Are Search Equilibria Competitive?

by

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Abstract

This study analyzes the equilibrium of a search-and-bargaining model with heterogenous goods and agents. The model allows for both explicit and implicit search costs as frictions defining the economy. We investigate the relationship between the steady-state search equilibrium of this economy as frictions become negligible and the (competitive) equilibrium of the frictionless counterpart of this economy. That the search-and-bargaining equilibria converge to the competitive one is well-established. Here, we present the converse of this analysis. We show that, under the assumption of strict supermodularity of the surplus function, for each competitive equilibrium, one can find a search-and-bargaining equilibrium that approximates the competitive equilibrium when frictions are small.

Keywords: Search and Bargaining Markets, Matching Markets, Competitive Equilibrium, Strategic Foundations.

Özet

Bu çalışmada, ayrışık mal ve ajanlarla tanımlı bir arama-ve-pazarlık piyasası modelinin dengesi analiz edilmektedir. Model, piyasayı etkileyen sürtünmeler olarak hem açık hem de örtülü maliyetlerin varlığına izin vermektedir. Çalışmada, sürtünmeler göz ardı edilebilir duruma gelirken bu modelin kararlı hâl (arama-ve-pazarlık) dengesi ile modelin sürtünmesiz karşılığının (rekabetçi) dengesi arasındaki ilişki araştırılmaktadır. Sürtünmeler kayboldukça arama-ve-pazarlık dengelerinin rekabetçi dengelere yakınsadığı zaten bilinmektedir. Bu çalışma ise söz konusu durumun çevriğini incelemektedir. Sonuçlar, kâr fonksiyonunun katı süpermodüleritesi varsayımı altında, her bir rekabetçi denge için, sürtünmeler yok oldukça söz konusu dengeye yakınsayan bir arama-ve-pazarlık dengesi bulunabileceğini göstermektedir.

Anahtar Sözcükler: Arama ve Pazarlık Piyasaları, Eşleşme Piyasaları, Rekabetçi Denge, Stratejik Temeller.

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Contents

Chapter 1

Introduction

The basic assumption of "competitive equilibrium" has traditionally been considered as central to the analysis of economic interactions, since the times of Adam Smith and León Walras. The competitive equilibrium framework is familiar, well-established and easy to deal with; moreover, it has its own attractive normative features. There is still one basic shortcoming of this framework, however, which is that this particular framework requires some strong assumptions in terms of the structure of the markets. Formally, it assumes that the markets are "efficient", or, to put it in a more appropriate way, "frictionless": for the Walrasian Equilibrium to hold, the agents must have complete information, there should be no external costs involved, and the agents must avoid strategic behavior. Nevertheless, markets are hardly frictionless in real life. The agents always have some asymmetric information, there are search costs involved, and the agents inevitably take part in some strategic behavior. It is then an interesting, and obviously necessary, question to ask whether the frictionless markets assumption is an adequate approximation for the real-life phenomenon that we observe every day.

To put this question in more concrete terms, consider the following example: assume a realistic model of a market place, where everyday transactions occur between buyers and sellers. Every day, buyers with certain valuations, as well as sellers with reservation prices, enter the market to meet a partner on the other side of the market and exercise a trade. The agents who meet a partner and agree upon the terms of the trade conduct the transfer leave the marketplace immediately. The question is: how would the equilibrium be realized in this market? Obviously, it is quite tempting to analyze this equilibrium in a Walrasian framework. Note, however, that the market is not necessarily frictionless. In real life, search is far from costless: there are implicit costs associated with finding the "perfect" match, as well as the costs arising due to the impatience of agents (i.e. the earlier an agent exercises a particular trade, the better off she is). Once these participation costs and impatience of agents are taken into consideration, however, the model in the hand is no longer Walrasian. On the other hand, when we assume that the costs and impatience are exactly zero, the behavior of the market in hand is well-known. The interesting question, therefore, is what happens when these frictions are arbitrarily small.

In this study, our aim is to inspect the relationship between equilibria prevailing in the marketplace with frictions ("search equilibria" hereafter) and those prevailing in frictionless environments ("competitive equilibria" hereafter).[1] The ultimate purpose of this study is two-fold: we analyze search equilibria as frictions become arbitrarily small, that is, we explore whether they converge to some competitive equilibria. Note that this sort of analysis has been extensively studied under several setups beforehand, and researchers have come up with different answers under different setups (see Literature Review). Yet the converse still remains as an open question. Hence, in the second part of this study, we address the question of whether one can always approximate a given competitive equilibrium with some search equilibria. Our answer is positive, under some restrictive assumptions: We show that, under the assumption of strict supermodularity of the surplus function, for each competitive equilibrium, one can find a search-and-bargaining equilibrium that approximates the competitive equilibrium when frictions are small. This, to the best

[1]The difference between the analysis of what happens when costs are zero, and what happens when they are close to, but not equal to, zero, can be distinguished by appreciating the difference between *limit theorems* and *theorems in the limit*, as in Gale (2000).

of our knowledge, is the first study that analyzes the converse question, and provides an essential step in the analysis of the strategic foundations of Walrasian core.

In addition to the intellectual curiosity that this particular study addresses, an additional significant explanation of why it is an interesting question can be constructed, based on the generically weak connections between the micro-level studies, which take strategic interactions into consideration, and the macro-level ones, which operate on the aggregate level, assuming no strategic interactions. In the macro-level models, the assumption of frictionless markets and competitive equilibrium are, in general, taken as primitives, and the analysis is conducted based on this primitive. It is therefore worthwhile to ask whether this basic assumption is sufficiently realistic, or, in other words, whether the competitive market taken as primitive can be approximated by a more realistic setup. We believe that this study will help economists provide a confident answer to this question, hence contributing to the connection between micro- and macro- level analysis.

Chapter 2

Literature Review

The rigorous analysis of the concept of the Walrasian core, in terms of its micro foundations and its relation with matching markets, started with the classic study by Shapley and Shubik (1972). In their framework, Shapley and Shubik take the point of view of a central planner who is attempting to solve a simple linear optimization problem in order to maximize the efficiency of matches formed in a market. Their results indicate that the *decentralized* equilibrium of competitive markets can be explained via the tools primarily generated for the study of *centralized* economies, thus establishing the bridge between two, apparently distant, areas of economical analysis. Note, however, that the introduction of decentralization into the model generates some sort of a pairwise interaction between the agents, thus a more assiduous analysis of what goes in these interactions, and what their implications on the macroeconomic variables of the economy would be, is needed. Earlier attempts on this aspect include the work by Mortensen (1979), Diamond (1981) and Diamond (1982). What these studies have in common is that their results suggest that the outcome of a decentralized market, where agents meet at random and conduct trade upon agreement, may fail to be Walrasian when the frictions vanish. On the other hand, the bargaining protocol these studies assume is the Nash Bargaining Solution, hence these earlier studies lack the strategic aspect, which is essential for the central concepts on the notion of friction. Rubinstein and Wolinsky (1985)'s classic study, in which they adopted Rubinstein's *strategic* bargaining solution proposed in Rubinstein (1982), fills this gap in the literature while still preserving the underlying flavor of earlier studies.

Rubinstein and Wolinsky's model is as follows: they assume a marketplace in which sellers of reservation price zero, and buyers of valuation one meet. Each period, agents meet randomly and start bargaining, as they do over how to share a surplus of size one using Rubinstein's bargaining model. In addition to agents' patience levels $\delta < 1$, one other friction which is at the core of the analysis is the exogenous matching probability of agents. Namely, each period, a seller (buyer) faces the (independent) probability $\alpha(\beta) \in (0, 1]$ of being matched with another buyer (seller), hence there is pressure on a matched buyer-seller pair in the sense that the bargaining will be carried into next period only with probability $(1-\alpha)(1-\beta)$. Rubinstein and Wolinsky establish that there is a unique, quasi-stationary, perfect equilibrium. The equilibrium price, as the impatience of agents disappear (i.e $\delta \to 1$), converge to a value that is a function of α and β. One could also construct an alternative, and more intuitive, setup in which α and β are determined by the number of agents present in the market, denoted by N_b and N_s, for buyers and sellers, respectively. Then, Rubinstein and Wolinsky argue, the price that prevails in the market, as $\delta \to 1$, is a function of $\frac{N_b}{N_s}$. [1] Realize that the results, elegantly established by Rubinstein and Wolinsky, constitute a solid critique on the competitive paradigm, because a Walrasian framework would simply predict that if $N_b > N_s$, the equilibrium price would be 1, and if $N_b < N_s$ it would be 0, regardless of the value of $\frac{N_b}{N_s}$. On the contrary, Rubinstein and Wolinsky demonstrate that the equilibrium price in a market, determined as a result of strategic interactions between buyers and sellers, can be non-Walrasian under an appropriate setup.

The most substantial response to Rubinstein and Wolinsky's approach, in favor of

[1] In particular, it is given by $\frac{N_b}{N_b+N_s}$.

the Walrasian paradigm, is proposed by Gale (1987). In his seminal work, Gale constructs a setup in which heterogenous buyers and sellers enter the market each period, and the matched buyer-seller pairs play an ultimatum bargaining game. Gale demonstrates that, as opposed to Rubinstein and Wolinsky (1985), the equilibrium price prevailing in the market converges to the Walrasian price as the impatience of agents disappear. Furthermore, Gale asserts that the apparent contradiction between his model and Rubinstein and Wolinsky (1985) is due to the notion of *equilibrium* adopted. He argues that the equilibrium concept used in Rubinstein and Wolinsky (1985) is a *stock equilibrium*, in which the market-clearing condition with respect to the agents currently present in the market is considered. This equilibrium concept, however, becomes increasingly less meaningful as the market size tends to infinity, because the market-clearing condition with respect to an infinite measure of agents is cumbersome and less than intuitive. Hence, Gale suggests, the proper equilibrium concept that should be adopted is *flow equilibrium*, in which the market-clearing condition with respect to the new entrants of the market is used.

Recent contributions to the literature include Mortensen and Wright (2002), which generalizes Gale's results for a broader set of matching technologies and bargaining rules. Satterthwaite and Shneyerov (2007) extends the framework by incorporating one-sided incomplete information about agents' types. Lauermann (2013) proposes a general characterization result for the convergence of equilibria of a dynamic matching-and-bargaining market to the Walrasian equilibrium, in which he replaces the *Pairwise Efficiency* condition of Shapley and Shubik (1972) by two conditions, *Weak Pairwise Efficiency* and *Weak Incentive Compatibility*, which are easier to analyze and can be used to investigate which conditions are violated in the models that fail to yield a Walrasian equilibrium in the limit. Manea (2011) considers the case for a generic class of matching and bargaining technologies, and shows the existence of an equilibrium, as well characterizing some sufficient conditions for the existence of a steady-state.

The model adopted in our study is most closely linked to Atakan (2010)'s model, in which the market setup does not only allow for heterogeneity of agents, but also heterogeneity of goods, as well as incomplete information. In particular, we do also allow for heterogeneity of goods and agents, and in addition to the explicit costs apparent in Atakan (2010)'s model, we assume that the agents are not perfectly patient, and hence add another dimension of friction into the model. This additional source of frictions turn out to be extremely crucial for the model, because the relative patience parameters of agents is the primary component of the model that explains the share of any surplus in any match, and thus any competitive equilibrium. In particular, we show that two agents share the surplus with respect to their relative patience parameters: in a setup where the surplus to be shared is 1, and where the buyer and the seller discount time with the instantaneous discount rates r_b and r_s, respectively, the buyer's share converges to $\frac{r_s}{r_b+r_s}$ as frictions become arbitrarily small. Realize that this is very much reminiscent of the result predicted in Rubinstein (1982), where, if a surplus of 1 to be shared, player 1 and 2's discount factors are δ_1 and δ_2, respectively, then player 1's share in the subgame perfect equilibrium whereas the frictions are becoming negligible emerges as:

$$\lim_{\Delta \to 0} x^*(\Delta) = \frac{\log \delta_2}{\log \delta_1 + \log \delta_2}$$

Therefore, another way to consider the findings of this study as an extension Rubinstein (1982)'s results into a search-and-bargaining model. Rubinstein (1982) demonstrates that, when two players are to bargain over a cake of unit size, the relative patience parameters determine the end agreement. Here, we show that this result is enlightening in terms of analyzing the equilibrium of a search-and-bargaining economy as well. The setup we construct will have the property of attaining a unique matching for sufficiently low frictions, that is, the trading partner of each agent will be unique and known in advance. What is not known in advance, however, is how the share of surplus is realized; in other words, we have a unique matching but a range of possible values in the equilibrium. Here, we show that, analogous to Rubinstein (1982), the choice of a particular valuation within the range can be achieved via the selection of relative patience parameters. In particular, we argue that, thanks to the unique matching, the search-and-bargaining equilibrium can be viewed as a collection of buyer and seller pairs bargaining (à la Rubinstein (1982)) over a certain surplus. As in Rubinstein (1982), the relative patiences determine what the end agreement is, and is sufficient to cover the whole range of equilibria.

Interested readers may consult Roth and Sotomayor (1992) for a detailed review of Shapley and Shubik (1972)'s model. Osborne and Rubinstein (1990) and Gale (2000) are distinguished examples of two excellent surveys in the literature of matching and bargaining markets.

Chapter 3

The Model

This study takes the following setup of marketplace as given: consider a well-defined set of buyers and sellers. The time is indexed by discrete indices, and every period, a certain measure of each type of buyers and sellers enter into the market, joining to the agents who are already in the market. Those who have once entered into the market start looking for a partner to exercise a trade, subject to some predefined matching technology.[1] Once a certain buyer and seller pair is formed, there is a certain surplus to be shared, which is a function of buyer and seller types. In order to share this available surplus, the buyer and the seller play an ultimatum bargaining game, where nature selects the proposer randomly. Those who agree upon the division of the surplus exercise the trade and leave the marketplace immediately, and those who fail to reach an agreement break their matches and return to the marketplace in order to meet new partners. There is complete information of agents, in the sense that both agents are able to observe their partner's types, as well as the distribution of all agents available in the market.[2]

[1]The most salient class of matching technology, which the one adopted in this study, is random matching, where the probability of matching a particular type of agent is proportional to the measure of that type currently in the market

[2]The requirement that agents can observe each other's histories is not necessary for our purposes, and observing the evolution of market distribution is also not required, since our analysis will mainly focus on the steady state.

Whereas the ultimatum bargaining game constitutes the strategic aspect of the model, we also need to impose some frictions as well. In particular, there are two dimensions of frictions which prevent this setup from being Walrasian. First, the agents need to pay some predetermined positive entry fee each period, in order to enter the market and take part in the search activity. One can justify this concept by imagining that this explicit costs represent either the implicit effort spent to find a better match in the market, or some other explicit costs that need to be paid such as transportation costs, etc. The other source of friction is the impatience of agents: each agent i discounts the gain obtained in a particular period t through multiplying it by a factor δ_i^t, where $\delta_i \in [0,1]$. Hence the parameters of frictions prevalent in the market can be fully characterized by the ordered pair $(c_i, \delta_i) \in \mathbb{R}_+ \times [0,1]$ for each agent i.

