ULTRACOLD SPIN-POLARIZED FERMI GASES WITH SPIN-ORBIT COUPLING

by

Selahattin Kalafat

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This is to certify that I have examined this copy of a master's thesis

by

Selahattin Kalafat

and have found that it is complete and satisfactory in all respects,

and that any and all revisions required by the final

examining committee have been made.

Thesis Committee Members:

Asst. Prof. Menderes Işkın (Advisor) .....

Assoc. Prof. Kaan Güven.....

Prof. Levent Demirel .....

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ABSTRACT

Superfluidity is one of the most remarkable phenomenon that we come across in many distinct fields of Physics. It was first found by H. K. Onnes in 1911 when he cooled the mercury sample below 4.2 Kelvin. He found that sample conducted electricity without dissipation called superconductivity. Both bosonic and fermionic system can exhibit superfluidity although their critical temperatures are different, and we focus on the fermionic superfluidity in the thesis.

In the main part of my thesis, we study ground-state properties of ultracold quantum Fermi gases with spin-orbit coupling. The formation of Cooper pairs and their condensation, i.e. the BCS theory which successfully describes many properties of some superconductors, are analyzed first. Then, we examine the BCS-BEC evolution, the generalization of the BCS theory, to pass the strong interaction regime. Atomic systems have the advantage of tunable interaction strength between fermions, and this permits the system to evolve from the BCS limit of loosely bound and largely overlapping Cooper pairs to the BEC limit of tightly bound bosonic molecules, i.e. the study of BCS-BEC evolution. Without this property, it is hopeless to reach the transition temperature of the superfluity in the experiment with cold atoms. In the main part of the thesis work, we study the effects of spin-orbit coupling on the BCS-BEC evolution problem, and analyze the ground state phase diagrams, excitation spectrum, and momentum distribution. In the last part, we also try to solve the system in an optical lattice.

ÖZET

Süperakışkanlık Fiziğin birçok alanında karşımıza çıkan önemli bir fenomendir. Ílk olarak H. K. Onnes tarafından 1911 yılında civa'yı 4.2 kelvinin altına indirince bulunmuştur. Civanın elektiriği sürtünmesiz ilettiğini bulmuştur. Fermionik ve bozonik sistemlerin ikiside bu süperakışkanlık özelliği gösterebilir ve bu tezde fermionik süperakışkanlığa odaklandık.

Bu tezin ana kısmında spin-orbit etkileşimli Fermi gazlarının en düşük enerji seviyesindeki özelliklerini inceledik. Ílk olarak Cooper çiftlerinin oluşumunu, onların yoğunlaşmasını ve bazı süperiletkenlerin özelliklerini açıklayan BCS teorisini analiz ettik. Daha sonra, güçlü etkileşim alanına geçmek için BCS teorisinin genellemesi olan BCS-BEC geçişini çalıştık. Atomik sistemler de fermionlar arası etkileşimi kontrol edebiliyoruz ve bu sayede BCS limit de zayıfça bağlanmış ve birbirlerinin üzerine örtüşmüş Cooper çiftlerinden BEC limit de sıkıca bağlı bozon moleküllerine geçebiliyoruz. Eğer atomlardaki bu özellik olmasaydı, ultra düşük sıcaklıktaki atomlar ile yapılan bu deneylerde superakışkan geçiş sıcaklığına erişmek imkansız olacaktı. Bu tezin ana bölümünde BCS-BEC geçişini spin-orbit etkileşiminide ekleyerek çalıştık ve en düşük enerji seviyesindeki faz diagramını, enerji spetkrumunu ve momentum dağılımını inceledik. Son bölümde, sistemi optik örgülerde çözmeye çalıştık.

To my family.

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1 Introduction

We came across much remarkable phenomena in many distinct fields of Physics in the last century. One of the fascinating problems that people have encountered was superfluidity which was first discovered by Heike Kamerling Onnes in 1911 when he cooled the mercury sample below 4.2 K using liquid helium-4 as a refrigerant. He discovered that sample conducted electricity without dissipation which is called superconductivity. Quantum theory was not developed until the 1920's. Therefore, physicists could not understand the mechanism behind superfluidity, and it took almost 50 years to find the microscopic theory of the problem.

Both bosonic and fermionic systems can exhibit superfluidity. Fermionic superfluidity can be found in neutron stars, nuclear matter, superconductors, helium-3, and ultracold atoms although their superfluid transition temperatures are distinct. After physicists discovered the bosonic superfluidty in laboratory in 1938 with helium-4, Fritz London suggested the connection between superfluidity and Bose-Einstein Condensation (BEC) which was first predicted by Albert Einstein [1]. Einstein realized that at sufficiently low temperatures, the macroscopic part of the bosons occupy the ground state called BEC [2,3]. After bosons condense to the lowest energy state, they can be described by one wave function. If there is no interaction between bosons, then all the bosons in the system go to the ground state at zero temperature. However, interactions reduce the condensation fraction even at absolute zero such as Helium-4 having 10 percent condensation fraction due to strong interaction between atoms. BEC occurs when the number of accessible states equal to the number of particles which is also the condition for quantum degenerate limit, low temperature and high density limit. Number of accessible states in volume Ω is approximately equal to $\frac{\Omega}{\lambda_T^3}$, where λ_T

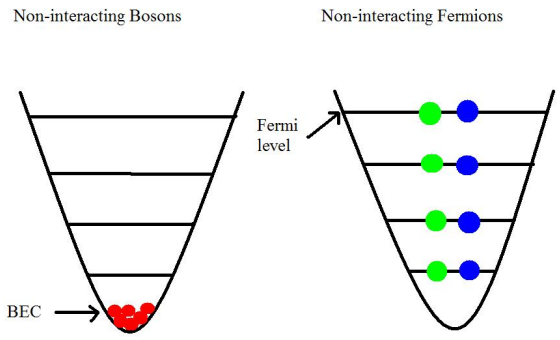

Figure 1: Quantum statistics for non-interacting bosons and fermions at zero temperature

is the thermal wavelength giving the extend of the particles wave function at temperature T. Therefore, critical temperature for BEC can be found from the condition $\frac{\Omega}{\lambda_T^3} = N$ which yields the $T_c = \frac{2\pi\hbar^2}{m}\left(\frac{N}{\Omega}\right)^{2/3}$. This relation gives the $3K$ for critical temperature of Helium-4 close to the actual value $2.2K$. Much microscopic understanding of bosonic superfluidty was developed by Nikolai Bogoliubov, and he studied the weakly interacting Bose gases. He concluded that BEC in ideal gas has a vanishing critical velocity, so it is not a superfluid. Therefore, BEC itself does not guarantee superfluidity. There must be interaction providing correlation which is the case in Helium-4.

After Fritz London suggested the connection between BEC and superfluidity, then a question came to mind. How do fermions condense to the lowest energy state at low temperatures in fermionic systems? It was known that the Pauli Principle blocks fermions to occupy the same quantum state. Moreover, degeneracy temperature for fermions is on the order of $\frac{E_F}{k_B}$ which is far above the observed T_c in experiment. Physicists thought the idea of tightly bound pairs of electrons, which can act as bosons and then condense to the ground state of the system. However, there is no known interaction overcoming the Coulomb interaction between electrons. In 1950, it was

found that there is an effective attractive interaction between electrons due to crystal vibration which causes superconductivity [4,5]. Then, question of fermion pairs was solved in 1956 by Leon Cooper, who showed that arbitrarily attractive interaction between fermions on top of the Fermi surface can lead to the formation of a bound state [6]. Bound fermions are called Cooper pairs. Then, this pair can undergo BEC like bosons. However, only one Cooper pair could not explain the whole mechanism of fermionic superfluidity. One year later, microscopic theory of superconductivity was discovered by John Bardeen, Cooper, and Robert Schrieffer. This famous theory is called the BCS theory which proposed a many particle wave function corresponding to the largely overlapping fermion pairs with zero center-of-mass momentum, zero angular momentum (s-wave), and zero total spin (singlet) [7]. In BCS theory, Cooper pairs are allowed for fermions with energies close to the Fermi energy. Besides BCS theory captured the critical temperatures of the superconductors found in laboratories before the 1980s, it also describes the superfluid phases of fermions in nuclear matter and neutron stars, where nuclear forces provide the glue for fermion pairs . However, another interesting story started in 1986 when high-Tc superconductors which have superfluid transition temperature above $100K$ were created [8]. Then, one of the significant goals is to find a room temperature superconductor for researchers. This high critical temperature could not be explained by BCS theory. In BCS theory, interaction between fermions is arbitrarily weak, so pairs are loosely bound and pairs are much larger than average separation. However, it seems that high-Tc superconductors have small Cooper pairs. Therefore, the BCS theory had to be generalized to pass to the strong interaction regime where the fermion pairs become tightly bound Bose molecules and can undergo BEC. Then, it seems to be that BCS and BEC theories are

the two end points of a more general theory. Anthony Leggett was the first physicist that studied the BCS-to-BEC evolution, and he showed that when the attraction was weak, a BCS superfluid appears, and when the attraction was strong, a BEC superfluid appears [9]. In contrast to the BCS theory, Cooper pairs are allowed for all fermions in the BCS-BEC evolution. We can pass from the BCS limit of loosely bound and largely overlapping Cooper pairs to the BEC limit of tightly bound bosonic molecules via increasing the attractive interaction between fermions, now achievable in the cold atom experiments.

Atomic BEC was first accomplished in the laboratory by Eric Cornell and Carl Wieman in Colorado in 1995. They accomplished it by cooling the Rb atoms to much lower temperatures than had been previously achieved. They used laser cooling and evaporation methods to cool the atoms to nanoKelvins. After this success, research on the ultracold atomic gases has been rapidly increasing. It is wise idea to deal with simple system rather than dense, strongly interacting systems like Helium-4. Ultracold atomic gases are dilute which means we can neglect the 3-body interaction. These ultracold systems have very low critical temperatures because they have low densities. However, if we scale their densities to the density of the electrons in metal, their critical temperatures are above the room temperature, so we can think the superfluid properties of ultracold atoms as high-temperature superconductors. Ultracold quantum gases are an ideal platform to investigate the new phases of matter since they allow us to control physical quantities experimentally. Atomic systems have the advantage of tunable interaction strength between fermions via Feshbach resonance allowing an examination of the whole BCS-BEC crossover experimentally [29]. The control over the interaction may show the physics of the strongly corre-

lated systems in condensed matter physics where the control is nonexistent such as neutron stars. We also control the density and temperature of these systems. Therefore, ultracold atoms have attracted much theoretical and experimental research for 15 years. Physicists have tried to find the exotic phases of matter for these systems. In 2003, three groups discovered the BEC of the tightly bound fermions, and eventually fermionic superfluidity was first realized in the experiment through the observation of vortices in 2005 at MIT using ultracold fermionic gases [10,11].

Almost all of the early works were focused on the balanced Fermi gases, where both components have equal number and mass. There is no phase transition going from BCS to BEC via increasing the interaction in balanced case, so this process is called BCS-BEC crossover [9,12]. The system is a superfluid in all crossover regime. This result was also realized in the experiments. Then, many of the theoretical works were concentrated on the population and mass imbalanced systems [13]. RF pulses, converting up fermions to down and vice versa, are used to create population imbalanced systems. It was found that BCS-BEC evolution is not a crossover, and there are new phases of matter. These phases are normal phase, phase separation and superfluid phases, also observed in the experiments [15,16]. There are also other research directions in the field. Fermionic systems in the precense of optical lattices, fermionic mixtures with three hyperfine state, and fermion mixtures in trap are some of the most attracted areas. It was found that new phases of matter can be seen when fermionic mixtures are put into the trap with different frequencies for each type of fermions even for balanced case [16]. Allowing mixtures of fermions with three different hyperfine states opens interesting results. There can be three types of Cooper pairs which are analogous to the case in quarks in the core of neutron stars.

Physicists have studied complicated systems with the hope of finding new phases of matter, and spin-orbit coupled Fermi gases is one of the hot research areas into the field. Spin-orbit coupling (SOC) interaction has attracted much research interest after it was achieved first with neutral atomic BEC by coupled two atomic spin states with a pair of lasers [17,18]. Recently, SOC effect in fermionic systems can be created in the experiment using the counterpropagating laser beams that flips the atomic spin states, and gives momentum to the atom [30,31]. The coupling between spin of the atoms and its center of mass motion leads to the SOC effect. Magnetic field generated by electrons motion couples to the electrons spin through the magnetic dipole interaction. Strength and direction of magnetic field depends on the electrons velocity producing correlation between electrons motion and spin states. This effect can usually be neglected for electron velocities which are not relativistic. SOC has been many consequences in to the variety of modern condensed matter problems. It is crucial for the spin Hall effect and topological insulators, and it contributes to the electronic properties of the materials. Some of the consequences of the system in ultracold atoms are followings. SOC increases the density of the states which favors to the Cooper pairing [19-28]. Therefore, SOC can induce the BCS-BEC evolution even if we keep the interparticle interaction fixed. Similar to the usual BCS-BEC crossover picture, there is no phase transition in BCS-BEC evolution for balanced system with increasing SOC effect [19-28]. Therefore, system is gapped superfluid for balanced case with or without SOC. For a population or mass imbalanced case, there are many exotic phases with SOC effect. Besides, gapped superfluid, normal phase and phase separation which are seen also without SOC, two types of gapless superfluid phases that their excitation energies vanish at same points in momentum

space has been found theoretically [19-28].

In this thesis, we analyze the Cooper problem first, and show the binding energy $E_B < 0$ of the one pair formation. Then, we introduce the second quantization formalism which we frequently use to deal with the many-body physics, and we find the many-body Hamiltonian in momentum space. In the first part of chapter 4, we introduce the famous BCS theory describing the properties of the some superconductors, and we solve the system with using variational method, the original method which Bardeen, Cooper, and Schrieffer were used in their original work. In the next part, we utilize mean-field approximation to obtain the results at finite temperatures, and we find number and gap equations describing the physical quantities of the system. BCS-BEC evolution, generalization of the BCS problem, is also examined. Then, we include SOC interaction to the system, and solve it using mean-field approximation in the whole BCS-BEC crossover regime in vacuum. Moreover, we construct the ground state phase diagrams of the system, excitation spectrum, and momentum distribution. In the last part, we studied the same system in an optical lattice.

2 Superfluidity in Fermionic Gases

In this chapter, we will analyze the fermionic systems leading mechanism behind the superfluidity. Before solving complicated systems, we start with the basic knowledge to understand the problem clearly.

2.1 Cooper Pairing

Fermions can not occupy the same quantum state due to Pauli blocking. After Fritz London suggested the superfluidity has a relation with BEC, main problem is how do fermionic systems condense to the ground state to create superfluidity. It was found by Leon Coopers that two fermions can form a pair on top of the non-interacting Fermi surface if there is arbitrarily attractive interaction between fermions [6]. In quantum mechanics, It is known that two fermions can always form a bound state in one and two dimension if there exists an attractive interaction between them. However, there is a minimum threshold interaction in three dimension that one should overcome to form a bound state for two fermions. Therefore, It seems that fermion pair can not be formed in three dimension for weakly attractive interaction. However, many-body effect comes into the problem because fermions' momenta are essentially confined to the narrow shell on top of the Fermi sea, so problem is effectively 2D. Therefore, fermions can form a bound state called Cooper pairs. Now, we show that system has a lower energy if Cooper pair is formed. We assume that there are same number of spin up and spin down fermions. Therefore, their Fermi energies $\epsilon_f = \frac{\hbar^2 k_f^2}{2m}$ are the same where $k_f = (6\pi^2 \frac{N}{V})^{\frac{1}{3}}$. Hamiltonian of the two fermions on top of the non-interacting Fermi sea is

$$\hat{H} = \frac{\hat{P_1}^2}{2m} + \frac{\hat{P_2}^2}{2m} + V(|\vec{r_1} - \vec{r_2}|) \tag{1}$$

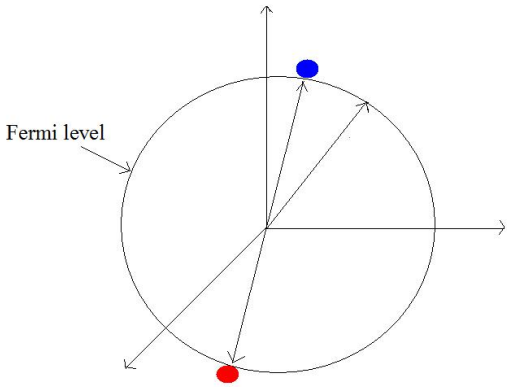

Figure 2: Two fermions having different intrinsic property labeled as blue and red on top of the non-interacting Fermi level.

