling period before the season ends by using absorbing states. On the other hand, this kind of model prevents us from understanding the effects of time on the optimal policy. Since we have a finite selling season, it is significant to understand how the optimal policy changes in time. Therefore, we consider a finite horizon problem and investigate the structural properties related to time. In addition to this, it is crucial to analyze the effects of the environment on the optimal policy in a random environment model. To our knowledge, our work is the only one which makes such an analysis in revenue management context. We make some intuitive assumptions on the environmental states to distinguish them, and we compare revenues and optimal policies for different environmental states. Moreover, we analyze the effects of varying problem parameters. We first investigate the effects of arrival probabilities and reward on the optimal policy. Then, we vary the transition matrix of the environmental process. In order to perform such an analysis, we use some assumptions which are made to compare different environments.

We further extend our study to the related dynamic pricing problem. In this model, we assume that there is only one type of product and firm needs to choose a pricing policy in order to maximize its revenue. We also investigate this problem in a randomly fluctuating demand environment. We provide the structure of the optimal pricing policy including the effects of time and inventory level.

Consumer behavior modeling is another widely studied topic of the revenue management. One of the consumer behavior models involves choosing from a given set of products offered to the consumer. This model considers the attitude of the customers towards to the changes in set of products offered. Moreover, we believe that demand for each product is affected by an external process. We model this environmental process by a Markov chain and consider the general discrete choice model of consumer behavior in such a randomly fluctuating demand environment. We provide the structure of the optimal policy including the effects of time and inventory level.

We also consider the case where the true state of the environmental process may not be directly observed. However, we assume that there is another process which provides us information about the true state of the environment. We have a belief vector which is actually the probability distribution for the true state of the environment and we update this belief vector at each period by using information up to that period. We call this the "observation process" and refer to the corresponding model as single resource capacity control problem under a partially observed demand environment. We provide structural results for the optimal policy.

The organization of this thesis is as follows. The next part contains relevant models in the literature. In Chapter 3, we first model a Markov modulated single resource capacity control problem and provide structural results, sensitivity analysis and numerical illustrations. We also model a dynamic pricing problem and its structural properties in a randomly fluctuating demand environment at the last section of this chapter. In Chapter 4, we introduce a choice model of consumer behavior in a fluctuating demand environment and its structural properties with a numerical example that illustrates these properties. We also provide a hidden Markov model in Chapter 5 with some structural results. Finally, we present some concluding remarks including some directions for future research in Chapter 6.

Chapter 2

LITERATURE REVIEW

Since the Airline Deregulation Act of 1978, there have been many studies on different topics of revenue management. In this thesis, we only study single resource allocation problem and dynamic pricing. We investigate some of the well-known models in these topics by considering a randomly fluctuating demand environment. In Section 1, we first provide the well-known literature related to the single resource allocation problem. Consumer behavior and pricing are other issues that firms need to consider in order to increase their revenues and we discuss these topics in Section 2. In addition to these topics, we provide the literature related to Markov modulated demand models in Section 3. There are some other topics such as network revenue management which are widely discussed in revenue management and we provide a brief overview on some of these topics in Section 4. Also a detailed overview for revenue management can be found in Talluri and van Ryzin [30] and Chiang et al. [11].

2.1 Static and Dynamic Single Resource Allocation Problems

Although many airline companies have connected flights, it has been always crucial to analyze the single leg problem. In such a problem, there is only one resource and there are multiple fares. The decision maker needs decide when to accept a request from a fare class or to reject that request. When the decision maker accepts a request from a specific fare class, then we may lose the chance of a request from a higher fare class afterwards. Therefore, it is not always optimal to accept a request. Especially, when one considers the fact that higher classes, such as business class, prefer to buy their tickets at a time close to departure time of the plane, then one can easily understand that it is not always optimal to accept a request from classes other than higher classes.

There are two main models in the study of single resource allocation problem. One of them is the static model in which different fare classes arrive at different, nonoverlapping time stages ordered in an increasing fare class prices. This type of model is first considered by Littlewood [24]. This model assumes that there are two fare classes with prices $p_1 > p_2$. Corresponding demands are D_1 and D_2 with distribution $F_1(\cdot)$ and $F_2(\cdot)$ respectively. In addition, this model assumes that second fare classes with price p_2 arrives first. The decision maker needs to decide a quantity, which is called a protection level, in order to leave that amount of the capacity for the first fare class. Littlewood shows that this protection level y^* should satisfy

$$y^* = F_1^{-1}\left(1 - \frac{p_2}{p_1}\right)$$

This fundamental result has been generalized for the case where there are more than two fare classes. However, customers may not arrive in an order where higher fare classes come at a later time and customers may arrive randomly. Such cases are considered in dynamic programming model of this problem. This is first analyzed by Lee and Hersh [23], and the structure of the optimal policy is investigated by Lautenbacher and Stidham [21]. It has been shown that there exists a threshold level for each fare class such that if the current inventory level is larger than this level, then it is optimal to accept a request from that fare class, otherwise it is optimal to reject the request.

In addition to such analysis, there are computational approaches to these models. Some of these computational approaches are EMSR-a (expected marginal seat revenue-version a) and EMSR-b (expected marginal seat revenue-version b) heuristics. These two heuristics are studied by Belobaba [7]. He considers the static problem for both approaches. In order to calculate a threshold level for a given fare class, Belobaba [7] uses the Littlewood rule and takes that fare class as the second demand and one of the remaining fare classes as the first demand in Littlewood rule. Consequently, there will be a threshold value by using the Littlewood rule for each remaining fare classes. Then the addition of these threshold levels will give us the approximate threshold level for the fare class that we initially want to calculate. In EMSR-b, Belobaba [7] considers the total demand of the remaining fare classes and calculates a weighted-average revenue for these classes. By using these values, he finds a threshold value.

2.2 Choice Model of Consumer Behavior and Dynamic Pricing

Consumer behavior modeling is another study of single resource allocation problems. In the previous models, the case where customer may change their decisions according to the current products offered by the firm is not considered. For example, one may prefer to buy an airplane ticket at most 100 dollars, however, if the firm offers a seat of 50 dollars worth, then he or she may change his/her idea. In the previous models, customers cannot change their preferences and they are willing to buy only one type of product. Shen and Su [27] provide a detailed overview of the consumer behavior modeling in revenue management. One of the consumer behavior models involves choosing from a given set of products offered to the consumer. This type of modeling is first investigated by [29] who refer to their model as general discrete choice model of consumer behavior. There are other studies based on the study of Talluri and van Ryzin. For example, Liu and van Ryzin [25] consider a similar model in a network setting. Zhang and Adelman [37] also provide a dynamic programming approximation to this network setting model.

Another issue that the firm needs to take into account is the price of its product. Extensive overview of pricing models are provided in Elmaghraby and Keskinocak [14], Yano and Gilbert [35]. Since the capacity is fixed in the airline industry, we only provide the literature related to the single-product dynamic pricing with no replenishment. For example, Gallego and van Ryzin [16] consider a continuous-time dynamic pricing model with a continuous demand. A discrete time version of this study has been studied by Bitran and Mondschein [9]. In addition to this study, Lazear [22] considers a model in which prices can be updated only at fixed time intervals. In such studies, structure of optimal pricing policy with respect to time and inventory level is investigated.

2.3 Markov Modulation and Hidden Markov Models

In some cases, external factors can provide an information about the distribution of the demand. These external factors are assumed to change according to a Markov chain. Such studies are called Markov modulated problems when there is a randomly fluctuating demand environment. We observe that fluctuating demand environment is widely accepted in inventory systems. For example, Song and Zipkin [28] argue that demand depends on external factors which they call the current state of the world. They also believe that this current state of the world can be described economic and financial conditions. Financial markets are often classified as "bull" and "bear" according to investment sentiments. Özekici and Parlar [26], and Gallego and Hu [15] provide other examples of Markov modulation of customer demand in inventory management. A fluctuating demand environment is also considered in dynamic pricing problems for inventory systems. Gayon et al. [17] investigate possible pricing strategies in inventory systems under a fluctuating demand environment.

Also work by Yim and Rajaram [36] is another paper that considers a pricing model with Markov modulated demand.

There may be some cases in which there is an uncertainty on this classification of states depending on various financial indicators. Thus, the true state of the market may not be known with certainty. We have an observation process that provides us information about the true state of environment. Such processes are called hidden Markov processes or partially observed Markov decision processes. For example, Treharne and Sox [31] consider an inventory system where demand is partially observed and supplier has a infinite capacity. Arifoğlu and Özekici [1] investigate a similar model where supplier has finite capacity and they show that optimal ordering policy is state-dependent base-stock policy. In these problems, the main decision is the replenishment amount; therefore, our hidden Markov model is significantly different than these problems. In addition to such studies, Aviv and Pazgal [2] provide a dynamic pricing model where demand is partially observed. They also show that optimal pricing policy depends on the belief vector which represents the probability distribution of true state of the current environment.

2.4 Other Research Topics in Revenue Management

There are also other topics which are significant in revenue management. For example, models that consider risk-sensitivity are appropriate to the risk-averse managers. Since demand is random, managers would want to minimize their risk too. Barz and Waldmann [6] provide a model in which they maximize the utility instead of the revenue of the firm. They show that the structural properties of dynamic model for the single resource allocation problem holds also in her model. Lan et al. [20] also provide a model for risk-averse managers in which they consider lower and upper bounds for the demand of each fare class. They claim that distribution of demand may not be easy to obtain, and only information that manager has can be lower and upper bounds. In addition to such studies, Birbil et al. [8] investigate the robust versions of static and dynamic models of single resource allocation problem that is discussed in Section 1.

Another important research topic in revenue management is network capacity control. Since this a multi-dimensional problem, most studies focus on approximations to this problem. For example, Kunnumkal and Topaloğlu [19] provides an approximation method for network revenue management problem with customer choice behavior by solving each flight leg as single-leg problem. Vulcano and van Ryzin [32] also study an approximation method for network revenue management under customer choice behavior by using a simulation-based method.

Chapter 3

SINGLE RESOURCE CAPACITY CONTROL PROBLEM

3.1 Model Formulation

We formulate a discrete time, finite horizon (T periods) MDP model of the admission control problem corresponding to single-leg capacity control.

Let $X_t \in \{1, 2, \cdots, M\}$ denote the randomly fluctuating external environment. $X = \{X_0, X_1, \cdots, X_T\}$ is assumed to be a Markov chain with transition matrix $\mathbf{P}$ where $p_{ij} = P\{X_{t+1} = j | X_t = i\}$. We assume that there is at most one arrival and that each arrival from a fare class can request a finite number of seats in each stage. The probability that fare class a arrives at any stage is denoted by r_{ja} when the current environment is j. The probability of no arrival in a given environment is denoted by r_{j0}. Therefore, $\sum_{a=0}^{N} r_{ja} = 1$ for any j. Non-stationary demand scenarios can be handled by defining appropriate environment and transition matrices. For each fare class a, suppose there is an upper bound B_a on the number of fare products requested. Let q_{jab} denote the probability that b units of inventory is requested given that current environment is j and the requested fare product is a.

In each stage t, the firm must choose the optimal number of seats to be sold for each fare class. We assume that customers accept the scenario of a partial satisfaction of their request. Brumelle and Walczak [10] showed that structural results on the optimal policy are not valid in case of acceptance or rejection of the whole demand when there is no environment process (Also, see Van Slyke and Young [34] and Çil et al. [12] for related issues). Therefore, we only analyze the case where customers accept the partial satisfaction of their requests. For each sold ticket, the reward is $c(a)$ if the fare product is a. The transition probabilities and reward function are assumed to be stationary and we suppose that the fare classes are ordered so that $c(a_1) \leq c(a_2)$ when $a_1 \leq a_2$. We let $\mathbb{Z}_+$ denote the set of positive integers and $\mathbb{R}$ denote the set of real numbers.

We also use the following notations:

$v_t(x,j)$ = expected maximum revenue from period t on, given that current inventory level is x and environment is j.

$\Delta v_t(x,j) = v_t(x,j) - v_t(x-1,j)$

$(x)^+ = \max\{x,0\}$

$U(b,x) = \{0,1,\cdots,\min\{b,x\}\}$

The optimal expected revenue and the admission control policy for this problem can be obtained by solving the following Bellman equation

$$v_t(x,j) = \sum_{a=1}^{N} r_{ja} \sum_{b=1}^{B_a} q_{jab} \max_{u\in U(b,x)} \left\{ \sum_{k=1}^{M} p_{jk} v_{t+1}(x-u,k) + c(a)u \right\} + r_{j0} \sum_{k=1}^{M} p_{jk} v_{t+1}(x,k) \tag{3.1}$$

with boundary conditions

$$\begin{aligned} v_t(0,j) &= 0 \text{ for } j = 1,2,...,M. \\ v_T(x,j) &= 0 \text{ for any } x \in \mathbb{Z}_+ \text{ and } j = 1,2,...,M. \end{aligned}$$

For obtaining structural results, the following equivalent representation that uses the definition of Δv_t turns out to be helpful

$$\begin{aligned} v_t(x,j) &= \sum_{a=1}^{N} r_{ja} \sum_{b=1}^{B_a} q_{jab} \max_{u\in U(b,x)} \left\{ c(a)u - \sum_{k=1}^{M} p_{jk} \left(\sum_{z=1}^{u} \Delta v_{t+1}(x+1-z,k) - v_{t+1}(x,k) \right) \right\} \\ &+ r_{j0} \sum_{k=1}^{M} p_{jk} v_{t+1}(x,k) \\ &= \sum_{a=1}^{N} r_{ja} \sum_{b=1}^{B_a} q_{jab} \max_{u\in U(b,x)} \left\{ \sum_{z=1}^{u} \left(c(a) - \sum_{k=1}^{M} p_{jk} \Delta v_{t+1}(x+1-z,k) \right) \right\} + \sum_{k=1}^{M} p_{jk} v_{t+1}(x,k) \end{aligned} \tag{3.2}$$

where the sum is set to be zero when $u = 0$.

3.2 Structural Properties

In this section, we investigate some structural properties of the Markov-modulated single-resource capacity control problem. To begin with, it is intuitive that if we have one more inventory, then expected revenue should be larger. Similarly, expected revenue should be larger if we have more time to go. These claims can be easily proven by induction on t. Second order properties are less trivial. Before proving the concavity of $v_t(x,j)$ in x, we first state a lemma by Lautenbacher and Stidham [21].

Lemma 3.1 *Suppose $g : \mathbb{Z}_+ \to \mathbb{R}$ is concave. Let $f : \mathbb{Z}_+ \to \mathbb{R}$ be defined by*

$$f(x) = \max_{\beta=0,1,\ldots,m} \{\beta p + g(x-\beta)\}$$

for any given $p \geq 0$, and nonnegative integer $m \leq x$. Then, f is concave in $x \geq 0$.

Using Lemma 3.1, we next establish the concavity of the value function in x for each environment.

Theorem 3.2 *$v_t(x,j)$ is a concave function in x for any environment j and time t.*

Proof. Since $v_T(x,j)$ is zero for any x and j we have the concavity of $v_T(x,j)$ in x for any j. Suppose that $v_{t+1}(x,j)$ is a concave function of x for any environment j. We can use lemma 3.1 by taking g as $\sum_{k=1}^{M} p_{jk} v_{t+1}(x,k)$, p as $c(a)$, and m as $\min\{b,x\}$ Therefore,

$$\max_{u \in U(b,x)} \left\{ \sum_{k=1}^{M} p_{jk} v_{t+1}(x-u,k) + c(a)u \right\} \tag{3.3}$$

is concave in x for any product a, batch size $0 \leq b \leq B_a$. Since equation (3.1) is positive linear combination of (3.3) and $v_{t+1}(x,k)$, we have the concavity of $v_t(x,j)$ in x for any environment j. ■

Theorem 3.2 establishes that $\Delta v_t(x,j)$ decreases as we increase the inventory level x. By considering (3.2), we can conclude that $c(a) - \sum_{k=1}^{M} p_{jk} \Delta v_{t+1}(x+1-z,k)$ is decreasing in z. Therefore, in (3.2) we should increase $u \leq \min\{b,x\}$ until $c(a) - \sum_{k=1}^{M} p_{jk} \Delta v_{t+1}(x+1-z,k)$ becomes negative or u is equal to $\min\{b,x\}$. Since $\Delta v_t(x,j)$ is decreasing in x for any j, there is a threshold level $l_t^{a,j}$ which is defined as

$$l_t^{a,j} = \min \left\{ x : c(a) \geq \sum_{k=1}^{M} p_{jk} \Delta v_{t+1}(x,k) \right\}. \tag{3.4}$$

Explicitly, $l_t^{a,j}$ is the maximum quantity for the inventory level such that if the current inventory level is less than $l_t^{a,j}$ it is optimal to reject any batch for fare class a in environment j. However, if the inventory on hand is greater than or equal to $l_t^{a,j}$, then demand for product a is satisfied until the inventory level drops to $l_t^{a,j} - 1$.