The "search equilibria" under consideration are those which arise in the steady-state, i.e. when the measures of agents in the market and agent strategies are independent of time. The equilibrium concept we are adopting is stationary subgame perfect equilibria, such that the strategy of each agent maximizes the discounted sum of payoffs after any possible path of play. We will be comparing this equilibria, which contains the steady-state measures and strategies of agents, to the "competitive equilibria" that arises in the competitive counterpart of this model, where $(c_i, \delta_i) = (0,1)$ for each agent i.

Now, let us move on with the formal treatment of the model.

3.1 Search-and-Bargaining Economy

3.1.1 The Search Economy

A search-and-bargaining economy (***search economy*** hereafter) consists of:

- A set of types of buyers B;

- A set of types of sellers S;
- A surplus function $f : B \times S \to \mathbb{R}^+$, which specifies the transferable utility generated when a certain buyer of type b and seller of type s are matched;
- A parameter $\beta \in (0,1)$ that specifies the probability that the buyer is assigned as proposer, given a match is formed (accordingly, the seller proposes with probability $(1-\beta) \in (0,1)$)[3];
- The time difference between two consecutive periods, denoted by $\Delta \geq 0$;
- The search costs $c_i = \Delta\kappa_i \geq 0$ for each $i \in B \cup S$, incurred by the buyers and sellers staying in the market each period;
- The patience parameters $\delta_i = e^{-r_i\Delta} \leq 1$ for each $i \in B \cup S$, specifying the rate at which the buyers and sellers discount their utilities.

Given $f = \{f_{bs}\}_{(b,s)\in B\times S}$, $\kappa = (\kappa_i)_{i\in B\cup S} \gg 0$ and $r = (r_i)_{i\in B\cup S} \geq 0$, a search economy is fully characterized by the tuple $\mathcal{S} = (B, S, f, \beta, \kappa, r, \Delta)$.

Remark 3.1. Realize that the two types of frictions in the market, the search costs and impatience, are both functions of Δ, and that $c_i \to 0$, $\delta_i \to 1$ as $\Delta \to 0$, i.e. taking Δ to 0 is sufficient to make these two types of frictions disappear. Hence, from now on, the phrases "as frictions disappear" and "as $\delta \to 0$" will be used interchangeably in this study, for these two statements are equivalent.

3.1.2 The Search Equilibrium

Population of agents and the random matching technology Each period, a unit measure of each type is born into the market. Suppose that the market is in steady state and let $l = \left(l_{b_1}, ..., l_{|B|}, l_{s_1}, ..., l_{|S|}\right) \in \mathbb{R}_+^{|B|+|S|}$ denote the steady state

[3]Alternatively, one can view β and $1-\beta$ as the bargaining powers of the buyer and the seller, respectively.

measure of buyers and sellers in the market. Based on this, we will now define some critical market statistics.

Let $L = \sum_{b \in B} l_b$ denote the total measure of buyers, $r = \sum_{s \in S} l_s / L$ denote the seller-to-buyer ratio, and $n(r) = \min\{1, r\}$ denote the market tightness. Also, let $p_b = l_b/L$ and $p_s = l_s/rL$ denote the frequency of type b buyers and type s sellers, respectively.

In such a market in steady state, the probability that a buyer finds a match is $n(r)$, and the symmetric probability for the seller is $n(r)/r$. The probabilities of each buyer b and seller s to be found, on the other hand, is proportional to their frequencies. Hence, the probability for any seller of meeting buyer b in a given period is given by $p_b n(r)/r$ and similarly the probability of meeting seller s is $p_s n(r)$. These measures are common knowledge to all agents.

Steady-state requirements For l to be part of any equilibrium, it should satisfy some steady-state requirements denoted below. Specifically, let m_{bs} denote the probability that b and s conduct the trade and leave the market, given that they are paired in a period and b proposes. Define m_{sb} symmetrically. For the market to remain in steady state, we need:

$$Lp_b\beta \sum\nolimits_S n(r)p_s m_{bs} + Lp_b(1-\beta)\sum\nolimits_S n(r)p_s m_{sb} \leq 1 \ \forall b \text{ and,}$$
$$Lrp_s\beta \sum\nolimits_B \frac{n(r)}{r} p_b m_{bs} + Lrp_s(1-\beta)\sum\nolimits_B \frac{n(r)}{r} p_b m_{sb} \leq 1 \ \forall s.$$

For the sake of simplicity, one can assume $m_{bs} = m_{sb}$.[4] Then the equations become:

$$Lp_b \sum\nolimits_S n(r)p_s m_{bs} \leq 1 \ \forall b \text{ and,} \tag{3.1}$$
$$Lrp_s \sum\nolimits_B \frac{n(r)}{r} p_b m_{bs} \leq 1 \ \forall s. \tag{3.2}$$

[4]The only case in which this symmetry assumption has bite is when the agents are indifferent between accepting each other. In this case, if $m_{bs} \neq m_{sb}$, then define a symmetric equilibrium with $\hat{m}_{bs} = \hat{m}_{sb} = \beta m_{bs} + (1-\beta)m_{sb}$.

These equations state that the number of type b buyers (or type s sellers) entering the market each period must be sufficient to compensate for those who form a match, trade and leave.

Agent behavior and strategies Let $\sigma = (\sigma_i)_{i \in B \cup S}$ denote a strategy profile. For any agent $b \in B$, σ_b specifies the following:

- In the first period, the strategy σ_b specifies the probability that agent b enters the market.
- If agents b enters the market, finds a match $s \in S$, and is designated as the proposer, σ_b specifies the price offer t.
- If agents b is designated as the responder, σ_b specifies the probability of accepting any offer t.

and similarly for any agent $s \in S$.

The per-period reward function for a buyer b paired in the current period with seller s is:

$$\pi_b(\sigma, s) = \begin{cases} -c_b + f_{bs} - t_{bs}(\sigma) & \text{proposal of } b \text{ accepted,} \\ -c_b + f_{bs} - t_{sb}(\sigma) & \text{proposal of } s \text{ accepted,} \\ -c_b & \text{proposal rejected,} \end{cases}$$

where $t_{bs}(\sigma)$ denotes the price offer made by buyer b to seller s given strategy σ. [5] If an agent does not enter the market or has made a transfer with her match in a prior period, then the agent's payoff for the period is equal to 0. Also, if a buyer (seller) is not paired in a period, then her payoff for the period is equal $-c_b$ ($-c_s$). Buyers and sellers choose a strategy to maximize the discounted sum of their payoffs. The solution concept adopted in this study is stationary subgame perfect equilibria, so that equilibrium strategies maximize the discounted sum of payoffs after any possible path of play.

[5] Symmetrically, a seller's payoff $\pi_s(\sigma, b)$ is equal to $-c_s + t_{bs}(\sigma)$, $-c_s + t_{sb}(\sigma)$, or $-c_S$ if she accepts the proposal, if her proposal is accepted, or if the proposal is rejected, respectively.

In a given search economy $\mathcal{S}$, the subgame perfect equilibrium of this economy (***search equilibrium*** herafter) is fully described by the tuple $\mathcal{E}(\mathcal{S}) = (l, \sigma)$, where the measure l satisfies the steady state equations (3.1) and (3.2), given that agents adopt strategy profile σ and, each σ_i is optimal after any subgame, given that agents use σ and the steady state measure is l.

It would be rightful to note that dealing with strategy profile σ would be quite burdensome. Hence, we need to adopt a different approach in characterizing the equilibrium of this search-and-matching market.

3.1.3 Alternative characterization of the equilibrium

To develop an alternative characterization of the search-and-matching equilibrium, we introduce the notion of *values*. Let $v = (v_i)_{i \in B \cup S} \in \mathbb{R}^{|B|+|S|}$ denote the expected values from entering the market, and also option values of remaining out of the market. In any search equilibrium, a buyer b offers seller s no more than $\delta_s v_s$ and a seller offers buyer b no more than $f_{bs} - \delta_b v_b$. Consequently, the values must satisfy the recursive equations:

$$v_b = \max\{-c_b + n(r)\beta \sum\nolimits_s p_s (f_{bs} - \delta_B v_b - \delta_S v_s)^+ + \delta_B v_b, 0\},$$
$$v_s = \max\{-c_s + \frac{n(r)}{r}(1-\beta) \sum\nolimits_b p_b (f_{bs} - \delta_B v_b - \delta_S v_s)^+ + \delta_S v_s, 0\},$$

where $(f_{bs} - \delta_B v_b - \delta_S v_s)^+ = \max\{f_{bs} - \delta_B v_b - \delta_S v_s, 0\}$.

Now that we have defined v, combine it with the previously defined l and $m = \{m_{bs}\}_{b,s \in B \times S}$ to characterize the equilibrium. The following four conditions are met by any tuple (l, m, v) in an equilibrium:

(i) **Individual rationality.** $v_i \geq 0$ *for all* $i \in B \cup S$*.*

(ii) **Efficient bargaining.** *If* $f_{bs} - \delta_b v_b - \delta_s v_s > 0$*, then* $m_{bs} = m_{sb} = 1$ *and if* $f_{bs} - \delta_b v_b - \delta_s v_s < 0$*, then* $m_{bs} = m_{sb} = 0$*.*

(*iii*) **Constant surplus.** *The valuations satisfy:*

$$-c_b + n(r)\beta \sum_S p_s m_{bs}(f_{bs} - \delta_b v_b - \delta_s v_s) \leq (1 - \delta_b)v_b \text{ for all } b$$
$$-c_s + \frac{n(r)}{r}(1 - \beta) \sum_b p_b m_{bs}(f_{bs} - \delta_b v_b - \delta_s v_s) \leq (1 - \delta_S)v_s \text{ for all } s$$

where the inequality holds with equality for i *with* $p_i > 0$.

(*iv*) **Steady state.** $l_b n(r) \sum_S m_{bs} p_s \leq 1$ *for all* $b \in B$, $l_s \frac{n(r)}{r} \sum_B m_{bs} p_b \leq 1$ *for all* $s \in S$*; and if* $v_i > 0$*, then the inequality for* $i \in B \cup S$ *holds with equality.*

Condition (1) holds because entry into the market is voluntary. Condition (2) follows since, in a random proposer game, any meeting between b and s with positive surplus results in a certain match. Condition (3) is a restatement of the Bellman equations for buyer and seller values. Condition (4) follows since the market is in steady state and all agents with strictly positive value enter the market.

The following proposition is stated in Atakan (2010) and proven for the case $\delta_i = 1$ for each $i \in B \cup S$. Our proof is almost identical, and is provided in the Appendix.

Proposition 3.2. *If* (l, m, v) *satisfy conditions* (1) *through* (4)*, then there exists a search equilibrium* (l, σ) *such that* m *and* v *are the equilibrium match probabilities and values.*

The proposition asserts that one can work with (l, m, v) instead of (l, σ) while analyzing the equilibrium of the search economy. Throughout the rest of this paper, we will be using the tuple (l, m, v) to specify $\mathcal{E}(\mathcal{S})$, and justifiably so thanks to this proposition.

3.2 The Competitive Economy

3.2.1 The Market Economy

Given a search economy $\mathcal{S} = (B, S, f, \beta, \kappa, r, \Delta)$, the competitive economy (***market economy*** hereafter) is the frictionless counterpart of this particular search economy, and denoted with $\mathcal{S}^* = (B, S, f)$.[6] As in the search-and-matching economy, the sets of buyers and sellers are denoted by B and S, respectively. Also similarly, f_{bs} denotes the transferable utility created when buyer b and seller s are matched.

3.2.2 The Competitive Equilibrium

For a market economy (B, S, f), we characterize the ***competitive equilibrium*** associated with this economy as the solution to the following assignment problem, which is a simple linear optimization problem, based on Shapley and Shubik (1972):

$$\max_{(q_{i,j})_{(i,j)\in B\times S}} \sum\nolimits_{B\times S} q_{bs} f_{bs}$$

subject to $(q) \in \mathbb{R}_+^{|B||S|}$ and

$$\sum\nolimits_{s\in S} q_{bs} \leq 1 \text{ for all } b, \qquad (\nu_b) \tag{3.3}$$

$$\sum\nolimits_{b\in B} q_{bs} \leq 1 \text{ for all } s, \qquad (\nu_s) \tag{3.4}$$

It is possible to interpret this problem as one of a social planner who attempts to maximize the total value created in a market economy by manually matching the members of the cohort who are born into the market each and every period. Alternatively, one can also write down the dual of this problem and analyze in terms of its dual counterpart:

[6] A useful method to imagine a market economy is to consider it as a search economy *in the limit*, where $\Delta = 0$.

$$\min_{\nu_b \in B, \nu_s \in S} \sum_B \nu_b + \sum_S \nu_s$$

subject to $(\nu) \in \mathbb{R}_+^{|B|+|S|}$ and

$$\nu_b + \nu_s \geq f_{bs} \text{ for all } (b,s) \in B \times S, \qquad (q_{bs}) \qquad (3.5)$$

Let $(q) = \{q_{bs}\}_{(b,s)\in B\times S}$ denote the solution to the primal problem, and let $(\nu) = \left(\nu_{b_1}, ..., \nu_{|B|}, \nu_{s_1}, ..., \nu_{|S|}\right)$ denote the solution to the dual. Then, given a market economy $\mathcal{S}^*$, the set of competitive equilibria associated with this economy is fully characterized by $\mathcal{E}^*(\mathcal{S}^*) = (q, \nu)$.

Now, having defined the basics of the model, let us go through some simple examples that will help us analyze the dynamics of the model.

Chapter 4

Initial Examples

In this chapter, we provide comprehensive solutions to some simple cases, just to improve our comprehension of the model. The strategy is as follows: for each case, we first characterize the set of competitive equilibria. We then proceed to consider the search economy counterpart of this particular case, and characterize the set of search equilibria that arises, especially when frictions in this setup begin to disappear. The comparison of both equilibria will follow, albeit being self-evident to some extent.

For the cases analyzed below, unless otherwise stated, we assume that the patience measures are common for both players ($\delta_b = \delta_s = \delta$ for each (b, s) pair.)

4.1 One-by-One Case

This is the simplest case: there is only one type of buyer and one type of seller ($|B| = |S| = 1$), who seek a transaction for only one type of good. For simplicity, the seller's reservation price is assumed to be zero and the buyer's valuation is one (hence the result $f_{bs} = 1$), but the results are generalizable.