After we use the center of mass and relative coordinates where $\vec{R} = \frac{\vec{r_1}+\vec{r_2}}{2}$ and $\vec{r} = \vec{r_1} - \vec{r_2}$, then Shrödinger equation $H\Psi = E\Psi$ becomes

$$[\frac{-\hbar^2}{2M} \bigtriangledown^2_{\mathbf{R}} - \frac{\hbar^2}{2\mu} \bigtriangledown^2_{\mathbf{r}} + V(|\mathbf{r}|)]\Psi(\mathbf{R},\mathbf{r}) = E\Psi(\mathbf{R},\mathbf{r}) \tag{2}$$

Where μ is the relative mass which is equal to $m/2$, and M is the center of mass that is equal to $2m$. We now use the separation of variables, and write the wave function as $\Psi(\mathbf{R},\mathbf{r}) = \phi(\mathbf{R})\psi(\mathbf{r})$. Then, we can separate the center of motion and the relative motion. For the center of motion part, equation (2) becomes $\bigtriangledown^2_{\mathbf{R}}\phi(\mathbf{R}) = \frac{-2ME_R}{\hbar^2}\phi(\mathbf{R})$. Then, wave function $\phi(\mathbf{R})$ of the center of the motion is a plane wave which is $e^{i\vec{K}\vec{R}}$ where energy of the center of the mass E_R is $\frac{\hbar^2K^2}{2M}$. Shrödinger equation for relative motion will be

$$[\frac{-\hbar^2}{2\mu} \bigtriangledown^2_{\mathbf{r}} + V(\mathbf{r})]\psi(\mathbf{r}) = E_r\psi(\mathbf{r}) \tag{3}$$

Where total energy E in equation (2) is equal to $E_R + E_r$. We want to write equation (3) in momentum space, so we take the Fourier transform of

the wave function as $\psi(r) = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{k}} a(\mathbf{k}) e^{i\mathbf{k}.\mathbf{r}}$. Then, if we multiply both sides of the equation (3) with $e^{-i\mathbf{k}^{'}.\mathbf{r}}$, and take the integral with respect to r. Equation (3) becomes

$$a(\mathbf{k})[\frac{\hbar^2 k^2}{2\mu} + E_{\mathbf{R}} - E] + \sum_{\mathbf{k}} a(\mathbf{k}) \frac{V(\mathbf{k}, \mathbf{k}^{'})}{\Omega} = 0 \tag{4}$$

Where integral $\int e^{i(\mathbf{k}-\mathbf{k}^{'}).\mathbf{r}} d^3r$ is equal to $\Omega \delta_{\mathbf{k},\mathbf{k}'}$, and $V(\mathbf{k}, \mathbf{k}^{'})$ that is equal to $\int V(\mathbf{r}) e^{i(\mathbf{k}-\mathbf{k}^{'}).\mathbf{r}} d^3r$, the Fourier transform of $V(\mathbf{r})$. We know define $\frac{\hbar^2 k^2}{2\mu} + E_R$ as 2ϵ. Attractive interaction between fermions assummed to be local, so we take the $V(\mathbf{r})$ as a Dirac delta function $\delta^3(\mathbf{r})$. Then, $V(\mathbf{k}, \mathbf{k}^{'})$ is equal to constant value $V_0 < 0$. Therefore, equation (4) becomes

$$a(\mathbf{k})[2\epsilon - E] + \frac{V_0}{\Omega} \sum_{\mathbf{k}} a(\mathbf{k}) = 0 \tag{5}$$

Now, let $\sum_{\mathbf{k}} a(\mathbf{k})$ is equal to constant c. Then, if we sum both sides over $\mathbf{k}$, equation (5) gives

$$-\frac{1}{V_0} = \frac{1}{\Omega} \sum_{\mathbf{k}} \frac{1}{2\epsilon - E} \tag{6}$$

We can replace the summation $\sum_{\mathbf{k}}$ with $\int D(\epsilon) d\epsilon$ where $D(\epsilon)$ is the density of states, number of states between energies ϵ and $\epsilon + d\epsilon$. Now, equation (6) can be written as

$$-\frac{1}{V_0} = \frac{1}{\Omega} \int \frac{D(\epsilon) d\epsilon}{2\epsilon - E} \tag{7}$$

Energy levels below the Fermi energy E_F are occupied by fermions, so they should not be included into the integration. We should take the integration in the interval $E_F < \epsilon < E_F + E_{cutoff}$. For superconductors, the natural cut-off is given by Debye frequency ω_D, corresponding the highest frequency at which ions can respond to an electron in crystal lattice. Since we have

$E_{cutoff} = \hbar\omega_D \ll E_F$, the integration is taken very narrow shell around the Fermi energy. Therefore, we can assume that density of states $D(\epsilon)$ is equal to density of states at Fermi energy $D(E_F)$ which is constant in the integration. E is the energy of the two fermions, so we can write it as $2E_f + E_B$ where $E_B < 0$ is binding energy of two fermions. Then, equation (7) becomes

$$-\frac{1}{V_0} = \frac{D(E_f)}{\Omega} \int_{E_f<\epsilon<E_f+\hbar\omega_D} \frac{d\epsilon}{2(\epsilon - E_f) - E_B} \tag{8}$$

Integral can be easly evaluated if we change the integral variable as $u = 2(\epsilon - E_f) - E_B$. Then, the binding energy will be

$$E_B = -2\hbar\omega_D e^{\frac{-2\Omega}{D(E_f)|V_0|}} < 0 \tag{9}$$

Result is following: binding energy depends exponentially on the interaction and $D(E_F)$, and there is a paired state for two weakly interacting fermions on top of the Fermi sea. The role of the constant density of states in 2D is here replaced by density of states at Fermi energy $D(E_F)$ due to Paulie blocking. Cooper pair at rest(opposite momentum with equal magnitude) has the largest binding energy. Thus, Fermi level is now unstable towards pairing because it decreases the total energy of the system. In this problem, we neglect the interaction between other fermions on the Fermi level. If we turn on all interaction, then system reorder itself into a new paired state which is described by the BCS theory.

2.2 Second Quantization

In quantum mechanics, wave function $\psi(\mathbf{r})$ is the most important physical quantity. $|\psi(\mathbf{r})|^2$ gives the probability of the particle at position $\mathbf{r}$, and it provides information about probability amplitude of position, momentum and other physical observables. If we put the wave function into the Shrödinger equation $H\psi = E\psi$, then we can find the energy of the system. Physical observables like position, momentum are operators acting to the state vectors in vector space. For non-interacting single particle system, Hamiltonian can be written as $\hat{H} = \frac{\hat{P}^2}{2m} + V_{ext}$, and wave function $\psi(\mathbf{r})$ is a function of 3 position coordinates. Shrödinger equation is not easy to solve exactly in general. There are few numbers of systems having exact solutions such as particle in a box, and harmonic oscillator. If we increase the number of particles in the system, and turn on the interaction between particles, then solving Shrödinger equation become even more difficult task. Therefore, using first quantized quantum mechanics for macroscopic systems having around Avogadro's number of particles is cumbersome.

Second quantization provides the general way to deal with macroscopic systems. Wave function $\psi(\mathbf{r})$ become a field operator $\hat{\psi}(\mathbf{r})$ that can create and annihilate the particles at position $\mathbf{r}$. Field operators obeys the fermionic and bosonic canonical commutation and anticommutation relations as

$$\begin{aligned} [\hat{\psi}(\mathbf{r}_1), \hat{\psi}^\dagger(\mathbf{r}_2)] &= \delta(\mathbf{r}_1 - \mathbf{r}_2) \qquad \text{for bosons} \\ \{\hat{\psi}(\mathbf{r}_1), \hat{\psi}^\dagger(\mathbf{r}_2)\} &= \delta(\mathbf{r}_1 - \mathbf{r}_2) \qquad \text{for fermions} \end{aligned} \tag{10}$$

where $[A, B] = AB - BA$, and $\{A, B\} = AB + BA$ commutation and anticommutation relations respectively. Complete many-body Hamiltonian in

second-quantiantized form can be written as

$$\hat{H} = \int d^3x \hat{\psi}^\dagger(\mathbf{x})[\frac{-\hbar^2}{2m} \bigtriangledown^2 - U(\mathbf{x})]\hat{\psi}(\mathbf{x}) + \frac{1}{2}\int d^3x_1 d^3x_2 : \hat{\rho}(\mathbf{x}_1)\hat{\rho}(\mathbf{x}_2) : V(\mathbf{x}_1 - \mathbf{x}_2) \tag{11}$$

where wavefunction $\psi(\mathbf{x})$ become an operator, and kinetic potential, and interaction terms are functions, not operators. Here, $\hat{\rho}(\mathbf{x})$ is a density operator which is equal to $\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})$, and ":.....:" denotes the normal ordering in the interaction term which denotes that all creation operators between the two colons must be ordered to lie to the left of all destruction operators.

2.2.1 Field Operators in Different Basis

We can write the field operators in different basis with utilizing the identity $\sum_n |n><n|$. For instance, fields operator in position basis is $\hat{\psi}(x) =< x|\psi >$. Then, we can write it as

$$\hat{\psi}(\mathbf{x}) = \sum_n < \mathbf{x}|n >< n|\psi > \tag{12}$$

where $< n|\psi >$ is the field operator in the new basis. Now, we can write equation (11) in momentum space with using field operators in momentum space basis with using relations below

$$\hat{\psi}(\mathbf{x}) = \frac{1}{\Omega}\sum_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}}\hat{C}_{\mathbf{k}} \qquad \text{and} \tag{13}$$
$$\hat{\psi}^\dagger(\mathbf{x}) = \frac{1}{\Omega}\sum_{\mathbf{k}} e^{-i\mathbf{k}.\mathbf{x}}\hat{C}^\dagger_{\mathbf{k}}$$

Then, kinetic energy term in equation (11) will be $\sum_{\mathbf{k},\mathbf{k}'} \frac{\hbar^2 k^2}{2m\Omega}\hat{C}^\dagger_{\mathbf{k}'}\hat{C}_{\mathbf{k}}\int d^3x e^{i(\mathbf{k}-\mathbf{k}').\mathbf{x}}$. The integral gives the Kronecker delta functiion $\delta_{\mathbf{k},\mathbf{k}'}$ which cancels one of

the summation term, and result is

$$\hat{H}_{kin} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \hat{C}^{\dagger}_{\mathbf{k}} \hat{C}_{\mathbf{k}} \tag{14}$$

where the term $\epsilon_{\mathbf{k}}$ is $\frac{\hbar^2 k^2}{2m}$, and $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$. The product $\hat{C}^{\dagger}_{\mathbf{k}} \hat{C}_{\mathbf{k}}$ is number operator $\hat{N}_{\mathbf{k}}$ that gives the number of particles at the state with momentum $\mathbf{k}$ if it acts to the state vector $|n_{\mathbf{k}}>$ as $\hat{N}_{\mathbf{k}}|n_{\mathbf{k}}>= n_{\mathbf{k}}|n_{\mathbf{k}}>$ where $n_{\mathbf{k}}$ is number of particles. Bosonic field operators in momentum basis act to the state in the same way as operators in harmonic oscillator, lowering and rising to the number of the particles in the state: $\hat{C}_{\mathbf{k}}|n_{\mathbf{k}}>= \sqrt{n_{\mathbf{k}}}|n_{\mathbf{k}}-1>$, $\hat{C}^{\dagger}_{\mathbf{k}}|n_{\mathbf{k}}>= \sqrt{n_{\mathbf{k}}+1}|n_{\mathbf{k}}+1>$, and $\hat{C}_{\mathbf{k}}|0>= 0$, where $|0>$ is vacuum state. For fermions, $n_{\mathbf{k}}$ is 0 or 1 due to Pauli blocking, so product $\hat{C}^{\dagger}_{\mathbf{k}} \hat{C}^{\dagger}_{\mathbf{k}}$ is zero for fermionic operators that is also consequence of the anticommutation relation for fermionic operators $\{\hat{C}^{\dagger}_{\mathbf{k}}, \hat{C}^{\dagger}_{\mathbf{k}}\}$=0

Now, let's write the interaction term which is

$$\hat{H}_{int} = \int d^3x_1 d^3x_2 \hat{\psi}^{\dagger}(\mathbf{x}_1)\hat{\psi}^{\dagger}(\mathbf{x}_2)V(\mathbf{x}_1 - \mathbf{x}_2)\hat{\psi}(\mathbf{x}_2)\hat{\psi}(\mathbf{x}_1) \tag{15}$$

Then, we write the Fourier transform of the operators in the same way as kinetic energy term. We use center of mass and relative coordinates as $\vec{R} = \frac{\vec{x_1}+\vec{x_2}}{2}$ and $\vec{r} = \vec{x_1} - \vec{x_2}$. When the particle interact at positions $\vec{x_1}$ and $\vec{x_2}$, momentum is conserved. Particle 1 comes with momentum $\mathbf{k}_1$ and gain momentum $\mathbf{q}$ from particle 2. Therefore;

$$\begin{aligned} &\text{particle 1} && \mathbf{k}_1 \rightarrow \mathbf{k}_1 + \mathbf{q} \\ &\text{particle 2} && \mathbf{k}_2 \rightarrow \mathbf{k}_2 - \mathbf{q} \end{aligned} \tag{16}$$

Using the relations $\Omega\delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}'_1+\mathbf{k}'_2} = \int d^3R e^{(\mathbf{k}'_1+\mathbf{k}'_2-\mathbf{k}_1-\mathbf{k}_2).\mathbf{R}}$ and Fourier transform of the interaction term which is $V(\mathbf{q}) = \int d^3r V(\mathbf{r}) e^{-i\mathbf{q}.\mathbf{r}}$, interaction term in momentum space will be

$$\hat{H}_{int} = \frac{1}{\Omega} \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} V(\mathbf{q}) \hat{C}^{\dagger}_{\mathbf{k}+\mathbf{q}} \hat{C}^{\dagger}_{\mathbf{k}'-\mathbf{q}} \hat{C}_{\mathbf{k}'} \hat{C}_{\mathbf{k}} \tag{17}$$

Therefore, total second-quantized many-body hamiltonian in momentum space is given by

$$\hat{H} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \hat{C}^{\dagger}_{\mathbf{k}} \hat{C}_{\mathbf{k}} + \frac{1}{\Omega} \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} V(\mathbf{q}) \hat{C}^{\dagger}_{\mathbf{k}+\mathbf{q}} \hat{C}^{\dagger}_{\mathbf{k}'-\mathbf{q}} \hat{C}_{\mathbf{k}'} \hat{C}_{\mathbf{k}} \tag{18}$$

This result will be widely used throughtout the thesis.

2.3 BCS Theory

After Leon Cooper showed that two fermions on top of the non-interacting Fermi sea can form a bound state if there is attractive interaction between them in 1956, one year later Bardeen, Cooper and Schrieffer found the many-body theory of the superconductivity. They used a variational wavefunction to explain the superconducting state of the metals at zero temperatures. Superconductivity is a ground state phenomenon whereas normal conductivity in metals occurs at excites states. Consider the following second-quantized Hamiltonian for two-component Fermi gas.

$$\hat{H} = \sum_{\mathbf{k},\alpha} (\epsilon_{\mathbf{k}} - \mu) \hat{C}^{\dagger}_{\mathbf{k},\alpha} \hat{C}_{\mathbf{k},\alpha} + \frac{V_0}{\Omega} \sum_{\mathbf{k},\mathbf{k}'} \hat{C}^{\dagger}_{\mathbf{k}'\uparrow} \hat{C}^{\dagger}_{-\mathbf{k}'\downarrow} \hat{C}_{-\mathbf{k}\downarrow} \hat{C}_{\mathbf{k}\uparrow} \tag{19}$$

Where the two components are labelled by spin index $\alpha = \uparrow\downarrow$, and $V_0 < 0$ is attractive interaction between fermions. We determine the ground state energy using variational wave function

$$|\psi_{BCS}> = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} \hat{C}^{\dagger}_{\mathbf{k}\uparrow} \hat{C}^{\dagger}_{-\mathbf{k}\downarrow}) |0> \tag{20}$$

where $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are variational parameters taken to be real for simplicity. This wave function describes the Bose-Einstein condensation of Cooper pairs. The operator $\hat{C}^{\dagger}_{\mathbf{k}\uparrow} \hat{C}^{\dagger}_{-\mathbf{k}\downarrow}$ creates the Cooper pair with opposite spin and momentum. Normalization condition $< \psi_{BCS} | \psi_{BCS} >= 1$ of the wave function yields

$$\prod_{\mathbf{k},\mathbf{k}'} < 0 | (u'_{\mathbf{k}} + v'_{\mathbf{k}} \hat{C}_{-\mathbf{k}'\downarrow} \hat{C}_{\mathbf{k}'\uparrow}) ((u_{\mathbf{k}} + v_{\mathbf{k}} \hat{C}^{\dagger}_{\mathbf{k}\uparrow} \hat{C}^{\dagger}_{-\mathbf{k}\downarrow}) |0> = 1 \tag{21}$$

Product of two creation and annihilation operators have zero expectation values, so $< \hat{C}^{\dagger}_{\mathbf{k}\uparrow}\hat{C}^{\dagger}_{-\mathbf{k}\downarrow} >$, and $< \hat{C}_{-\mathbf{k}'\downarrow}\hat{C}_{\mathbf{k}'\uparrow} >$ are equal to zero. Therefore,

$$\prod_{\mathbf{k},\mathbf{k}'} < 0|u_{\mathbf{k}}u'_{\mathbf{k}} + v_{\mathbf{k}}v'_{\mathbf{k}}\hat{C}_{-\mathbf{k}'\downarrow}\hat{C}_{\mathbf{k}'\uparrow}\hat{C}^{\dagger}_{\mathbf{k}\uparrow}\hat{C}^{\dagger}_{-\mathbf{k}\downarrow}|0 >= 1 \tag{22}$$