Hence the optimal decision for product a at stage t and environment j, when demand is b, is

$$u^* = \min\left\{\left(x - l_t^{a,j} + 1\right)^+, b\right\}. \tag{3.5}$$

Theorem 3.2 implies that optimal admission control policies are of threshold (or booking limit) type as in standard single-resource capacity control. The difference in this case is that the thresholds now depend on the current state of the environment. Nevertheless, such policies are relatively easy to implement.

Since optimal thresholds are determined by the marginal value function via (3.4), we next investigate the structure of this function. First, we analyze how the marginal value function changes in time. Next proposition states a result on the marginal value of one additional inventory over time.

Proposition 3.3 $\Delta v_{t+1}(x,j) \leq \Delta v_t(x,j)$ *for any inventory level* x*, environment* j *and time* t*.*

Proof. Since $v_t(x,j)$ is increasing in x, $\Delta v_t(x,j) \geq 0$. Also $\Delta v_T(x,j) = 0$, which implies that $\Delta v_T(x,j) \leq \Delta v_{T-1}(x,j)$. Suppose $\Delta v_{t+2}(x,j) \leq \Delta v_{t+1}(x,j)$ for any environment j and inventory level x. Consider the following inequality,

$$\begin{gathered}
\max_{u_1 \in U(b,x)} \left\{\sum_{k=1}^{M} p_{jk} v_{t+2}(x-u_1,k) + c(a)\,u_1\right\} - \max_{u_2 \in U(b,x-1)} \left\{\sum_{k=1}^{M} p_{jk} v_{t+2}(x-1-u_2,k) + c(a)\,u_2\right\} \\
\leq \\
\max_{u_3 \in U(b,x)} \left\{\sum_{k=1}^{M} p_{jk} v_{t+1}(x-u_3,k) + c(a)\,u_3\right\} - \max_{u_4 \in U(b,x-1)} \left\{\sum_{k=1}^{M} p_{jk} v_{t+1}(x-1-u_4,k) + c(a)\,u_4\right\}
\end{gathered} \tag{3.6}$$

for any a, and batch size $0 \leq b \leq B_a$. It is sufficient to show that this inequality holds, in order to conclude that $\Delta v_{t+1}(x,j) \leq \Delta v_t(x,j)$, since the remaining terms in $\Delta v_t(x,j) - \Delta v_{t+1}(x,j)$ are clearly positive by using the induction hypothesis.

Let u_i^* be the optimal value of u_i in (3.6). We should note that $l_{t+1}^{a,j} \leq l_t^{a,j}$ for any product a and environment j. This can be easily seen by considering the induction hypothesis and (3.4). As a result, we have $u_3^* \leq u_1^*$. Also, we know that $u_1^* - u_2^*$ is either 1 or zero. Same reasoning is valid for $u_3^* - u_4^*$. If they are equal, then this is possible only either $u_1^* = u_2^* = 0$ or $u_1^* = u_2^* = b$.

Therefore, there are six cases we need to consider for the possible values of $u_1^*, u_2^*, u_3^*, u_4^*$.

Case	$(u_1^*, u_2^*, u_3^*, u_4^*)$	Inequality (3.6) simplifies to
1	$(0,0,0,0)$	$\sum_{k=1}^{M} p_{jk}\Delta v_{t+2}(x,k) \leq \sum_{k=1}^{M} p_{jk}\Delta v_{t+1}(x,k)$
2	$(y_2+1, y_2, 0, 0)$	$c(a) \leq \sum_{k=1}^{M} p_{jk}\Delta v_{t+1}(x,k)$
3	$(b,b,0,0)$	$\sum_{k=1}^{M} p_{jk}\Delta v_{t+2}(x-b,k) \leq \sum_{k=1}^{M} p_{jk}\Delta v_{t+1}(x,k)$
4	(y_2+1, y_2, y_1+1, y_1)	$c(a) \leq c(a)$
5	(b,b,y_1+1,y_1)	$\sum_{k=1}^{M} p_{jk}\Delta v_{t+2}(x-b,k) \leq c(a)$
6	(b,b,b,b)	$\sum_{k=1}^{M} p_{jk}\Delta v_{t+2}(x-b,k) \leq \sum_{k=1}^{M} p_{jk}\Delta v_{t+1}(x-b,k)$

Here, y_1 and y_2 are integers such that $0 \leq y_1 \leq y_2 \leq b-1$. Case 1 and 6 are true due to the induction hypothesis. Also case 4 is automatically true. In case 2, suppose that $\sum_{k=1}^{M} p_{jk}\Delta v_{t+1}(x,k) < c(a)$, then we should accept at least one customer when current inventory level is x at stage t but $u_3^* = 0$. Therefore, inequality in case 2 is true. In case 5, suppose that $\sum_{k=1}^{M} p_{jk}\Delta v_{t+2}(x-b,k) > c(a)$. Then, at time $t+1$, accepted batch size is less than $b-1$ when current inventory level is x. However $u_2^* = b$, which means that the inequality in case 5 is also true. In case 3, we have $c(a) \leq \sum_{k=1}^{M} p_{jk}\Delta v_{t+1}(x,k)$ since $u_3^* = 0$. Also we have $\sum_{k=1}^{M} p_{jk}\Delta v_{t+2}(x-b,k) \leq c(a)$ since $u_2^* = b$. Note that, these inequalities can be shown by using the methodology used in case 2 and 5. Hence, we have the inequality of case 3. Consequently, Δv_t decreases in t. ■

Please note that Theorem 3.2 and Proposition 3.3 extend the corresponding results in Aydın et al. [3] to a setting with multiple environments. With regard to Proposition 3.3, a corresponding results in Aydın et al. [3] establishes that admission thresholds decrease as time increases when there is a single environment . When there are multiple environment states, the environment also changes over time; hence, we cannot guarantee the decrease of the thresholds over time when the environment changes. On the other hand, if the environment does not change, then we can establish the admission threshold should decrease in the next period. This result follows from comparing

$$\sum_{k=1}^{M} p_{jk}\Delta v_{t+1}(x,k) \leq \sum_{k=1}^{M} p_{jk}\Delta v_{t}(x,k)$$

which is obviously true by Proposition 3.3. Therefore, $l_t^{a,j} \geq l_{t+1}^{a,j}$ for any fare class a, time t and environment j.

Since demand varies according to the environment, optimal threshold levels change with the environment. To better understand the effects of the environment on the optimal thresholds, we must classify and order the environments. To this end, we need some assumptions on arrival probabilities and the transition matrix of the environmental process. The following classification is useful for this purpose.

Definition 3.4 *A Markov chain is said to be IFR (Increasing Failure Rate) if the rows of its transition probability matrix are in increasing stochastic order, i.e.,*

$$f(i) = \sum_{j=k}^{M} p_{ij}$$

is nondecreasing in i for all $k = 1, ..., M$. Similarly, a matrix X is said to be IFR if the rows of X are in increasing stochastic order.

In reliability theory, life distribution classifications, like IFR, play a crucial role in identifying the structure of optimal maintenance policies. This usually leads to optimal threshold policies since the IFR property implies the increasing marginal deterioration of the system. An example is the age replacement policy which states that the system is replaced as soon as its age exceeds a critical level. The reader is referred to Barlow and Porschan [4] for basic concepts on life distribution classifications, and Keilson and Kesten [18] for classifications of Markov chains using their transition matrices. In our context, we need to impose similar restrictions on the environmental process so that the state becomes more or less "desirable" in generating revenue.

Let $\mathbf{R}$ be a matrix such that $R_{j,a} = r_{ja}$, and suppose $\mathbf{R}$ is IFR. This implies that environments are ordered in terms of the arrival probability of customers from higher fare classes. For example, suppose we have 2 environments, then the second environment is said to be "better" than the first one if it is more probable to have a demand for higher reward products in the second environment.

Let $B = \max\{B_a : a = 1, 2, \cdots, N\}$ and set $q_{jab} = 0$ for any $B_a < b \leq B$. Also, let $\mathbf{Q}$ denote a 3 dimensional matrix whose (j, a, b)th component is q_{jab} as defined above. Then we define 2 dimensional submatrices of $\mathbf{Q}$ where we fix one component of $\mathbf{Q}$. Let the fixed component be denoted as a superscript while the other components are denoted by subscripts. We also assume that the matrix $\mathbf{Q}^{(\mathbf{a})}_{\mathbf{jb}}$ is IFR for a fixed a. Finally, we also assume that $\mathbf{Q}^{(\mathbf{i})}_{\mathbf{ab}}$ is IFR for fixed i.

Last, we need a condition on the transition matrix $\mathbf{P}$ of environment process. We assume that $\mathbf{P}$ is IFR. This is also plausible. If the index of an environment i is higher than another environment j, then we call i a "better" environment than j by the explanation above. Since environment i is better than j, it is more likely for environment i to make a transition to an environment that is better than an arbitrary given environment. Intuitively, the probability that the current environment will transition in the future to a better environment increases as the level of the current environment increases. We now summarize all of these conditions.

Condition 3.5 *(1)* $\mathbf{P}$ *is IFR.*

(2) $\mathbf{R}$ *is IFR.*

(3) $\mathbf{Q_{jb}^{(a)}}$ *is IFR for any product* a*.*

(4) $\mathbf{Q_{ab}^{(j)}}$ *is IFR for any environment* j*.*

The above condition imposes an order on the environments. This order is a minimal requirement for obtaining structural results as a function of the environment. When the condition holds, j is a more favorable environment than i where $i \leq j$. Let us discuss the modeling implications of Condition 3.5. Condition 3.5 (1) concerns the environment transitions. The environment transitions need to have a smoothness property where better current environments are likelier to lead to better future environments which seems natural for most applications. Condition 3.5 (2) can be viewed as a consequence of the environment classification where better current states have a more favorable demand arrival distribution. Without this condition, environment states do not necessarily have a natural order which prevents monotonicity. Condition 3.5 (3) states that batch sizes are likelier to be larger in better environments which also appears natural. Condition 3.5 (4) imposes constraints on the demand batch size as a function of the class of customers. This condition is automatically satisfied for the frequently encountered case of unit demand arrivals (see Talluri and van Ryzin [30]) and for the case where the batch sizes are not class dependent.

We first investigate the expected maximum revenue from period t on for different environments at stage t under Condition 3.5. In particular, in the next proposition, we establish that the maximum expected revenue increases when the environment gets better .

Proposition 3.6 *Under Condition 3.5,* $v_t(x,i) \leq v_t(x,j)$ *for any inventory level* x*, environment* $i \leq j$ *and time* t*.*

Proof. For $t = T$ we have the result trivially since $v_T(x,i) = 0$ for any inventory level x and environment i. Suppose $v_{t+1}(x,i) \leq v_{t+1}(x,j)$ for any $i \leq j$. We provide some definitions to make the proof clearer. Let

$$W(a,b,j) = \max_{u\in U(b,x)} \left\{ \sum_{k=1}^{M} p_{jk} v_{t+1}(x-u,k) + c(a)u \right\}$$

and

$$S(a,j) = \sum_{b=1}^{B} q_{jab} W(a,b,j)$$

$$S(0,j) = \sum_{k=1}^{M} p_{jk} v_{t+1}(x,k)$$

for $a = 1,2,...N$. Then, we need to show the following

$$\sum_{a=0}^{N} r_{ia} S(a,i) \leq \sum_{a=0}^{N} r_{ja} S(a,j)$$

First of all, it is clear that $W(a,b,j)$ is nondecreasing in b, and a. Also, since $\mathbf{P}$ is IFR, by the induction hypothesis we know that $W(a,b,j)$ is nondecreasing in j. Hence $S(a,j)$ is nondecreasing in j, because $\mathbf{Q}^{(\mathbf{a})}_{\mathbf{jb}}$ is IFR. Also, by induction hypothesis we know $S(0,j)$ is nondecreasing in j. We also need to show that $S(a,j)$ is a nondecreasing function in a. Take $a \in \{1,2,\cdots,N\}$, since $W(a,b,j)$ is nondecreasing in a and $\mathbf{Q}^{(\mathbf{j})}_{\mathbf{ab}}$ is IFR, we know that $S(a,j)$ is nondecreasing in the domain $\{1,2,...,N\}$. It is also easy to show that $S(0,j) \leq S(1,j)$ hence $S(a,j)$ is nondecreasing in a. Since $S(a,i) \leq S(a,j)$ and $S(a,j)$ is a nondecreasing function in a, we have $v_t(x,i) \leq v_t(x,j)$ by using the IFR property of $\mathbf{R}$. ■

From a practical perspective, Proposition 3.6 states that better starting environments lead to better expected revenues. Second, we consider the effect of the environment on the expected marginal value of one additional inventory. This value is important in understanding the structure of threshold values in different environments.

Proposition 3.7 *Under Condition 3.5, $\Delta v_t(x,i) \leq \Delta v_t(x,j)$ for any inventory level x, environment $i \leq j$ and time t.*

Proof. Clearly, we have $\Delta v_T(x,i) = \Delta v_T(x,j) = 0$. Suppose $\Delta v_{t+1}(x,i) \leq \Delta v_{t+1}(x,j)$ for any $i \leq j$. Let

$$W(a,b,j) = \max_{u\in U(b,x)} \left\{ \sum_{k=1}^{M} p_{jk} v_{t+1}(x-u,k) + c(a)u \right\} - \max_{u\in U(b,x-1)} \left\{ \sum_{k=1}^{M} p_{jk} v_{t+1}(x-1-u,k) + c(a)u \right\}$$

for $a = 1, 2, ..., N$. Also define,

$$S(a,j) = \sum_{b=1}^{B} q_{jab} W(a,b,j)$$

$$S(0,j) = \sum_{k=1}^{M} p_{jk} \left(v_{t+1}(x,k) - v_{t+1}(x-1,k)\right)$$

for $a = 1, 2, ..., N$. After making these definitions, we need to show

$$\sum_{a=0}^{M} r_{ia} S(a,i) \leq \sum_{a=0}^{M} r_{ja} S(a,j)$$

for any environment $i \leq j$. First of all we will show $W(a,b,i) \leq W(a,b,j)$ for any $a \in \{1, 2, ..., N\}$ and b. Let u_1^* be the optimal decision when current inventory is x and environment is i, and let u_2^* be the optimal decision when the current inventory is $x-1$ and environment is i (u_3^* and u_4^* are also defined in a similar fashion for environment j). Using the same reasoning that we used in the proof of the Proposition 3.3, we have the following relations and results by noting that $l_t^{a,j} \geq l_t^{a,i}$.

C	$(u_1^*, u_2^*, u_3^*, u_4^*)$	Inequality $W(a,b,i) \leq W(a,b,j)$ simplifies to
1	$(0,0,0,0)$	$\sum_{k=1}^{M} p_{ik} \Delta v_{t+1}(x,k) \leq \sum_{k=1}^{M} p_{jk} \Delta v_{t+1}(x,k)$
2	$(y_2+1, y_2, 0, 0)$	$c(a) \leq \sum_{k=1}^{M} p_{jk} \Delta v_{t+1}(x,k)$
3	$(b,b,0,0)$	$\sum_{k=1}^{M} p_{ik} \Delta v_{t+1}(x-b,k) \leq \sum_{k=1}^{M} p_{jk} \Delta v_{t+1}(x,k)$
4	(y_2+1, y_2, y_1+1, y_1)	$c(a) \leq c(a)$
5	(b, b, y_1+1, y_1)	$\sum_{k=1}^{M} p_{ik} \Delta v_{t+1}(x-b,k) \leq c(a)$
6	(b,b,b,b)	$\sum_{k=1}^{M} p_{ik} \Delta v_{t+1}(x-b,k) \leq \sum_{k=1}^{M} p_{jk} \Delta v_{t+1}(x-b,k)$

Here, y_1 and y_2 are integers such that $0 \leq y_1 \leq y_2 \leq b-1$. Note that case 6 and case 1 are obviously true due to the induction hypothesis and the IFR property of $\mathbf{P}$. The remaining cases are true as explained in the proof of Proposition 3.3.

Secondly, we will show that $W(a,b,i)$ is nondecreasing in ordered quantity b for any i and $a \in \{1, 2, ..., N\}$. Take $1 \leq b < B$. Let u_1^* be the optimal decision when current inventory is x and ordered quantity is b, and let u_2^* be the optimal decision when the current inventory is $x-1$ and ordered quantity is b (u_3^* and u_4^* are also defined in a similar fashion for ordered quantity $b+1$). We have 4 cases,

C	$(u_3^* u_4^*)$	Results	$W(a,b,i) < W(a,b+1,i)$ reduces to
1	$(0,0)$	$(u_1^*, u_2^*) = (0,0)$	$\sum_{k=1}^{M} p_{ik} \Delta v_{t+1}(x,k) \leq \sum_{k=1}^{M} p_{ik} \Delta v_{t+1}(x,k)$
2	$(b+1, b+1)$	$x - l_t^{a,j} \geq b+1$ $(u_1^*, u_2^*) = (b,b)$	$\sum_{k=1}^{M} p_{ik} \Delta v_{t+1}(x-b,k) \leq \sum_{k=1}^{M} p_{ik} \Delta v_{t+1}(x-b-1,k)$
3	$(b+1, b)$	$x - l_t^{a,j} = b$ $(u_1^*, u_2^*) = (b,b)$	$\sum_{k=1}^{M} p_{ik} \Delta v_{t+1}(x-b,k) \leq c(a)$
4	$(y, y-1)$	$(u_1^*, u_2^*) = (y, y-1)$	$c(a) \leq c(a)$

where $1 \leq y < b+1$. Case 1 and 4 are obviously true since right-hand side and left-hand side are equal in both cases. Case 2 is also true since $\Delta v_{t+1}(x)$ is nonincreasing in x. In case 3, suppose $\sum_{k=1}^{M} p_{ik} \Delta v_{t+1}(x-b,k) > c(a)$. Then this fact contradicts with $l_t^{a,j} = x - b$.