4.1.1 Competitive Equilibrium

The competitive equilibrium of this particular case, which corresponds to the set of assignment game solutions, $(q_{bs}; (\nu_b, \nu_s))$, needs the satisfy the following:

$$q_{bs} = 1$$
$$\nu_b \in [0,1]$$
$$\nu_s = 1 - \nu_b$$

In other words, we have: $\mathcal{E}^*(\mathcal{S}^*) = \{(1; (\nu_b, \nu_s)) : \nu_b \in [0,1], \nu_s = 1 - \nu_b\}$

4.1.2 Search Equilibrium

The set of search equilibria is characterized by the tuple $((l_b, l_s); m_{bs}; (v_b, v_s))$. Remember that, given this tuple, market size is given by l_b, and the seller-to-buyer ratio is:

$$r = \frac{l_s}{L} = \frac{l_s}{l_b}$$

As noted before, parameters of the market equilibrium vector $((l_b, l_s); m_{bs}; (v_b, v_s))$ must satisfy the following:

(i) **Individual rationality.** $v_b, v_s \geq 0$.

(ii) **Efficient bargaining.** *If* $1 - \delta v_b - \delta v_s > 0$, *then* $m_{bs} = m_{sb} = 1$ *and if* $1 - \delta v_b - \delta v_s < 0$, *then* $m_{bs} = m_{sb} = 0$.

(iii) **Constant surplus.** *The valuations satisfy:*

$$-c_B + n(r)\beta p_s m_{bs}(1 - \delta v_b - \delta v_s) \leq (1-\delta)v_b$$
$$-c_S + \frac{n(r)}{r}(1-\beta)p_b m_{bs}(1 - \delta v_b - \delta v_s) \leq (1-\delta)v_s$$

where the inequality holds with equality for i *with* $p_i > 0$.

(*iv*) **Steady state.** $l_b n(r) m_{bs} p_s \leq 1$, $l_s \frac{n(r)}{r} m_{bs} p_b \leq 1$; *and if* $v_i > 0$, *then the inequality for* $i \in \{b, s\}$ *holds with equality.*

Now, we'll proceed to obtain the values of these equilibrium parameters when frictions disappear. The first thing to realize is that there always exists some trivial equilibrium, in which no buyers and no sellers enter the market (this applies to any model we can construct under this setup). To get rid of this unpleasant equilibrium, one could simply assume that $m_{bs} > 0$ and proceed with the analysis; yet, that would essentially mean putting constraints on the endogenous variables arising in the equilibrium. A more elegant approach would be assuming that agents are born into the market at the first period they are introduced into the model, rather than making the initial choice of whether to enter the market or not. This means that each period, a unit measure of each type of agent are born into the market, i.e. in the steady state, there is at least a unit measure of each type in the market. Thus, we implicitly impose the following condition:

(v) **No trivial equilibrium.** $l_b, l_s \geq 1$.[1]

The proposition establishes that this condition indeed eliminates the possibility of a trivial equilibrium is provided and proven in the Appendix.

Now, having established that there is no trivial equilibrium in the steady-state, we can confidently inspect the non-trivial steady-state equilibrium. First, realize that, as an auxiliary implication of condition *(v)*, it is certain that $p_b = p_s = 1$ (we'll refer to this later on).

Let us continue by stating a few lemmas.

Lemma 4.1. *In the equilibrium, it is impossible that* $v_b = v_s = 0$ *as* $\Delta \to 0$.

Proof. Suppose, to get a contradiction, that $v_b = v_s = 0$. Then, manipulating conditions 1 through 3, we need to have $0 = max\{-c_b + n(r)\beta\}$ and $0 = max\{-c_s +$

[1] Imposing this condition also requires replacing the last part of condition 3 with: "... where the inequality holds with equality for i with $l_i > 1$."

$\frac{n(r)}{r}(1-\beta)\}$, which is translated into the conditions: $c_b \geq n(r)\beta$ and $c_s \geq \frac{n(r)}{r}(1-\beta)$. Because the choice of Δ, and consequently the choice of $(c_b, c_s) = (\Delta\kappa_b, \Delta\kappa_s)$, is arbitrary, we may then conclude that $n(r)\beta = \frac{n(r)}{r}(1-\beta) = 0$.Given the definitions of $n(r)$ and $\frac{n(r)}{r}$, this is clearly impossible, indicating the desired contradiction. □

Based on this lemma, we can prove one of our critical findings about this setup:

Lemma 4.2. *In the equilibrium, $m_{bs} = 1$.*

Proof. By the previous lemma, we know that at least one element of the (v_b, v_s) pair is strictly positive. Without loss of generality, assume $v_b > 0$. Then, by condition 3, we obtain: $(1-\delta)v_b = -c_b + n(r)\beta max\{0, (1-\delta v_b - \delta v_s)\}$, which yields: $max\{0, (1-\delta v_b - \delta v_s)\} = \frac{c_b+(1-\delta)v_b}{n(r)\beta}$. Clearly, the right-hand side is strictly positive, so we obtain: $max\{0, (1-\delta v_b - \delta v_s)\} > 0$. This implies $1-\delta v_b - \delta v_s > 0$, which, by condition 2, implies the desired result: $m_{bs} = 1$. □

Another interesting implication of lemma 4.1 lies at the ease it provides while simplifying condition 4 characterizing the equilibrium. First, realize that the two inequalities within the condition, namely, $l_b n(r) m_{bs} p_s \leq 1$ and $l_s \frac{n(r)}{r} m_{bs} p_b \leq 1$ are equivalent to each other in a 1-to-1 setup; to see this, one only needs to take into account that $r = \frac{l_s}{l_b}$ in a one buyer-one seller market. Second, as a corollary of lemma 4.1, $v_i > 0$ for at least one $i \in \{b, s\}$, therefore, the inequalities must hold with equality. Finally, a simple case-by-case analysis yields that the equalities force either l_b or l_s to be exactly equal to one. This conclusion simplifies condition 4 tremendously.

Now, combining our findings that $m_{bs} = 1$ and $p_b = p_s = 1$, as well as the simplified condition 4, we can proceed with the analysis of steady-state market equilibrium. Substitute these findings into the set of conditions defining the equilibrium to get the following three, revised, conditions:

(*i*) $v_b, v_s \geq 0$; $l_b, l_s \geq 1$.

(*ii*) *The valuations satisfy:*

$$-c_B + n(r)\beta(1-\delta v_b - \delta v_s) \leq (1-\delta)v_b$$
$$-c_S + \frac{n(r)}{r}(1-\beta)(1-\delta v_b - \delta v_s) \leq (1-\delta)v_s$$

where the inequality holds with equality for i with $l_i > 1$.

(*iii*) $l_b = 1$ *or* $l_s = 1$.

Having defined the simplest version of conditions defining the steady-state equilibrium, we can now proceed with the analysis. The first task will be defining an upper bound on the market size, or, to put it more clearly, finding the maximum rate with which the market size can grow.

Begin by realizing that, by condition 3, one side of the market is already under control. Therefore, what we need to ensure that the long side of the market can not tend to infinity unboundedly, whereas the measure of the short side is kept constant at 1. We'll conduct a case-by-case analysis for this.

Case 1. When $l_b = 1$.

Begin by realizing that condition 1 directly implies that $l_s \geq l_b = 1$, thus, $r = \frac{l_s}{l_b} \geq 1$, and $n(r) = 1$, $\frac{n(r)}{r} = \frac{l_b}{l_s} = \frac{1}{l_s}$.

When $l_s = 1$, the market size is clearly under control, so this is hardly an interesting case at all. Therefore, one can simply assume that $l_s > 1$, which, by condition 2, implies that the equality $-c_S + \frac{1}{l_s}(1-\beta)(1-\delta v_b - \delta v_s) = (1-\delta)v_s$ is always satisfied. There exists two possibilities to consider:

When $v_b = 0$**.** In this case, the equality simplifies to: $-c_S + \frac{1}{l_s}(1-\beta)(1-\delta v_s) = (1-\delta)v_s$, which, with simple algebra, yields:

$$v_s = \frac{-c_s + \frac{1-\beta}{l_s}}{1-\delta+\delta\frac{1-\beta}{l_s}} \tag{4.1}$$

So, how fast can $l_s \to \infty$? Clearly, there are two conditions that needs to be satisfied. The first is condition 1, which says $v_s \geq 0$. The second is the assumption we made at the beginning, i.e. $v_b = 0$. One could let l_s tend to infinity as fast as possible, without violating these two.

Regarding the nonnegativity constraint of v_s, realize that $v_s \geq 0 \Rightarrow (1-\delta)v_s = -c_S + \frac{1-\beta}{l_s}(1-\delta v_s) \geq 0 \Rightarrow (1-\delta v_s) \geq \frac{c_s l_s}{1-\beta}$.

Regarding the constraint $v_b = 0$, one only needs to ensure that expected payoff for s at each period should be at most zero, i.e. $-c_b + \beta(1-\delta v_s) \leq 0$, which implies: $(1-\delta v_s) \leq \frac{c_b}{\beta}$.

Combining the two, we obtain the condition: $\frac{c_b}{\beta} \geq (1-\delta v_s) \geq \frac{c_s l_s}{1-\beta}$, which translates into: $l_s \leq \frac{c_b/\beta}{c_s/(1-\beta)}$. Realize that, as $\Delta \to 0$, this quantity converges to $\frac{\kappa_b/\beta}{\kappa_s/(1-\beta)}$. [2]

When $v_b > 0$. In this case, the condition $(1-\delta)v_b \geq -c_B + n(r)\beta(1-\delta v_b - \delta v_s)$ must also hold with equality. We have two equalities that must jointly be satisfied:

$$\begin{aligned}(1-\delta)v_b &= -c_B + \beta(1-\delta v_b - \delta v_s)\\ (1-\delta)v_s &= -c_S + \frac{1}{l_s}(1-\beta)(1-\delta v_b - \delta v_s)\end{aligned}$$

Along with two nonnegativity constraints: $v_b \geq 0$, $v_s \geq 0$.

Simple algebra shows that the two nonnegativity constraints can be stated as: $\delta(v_b + v_s) \leq \min\{1 - \frac{c_b}{\beta}, 1 - \frac{c_s l_s}{1-\beta}\}$. Furthermore, using the equalities, and by the help of some manipulation, one can show that:

$$\delta(v_b + v_s) = \frac{-c_b - c_s + \beta + \frac{1-\beta}{l_s}}{\frac{1-\delta}{\delta} + \beta + \frac{1-\beta}{l_s}}$$

[2] In a sense, one could argue that this market setup only makes sense when $\frac{c_b/\beta}{c_s/(1-\beta)} \to \frac{\kappa_b/\beta}{\kappa_s/(1-\beta)} \in (0, \infty)$, i.e. when the ratio of two cost measures remain comparable in the limit. Under this assumption, the upper bound we have found for l_s is indeed quite tight.

Hence the inequality:

$$\frac{-c_b - c_s + \beta + \frac{1-\beta}{l_s}}{\frac{1-\delta}{\delta} + \beta + \frac{1-\beta}{l_s}} \leq \min\{1 - \frac{c_b}{\beta}, 1 - \frac{c_s l_s}{1-\beta}\}$$

follows, which implies the inequality: [3]

$$l_s \leq \frac{c_b/\beta}{c_s/(1-\beta)} + \frac{1-\delta}{\delta}\frac{1-\beta}{\beta}\frac{1}{c_s}$$

where, as $\Delta \to 0$, the right hand-side converges to: $\frac{\kappa_b/\beta}{\kappa_s/(1-\beta)} + \frac{1-\beta}{\beta}\frac{r}{\kappa_s}$, a well-defined upper bound.

Realize that in any case, we have accomplished to achieve an upper bound on l_s. Now, it's time to proceed with the other case.

Case 2. When $l_s = 1$.

The analysis of this case is entirely symmetric with the argument provided above, hence, we omit the algebra and proceed with the results.

When $v_s = 0$. In this case, we obtain the inequality: $l_b \leq \frac{\kappa_s/(1-\beta)}{\kappa_b/\beta}$ as $\Delta \to 0$.

When $v_s > 0$. In this case, we obtain the inequality:

$$\frac{-c_b - c_s + (1-\beta) + \frac{\beta}{l_b}}{\frac{1-\delta}{\delta} + (1-\beta) + \frac{\beta}{l_b}} \leq \min\{1 - \frac{c_b l_b}{\beta}, 1 - \frac{c_s}{1-\beta}\}$$

which implies the upper bound:

$$l_b \leq \frac{c_s/(1-\beta)}{c_b/\beta} + \frac{1-\delta}{\delta}\frac{\beta}{1-\beta}\frac{1}{c_b}$$

where, as $\Delta \to 0$, the right hand-side converges to: $\frac{\kappa_s/(1-\beta)}{\kappa_b/\beta} + \frac{\beta}{1-\beta}\frac{r}{\kappa_b}$.

[3]The algebra is a little lengthy, yet straightforward. Begin by observing that the right hand-side of inequality changes depending on whether $l_s \frac{c_s}{1-\beta} \geq \frac{c_b}{\beta}$ holds or not. If $\frac{c_b}{\beta} \geq l_s \frac{c_s}{1-\beta}$, then, the usual bound $l_s \leq \frac{c_b/\beta}{c_s/(1-\beta)}$ applies. If $l_s \frac{c_s}{1-\beta} > \frac{c_b}{\beta}$, then, simple algebra yields the inequality: $l_s < \frac{\delta c_b(1-\beta)+(1-\delta)(1-\beta)}{(1-\delta)c_s+\beta\delta c_s}$. The $(1-\delta)c_s$ term in the denominator is negligible, and omitting it preserves the inequality. The results follows.

Having established an upper bound on the size of the market, now, rest of the analysis is relatively straightforward. For the rest of the analysis, assume that $\frac{\kappa_b}{\kappa_s} = \frac{\beta}{1-\beta} \Rightarrow \frac{\kappa_b/\beta}{\kappa_s/(1-\beta)} = 1$, i.e. the market is constructed such that it is "fair" towards both sides.[4] Referring to the analysis above, clearly, this puts a very neat upper bound on the market size, namely, it supports the symmetric case $l_s = l_b = 1$ in the steady-state search equilibrium. [5] Note that this also implies: $n(r) = \frac{n(r)}{r} = 1$.

So, we are done with the equilibrium measures. But what about the valuations (v_b, v_s), especially when the frictions disappear (i.e. when $\Delta \to 0$)? For the rest of the analysis, assume $v_b \neq 0$ and $v_s \neq 0$. [6] Before we prove the actual result, we need a preliminary lemma.

Lemma 4.3. *In the equilibrium, $v_b + v_s = 1 - c_b - c_s$.*

Proof. Taking the fact that $v_b, v_s > 0$ into account, using the conditions defining v_b and v_s, along with our previous findings that $m_{bs} = 1$ and $n(r) = \frac{n(r)}{r} = 1$, we obtain:

$$(1-\delta)v_b = -c_b + \beta(1 - \delta v_b - \delta v_s)$$
$$(1-\delta)v_s = -c_s + (1-\beta)(1 - \delta v_b - \delta v_s)$$

[4] The results would extend to the case where the market is "unfair." However, the condition that $\frac{\kappa_b/\beta}{\kappa_s/(1-\beta)} \in (0, \infty)$ is still crucial, i.e. one can not allow the market to become *terribly* unfair: the ratio of normalized cost measures should remain bounded away from zero and infinity. But the setup is constructed such that $\kappa_i > 0$ for each i, so we can be confident that this unpleasant case does not occur.