If we use the anticommutation relation of fermionic operator $\{\hat{C}_{\mathbf{k}'\uparrow}, \hat{C}^{\dagger}_{\mathbf{k}\uparrow}\} = \delta_{\mathbf{k},\mathbf{k}'}$, then normalization condition becomes

$$\prod_{k}(u^2_{\mathbf{k}} + v^2_{\mathbf{k}}) = 1 \tag{23}$$

Therefore, $u^2_{\mathbf{k}} + v^2_{\mathbf{k}} = 1$ which quarantee the probability condition. Whereas $u^2_{\mathbf{k}}$ gives the probability of the unoccupied paired state, $v^2_{\mathbf{k}}$ gives the probability of the occupied paired state. We will find the gap equation from minimizing the ground state energy $< \psi_{BCS}|\hat{H}|\psi_{BCS} >$. Therefore, it is beneficial to find the expectation values of the operators $< \psi_{BCS}|\hat{C}^{\dagger}_{\mathbf{k},\alpha}\hat{C}_{\mathbf{k},\alpha}|\psi_{BCS} >$ and $< \psi_{BCS}|\hat{C}_{\mathbf{k},\downarrow}\hat{C}_{-\mathbf{k},\uparrow}|\psi_{BCS} >$. First expectation value can be found as

$$\prod_{\mathbf{k},\mathbf{k}',\mathbf{k}''} < 0|(u'_{\mathbf{k}} + v'_{\mathbf{k}}\hat{C}_{-\mathbf{k}'\downarrow}\hat{C}_{\mathbf{k}'\uparrow})(\hat{C}^{\dagger}_{\mathbf{k},\alpha}\hat{C}_{\mathbf{k},\alpha})((u''_{\mathbf{k}} + v''_{\mathbf{k}}\hat{C}^{\dagger}_{\mathbf{k}''\uparrow}\hat{C}^{\dagger}_{-\mathbf{k}''\downarrow})|0 > \tag{24}$$

Expectation values of the operator products $< \hat{C}^{\dagger}_{\mathbf{k},\alpha}\hat{C}_{\mathbf{k},\alpha} >$, $< \hat{C}^{\dagger}_{\mathbf{k},\alpha}\hat{C}_{\mathbf{k},\alpha}\hat{C}^{\dagger}_{\mathbf{k}''\uparrow}\hat{C}^{\dagger}_{-\mathbf{k}''\downarrow} >$, and $< \hat{C}_{-\mathbf{k}'\downarrow}\hat{C}_{\mathbf{k}'\uparrow}\hat{C}^{\dagger}_{\mathbf{k},\alpha}\hat{C}_{\mathbf{k},\alpha} >$ gives zero. Only one term makes a contribution which is

$$\prod_{\mathbf{k},\mathbf{k}',\mathbf{k}''} < 0|v'_{\mathbf{k}}v''_{\mathbf{k}}\hat{C}_{-\mathbf{k}'\downarrow}\hat{C}_{\mathbf{k}'\uparrow}\hat{C}^{\dagger}_{\mathbf{k},\alpha}\hat{C}_{\mathbf{k},\alpha}\hat{C}^{\dagger}_{\mathbf{k}''\uparrow}\hat{C}^{\dagger}_{-\mathbf{k}''\downarrow}|0 > \tag{25}$$

After we use the result in (23) and the anticommutation relation of the operators, expactation value of the product $\hat{C}^{\dagger}_{\mathbf{k},\alpha}\hat{C}_{\mathbf{k},\alpha}$ will be

$$< \psi_{BCS}|\hat{C}^{\dagger}_{\mathbf{k},\alpha}\hat{C}_{\mathbf{k},\alpha}|\psi_{BCS} >= v^2_k \tag{26}$$

Using the same argument above, we can find the expectation value of the operator $< \psi_{BCS}|\hat{C}_{\mathbf{k},\downarrow}\hat{C}_{-\mathbf{k},\uparrow}|\psi_{BCS} >$ that is

$$< \psi_{BCS}|\hat{C}_{\mathbf{k},\downarrow}\hat{C}_{-\mathbf{k},\uparrow}|\psi_{BCS} >= u_{\mathbf{k}}v_{\mathbf{k}} \tag{27}$$

Now, we have to find the ground state energy $< \psi_{BCS}|\hat{H}|\psi_{BCS} >$ to minimize it. If we take the expectation value of the Hamiltonian, it will be

$$\sum_{\mathbf{k},\alpha}(\epsilon_{\mathbf{k}} - \mu) < \hat{C}^{\dagger}_{\mathbf{k},\alpha}\hat{C}_{\mathbf{k},\alpha} > + \frac{V_0}{\Omega}\sum_{\mathbf{k},\mathbf{k}'} < \hat{C}^{\dagger}_{-\mathbf{k}\uparrow}\hat{C}^{\dagger}_{\mathbf{k}\downarrow}\hat{C}_{\mathbf{k}'\downarrow}\hat{C}_{-\mathbf{k}'\uparrow} > \tag{28}$$

Kinetic energy term is easy to deal with because we found the $< \hat{C}^{\dagger}_{\mathbf{k},\alpha}\hat{C}_{\mathbf{k},\alpha} >$ above which is equal to $v^2_{\mathbf{k}}$. We use the Wick's theorem to write the four product operator in interaction part in terms of two product operators as

$$< \hat{C}^{\dagger}_{-\mathbf{k}\uparrow}\hat{C}_{-\mathbf{k}'\uparrow}\hat{C}^{\dagger}_{\mathbf{k}\downarrow}\hat{C}_{\mathbf{k}'\downarrow} >=< \hat{C}^{\dagger}_{-\mathbf{k}\uparrow}\hat{C}_{\mathbf{k}'\downarrow} >< \hat{C}_{-\mathbf{k}'\uparrow}\hat{C}^{\dagger}_{\mathbf{k}\downarrow} > - < \hat{C}^{\dagger}_{-\mathbf{k}\uparrow}\hat{C}^{\dagger}_{\mathbf{k}\downarrow} >< \hat{C}_{-\mathbf{k}'\uparrow}\hat{C}_{\mathbf{k}'\downarrow} > \tag{29}$$

First term is zero, so expectation of four product operators can be written as

$$< \hat{C}^{\dagger}_{-\mathbf{k}\uparrow}\hat{C}_{-\mathbf{k}'\uparrow}\hat{C}^{\dagger}_{\mathbf{k}\downarrow}\hat{C}_{\mathbf{k}'\downarrow} >=< \hat{C}^{\dagger}_{-\mathbf{k}\uparrow}\hat{C}^{\dagger}_{\mathbf{k}\downarrow} >< \hat{C}_{\mathbf{k}'\downarrow}\hat{C}_{-\mathbf{k}'\uparrow} > \tag{30}$$

We found before that $< \hat{C}_{\mathbf{k}'\downarrow}\hat{C}_{-\mathbf{k}'\uparrow} >$ is $u_{\mathbf{k}'}v_{\mathbf{k}'}$, and $< \hat{C}^{\dagger}_{-\mathbf{k}\uparrow}\hat{C}^{\dagger}_{\mathbf{k}\downarrow} >$ is the complex conjeguate of the $u_{\mathbf{k}}v_{\mathbf{k}}$ that is also equal to itself because we take the $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ real for simplicity. Therefore, ground state energy of the system becomes

$$E =< \hat{H} >= 2\sum_{\mathbf{k}}(\epsilon_{\mathbf{k}} - \mu)v^2_{\mathbf{k}} + \frac{V_0}{\Omega}\sum_{\mathbf{k},\mathbf{k}'} u_{\mathbf{k}}v_{\mathbf{k}}u_{\mathbf{k}'}v_{\mathbf{k}'} \tag{31}$$

We may write $u_{\mathbf{k}} = \sin\theta_{\mathbf{k}}$ and $v_{\mathbf{k}} = \cos\theta_{\mathbf{k}}$ and then minimize the ground state energy with respect to $\theta_{\mathbf{k}}$. With these changes, ground state energy can be written as

$$E = \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - \mu)(1 + \cos 2\theta_{\mathbf{k}}) + \frac{V_0}{4\Omega} \sum_{\mathbf{k},\mathbf{k}'} \sin 2\theta_{\mathbf{k}} \sin 2\theta_{\mathbf{k}'} \tag{32}$$

Now, we take the variational derivative of E with respect to $\theta_{\mathbf{k}}$ and set to zero for minimization condition. When we take the variational derivative with respect to $\theta_{\mathbf{k}}$, Kroneker delta function emerges, and cancels one of the summation. Then, result will be

$$(\epsilon_{\mathbf{k}} - \mu) \sin 2\theta_{\mathbf{k}} - \frac{V_0 \cos 2\theta_{\mathbf{k}}}{2\Omega} \sum_{\mathbf{k}'} \sin 2\theta_{\mathbf{k}'} = 0 \tag{33}$$

We define the gap $\triangle$ as $-\frac{V_0}{2\Omega} \sum_{\mathbf{k}} \sin 2\theta_{\mathbf{k}'}$. $\triangle$ is interpreted as energy needed to break up a Cooper pair, which means that it costs a certain amount of energy to excite BCS ground state. With defining gap $\triangle$, minimizing condition above becomes

$$(\epsilon_{\mathbf{k}} - \mu) \sin 2\theta_{\mathbf{k}} + \triangle \cos 2\theta_{\mathbf{k}} = 0 \tag{34}$$

If we divide both sides by $\cos 2\theta_k$, then result will be

$$\tan 2\theta_{\mathbf{k}} = \frac{-\triangle}{\epsilon_{\mathbf{k}} - \mu} \tag{35}$$

Then, $\sin 2\theta_{\mathbf{k}}$ becomes $\frac{\triangle}{\sqrt{(\epsilon_{\mathbf{k}}-\mu)^2+\triangle^2}}$, and if we put it into the gap $\triangle$, gap equation will be

$$-\frac{1}{V_0} = \frac{1}{\Omega} \sum_{\mathbf{k}} \frac{1}{2\sqrt{(\epsilon_{\mathbf{k}} - \mu)^2 + \triangle^2}} \tag{36}$$

Number of particles into the system can be found from the formula

$$N = \sum_{\mathbf{k},\alpha} < \psi_{BCS}|\hat{C}^{\dagger}_{\mathbf{k},\alpha}\hat{C}_{\mathbf{k},\alpha}|\psi_{BCS} > \tag{37}$$

We know that the expactation value $< \psi_{BCS}|\hat{C}^{\dagger}_{\mathbf{k},\alpha}\hat{C}_{\mathbf{k},\alpha}|\psi_{BCS} >$ is $v^2_{\mathbf{k}}$. Therefore, number equation becomes

$$N = 2\sum_{\mathbf{k}} v^2_{\mathbf{k}} \tag{38}$$

We know that $u^2_{\mathbf{k}} + v^2_{\mathbf{k}} = 1$ and, we defined the $u_{\mathbf{k}} = \sin\theta_{\mathbf{k}}$ and $v_{\mathbf{k}} = \cos\theta_{\mathbf{k}}$, so $v^2_{\mathbf{k}} - u^2_{\mathbf{k}} = \cos 2\theta_{\mathbf{k}} = \frac{-(\epsilon_{\mathbf{k}}-\mu)}{\sqrt{(\epsilon_{\mathbf{k}}-\mu)^2+\triangle^2}}$. Therefore, we can find the $u^2_{\mathbf{k}}$ and $v^2_{\mathbf{k}}$ which are

$$\begin{aligned} v^2_{\mathbf{k}} &= \frac{1}{2}(1 - \frac{\epsilon_{\mathbf{k}} - \mu}{E_{\mathbf{k}}}) \\ u^2_{\mathbf{k}} &= \frac{1}{2}(1 + \frac{\epsilon_{\mathbf{k}} - \mu}{E_{\mathbf{k}}}) \end{aligned} \tag{39}$$

Where $E_{\mathbf{k}} = \sqrt{(\epsilon_{\mathbf{k}} - \mu)^2 + \triangle^2}$ is an excitation energy. Number, and gap equations will be

$$\text{Number equation} \qquad N = \sum_{\mathbf{k}}(1 - \frac{\epsilon_{\mathbf{k}} - \mu}{E_{\mathbf{k}}}) \tag{40}$$

$$\text{Gap equation} \qquad -\frac{1}{V_0} = \frac{1}{\Omega}\sum_{\mathbf{k}} \frac{1}{2E_{\mathbf{k}}} \tag{41}$$

These two equations are coupled and they can be solved self-consistently. Two unknowns chemical potential μ and gap $\triangle$ are found numerically by Matlab. However, we have to scale two equations to solve them numerically. We choose the energy scale as the Fermi energy ϵ_f of $\frac{N}{2}$ fermions such that $N = \frac{k_f^3\Omega}{3\pi^2}$. We can change the summation $\sum_{\mathbf{k}}$ with integral as $\sum_{\mathbf{k}} \rightarrow$

$\Omega \int \frac{d^3k}{2\pi^3}$. Moreover, we will replace V_0 by the scattering length a_s using the relation below

$$\frac{1}{V_0} = \frac{m}{4\pi\hbar^2 a_s} - \frac{m}{\hbar^2}\int \frac{d^3k}{2\pi^3}\frac{1}{\mathbf{k}^2} \tag{42}$$

which is the consequence of the scattering theory [19]. Therefore, our gap and number equations become

$$\text{Number equation} \qquad N = \Omega \int \frac{d^3k}{2\pi^3}\left(1 - \frac{\varepsilon_{\mathbf{k}}}{E_{\mathbf{k}}}\right) \tag{43}$$

$$\text{Gap equation} \qquad -\frac{m}{4\pi\hbar^2 a_s} = \int \frac{d^3k}{2\pi^3}\left(\frac{1}{2E_{\mathbf{k}}} - \frac{1}{\epsilon_{\mathbf{k}}}\right) \tag{44}$$

There is no θ and ϕ dependence inside the integrals, so we can write integral $\int d^3k$ as $\int 4\pi k^2 dk$. We scale the $\mathbf{k}$ with k_f as $\mathbf{k} = \widetilde{k}k_f$ where $\widetilde{k}$ is now dimensionless parameter. Then, our scaled equations will be

$$\text{Number equation} \qquad -\frac{\pi}{2k_f a_s} = \int d\widetilde{k}\left[\frac{\widetilde{k}^2 - \sqrt{(\widetilde{k}^2 - \widetilde{\mu})^2 + \widetilde{\triangle}^2}}{\sqrt{(\widetilde{k}^2 - \widetilde{\mu})^2 + \widetilde{\triangle}^2}}\right] \tag{45}$$

$$\text{Gap equation} \qquad \frac{2}{3} = \int d\widetilde{k}\widetilde{k}^2\left[\frac{\sqrt{(\widetilde{k}^2 - \widetilde{\mu})^2 + \widetilde{\triangle}^2} - \widetilde{k}^2 + \widetilde{\mu}}{\sqrt{(\widetilde{k}^2 - \widetilde{\mu})^2 + \widetilde{\triangle}^2}}\right]$$

where $\widetilde{\mu} = \frac{\mu}{\epsilon_f}$, and $\widetilde{\triangle} = \frac{\triangle}{\epsilon_f}$

2.3.1 BCS-BEC Crossover

BCS-BEC crossover is simple one channel model, where the scattering length a_s is the parameter tuning the interaction. For arbitrarily weak interaction, we expect the formation of Cooper pairs due to many-body effect, and they condensate to the ground state as described by BCS theory. For strong interaction, pairing is not a many-body effect, and it can be formed even in vacuum. Here, we expect the condensation of the tightly bound fermion

pairs to the ground state. Anthony Leggett realized that going from BCS limit to the BEC limit is smooth which means there is no phase transition, and system is gapped superfluid through the evolution regime for balanced case [9]. We can pass from BCS limit to the BEC limit just by increasing the interaction. Equations in the (45), we can think the $1/k_f a_s$ as the interaction. $k_f a_s \to 0_-$ is the BCS limit of weak interaction, and $k_f a_s \to 0_+$ is the BEC limit of strong interaction. In the BCS limit, Cooper pairs are formed and condensate at the same time. Pair distance is much larger than the average separation of the atoms. Therefore, Cooper pairs are largely overlapping. As we increase the $1/k_f a_s$ from $-\infty$ to the $+\infty$, we pass from the BCS regime to the BEC regime. In the BEC limit, Cooper pairs become Bose molecules that condensate to the ground state, and they repulse each others. Therefore, we call this regime as weakly interacting Bose gas, and Gross-Pitaevskii equations describing the ground state of interacting Bose gases can be used in the BEC regime of the evolution. In between these two extremes, there is a regime where pair size comparable to the interparticle spacing. We called this regime as crossover, and fermions are strongly interacting within the crossover. The properties of the system in this region may give insight to understand the strongly interacting fermionic systems such as neutron stars, and high-temperature superconductors. The resonance where $1/k_f a_s = 0$ is called unitarity where the interaction is just enough to bind particles in free space, so bound length of the molecule is infinite $a_s = \infty$. Crossover takes place around the unitarity. Therefore, we change the value of the $1/k_f a_s$ from -1 to +1 for examine the crossover region. We can find the gap and chemical potential of the system in the crossover regime with solving coupled equation (45). $\widetilde{T}$, and $1/k_f a_s$ are the parameters that we can control. We take $\widetilde{T} = 0$ to examine the ground state

of the system.