Clearly, $S(0,i) \leq S(0,j)$ since $\mathbf{P}$ is IFR. Since $W(a,b,i) \leq W(a,b,j)$ and $W(a,b,i)$ is nondecreasing in ordered quantity b for any i and $a \in \{1,2,...,N\}$, by using the IFR property of $\mathbf{Q_{jb}^{(a)}}$ we have

$$S(a,i) \leq S(a,j)$$

for any $a \in \{0,1,...,N\}$. Now, it is sufficient to show that $S(a,j)$ is nondecreasing in a to show $\Delta v_t(x,i) \leq \Delta v_t(x,j)$ because we can use the IFR property of $\mathbf{R}$ to conclude our result. Take $a_1, a_2 \in \{1,2,...,N\}$ with $a_1 \leq a_2$. Since $c(a_1) \leq c(a_2)$, we know that $l_t^{a_1,j} \geq l_t^{a_2,j}$. Also, we have already shown that $W(a,b,j)$ is nondecreasing in b. It is sufficient to prove $W(a_1,b,j) \leq W(a_2,b,j)$, then we can use the IFR property of $\mathbf{Q_{ab}^{(j)}}$ and $W(a,b,j)$'s being nondecreasing in b to conclude that $S(a_1,j) \leq S(a_2,j)$. Let u_1^* be the optimal decision when current inventory is x and product type is a_1, and let u_2^* be the optimal decision when the current inventory is $x-1$ and product type is a_1 (u_3^* and u_4^* are also defined in a similar fashion for product type a_2). Then, there are six cases for the values of $(u_1^*, u_2^*, u_3^*, u_4^*)$ as before.

C	$(u_1^*, u_2^*, u_3^*, u_4^*)$	Inequality $W(a_1, b, i) \leq W(a_2, b, i)$ simplifies to
1	$(0,0,0,0)$	$\sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x,k) \leq \sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x,k)$
2	$(0,0,y_1+1,y_1)$	$\sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x,k) \leq c(a_2)$
3	$(0,0,b,b)$	$\sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x,k) \leq \sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x-b,k)$
4	(y_2+1,y_2,y_1+1,y_1)	$c(a_1) \leq c(a_2)$
5	(y_2+1,y_2,b,b)	$c(a_1) \leq \sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x-b,k)$
6	(b,b,b,b)	$\sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x-b,k) \leq \sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x-b,k)$

Here, y_1 and y_2 are integers such that $0 \leq y_2 \leq y_1 \leq b-1$. Note that in case 1 and 6, right hand sides and left hand sides are identical. Case 4 is true since $c(a_1) \leq c(a_2)$. In case 2, suppose $\sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x,k) > c(a_2)$ then we should not sell any product of type a_2 when current inventory level is x at time t, but $u_3^* = y_1 + 1 \geq 1$. Case 3 is also true since $\Delta v_{t+1}(x)$ is nonincreasing in x. In case 5, suppose $c(a_1) > \sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x-b,k)$. Since $y_2 = u_2^* \leq b-1$, we have $x - b + 1 \leq l_t^{a,j}$ and this result contradicts with our assumption. Therefore $S(a_1, i) \leq S(a_2, i)$ for $a_1, a_2 \in \{1, 2, ..., N\}$. Also, we need to show $S(0,i) \leq S(1,i)$. We have the following inequality since $W(1,b,i)$ is nondecreasing in b.

$$
\begin{aligned}
W(1,1,i) &= \max_{u_1 \in \{0,1\}} \left\{ \sum_{k=1}^{M} p_{ik} v_{t+1}(x-u,k) + c(1)u \right\} - \max_{u_2 \in \{0,1\}} \left\{ \sum_{k=1}^{M} p_{ik} v_{t+1}(x-1-u,k) + c(1)u \right\} \\
&\leq S(1,i) = \sum_{b=1}^{B} q_{j1b} W(1,b,i)
\end{aligned}
$$

It is sufficient to show $W(1,1,i) \geq S(0,i)$. Note that we have $u_1^* \geq u_2^*$. Therefore, we have 3 cases,

C	(u_1^*, u_2^*)	Inequality $W(1,1,i) \geq S(0,i)$ reduces to
1	$(1,1)$	$\sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x-1,k) \geq \sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x,k)$
2	$(0,0)$	$\sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x,k) \geq \sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x,k)$
3	$(1,0)$	$c(1) \geq \sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x,k)$

Case 2 is obviously true, also case 1 is true since $\Delta v_{t+1}(x)$ is nonincreasing in x. In case 3 suppose $\sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x,k) > c(a)$ but this contradicts with $u_1^* = 1$.

Hence, $S(a_1,i) \le S(a_2,i)$ for $a_1, a_2 \in \{0,1,...,N\}$ and $S(a,i) \le S(a,j)$ for $i \le j$. Since $\mathbf{R}$ is IFR, we have

$$\Delta v_t(x,i) = \sum_{a=0}^{M} r_{ia} S(a,i) \le \sum_{a=0}^{M} r_{ja} S(a,j) = \Delta v_t(x,j)$$

■

Let us discuss the implication of this proposition. Since the admission policy is determined by the structure of the difference $\Delta v_t(x,j)$, and since this difference increases in j, we can conclude that $l_t^{a,j}$ increases in j. Since the demand for a more valuable product will increase in probability as the environment gets better, it is optimal to protect the stock more in a better environment. For implementation purposes, this implies that the optimal admission thresholds are non-decreasing in more favorable environments. By using Propositions 3.3 and 3.7, we have the following immediate result that extends the property in Proposition 3.7 to different time periods.

Corollary 3.8 *Under Condition 3.5, $\Delta v_{t+1}(x,i) \le \Delta v_t(x,j)$ for any inventory level x, environment $i \le j$ and time t.*

By this corollary, we know that the threshold level of a product in a given stage will decrease in the next stage if the environment of the next stage is worse than the one in the previous stage. However, we cannot guarantee the decrease of the threshold if the environment of the next stage is better than the one in the previous stage. This is explored further in Section 4.

3.3 Sensitivity Analysis

In this section, we will provide results on the sensitivity of the structural properties on the model parameters. A recent paper by Çil et al. [13] presents a general approach for this type of analysis and Aydın et al. [3] presents corresponding results for a standard single-leg capacity control problem.

First, by setting $z_{jab} = q_{jab} r_{ja}$, we will use the following equivalent form of our problem

$$v_t(x,j) = \sum_{a=1}^{N} \sum_{b=1}^{B_a} z_{jab} \max_{u \in U(b,x)} \left\{ \sum_{k=1}^{M} p_{jk} v_{t+1}(x-u,k) + c(a)u \right\} + r_{j0} \sum_{k=1}^{M} p_{jk} v_{t+1}(x,k) \quad (3.7)$$

with boundary conditions $v_t(0, j) = 0$ and $v_T(x, j) = 0$ for all x and t. We show the effects of changing components of arrival probabilities ($\mathbf{Z}$), transition matrix ($\mathbf{P}$), and reward function (c). We will change a component of these matrices or the reward function by a small amount and explore the effects of this change under some specific conditions.

We first provide the results on the effects of varying the arrival probabilities. Aydın et al. [3] also considers a similar study on the effects of parameters, where only the fictitious event probability is decreased when a given arrival probability is increased. We employ a more general approach and consider decreasing any other arrival probability.

Let us increase z_{iab} by $\epsilon \geq 0$ for a given environment i, class $a \geq 1$ and batch size b. In order to have a valid probability distribution we will reduce $z_{ia_2b_2}$ by ϵ where $1 \leq a_2 \leq a$ and $b_2 \leq b$. Here, ϵ should be small enough in order to have both $z_{iab} + \epsilon$ and $z_{ia_2b_2} - \epsilon$ lie in the interval $[0, 1]$. Let $v_t^\epsilon(x, j)$ be the value function for the modified system.

Proposition 3.9 $v_t^\epsilon(x, j) \geq v_t(x, j)$ *for any environment* j*, time* t *and inventory level* x*.*

Proof. Clearly $v_T^\epsilon(x, j) = v_T(x, j) = 0$, suppose $v_{t+1}^\epsilon(x, j) \geq v_{t+1}(x, j)$ for any environment j and inventory level x. For a given product a, and amount $b \in \{1, 2, ..., B_a\}$, by using the induction hypothesis we know

$$\sum_{k=1}^{M} p_{jk} v_{t+1}(x-u, k) + c(a)u \leq \sum_{k=1}^{M} p_{jk} v_{t+1}^\epsilon(x-u, k) + c(a)u.$$

for any $0 \leq u \leq \min\{b, x\}$. Hence we have

$$\max_{u \in U(b,x)} \left\{ \sum_{k=1}^{M} p_{jk} v_{t+1}(x-u, k) + c(a)u \right\} \leq \max_{u \in U(b,x)} \left\{ \sum_{k=1}^{M} p_{jk} v_{t+1}^\epsilon(x-u, k) + c(a)u \right\} \quad (3.8)$$

Consider any environment $j \neq i$. Then $v_t^\epsilon(x, j) \geq v_t(x, j)$ which is clear from inequality (3.8). When we consider i as an environment, it is sufficient to show the following

$$\max_{u \in U(b_2,x)} \left\{ \sum_{k=1}^{M} p_{ik} v_{t+1}^\epsilon(x-u, k) + c(a_2)u \right\} \leq \max_{u \in U(b,x)} \left\{ \sum_{k=1}^{M} p_{ik} v_{t+1}^\epsilon(x-u, k) + c(a)u \right\}$$

Note that, since $b_2 \leq b$ and $a_2 \leq a$ we know

$$\max_{u \in U(b_2,x)} \left\{ \sum_{k=1}^{M} p_{ik} v_{t+1}^\epsilon(x-u, k) + c(a_2)u \right\} \leq \max_{u \in U(b,x)} \left\{ \sum_{k=1}^{M} p_{ik} v_{t+1}^\epsilon(x-u, k) + c(a_2)u \right\}$$

and

$$\max_{u\in U(b,x)}\left\{\sum_{k=1}^{M}p_{jk}v_{t+1}^{\epsilon}(x-u,k)+c(a_2)u\right\}\leq\max_{u\in U(b,x)}\left\{\sum_{k=1}^{M}p_{jk}v_{t+1}^{\epsilon}(x-u,k)+c(a)u\right\}$$

Hence we have the result. ■

Proposition 3.9 formalizes that increased demand from more valuable classes improves expected revenues. Next, we consider the effects of varying arrival probabilities on the expected marginal value of one additional inventory, since admission thresholds are determined by this value. The marginal value for the modified system is denoted by $\Delta v_t^{\epsilon}(x,j)$.

Proposition 3.10 *$\Delta v_t^{\epsilon}(x,j)\geq\Delta v_t(x,j)$ for any environment j, time t and inventory level x.*

Proof. Clearly, $\Delta v_T^{\epsilon}(x,j)=\Delta v_T(x,j)=0$. Suppose $\Delta v_{t+1}^{\epsilon}(x,j)\geq\Delta v_{t+1}(x,j)$ for any environment j and inventory level x. Consider any environment $j\neq i$. Then as done in proposition 3.3, it can be shown that $\Delta v_t^{\epsilon}(x,j)\geq\Delta v_t(x,j)$. When we consider i as an environment, it is sufficient to show the following inequality,

$$\max_{u\in U(b_2,x)}\left\{\sum_{k=1}^{M}p_{ik}v_{t+1}^{\epsilon}(x-u,k)+c(a_2)u\right\}-\max_{u\in U(b_2,x-1)}\left\{\sum_{k=1}^{M}p_{ik}v_{t+1}^{\epsilon}(x-1-u,k)+c(a_2)u\right\}$$

$$\leq$$

$$\max_{u\in U(b,x)}\left\{\sum_{k=1}^{M}p_{ik}v_{t+1}^{\epsilon}(x-u,k)+c(a)u\right\}-\max_{u\in U(b,x-1)}\left\{\sum_{k=1}^{M}p_{ik}v_{t+1}^{\epsilon}(x-1-u,k)+c(a)u\right\}$$

Note that the right and left hand sides are similar to the definition of $W(a,b,j)$ in the proof of proposition 3.7 , and W is nondecreasing in b. (None of the IFR properties are used to show this, hence the same proof is also valid in here.) Therefore, it is sufficient to show,

$$\max_{u\in U(b,x)}\left\{\sum_{k=1}^{M}p_{ik}v_{t+1}^{\epsilon}(x-u,k)+c(a_2)u\right\}-\max_{u\in U(b,x-1)}\left\{\sum_{k=1}^{M}p_{ik}v_{t+1}^{\epsilon}(x-1-u,k)+c(a_2)u\right\}$$

$$\leq$$

$$\max_{u\in U(b,x)}\left\{\sum_{k=1}^{M}p_{ik}v_{t+1}^{\epsilon}(x-u,k)+c(a)u\right\}-\max_{u\leq U(b,x)}\left\{\sum_{k=1}^{M}p_{ik}v_{t+1}^{\epsilon}(x-1-u,k)+c(a)u\right\}$$

Also we know that W is nondecreasing in a as done in the proof of proposition 3.7. (Again the IFR properties are not used to show this.) Hence, $\Delta v_t^{\epsilon}(x,j)\geq\Delta v_t(x,j)$ for any environment j and inventory level x. ■

Since expected marginal value of one additional inventory is greater in the modified system, the threshold level of the modified system for a given product, time and environment is greater than the one of the original model. In other words, $l_t^{a,j} \leq l_{t,\epsilon}^{a,j}$ where $l_{t,\epsilon}^{a,j}$ denotes the threshold level in the modified system. Please note that an increase in some arrival probability at a given environment i causes the admission thresholds in all environments j to increase. Propositions 3.9 and 3.10 extend the corresponding results in Aydın et al. [3] to multiple environment states.

Second, we analyze the effects of changing a component of $\mathbf{P}$ which is assumed to be IFR. Suppose we increase p_{ij} by $\epsilon \geq 0$. To have a valid distribution, we need to reduce another component in the ith row of $\mathbf{P}$ with a column index smaller than j by ϵ. Again, ϵ should be small enough to make the changed components lie in $[0,1]$ interval. These changes must preserve the IFR property of $\mathbf{P}$. Let the modified solution be denoted by $v_t^\epsilon(x,j)$ and the transition probability matrix by $\mathbf{P}^\epsilon$. We have only changed the ith row of $\mathbf{P}$, hence the remaining rows of $\mathbf{P}^\epsilon$ are identical to $\mathbf{P}$. First, we compare the expected revenue of these two systems.

Proposition 3.11 *Under Condition 3.5, $v_t^\epsilon(x,j) \geq v_t(x,j)$ for any environment j, time t and inventory level x.*

Proof. Clearly, $v_T^\epsilon(x,j) = v_T(x,j) = 0$. Suppose $v_{t+1}^\epsilon(x,j) \geq v_{t+1}(x,j)$. It is easy to verify $v_t^\epsilon(x,j) \geq v_t(x,j)$ when $j \neq i$ since components of $\mathbf{P}$ remain same except the i^{th} row. Now, suppose that the current environment is i. Take any $a \in \{1,2,...,N\}$ and $1 \leq b \leq B_a$. It is sufficient to show

$$\max_{u\in U(b,x)}\left\{\sum_{k=1}^{M} p_{jk}v_{t+1}(x-u,k)+c(a)u\right\} \leq \max_{u\in U(b,x)}\left\{\sum_{k=1}^{M} p_{jk}^\epsilon v_{t+1}^\epsilon(x-u,k)+c(a)u\right\}$$

By proposition 3.6, we know that $v_{t+1}(x-u,k)$ and also $v_{t+1}^\epsilon(x-u,k)$ are nondecreasing function in k. Therefore,

$$\sum_{k=1}^{M} p_{jk}v_{t+1}(x-u,k) \leq \sum_{k=1}^{M} p_{jk}^\epsilon v_{t+1}^\epsilon(x-u,k)$$

for any $u \in \{0,1,...,\min\{b,x\}\}$ $\blacksquare$

Proposition 3.11 establishes that a better environment probability transition matrix leads to higher expected revenues. For practical purposes, this implies that more favorable forecasts of future demand environments results in improved expected revenues.

Next, we investigate the effects of such a change on the optimal policy As done in the previous analysis on $\mathbf{Z}$, we focus on the expected marginal value of one additional inventory level. The marginal value of the modified system is denoted by $\Delta v_t^{\epsilon}(x,j)$. In the next proposition we show that marginal value of the modified system is greater than the original system.