[5] There are also other equilibria with features $1 = l_s < l_b < 1 + \frac{r}{\kappa_s}$ and $1 = l_b < l_s < 1 + \frac{r}{c_b}$. The case $l_b = l_s = 1$ is considered for simplicity, yet, it would not affect the results dramatically. An analysis of the asymmetric market equilibrium is provided in the Appendix.

[6] The case in which one of them is zero is indeed much easier to analyze. Suppose, without loss of generality, that $v_b = 0$. Then, using equation defining v_s, we have:

$$v_s = \frac{-c_s + \frac{1-\beta}{l_s}}{1 - \delta + \delta\frac{1-\beta}{l_s}}$$

which, with some manipulation, yields: $v_s = 1 - \frac{c_s + \beta(1-\delta)}{1-\delta\beta}$. Clearly, as the frictions disappear, $v_s \to 1$.

adding the two equations, we obtain:

$$(1-\delta)(v_b+v_s) = -c_b - c_s + 1 - \delta(v_b+v_s)$$

which yields,

$$(v_b+v_s) = 1 - c_b - c_s.$$

□

This lemma provides us with great ease in computing what happens to (v_b, v_s) as $c \to 0$ and $\delta \to 1$. Remember that the conditions defining v_b and v_s were:

$$(1-\delta)v_b = -c_b + \beta(1 - \delta v_b - \delta v_s)$$
$$(1-\delta)v_s = -c_s + (1-\beta)(1 - \delta v_b - \delta v_s)$$

Rearranging, we obtain:

$$v_b = \frac{-c_b}{1-\delta} + \beta\frac{1-\delta v_b - \delta v_s}{1-\delta}$$
$$v_s = \frac{-c_s}{1-\delta} + (1-\beta)\frac{1-\delta v_b - \delta v_s}{1-\delta}$$

Where we know, by Lemma 4.3, that $v_b + v_s = 1 - c_b - c_s$. Substitute to get:[7]

$$v_b = \frac{-c_b}{1-\delta} + \beta\frac{1-\delta+\delta(c_b+c_s)}{1-\delta}$$
$$v_s = \frac{-c_s}{1-\delta} + (1-\beta)\frac{1-\delta+\delta(c_b+c_s)}{1-\delta}$$

Finally, utilizing the fact that $\frac{c_b}{c_s} = \frac{\beta}{1-\beta}$, we obtain the elegant result:

$$v_b = \beta - c_b$$
$$v_s = (1-\beta) - c_s$$

[7]These equalities indeed can be derived by direct solution of the two equalities. Some long and cumbersome calculations directly yield: $v_b = \frac{-c+2\beta\delta c+\beta(1-\delta)}{1-\delta}$, which is the result we derived. However, we think that the approach adopted here is more versatile, and easily generalizable to more complex cases.

which implies:

$$\lim_{\Delta \to 0} v_b = \beta$$
$$\lim_{\Delta \to 0} v_s = 1 - \beta$$

Now we are ready to state the crucial result in this setup.

Corollary 4.4. *For any competitive equilibrium values of this economy, $\nu_b \in [0,1], \nu_s = 1 - \nu_b$, we can find a search equilibrium values (v_b, v_s) that converge to this solution as frictions disappear (i.e. as $\Delta \to 0$).*

Proof. The proof follows from the last equation derived. Set any $\delta_b = \delta_s = \delta$, pick $\beta = \nu_b$, $1 - \beta = \nu_s$ and set κ_b, κ_s such that $\frac{\kappa_b}{\kappa_s} = \frac{\beta}{1-\beta}$. □

Now, let's consider an alternative setup, in which we do not choose β, but rather choose the respective patience parameters. Suppose the buyers and sellers have different levels of instantaneous rate of discounting, $r_b \geq 0$ and $r_s \geq 0$. This implies we have two levels of patience, $\delta_s = e^{-r_s \Delta}$ and $\delta_b = e^{-r_b \Delta}$, both converging to 1 in the limit. Note that to protect market fairness, we need to make sure that the ratio of cost measures, $\frac{c_b}{c_s}$, is equal to $\frac{\beta}{1-\beta}$. [8] Then, equations defining v_b and v_s translate into:

$$(1 - \delta_b)v_b = -c_b + \beta(1 - \delta_b v_b - \delta_s v_s)$$
$$(1 - \delta_s)v_s = -c_s + (1 - \beta)(1 - \delta_b v_b - \delta_s v_s)$$

Remember that we have derived: $1 - c_b - c_s = v_b + v_s$, and substitute to get: $1 - \delta_b v_b - \delta_s v_s = 1 - \delta_b v_b - \delta_s(1 - v_b - c_b - c_s) = 1 - \delta_s(1 - c_b - c_s) + (\delta_s - \delta_b)v_b$, or, symmetrically: $1 - \delta_b v_b - \delta_s v_s = 1 - \delta_b(1 - c_b - c_s) + (\delta_b - \delta_s)v_s$. Now, we can

[8] Another alternative setup would be keeping δ's equal and playing with cost parameters, allowing the market fairness to change. That would also enable us to receive the same results.

obtain:

$$v_b = \frac{-c_b}{1-\beta\delta_s-(1-\beta)\delta_b} + \beta\frac{1-\delta_s+\delta_s(c_b+c_s)}{1-\beta\delta_s-(1-\beta)\delta_b}$$
$$v_s = \frac{-c}{1-\beta\delta_s-(1-\beta)\delta_b} + (1-\beta)\frac{1-\delta_b+\delta_b(c_b+c_s)}{1-\beta\delta_s-(1-\beta)\delta_b}$$

For expositional simplicity, define the weighted average of patience levels of traders as: $\delta_{avg} := \beta\delta_s + (1-\beta)\delta_b$. Realize that we have obtained:

$$v_b = \frac{-c_b}{1-\delta_{avg}} + \beta\frac{1-\delta_s+\delta_s(c_b+c_s)}{1-\delta_{avg}}$$
$$v_s = \frac{-c_s}{1-\delta_{avg}} + (1-\beta)\frac{1-\delta_b+\delta_b(c_b+c_s)}{1-\delta_{avg}}$$

Which, using the fact that $\frac{c_b}{c_s} = \frac{\beta}{1-\beta}$, implies:

$$v_b = (\beta - c_b)\frac{1-\delta^s}{1-\delta_{avg}}$$
$$v_s = ((1-\beta)-c_s)\frac{1-\delta^b}{1-\delta_{avg}}$$

where, using the definitions of δ_b and δ_s and L'Hôpital's Rule, it can be shown that,

$$\lim_{\Delta\to 0} v_b = \beta\frac{r_s}{\beta r_s + (1-\beta)r_b}$$
$$\lim_{\Delta\to 0} v_s = (1-\beta)\frac{r_b}{\beta r_s + (1-\beta)r_b}$$

Which gives us the desired result:

Corollary 4.5. *For any competitive equilibrium values of the economy,* $\nu_b \in [0,1], \nu_s = 1-\nu_b$*, we can find a search equilibrium values* (v_b, v_s) *that converge to this solution as frictions disappear (i.e. as* $\Delta \to 0$.*)*

Proof. The proof follows from the last equalities derived. Given a particular value of $\beta \in [0,1]$, it will be sufficient to choose (δ_b, δ_s) (or, more properly, (r_b, r_s)) accordingly. □

Let's review our finding by considering two concrete examples.

Example 4.1. *Suppose $\beta = 1/2$, and the competitive equilibrium values is given as: $(\nu_b, \nu_s) = (1/8, 7/8)$. Then, to ensure that $(v_b, v_s) \to (\nu_b, \nu_s)$, it will be sufficient to choose: $\{(\delta_s)_n\} = \{1 - \frac{1}{n}\}, \{(\delta_b)_n\} = \{1 - \frac{7}{n}\}, \{(c_b)_n\} = \{(c_s)_n\} = \{\frac{1}{n}\}$.*

Another way to state this result is that, when $\beta = 1/2$, to get the competitive equilibrium values where $\{\Delta\}_n \to 0$, one needs to pick (r_b, r_s) such that $\frac{r_b}{r_s} = 7$.

Example 4.2. *Suppose $\beta = 1/2$, and the solution of the assignment game is given as: $(\nu_b, \nu_s) = (1/4, 3/4)$. Then, to ensure that $(v_b, v_s) \to (\nu_b, \nu_s)$, it will be sufficient to choose: $\{(\delta_s)_n\} = \{1 - \frac{1}{n}\}, \{(\delta_b)_n\} = \{1 - \frac{3}{n}\}, \{(c_b)_n\} = \{(c_s)_n\} = \{\frac{1}{n}\}$.*

Another way to state this result is that, when $\beta = 1/2$, to get the competitive equilibrium values where $\{\Delta\}_n \to 0$, one needs to pick (r_b, r_s) such that $\frac{r_b}{r_s} = 3$.

Realize the particular difference between the two example cases. In the second case, $(\delta_b)_n$ is higher (or r_b is lower) compared to the first case, whereas everything else remains same. This change results in a higher limit value for v_b. Therefore, the examples successfully demonstrate the phenomenon "being relatively more patient makes you better off", which is intuitively appealing, and in line with Rubinstein (1982).

4.1.3 Discussion

The main result obtained in the solution of this example, Corollary 4.5, shows that the initial results are encouraging: one could possibly cover the whole range of competitive equilibrium by setting $\beta = 1 - \beta = 1/2$, $c_b = c_s$ (i.e. $\kappa_b = \kappa_s$), and just picking the relative patience parameters δ_b, δ_s (r_b, r_s) properly. This is the result that we'll be seeking the generalization of in the following chapter.

4.2 One-by-Two Case

The strategic interactions of this case is more interesting: there is only one type of buyer but two types of sellers (types 1 and 2),competing with each other to make the transaction with the buyer (i.e. $|B| = 1$ and $|S| = 2$). For simplicity, we assume that $f_{b1} = 1$ and $f_{b2} = 0.5$, but the results are again generalizable.

4.2.1 Competitive Equilibrium

The competitive equilibrium of this case, which corresponds to the set of assignment game solutions, $((q_{b1}, q_{b2}); (\nu_b, \nu_1, \nu_2))$, are characterized by the following:

$$q_{b1} = 1, q_{b2} = 0$$
$$\nu_b \in [0.5, 1]$$
$$\nu_1 = 1 - \nu_b$$
$$\nu_2 = 0$$

In other words, we have: $\mathcal{E}^*(\mathcal{S}^*) = \{((1, 0); (\nu_b, \nu_1, 0)) : \nu_b \in [0.5, 1], \nu_1 = 1 - \nu_b\}$

4.2.2 Search Equilibrium

The set of search equilibria is characterized by the tuple $((l_b, l_1, l_2); (m_{b1}, m_{b2}); (v_b, v_1, v_2))$. Again, the equilibrium vector $((l_b, l_1, l_2); (m_{b1}, m_{b2}); (v_b, v_1, v_2))$ must satisfy the conditions that define a search equilibrium.

As usual, we start by investigating the values of (l_b, l_1, l_2) in the equilibrium. Keep in mind that ultimate purpose is to show that the equilibrium converges to the assignment equilibrium, as frictions disappear. Since, in the assignment equilibrium, the only trade occurs between the buyer and seller 2, this establishes a good candidate strategy of first proving that $m_{b2} = 0$. If one can show that $m_{b2} = 0$, rest of the analysis is straightforward: such a finding implies that seller 2 is effectively kicked out of the market; then, we are faced with a case that is simply equivalent of the one-by-one case analyzed above.

Realize that we can investigate this setup under two different cases. The first one is the case in which $m_{b2} < 1$. This case is particularly easy to deal with, because the assumption that $m_{b2} < 1$ implies $v_2 \leq 0$ (because there is no possibility of trading, and obtaining a strictly positive gain, for seller 2). Therefore, seller 2 does not find it profitable to remain in the market, and we have $l_2 = 1$. Rest of the analysis is exactly the same as the case with 1 buyer and 1 seller, which is examined above.

So, let's consider the case in which $m_{b2} = 1$, and see whether such an equilibrium can exist. First, we need a few useful results.

Lemma 4.6. *In this particular setup, if buyer trades with seller 2 with probability 1 in the equilibrium, then she trades with seller 1 with probability one, too. (i.e.* $m_{b2} = 1 \Rightarrow m_{b1} = 1$*).*

We provide two alternative proofs.

Proof. **(1)** Suppose that $m_{b2} = 1$. Note that this implies $0.5 - \delta v_2 - \delta v_b \geq 0$, by the efficient bargaining condition.

Note that if we can show: $1 - \delta v_1 - \delta v_b > 0$, which implies $m_{b1} = 1$, then we are done. Suppose, to get a contradiction, that $1 - \delta v_1 - \delta v_b \leq 0$. Substituting this into the equation defining v_1, we obtain: $v_1 = max\{0, -c_{s1} + \delta v_1\}$. Clearly, the unique solution for this equality is: $v_1 = 0$. Then, since we have $1 - \delta v_1 - \delta v_b \leq 0$, this implies: $\delta v_b \geq 1$. But realize that this implies: $0.5 - \delta v_2 - \delta v_b < 0$ for every $v_2 \geq 0$, a contradiction. □

For the second proof, we need a new definition.

Definition 4.7. A set of valuations (v_1, v_2) are *incentive-compatible* if $|v_1 - v_2| \leq |f_{b1} - f_{b2}|$.

Now, it should be trivial to see why incentive-compatibility is a crucial concept for us. Clearly, a set of valuations cannot constitute an equilibrium for the matching game if they are not incentive-compatible. This is because violation of incentive-compatibility indicates a profitable deviation for one of the parties. Consequently, if incentive-compatibility is violated, we have one of the sellers imitating to be the other type, which leaves us effectively with only one type of sellers, hence with the model that we examined previously. Having defined this concept, we can now proceed with the second proof.

Proof. **(2)** Suppose that $m_{b,2} = 1 \Rightarrow 0.5 - \delta v_2 - \delta v_b \geq 0$. Furthermore, since we are looking for valuations in the equilibrium, they must satisfy incentive-compatibility, i.e. we have: $v_1 - v_2 \leq 0.5$. Because $\delta < 1$, this also implies: $\delta(v_1 - v_2) < 0.5$. All in all, we have:

$$\delta v_2 + \delta v_b \leq 0.5$$
$$\delta v_1 - \delta v_2 < 0.5$$

Add the two inequalities to get: $\delta v_1 + \delta v_b < 1 \Rightarrow m_{b1} = 1$. □

Having derived the relationship between m_{b2} and m_{b1}, now we can proceed to obtain some results on equilibrium measures.

Lemma 4.8. *In an equilibrium, if* $m_{b2} = 1$*, then we have:* $l_b \leq l_1 + l_2$*.*

Proof. Suppose not. Then, $l_b > l_1 + l_2 \Rightarrow r < 1$. This implies $n(r) = min\{1, r\} < 1$, and $\frac{n(r)}{r} = 1$ (i.e. each seller in the market definitely finds a match in every period). Now, combine this information with what we already know about the market: $m_{b2} = 1$ and, by previous Lemma, $m_{b1} = 1$. This piece of information suggests that every seller of each type who is matched immediately trades and leaves the market.