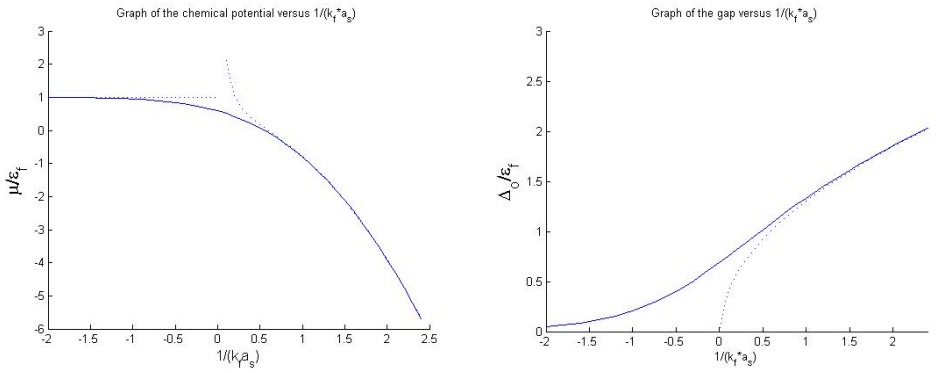

Figure 3: Chemical potential and gap in the BCS-BEC crossover as a function of interaction parameter $1/k_f a_s$

In BCS weak interaction limit , we have $k_f a_s \to 0_-$. If we insert it into the equations (45), then chemical potential and gap will be

$$\begin{aligned} \mu &\approx E_f \\ \triangle &\approx \frac{8}{e^2} E_f e^{-\pi/2k_f|a_s|} \end{aligned} \tag{46}$$

Chemical potential is energy needed to add one more particle to the system. BCS theory implicitly assumes that there are equal number of spin up and spin down particles into the system. Therefore, chemical potential is the energy cost for adding a spin up atom if at the same time spin down atom is added. First equation tells that in BCS limit, adding a spin up and spin down particle to the system costs Fermi energy per particle. In the BCS limit, Pauli blocking dominates over the interaction, so we can add particle only at the Fermi level. Second equation is the superfluid gap of the BCS theory, and we should compare it with the binding energy of the single Cooper pairs in (9). However, we should replace the V_0 by scattering length using the relation (42). Then, binding energy of the single Cooper pair will

be

$$E_B \approx \frac{8}{e^2} E_f e^{-\pi/k_f|a_s|} \tag{47}$$

Now, if we compare the superfluid gap and binding energy, we see that gap is exponentially larger. This difference comes from the fact that in the BCS problem not only the fermions above the Fermi level contribute the pairing, but also fermions below it make a contribution. Due to this difference, particles now takes part in the pairing, but gap is exponentially small compare to the Fermi energy, so Cooper pairs are fragile. It seems to be very difficult to reach the superfluidity in Fermi gases. However, Feshbach resonance allow us to tune the interaction and pass to the strong interaction regime where $1/k_f|a_s| < 1$. In this region, Cooper pairs become tightly bounded, so there is a good chance to achieve superfluidity. If we try to understand the sense of the scale, Fermi energy of the dilute lithium gas is the order of micro kelvin $\sim 10^{-6}K$, corresponding to $1/k_f a_s = 4000$, so the superfluid gap $\triangle$ is very small, $\frac{\triangle}{k_B} \approx 10^{-30}K$ in the BCS limit. Therefore, we can think that it is impossible to observe the superfluidity. However, Feshbach resonance allows us to tune the interaction and brings the system into the strong interaction regime. In this regime, gap is around 200 nanokelvin for $k_f|a_s| = 1$, and such temperatures are now achieved in experiments on ultracold atomic gases.

In BEC strong interaction limit , we have $k_f a_s \to 0_+$. If we insert it into the equations (45), then chemical potential and gap will be

$$\mu \approx \frac{-\hbar^2}{2ma_s^2} + \frac{\pi\hbar^2 a_s n}{m} \tag{48}$$

$$\triangle \approx \sqrt{\frac{16}{3\pi}} \frac{E_f}{\sqrt{k_f a_s}} \tag{49}$$

First term is the binding energy of the tightly bound molecules, and second

term is the repulsive interaction between molecules in the gas.

2.4 Finite Temperatures

In order to analyze the system at finite temperature, we have to add the fluctuations to the BCS variational ground state. This can be done by Bogoliubov and Valatin transformation. We do not have a variational wave-function to describe the system at finite temperature. We will utilize the quantum statistical physics to solve the system. Our main aim is to find the thermodynamic potential that we can find any physical quantities from it. Bogoliubov and Valatin transformation allow us to write the Hamiltonian of the systems in terms of non-interacting quasi-particles diagonalizing the original many-body Hamiltonian. Our Hamiltonian is

$$\hat{H} = \sum_{\mathbf{k},\alpha} (\epsilon_{\mathbf{k},\alpha} - \mu_\alpha) \hat{C}^\dagger_{\mathbf{k},\alpha} \hat{C}_{\mathbf{k},\alpha} + \frac{V_0}{\Omega} \sum_{\mathbf{k},\mathbf{k}'} \hat{C}^\dagger_{\mathbf{k}\uparrow} \hat{C}^\dagger_{-\mathbf{k}\downarrow} \hat{C}_{-\mathbf{k}'\downarrow} \hat{C}_{\mathbf{k}'\uparrow} \tag{50}$$

which describes the Cooper pairs at rest $\mathbf{q} = 0$. Now, we have to threat the interaction part, and write the four product field operators in terms of two product operators. This is done by mean-field approximation. We write the operator in terms of its average and its fluctuation as $\hat{a} =< \hat{a} > +\delta\hat{a}$. Therefore, the product of the two operators can be written as

$$\hat{a}\hat{b} = \hat{a} < \hat{b} > +\hat{b} < \hat{a} > - < \hat{a} >< \hat{b} > \tag{51}$$

where we neglect the term $\delta\hat{a}\delta\hat{b}$. We can see that two product operator can be written in terms of a one operator. Now, lets define the operators $B^\dagger_{\mathbf{k}} = \hat{C}^\dagger_{\mathbf{k}\uparrow} \hat{C}^\dagger_{-\mathbf{k}\downarrow}$ and $B_{\mathbf{k}'} = \hat{C}_{-\mathbf{k}'\downarrow} \hat{C}_{\mathbf{k}'\uparrow}$. Then, interaction part of the Hamiltonian become

$$\hat{H}_{int} = -\frac{|V_0|}{\Omega} \sum_{\mathbf{k},\mathbf{k}'} [B^\dagger_{\mathbf{k}} < B_{\mathbf{k}'} > +B_{\mathbf{k}'} < B^\dagger_{\mathbf{k}} > - < B_{\mathbf{k}'} >< B^\dagger_{\mathbf{k}} >] \tag{52}$$

with using mean field approximation. Now, lets define the interaction stregth $g > 0$ that is equal to $\frac{|V_0|}{\Omega}$, and define the gap as $\triangle = g \sum_{\mathbf{k}} < \hat{C}_{\mathbf{k}\uparrow} \hat{C}_{-\mathbf{k}\downarrow} >$. Then, interaction part will be

$$\hat{H}_{int} = \sum_{\mathbf{k}} (\triangle \hat{C}^{\dagger}_{\mathbf{k}\uparrow} \hat{C}^{\dagger}_{-\mathbf{k}\downarrow} + \triangle^{*} \hat{C}_{-\mathbf{k}\downarrow} \hat{C}_{\mathbf{k}\uparrow}) + \triangle \sum_{\mathbf{k}} < \hat{C}^{\dagger}_{-\mathbf{k}\downarrow} \hat{C}^{\dagger}_{\mathbf{k}\uparrow} > \tag{53}$$

The summation $\sum_{\mathbf{k}} < \hat{C}^{\dagger}_{-\mathbf{k}\downarrow} \hat{C}^{\dagger}_{\mathbf{k}\uparrow} >$ is equal to $\frac{\triangle^{*}}{g}$, so Hamiltonian is

$$\hat{H} = \sum_{\mathbf{k},\alpha} (\epsilon_{\mathbf{k},\alpha} - \mu_{\alpha}) \hat{C}^{\dagger}_{\mathbf{k},\alpha} \hat{C}_{\mathbf{k},\alpha} + \sum_{\mathbf{k}} (\triangle \hat{C}^{\dagger}_{\mathbf{k}\uparrow} \hat{C}^{\dagger}_{-\mathbf{k}\downarrow} + \triangle^{*} \hat{C}_{-\mathbf{k}\downarrow} \hat{C}_{\mathbf{k}\uparrow}) + \frac{|\triangle|^2}{g} \tag{54}$$

Now, we are ready to write the Hamiltonian in terms of the matrixes and we can diagonalize it with using the Bogoliubov and Valatin transformation, and find the new basis called quasi-particles. Hamiltonian can be written as

$$\hat{H} = \sum_{\mathbf{k}} (h_{\mathbf{k}} + \varepsilon_{\mathbf{k}\downarrow}) + \frac{|\triangle|^2}{g} \tag{55}$$

where $h_{\mathbf{k}}$ is

$$h_{\mathbf{k}} = \begin{pmatrix} C^{\dagger}_{\mathbf{k}\uparrow} & C_{-\mathbf{k}\downarrow} \end{pmatrix} \begin{pmatrix} \varepsilon_{\mathbf{k}\uparrow} & \triangle \\ \triangle^{*} & -\varepsilon_{\mathbf{k}\downarrow} \end{pmatrix} \begin{pmatrix} C_{\mathbf{k}\uparrow} \\ C^{\dagger}_{-\mathbf{k}\downarrow} \end{pmatrix} \tag{56}$$

and energy dispersion relation is $\epsilon_{\mathbf{k}\uparrow} = \epsilon_{\mathbf{k}\downarrow} = \frac{\hbar^2 k^2}{2m} = \epsilon_{\mathbf{k}}$, $\varepsilon_{\mathbf{k}\uparrow} = \epsilon_{\mathbf{k}} - \mu_{\uparrow}$, and $\varepsilon_{\mathbf{k}\downarrow} = \epsilon_{\mathbf{k}} - \mu_{\downarrow}$. To diagonalize the 2 by 2 matrix in h_k, we multiply the both sides of the matrix with $A^{\dagger}A$ where the matrix A is unitary matrix which is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad\qquad |a|^2 + |b|^2 = 1 = |c|^2 + |d|^2 \tag{57}$$

where a,b,c,d are complex numbers in general. This transformation preserves the eigenvalues of the original matrix, and new quasi-particle operators must obey the same anticommutation relations of the original operators $C_{\mathbf{k}\alpha}$ and $C^{\dagger}_{\mathbf{k},\alpha}$. Eigenvalues of the original matrix $\begin{pmatrix} \varepsilon_{\mathbf{k}\uparrow} & \triangle \\ \triangle^* & -\varepsilon_{\mathbf{k}\downarrow} \end{pmatrix}$ can be easly found from the determinant below

$$\det \begin{pmatrix} \varepsilon_{\mathbf{k}\uparrow} - \lambda & \triangle \\ \triangle^* & -\varepsilon_{\mathbf{k}\downarrow} - \lambda \end{pmatrix} = 0 \tag{58}$$

which yields the characteristic polyinomial below

$$\lambda^2 - \lambda(\varepsilon_{\mathbf{k}\uparrow} - \varepsilon_{\mathbf{k}\downarrow}) - \varepsilon_{\mathbf{k}\uparrow}\varepsilon_{\mathbf{k}\downarrow} - |\triangle|^2 = 0 \tag{59}$$

Eigenvalues of the matrix are the roots of the equation above which are

$$\lambda_{\pm} = -h \pm \sqrt{\varepsilon^2_{\mathbf{k}} + |\triangle|^2} \tag{60}$$

where $h = \frac{\mu_\uparrow - \mu_\downarrow}{2}$ for population imbalance case, and $\varepsilon_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$, $\mu = \frac{\mu_\uparrow + \mu_\downarrow}{2}$. New quasi-particle operators are

$$\begin{aligned} \gamma_{\mathbf{k}\uparrow} &= aC_{\mathbf{k}\uparrow} + bC^{\dagger}_{-\mathbf{k}\downarrow} \\ \gamma^{\dagger}_{-\mathbf{k}\downarrow} &= cC_{\mathbf{k}\uparrow} + dC^{\dagger}_{-\mathbf{k}\downarrow} \end{aligned} \tag{61}$$

New operators $\gamma_{\mathbf{k}\uparrow}$, $\gamma^{\dagger}_{-\mathbf{k}\downarrow}$ obey the anti-commutation relation $\{\gamma_{\mathbf{k}\alpha}, \gamma^{\dagger}_{\mathbf{k}'\alpha'}\} = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\alpha\alpha'}$. The relation $\{\gamma_{\mathbf{k}\uparrow}, \gamma_{-\mathbf{k}\downarrow}\} = 0$ yields the restriction on complex numbers a, b, c, d.

$$ac^* + bd^* = 0 \tag{62}$$

The condition $\{\gamma^{\dagger}_{\mathbf{k}\uparrow}, \gamma^{\dagger}_{-\mathbf{k}\downarrow}\} = 0$ also gives the $a^*c + b^*d = 0$. In addition to the these conditions, $\{\gamma_{\mathbf{k}\uparrow}, \gamma^{\dagger}_{\mathbf{k}\uparrow}\} = 1$ and $\{\gamma_{-\mathbf{k}\downarrow}, \gamma^{\dagger}_{-\mathbf{k}\downarrow}\} = 1$ gives the condition

$$|a|^2 + |b|^2 = 1 = |c|^2 + |d|^2 \tag{63}$$

This is also the condition for unitary matrix. Therefore, we choose the transformation matrix A as

$$A = \begin{pmatrix} u_{\mathbf{k}} & -v_{\mathbf{k}} \\ v^*_{\mathbf{k}} & u^*_{\mathbf{k}} \end{pmatrix} \tag{64}$$

So, our new operators that diagonalize the matrix in original Hamiltonian are

$$\begin{aligned} \gamma_{\mathbf{k}\uparrow} &= u_{\mathbf{k}} C_{\mathbf{k}\uparrow} - v_{\mathbf{k}} C^{\dagger}_{-\mathbf{k}\downarrow} \\ \gamma^{\dagger}_{-\mathbf{k}\downarrow} &= v^*_{\mathbf{k}} C_{\mathbf{k}\uparrow} + u^*_{\mathbf{k}} C^{\dagger}_{-\mathbf{k}\downarrow} \end{aligned} \tag{65}$$

Now, we can write the Hhamiltonian in diagonal form as

$$\hat{H} = \sum_{\mathbf{k}} \left[\begin{pmatrix} \gamma^{\dagger}_{\mathbf{k}\uparrow} & \gamma_{-\mathbf{k}\downarrow} \end{pmatrix} \begin{pmatrix} E_{\mathbf{k}\uparrow} & 0 \\ 0 & -E_{\mathbf{k}\downarrow} \end{pmatrix} \begin{pmatrix} \gamma_{\mathbf{k}\uparrow} \\ \gamma^{\dagger}_{-\mathbf{k}\downarrow} \end{pmatrix} + \varepsilon_{\mathbf{k}\downarrow} \right] + \frac{|\triangle|^2}{g} \tag{66}$$

Then, if we write the Hamiltonian in the original form same as in equation (50), it will be in the form of non-interacting quasi-particles as

$$\hat{H} = \sum_{\mathbf{k},\alpha} E_{\mathbf{k},\alpha} \gamma^{\dagger}_{\mathbf{k},\alpha} \gamma_{\mathbf{k},\alpha} + \sum_{\mathbf{k}} (\varepsilon_{\mathbf{k}\downarrow} - E_{\mathbf{k},\downarrow}) + \frac{|\triangle|^2}{g} \tag{67}$$

Where $E_{\mathbf{k},\uparrow}$ and $E_{\mathbf{k}\downarrow}$ are the eigenvalues of the Hamiltonian. Unitary transformation preserves the eigenvalues. Therefore, $E_{\mathbf{k},\uparrow}$ and $E_{\mathbf{k}\downarrow}$ will be

$$\begin{aligned} E_{\mathbf{k}\uparrow} &= -h + \sqrt{\varepsilon_{\mathbf{k}}^2 + \triangle^2} \\ E_{\mathbf{k}\downarrow} &= h + \sqrt{\varepsilon_{\mathbf{k}}^2 + \triangle^2} \end{aligned} \tag{68}$$

For BCS theory $\mu_\uparrow = \mu_\downarrow$ (population balanced), so $h = 0$. Then, excitation energy will be

$$E_{\mathbf{k}\uparrow} = E_{\mathbf{k}\downarrow} = \sqrt{\varepsilon_{\mathbf{k}}^2 + \triangle^2} = E_{\mathbf{k}} \tag{69}$$

Now, we can find the grand canonical partition function of the system Q which is equal to $Tr\left(e^{-\beta\hat{H}}\right)$. We use grand canonical ensemble because number of the particles in the system is not fixed, and it is an intrinsic property. Trace is independent of the basis. Therefore, we can write the Q in energy basis as

$$Q = Tr\left(e^{-\beta\hat{H}}\right) = \sum_{\mathbf{k},\alpha} < \mathbf{k},\alpha|e^{-\beta\hat{H}}|\mathbf{k},\alpha > \tag{70}$$