Proposition 3.12 *Under Condition 3.5,* $\Delta v_t^{\epsilon}(x,j) \geq \Delta v_t(x,j)$ *for any environment* j*, time* t *and inventory level* x*.*

Proof. Clearly, $\Delta v_T^{\epsilon}(x,j) = \Delta v_T(x,j) = 0$. Suppose $\Delta v_{t+1}^{\epsilon}(x,j) \geq \Delta v_{t+1}(x,j)$ for any environment j and inventory level x. Consider any environment $j \neq i$, then it is easy to verify $\Delta v_t^{\epsilon}(x,j) \geq \Delta v_t(x,j)$ as done in the proof of proposition 3.3, because j^{th} row of $\mathbf{P}$ and $\mathbf{P}^{\boldsymbol{\epsilon}}$ are identical. When the environment is i, it is sufficient to show

$$\max_{u_1 \in U(b,x)} \left\{ \sum_{k=1}^{M} p_{ik} v_{t+1}(x-u_1,k) + c(a)\, u_1 \right\} - \max_{u_2 \in U(b,x-1)} \left\{ \sum_{k=1}^{M} p_{ik} v_{t+1}(x-1-u_2,k) + c(a)\, u_2 \right\}$$
$$\leq$$
$$\max_{u_3 \in U(b,x)} \left\{ \sum_{k=1}^{M} p_{ik}^{\epsilon} v_{t+1}^{\epsilon}(x-u_3,k) + c(a)\, u_3 \right\} - \max_{u_4 \in U(b,x-1)} \left\{ \sum_{k=1}^{M} p_{ik}^{\epsilon} v_{t+1}^{\epsilon}(x-1-u_4,k) + c(a)\, u_4 \right\}$$

for any $1 \leq a \leq N$, $1 \leq b \leq B_a$. Since $\Delta v_{t+1}^{\epsilon}(x,j) \geq \Delta v_{t+1}(x,j)$ and $\Delta v_t^{\epsilon}(x,k)$ is nondecreasing in k,

$$\sum_{k=1}^{M} p_{ik}^{\epsilon} \Delta v_{t+1}^{\epsilon}(x,k) \geq \sum_{k=1}^{M} p_{ik} \Delta v_{t+1}(x,k)$$

Then $l_{t,\epsilon}^{a,j} \geq l_t^{a,j}$. Let u_i^* be the optimal value of u_i in the inequality above. As a result, we have $u_3^* \leq u_1^*$. Also, we know that $u_1^* - u_2^*$ is either 1 or zero. The same reasoning is valid for $u_3^* - u_4^*$. If they are equal, then this is possible only either $u_1^* = u_2^* = 0$ or $u_1^* = u_2^* = b$.

Therefore, there are six cases we need to consider for the possible values of $u_1^*, u_2^*, u_3^*, u_4^*$.

Case	$(u_1^*, u_2^*, u_3^*, u_4^*)$	Inequality simplifies to
1	$(0,0,0,0)$	$\sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x,k) \le \sum_{k=1}^{M} p_{ik}^{\epsilon}\Delta v_{t+1}^{\epsilon}(x,k)$
2	$(y_2+1, y_2, 0, 0)$	$c(a) \le \sum_{k=1}^{M} p_{ik}^{\epsilon}\Delta v_{t+1}^{\epsilon}(x,k)$
3	$(b,b,0,0)$	$\sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x-b,k) \le \sum_{k=1}^{M} p_{ik}^{\epsilon}\Delta v_{t+1}^{\epsilon}(x,k)$
4	(y_2+1, y_2, y_1+1, y_1)	$c(a) \le c(a)$
5	(b,b,y_1+1,y_1)	$\sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x-b,k) \le c(a)$
6	(b,b,b,b)	$\sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x-b,k) \le \sum_{k=1}^{M} p_{ik}^{\epsilon}\Delta v_{t+1}^{\epsilon}(x-b,k)$

Here, y_1 and y_2 are integers such that $0 \le y_1 \le y_2 \le b-1$. Case 1 is obviously true as shown above. In case 5, suppose $\sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x-b,k) > c(a)$, then accepted batch size should be less than b at time t when current inventory level is x, but this result contradicts with $u_2^* = b$. Similarly, in case 2, suppose $c(a) > \sum_{k=1}^{M} p_{ik}^{\epsilon}\Delta v_{t+1}^{\epsilon}(x,k)$. Then, we need to satisfy at least one of the requested amount at time t in modified system when current inventory level is x, but we have $u_3^* = 0$. In case 3, we have $c(a) \le \sum_{k=1}^{M} p_{ik}^{\epsilon}\Delta v_{t+1}^{\epsilon}(x,k)$ since $u_3^* = 0$. Also we have $\sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x-b,k) \le c(a)$ since $u_2^* = b$. Note that, these inequalities can be shown by using the methodology used in case 2 and 5. Hence we have the inequality of case 3. Case 6 is also true by the induction assumption and the fact that $\Delta v_{t+1}^{\epsilon}(x,k)$ is nondecreasing function of k. Therefore, we have $\Delta v_t^{\epsilon}(x,j) \ge \Delta v_t(x,j)$ for any environment j, time t and inventory level x. ■

As explained before, since the expected marginal value of an additional inventory is greater in the modified system, threshold level of the modified system is greater than the one of the original system. (i.e., $l_{t,\epsilon}^{a,j} \ge l_t^{a,j}$). A more advantageous environment transition structure leads to higher admission thresholds for all environments.

Now, we investigate the sensitivity of the marginal value function in the reward of each fare class. We will increase the reward of a specific product and try to see its impact. We define $\Delta v_{t+1}^{\epsilon}(x,k)$ as the marginal value of an additional inventory. We have the following proposition about the effects of reward on the marginal value.

Proposition 3.13 *$\Delta v_t^{\epsilon}(x,j) \ge \Delta v_t(x,j)$ if $c(N)$ is increased by $\epsilon \ge 0$.*

Proof. We denote the modified function of price as $c_\epsilon(a)$ where $c(a)$ and $c_\epsilon(a)$ are identical expect $a = N$. At the terminal stage we trivially have $\Delta v_T^\epsilon(x,j) = \Delta v_T(x,j) = 0$ or any inventory level x and environment j. Suppose $\Delta v_{t+1}^\epsilon(x,j) \geq \Delta v_{t+1}(x,j)$ for $\forall x, j$. It is sufficient to show,

$$\max_{u_1 \in U(b,x)} \left\{ \sum_{k=1}^{M} p_{ik} v_{t+1}(x-u_1,k) + c(a)\,u_1 \right\} - \max_{u_2 \in U(b,x-1)} \left\{ \sum_{k=1}^{M} p_{ik} v_{t+1}(x-1-u_2,k) + c(a)\,u_2 \right\}$$
$$\leq$$
$$\max_{u_3 \in U(b,x)} \left\{ \sum_{k=1}^{M} p_{ik} v_{t+1}^\epsilon(x-u_3,k) + c_\epsilon(a)\,u_3 \right\} - \max_{u_4 \in U(b,x-1)} \left\{ \sum_{k=1}^{M} p_{ik} v_{t+1}^\epsilon(x-1-u_4,k) + c_\epsilon(a)\,u_4 \right\} \tag{3.9}$$

for any $1 \leq a \leq N$, $1 \leq b \leq B_a$. When $a \neq N$, then this inequality is true by a similar proof to that of proposition 3.3. In case of $a = N$, we know that threshold level is always 0, therefore, optimal quantity is $u_3^* = \min\{b, x\}$ when inventory level is x. Similarly $u_1^* = \min\{b, x\}$, $u_2^* = \min\{b, x-1\} = u_4^*$.

Case	$(u_1^*, u_2^*, u_3^*, u_4^*)$	Inequality (3.9) simplifies to
1	(b,b,b,b)	$\sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x-b,k) \leq \sum_{k=1}^{M} p_{ik}\Delta v_{t+1}^\epsilon(x-b,k)$
2	$(0,0,0,0)$	$\sum_{k=1}^{M} p_{ik}\Delta v_{t+1}(x,k) \leq \sum_{k=1}^{M} p_{ik}\Delta v_{t+1}^\epsilon(x,k)$
3	$(x, x-1, x, x-1)$	$c(N) \leq c_\epsilon(N)$

Case 3 is true due to the increase in $c(N)$. Case 1 and 2 are also true by the induction hypothesis. ■

Proposition 3.13 establishes increasing the reward of the highest class leads to higher admission thresholds: $l_{t,\epsilon}^{a,j} \geq l_t^{a,j}$. As before, somewhat surprisingly, a positive perturbation of $c(N)$ requires a stronger protection for class N and therefore has a non-decreasing effect for all admission thresholds. Please note a corresponding result exists in Aydın et al. [3] for the case with a single environment state.

We have also investigated the effect of increasing the reward of any other product rather than the one with the highest reward. It is not always true that the marginal value of an additional inventory in a modified system is greater than the one in the original system or vice versa. We have a counter-example in the next section.

3.4 Numerical Illustrations

In our illustrations, we assume that an arrival customer demands only one product, this implies that $B_a = 1$ for any product a. First, we illustrate that the threshold level decreases as time increases and increases as the environment gets better. The transition matrix, reward vector, and arrival probability matrix are respectively:

$$\mathbf{P} = \begin{bmatrix} 0.95 & 0.05 \\ 0.05 & 0.95 \end{bmatrix} \qquad c\,(a) = \begin{cases} 0 & \text{if } \ a = 0 \\ 50 & \text{if } \ a = 1 \\ 100 & \text{if } \ a = 2 \\ 200 & \text{if } \ a = 3 \end{cases} \qquad \mathbf{R} = \begin{bmatrix} 0.7 & 0.2 & 0.1 & 0 \\ 0.1 & 0.2 & 0.2 & 0.5 \end{bmatrix} \tag{3.10}$$

with planning horizon $T = 500$. Note that 0 in vector c stands for the reward of the fictitious event. We only show the last 10 threshold levels in our tables. Note that $\mathbf{R}$ has the IFR property, hence we can label the first row of $\mathbf{R}$ as a bad environment and the second row as a good environment. Threshold levels for fare class 1 (with reward 50) and 2 (with reward 100) are given in Table 3.1. Recall that $l_t^{a,j}$ stands for the threshold level of fare class a at time t in environment j. As we expect, the threshold level decreases as time increases for any environment and the threshold level of a better environment is higher at any given time. Also we know that the threshold level for fare class 3 (with reward 200) is always 1 for any environment and time. Since, we always accept a request for the fare class with the highest reward. Finally, recall that Corollary 1 established that $\Delta v_{t+1}(x, i) \leq \Delta v_t(x, j)$ for $i \leq j$. It can be observed from Table 1 that the condition $i \leq j$ is crucial. In fact, we observe that $l_6^{1,1} \leq l_7^{1,2}$. Therefore, the threshold is not necessarily monotone in all cases.

Table 3.1: Threshold levels for fare classes 1 and 2 in both environments

Time t	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$l_t^{1,1}$	4	4	3	3	3	2	2	1	1	1
$l_t^{1,2}$	8	8	7	6	5	5	4	3	2	1
$l_t^{2,1}$	2	1	1	1	1	1	1	1	1	1
$l_t^{2,2}$	6	5	5	4	4	3	3	2	2	1

Next, we investigate the effects of changing the parameters of the problem. Suppose that we decrease the arrival probability of fare class 3 from 0.5 to 0.1 and increase the arrival probability of fare class 1 from 0.2 to 0.6 in environment 2, then we compare the threshold levels for fare classes 1 and 2 in both systems. See Tables 3.2 and 3.3 for these threshold levels. As expected, threshold levels for fare classes 1 and 2 are smaller in the modified system in any environment.

Table 3.2: Threshold levels of fare class 1 (Change in arrival probability of fare class 1)

Time t	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$l_t^{1,1}$	4	4	3	3	3	2	2	1	1	1
$l_{t,\epsilon}^{1,1}$	3	3	3	2	2	2	1	1	1	1
$l_t^{1,2}$	8	8	7	6	5	5	4	3	2	1
$l_{t,\epsilon}^{1,2}$	5	5	5	4	4	3	3	2	2	1

Table 3.3: Threshold levels of fare class 2 (Change in arrival probability of fare class 1)

Time t	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$l_t^{2,1}$	2	1	1	1	1	1	1	1	1	1
$l_{t,\epsilon}^{2,1}$	1	1	1	1	1	1	1	1	1	1
$l_t^{2,2}$	6	5	5	4	4	3	3	2	2	1
$l_{t,\epsilon}^{2,2}$	2	2	2	2	2	2	1	1	1	1

We also change the entries of $\mathbf{P}$ while the modified $\mathbf{P}$ matrix still has the IFR property. Suppose that we have the following modified $\mathbf{P}$

$$\mathbf{P}^{\epsilon} = \begin{bmatrix} 0.05 & 0.95 \\ 0.05 & 0.95 \end{bmatrix} \tag{3.11}$$

which is obtained by changing the first row of the transition matrix. The threshold levels for fare class 2 in the modified system is greater than the one in the original system as shown in Table 3.4 for environment 1.

Further, we increase the reward of the third fare class, which is the most expensive one, from 200 to 250. Threshold levels for fare classes 1 and 2 are given in the Tables 3.5 and 3.6.

Table 3.4: Threshold levels of fare class 2 in environment 1 (Change in transition matrix)

Time t	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$l_t^{2,1}$	2	1	1	1	1	1	1	1	1	1
$l_{t,\epsilon}^{2,1}$	7	6	5	5	4	4	3	2	2	1

Table 3.5: Threshold levels of fare class 1 (Change in price of fare class 1)

Time t	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$l_t^{1,1}$	4	4	3	3	3	2	2	1	1	1
$l_{t,\epsilon}^{1,1}$	5	4	4	3	3	2	2	1	1	1
$l_t^{1,2}$	8	8	7	6	5	5	4	3	2	1
$l_{t,\epsilon}^{1,2}$	9	8	7	6	6	5	4	3	2	1

Table 3.6: Threshold levels of fare class 2 (Change in price of fare class 1)

Time t	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$l_t^{2,1}$	2	1	1	1	1	1	1	1	1	1
$l_{t,\epsilon}^{2,1}$	3	2	2	2	1	1	1	1	1	1
$l_t^{2,2}$	6	5	5	4	4	3	3	2	2	1
$l_{t,\epsilon}^{2,2}$	6	6	5	5	4	4	3	3	2	1

The threshold levels for fare classes 1 and 2 of the modified system are greater in both environments. We also provide a counter-example for the case when the reward of any other fare class rather than the most expensive one is changed. Suppose that we change the reward of fare class 2 from 100 to 150. The threshold levels of fare class 1 in environment 1 and fare class 2 in environment 2 are given in Table 3.7.

Note that the threshold levels of product one increase; however, the threshold levels of fare class two decrease. Therefore, it is not always true that expected marginal value of an additional inventory decreases (or increases) as we increase the reward of a fare class which is not the most expensive.

Remember that when $\mathbf{R}$ is IFR, we can order the environments. In addition to this property, if $\mathbf{P}$ is IFR, we know that $l_t^{a,j} \leq l_t^{a,i}$ whenever $j \leq i$. However, we cannot conclude

Table 3.7: Threshold levels for fare classes 1 and 2 (Change in price of product 2)

Time t	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$l_t^{1,1}$	4	4	3	3	3	2	2	1	1	1
$l_{t,\epsilon}^{1,1}$	5	4	4	3	3	2	2	2	1	1
$l_t^{2,2}$	6	5	5	4	4	3	3	2	2	1
$l_{t,\epsilon}^{2,2}$	4	4	4	3	3	3	2	2	1	1

the same result when $\mathbf{P}$ is not IFR. We have the following counter-example to show this claim. We use the same problem parameters except the matrix

$$\mathbf{P} = \begin{bmatrix} 0.05 & 0.95 \\ 0.95 & 0.05 \end{bmatrix} \tag{3.12}$$

which is not IFR anymore. The threshold levels for fare class 1 (with price 50) are given in the Table 3.8. Even though environment 2 can be considered better than environment 1, threshold levels of fare class 1 at times 3, 5, 7 and 9 in environment 1 are greater than those in environment 2.

Table 3.8: Threshold levels for fare class 1 when P is not IFR

Time t	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$l_t^{1,1}$	6	5	5	4	4	3	3	2	2	1
$l_t^{1,2}$	6	5	4	4	3	3	2	2	1	1

3.5 The Efficiency of the Environment-Based Model

To assess the performance of our environment based model, we consider a 2-environment problem in which arriving customers demand only one product at a time. In this setting, we compare the expected revenues from our model to a simple but reasonable benchmark approach where the system manager incorrectly believes that the system will always remain in one of the environment states (i.e. the environment will not fluctuate). In this case, the manager solves a simpler standard dynamic program to find the optimal admission policy.

To implement the benchmark approach, let us define $w_t^j(x)$, the maximum expected total revenue when the environment j is the environment believed to be true by the manager. The corresponding optimal policy can be formulated by the Bellman equation

$$w_t^j(x) = \sum_{a=1}^{N} r_{ja} \max\left\{w_{t+1}^j(x-1) + c(a), w_{t+1}^j(x)\right\} + r_{j0} w_{t+1}^j(x) \tag{3.13}$$

with boundary conditions $w_T^j(x) = 0$ and $w_t^j(0) = 0$ for all x and t. R and P are as given in (3.10) and $c(0) = 0$, $c(1) = 50$, $c(3) = 200$ and we vary $c(2)$ between 65 and 185 (using a step-size of 30).