Remember that by the structure of the setup, each period, there exists at least 1 measure of type 1 sellers and 1 measure of type 2 sellers in the market each period.

Hence, since all sellers find a match, each period, at least 1 measure of type 1 and 1 measure of type 2 sellers trade and leave the market each period. Obviously, the only type they can form a trading pair with is the buyer. Consequently, we obtain that each period, a total of 2 measure of buyers leave the market. Given that the supply of buyers entering the market each period is 1, this is clearly not sustainable.

Let's interpret this argument in terms of our own notation, and referring to the equations that define the search equilibrium. Recognizing that $n(r) = \frac{l_1+l_2}{l_b} < 1$ and $\frac{n(r)}{r} = 1$, the steady-state requirement for seller becomes: $1 - l_b \frac{l_1+l_2}{l_b}(p_1 m_{b,1} + p_2 m_{b,2}) \geq 0$. Furthermore, remember the definition of p_1 and p_2, along with the finding $m_{b1} = m_{b2} = 1$ to substitute, and get: $1 - l_b \frac{l_1+l_2}{l_b}(\frac{l_1}{l_1+l_2} + \frac{l_2}{l_1+l_2}) \geq 0$. The inequality therefore simplifies to:

$$1 - (l_1 + l_2) \geq 0$$

Considering that in any steady state $l_1 \geq 1$ and $l_2 \geq 1$, this is clearly impossible, a contradiction. □

By the help of Lemma 4.8, we begin to understand how equilibrium looks like when $m_{b2} = 1$. We know: $l_b \leq l_1 + l_2$ in the equilibrium, which also implies: $r > 1$ and thus $n(r) = 1$, $\frac{n(r)}{r} = \frac{l_b}{l_1+l_2}$. Furthermore, $m_{b1} = m_{b2} = 1$, indicating that each buyer-seller pair who are matched immediately trade and leave the market. Considering the fact that p_1 and p_2 are proportional to the respective measures of the corresponding seller types, we can immediately conclude that $\frac{l_1}{l_2} \nrightarrow 0$, nor $\frac{l_2}{l_1} \nrightarrow 0$ in any equilibrium in which $m_{b2} = 1$, if such an equilibrium exists.

Now, given these findings, let's write down the steady-state requirements for the case $m_{b2} = 1$, from which the impossibility of such an equilibrium will follow.

$$l_1 \frac{l_b}{l_1 + l_2} \leq 1 \tag{4.2}$$

$$l_2 \frac{l_b}{l_1 + l_2} \leq 1 \tag{4.3}$$

$$l_b\left(\frac{l_1}{l_1 + l_2} + \frac{l_2}{l_1 + l_2}\right) \leq 1 \tag{4.4}$$

Where (4.2) holds with equality if $v_1 > 0$, and similarly for (4.3) if $v_2 > 0$, as for (4.4) if $v_b > 0$. Indeed, we will show that (4.4) must hold with equality in such a case, at least for sufficiently small frictions.

Lemma 4.9. *In any equilibrium where $m_{b2} = 1$, we must have: $v_b > 0$ as $\Delta \to 0$.*

Proof. We do already know that $m_{b1} = m_{b2} = 1$, and $1 - \delta v_1 - \delta v_b > 0$, $0.5 - \delta v_2 - \delta v_b \geq 0$. The expected gain from entering the market in a period for the buyer is: $-c_b + \frac{l_1}{l_1+l_2}\beta(1 - \delta v_1 - \delta v_b) + \frac{l_2}{l_1+l_2}\beta(0.5 - \delta v_2 - \delta v_b)$. Clearly, for sufficiently low c_b, this term is strictly positive, and v_b is nonzero. □

Now that we have shown $v_b > 0$ in an equilibrium where $m_{b2} = 1$, one can conclude that (4.4) holds with equality. Our last lemma is the final step towards proving the impossibility of such an equilibrium.

Lemma 4.10. *In any equilibrium where $m_{b2} = 1$, we must have: $v_1 = v_2 = 0$.*

Proof. Without loss of generality, suppose $v_1 > 0$. Then, (4.2) must hold with equality. Given that, by previous lemma, (4.4) also holds with strict equality, one can subtract (4.2) from (4.4) to obtain: $l_2 \frac{l_b}{l_1+l_2} = 0$. But remember, if (4.2) holds with equality, then $l_1 \frac{l_b}{l_1+l_2} = 1$. These last two equations coexist only if $l_2 = 0$[9], which is a contradiction with the fact that each period, at least one measure of each type enters the market. □

[9] Another possibility is that $\frac{l_2}{l_1} \to 0$, but we already had the discussion that this is impossible.

The final result that we have derived is clearly inconsistent with our previous findings that $1 - \delta v_1 - \delta v_b > 0$, $0.5 - \delta v_2 - \delta v_b > 0$. Therefore, we have a contradiction which indicates the impossibility of such an equilibrium.

Corollary 4.11. *In the setup with one buyer and two sellers, there is no matching equilibrium where $m_{b2} = 1$.*

Now that we have eliminated all the implausible equilibria, we can say that in any matching equilibrium of this setup, $m_{b2} = 0$. This, in turn, implies that the following conditions are satisfied, at least as friction disappear (i.e $\Delta \to 0$).

$$0.5 - \delta v_b \leq 0 \Rightarrow \delta v_b \geq 0.5$$
$$1 - \delta v_1 - \delta v_b > 0$$
$$v_2 = 0$$

It is easy to see that the set of search equilibria, therefore, converges to the competitive equilibria:

$$\nu_b \in [0.5, 1]$$
$$\nu_1 = 1 - \nu_b$$
$$\nu_2 = 0$$

as the frictions disappear. One can conduct an analysis similar to the one provided at the end of previous section to convince herself that for each competitive equilibrium values (ν_b, ν_1, ν_2), there exists a search equilibrium values (v_b, v_1, v_2) that converges to (ν_b, ν_1, ν_2) as $\Delta \to 0$.

4.2.3 Discussion

The point that needs to be emphasized on the analysis of this case is that, even if there are different numbers of types of buyers and sellers in the market (i.e. if B

and S have different cardinalities) one can still simplify the case into one where the number of types are equal, using proper justifications. This implies that having the assumption $|B| = |S|$ is less restrictive than it seems.

4.3 Two-by-Two Case

This again is an interesting case, with its richness in strategic interactions and the insights it contains. We will begin by a relatively simpler setting with two buyers and two sellers: we will assume the market is *separable*, in the sense that there is a unique perfect match for each type of buyer and seller, and trade with remaining types are not desirable. Suppose there are two types of buyers and two types of sellers (i.e. $|B| = |S| = 2$), where:

$$f_{11} = f_{22} = 1$$
$$f_{12} = f_{21} = 0$$

Once more, unless otherwise stated, assume that $\delta_{b_1} = \delta_{s_1} = \delta_{b_2} = \delta_{s_2} = \delta$ (i.e. $r_{b_1} = r_{s_1} = r_{b_2} = r_{s_2} = \delta$), $c_{b_1} = c_{b_2} = c_B$ (i.e. $\kappa_{b_1} = \kappa_{b_2} = \kappa_B$) and $c_{s_1} = c_{s_2} = c_S$ (i.e. $\kappa_{s_1} = \kappa_{s_2} = \kappa_S$.)

4.3.1 Competitive Equilibrium

The competitive equilibrium of this case, $((q_{b1,s1}, q_{b1,s2}, q_{b2,s1}, q_{b2,s2});\ (\nu_{b1}, \nu_{b2}, \nu_{s1}, \nu_{s2}))$, are characterized by the following:

$$\begin{aligned} q_{11} &= q_{22} = 1 \\ q_{12} &= q_{21} = 0 \\ \nu_{b1} &\in [0,1] \\ \nu_{s1} &= 1 - \nu_{b1} \\ \nu_{b2} &\in [0,1] \\ \nu_{s2} &= 1 - \nu_{b2} \end{aligned}$$

In other words, we have: $\mathcal{E}^*(\mathcal{S}^*) = \{(1,0,0,1); (\nu_{b1}, \nu_{b2}, \nu_{s1}, \nu_{s2})) : \nu_{b1} \in [0,1], \nu_{s1} = 1 - \nu_{b1}, \nu_{b2} \in [0,1], \nu_{s2} = 1 - \nu_{b2}\}$

4.3.2 Search Equilibrium

The set of search equilibria is characterized by the tuple $((l_{b1}, l_{b2}, l_{s1}, l_{s2}); (m_{11}, m_{12}, m_{21}, m_{22});$ $(v_{b1}, v_{b2} v_{s1}, v_{s2}))$. Parameters of the market equilibrium vector $(q; m; v)$, as usual, must satisfy the conditions defining a search equilibrium.

We begin calculating the values of parameters in the equilibrium vector by first analyzing m. The first lemma states that the only trade occurs between the agents who are the "ideal matches" of each other.

Lemma 4.12. *In the equilibrium,* $m_{12} = m_{21} = 0$ *as* $\Delta \to 0$.

Proof. Suppose, to get a contradiction, that $m_{12} \neq 0$ (there is no loss of generality in this assumption). By the efficient bargaining condition, this implies that we need to have $v_{b1} = v_{s2} = 0$ (otherwise the efficient bargaining condition would imply that $m_{12} = 0$.) Realize that at least one of the remaining two types must have a nonzero valuation, then. (Otherwise, if each type in the market receives zero payoff, writing constant surplus condition for all types and adding gives $n(r)\beta \leq c_B = \Delta\kappa_B$ and $\frac{n(r)}{r}(1-\beta) \leq c_S = \Delta\kappa_s$. Clearly, these two inequalities cannot be satisfied simultaneously as $\Delta \to 0$.) Without loss of generality, suppose $v_{s1} \neq 0$ (which, by individual rationality condition, implies $v_{s1} > 0$. This has two implications: first, by efficient bargaining condition, $m_{21} = 0$. Second, by the steady-state condition for s_1 holds with equality.) The steady-state conditions for b_1 and s_1 are:

$$l_{b1}n(r)(m_{11}p_{s1} + m_{12}(1-p_{s1})) \leq 1$$
$$l_{s1}\frac{n(r)}{r}m_{11}p_{b1} = 1$$

Realizing that $l_{b1}n(r)m_{11}p_{s1} = l_{s1}\frac{n(r)}{r}m_{11}p_{b1}$, one could easily deduce that, for the first inequality to be satisfied, it must be the case that $m_{12} = 0$,[10] a contradiction. □

Now, having derived the result that there will be no "cross-trade," it remains to show that trade between ideal matches definitely occurs.

Lemma 4.13. *In the equilibrium,* $m_{11} = m_{22} = 1$ *as* $\Delta \to 0$.

Proof. Suppose, to get a contradiction, that $m_{11} < 1$ (without loss of generality.) By the efficient bargaining condition, this implies: $f_{11} - \delta v_{b1} - \delta v_{s1} \leq 0 \Rightarrow (v_{b1} + v_{s1}) \geq \frac{1}{\delta}$. This, in turn, implies that $v_{b1} > 0$ or $v_{s1} > 0$. Either way, at least one of the steady-state conditions associated with these types must hold with equality, and since $l_{b1}n(r)m_{11}p_{s1} = l_{s1}\frac{n(r)}{r}m_{11}p_{b1}$, we can be sure that both steady state conditions hold with equality. Rearranging terms in these steady state conditions,

[10] Another alternative is to let $p_{s1} = 1$, or, to let l_{s1} tend to infinity. This is not possible, either. To see this, attempt to write down a case where $v_{b1} = v_{s2} = 0$ and l_{s1} tends to infinity. By the equality defining v_{s1}, we get $-c_S + \frac{n(r)}{r}(1-\beta)(p_{b1})(1-\delta v_{s1}) = (1-\delta)v_{s1} \geq 0$ (equality because $l_{s1} > 1$.) For the same inequality for v_{b1}, we get: $0 = (1-\delta)v_{b1} \geq -c_B + n(r)\beta p_{s1}(1-\delta v_{s1})$. Combining, we obtain the inequality: $\frac{c_B/\beta}{c_S/(1-\beta)} \geq r\frac{p_{s1}}{p_{b1}}$. If p_{b1} is bounded away from zero, then r must converge towards infinity, and the inequality cannot be satisfied. Otherwise, conducting the symmetric analysis for the $b_2 - s_2$ pair yields the inequality: $\frac{c_B/\beta}{c_S/(1-\beta)} \leq r\frac{p_{s2}}{p_{b2}}$. Since $p_{s2} = 0$, with r and p_{b2} being finite numbers, this cannot hold, a contradiction.

one obtains $m_{11} = \frac{1}{\frac{l_{b1}l_{s1}}{l_{s1}+l_{s2}}n(r)} = \frac{1}{\frac{l_{b1}l_{s1}}{l_{b1}+l_{b2}}\frac{n(r)}{r}} < 1$. Now, this inequality implies that $l_{b1} > 1$ or $l_{s1} > 1$ (otherwise, if $l_{b1} = l_{s1} = 1$, plugging them into the inequality gives $\frac{l_{s1}+l_{s2}}{n(r)} = \frac{l_{b1}+l_{b2}}{n(r)/r} < 1$. Taking into account that $l_i \geq 1$ for each i, this is clearly impossible.) But then, by the constant surplus condition, at least one of the inequalities denoting expected payoffs of b_1 and s_1 must hold with equality. Suppose, without loss of generality, that it is b_1. Therefore, we must have:

$$-c_B + n(r)p_{s1}(m_{11}(1 - \delta v_{b1} - \delta v_{s1}) + m_{12}(0 - \delta v_{b1} - \delta v_{s2}) = (1 - \delta)v_{b1}$$

We have already shown that $m_{12} = 0$ and $1 - \delta v_{b1} - \delta v_{s1} \leq 0$. Substituting gives: $-c_B = (1 - \delta)v_{b1}$, which, by the individual rationality condition, indicates a contradiction. □

Since we have identified that $(m_{11}, m_{12}, m_{21}, m_{22}) = (1, 0, 0, 1)$ in an equilibrium, now, we continue by deriving the equilibrium values of $(l_{b_1}, l_{b_2}2, l_{s_1}, l_{s_2})$. Substituting the values of m_{bs} into the constant surplus and steady state conditions, we have the following inequalities for each buyer-seller pair:

$$-c_B + n(r)\beta p_s(1 - \delta v_b - \delta v_s) \leq (1 - \delta)v_b$$
$$-c_S + \frac{n(r)}{r}(1 - \beta)p_b(1 - \delta v_b - \delta v_s) \leq (1 - \delta)v_s$$

where the inequality holds with equality for i with $l_i > 1$.

$$l_b n(r)p_s = l_s \frac{n(r)}{r} p_b \leq 1$$

where the inequality holds with equality for i with $v_i > 0$.