Where $\beta = \frac{1}{k_B T}$ and $\hat{H}|\mathbf{k},\alpha >= \sum_{\mathbf{k},\alpha} E_{\mathbf{k},\alpha} n_{\mathbf{k},\alpha}|\mathbf{k},\alpha >$ is an eigenvalue equation of the Hamiltonian $\sum_{\mathbf{k},\alpha} E_{\mathbf{k},\alpha}\gamma^\dagger_{\mathbf{k},\alpha}\gamma_{\mathbf{k},\alpha}$ in equation (67) . Other parts in Hamiltonian are constants. Therefore,

$$Q = \sum_{\{n_{\mathbf{k}}\}} e^{-\beta\sum_{\mathbf{k}\alpha} E_{\mathbf{k},\alpha} n_{\mathbf{k},\alpha} - \beta\sum_{\mathbf{k}}\left(\varepsilon_{\mathbf{k}\downarrow} - E_{\mathbf{k},\downarrow}\right) - \beta\frac{|\triangle|^2}{g}} \tag{71}$$

Where $\{n_{\mathbf{k}}\}$ denotes all possible microstates, and $n_{\mathbf{k}}$ denotes the occupation number of states which is 0 or 1 for fermions. Then,

$$Q = e^{-\beta\sum_{\mathbf{k}}\left(\varepsilon_{\mathbf{k}\downarrow} - E_{\mathbf{k},\downarrow}\right) - \beta\frac{|\triangle|^2}{g}} \prod_{\mathbf{k},\alpha}\left(\sum_{n_{\mathbf{k}}} e^{-\beta E_{\mathbf{k},\alpha} n_{\mathbf{k}}}\right) \tag{72}$$

We put the 0 and 1 for occupation number $n_{\mathbf{k}}$, and eventually grand canonical partition function of the system becomes

$$Q = e^{-\beta \sum_{\mathbf{k}} (\varepsilon_{\mathbf{k}\downarrow} - E_{\mathbf{k},\downarrow}) - \beta \frac{|\triangle|^2}{g}} \prod_{\mathbf{k},\alpha} \left(1 + e^{-\beta E_{\mathbf{k},\alpha}}\right) \tag{73}$$

Now, we are ready to find the thermodynamic potential Ω which is equal to $-k_B T \ln Q$. Using the property $\ln(ab) = \ln a + \ln b$, and writing the exponential term in terms of $tanh$ function as $e^{-\frac{E_{\mathbf{k},\alpha}}{T}} = \frac{1-\tanh\frac{E_{\mathbf{k},\alpha}}{2T}}{1+\tanh\frac{E_{\mathbf{k},\alpha}}{2T}}$, potential will be

$$\Omega = \sum_{\mathbf{k}} (\varepsilon_{\mathbf{k}\downarrow} - E_{\mathbf{k},\downarrow}) + \frac{|\triangle|^2}{g} + T \sum_{\mathbf{k},\alpha} \ln(\frac{1 + \tanh(\frac{E_{\mathbf{k},\alpha}}{2T})}{2}) \tag{74}$$

We can get any physical quantities from potential Ω which is also equal to $E - TS - \mu N$ where S is entropy of the system. Differantial of the potential is

$$d\Omega = -SdT - PdV - Nd\mu \tag{75}$$

where $dE = TdS + \mu dN - PdV$. Then, physical quantities S, P and N can be found from the relations below

$$N = -\frac{\partial \Omega}{\partial \mu}|_{T,V} \qquad S = -\frac{\partial \Omega}{\partial T}|_{\mu,V} \qquad P = -\frac{\partial \Omega}{\partial V}|_{T,\mu} \tag{76}$$

For population balanced case, $\varepsilon_{\mathbf{k}\downarrow} = \varepsilon_{\mathbf{k}} = (\epsilon_{\mathbf{k}} - \mu)$, and $E_{\mathbf{k},\downarrow} = E_{\mathbf{k}} = \sqrt{\varepsilon_{\mathbf{k}}^2 + \triangle^2}$. Therefore, thermodynamic potential for population balanced case will be

$$\Omega = \sum_{\mathbf{k}} (\varepsilon_{\mathbf{k}} - E_{\mathbf{k}}) + \frac{|\triangle|^2}{g} + 2T \sum_{k} \ln(\frac{1 + \tanh(\frac{E_{\mathbf{k}}}{2T})}{2}) \tag{77}$$

We can obtain the gap equation from the condition $\frac{\partial\Omega}{\partial\triangle}=0$ which yields

$$\frac{2\left|\triangle\right|}{g}-\sum_{\mathbf{k}}\frac{\partial E_{\mathbf{k}}}{\partial\left|\triangle\right|}+\sum_{\mathbf{k}}\frac{\partial E_{\mathbf{k}}}{\partial\left|\triangle\right|}\frac{2}{e^{\frac{E_{\mathbf{k}}}{T}}+1}=0 \tag{78}$$

We want to write $\frac{2}{e^{\frac{E_{\mathbf{k}}}{T}}+1}$ in terms of tanh function as $1-\tanh\frac{E_{\mathbf{k}}}{2T}$, and $\frac{\partial E_{\mathbf{k}}}{\partial|\triangle|}=\frac{|\triangle|}{E_{\mathbf{k}}}$. Therefore, gap equation will be

$$\frac{1}{g}=\sum_{\mathbf{k}}\frac{\tanh\left(\frac{E_{\mathbf{k}}}{2T}\right)}{2E_{\mathbf{k}}} \tag{79}$$

We change the interaction term V_0 with scattering length a_s with using relation in (42) as usual. Eventually, gap equation becomes

$$\frac{-m}{4\pi\hbar^2a_s}=\int\frac{d^3k}{2\pi^3}\frac{1}{2E_{\mathbf{k}}}\left[\tanh\left(\beta E_{\mathbf{k}}/2\right)-\frac{1}{2\epsilon_{\mathbf{k}}}\right] \tag{80}$$

at finite temperatures.

2.4.1 Critical Temperatures

Now, we want to find the temperature T^* when pair formation stars, In the BCS limit, pairs are formed and condensate at the same temperature, so $T^*=T_C$. However, it is not true for strong interaction regime. In the BEC side, $T_C<T^*$, so pairs are first formed, then condensate to the ground state. Temperatures between these two values T^*, T_C, there are pairs not condensate to the ground state. In order to find T^*, we want to solve the two coupled equations which are number and gap equations with setting $\triangle=0$. Above the temperature T^*, we have a normal Fermi gas, so number

of the particles in the system is given by Fermi-Dirac distribution.

$$n = 2\int \frac{d^3k}{2\pi^3}\frac{1}{1+e^{\beta\xi_{\mathbf{k}}}} \tag{81}$$

This has to be solved simultaneously with gap equation. We set $\triangle = 0$, and in the BCS limit we insert $\mu = E_F$, and $k_f a_s = 0^-$ in the gap equation. Then, critical temperature for superfluidity in the BCS limit is

$$T^*_{BCS} = \frac{e^\gamma}{\pi}\triangle_0 \tag{82}$$

where e^γ=1.78, and $\triangle_0$ is the superfluid gap at zero temperatures in the BCS limit given in (46). In the BEC limit where $k_f a_s = 0^+$, and chemical potential $\mu = \frac{-\hbar^2}{2ma_s^2} = \frac{-E_b}{2}$ where E_b is binding energy, temperature T^* will be

$$T^*_{BEC} = \frac{|E_b|}{2\left(\ln\frac{|E_b|}{E_f}\right)^{3/2}} \tag{83}$$

As we said before this is not the critical temperatures for superfluid transition, but it is a temperature when pairs start to form. We can find the critical temperature T_C from the BEC transition temperature of non-interacting gas of density $n/2$, and mass $2m$ that is

$$T_C = \frac{\pi\hbar^2}{m}\left(\frac{n}{2\zeta\left(\frac{3}{2}\right)}\right)^{2/3} \approx 0.2E_f \tag{84}$$

This approximation holds for weakly interacting Bose gases with a small error.

2.5 Polarized Fermi Gases with Spin-Orbit Coupling Interaction

After spin-orbit coupling(SOC) has been realized recently in Bose gases by using the varying laser fields, effects of the SOC have been studied for two-component Fermi gases [17,18]. Utilizing the technique of Feshbach resonance , the spin-orbit coupled ultracold atoms provide a clean environment to investigate the new phases of the condensed matter systems. In the presence of the SOC, Fermi surface is changed, and many interesting phases become possible. SOC enriches the ground state phases of the two-component Fermi gas. In the absence of the SOC, mean-field approximation is adequate to describe the correct physics of the whole BCS-BEC evolution regime, and now we assume that it is also sufficient in the presence of the SOC. We consider the uniform polarized two-component Fermi gas with SOC, which is described by the Hamiltonian:

$$\hat{H} = \hat{H}_0 + \hat{H}_{SO} + \hat{H}_{int} \tag{85}$$

where $\hat{H}_0$ is kinetic energy term, $\hat{H}_{SO}$ is the spin-orbit interaction, and $\hat{H}_{int}$ is the s-wave interaction between two fermionic species. They are

$$\begin{aligned}
\hat{H}_0 &= \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k},\sigma} \hat{C}^{\dagger}_{\mathbf{k},\sigma} \hat{C}_{\mathbf{k},\sigma} \\
\hat{H}_{SO} &= \alpha \sum_{\mathbf{k}} \mathbf{k} \left(e^{i\theta_k} \hat{C}^{\dagger}_{\mathbf{k},\uparrow} \hat{C}_{\mathbf{k},\downarrow} + h.c \right) \\
\hat{H}_{int} &= -g \sum_{\mathbf{k},\mathbf{k}'} \hat{C}^{\dagger}_{\mathbf{k}\uparrow} \hat{C}^{\dagger}_{-\mathbf{k}\downarrow} \hat{C}_{-\mathbf{k}'\downarrow} \hat{C}_{\mathbf{k}'\uparrow}
\end{aligned}$$

where $\xi_{\mathbf{k},\sigma} = \frac{\hbar^2 k^2}{2m} - \mu_\sigma$, $\hat{C}^{\dagger}_{\mathbf{k},\sigma}$ $\left(\hat{C}_{\mathbf{k},\sigma} \right)$ denotes the creation(annihilation) operators for a fermion with momentum $\mathbf{k} = (k_x, k_y, k_z)$ and spin $\sigma = \{\uparrow, \downarrow\}$, $\alpha \geq$

0 is the strength of the Rashba type spin-orbit coupling, $\theta_k = arg(k_x + ik_y)$, and $g \geq 0$ is the strength of the s-wave interaction which is $\frac{|V_0|}{\Omega}$. Our main aim is to find the thermodynamic potential of the system. Then, we can get any physical quantities from it. In order to find potential, we have to write the Hamiltonian in terms of the non-interacting quasi-particles, diagonalizing the Hamiltonian. We threat the interaction term $\hat{H}_{int}$ using mean-field approximation through the same way that we did in section 4.2. We write the field operators in terms of its average and fluctuation. Then, interaction part will be

$$\hat{H}_{int} = -\sum_{\mathbf{k}} \left(\triangle \hat{C}^{\dagger}_{-\mathbf{k},\downarrow} \hat{C}^{\dagger}_{\mathbf{k},\uparrow} + \triangle^{*} \hat{C}_{\mathbf{k},\uparrow} \hat{C}_{-\mathbf{k},\downarrow} \right) + \frac{|\triangle|^2}{g} \tag{86}$$

where $\triangle$ is the pairing energy that is equal to $g \sum_{\mathbf{k}} < \hat{C}_{\mathbf{k},\uparrow} \hat{C}_{-\mathbf{k},\downarrow} >$. Momentum summation in Hamiltonian compass both negative and positive $\mathbf{k}$ values. Therefore, we can add opposite sign of the k terms into the Hamiltonian and divide it by 2 . This process does not change anything, but allows us to write the Hamiltonian in matrix form. Then, our Hamiltonian will be

$$\begin{aligned}
\hat{H}_0 &= \frac{1}{2} \sum_{\mathbf{k}} \left(\xi_{\mathbf{k},\uparrow} \hat{C}^{\dagger}_{\mathbf{k},\uparrow} \hat{C}_{\mathbf{k},\uparrow} + \xi_{\mathbf{k},\uparrow} \hat{C}^{\dagger}_{-\mathbf{k},\uparrow} \hat{C}_{-\mathbf{k},\uparrow} + \xi_{\mathbf{k},\downarrow} \hat{C}^{\dagger}_{-\mathbf{k},\downarrow} \hat{C}_{-\mathbf{k},\downarrow} + \xi_{\mathbf{k},\downarrow} \hat{C}^{\dagger}_{\mathbf{k},\downarrow} \hat{C}_{\mathbf{k},\downarrow} \right) \\
\hat{H}_{SO} &= \frac{1}{2} \sum_{\mathbf{k}} \left(S_{\mathbf{k}} \hat{C}^{\dagger}_{\mathbf{k},\uparrow} \hat{C}_{\mathbf{k},\downarrow} - S_{\mathbf{k}} \hat{C}^{\dagger}_{-\mathbf{k},\uparrow} \hat{C}_{-\mathbf{k},\downarrow} + h.c \right) \\
\hat{H}_{int} &= -\frac{1}{2} \sum_{\mathbf{k}} \left(\triangle \hat{C}^{\dagger}_{-\mathbf{k},\downarrow} \hat{C}^{\dagger}_{\mathbf{k},\uparrow} + \triangle^{*} \hat{C}_{\mathbf{k},\uparrow} \hat{C}_{-\mathbf{k},\downarrow} + \triangle \hat{C}^{\dagger}_{\mathbf{k},\downarrow} \hat{C}^{\dagger}_{-\mathbf{k},\uparrow} + \triangle^{*} \hat{C}_{-\mathbf{k},\uparrow} \hat{C}_{\mathbf{k},\downarrow} \right) \\
&\quad + \frac{|\triangle|^2}{g}
\end{aligned}$$

where $S_{\mathbf{k}} = \alpha \left(k_x + ik_y \right)$ is the Rashba type spin-orbit fields. Now, we can write Hamiltonian $\hat{H}$ in the matrix form in the basis $\Psi^{\dagger}_{\mathbf{k}} = \left(\hat{C}^{\dagger}_{\mathbf{k},\uparrow} \hat{C}^{\dagger}_{\mathbf{k},\downarrow} \hat{C}_{-\mathbf{k},\uparrow} \hat{C}_{-\mathbf{k},\downarrow} \right)$

as

$$\hat{H} = \frac{1}{2}\sum_{\mathbf{k}} \Psi_{\mathbf{k}}^{\dagger} \begin{pmatrix} \xi_{\mathbf{k},\uparrow} & S_{\mathbf{k}} & 0 & \triangle \\ S_{\mathbf{k}}^{*} & \xi_{\mathbf{k},\downarrow} & -\triangle & 0 \\ 0 & -\triangle^{*} & -\xi_{\mathbf{k},\uparrow} & S_{\mathbf{k}}^{*} \\ \triangle^{*} & 0 & S_{\mathbf{k}} & -\xi_{\mathbf{k},\downarrow} \end{pmatrix} \Psi_{\mathbf{k}} + \frac{1}{2}\sum_{\mathbf{k},\sigma} \xi_{\mathbf{k},\sigma} + \frac{|\triangle|^2}{g} \quad (87)$$

Now, we have to diagonalize the Hamiltonian, and write it in terms of non-interacting quasi-particles. After applying the Bogoliubav and Valatin transformation in the same way that we did in section 4.2, Hamiltonian of the system will be

$$\hat{H} = \frac{1}{2}\sum_{\mathbf{k},\lambda} E_{\mathbf{k},\lambda}\gamma_{\mathbf{k},\lambda}^{\dagger}\gamma_{\mathbf{k},\lambda} + \sum_{\mathbf{k}} \xi_{\mathbf{k},+} + \frac{|\triangle|^2}{g} \quad (88)$$

where $\gamma_{\mathbf{k},\lambda}^{\dagger} = (\gamma_{\mathbf{k},\uparrow}^{\dagger}, \gamma_{\mathbf{k},\downarrow}^{\dagger}, \gamma_{-\mathbf{k},\uparrow}, \gamma_{-\mathbf{k},\downarrow})$, $\lambda = (1,2,3,4)$ labels the quasiparticle/quasihole excitation energies $E_{\mathbf{k}\lambda}$, and $\xi_{\mathbf{k},s} = \epsilon_{\mathbf{k},s} - \mu_s$ where $\mu_s = \frac{\mu_\uparrow + s\mu_\downarrow}{2}$, and s denotes $\pm$. Excitation energies of the quasiparticles/quasiholes are eigenvalues of the Hamiltonian matrix above. We can obtain the partition function of the system using formula $Tr\left(e^{-\beta\hat{H}}\right)$. Trace is basis invariant, so we use the energy basis. Then

$$Q = \sum_{\{n_{\mathbf{k}}\}} e^{-\frac{\beta}{2}\sum_{\mathbf{k}\lambda} E_{\mathbf{k},\lambda} n_{\mathbf{k},\lambda} - \beta\sum_{\mathbf{k}} \xi_{\mathbf{k},+} - \beta\frac{|\triangle|^2}{g}} \quad (89)$$

which is similar to the case in section 4.2 . Using the same argument that we did before, partition function of the system will be

$$Q = e^{-\beta\sum_{\mathbf{k}} \xi_{\mathbf{k},+} - \beta\frac{|\triangle|^2}{g}} \prod_{\mathbf{k},\lambda} \left(1 + e^{-\beta E_{\mathbf{k},\lambda}/2}\right) \quad (90)$$