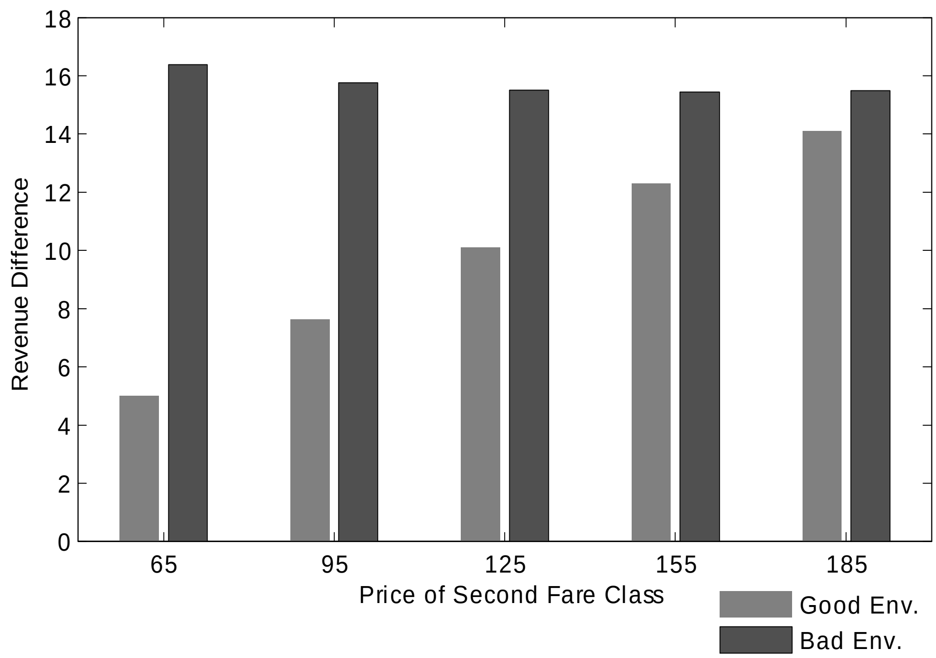
For each environment state, we compute the optimal admission policy and use this policy in our environment-based model and calculate the corresponding expected revenue for an initial inventory level of 200 and planning horizon of 500 starting with environment 1. In addition, we compute the expected optimal revenue using the environment-dependent model for the same parameters. Figure 3.1 reports the the percentage differences in expected revenues due to using a simpler model for different values of $c(2)$. It can be observed that the difference is consistently over 15% when the manager employs the good environment state (maybe due to optimistic expectations). On the other hand, the difference varies significantly and appears to be an increasing as a function of $c(2)$ when the manager employs the bad environment state.

Next, we explore how the benefits of the environment-based model are affected by the demand profile similarity or dissimilarity in different environments. We use c and $\mathbf{P}$ as given in (3.10) and we define $\mathbf{R}(\epsilon)$ (where $\epsilon = 0$, 0.1, 0.2, 0.3, 0.4) as follows:

$$\mathbf{R}(\epsilon) = \begin{bmatrix} 0.7-\epsilon & 0.2 & 0.1 & 0+\epsilon \\ 0.1 & 0.2 & 0.2 & 0.5 \end{bmatrix}.$$

Please note that increasing the value of ϵ makes the demand profiles in the two environments more similar. Therefore, when $\epsilon = 0$ the demand profiles are very different from each other and when $\epsilon = 0.4$, the demand profiles are fairly similar. For each ϵ, we repeat the same investigation as above and compare the revenues in an environment-based model with a fixed environment model. The percentage differences as a function of the environment are reported in Table 3.9. As can be observed from the table, there is significant value in using the environment-based model when the demand profiles are different but as expected this value diminishes as the demand profiles of the different environments become similar.

Figure 3.1: Effects of Environment



Finally, we investigate the effect of total demand rate difference between the environments. The situation in mind we have is external factors that affect the aggregate demand rate in varying degrees. In particular, if the demand rate in the first environment for a given demand class a $(a = 1,\ 2,\ 3)$ is r_{1a}, then the corresponding demand rate in the second environment is αr_{1a} where $0 < \alpha < 1$. For the numerical experimentation, we use c as given in (3.10) and the other parameters are given below.

$$\mathbf{R}(\alpha) = \begin{bmatrix} 1-0.9\alpha & 0.4\alpha & 0.3\alpha & 0.2\alpha \\ 0.1 & 0.4 & 0.3 & 0.2 \end{bmatrix},\ \mathbf{P_1} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix},\ \mathbf{P_2} = \begin{bmatrix} 0.7 & 0.3 \\ 0.1 & 0.9 \end{bmatrix},\ \mathbf{P_3} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

Using the above parameters, we experiment with three levels of α and repeat the earlier experimentation by comparing the revenues using the environment-based dynamic program versus revenues obtained by solving simpler single environment models. Please note that we also use three different transition matrices. The results are reported in Table 3.10. Once

Table 3.9: Percentage difference as function of R

ϵ	**0**	**0.1**	**0.2**	**0.3**	**0.4**
Good Env.	8.06	4.69	1.78	0.26	0.0017
Bad Env.	15.71	11.63	5.46	3.57	0.0060

again, the benefits of using the environment-based model are important when the demand profiles (aggregate demand rates in this experiment) are different.

Table 3.10: Percentage difference when the total demand rate fluctuates

	$\mathbf{P_1}$		$\mathbf{P_2}$		$\mathbf{P_3}$	
α	Good Env.	Bad Env.	Good Env.	Bad Env.	Good Env.	Bad Env.
0.25	3.56	8.15	0.20	17.34	3.62	8.49
0.5	0.62	3.18	0.07	5.13	0.51	3.10
0.75	0.07	0.06	0.01	0.13	0.06	0.06

3.6 Extension: Markov Modulated Dynamic Pricing

In this section, we extend our investigation to a corresponding dynamic pricing problem. A similar continuous-time problem with replenishment has been explored by Gayon et al. [17]. In dynamic pricing, customers are not segmented to different classes but they have different purchasing probabilities as a function of the offered price. The goal is to find the price to charge in a given state to maximize the expected revenue. We assume that there is only one customer in each stage and his willingness to pay is a random variable which depends on the current environment. If the current environment is j then the price he is willing to pay W_j has a distribution $F_j(v) = P\{W_j \leq v\}$. The distribution function is assumed to be differentiable and we denote the density by $f_j(p)$. We also assume that the distribution function has an inverse F_j^{-1}. Let $v_t(x, j)$ be the expected maximum revenue from period t on, given that current inventory level is x and environment is j. The manager needs to choose the price of the product in each stage t with a given environment j and inventory level x. Therefore, we now have the following Bellman equation

$$v_t(x,j) = \max_{p\geq 0}\left\{(1-F_j(p))\left(p+\sum_{k=1}^{N} p_{jk} v_{t+1}(x-1,k)\right) + F_j(p)\sum_{k=1}^{N} p_{jk} v_{t+1}(x,k)\right\}$$

with $v_T(x,j) = 0$ and $v_t(0,j) = 0$ as boundary conditions. Since distribution function is one-to-one, there exists a unique p such that $d = \bar{F}_j(p) = 1 - F_j(p)$ for any $0 \leq d \leq 1$. Therefore, we have the following equivalent formulation

$$v_t(x,j) = \max_{0\leq d\leq 1}\left\{d\left(p_j(d)+\sum_{k=1}^{N} p_{jk} v_{t+1}(x-1,k)\right) + (1-d)\sum_{k=1}^{N} p_{jk} v_{t+1}(x,k)\right\}$$

where $p_j(d) = F_j^{-1}(1-d)$. By using $\Delta v_t(x,j) = v_t(x,j) - v_t(x-1,j)$, we have

$$v_t(x,j) = \max_{0\leq d\leq 1}\left\{d p_j(d) - d\sum_{k=1}^{N} p_{jk}\Delta v_{t+1}(x,k)\right\} + \sum_{k=1}^{N} p_{jk} v_{t+1}(x,k) \tag{3.14}$$

In (3.14), $dp_j(d)$ is the expected revenue during the current stage. Let $H_j(d)$ be the derivative of $dp_j(d)$ with respect to d, we make the following standard assumption as in Talluri and van Ryzin [30].

Condition 3.14 *For any environment j, $H_j(d)$ is a decreasing function in d, and this condition also implies that $H_j\left(\bar{F}_j(p)\right) = p - (1-F_j(p))/f_j(p)$ is increasing function of p.*

By using this condition, we know that inner part of the maximization problem is a concave function in d, therefore; the optimal solution can be found by using

$$H_j(d^*) = \sum_{k=1}^{N} \Delta v_{t+1}(x,k) \tag{3.15}$$

For the rest of this section, we assume that $d^* \in (0,1)$. To gain insights on the the structure of the optimal pricing policy we need to investigate the structure of Δv_t. First, we show that marginal revenue decreases as we have more inventory.

Proposition 3.15 *$\Delta v_t(x,i)$ is a decreasing function of x for any environment i and time t.*

Proof. We know that $v_T(x,j) = 0$, therefore $\Delta v_T(x,j) = 0$. Assume that Δv_{t+1} is increasing function of x. Then

$$\begin{aligned}
\Delta v_t(x-1,j) - \Delta v_t(x,j) &= \sum_{k=1}^{N} p_{jk}\Delta v_{t+1}(x-1,k) - \sum_{k=1}^{N} p_{jk}\Delta v_{t+1}(x,k) \\
&+ \max_{0\le d\le 1}\left\{dp_j(d) - d\sum_{k=1}^{N} p_{jk}\Delta v_{t+1}(x-1,k)\right\} \\
&- \max_{0\le d\le 1}\left\{dp_j(d) - d\sum_{k=1}^{N} p_{jk}\Delta v_{t+1}(x-2,k)\right\} \\
&- \max_{0\le d\le 1}\left\{dp_j(d) - d\sum_{k=1}^{N} p_{jk}\Delta v_{t+1}(x,k)\right\} \\
&+ \max_{0\le d\le 1}\left\{dp_j(d) - d\sum_{k=1}^{N} p_{jk}\Delta v_{t+1}(x-1,k)\right\}
\end{aligned}$$

Let d_1 be optimal solution for $\max_{0\le d\le 1}\left\{dp_j(d) - d\sum_{k=1}^{N} p_{jk}\Delta v_t(x,k)\right\}$ and d_2 be optimal solution for $\max_{0\le d\le 1}\left\{dp_j(d) - d\sum_{k=1}^{N} p_{jk}\Delta v_t(x-2,k)\right\}$, then we have

$$\begin{aligned}
\Delta v_t(x-1,j) - \Delta v_t(x,j) &\ge \sum_{k=1}^{N} p_{jk}\Delta v_{t+1}(x-1,k) - \sum_{k=1}^{N} p_{jk}\Delta v_{t+1}(x,k) \\
&+ d_1 p_j(d_1) - d_1\sum_{k=1}^{N} p_{jk}\Delta v_{t+1}(x-1,k) \\
&- d_2 p_j(d_2) + d_2\sum_{k=1}^{N} p_{jk}\Delta v_{t+1}(x-2,k) \\
&- d_1 p_j(d_1) + d_1\sum_{k=1}^{N} p_{jk}\Delta v_{t+1}(x,k) \\
&+ d_2 p_j(d_2) - d_2\sum_{k=1}^{N} p_{jk}\Delta v_{t+1}(x-1,k)
\end{aligned}$$

After cancellations and rearranging the terms, we have

$$\begin{aligned}
\Delta v_t(x-1,j) - \Delta v_t(x,j) &\ge (1-d_1)\sum_{k=1}^{N} p_{jk}\left(\Delta v_{t+1}(x-1,k) - \Delta v_{t+1}(x,k)\right) \\
&+ d_2\sum_{k=1}^{N} p_{jk}\left(\Delta v_{t+1}(x-2,k) - \Delta v_{t+1}(x-1,k)\right)
\end{aligned}$$

Since $0 \leq d_1, d_2 \leq 1$, right hand side of the last inequality is greater than 0 by using the induction hypothesis. Hence we have the result. ∎

Under Condition 3.14, Proposition 3.15 implies that the optimal prices are non-increasing in the inventory on hand. Next, explore the effect of time on the marginal revenue.

Proposition 3.16 *$\Delta v_t(x,i)$ is a decreasing function of t for any environment i and inventory level x.*

Proof. We know that $\Delta v_T(x,j) = 0$. Also, one can easily show that $v_t(x,j)$ is an increasing function of x by using induction. Therefore, $\Delta v_T(x,j) \leq \Delta v_{T-1}(x,j)$. Assume that $\Delta v_{t+1}(x,j) \leq \Delta v_t(x,j)$ for any inventory level x and environment j. Then

$$\begin{aligned}
\Delta v_{t-1}(x,j) - \Delta v_t(x,j) &= \sum_{k=1}^{N} p_{jk} \Delta v_t(x,k) - \sum_{k=1}^{N} p_{jk} \Delta v_{t+1}(x,k) \\
&\quad + \max_{0 \leq d \leq 1} \left\{ d p_j(d) - d \sum_{k=1}^{N} p_{jk} \Delta v_t(x,k) \right\} \\
&\quad - \max_{0 \leq d \leq 1} \left\{ d p_j(d) - d \sum_{k=1}^{N} p_{jk} \Delta v_t(x-1,k) \right\} \\
&\quad - \max_{0 \leq d \leq 1} \left\{ d p_j(d) - d \sum_{k=1}^{N} p_{jk} \Delta v_{t+1}(x,k) \right\} \\
&\quad + \max_{0 \leq d \leq 1} \left\{ d p_j(d) - d \sum_{k=1}^{N} p_{jk} \Delta v_{t+1}(x-1,k) \right\}
\end{aligned}$$

Let d_2 be optimal solution for $\max_{0 \leq d \leq 1} \left\{ d p_j(d) - d \sum_{k=1}^{N} p_{jk} \Delta v_t(x-1,k) \right\}$ and d_3 be the optimal solution for $\max_{0\leq d\leq 1}\left\{dp_j(d)-d\sum_{k=1}^{N}p_{jk}\Delta v_{t+1}(x,k)\right\}$. Then

$$\begin{aligned}\Delta v_{t-1}(x,j)-\Delta v_t(x,j) &\geq \sum_{k=1}^{N}p_{jk}\left(\Delta v_t(x,k)-\Delta v_{t+1}(x,k)\right)\\ &+d_3p_j(d_3)-d_3\sum_{k=1}^{N}p_{jk}\Delta v_t(x,k)\\ &-d_2p_j(d_2)+d_2\sum_{k=1}^{N}p_{jk}\Delta v_t(x-1,k)\\ &-d_3p_j(d_3)+d_3\sum_{k=1}^{N}p_{jk}\Delta v_{t+1}(x,k)\\ &+d_2p_j(d_2)-d_2\sum_{k=1}^{N}p_{jk}\Delta v_{t+1}(x-1,k)\end{aligned}$$

After cancellations and rearranging the terms we have

$$\begin{aligned}\Delta v_{t-1}(x,j)-\Delta v_t(x,j) &\geq (1-d_3)\sum_{k=1}^{N}p_{jk}\left(\Delta v_t(x,k)-\Delta v_{t+1}(x,k)\right)\\ &+d_2\sum_{k=1}^{N}p_{jk}\left(\Delta v_t(x-1,k)-\Delta v_{t+1}(x-1,k)\right)\end{aligned}$$

By using the induction hypothesis and $0\leq d_2,d_3\leq 1$, we have the result. ■

Proposition 3.16 provides further insights on the structure of the optimal pricing policy. Under Condition 3.14, Proposition 3.16 implies that the optimal prices are non-increasing in the remaining time for the same inventory level and environment. While the optimal price paths need to be non-increasing in general, they are so when the environment does not fluctuate.

3.7 Conclusion and Future Research

We investigated a single resource capacity control problem with a fluctuating demand environment. Modeling fluctuating demand through a Markov-modulated environment process is widely accepted in the inventory control literature. But there has not been much work on such models in capacity control problems rooted in revenue management.

We were able to provide a fairly complete set of structural results on the optimal admission policy under a Markov-modulated demand process. The structural results comprise the existence of environment-based thresholds but also extend to the effect of the time, environments and various problem parameters. Some extensions of the model follow relatively easily as in the dynamic pricing case presented in Section 3.6. Other extensions such as network revenue management merit further research. Another interesting and challenging line of extension is to consider uncertain environment transition rates.

Chapter 4

CHOICE MODEL OF CONSUMER BEHAVIOR

4.1 Model Formulation

We formulate a discrete time, finite horizon (T periods) Markov decision process (MDP) model of the general discrete choice model of consumer behavior under a fluctuating demand environment. Let $X_t \in \{1, 2, \cdots, M\}$ denote the environment. $X = \{X_0, X_1, \cdots, X_T\}$ is assumed to be a Markov chain with transition matrix P where $p_{ij} = P\{X_{t+1} = j | X_t = i\}$. We assume that there is at most one arrival during each time interval. We denote the probability of arrival in environment j by λ_j.