Our initial aim is to find an upper bound for l_b and l_s, as we did in the 1-by-1 setup. To achieve this goal, we make some simplifying assumption that we will relax later. For now, assume that the market equilibrium is *symmetric* in the sense that $l_{b_1} = l_{b_2} = l_b$ and $l_{s_1} = l_{s_2} = l_s$. Thanks to this assumption, we can substitute each probability with 1/2, and furthermore, we can use the equality $r = \frac{l_s}{l_b}$.

Now, can we find an upper bound on the equilibrium measures l_b and l_s. Firstly, a case-by-case analysis of the steady state condition shows that there is already an upper bound on the shorter side of the market. That is, if $l_s \leq l_b$, we need to have $l_s \leq 2$ and if $l_b \leq l_s$, we need to have $l_b \leq 2$.

But what about the longer side? Suppose that we have $l_b \geq l_s$ (i.e. buyers are at the longer side of the market) and the shorter side has already achieved its upper bound (i.e. $l_s = 2$.)[11] Clearly, since $1 < l_s \leq l_b$, by the constant surplus condition, both inequalities defining v_b and v_s must hold as equality. Then we must have:

$$(1-\delta)v_b = -c_B + \frac{l_s}{l_b}\frac{\beta}{2}(1 - \delta v_b - \delta v_s) \geq 0$$
$$(1-\delta)v_s = -c_S + \frac{1-\beta}{2}(1 - \delta v_b - \delta v_s) \geq 0$$

Again, simple algebra shows that the two nonnegativity constraints can be stated as: $\delta(v_b + v_s) \leq \min\{1 - \frac{c_B l_b}{\beta}, 1 - \frac{2c_S}{1-\beta}\}$. Furthermore, some algebraic manipulation on the equalities show that:

$$\delta(v_b + v_s) = \frac{-c_B - c_S + \frac{\beta}{l_b} + \frac{1-\beta}{2}}{\frac{1-\delta}{\delta} + \frac{\beta}{l_b} + \frac{1-\beta}{2}}$$

Therefore, the inequality:

$$\frac{-c_B - c_S + \frac{\beta}{l_b} + \frac{1-\beta}{2}}{\frac{1-\delta}{\delta} + \frac{\beta}{l_b} + \frac{1-\beta}{2}} \leq \min\{1 - \frac{c_B l_b}{\beta}, 1 - \frac{2c_S}{1-\beta}\}$$

[11]This is not a very restrictive assumption, and is merely for computational ease. The analysis would not really differ from the one presented here if we assumed the general case $l_s \in (1, 2]$–one just needs to replace 2 by l_s in the bound above. When $l_s = 1$, however, different forces might still be at work, because the constant surplus condition does not necessarily hold any more. It can be shown, however, that the standard upper bound $\frac{\kappa_S/(1-\beta)}{\kappa_B/\beta}$ is valid for l_b. The proof is essentially the same as in the one provided in 1-by-1 setup.

follows. Using this inequality, one can obtain the upper bound on l_b:[12]

$$l_b \leq 2(\frac{c_S/1-\beta}{c_B/\beta} + \frac{1-\delta}{\delta}\frac{\beta}{1-\beta}\frac{1}{c_B})$$

where, as $\Delta \to 0$, the right hand side converges to: $2(\frac{\kappa_S/1-\beta}{\kappa_B/(\beta)} + \frac{\beta}{1-\beta}\frac{r}{\kappa_B})$, a well-defined upper bound.

The analysis provided so far is useful in the sense that it gives hints favoring the idea that the 2-by-2 setup is far from problems, too. However, in order to feel confident on this conclusion, one needs a general treatment of the setup; for instance, we need to drop the initial assumption of *symmetry* of the market. This is what we do in the following pages.

Let's begin with a recap of the recent findings in the setup. The derivation of $(m_{11}, m_{12}, m_{21}, m_{22}) = (1, 0, 0, 1)$ does not only simplify the constant surplus condition tremendously: it also has a quite elegant implication on the steady-state condition. Thanks to the finding that $m_{ij} = 0$ for $i \neq j$, we are now sure that for agents who are perfect matches of each other, the steady-state conditions are equivalent to each other, i.e. we can express both steady-state conditions as:

$$l_{b_i} n(r) p_{s_i} = l_{s_i} \frac{n(r)}{r} p_{b_i} \leq 1$$

where the inequality holds with equality for i with $v_i > 0$.

Therefore, the conditions defining the equilibrium can be revised as follows as each buyer-seller pair (b_i, s_i):

(*i*) $v_{b_i}, v_{s_i} \geq 0$, $l_{b_i}, l_{s_i} \geq 1$.

[12] Again, the algebra is very similar to the 1-by-1 case. The right hand-side depends on which one of the quantities $\frac{c_B l_b}{\beta}$ and $\frac{2c_S}{1-\beta}$ is larger. If $\frac{c_B l_b}{\beta} \leq l_s \frac{2c_S}{1-\beta}$, then, the usual bound $l_s \leq 2\frac{c_S/1-\beta}{c_B/(\beta)}$ applies. If $\frac{c_B l_b}{\beta} > \frac{2c_S}{1-\beta}$, then, simple algebra yields the inequality: $l_b < \frac{2\beta\delta c_S + 2\beta(1-\delta)}{2(1-\delta)c_B + \delta(1-\beta)c_B}$. The $2(1-\delta)c_B$ term in the denominator is negligible, and omitting it preserves the inequality.

(ii)

$$-c_B + n(r)\beta p_{s_i}(1 - \delta v_{b_i} - \delta v_{s_i}) \le (1-\delta)v_{b_i}$$
$$-c_S + \frac{n(r)}{r}(1-\beta)p_{b_i}(1 - \delta v_{b_i} - \delta v_{s_i}) \le (1-\delta)v_{s_i}$$

where the inequality holds with equality for i with $l_i > 1$.

(iii)

$$l_{b_i} n(r) p_{s_i} = l_{s_i}\frac{n(r)}{r}p_{b_i} \le 1$$

where the inequality holds with equality if $v_{b_i} > 0$ or $v_{s_i} > 0$.

We will continue by eliminating some unpleasant possibilities. Begin by realizing that if at least one of the valuations is nonzero, the steady state condition binds and we can make a significant progress in terms of analyzing the market size. This is because the steady-state condition, when binds, indeed gives l_{s_i} as a function of p_{b_i} (and vice versa), and we would have taken one side of the market under control. Note, however, the condition $v_{b_i} > 0$ or $v_{s_i} > 0$ must be satisfied to make this progress. We now show that this is necessarily the case in an equilibrium.

Lemma 4.14. *In the equilibrium, at least one side of each buyer-seller pair has nonzero valuation ($v_{b_i} > 0$ or $v_{s_i} > 0$ for each $i = \{1, 2\}$.)*

Proof. Suppose, to get a contradiction, that $v_{b_1} = v_{s_1} = 0$. Plugging these values into condition (iii) above, we obtain: $n(r)p_{s_1} \le \frac{c_B}{\beta}$ and $\frac{n(r)}{r}p_{b_1} \le \frac{c_B}{1-\beta}$. For simplicity, assume that the buyers are at the longer side of the market (the analysis of the alternative case is essentially the same as this one.) Furthermore, as we do usually, assume that $\frac{\kappa_B}{\beta} = \frac{\kappa_S}{1-\beta}$. Then these two inequalities imply that $\frac{l_{b_1}}{l_{b_1}+l_{b_2}}$ and $\frac{l_{s_1}}{l_{b_1}+l_{b_2}}$ are bounded above by $(c_B + c_S)$.

Assuming $l_{b_1} = l_{s_1} = l_1$,[13] we obtain: $l_{b_2} \geq \frac{1-c_B-c_S}{c_B+c_S}$, i.e. l_{b_2} must grow unboundedly in the order of $\frac{1}{c} = \frac{1}{\Delta\kappa}$, thus, in the order of $\frac{1}{\Delta}$. First, realize that this immediately implies $l_{b_2} > 1$, and therefore the constant surplus condition with respect to b_2 must hold with equality. For the (b_2, s_2) pair, we then have:

$$(1-\delta)v_{b_2} = -c_B + \frac{l_{s_2}}{l_{b_1} + l_{b_2}}(\beta)(1 - \delta v_{b_2} - \delta v_{s_2})$$
$$(1-\delta)v_{s_2} \geq -c_S + \frac{l_{b_2}}{l_{b_1} + l_{b_2}}(1-\beta)(1 - \delta v_{b_2} - \delta v_{s_2})$$

These two conditions together imply:

$$\frac{l_{b_2}}{l_{s_2}} \leq \frac{(1-\delta)v_{s_2} + c_S}{(1-\delta)v_{b_2} + c_B}\frac{\beta}{1-\beta}$$

with equality if $l_{s_2} > 1$.

Take a moment to appreciate that, as a result of this inequality, l_{s_2} remains comparable to l_{b_2}, and consequently, l_{s_2} converges to infinity in the order of $\frac{1}{\Delta}$. Now, we have l_{b_2} and l_{s_2}, both growing unboundedly. Note, however, by the steady-state condition of (b_2, s_2) pair, we must have: $\frac{l_{s_2} l_{b_2}}{l_{b_1}+l_{b_2}} = 1$. Taking into account that l_{b_1}/l_{b_2} is close to zero, this is clearly a contradiction. □

Given this lemma, now, we know that at least one agent in each match has strictly positive valuation, and consequently, the steady-state condition holds with equality for each match. Assuming (without loss of generality) that the buyers are at the longer side of the market, this translates into: $l_{s_i} p_{b_i} = 1$ for each i. It naturally follows that $l_{s_i} = \frac{1}{p_{b_i}}$ for each i, i.e. one side of the market is determined as a function of the other, as we had conjectured.

Since $p_i \in [0,1]$ for each i, $\frac{1}{p_i} \in [1, \infty]$ and thus no trivial equilibrium condition is also automatically satisfied with this equality. Now, if we can show that $p_{b_i} \in (0,1)$, this equality also gives us $l_{s_i} > 1$, which in turn implies that the condition defining v_{s_i} holds with equality. Therefore, there is a strong motivation behind proving that

[13] Again, a simplifying assumption. Relaxing it would not change the results dramatically.

$p_{b_i} \in (0,1)$, or, more intuitively, proving that l_{b_1} and l_{b_2} remain comparable in the equilibrium. Next lemma shows that this is actually the case.

Lemma 4.15. *In the equilibrium,* $0 < \frac{l_{b_1}}{l_{b_2}} < \infty$.

Proof. To get a contradiction, suppose (without loss of generality) that $\frac{l_{b_1}}{l_{b_2}} \to \infty$, which, in turn, implies $p_{b_1} = 1$, $p_{b_2} = 0$. Therefore, we need to have $l_{s_1} = 1/p_{b_1} = 1$. On the other hand, $\frac{l_{b_1}}{l_{b_2}} \to \infty$, along with $l_{b_2} \geq 1$, clearly implies $l_{b_1} > 1$; therefore, the condition defining v_{b_1} holds with equality. Following the same strategy as the one used in the previous proof, one can show that:

$$\frac{l_{b_1}}{l_{s_1}} \leq \frac{(1-\delta)v_{s_1} + c_S}{(1-\delta)v_{b_1} + c_B} \frac{\beta}{1-\beta}$$

Again, this implies that l_{s_1} is comparable to l_{b_1}, which grows unboundedly. Yet we had arrived at the conclusion that $l_{s_1} = 1$, a contradiction. □

By the help of this lemma, we can vaguely characterize how the market measures in the equilibrium look like. We know that l_{b_1} and l_{b_2} remain comparable relative to each other, i.e. $p_{b_1}, p_{b_2} \in (0,1)$. Furthermore, the equality $l_{s_i} = \frac{1}{p_{b_i}}$ takes the equilibrium measures of shorter side under control. The only concern left, therefore, regards whether L_b grows uncontrollably relative to L_s. To evaluate this possibility, we go back to the equations defining the valuations of agents.

Since $p_{b_i} \in (0,1)$, $l_{s_i} > 1$ for each i, and thus the conditions defining the valuations of sellers hold with equality. Hence we have:

$$(1-\delta)v_{b_i} \geq -c_B + \frac{l_{s_i}}{l_{b_1} + l_{b_2}}(\beta)(1 - \delta v_{b_i} - \delta v_{s_i})$$

$$(1-\delta)v_{s_i} = -c_S + \frac{l_{b_i}}{l_{b_1} + l_{b_2}}(1-\beta)(1 - \delta v_{b_i} - \delta v_{s_i})$$

where the first one holds with equality if $l_{b_i} > 1$.

Let's assume for a moment that $l_{b_i} > 1$, because we are trying to come up with an upper bound on l_{b_i}. When the quantity under consideration is equal to 1, clearly, this is not a very interesting pursuit. Then we can confidently assume that both conditions hold with equality, which yields:

$$\frac{l_{b_i}}{l_{s_i}} = \frac{(1-\delta)v_{s_i} + c_S}{(1-\delta)v_{b_i} + c_B} \frac{\beta}{1-\beta}$$

Using the fact that $l_{s_i} = \frac{1}{p_{b_i}} = \frac{l_{b_1}+l_{b_2}}{l_{b_i}}$, this translates into: $\frac{l_{b_i}^2}{l_{b_1}+l_{b_2}} = \frac{(1-\delta)v_{s_i}+c_S}{(1-\delta)v_{b_i}+c_B} \frac{\beta}{1-\beta}$. Realize that, if one assumes $\frac{c_B}{\beta} = \frac{c_S}{1-\beta}$ as usual, then the right-hand of the equality can be simplified to: $\frac{1+\frac{1-\delta}{c_S}v_{s_i}}{1+\frac{1-\delta}{c_B}v_{b_i}}$, which, as $\Delta \to 0$, converges to $\frac{1+\frac{r}{\kappa_S}v_{s_i}}{1+\frac{r}{\kappa_B}v_{b_i}}$. Clearly, this is a quantity that heavily depends on $\frac{r}{\kappa_i}$, which is hardly surprising.

To take the analysis one step further, define $\gamma_i = \frac{1+\frac{r}{\kappa_B}v_{b_i}}{1+\frac{r}{\kappa_S}v_{s_i}}$. Then the equality becomes: $\frac{l_{b_1}+l_{b_2}}{l_{b_i}^2} = \gamma_i$, or, equivalently, we have the following set of equalities satisfied as $\Delta \to 0$:

$$l_{b_1} = \gamma_2 l_{b_2}^2 - l_{b_2}$$
$$l_{b_2} = \gamma_1 l_{b_1}^2 - l_{b_1}$$

There is a unique solution to this set of nonlinear equalities, whose solution depends on the values of γ's. If we assume that $\gamma_1 = \gamma_2 = \gamma$, then the unique solution is at $l_{b_1} = l_{b_2} = \frac{2}{\gamma}$. Yet, as mentioned above, the value of γ, and thus the values of equilibrium measures depend on the ratio between r and κ. In the two extreme cases,

When $\frac{r}{\kappa} = 0$, we have $\gamma_1 = \gamma_2 = 1$ and the equilibrium measures are forced to be $l_{b_1} = l_{b_2} = 2$. We encounter a perfectly symmetric market with $l_i = 2$ for each i.