Thermodynamic potential of the system is given by the relation $\Omega = -T\ln(Q)$. Using the property that $\ln(ab) = \ln a + \ln b$, and $e^{-\frac{E_{\mathbf{k},\lambda}}{T}} = \frac{1-\tanh\frac{E_{\mathbf{k},\lambda}}{2T}}{1+\tanh\frac{E_{\mathbf{k},\lambda}}{2T}}$ potential will be

$$\Omega = \frac{|\triangle|^2}{g} + \sum_{\mathbf{k}} \xi_{\mathbf{k},+} + \frac{T}{2}\sum_{\mathbf{k},\lambda} \ln\left(\frac{1+\tanh\left(\frac{E_{\mathbf{k},\lambda}}{2T}\right)}{2}\right) \tag{91}$$

where we take $k_B = 1$. Following from the usual precedure, we obtain the gap equation from the condition $\frac{\partial\Omega}{\partial\triangle} = 0$ that is

$$\frac{2\,|\triangle|}{g} = \frac{1}{4}\sum_{\mathbf{k},\lambda} \frac{\partial E_{\mathbf{k},\lambda}}{\partial\,|\triangle|}\left(\tanh\left(E_{\mathbf{k},\lambda}/2T\right) - 1\right) \tag{92}$$

Number equations are found from the relation $N_\uparrow + sN_\downarrow = \frac{\partial\Omega}{\partial\mu_s}$, so they are

$$N_\uparrow \pm N_\downarrow = \sum_{\mathbf{k}} \frac{(1\pm 1)}{2} + \frac{1}{4}\sum_{\mathbf{k},\lambda} \frac{\partial E_{\mathbf{k},\lambda}}{\partial\mu_\pm}\left(\tanh\left(E_{\mathbf{k},\lambda}/2T\right) - 1\right) \tag{93}$$

We can write the summation $\sum_{\mathbf{k}} = \frac{1}{4}\sum_{\mathbf{k},\lambda}$. Then, number equations will be

$$N_\uparrow \pm N_\downarrow = \frac{1}{4}\sum_{\mathbf{k},\lambda}\left[\frac{(1\pm 1)}{2} + \frac{\partial E_{\mathbf{k},\lambda}}{\partial\mu_\pm}\left(\tanh\left(E_{\mathbf{k},\lambda}/2T\right) - 1\right)\right] \tag{94}$$

These equations are self-consistency equations of the system, and there are 3 equations and 3 unknowns which are $\triangle, \mu_\uparrow, \mu_\downarrow$. These coupled equations can be solved numerically. We check the stability of the mean-field solution using the curvature criterion [19], which says that $\frac{\partial^2\Omega}{\partial|\triangle|^2}$ have to be positive. When the curvature $\frac{\partial^2\Omega}{\partial|\triangle|^2}$ is negative, mean-field solution does not correspond the minimum of the Ω, and phase separation state is favored. However, solution having positive curvature may corresponds the metastable state. This causes minor quantitative errors in the phase diagram in 3 dimension. Curvature

criterion is

$$\frac{\partial^2\Omega}{\partial\left|\triangle\right|^2}=\frac{1}{4}\sum_{\mathbf{k},\lambda}\left[\left(\frac{1}{\left|\triangle\right|}\frac{\partial E_{\mathbf{k},\lambda}}{\partial\left|\triangle\right|}-\frac{\partial E_{\mathbf{k},\lambda}^2}{\partial\left|\triangle\right|^2}\right)\left(X_{\mathbf{k},\lambda}-1\right)-\frac{1}{2}\frac{\partial E_{\mathbf{k},\lambda}^2}{\partial\left|\triangle\right|^2}Y_{\mathbf{k},\lambda}\right] \quad (95)$$

where $X_{\mathbf{k},\lambda}=\tanh\left(E_{\mathbf{k},\lambda}/2T\right)$, and $Y_{\mathbf{k},\lambda}=sech^2(E_{\mathbf{k},\lambda}/2T)$. We need to find the excitation energies of the system $E_{\mathbf{k}.\lambda}$ which are eigenvalues of the Hamiltonian matrix. It is almost impossible to find the eigenvalues of the Hamiltonian matrix analytically. Therefore, we use computer program Matematica to find $E_{\mathbf{k}.\lambda}$ that is

$$E_{\mathbf{k},\lambda}=s_\lambda\sqrt{\xi_{\mathbf{k},+}^2+\xi_{\mathbf{k},-}^2+\left|\triangle\right|^2+\left|S_{\mathbf{k}}\right|^2+2p_\lambda A_{\mathbf{k}}} \quad (96)$$

where $A_{\mathbf{k}}=\sqrt{\xi_{\mathbf{k},-}^2\left(\xi_{\mathbf{k},+}^2+\left|\triangle\right|^2\right)+\left|S_{\mathbf{k}}\right|^2\xi_{\mathbf{k},+}^2}$, $\left|S_{\mathbf{k}}\right|^2=\alpha^2\left(k_x^2+k_y^2\right)=\alpha^2k_\perp^2$, $s_1=s_2=+$, and $s_3=s_4=-$. Therefore, $E_{\mathbf{k},3}=-E_{\mathbf{k},1}$, and $E_{\mathbf{k},4}=-E_{\mathbf{k},2}$. If we put this results into the gap and number equations, and use the property that $\tanh\left(-x\right)=-\tanh(x)$, gap and number equations reduce to

$$\begin{aligned}2\frac{\left|\triangle\right|}{g}&=\frac{1}{2}\sum_{\mathbf{k},s}\frac{\partial E_{\mathbf{k},s}}{\partial\left|\triangle\right|}\tanh\left(\frac{E_{\mathbf{k},s}}{2T}\right) \quad (97)\\ N_\uparrow\pm N_\downarrow&=\frac{1}{2}\sum_{\mathbf{k},s}\left[\frac{(1\pm1)}{2}+\frac{\partial E_{\mathbf{k},s}}{\partial\mu_\pm}\tanh\left(\frac{E_{\mathbf{k},s}}{2T}\right)\right]\end{aligned}$$

where $E_{\mathbf{k},+}=E_{\mathbf{k},1}$, and $E_{\mathbf{k},-}=E_{\mathbf{k},2}$. Now, we have to find the partial derivatives in equations above. $\frac{\partial E_{\mathbf{k},s}}{\partial\left|\triangle\right|}=\frac{\left|\triangle\right|}{E_{\mathbf{k},s}}\left(1+s\frac{\xi_{\mathbf{k},-}^2}{A_{\mathbf{k}}}\right)$, so gap equation will be

$$\frac{1}{g}=\frac{1}{4}\sum_{\mathbf{k}}\left[\frac{\left(1+\xi_{\mathbf{k},-}^2/A_{\mathbf{k}}\right)}{E_{\mathbf{k},1}}\tanh\left(\frac{E_{\mathbf{k},1}}{2T}\right)+\frac{\left(1-\xi_{\mathbf{k},-}^2/A_{\mathbf{k}}\right)}{E_{\mathbf{k},2}}\tanh\left(\frac{E_{\mathbf{k},2}}{2T}\right)\right] \quad (98)$$

Partial derivative $\frac{\partial E_{\mathbf{k},s}}{\partial \mu_-} = -\frac{\xi_{\mathbf{k},-}}{E_{\mathbf{k},s}}\left(1 + s\frac{\left(\xi^2_{\mathbf{k},+}+|\triangle|^2\right)}{A_{\mathbf{k}}}\right)$, so one of the number equations become

$$N_\uparrow - N_\downarrow = \frac{1}{2}\sum_{\mathbf{k}} \frac{-\xi_{\mathbf{k},-}}{E_{\mathbf{k},1}}\left(1 + \frac{\left(\xi^2_{\mathbf{k},+} + |\triangle|^2\right)}{A_{\mathbf{k}}}\right)\tanh\left(\frac{E_{\mathbf{k},1}}{2T}\right) \tag{99}$$
$$+ \frac{1}{2}\sum_{\mathbf{k}} \frac{-\xi_{\mathbf{k},-}}{E_{\mathbf{k},2}}\left(1 - \frac{\left(\xi^2_{\mathbf{k},+} + |\triangle|^2\right)}{A_{\mathbf{k}}}\right)\tanh\left(\frac{E_{\mathbf{k},2}}{2T}\right)$$

Partial dervative $\frac{\partial E_{\mathbf{k},s}}{\partial \mu_+} = -\frac{\xi_{\mathbf{k},+}}{E_{\mathbf{k},s}}\left(1 + s\frac{\left(\xi^2_{\mathbf{k},-}+|S_{\mathbf{k}}|^2\right)}{A_{\mathbf{k}}}\right)$, so other number equation will be

$$N_\uparrow + N_\downarrow = \frac{1}{2}\sum_{\mathbf{k}}\left[2 - \frac{\xi_{\mathbf{k},+}}{E_{\mathbf{k},1}}\left(1 + \frac{\left(\xi^2_{\mathbf{k},-} + |S_{\mathbf{k}}|^2\right)}{A_{\mathbf{k}}}\right)\tanh\left(\frac{E_{\mathbf{k},1}}{2T}\right)\right] \tag{100}$$
$$- \frac{1}{2}\sum_{\mathbf{k}}\left[\frac{\xi_{\mathbf{k},+}}{E_{\mathbf{k},2}}\left(1 - \frac{\left(\xi^2_{\mathbf{k},-} + |S_{\mathbf{k}}|^2\right)}{A_{\mathbf{k}}}\right)\tanh\left(\frac{E_{\mathbf{k},2}}{2T}\right)\right]$$

In order to solve the 3 coupled equations numerically, we have to scale them. We choose the energy scale as the Fermi energy ϵ_f of $\frac{N}{2}$ fermions such that $N = \frac{k_f^3\Omega}{3\pi^2}$ the same as before. We scale the $k_\perp$, and k_z with k_f as $k_\perp = \widetilde{k_\perp}k_f$, and $k_z = \widetilde{k_z}k_f$ where $\widetilde{k_\perp}, \widetilde{k_z}$ are now unitless. Scaled excitation energies will be $E_{\mathbf{k},s} = \epsilon_f\widetilde{E_{k,s}}$, and $A_{\mathbf{k}} = \epsilon_f^2\widetilde{A_k}$ where

$$\widetilde{E_{k,s}} = \sqrt{\widetilde{r}^2 + \left(\frac{\widetilde{\mu_\downarrow} - \widetilde{\mu_\uparrow}}{2}\right)^2 + \left|\widetilde{\triangle}\right|^2 + \left(\frac{\alpha\widetilde{k_\perp}2m}{k_f}\right)^2 + s2\widetilde{A_k}}$$
$$\widetilde{A_k} = \sqrt{\left(\frac{\widetilde{\mu_\downarrow} - \widetilde{\mu_\uparrow}}{2}\right)^2\left(\widetilde{r}^2 + |\triangle|^2\right) + \left(\frac{\alpha\widetilde{k_\perp}2m}{k_f}\right)^2\widetilde{r}^2}$$

and $\widetilde{r} = \left(\left(\widetilde{k_\perp}^2 + \widetilde{k_z}^2\right) - \frac{\left(\widetilde{\mu_\uparrow}+\widetilde{\mu_\downarrow}\right)}{2}\right)$. When we write the summation $\sum_{\mathbf{k}}$ in terms of integral, we have to be careful now. The terms in the integral

depends on the $k_\perp$ and and k_z where $k^2 = k_\perp^2 + k_z^2$, so we will use the cylindrical coordinates to evaluate integrals as $\sum_k = \Omega \int\int \frac{2\pi dk_\perp dk_z}{2\pi^3}$. Therefore, our scaled equations will be

$$\begin{aligned}
\frac{-2\pi}{k_f a_s} &= \int_0^\infty \int_{-\infty}^\infty \widetilde{k_\perp} d\widetilde{k_\perp} d\widetilde{k_z} f_1(\widetilde{k_\perp}, \widetilde{k_z}, \widetilde{\mu_\uparrow}, \widetilde{\mu_\downarrow}, \widetilde{\triangle}) \\
\frac{8\left(N_\uparrow - N_\downarrow\right)}{3N} &= \int_0^\infty \int_{-\infty}^\infty \widetilde{k_\perp} d\widetilde{k_\perp} d\widetilde{k_z} f_2(\widetilde{k_\perp}, \widetilde{k_z}, \widetilde{\mu_\uparrow}, \widetilde{\mu_\downarrow}, \widetilde{\triangle}) \\
\frac{8}{3} &= \int_0^\infty \int_{-\infty}^\infty \widetilde{k_\perp} d\widetilde{k_\perp} d\widetilde{k_z} f_3(\widetilde{k_\perp}, \widetilde{k_z}, \widetilde{\mu_\uparrow}, \widetilde{\mu_\downarrow}, \widetilde{\triangle})
\end{aligned} \tag{101}$$

where f_1, f_2, and f_3 are functions of 5 variables. Moreover, we replaced the interaction term V_0 for gap equation by the scattering length a_s using the relation in (42). The functions inside the integrals above are the scaled functions from the number and gap equations where $\widetilde{T} = \frac{T}{\epsilon_f}, \widetilde{\mu}_{\uparrow,\downarrow} = \frac{\mu_{\uparrow,\downarrow}}{\epsilon_f}$, and $\widetilde{\triangle} = \frac{\triangle}{\epsilon_f}$. These 3 coupled equations can be solved numerically using Matlab fsolve function, and we can get unknowns $\widetilde{\mu_\uparrow}, \widetilde{\mu_\downarrow}$, and $\widetilde{\triangle}$ for each solution. When we write the numeric program, polarization $\frac{\left(N_\uparrow - N_\downarrow\right)}{N}$, interaction $\frac{1}{k_f a_s}$, SOC term $\frac{\alpha m}{k_f}$, and temperature $\widetilde{T}$ are the parameters that we can control them, and we are able to construct the phase diagrams of the systems. We want to find the ground state phase diagrams of the system, so we will take the $\widetilde{T}$=0 in the numeric code. If we set the polarization, and SOC term to the zero, we reproduce the same results in Figure 3.

2.5.1 Conditions For Gapless Superfluid

We know that superfluid phase of the system is gapped without SOC effect. This means that excitation energy of the system does not equal to zero in any case. In chapter 4, we calculated $E_{\mathbf{k}}$ equal to $\sqrt{\xi_{\mathbf{k}}^2 + \triangle^2}$ for population balanced case. Therefore, system is gapped superfluid through the BCS-BEC evolution. However, picture is different when we add the SOC interaction to the system. Excitation energies of the system are zero in certain conditions, and we will find these conditions. We found excitation energy using computer program Matematica, and we know that $E_{\mathbf{k},3} = -E_{\mathbf{k},1}$, and $E_{\mathbf{k},4} = -E_{\mathbf{k},2}$. Therefore, it is enough for us to look at just 2 excitation energies for gapless condition. They are

$$E_{\mathbf{k},s} = \sqrt{\xi_{\mathbf{k},+}^2 + \xi_{\mathbf{k},-}^2 + |\triangle|^2 + |S_{\mathbf{k}}|^2 + 2sA_{\mathbf{k}}} \tag{102}$$

where $E_{\mathbf{k},+} = E_{\mathbf{k},1}$, and $E_{\mathbf{k},-} = E_{\mathbf{k},2}$. We can easly see that $E_{\mathbf{k},1} > 0$, so it is gapped. However, $E_{\mathbf{k},2}$ can be zero under certain conditions. Condition for gapless $E_{\mathbf{k},2}$ is:

$$\xi_{\mathbf{k},+}^2 + \xi_{\mathbf{k},-}^2 + |\triangle|^2 + |S_{\mathbf{k}}|^2 = 2A_{\mathbf{k}} \tag{103}$$

This yields 2 conditions which are

$$|S_{\mathbf{k}}| = 0 \qquad \text{and} \qquad \xi_{\mathbf{k},\uparrow}\xi_{-\mathbf{k},\downarrow} + |\triangle|^2 = 0 \tag{104}$$

$|S_{\mathbf{k}}| = \alpha\sqrt{k_x^2 + k_y^2} = 0$, so $k_x = k_y = 0$. This means that gapless $E_{\mathbf{k},2}$ occurs at real k_z values in momentum space. Second condition gives the restriction

on k_z values which is

$$k_{z,s}^2 = 2m\mu_+ + 2ms\sqrt{\mu_-^2 - |\triangle|^2} \tag{105}$$

There are 2 possibilities. k_z might have 2 solutions, and 4 solutions in k-space. These 2 distinct conditions correspond to the different phases. We define these phases as GSF(I) and GSF(II). Excitation energy $E_{k,2}$ equal to zero at 2 k_z points for GSF(I), and at 4 k_z points for GSF(II). These conditions are

$$\begin{array}{llll} |\mu_-| > |\triangle|\,, \mu_+ \geq 0 & \text{and} & \mu_\uparrow \mu_\downarrow \geq -|\triangle|^2 & \text{for GSF(II)} \\ |\mu_-| > |\triangle| & \text{and} & \mu_\uparrow \mu_\downarrow \leq -|\triangle|^2 & \text{for GSF(I)} \end{array}$$

We will use these conditions to separate the different phases of the system when we construct the phase diagrams.