Each customer makes a decision according to the current environment and set of products offered by the firm. Therefore, firm's objective is to choose the optimal set of products to offer to maximize its expected revenue. Let N be the finite set of all products that can be offered by the firm. Let $P_a^j(S)$ denote the probability that a product of type $a \in S$ has been chosen in environment j given that the set of offered products is S. Similarly, we define $P_0^j(S)$ as the probability of no purchase in environment j when the firm offers product set $S \subseteq N$.

For each product a that is sold the reward is $c(a)$. The transition probabilities and reward function are assumed to be stationary and we suppose without loss of generality that the fare classes are ordered so that $c(a_1) \leq c(a_2)$ when $a_1 \leq a_2$. We let $\mathbb{R} = (-\infty, +\infty)$ denote the set of real numbers and $\mathbb{R}_+ = [0, +\infty)$ denote the set of positive real numbers.

We also use the following notations:

$v_t(x, j) =$ expected maximum revenue from period t until period T given that the current inventory level is x and the environment is j.

$\Delta v_t(x, k) =$ $v_t(x, k) - v_t(x-1, k)$

The optimal solution to this problem can be obtained by solving the following Bellman equation

$$\begin{aligned} v_t(x,j) &= \max_{S\subseteq N}\{\sum_{a\in S}\lambda_j P_a^j(S)\left(c(a)+\sum_{k=1}^{M}p_{jk}v_{t+1}(x-1,k)\right) \\ &\quad +\left(\lambda_j P_0^j(S)+1-\lambda_j\right)\sum_{k=1}^{M}p_{jk}v_{t+1}(x,k)\} \\ &= \max_{S\subseteq N}\left\{\sum_{a\in S}\lambda_j P_a^j(S)\left(c(a)-\sum_{k=1}^{M}p_{jk}\Delta v_{t+1}(x,k)\right)\right\}+\sum_{k=1}^{M}p_{jk}v_{t+1}(x,k) \end{aligned} \tag{4.1}$$

with the following boundary conditions

$$\begin{aligned} v_t(0,j) = 0 &\quad \text{for any environment } j,\ t=1,\cdots,T, \\ v_T(x,j) = 0 &\quad \text{for any inventory level } x, \text{ environment } j. \end{aligned}$$

Talluri and van Ryzin [29] suggest the reformulation for the choice model in a way that uses the total probability of purchase and the total expected revenue when the offered product set is S. We adapt this reformulation to the environment based model and write

$$v_t(x,j) = \lambda_j \max_{S\subseteq N}\{(R^j(S) - Q^j(S)\sum_{k=1}^{M}p_{jk}\Delta v_{t+1}(x,k)\} + \sum_{k=1}^{M}p_{jk}v_{t+1}(x,k) \tag{4.2}$$

where

$$Q^j(S) = \sum_{a\in S}P_a^j(S) = 1 - P_0^j(S) \tag{4.3}$$

and

$$R^j(S) = \sum_{a\in S}c(a)P_a^j(S). \tag{4.4}$$

Note that $Q^j(S)$ and $R^j(S)$ respectively denote the probability that a product will be purchased and the expected revenue if S is offered in environment j. To understand the structure of the optimal sets, we now define environment based efficient sets.

Definition 4.1 *A set T is $j-$inefficient if there exists probabilities $\alpha(S)$ for any $S\subseteq N$ with $\sum_{S\subseteq N}\alpha(S)=1$ such that*

$$Q^j(T) \geq \sum_{S\subseteq N}\alpha(S)Q^j(S) \quad \textit{and} \quad R^j(T) < \sum_{S\subseteq N}\alpha(S)R^j(S).$$

Otherwise, T is $j-$efficient.

The intuition behind the definition of an environment based inefficient set is similar to the interpretation of inefficient sets in Talluri and van Ryzin [29]. A set T is j-inefficient if other sets $S \subseteq N$ exist, such that the combination of their corresponding expected revenues is strictly greater than the expected revenue of T (*i.e.*, $R^j(T)$), but the combination of their corresponding probability of purchase is less than $Q^j(T)$.

Talluri and van Ryzin [29] show that it is never optimal to offer an inefficient set. We also have the same property and this can be shown by using the following version of Proposition 1 proven in Talluri and van Ryzin [29].

Proposition 4.2 *A set T is $j-$efficient if and only if, for some value $v \geq 0$, T is an optimal solution to*

$$\max_{S \subseteq N} \left\{ R^j(S) - vQ^j(S) \right\}.$$

By using this proposition and the fact that $\sum_{k=1}^{M} p_{jk} \Delta v_{t+1}(x,k) \geq 0$, we have the following important result.

Proposition 4.3 *An $j-$inefficient set cannot be optimal to (4.2).*

Since N is a finite set, we have finite number of efficient sets in each environment. Talluri and van Ryzin [29] argued that efficient sets can be ordered such that both expected revenues and probabilities of purchase increases in order such that

$$Q^j\left(S_1^j\right) \leq Q^j\left(S_2^j\right) \leq ... \leq Q^j\left(S_k^j\right) \Rightarrow R^j\left(S_1^j\right) \leq R^j\left(S_2^j\right) \leq ... \leq R^j\left(S_k^j\right)$$

where S_n^j corresponds to the nth efficient set in environment j and k is the total number of such sets. Talluri and van Ryzin [29] show this result by using the following version of Lemma 2 which is also valid for our problem.

Lemma 4.4 *The efficient frontier $\bar{R}^j : [0,1] \rightarrow \mathbb{R}$ defined by*

$$\bar{R}^j(q) = \max \left\{ \begin{array}{c} \sum_{S \subseteq N} \alpha(S) R^j(S) : \sum_{S \subseteq N} \alpha(S) Q^j(S) \leq q, \\ \sum_{S \subseteq N} \alpha(S) = 1, \alpha(S) \geq 0 \end{array} \right\}$$

is concave increasing in q.

4.2 Structural Properties

In this section, we obtain structural properties of the optimal policy for the choice model of consumer behavior under a Markov modulated demand. We show the monotonicity results corresponding to the structure of the optimal policy. First, we need some preliminary results before stating them.

Lemma 4.5 *If $R^j\left(S_l^j\right) - Q^j(S_l^j)v_0 \geq R^j\left(S_k^j\right) - Q^j(S_k^j)v_0$ for some $v_0 \geq 0$ and environments $l > k$, then*

$$R^j\left(S_l^j\right) - Q^j(S_l^j)v \geq R^j\left(S_k^j\right) - Q^j(S_k^j)v$$

for any $0 \leq v \leq v_0$.

Proof. $R^j\left(S_l^j\right) - Q^j(S_l^j)v_0 \geq R^j\left(S_k^j\right) - Q^j(S_k^j)v_0$ is equivalent to

$$R^j\left(S_l^j\right) - R^j\left(S_k^j\right) \geq v_0\left(Q^j(S_l^j) - Q^j(S_k^j)\right).$$

Since $l > k$, we have $Q^j(S_l^j) \geq Q^j(S_k^j)$. Then, we have the following inequality by using $0 \leq v \leq v_0$,

$$v_0\left(Q^j(S_l^j) - Q^j(S_k^j)\right) \geq v\left(Q^j(S_l^j) - Q^j(S_k^j)\right).$$

Hence, we have the desired result. ■

Let $k_{j,t}^*(x)$ be the index of the efficient set that is optimal in environment j at time t with an inventory level x. In case of equivalence, we take the set with the largest index. We have the following proposition to understand the structure of the optimal policy.

Proposition 4.6 *$k_{j,t}^*(x)$ is decreasing as $\sum_{k=1}^{M} p_{jk}\Delta v_{t+1}(x,k)$ is increasing.*

Proof. Let v^j denote $\sum_{k=1}^{M} p_{jk}\Delta v_{t+1}(x,k)$. Consider $0 \leq v_1^j \leq v_2^j$, and let k_i be the index among efficient sets such that it solves $\max_k \left\{R^j\left(S_k^j\right) - Q^j\left(S_k^j\right)v_i^j\right\}$ for $i = 1, 2$. Suppose $k_1 \leq k_2$, then we have

$$R^j\left(S_{k_2}^j\right) - Q^j\left(S_{k_2}^j\right)v_2^j \geq R^j\left(S_{k_1}^j\right) - Q^j\left(S_{k_1}^j\right)v_2^j$$

since k_2 is an optimal efficient set for $\max_k \left\{R^j\left(S_k^j\right) - Q^j\left(S_k^j\right)v_2^j\right\}$. By using $0 \leq v_1^j \leq v_2^j$ and the previous lemma

$$R^j\left(S_{k_2}^j\right) - Q^j\left(S_{k_2}^j\right)v_1^j \geq R^j\left(S_{k_1}^j\right) - Q^j\left(S_{k_1}^j\right)v_1^j.$$

However, this inequality contradicts the optimality of k_1 for $\max_k \left\{ R^j \left(S_k^j \right) - Q^j \left(S_k^j \right) v_1^j \right\}$.
■

Let $S_t^* (x, j)$ denote the optimal set that solves (4.1) so that

$$S^* = \{S_t^* (x, j) ;\ x = 0, 1, \cdots, N,\ j = 1, 2, \cdots, M,\ t = 0, 1, \cdots, T\}$$

is the optimal policy. We have the following proposition to show the effects of the current inventory level on the optimal policy.

Proposition 4.7 *$v_t(x, j)$ is a concave function of x for any environment j and time t, i.e.,* $\Delta v_t (x, j) \leq \Delta v_t (x - 1, j)$.

Proof. Clearly $\Delta v_T (x - 1, j) = 0$ for any x. By induction, suppose $\Delta v_{t+1} (x, j) \leq \Delta v_{t+1} (x - 1, j)$ for any x and j. Then,

$$\begin{aligned}
\Delta v_t (x-1, j) - \Delta v_t (x, j) \quad = \quad & \sum_{k=1}^{M} p_{jk} \Delta v_{t+1} (x - 1, k) - \sum_{k=1}^{M} p_{jk} \Delta v_{t+1} (x, k) \\
& + \sum_{k=1}^{M} p_{jk} \sum_{a \in S_t (x-1,j)} \lambda_j P_a^j (S_t^* (x - 1, j)) (c(a) - \Delta v_{t+1} (x - 1, k)) \\
& - \sum_{k=1}^{M} p_{jk} \sum_{a \in S_t (x-2,j)} \lambda_j P_a^j (S_t^* (x - 2, j)) (c(a) - \Delta v_{t+1} (x - 2, k)) \\
& - \sum_{k=1}^{M} p_{jk} \sum_{a \in S_t (x,j)} \lambda_j P_a^j (S_t^* (x, j)) (c(a) - \Delta v_{t+1} (x, k)) \\
& + \sum_{k=1}^{M} p_{jk} \sum_{a \in S_t (x-1,j)} \lambda_j P_a^j (S_t^* (x - 1, j)) (c(a) - \Delta v_{t+1} (x - 1, k))
\end{aligned}$$

Since $S_t^*(x,j)$ is the optimal set when inventory level is x, and the environment is j at time t, any other set will be worse than this set. Hence, we have

$$\begin{aligned}\Delta v_t(x-1,j)-\Delta v_t(x,j) \geq\ & \sum_{k=1}^{M} p_{jk}\Delta v_{t+1}(x-1,k)-\sum_{k=1}^{M} p_{jk}\Delta v_{t+1}(x,k)\\ &+\sum_{k=1}^{M} p_{jk}\sum_{a\in S_t(x-2,j)}\lambda_j P_a^j\left(S_t^*(x-2,j)\right)\left(c(a)-\Delta v_{t+1}(x-1,k)\right)\\ &-\sum_{k=1}^{M} p_{jk}\sum_{a\in S_t(x-2,j)}\lambda_j P_a^j\left(S_t^*(x-2,j)\right)\left(c(a)-\Delta v_{t+1}(x-2,k)\right)\\ &-\sum_{k=1}^{M} p_{jk}\sum_{a\in S_t(x,j)}\lambda_j P_a^j\left(S_t^*(x,j)\right)\left(c(a)-\Delta v_{t+1}(x,k)\right)\\ &+\sum_{k=1}^{M} p_{jk}\sum_{a\in S_t(x,j)}\lambda_j P_a^j\left(S_t^*(x,j)\right)\left(c(a)-\Delta v_{t+1}(x-1,k)\right)\end{aligned}$$

After some cancellations, we obtain

$$\begin{aligned}\Delta v_t(x-1,j)-\Delta v_t(x,j) \geq\ & \sum_{k=1}^{M} p_{jk}\Phi(x,k)\\ &+\sum_{k=1}^{M} p_{jk}\sum_{a\in S_t(x-2,j)}\lambda_j P_a^j\left(S_t^*(x-2,j)\right)\Phi(x-1,k)\\ &-\sum_{k=1}^{M} p_{jk}\sum_{a\in S_t(x,j)}\lambda_j P_a^j\left(S_t^*(x,j)\right)\Phi(x,k)\end{aligned}$$

where $\Phi(x,k)=\Delta v_{t+1}(x-1,k)-\Delta v_{t+1}(x,k)$. This further leads to

$$\begin{aligned}\Delta v_t(x-1,j)-\Delta v_t(x,j) \geq\ & \sum_{k=1}^{M} p_{jk}\sum_{a\in S_t(x-2,j)}\lambda_j P_a^j\left(S_t^*(x-2,j)\right)\Phi(x-1,k)\\ &+\sum_{k=1}^{M} p_{jk}\left(1-\sum_{a\in S_t(x,j)}\lambda_j P_a^j\left(S_t^*(x,j)\right)\right)\Phi(x,k)\end{aligned}$$

after some mathematical manipulations. The right hand side of the inequality is positive by using the induction hypothesis and the fact that $\sum_{a\in S_t(x,j)}\lambda_j P_a^j\left(S_t^*(x,j)\right)\leq 1$. ■

Proposition 4.8 $\Delta v_{t+1}(x,j)\leq\Delta v_t(x,j)$ *for any inventory level* x*, environment* j *and time* t*.*

Proof. Clearly $\Delta v_T(x,j) = 0 \le \Delta v_{T-1}(x,j)$ for any x. By induction, suppose $\Delta v_{t+2}(x,j) \le \Delta v_{t+1}(x,j)$ for any x and j. Then,

$$\begin{aligned}
\Delta v_t(x,j) - \Delta v_{t+1}(x,j) \quad = \quad & \sum_{k=1}^{M} p_{jk} \Delta v_{t+1}(x,k) - \sum_{k=1}^{M} p_{jk} \Delta v_{t+2}(x,k) \\
& + \sum_{k=1}^{M} p_{jk} \sum_{a \in S_t(x,j)} \lambda_j P_a^j (S_t^*(x,j)) (c(a) - \Delta v_{t+1}(x,k)) \\
& - \sum_{k=1}^{M} p_{jk} \sum_{a \in S_t(x-1,j)} \lambda_j P_a^j (S_t^*(x-1,j)) (c(a) - \Delta v_{t+1}(x-1,k)) \\
& - \sum_{k=1}^{M} p_{jk} \sum_{a \in S_{t+1}(x,j)} \lambda_j P_a^j \left(S_{t+1}^*(x,j)\right) (c(a) - \Delta v_{t+2}(x,k)) \\
& + \sum_{k=1}^{M} p_{jk} \sum_{a \in S_{t+1}(x-1,j)} \lambda_j P_a^j (S_t^*(x-1,j)) (c(a) - \Delta v_{t+2}(x-1,k))
\end{aligned}$$

Since $S_t^*(x,j)$ is the optimal set when inventory level is x, and the environment is j at time t, any other set will be worse than this set. Hence, we have

$$\begin{aligned}
\Delta v_t(x,j) - \Delta v_{t+1}(x,j) \quad \ge \quad & \sum_{k=1}^{M} p_{jk} \Delta v_{t+1}(x,k) - \sum_{k=1}^{M} p_{jk} \Delta v_{t+2}(x,k) \\
& + \sum_{k=1}^{M} p_{jk} \sum_{a \in S_{t+1}(x,j)} \lambda_j P_a^j \left(S_{t+1}^*(x,j)\right) (c(a) - \Delta v_{t+1}(x,k)) \\
& - \sum_{k=1}^{M} p_{jk} \sum_{a \in S_t(x-1,j)} \lambda_j P_a^j (S_t^*(x-1,j)) (c(a) - \Delta v_{t+1}(x-1,k)) \\
& - \sum_{k=1}^{M} p_{jk} \sum_{a \in S_{t+1}(x,j)} \lambda_j P_a^j \left(S_{t+1}^*(x,j)\right) (c(a) - \Delta v_{t+2}(x,k)) \\
& + \sum_{k=1}^{M} p_{jk} \sum_{a \in S_t(x-1,j)} \lambda_j P_a^j (S_t^*(x-1,j)) (c(a) - \Delta v_{t+2}(x-1,k))
\end{aligned}$$

After some cancellations, we obtain

$$\begin{aligned}\Delta v_t(x,j)-\Delta v_{t+1}(x,j) \quad \geq \quad & \sum_{k=1}^{M} p_{jk}\left(\Delta v_{t+1}(x,k)-\Delta v_{t+2}(x,k)\right) \\ & -\sum_{k=1}^{M} p_{jk} \sum_{a\in S_{t+1}(x,j)} \lambda_j P_a^j\left(S_{t+1}^*(x,j)\right)\Phi(x,k) \\ & +\sum_{k=1}^{M} p_{jk} \sum_{a\in S_t(x-1,j)} \lambda_j P_a^j\left(S_t^*(x-1,j)\right)\Phi(x-1,k)\end{aligned}$$

where $\Phi(x,k)=\Delta v_{t+1}(x,k)-\Delta v_{t+2}(x,k)$. This is equivalent to

$$\begin{aligned}\Delta v_t(x,j)-\Delta v_{t+1}(x,j) \quad \geq \quad & \sum_{k=1}^{M} p_{jk}\left(1-\sum_{a\in S_{t+1}(x,j)} \lambda_j P_a^j\left(S_{t+1}^*(x,j)\right)\right)\Phi(x,k) \\ & +\sum_{k=1}^{M} p_{jk} \sum_{a\in S_t(x-1,j)} \lambda_j P_a^j\left(S_t^*(x-1,j)\right)\Phi(x-1,k)\end{aligned}$$

The right hand side of the inequality is positive by using the induction hypothesis and $\sum_{a\in S_{t+1}(x,j)} \lambda_j P_a^j\left(S_{t+1}^*(x,j)\right)\leq 1$. ■

We have the following corollary for the structure of the optimal policy. This corollary is proven by using Proposition 4.6, 4.7, and 4.8.