When $\frac{r}{\kappa} = \infty$, things get more complicated. We have a case where $\frac{l_{b_1}+l_{b_2}}{l_{b_i}^2} = \frac{v_{b_i}/\beta}{v_{s_i}/1-\beta}$. But what are the valuations? They do, in turn, depend on the equilibrium measures, so there is some cyclicity in the argument, which makes this case hard to deal with. Still, there is an upper limit on the rate with which l_{b_i} can grow. Following the analysis conducted in the symmetric case, and assuming $\frac{\kappa_B}{\beta} = \frac{\kappa_S}{1-\beta}$, it can be shown that $l_{b_i} \leq l_{s_i}(1 + \frac{\beta}{1-\beta}\frac{1-\delta}{\delta}\frac{1}{c_B})$. Obviously, the term $\frac{1-\delta}{c_B}$ places an upper bound on the growth rate of $\frac{l_{b_i}}{l_{s_i}}$. Yet, due to the complications in the analysis of this case, it is sensible to assume that $\kappa_i > 0$ for each i. Thanks to this assumption, the structure of the setup now does not allow for the case $\frac{r}{\kappa} = \infty$, and everything about the equilibrium measures are well-defined.

This wraps up the discussion about equilibrium measures. The only things that remains to be analyzed is the behavior of equilibrium valuations. The analysis should be familiar to the reader by now, and thus we only present the results. In one extreme case where $\frac{r}{\kappa} = 0$, the equilibrium measures are already tightly set. It can be shown that, in this case:

$$v_{b_i} = \frac{\beta}{2-\delta} - \frac{2c_B}{2-\delta}$$
$$v_{s_i} = \frac{1-\beta}{2-\delta} - \frac{2c_S}{2-\delta}$$

which clearly converge to β and $(1-\beta)$, respectively, as frictions disappear. In the other extreme, when $\frac{r}{\kappa} = \infty$, the choice of equilibrium measures l_{b_i} and l_{s_i} are free, as long as they remain within the limits set by the upper bound mentioned above. Nevertheless, one can show that, as the frictions disappear,

$$v_{b_i} \to \frac{\beta/l_{b_i}}{\beta/l_{b_i} + (1-\beta)/l_{s_i}}$$
$$v_{s_i} \to \frac{(1-\beta)/l_{s_i}}{\beta/l_{b_i} + (1-\beta)/l_{s_i}}$$

which obviously makes sense, since the surplus is shared according to the bargaining power and the equilibrium measures (when the bargaining powers are equal, the model again unsurprisingly collapses to that of Rubinstein and Wolinsky (1985).)

4.3.3 Discussion

One critical point that needs to be kept in mind in the analysis of this case is that, if the surplus function is generated in a way that would enforce a unique matching, then the analysis of the market turns out to be less complicated than what would be otherwise: one can begin the analysis by asserting that the unique matching would eventually be realized (with sufficiently small frictions) and then show that the market is essentially the same as a setup where there are a group of bargaining pairs. This is the approach that we will generalize in the proof of the main theorem.

The second critical point that one can learn from this analysis that, when $\frac{r_i}{\kappa_i} = \infty$ for some i (which occurs when $\kappa_i = 0$) the analysis of the case turns becomes challenging. Intuitively, this is because when $\kappa_i = 0$ for some i, $c_i = 0$ and remaining in the market is not costly for the agent, even if she is not able to find a match to trade with. Therefore, that agent might prefer to stay in the market infinitely long (which yields zero payoff) and we might encounter a type which dominates one side of the market even though she doesn't trade, thus forestalling other matches from finding each other and trading. Having the assumption "$\kappa_i > 0$ for each $i \in B \cup S$" is a simple and effective solution to this unpleasant possibility.

Now, having observed the important points behind the analysis of simple cases, one can continue with the treatment of the general model. We first present the assumptions that we will retain in the general setup.

Chapter 5

Assumptions

Below is a list of the assumptions on the generalized setup, that we retain throughout this study.

Assumption 5.1. B and S are finite sets, and $|B| = |S|$.

The first part of the assumption (finiteness of types and equal cardinality of type sets) is made for simplicity, the setup can easily be extended to a model with a continuum of agents. The second part of the assumption (same cardinality) is also not that restrictive; as we have already demonstrated in Section 4.2, the cases where $|B| \neq |S|$ can be converted into ones with $|B| = |S|$ with proper adjustments.

Assumption 5.2. The surplus function $f : B \times S \to \mathbb{R}^+$ is strictly supermodular, i.e. if $b_1 > b_2$ and $s_1 > s_2$, then $f_{b_1,s_1} + f_{b_2,s_2} > f_{b_1,s_2} + f_{b_2,s_1}$. Furthermore, for each type $b_i \in B$ and $s_i \in S$, $f_{ii} > 0$.

The first part of this assumption is quite crucial for the heart of this study; eventually, we will use this assumption to impose a particular matching, namely *perfect assortative matching*, in the search equilibrium as friction disappear. A simplified version of the strategy of imposing a particular matching is already provided in Section 4.3. The reasoning behind the second part is, to some extent, obvious: we don't want to have any *redundant* types in the market, hence it is not very demanding to ask that for each buyer (seller) i, there is a seller (buyer) j such that $f_{ij} > 0$ ($f_{ji} > 0$). All we do is to ask that one of these *nonzero* types happen to be the corresponding type.

Assumption 5.3. $l_i \geq 1$ for each $i \in B \cup S$.

This is essentially an assumption about the structure of the market: we assume that the agents are born *into* the market at the first period they appear. Indeed, it serves a very practical purpose: it eliminates the possibility of a trivial search equilibrium, where no one enters into the market and no trade occurs (such an equilibrium always exists if we drop this Assumption.) Remember that this Assumption was also made in Section 4.1, and, as expressed in there, the Proposition that proves that this assumption indeed serves its purpose is provided and proven in the Appendix.

Now, having established the basic blocks of the setup, we can continue with the core theorem of this study that investigates the relationship between search equilibria and competitive equilibria.

Chapter 6

Main Theorem and Proof

The following is the central result of this study.

Theorem 6.1. *Let $\mathcal{S}$ be a search economy and $\mathcal{S}^*$ be its frictionless counterpart. Accordingly, let $\mathcal{E}(\mathcal{S})$ be the set of search equilibria and $\mathcal{E}^*(\mathcal{S}^*)$ be the set of competitive equilibria. Then the following are true:*

- *For each $e \in \mathcal{E}(\mathcal{S})$, $e \to e^*$ for some $e^* \in \mathcal{E}^*(\mathcal{S}^*)$ as $\Delta \to 0$.*
- *Set $\beta = 1/2$, and assume a uniform search cost for each type ($\kappa_b = \kappa_s = \kappa$.) For each $e^* \in \mathcal{E}^*(\mathcal{S}^*)$, there exists a vector of relative patience parameters $\tilde{r}$ such that for each $\mathcal{S} = (\mathcal{S}^*, 1/2, \kappa, \tilde{r}, \Delta)$ with any $\kappa > 0$, there exists some $e \in \mathcal{E}(\mathcal{S})$ such that $e \to e^*$ as $\Delta \to 0$.*

Remark 6.2. The second part of the Theorem can be relaxed in multiple ways. Setting $\beta = 1/2$ is made for convenience; one can choose any $\beta \in (0, 1)$, and the proof works identically with any cost vector κ with the restriction that $\frac{\kappa_{b_i}}{\kappa_{s_i}} = \frac{\beta}{1-\beta}$ for any i. Dropping this restriction and allowing any vector $\kappa \gg 0$ is also possible, but the analysis gets more complicated. For the purposes of this study, demonstrating that there exists *some* search equilibria that approximates any competitive equilibrium is sufficient, and we prefer to stick to this version of Theorem.

The proof of this theorem utilizes and combines several Propositions throughout the process. The basic strategy is as follows: we first demonstrate that strict supermodularity of surplus function imposes a *perfect assortative matching* in the equilibrium, i.e. each buyer of type i trades with only seller i and vice versa. Then, we show that, as an implication of having two types of frictions which remain comparable to each other, the equilibrium steady-state measures of each type remain comparable to their counterparts, i.e. l_{b_i} and l_{s_i} are comparable to each other for each i. This fact, combined with the steady-state condition, implies that probability of finding each type remains bounded away from zero. Consequently, the connection between the search economy and market economy, where the latter is characterized by Shapley and Shubik (1972) becomes apparent.

Now, let's proceed by proving the first claim, which is established on the connection between strictly supermodularity and perfect assortative matching.

6.1 Perfect Assortative Matching

By perfect assortative matching, we insinuate that any agent of type i trades only with the corresponding agent of type i in the equilibrium. In our notation, this corresponds to the observation that the trading possibility of types which do not correspond to each other are zero. Formalization of this idea is as follows:

Definition 6.3. A search equilibrium $e \in \mathcal{E}(\mathcal{S})$ admits **perfect assortative matching** if: $m_{b_i,s_j} = 0$ for any $i \neq j$ and $m_{b_i,s_i} = 1$ for each i.

The following Proposition, whose proof is provided in the Appendix, is pivotal to this section of study.

Proposition 6.4. *Take any search economy $\mathcal{S}$, and consider the equilibrium $e \in \mathcal{E}(\mathcal{S})$ as $\Delta \to 0$. There exists $\underline{\Delta} > 0$ such that, for all $\Delta < \underline{\Delta}$, e admits perfect assortative matching.*

Some elaborate analysis of Proposition 6.4 shows that the result it suggests is indeed quite intuitive. We demonstrate that, when the economy under question attains sufficiently low frictions, there is a unique matching it admits: in that unique matching, the buyer of type 1 trades with only seller 1, buyer 2 only with seller 2, and similarly for each pair up to type n. The critical point behind this result is Assumption 5.2. Remember that strict supermodularity of surplus function has the interpretation that *high* types should better be matched with *high* types, and *low* types similarly with *low* ones. By imposing this condition on the surplus function, we implicitly require that every agent should better be matched with her corresponding type, and thus the result of perfect assortative matching follows.

It is worth noting that the relationship between a strictly supermodular surplus function and assortative matching is discovered long before: Becker (1973) is the first study to discover that a strictly supermodular production function enforces assortative matching in a frictionless search market. Shimer and Smith (2000) demonstrates that the result might might fail to extend to a model with frictions, when the only type of frictions available is the patience of agents. Atakan (2006), on the other hand, shows that Becker (1973)'s result continues to hold in a model where the only type of friction is search costs, which need to be paid explicitly. This study is, to the best of our knowledge, the first one to involve a model where there are both implicit costs (patience) and explicit search costs. Proposition 6.4 therefore has the alternative interpretation that Becker (1973)'s result still holds in such a model. Considering the setup, it is trivial to realize that there is always a nonnegligible explicit component of search costs, hence it should not be surprising to observe that Atakan (2006)'s result extends to this model. Nevertheless, it is worthwhile to note that the previous papers in the literature involve a continuum of agents rather than a finite set of types, hence the definition of assortative matching and strategy of the proof differs between those papers and this study.

Now, having been done with the analysis of trading probabilities, we can move with the analysis of steady-state measures.

6.2 Market Size is Bounded

This section contains one of the most critical results of this study: we show that there is a uniform upper bound on the size of the market. The following Proposition, whose proof is in the Appendix, asserts this.

Proposition 6.5. *In any search equilibrium $\mathcal{E}(\mathcal{S})$, the market size parameters (L and Lr) remain finite as the frictions disappear.*

The crucial point in understanding this Proposition and why it makes sense hinges on recognizing that, for the market size to diverge to infinity, there must be some types who remain in the market infinitely long. This means that some types enter the market, and even though they are unable to find their corresponding types, they keep on waiting each period. Note, however, the two types of frictions are operative: the implicit cost (discounting the payoff by r_i) and the explicit cost (paying the search cost $\Delta\kappa_i$) each period. As $\Delta \to 0$, both frictions disappear, but if the explicit component is always nonnegligible when compared to the implicit one, then remaining in the market becomes not profitable, and thus not individually rational. The model we construct satisfies this property, hence it should not be surprising to see that the observation $\frac{\kappa_i}{r_i} > 0$ for each i lies at the heart of the proof. Thanks to comparability of two types of frictions, no one prefers to stay in the market infinitely long, thus the market size does not explode.

The crucial result that we will use in establishing the connection between search equilibria and competitive equilibria follows quite easily from this Proposition.

Corollary 6.6. *In any search equilibrium $\mathcal{E}(\mathcal{S})$, the probability of each type to be found remains bounded away from zero ($p_i > 0$ for each i.)*

Proof. Realize that $p_{b_i} = \frac{l_{b_i}}{L}$ and $p_{s_i} = \frac{l_{s_i}}{Lr}$. By the No Trivial Equilibrium condition, the numerators are greater than 1, and by Proposition 6.5, the denominators are bounded. □

It needs to be emphasized that Corollary 6.6 has encouraging implications for the analysis of search equilibrium. This is because, if any agent can be found with a strictly positive probability, then waiting for the "ideal partner" would just require an agent to be patient enough, which is eventually the case when $\Delta \to 0$. This implies that Corollary 6.6 might constitute an important step in proving that a search equilibrium converges to a competitive one as frictions disappear. We show that this is indeed the case in the next section.

6.3 Search Equilibria Becomes Competitive

The following result lies at the heart of this study: the connection between any equilibrium $e \in \mathcal{E}(\mathcal{S})$ in a search economy and the competitive equilibrium in its frictionless counterpart, $e^* \in \mathcal{E}^*(\mathcal{S}^*)$. It proves the first part of Theorem 6.1 and a significant portion of the second part.

Consider any search-and-matching equilibrium $e = (l, m, v)$. Define the measure of (b, s) pairs who trade and leave the market at each period as: $\hat{q}_{bs} = Ln(r)p_b p_s m_{bs}$. The following proposition draws the parallel between two equilibria.

Proposition 6.7. *For any search-and-matching equilibrium $e = (l, m, v) \in \mathcal{E}(\mathcal{S})$ of an economy $\mathcal{S} = (B, S, f, \beta, \kappa, r, \Delta)$, the implied measures and values $(\hat{q}, v)$ constitute an assignment equilibrium of $\mathcal{S}^* = (B, S, f)$ as $\Delta \to 0$.*

The main idea in the proof of this Proposition, which is provided in the Appendix, is as follows: Since Shapley and Shubik (1972) completely characterizes the equilibrium of a frictionless economy, it is sufficient to show that $(\hat{q}, v)$ satisfies the conditions specified by this characterization. Namely, we show that $\hat{q}$ constitutes a solution to the primal problem and v constitutes a solution to the dual problem. It is worth reminding that Corollary 6.6 is crucial for the proof, as expected.