2.5.2 Ground State Phase Diagrams

Without SOC $\alpha \to 0$, number, and gap equations in (101) are sufficient to describe the BCS-BEC crossover, and we assume that mean-field formalism is also applicable to the system with SOC interaction. Then, we analyze the ground state phase diagram of population-imbalanced as a function of SOC parameter. There are 3 phases in the ground state which are normal state(N), uniform superfluid, and non-uniform superfluid(Phase Seperation). Normal phase is charecterized by $\triangle = 0$. Uniform, and non-uniform superfluid phases are described by $\frac{\partial^2\Omega}{\partial|\triangle|^2} > 0$, and $\frac{\partial^2\Omega}{\partial|\triangle|^2} < 0$ respectively with $\triangle \neq 0$. Futhermore, Besides the gapped superfluid(SF), there are two different regions in the uniform superfluid phase which are GSF(I) and GSF(II). These two distinct regions correspond to the gapless superfluid

with different momentum space topology of quasiparticle excitation energy spectrum. Depending of the number of zeros of $E_{\mathbf{k},2}$, we distinguish the 2

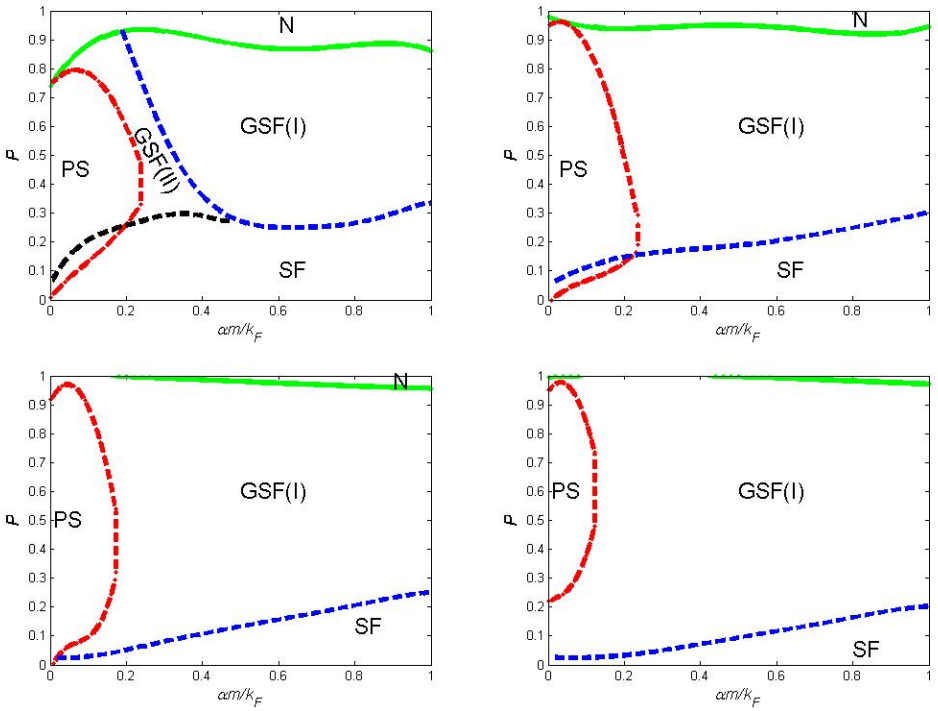

Figure 4: Ground state phase diagrams of the system as a function of $P = \frac{N_\uparrow - N_\downarrow}{N}$ and α, where $1/k_f a_s$ set to the 0, 0.5, 1 and 1.5 respectively

distinct GSF phases: GSF(I) where $E_{\mathbf{k},2}$ has 2 zeros, and GSF(II) where $E_{\mathbf{k},2}$ has 4 zeros in the momentum space spectrum shown in Figure 3. Trivial superfluid phase(SF) corresponds to the phase where both $E_{\mathbf{k},1}$ and $E_{\mathbf{k},2}$ are gapped. In the phase diagram, we see that SOC stabilizes the GSF phase against normal phase N at low population-imbalanced P, and destabilizes the GSF against normal phase N at high P. Competition between population-imbalanced and SOC stabilize the GSF and SF against phase separation(PS). Therefore, at any P, system will be reach GSF or SF phase if we increase the SOC parameter α. Furthermore, SOC stabilizes the GSF phase around unitarity although system is unstable against phase separation

without SOC effect $\alpha \to 0$. Therefore, GSF phase is stable for large parameter region, and this may allows us to realize the GSF phase with coldatoms in the experiments. There is a phase transition going from GSF(I) to GSF(II).

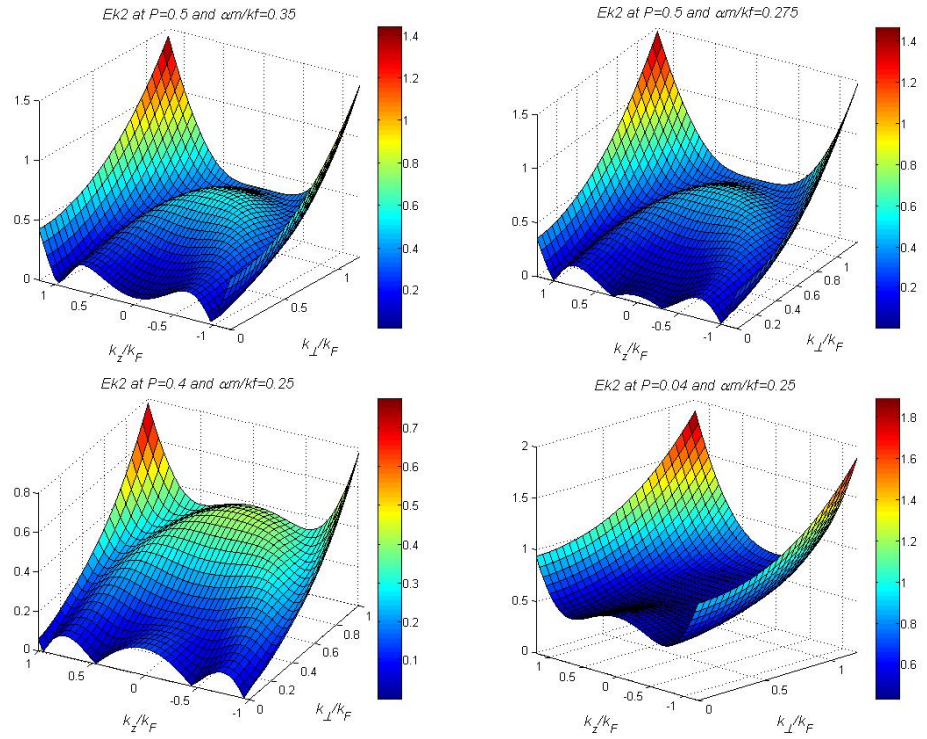

Figure 5: Quasi-particle excitation spectrum of Rashba type SOC for GSF(I), GSF(II), and SF phase.

We can see the signature of the transition from the sharp difference of the momentum distributions which is shown in the figure 4.

$$N_\uparrow = \sum_{\mathbf{k}} n_{\mathbf{k},\uparrow} \qquad \text{and} \qquad N_\downarrow = \sum_{\mathbf{k}} n_{\mathbf{k},\downarrow} \tag{106}$$

We see that major chages occur for $n_{\mathbf{k},\downarrow}$ at the GSF(I) and GSF(II) transition boundary, where the sharp peak disappear suddenly when the transition happens.

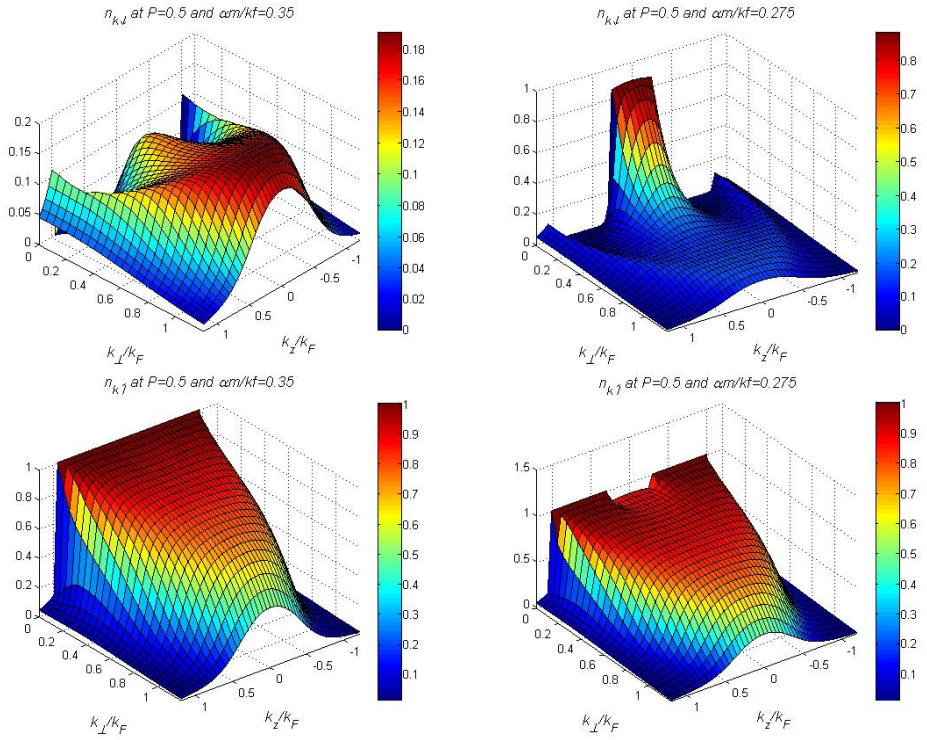

Figure 6: Momentum Distribution of the system for GSF(I), and GSF(II).

2.6 System in Optical Lattice

Ultracold gases in an optical lattice have attracted much research interest recently. Physicists believe that ultracold fermionic gases confined into the optical lattice with population imbalance and tunable interaction might provide a significant model to understand more about the High-Tc superconductivity. Therefore, besides solving system in vacuum that we did it in section 4.3, we also want to solve the system in optical lattice. In the BCS limit, lattice and continuum model describe the same physics since size of Cooper pairs is much larger than the lattice spacing. As we increase the interaction, and pair size becomes comparable to the lattice spacing, then difference starts between these two models. In the BEC limit, there can only be single Bose molecule on any lattice site with high binding energy. They become heavier with increasing the interaction, and critical temperature for BEC drops to the zero whereas critical temperature for BEC is finite in continuum case. We consider a polarized Fermi gases with SOC interaction in 2D square lattice. Hopping contribution of the Hamiltonian may be written as

$$\hat{H}_{KIN+SO} = -t\sum_{\mathbf{i}} \begin{pmatrix} C^{\dagger}_{\mathbf{i}+\hat{x}\uparrow} & C^{\dagger}_{\mathbf{i}+\hat{x}\downarrow} \end{pmatrix} \begin{pmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{pmatrix} \begin{pmatrix} C_{\mathbf{i}\uparrow} \\ C_{\mathbf{i}\downarrow} \end{pmatrix} - t\sum_{\mathbf{i}} \begin{pmatrix} C^{\dagger}_{\mathbf{i}+\hat{y}\uparrow} & C^{\dagger}_{\mathbf{i}+\hat{y}\downarrow} \end{pmatrix} \begin{pmatrix} cos\beta & isin\beta \\ isin\beta & cos\beta \end{pmatrix} \begin{pmatrix} C_{\mathbf{i}\uparrow} \\ C_{\mathbf{i}\downarrow} \end{pmatrix} + h.c$$

where summation $\mathbf{i}$ counts the all lattice sites, and $C^{\dagger}_{\mathbf{i}\uparrow}$,$C_{\mathbf{i}\uparrow}$ creates and annihilates the fermions on lattice site $\mathbf{i}$ respectively [20]. Complete Hamiltonian can be obtain with including the interaction term. After we treat the interaction term using mean-field approximation same as in section 4.2, complete

Hamiltonian will be

$$\hat{H} = \hat{H}_{KIN+SO} + \sum_{\mathbf{i}} \triangle \left[C^{\dagger}_{\mathbf{i}\uparrow} C^{\dagger}_{\mathbf{i}\downarrow} + C_{\mathbf{i}\downarrow} C_{\mathbf{i}\uparrow} \right] - \sum_{\mathbf{i}} \left[(\mu + h)\, C^{\dagger}_{\mathbf{i}\uparrow} C_{\mathbf{i}\uparrow} + (\mu - h)\, C^{\dagger}_{\mathbf{i}\downarrow} C_{\mathbf{i}\downarrow} \right] \tag{107}$$

where $\triangle = g < C_{\mathbf{i}\downarrow} C_{\mathbf{i}\uparrow} >$ is the BSC pairing energy which we take it real for simplicity, and $g > 0$ is the attractive interaction between fermions and constant term in Hamiltonian is neglected. Pairing energy $\triangle$ is not an external parameter, we have to find it self-consistently same as before. h is the population imbalance term, $(\mu_{\uparrow} - \mu_{\downarrow})/2$. Hamiltonian above is written in the real space. Now, we want to write the Hamiltonian in momentum space. In order to do that, we need to write creation, and annihilation operators in terms of their Fourier transforms as

$$C^{\dagger}_{\mathbf{i}\uparrow} = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{k}} e^{-i\mathbf{k}.\mathbf{i}} C^{\dagger}_{\mathbf{k}\uparrow} \qquad \text{and} \qquad C^{\dagger}_{\mathbf{i}\downarrow} = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{k}'} e^{i\mathbf{k}.'\mathbf{i}} C^{\dagger}_{\mathbf{k}\downarrow}$$

We put these transformations into the Hamiltonian above, and use the relation $\sum_{\mathbf{i}} e^{-i(\mathbf{k}-\mathbf{k}').\mathbf{i}} = N_s \delta_{\mathbf{k},\mathbf{k}'}$, where N_s is the total number of sites in 2D lattice. Then, we can write the Hamiltonian in momentum space, and it can also be written in the matrix form in the basis $\Psi^{\dagger}_{\mathbf{k}} = \left(\hat{C}^{\dagger}_{\mathbf{k},\uparrow} \hat{C}^{\dagger}_{\mathbf{k},\downarrow} \hat{C}_{-\mathbf{k},\uparrow} \hat{C}_{-\mathbf{k},\downarrow} \right)$ same as before. Hamiltonian will be

$$\hat{H} = \frac{1}{2} \sum_{\mathbf{k}} \Psi^{\dagger}_{\mathbf{k}} H_{\mathbf{k}} \Psi_{\mathbf{k}} + \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} + N \frac{|\triangle|^2}{g} \tag{108}$$

where $H_{\mathbf{k}}$ 4 by 4 matrix that is

$$H_{\mathbf{k}} = \begin{pmatrix} \epsilon_{\mathbf{k}} - h & S^*_{\mathbf{k}} & 0 & \triangle \\ S_{\mathbf{k}} & \epsilon_{\mathbf{k}} + h & -\triangle & 0 \\ 0 & -\triangle & -(\epsilon_{\mathbf{k}} - h) & S_{\mathbf{k}} \\ \triangle & 0 & S^*_{\mathbf{k}} & -(\epsilon_{\mathbf{k}} + h) \end{pmatrix}$$

The dispersion relation and SOC term read

$$\epsilon_{\mathbf{k}} = -2t\left(\cos\alpha\cos k_x + \cos\beta\cos k_y\right) - \mu$$
$$S_{\mathbf{k}} = 2t\left(\sin\beta\sin k_y + i\sin\alpha\sin k_x\right)$$

Now, we need to diagonalize the Hamiltonian , and write it in terms of new quasi-particle/quasi-hole creation and annihilation operators $(\gamma^\dagger_{\mathbf{k},\uparrow}, \gamma^\dagger_{\mathbf{k},\downarrow}, \gamma_{-\mathbf{k},\uparrow}, \gamma_{-\mathbf{k},\downarrow})$ using the Bogoliubav and Valatin transformation which allows us to diagonalize the matrix $H_{\mathbf{k}}$ with preseving the eigenvalues same as before. Hamiltonian of the system will be

$$\hat{H} = \frac{1}{2}\sum_{\mathbf{k},\lambda} E_{\mathbf{k},\lambda}\gamma^\dagger_{\mathbf{k},\lambda}\gamma_{\mathbf{k},\lambda} + \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} + N\frac{|\triangle|^2}{g} \tag{109}$$

where $E_{\mathbf{k},\lambda}$ are the eigenvalues of Hamiltonian matrix $H_{\mathbf{k}}$. Then, we can obtain the partition function of the system with using the relation $Q = Tr\left(e^{-\beta\hat{H}}\right)$ that is

$$Q = e^{-\beta\sum_{\mathbf{k}}\epsilon_{\mathbf{k}} - \beta N\frac{|\triangle|^2}{g}} \prod_{\mathbf{k},\lambda}\left(1 + e^{-\beta E_{\mathbf{k},\lambda}/2}\right) \tag{110}$$

Thermodynamic potential of the system can be found from the Q via the relation $\Omega = -T\ln(Q)$ which is

$$\Omega = N\frac{|\triangle|^2}{g} + \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} + \frac{T}{2}\sum_{\mathbf{k},\lambda} \ln\left(\frac{1+\tanh\left(\frac{E_{\mathbf{k},\lambda}}{2T}\right)}{2}\right) \tag{111}$$

Eigenvalues of the Hamiltonian matrix $H_{\mathbf{k}}$ can be found from computer program Matematica.