Corollary 4.9 $k_{j,t}^*(x)$ *increases as we increase the inventory level* x*, and it increases as* t *increases, for any environment* j*.*

Let us discuss the implications of this corollary. As time increases, probability of purchase and expected revenue from the optimal set of products increase. In addition to that, these values decrease as inventory level decreases. Since firm manager will want to offer products such that probability of purchase is high when there is more inventory or less time to the end of the selling season.

4.3 Numerical Illustration

Suppose that there are 2 environments and 3 products which are labeled as K, L and M. Then, we can offer 8 possible sets of products and we label these sets by numbers from 1 to 8. The demand probabilities $P_a^j(S)$ are provided in Table 4.1.

We further suppose that the prices of the products are

Product a	K	L	M
Price $c(a)$	100	300	1000

and the arrival probabilities are

Environment j	1	2
Arrival probability λ_j	0.8	0.9

with planning horizon $T = 11$.

Table 4.1: Demand Probabilities

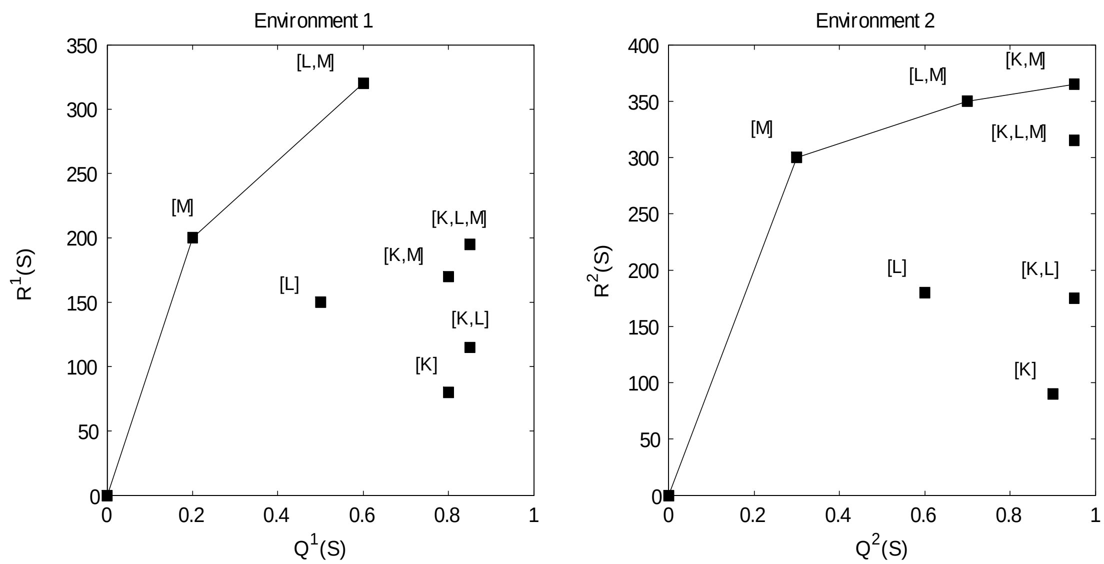
Set	Label	1	2	3	4	5	6	7	8
	a	$\varnothing$	$\{K\}$	$\{L\}$	$\{M\}$	$\{K, L\}$	$\{K, M\}$	$\{L, M\}$	$\{K, L, M\}$
$P_a^1(S)$	K	0	0.8	0	0	0.7	0.7	0	0.65
	L	0	0	0.5	0	0.15	0	0.4	0.1
	M	0	0	0	0.2	0	0.1	0.2	0.1
	0	1	0.2	0.5	0.8	0.15	0.2	0.4	0.15
	Label	1	2	3	4	5	6	7	8
	a	$\varnothing$	$\{K\}$	$\{L\}$	$\{M\}$	$\{K, L\}$	$\{K, M\}$	$\{L, M\}$	$\{K, L, M\}$
$P_a^2(S)$	K	0	0.9	0	0	0.55	0.65	0	0.55
	L	0	0	0.6	0	0.4	0	0.5	0.2
	M	0	0	0	0.3	0	0.3	0.2	0.2
	0	1	0.1	0.4	0.7	0.05	0.05	0.3	0.05

To find the efficient sets, we first find $R^j(S)$ and $Q^j(S)$ using these parameters by (4.3) and (4.4). Hence, we have

Set Label	1	2	3	4	5	6	7	8
	$\varnothing$	$\{K\}$	$\{L\}$	$\{M\}$	$\{K, L\}$	$\{K, M\}$	$\{L, M\}$	$\{K, L, M\}$
$R^1(S)$	0	80	150	200	115	170	320	195
$Q^1(S)$	0	0.8	0.5	0.2	0.85	0.8	0.6	0.85
$R^2(S)$	0	90	180	300	175	365	350	315
$Q^2(S)$	0	0.9	0.6	0.3	0.95	0.95	0.7	0.95

Then, we plot the scatter diagram of the corresponding values $R^j(S)$ and $Q^j(S)$ for any offer set S and $j = 1, 2$ to determine efficient sets for both environments. We find that $\{M\}$ and $\{L, M\}$ are efficient sets in environment 1 and $\{M\}$, $\{L, M\}$ and $\{K, M\}$ in environment 2 by considering Figure 4.1. We now suppose that the transition matrix of the

Figure 4.1: Effects of Environment - Consumer Choice Model



environmental process is

$$P = \begin{bmatrix} 0.95 & 0.05 \\ 0.05 & 0.95 \end{bmatrix}.$$

To find the optimal set for any inventory level, environment and time, we use the labels defined for any set in each environment. We search among all sets, in order to show that inefficient sets cannot be optimal. Also note that efficient sets of environment 1 are $\{M\}$ and $\{L, M\}$ which are labeled 4 and 6 respectively. Since $R^1(\{M\}) \le R^1(\{L, M\})$, we label $\{M\}$ as the 1st efficient set, and $\{L, M\}$ the 2nd efficient set. Similarly, the efficient sets of environment 2 are $\{M\}, \{L, M\}$ and $\{K, M\}$ which are labeled 4, 7 and 6 respectively. Since $R^1(\{M\}) \le R^1(\{L, M\}) \le R^1(\{K, M\})$, we label $\{M\}, \{L, M\}$, and $\{K, M\}$ the 1st, 2nd and 3rd efficient sets respectively. We use both labels by $i|j$ where i is the index among all sets, j is the index among efficient sets to show the optimal sets. Table 4.2 shows the

Table 4.2: Environment 1 - Optimal Sets

Env. 1	Time $\xrightarrow{}$	1	2	3	4	5	6	7	8	9	10
	1	4|1	4|1	4|1	4|1	4|1	4|1	4|1	4|1	7|2	7|2
	2	4|1	4|1	4|1	4|1	4|1	7|2	7|2	7|2	7|2	7|2
$\uparrow$	3	4|1	4|1	7|2	7|2	7|2	7|2	7|2	7|2	7|2	7|2
x	4	7|2	7|2	7|2	7|2	7|2	7|2	7|2	7|2	7|2	7|2
$\downarrow$	5	7|2	7|2	7|2	7|2	7|2	7|2	7|2	7|2	7|2	7|2
	6	7|2	7|2	7|2	7|2	7|2	7|2	7|2	7|2	7|2	7|2
	7	7|2	7|2	7|2	7|2	7|2	7|2	7|2	7|2	7|2	7|2
	8	7|2	7|2	7|2	7|2	7|2	7|2	7|2	7|2	7|2	7|2

optimal set for a given inventory level and time when the current environment is the first one.

As we expect, only 4th and 7th sets are optimal, and none of the inefficient sets is optimal in environment 1. For a given inventory level, the index of the optimal set increases from 1 to 2 as time increases. Similarly, for a given time, we observe an increase in the index of the optimal set as inventory level increases. The optimal sets for environment 2 is provided in Table 4.3.

Table 4.3: Environment 2 - Optimal Sets

Env. 2	Time $\xrightarrow{}$	1	2	3	4	5	6	7	8	9	10
	1	4|1	4|1	4|1	4|1	4|1	4|1	4|1	4|1	4|1	6|3
	2	4|1	4|1	4|1	4|1	4|1	4|1	4|1	4|1	6|3	6|3
$\uparrow$	3	4|1	4|1	4|1	4|1	4|1	4|1	7|2	6|3	6|3	6|3
x	4	4|1	4|1	4|1	4|1	7|2	7|2	6|3	6|3	6|3	6|3
$\downarrow$	5	4|1	4|1	7|2	7|2	6|3	6|3	6|3	6|3	6|3	6|3
	6	7|2	7|2	7|2	6|3	6|3	6|3	6|3	6|3	6|3	6|3
	7	7|2	7|2	6|3	6|3	6|3	6|3	6|3	6|3	6|3	6|3
	8	6|3	6|3	6|3	6|3	6|3	6|3	6|3	6|3	6|3	6|3

Recall that we have 3 efficient sets in environment 2. We observe that only these sets are optimal and monotonicity results corresponding to the structure of the optimal policy also hold in environment 2.

4.4 The Efficiency of the Environment Based Model

To evaluate the efficiency of our environment based model, we mix the probabilities of each product demanded so that $P_a(S) = qP_a^1(S) + (1-q)\ P_a^2(S)$ where $q \in [0,1]$ is the probability mixture. Similarly, we mix the arrival probabilities by $\lambda = q\ \lambda_1 + (1-q)\ \lambda_2$. We use the same problem parameters as we did in the preceding section. In addition to that, we take the horizon length $T = 500$ and capacity 200. We consider the optimal policy of the following new problem which is independent of the environment. The Bellman equation is

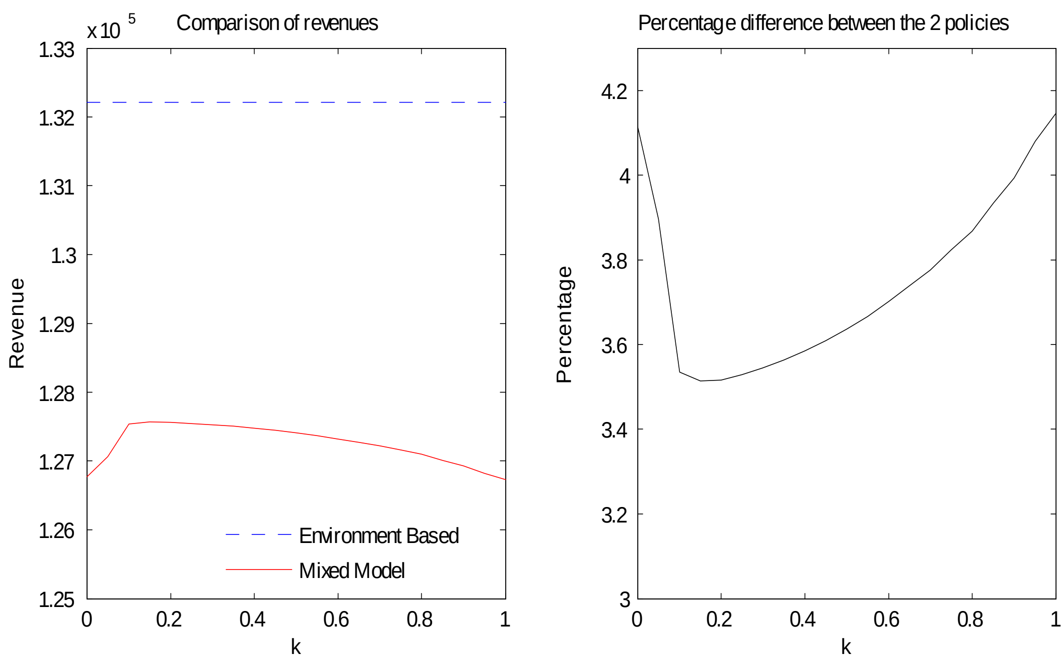
$$\begin{aligned} w_t(x) &= \max_{S \subseteq N}\{\sum_{a \in S} \lambda P_a(S)(c(a) + w_{t+1}(x-1)) \\ &\quad + (\lambda P_0(S) + 1 - \lambda)\, w_{t+1}(x)\} \end{aligned}$$

with boundary conditions $w_T(x) = 0$ and $w_t(0) = 0$ for all x and t.

By using this model, we obtain a policy that does not depend on the environment. This gives a non-optimal policy for our environment based model. We use this policy in our environment model and we calculate the corresponding revenue function value for time 0 and inventory level 200 starting with environment 1. We compare this value with the revenue function value for time 0, inventory level 200 and environment 1 by using the optimal policy of the environment based model. This comparison and difference is given for different values of q in Figure 4.2.

In the environment based model, the inventory manager makes full use of the information that is available at any time. The optimal product sets that are offered depend on the environment. In the mixed model, however, the information is either ignored or unavailable. Instead, the manager considers a simplified model where parameters for the next period are the same as those of environment 1 with a mixing probability q or environment 2 with probability $1-q$. Figure 4.2 clearly demonstrates the benefit of our environment based model.

Figure 4.2: Effects of Environment - Consumer Choice Model



4.5 Conclusion

In this chapter, we have established the structural properties of general discrete choice model of consumer behavior with a randomly fluctuating demand environment. We focused on the structure of the optimal policy and showed that main results of the general discrete choice model of consumer behavior also hold in a randomly fluctuating demand environment. In other words, we showed that only efficient sets can be optimal and the index of the optimal efficient set has a monotonic structure in time and inventory level. Moreover, we have illustrated the structural results and show the efficiency of the environment based model. A worthwhile extension would be to investigate the model in continuous time.

Chapter 5

PARTIALLY OBSERVED DEMAND ENVIRONMENT

We formulate a discrete time, finite horizon (duration T) partially observable Markov decision process (POMDP) model of single resource capacity control problem. We assume that there is at most one arrival in each period. Let $X_t \in \{1, 2, \cdots, M\}$ denote the environment at period t. $X = \{X_0, X_1, \cdots, X_T\}$ is assumed to be a Markov chain with transition matrix P where $p_{ij} = P\{X_{n+1} = j | X_n = i\}$. Let $D_t \in \{0, 1, 2, \cdots, N\}$ denote the type of fare class that arrives at period t. If $D_t = 0$, then there is no arrival at that period. Probability that fare class of d arrives is denoted by r_{jd}, and probability of no arrival is denoted by r_{j0} when the current environment is j. Therefore, $\sum_{a=0}^{N} r_{ja} = 1$ for any j. Let $B_t \in \left\{1, 2, \cdots, B^d\right\}$ the number of fare products requested when fare class d arrives at period t. Probability that b units of the product is requested is denoted by q_{jdb} given that the current environment is j and fare class is d. Clearly, if $D_t = 0$ then $B_t = 0$ since $B^0 = 0$ trivially. Let $Y_t \in \digamma$ denote the observation at period t where $\digamma$ is a finite set of all possible observations. We suppose that Y gives partial information about the process X and realization of process Y depends on the true state of the environment X such that

$$E(k, y) = P\{Y_t = y | X_t = k\}$$

for some so-called emission matrix E. For each product sold, the reward is $c(y, d)$ if observation is y, and fare class is d. The transition probabilities and reward function are assumed to be stationary and we suppose that fare classes are ordered so that $c(y, d_1) \leq c(y, d_2)$ when $d_1 \leq d_2$. Note that $c(y, 0) = 0$.

We will use the following notations:

$\bar{D}_t =$ $(D_0, D_1, \cdots, D_t)$

$\bar{B}_t =$ $(B_0, B_1, \cdots, B_t)$

$\bar{Y}_t =$ $(Y_0, Y_1, \cdots, Y_t)$

$\pi_t =$ (belief vector) conditional distribution of X_t given $\bar{Y}_t$, $\bar{D}_{t-1}$ and $\bar{B}_{t-1}$.