$$E_{\mathbf{k},\lambda} = s_\lambda \sqrt{\epsilon_{\mathbf{k}}^2 + h^2 + |\triangle|^2 + |S_{\mathbf{k}}|^2 + 2p_\lambda A_{\mathbf{k}}} \tag{112}$$

where $A_{\mathbf{k}} = \sqrt{h^2\left(\epsilon_{\mathbf{k}}^2 + |\triangle|^2\right) + |S_{\mathbf{k}}|^2 \epsilon_{\mathbf{k}}^2}$, $s_1 = s_2 = +$, and $s_3 = s_4 = -$. Therefore, $E_{\mathbf{k},3} = -E_{\mathbf{k},1}$, and $E_{\mathbf{k},4} = -E_{\mathbf{k},2}$. Condition for zeros of $E_{\mathbf{k},2}$ is given by

$$\epsilon_{\mathbf{k}}^2 + h^2 + |\triangle|^2 + |S_{\mathbf{k}}|^2 = 2A_{\mathbf{k}} \tag{113}$$

This yields 2 conditions which are

$$|S_{\mathbf{k}}| = 0 \qquad \text{and} \qquad \epsilon_{\mathbf{k}}^2 - h^2 + |\triangle|^2 = 0 \tag{114}$$

For $\alpha = \beta = \pi/4$, $|S_{\mathbf{k}}| = 2t\sqrt{\sin^2 k_x + \sin^2 k_y} = 0$, so $k_x = k_y = n\pi$, and $-\sqrt{2}t = \mu_+ \pm \sqrt{\mu_-^2 - |\triangle|^2}$ are the condition for zeros of $E_{k,3}$. We obtain the number and gap equations using the relations $N_\uparrow + sN_\downarrow = \frac{\partial\Omega}{\partial\mu_s}$, and

$\frac{\partial\Omega}{\partial\triangle} = 0$ which yields following results.

$$\frac{N}{g} = \frac{1}{4}\sum_{\mathbf{k}}\left[\frac{\left(1+h^2/A_{\mathbf{k}}\right)}{E_{\mathbf{k},1}}\tanh\left(\frac{E_{\mathbf{k},1}}{2T}\right) + \frac{\left(1-h^2/A_{\mathbf{k}}\right)}{E_{\mathbf{k},2}}\tanh\left(\frac{E_{\mathbf{k},2}}{2T}\right)\right] \tag{115}$$

$$N_\uparrow - N_\downarrow = \frac{1}{2}\sum_{\mathbf{k}}\frac{h}{E_{\mathbf{k},1}}\left(1+\frac{\left(\epsilon_{\mathbf{k}}^2+|\triangle|^2\right)}{A_{\mathbf{k}}}\right)\tanh\left(\frac{E_{\mathbf{k},1}}{2T}\right) \tag{116}$$
$$+\frac{1}{2}\sum_{\mathbf{k}}\frac{h}{E_{\mathbf{k},2}}\left(1-\frac{\left(\epsilon_{\mathbf{k}}^2+|\triangle|^2\right)}{A_{\mathbf{k}}}\right)\tanh\left(\frac{E_{\mathbf{k},2}}{2T}\right)$$

$$N_\uparrow + N_\downarrow = \frac{1}{2}\sum_{\mathbf{k}}\left[2-\frac{\epsilon_{\mathbf{k}}}{E_{\mathbf{k},1}}\left(1+\frac{\left(h^2+|S_{\mathbf{k}}|^2\right)}{A_{\mathbf{k}}}\right)\tanh\left(\frac{E_{\mathbf{k},1}}{2T}\right)\right] \tag{117}$$
$$-\frac{1}{2}\sum_{\mathbf{k}}\left[\frac{\epsilon_{\mathbf{k}}}{E_{\mathbf{k},2}}\left(1-\frac{\left(h^2+|S_{\mathbf{k}}|^2\right)}{A_{\mathbf{k}}}\right)\tanh\left(\frac{E_{\mathbf{k},2}}{2T}\right)\right]$$

In order to solve the 3 coupled equations numerically, we have to scale them. We choose the energy scale as the hopping energy t. Scaled excitation energies will be $E_{\mathbf{k},s} = t\widetilde{E_{k,s}}$, and $A_k = t^2\widetilde{A_k}$ where

$$\widetilde{E_{k,s}} = \sqrt{\widetilde{\epsilon_k}^2 + \left(\frac{\widetilde{\mu_\downarrow}-\widetilde{\mu_\uparrow}}{2}\right)^2 + \left|\widetilde{\triangle}\right|^2 + \widetilde{|S_k|}^2 + s2\widetilde{A_k}}$$
$$\widetilde{A_k} = \sqrt{\left(\frac{\widetilde{\mu_\downarrow}-\widetilde{\mu_\uparrow}}{2}\right)^2\left(\widetilde{\epsilon_k}^2+|\triangle|^2\right) + \widetilde{\epsilon_k}^2\widetilde{|S_k|}^2}$$

where the term $\widetilde{\epsilon_k} = -\left(2\cos\alpha\cos k_x + 2\cos\beta\cos k_y + \frac{(\widetilde{\mu_\uparrow}+\widetilde{\mu_\downarrow})}{2}\right)$, $\widetilde{S_k} = 2\left(\sin\beta\sin k_y + i\sin\alpha\sin k_x\right)$, and $E_{\mathbf{k},+(-)} = E_{\mathbf{k},1(2)}$. When we write the summation $\sum_{\mathbf{k}}$ in terms of integral, we have to be careful now. There is no cylindrical symmetry in the problem, so we need to use cartesian coordinates different from the case in section 4.3, $\sum_{\mathbf{k}} = N\int\int\frac{dk_x dk_y}{(2\pi)^2}$. We should take

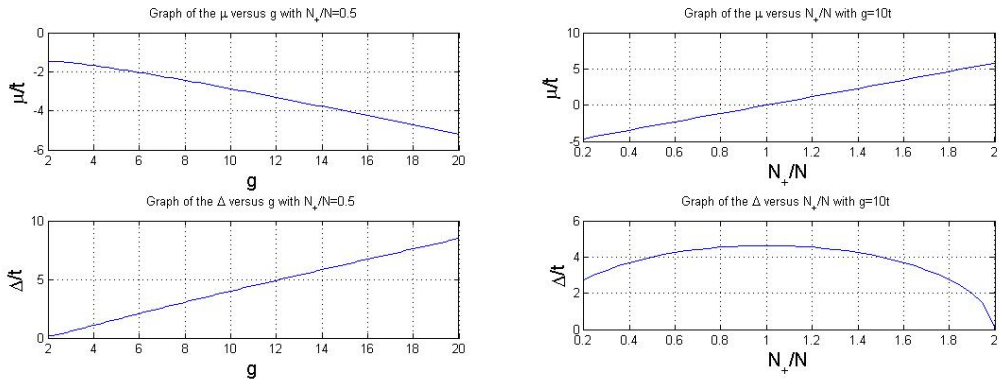

Figure 7: Superfluid gap and chemical potential for balanced case in an 2D optical lattice.

the integral in the first Brilliouin zone, a uniquely defined primitive cell in the momentum space. Therefore, our scaled equations will be

$$\begin{aligned}
\frac{16\pi^2}{\widetilde{g}} &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} dk_x dk_y f_1(k_x, k_y, \widetilde{\mu_\uparrow}, \widetilde{\mu_\downarrow}, \widetilde{\triangle}) \\
\frac{8\pi^2\left(N_\uparrow - N_\downarrow\right)}{N} &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} dk_x dk_y f_2(k_x, k_y, \widetilde{\mu_\uparrow}, \widetilde{\mu_\downarrow}, \widetilde{\triangle}) \\
\frac{8\pi^2 N_+}{N} &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} dk_x dk_y f_3(k_x, k_y, \widetilde{\mu_\uparrow}, \widetilde{\mu_\downarrow}, \widetilde{\triangle})
\end{aligned} \tag{118}$$

where f_1, f_2, and f_3 are functions of 5 variables, and $N_+ = N_\uparrow + N_\downarrow$. These functions are scaled functions from the number and gap equations where $\widetilde{T} = \frac{T}{t}, \widetilde{\mu}_{\uparrow,\downarrow} = \frac{\mu_{\uparrow,\downarrow}}{t}$, and $\widetilde{\triangle} = \frac{\triangle}{t}$. We can obtain any physical quantities from solving these 3 coupled equations self-consistently. We will find the superfluid gap $\triangle$, and chemical potential μ for balanced case($\mu_\uparrow = \mu_\downarrow$) in the ground state $T = 0$ shown in the figure 7. Now, we construct the ground state phase diagrams of the system. Firstly, we look at the phase diagrams of population imbalance P versus interaction g at $T = 0$, $F = \frac{N_\uparrow + N_\downarrow}{N} = 1$, and $\alpha = \beta = \pi/4, \pi/6$.

In these phase diagrams, there are 3 phases which are normal phase is

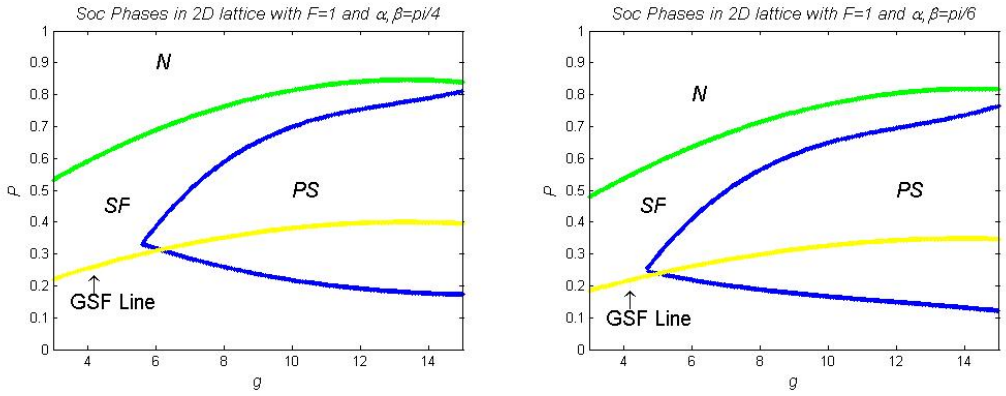

Figure 8: Ground state phase diagrams of the system as a function of P and g in 2D optical lattice, where $\alpha = \beta$ set to the $\pi/4$ and $\pi/6$ respectively.

charecterized by $\triangle = 0$, uniform and non-uniform superfluid(Phase Separation) as before. Uniform, and non-uniform superfluid phases are described by $\frac{\partial^2\Omega}{\partial|\triangle|^2} > 0$, and $\frac{\partial^2\Omega}{\partial|\triangle|^2} < 0$ respectively with $\triangle \neq 0$. There are 2 regions in uniform superfluid phase that are gapped and gapless superfluid phases. In the gapless superfluid region, excitation energy of the system vanishes at the some points in momentum space shown in the figure 10. In 2 dimension, gapless superfluid phase does not occupy the region, and it occurs just on the line shown in the figure above. We see that if we increase the SOC interaction, system become superfluid at the larger parameter region, and SOC also stabilizes the superluid phase allowing the realization of these phases in the experiments in the future. Secondly, we look at the phase diagrams of population imbalance P versus SOC interaction α, β at $T = 0$, $F = \frac{N_\uparrow + N_\downarrow}{N} = 1$, and $g = 6, 12t$.

As we increase the interaction g, we see from the phase diagram above that Cooper pairing starts to form for large parameter region and creates superfluid phase, and increasing interaction also stabilizes the both gapped and gapless superfluid phase. However, we know that stability condition correctly discards the unstable solutions, but metastable solutions may still

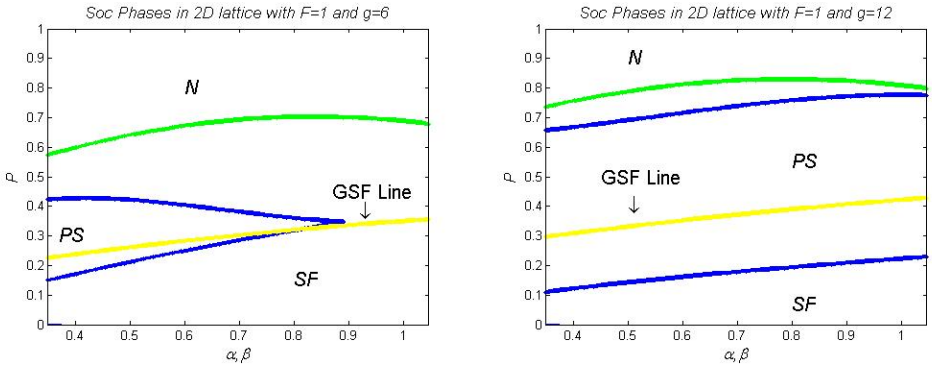

Figure 9: Ground state phase diagrams of the system as a function of P and α, β in 2D optical lattice, where g set to the $6t$ and $12t$ respectively.

survive in the phase diagrams, phase diagrams have minor quantitative errors in the phase diagrams boundaries especially in 2 dimension.

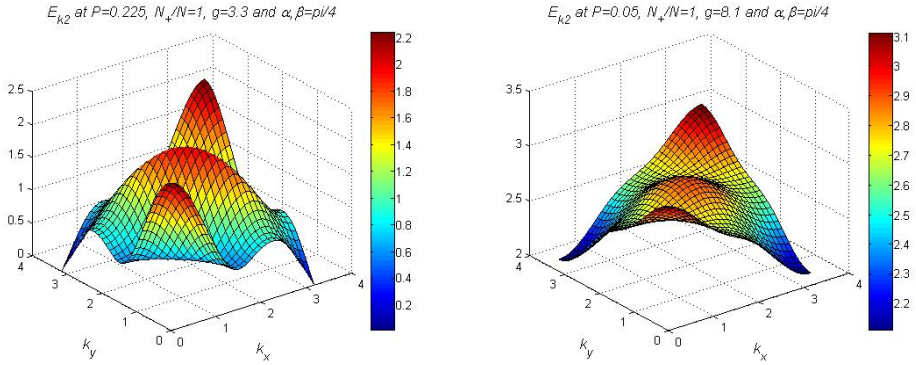

Figure 10: Excitation spectrum of the system in gapless and gapped superfluid phases in 2D optical lattice respectively.

3 Conclusion and Outlook

3.1 Conclusion

In this thesis, we focused on the fermionic superfluidity that can be seen in superconductors, neutron stars, nuclear matter, and ultracold atoms vs. We first studied that fermions with opposite spins and momenta on top of the non-interacting Fermi level can form a bound state if there is an arbitrarily attractive interaction between them [6]. Bound fermions are called Cooper pairs that can condense to the ground state like bosons. In superconductors, crystal vibration (phonon) causes the induced attractive interaction between electrons [5]. Then, we studied the BCS theory that proposes variational wavefunction describing the Cooper pairs, loosely bound and largely overlapping, at ground state, and this theory is successful to explain critical temperatures of the some superconductors. However, it fails to explain the high-Tc superconductors, and ultracold fermionic systems having small Cooper pairs. Therefore, we generalized BCS theory to analyze the strong interaction regime. If we increase the interaction between fermions, we can smoothly pass from BCS limit of loosely bound and largely overlapping Cooper pairs to the BEC limit of tightly bound Bose molecules. There is no phase transition in this process, so it is called BCS-BEC crossover for balanced case, where both components have equal number and mass. However, for imbalanced case, Fermi levels of the fermionic species are distinct, and we need stronger interaction to bound fermions. Therefore, system is normal Fermi gas in the BCS limit. As the attractive interaction increases, distinct fermion species gain enough energy to pair with each other but repel the excess atoms leading the phase separation between superfluid paired fermions and unpaired, excess fermions. As interaction is increased further,

the pairs become tightly bound and repel excess fermions less, so system is mixed with superfluid and excess fermions. Therefore, there are 3 phases for imbalanced case which are normal phase, phase separation and superfluid phase.

Physicists study more complicated systems to find new phases of matter, and spin-orbit coupled (SOC) atomic Fermi gas is one of the most attracted system after it was recently achieved first with bosonic than fermionic systems [17, 18]. We constructed phase diagrams of the system with SOC, and we found that SOC enriches the ground state phases. Besides normal phase (N), phase separation (PS), and superfluid phase (SF), there are two new gapless superfluid phase, GSF(I) and GSF(II). Moreover, SOC stabilizes the GSF phase against normal phase N at low population-imbalanced P, and destabilizes the GSF against normal phase N at high P. Competition between population-imbalanced and SOC stabilize the GSF and SF against phase separation (PS). Therefore, at any P, system will reach GSF or SF phase if we increase the SOC parameter α. SOC stabilizes the GSF phase around unitarity although system is unstable against phase separation without SOC effect $\alpha \to 0$. Therefore, GSF phase is stable for large parameter region, and this may allows us to realize the GSF phase with coldatoms in the experiments. These new phases might be seen also in the laboratory in the future.

In the last part, we solve the system in 2D optical lattice, and construct the ground state phase diagrams. We also found stable new phase in the ground state with SOC effect. However, this phase can be seen on the line in 2 dimension whereas it occupy the region in 3 dimension in the ground state phase diagrams. Increasing SOC effect stabilizes the phases, so it allows to the realization of these phases in the experiments.

3.2 Outlook

It is significant to extend the analysis given in the thesis. First, we need to solve the system in 3D optical lattice and construct the phase diagrams of the systems which are more reliable. We can also solve the system for finite momentum pairs that we check the stability of the phases at finite momentum pairing rather than Cooper pairs at rest. Furthermore, rather than solving uniform system, we can also add a harmonic trap, and investigate the effect of the trap on the system.

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