$v_t(x, y, d, b, \pi) =$ expected maximum revenue from period t until period T given that the current inventory level is x and belief vector is π, and fare class is d with a requested amount of b, and observation is y.

$\Delta v_t(x, y, d, b, \pi) =$ $v_t(x, y, d, b, \pi) - v_t(x-1, y, d, b, \pi)$

$U(b, x) =$ $\{0, 1, \cdots, \min\{b, x\}\}$

We suppose that the initial belief vector π_0 is known. By using Bayesian updating we have

$$\pi_{n+1}^k = P\left\{X_{n+1} = k | \bar{D}_n, \bar{B}_n, \bar{Y}_{n+1}\right\} = T^k_{\pi_n | D_n, B_n, Y_{n+1}}$$

where

$$\begin{aligned}
T^k_{\pi|d,b,y} &= P\left\{X_{n+1} = k | \bar{D}_{n-1}, D_n = d, \bar{B}_{n-1}, B_n = b, \bar{Y}_n, Y_{n+1} = y\right\} \\
&= \frac{P\left\{X_{n+1} = k, \bar{D}_{n-1}, D_n = d, \bar{B}_{n-1}, B_n = b, \bar{Y}_n, Y_{n+1} = y\right\}}{P\left\{\bar{D}_{n-1}, D_n = d, \bar{B}_{n-1}, B_n = b, \bar{Y}_n, Y_{n+1} = y\right\}}
\end{aligned}$$

$$= \frac{\sum_{j=1}^{M} P\left\{X_{n+1} = k, \bar{D}_{n-1}, D_n = d, \bar{B}_{n-1}, B_n = b, \bar{Y}_n, Y_{n+1} = y, X_n = j\right\}}{\sum_{j=1}^{M}\sum_{i=1}^{M} P\left\{\bar{D}_{n-1}, D_n = d, \bar{B}_{n-1}, B_n = b, \bar{Y}_n, Y_{n+1} = y, X_n = j, X_{n+1} = i\right\}}$$

$$= \frac{\sum_{j=1}^{M} E(k, y)\, q_{jdb} r_{jd} p_{jk} \pi^j}{\sum_{j=1}^{M}\sum_{i=1}^{M} E(i, y)\, q_{jdb} r_{jd} p_{ji} \pi^j}$$

for $a = 1, 2, \cdots, N$.

Assuming inventory level at the beginning of period t is x, observation at the beginning of period t is y, fare class is d with a demand of b, and current conditional distribution of true state of environment π, optimal solution to this problem can be obtained by solving the following Bellman equation

$$v_t(x, y, d, b, \pi) = \max_{u \in U(b,x)} \left\{ c(y, d)\, u + \sum_{j=1}^{M} \pi^j H^j_{v_{t+1}}(x - u, d, b, \pi) \right\} \tag{5.1}$$

where

$$
\begin{aligned}
H_f^j(x,d,b,\pi) &= \sum_{k=1}^{M}\sum_{z\in F} p_{jk}E(k,z)\, I_f^j\left(x,z,T_{\pi|d,b,z}\right)\\
I_f^j(x,z,\pi) &= \sum_{e=1}^{N}\sum_{a=1}^{B^e} r_{je}q_{jea}f(x,z,e,a,\pi) + r_{j0}f(x,z,0,0,\pi)
\end{aligned}
$$

with the following boundary conditions

$$
\begin{aligned}
v_t(0,d,b,\pi) &= 0 \text{ for any } \pi,\ d,\ b.\\
v_T(x,d,b,\pi) &= 0 \text{ for any } x,\ \pi,\ d, \text{ and } b.
\end{aligned}
$$

By using the definition of Δv_t, this model is equivalent to

$$
\begin{aligned}
v_t(x,y,d,b,\pi) &= \max_{u\in U(b,x)}\left\{\sum_{k=1}^{u}\left(c(y,d) - \sum_{j=1}^{M}\pi^j H_{\Delta v_{t+1}}^j(x+1-k,d,b,\pi)\right)\right\} \qquad (5.2)\\
&\quad + \sum_{j=1}^{M}\pi^j H_{v_t}^j(x,d,b,\pi)
\end{aligned}
$$

where the sum is zero if $u=0$.

5.1 Structural Properties

In this section, we obtain structural properties of this model. Intuitively, if we have one more inventory, then expected revenue should be larger. Similarly, expected revenue should be larger if we have more time to go. These claims can be easily shown by using induction. Another important structural property is the concavity of $v_t(x,d,b,\pi)$ in x.

Theorem 5.1 *$v_t(x,y,d,b,\pi)$ is a concave function in x for any fare class d, demand b, belief vector π, and time t.*

Proof. Since $v_T(x,y,d,b,\pi)$ is zero for any x and π, we have the concavity of $v_T(x,y,d,b,\pi)$ for any π. Suppose that $v_{t+1}(x,y,d,b,\pi)$ is a concave function of x for any y, d, b, and π. We can use Lemma 3.1 by taking g as $\sum_{j=1}^{M}\pi^j H_{v_{t+1}}^j(x-u,d,b,\pi)$ and m as $\min\{b,x\}$. Note that $I_{v_{t+1}}^j\left(x-u,z,T_{\pi|d,b,y}\right)$ is concave for any $u\in U(b,x)$ and $y\in F$ since $I_{v_{t+1}}^j$ is a positive linear combination of v_{t+1}. Similarly, $H_{v_{t+1}}^j$ is also a concave function in x. Therefore, we have the concavity of function g by using the fact that $\pi^j \geq 0$. Hence $v_t(x, y, d, b, \pi)$ defined by (5.1) is concave in the current inventory level x. ■

Theorem 5.1 establishes that $\Delta v_t(x, \pi)$ decreases as we increase the inventory level x. By considering (5.2), we can conclude that

$$c(y, d) - \sum_{j=1}^{M} \pi^j H^j_{\Delta v_{t+1}}(x + 1 - k, d, b, \pi) \tag{5.3}$$

is decreasing in k. Therefore, in (5.2), we should increase u until (5.3) becomes negative or it is equal to $\min\{b, x\}$. As a result, there is a threshold level

$$l_t^{d,b,y,\pi} = \min\left\{ x : c(y, d) \geq \sum_{j=1}^{M} \pi^j H^j_{\Delta v_{t+1}}(x, d, b, \pi) \right\} \tag{5.4}$$

which is the maximum quantity for the inventory level such that if the current inventory level is less than or equal to $l_t^{d,b,y,\pi}$ it is optimal to reject any demand for fare class d when amount b is requested, and y is observed with a belief vector π at period t. However, if the inventory on hand is greater than $l_t^{d,b,y,\pi}$, then demand for fare class d is satisfied until the inventory level drops to $l_t^{d,b,y,\pi}$ or the whole demand is satisfied. Hence the optimal decision at period t is

$$u^* = \min\left\{ \left(x - l_t^{d,b,y,\pi} + 1\right)^+, b \right\} \tag{5.5}$$

when b units of product d is demanded while the observation is y and belief vector is π. We next investigate the structure of the marginal value function. By using the following proposition, we will be able to conclude that the marginal value of one additional inventory is a non-increasing function in time t.

Proposition 5.2 Δv_t *is a decreasing function of* t.

Proof. Since $v_t(x, y, d, b, \pi)$ is increasing in x, $\Delta v_t(x, y, d, b, \pi) \geq 0$. Also $\Delta v_T(x, j) = 0$, which implies that $\Delta v_T(x, y, d, b, \pi) \leq \Delta v_{T-1}(x, y, d, b, \pi)$. Suppose $\Delta v_{t+2}(x, y, d, b, \pi) \leq \Delta v_{t+1}(x, y, d, b, \pi)$ for any inventory level x, observation y, product type d with an amount b requested and belief vector π. Consider the following inequality

$$\max_{u_1 \in U(b,x)} \left\{ c(y,d)\,u + \sum_{j=1}^{M} \pi^j H^j_{v_{t+2}}(x-u,d,b,\pi) \right\} - \max_{u_2 \in U(b,x-1)} \left\{ c(y,d)\,u + \sum_{j=1}^{M} \pi^j H^j_{v_{t+2}}(x-1-u,d,b,\pi) \right\} \tag{5.6}$$

$$\leq$$

$$\max_{u_3 \in U(b,x)} \left\{ c(y,d)\,u + \sum_{j=1}^{M} \pi^j H^j_{v_{t+1}}(x-u,d,b,\pi) \right\} - \max_{u_4 \in U(b,x-1)} \left\{ c(y,d)\,u + \sum_{j=1}^{M} \pi^j H^j_{v_{t+1}}(x-1-u,d,b,\pi) \right\}$$

Let u_i^* be the optimal value of u_i in the inequality above. We should note that $l_{t+1}^{d,b,y,\pi} \leq l_t^{d,b,y,\pi}$ for any d, b, y, π. This can be easily seen by considering the induction hypothesis and (5.4). As a result, we have $u_3^* \leq u_1^*$ and $u_4^* \leq u_2^*$. Also, we know that $u_1^* - u_2^*$ is either 1 or zero. Same reasoning is valid for $u_3^* - u_4^*$. If they are equal, then this is possible only either $u_1^* = u_2^* = 0$ or $u_1^* = u_2^* = b$. Therefore, there are six cases we need to consider for the possible values of u_1^*, u_2^*, u_3^*, u_4^*.

Case	$(u_1^*, u_2^*, u_3^*, u_4^*)$	Inequality (5.6) simplifies to
1	$(0,0,0,0)$	$\sum_{j=1}^{M} \pi^j H^j_{\Delta v_{t+2}}(x,d,b,\pi) \leq \sum_{j=1}^{M} \pi^j H^j_{\Delta v_{t+1}}(x,d,b,\pi)$
2	$(y_2+1, y_2, 0, 0)$	$c(y,d) \leq \sum_{j=1}^{M} \pi^j H^j_{\Delta v_{t+1}}(x,d,b,\pi)$
3	$(b,b,0,0)$	$\sum_{j=1}^{M} \pi^j H^j_{\Delta v_{t+2}}(x-b,d,b,\pi) \leq \sum_{j=1}^{M} \pi^j H^j_{\Delta v_{t+1}}(x,d,b,\pi)$
4	(y_2+1, y_2, y_1+1, y_1)	$c(y,d) \leq c(y,d)$
5	(b,b,y_1+1,y_1)	$\sum_{j=1}^{M} \pi^j H^j_{\Delta v_{t+2}}(x-b,d,b,\pi) \leq c(y,d)$
6	(b,b,b,b)	$\sum_{j=1}^{M} \pi^j H^j_{\Delta v_{t+2}}(x-b,d,b,\pi) \leq \sum_{j=1}^{M} \pi^j H^j_{\Delta v_{t+1}}(x-b,d,b,\pi)$

Here, y_1 and y_2 are integers such that $0 \leq y_1 \leq b-1$ and $y_1 \leq y_2 \leq b-1$. Case 1 and 6 are true due to the induction hypothesis. Also, case 4 is obviously true. In case 2, suppose that $c(y,d) > \sum_{j=1}^{M} \pi^j H^j_{\Delta v_{t+1}}(x,d,b,\pi)$, then we should accept at least one customer when current inventory level is x at period t. But $u_3^* = 0$, and inequality in case 2 is true. In case 5, suppose that $\sum_{j=1}^{M} \pi^j H^j_{\Delta v_{t+2}}(x-b,d,b,\pi) > c(y,d)$, but this contradicts with $u_2^* = b$. In case 3, since $u_3^* = 0$, we have $c(y,d) \leq \sum_{j=1}^{M} \pi^j H^j_{\Delta v_{t+1}}(x,d,b,\pi)$. This can be shown by using the same reasoning used in case 2. Similarly, we have $\sum_{j=1}^{M} \pi^j H^j_{\Delta v_{t+2}}(x-b,d,b,\pi) \leq c(y,d)$ since $u_2^* = b$. Hence we have the inequality of case 3. Consequently, Δv_t decreases in t. ∎

We analyze the structure of threshold levels by using this proposition. Since the marginal value of one additional inventory is a non-increasing function in time t, we conclude that

$$l_{t+1}^{d,b,y,\pi} \leq l_t^{d,b,y,\pi}$$

which means that threshold level is non-increasing in time by considering (5.4).

5.2 Numerical Illustration

In our illustration, we assume that customer demands only one product and Bayesian updating is done only according to the observation. We also assume that there are two types of observations, two environments and two fare-classes. The cost vector, transition matrix, and arrival probability matrix are

$$P = \begin{bmatrix} 0.3 & 0.7 \\ 0.1 & 0.9 \end{bmatrix}, \qquad c = \begin{bmatrix} 0 & 100 & 400 \\ 0 & 200 & 500 \end{bmatrix}, \quad R = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

with a planning horizon $T = 8$ and capacity 5. Furthermore, we take arbitrarily $\pi_0 =$ [0.5 0.5] as the initial belief vector. Note that 0 in the vector c stands for the revenue of the fictitious event. Suppose that we observe state 2 and fare-class 2 arrives in the first period. Therefore, we have a customer who is willing to pay 500 dollars in the initial stage.

For simplicity, we assume that Bayesian updating is done only according to the observation. Therefore, we have

$$\begin{aligned} T^k_{\pi|d,b,y} &= P\left\{X_{n+1} = k|\bar{Y}_n, Y_{n+1} = y\right\} \\ &= \frac{\sum\limits_{j=1}^{M} E\left(k,y\right) p_{jk} \pi^j}{\sum\limits_{j=1}^{M} \sum\limits_{i=1}^{M} E\left(i,y\right) p_{ji} \pi^j}. \end{aligned}$$

for this illustration. Hence the optimal decision has the form

$$u^* = \min\left\{\left(x - l_t^{d,b,y,\pi} + 1\right)^+, b\right\}.$$

where $l_t^{d,b,y,\pi}$ is actually independent of b. Since belief vector is updated by the observations, we have 2^{t-1} different belief vectors at time t due to the different sequences of observations. In our numerical illustration, our purpose is to observe the effect of the emission matrix

Table 5.1: Optimal expected revenue with different values of emission matrix

i	0	1	2	3	4	5
Expected Revenue	1845.9	1874	1901.9	1929.3	1955.9	1981.2

Table 5.2: Optimal threshold levels for fare class 1 with different values of emission matrix

i	0	1	2	3	4	5
Threshold	4	4	4	5	5	5

on the optimal policy and optimal expected revenue. Therefore, we change the emission matrix by taking

$$E_i = \begin{bmatrix} 0.5 + 0.1i & 0.5 - 0.1i \\ 0.5 - 0.1i & 0.5 + 0.1i \end{bmatrix}$$

where $i = 0, 1, \cdots, 5$. Note that the precision of the observation increases as i increases. As a matter of fact, we have perfect observation on the true state of the environment when $i = 5$. Optimal expected revenues for each case are in Table 1.

Suppose that we have observed $(1, 2, 1)$ as a sequence of observations up to the third period. Then, the threshold values for fare-class 1 at the third stage (with a price of 100 in observation 1 and 200 in observation 2) are provided in Table 2 for different values of the emission matrix.

Chapter 6

CONCLUSIONS

In this thesis, we explore dynamic single resource allocation problems when demand is randomly fluctuating. We first model the classical single resource allocation problem and provide structural result for the optimal admission policy. We show that well-known results of classical model also hold in our Markov modulated model. For example, we show that optimal policy is threshold type and the threshold level of each product decreases in time. The focus point of this part is the effect of environment on the optimal policy. First, we provide some intuitive assumptions in order to distinguish environments, then we compare the expected marginal revenues in different environments. We show that threshold level increases when we have a better environment. In this part, we provide a sensitivity analysis, and we observe the effects of each parameter on the expected revenue and expected marginal revenue. We also consider a pricing model in which demand randomly fluctuates. We provide some useful properties including the structure of the optimal policy.

In the second part in this thesis, we investigate a general discrete choice model of consumer behavior in a randomly fluctuating demand environment. We first distinguish each set by considering whether it is efficient or not. The definition of efficiency is environment dependent. Therefore, a set may be efficient in one environment although it is not efficient in another environment. We show that inefficient set in one environment cannot be optimal in that environment. Then we show that we can order these efficient sets in their expected probability of purchase and optimal set number among this order is monotonic in the remaining level of inventory and time.

In the third of part in this thesis, we consider a Hidden Markov version of the model we provide in the first part. In this model, we investigate the case where true state of the environment cannot be directly observed. There is a belief vector for the true state of environment and this belief vector is updated according to the observation that is provided through an external process. In this model, we also show that optimal policy is threshold type and threshold level decreases in time.

In each part, we provide numerical examples to illustrate our structural results. We also use some techniques to show the efficiency of our environment based models with respect to the classical models. We show that our environment based models lead to a significant increase in the expected revenue. In addition, we observe an approximation method to our environment based model by considering the limiting distribution of the environment process.

This line of research can be extended in several directions by future studies. An interesting future research can involve a model in which the firm is risk-sensitive and demand randomly fluctuates. Another research direction can consider a continuous demand where this demand also fluctuates. Also our models can be investigated when the firm can perform hedging in financial markets.

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