Given a frictionless economy $\mathcal{S}^*$, let $\mathcal{L}(\mathcal{S}^*)$ denote the set of set of all limit points of the equilibria of the economies that extend $\mathcal{S}^*$. Note that Proposition 6.7 asserts:

$\mathcal{L}(\mathcal{S}^*) \subseteq \mathcal{E}^*(\mathcal{S}^*) = \mathcal{E}(\mathcal{S}^*)$, i.e. the correspondence $\mathcal{E}(\mathcal{S})$ is upper hemi-continuous at the limit, hence the first part of the theorem is established. One can now proceed with the converse analysis.

6.4 Each Competitive Equilibrium is Approximated

It is worth noting that, for any competitive equilibrium $e^* = (q, \nu)$ of a frictionless economy, q is well-defined and the convergence q is proven by Proposition 6.4. Therefore, the only parameter whose range needs to be covered is ν. By the discussion provided in the previous section (most notably, in the proof of Lemma F.1), we know that $\nu \in \mathbb{R}^{|B|+|S|}$ is completely characterized by:

$$\nu \in \mathbb{R}^{|B|+|S|} \textit{ such that } \nu_{b_i} + \nu_{s_j} \geq f_{ij} \ \forall (b_i, s_j) \in B \times S$$
$$\textit{with equality if } i = j.$$

One only needs to show that the whole range of (ν_{b_i}, ν_{s_i}) can be covered via proper selection of frictions. The following Proposition asserts this, and it is even in a stronger form than required: it demonstrates that, without the need to pick κ_i's, picking (r_{b_i}, r_{s_i}) properly for each i would suffice. Hence, for the rest of the analysis, we assume that a single parameter $\kappa > 0$ specifies all the search costs, i.e. $\kappa_i = \kappa$ for each $i \in B \cup S$.

Proposition 6.8. *For each $e^* \in \mathcal{E}^*(\mathcal{S}^*)$, there exists a vector of patience parameters $(\tilde{r})$ such that given $\mathcal{S} = (\mathcal{S}^*, 1/2, \kappa, \tilde{r}, \Delta)$ with any $\kappa > 0$, there exists an $e \in \mathcal{E}(\mathcal{S})$ such that $e \to e^*$ as $\Delta \to 0$.*

Proof. Take any pair (ν_{b_i}, ν_{s_i}) which is a part of e^*. The strategy of the proof is to construct a search equilibrium with $v_{b_i} \to \nu_{b_i}$ and $v_{s_i} \to \nu_{s_i} = f_{ii} - \nu_{b_i}$.

We will construct the simplest symmetric search equilibrium with $l_i = n$ for each $i \in B \cup S$ (which directly implies: $p_i = \frac{1}{n}$ for each i.) Realize that, thanks to this choice of steady-state measures, the steady-state conditions for each pair definitely holds, hence one basically needs to consider the constant surplus conditions.

Set $\beta = 1 - \beta = 1/2$, realize that, by the choice of steady-state measures, $r = 1$. Furthermore, since $l_{b_i} = l_{s_i} = n > 1$, the constant surplus conditions hold with equality, which are:

$$\begin{aligned}(1-\delta_{b_i})v_{b_i} &= -c_{b_i} + \frac{l_{s_i}}{2L}(f_{ii} - \delta_{b_i}v_{b_i} - \delta_{s_i}v_{s_i})\\ (1-\delta_{s_i})v_{s_i} &= -c_{s_i} + \frac{l_{b_i}}{2L}(f_{ii} - \delta_{b_i}v_{b_i} - \delta_{s_i}v_{s_i})\end{aligned}$$

Direct solution of the equalities yields the following closed-form solutions:

$$\begin{aligned}v_{b_i} &= \frac{f_{ii}\frac{l_{s_i}}{2L} - c_{b_i} + f_{ii}\frac{\delta_{s_i}}{2L}\frac{1}{1-\delta_{s_i}}(c_{s_i}l_{s_i} - c_{b_i}l_{b_i})}{1 - \delta_{b_i} + \delta_{b_i}\frac{l_{s_i}}{2L} + \frac{1-\delta_{b_i}}{1-\delta_{s_i}}\delta_{s_i}\frac{l_{b_i}}{2L}}\\ v_{s_i} &= \frac{f_{ii}\frac{l_{b_i}}{2L} - c_{s_i} + f_{ii}\frac{\delta_{b_i}}{2L}\frac{1}{1-\delta_{b_i}}(c_{b_i}l_{b_i} - c_{s_i}l_{s_i})}{1 - \delta_{s_i} + \delta_{s_i}\frac{l_{b_i}}{2L} + \frac{1-\delta_{s_i}}{1-\delta_{b_i}}\delta_{b_i}\frac{l_{s_i}}{2L}}\end{aligned}$$

where, using the equalities $c_{b_i} = \Delta\kappa_{b_i}$, $c_{s_i} = \Delta\kappa_{s_i}$, $\delta_{b_i} = e^{-r_{b_i}\Delta}$ and $\delta_{s_i} = e^{-r_{s_i}\Delta}$, converge to the following values:

$$\begin{aligned}\lim_{\Delta\to 0} v_{b_i} &= f_{ii}\frac{l_{s_i} + \frac{1}{r_{s_i}}(\kappa_{s_i}l_{s_i} - \kappa_{b_i}l_{b_i})}{l_{s_i} + \frac{r_{b_i}}{r_{s_i}}l_{b_i}}\\ \lim_{\Delta\to 0} v_{b_i} &= f_{ii}\frac{l_{b_i} + \frac{1}{r_{b_i}}(\kappa_{b_i}l_{b_i} - \kappa_{s_i}l_{s_i})}{l_{b_i} + \frac{r_{s_i}}{r_{b_i}}l_{s_i}}\end{aligned}$$

One can now set $\kappa_{b_i} = \kappa_{s_i} = \kappa$ and use $l_{b_i} = l_{s_i}$ to obtain the simplifications:

$$\begin{aligned}\lim_{\Delta\to 0} v_{b_i} &= f_{ii}\frac{r_{s_i}}{r_{b_i} + r_{s_i}}\\ \lim_{\Delta\to 0} v_{b_i} &= f_{ii}\frac{r_{b_i}}{r_{b_i} + r_{s_i}}\end{aligned}$$

Realize that, not very surprisingly, these values correspond to the shares that the parties obtain in Rubinstein (1982).[1] One now has the freedom to pick $(r_{b_i}, r_{s_i}) \geq (0, 0)$ to ensure that $\lim_{\Delta \to 0}(v_{b_i}, v_{s_i}) = (\nu_{b_i}, \nu_{s_i})$. □

This Proposition, therefore, establishes the second part of Theorem and demonstrates that $\mathcal{E}(\mathcal{S}^*) \subseteq \mathcal{L}(\mathcal{S}^*)$. Combined with the argument in the previous section, this means $\mathcal{E}(\mathcal{S}^*) = \mathcal{L}(\mathcal{S}^*)$, and thus, the correspondence $\mathcal{E}(\mathcal{S})$ is both upper and lower hemi-continuous (i.e. continuous) at the limit.

[1] The original work indeed uses δ's directly and shows that $v_1 = \frac{1-\delta_2}{1-\delta_1\delta_2}$. Note, however, that taking $\delta_i = e^{-r_i\Delta}$, one can easily see that $v_1 \to \frac{r_2}{r_1+r_2}$ as $\Delta \to 0$.

Chapter 7

Conclusion

This study establishes the continuity of search equilibrium correspondence under the assumption of strictly supermodular surplus function, and, to the best of our knowledge, the first one to come up with the converse analysis of Gale (1987). One particular dimension which these results will be improved upon is the restrictiveness of the strict supermodularity assumption. It does not take so long to realize that the basic use of strict supermodularity assumption is to impose a particular and unique matching in the equilibrium. Therefore, it seems intuitively plausible that the less restrictive assumption of "unique matching in the limit" will also be sufficient for our purposes. This is indeed the current line of research we are working upon. On the other hand, when the matching in the limit fails to be unique, the extension of the results into a more general setting does not occur so trivially. This is because, in the current setup, the result of perfect assortative matching enables us to deduce that the agents have no outside options rather than their corresponding partners. Therefore, one can simply proceed as if all the bargaining process between b_i and s_i occurs on the question of how the sharing of f_{ii} will be realized between the two agents, totally isolated from the results of other bargainings in the market. This, unfortunately, is no longer the case when the market in hand does not have a unique matching: the agents possibly have outside options that are nonzero, hence a more assiduous analysis of what goes on in the other matchings becomes relevant. [1] The arguments presented here might need a substantial revision, if one needs to develop an ultimate theory on the strategic foundations of general equilibrium.

Another direction in which this study can be improved upon might be adding another type of friction, namely, the friction of incomplete information into the model and observe how the findings are affected. This, obviously, constitutes a potential improvement in terms of attaining a more realistic market model to serve as the foundation for the frictionless market. This is another area worth exploring, and is another line of research we are working upon.

[1] It is also worth noting that, whereas the assumption of unique matching can be considered realistic in some markets, it might not be so in the others. For instance, a market with a homogenous good, where the sellers are characterized by their reservation prices, buyers by their valuations, and the surplus generated is considered as the difference between these two values, fails to have a unique matching.

Appendix A

Proof of Proposition 3.2

Proof. Given (l, m, v), for each $i \in B \cup S$, define the strategy σ_i such that the proposer makes an offer such that she leaves $\delta_j v_j$ to the responder, if $f_{ij} - \delta_i v_i - \delta_j v_j \geq 0$, and leaves $f_{ij} - \delta_i v_i$ if $f_{ij} - \delta_i v_i - \delta_j v_j < 0$. Given this profile, the responder accepts the offer with probability m_{ij}.

With this strategy profile, all agents of type b (who have not traded yet) remain in the market if $v_b > 0$. Otherwise, among the unit measure of type b agents who are born into the market, $l_b n(r) \sum_S p_s(\beta m_{bs} + (1 - \beta) m_{sb})$ conduct a trade and leave, and the rest leaves without any trade.

It is trivial to check that the strategies are sub-game perfect, solves the maximization problem for each agent, the market remains in steady-state and v specifies the value for each type under σ and l. $\square$

Appendix B

Analysis of the Asymmetric 1-by-1 Market Equilibrium

Remember that we are after the analysis of a search equilibrium that is *asymmetric* (i.e. $l_b \neq l_s$ in the steady-state.) As in the text, assume $v_b > 0$ and $v_s > 0$. Furthermore, for simplicity, assume that $\beta = 1 - \beta = 1/2$ and $c_B = c_S = c$ (i.e. $\kappa_B = \kappa_S = \kappa$.) Finally, since the know that condition *(iii)* of 1-by-1 market equilibrium, stating "$l_b = 1$ or $l_s = 1$" must hold, assume, without loss of generality, that $l_b = 1$ and $l := l_s > 1$. Then, the following equalities defining the equilibrium valuations must hold:

$$(1-\delta)v_b = -c + \frac{1}{2}(1 - \delta v_b - \delta v_s) \tag{B.1}$$

$$(1-\delta)v_s = -c + \frac{1}{2l}(1 - \delta v_b - \delta v_s) \tag{B.2}$$

Adding the two equations and some algebra yields:

$$v_b + v_s = \frac{-2c + \frac{l+1}{2l}}{1 - \delta + \delta\frac{l+1}{2l}} \to 1$$

which, after some more algebra, also yields:

$$1 - \delta v_b - \delta v_s = \frac{1 - \delta + 2\delta c}{1 - \delta + \delta \frac{l+1}{2l}}$$

Plugging this back into equation B.1, we obtain:

$$\begin{aligned} v_b &= -\frac{c}{1-\delta} + \frac{1}{2}\frac{1 + \frac{2\delta c}{1-\delta}}{1 - \delta + \delta\frac{l+1}{2l}} \\ &= -\frac{c}{1-\delta} + \frac{\frac{1}{2} + \frac{\delta c}{1-\delta}}{1 + \frac{\delta}{1-\delta}\frac{l+1}{2l}}\frac{1}{1-\delta} \\ &= \frac{1}{1-\delta}\frac{\frac{1}{2} - c + \frac{l-1}{2l}\frac{\delta c}{1-\delta}}{1 + \frac{\delta}{1-\delta}\frac{l+1}{2l}} \\ &= \frac{l + c\delta(l-1) - 2lc}{(1-\delta)2l + \delta(l+1)} \end{aligned}$$

which, clearly converges to $\frac{l}{l+1}$ as $\Delta \to 0$.[1] Conducting the symmetric calculations, it can be shown that

$$v_s = \frac{1 - c\delta(l-1) - 2lc}{(1-\delta)2l + \delta(l+1)}$$

which converges to $\frac{1}{l+1}$ (smaller than the limit value of v_b, when one takes into account that $l > 1$.) The findings also make sense: they suggest that the long side of the market receives a smaller surplus in the limit. When everything else is symmetric (as in this example), the ratio of these valuations is only a function of how *asymmetric* a market is. Note that, not surprisingly, the results we found are the same as in Rubinstein and Wolinsky (1985), because the model collapses to that of Rubinstein and Wolinsky (1985) when, say, $c = 0$ (or more generally, when $\frac{c}{1-\delta} \to 0$.)

[1]The argument indeed requires a little bit more construction. The best option seems to be a case-by-case analysis. When $\frac{c}{1-\delta} \to \frac{\kappa}{r} = \infty$, the limit we constructed on l, $1 + \frac{r}{\kappa}$, forces l to be arbitrarily close to one. When $\frac{c}{1-\delta} \to \frac{\kappa}{r} = 0$, l might tend towards infinity, but the $\frac{l-1}{2l}\frac{\delta c}{1-\delta}$ term in the numerator can be shown to be converging towards a finite number (zero, if l becomes too big.) In any case, this complicated case is already omitted thanks to the assumption that $\kappa > 0$.

The case is essentially very similar, albeit a little bit more complicated, when the model is constructed less symmetrically (i.e. when $\beta \neq 1/2$.) The conjecture is as follows: one could still show that $v_b + v_s \to 1$ in such a case. Furthermore, some careful inspection of the set of equations defining v_b and v_s should show that the ratio of these two values, $\frac{v_b}{v_s}$, converges to $\frac{\beta}{(1-\beta)/l_s}$. [2] These two facts altogether imply that $v_b \to \frac{\beta l_s}{\beta l_s + 1 - \beta}$ and $v_s \to \frac{1-\beta}{\beta l_s + 1 - \beta}$. Thus, the ratio of two valuations is again a function of two sources of asymmetry in the market: *(i)* the asymmetry originating from the structure of the market ($\beta \neq 1 - \beta$) and *(ii)* the asymmetry originating from the measures of agents present in the market ($l_b \neq l_s$.)

[2] When $\frac{c_i}{1-\delta} \to \frac{\kappa_i}{r} = 0$, one can see this immediately. Otherwise, requires some more algebra.

Appendix C

No Trivial Equilibrium Assumption

The following proposition suggests that the No Trivial Equilibrium condition indeed eliminates the possibility of a trivial equilibrium. It is provided for the case $|B| = |S| = 1$, yet it is easily generalizable.

Proposition C.1. *If Assumption 5.3 holds, then, in the steady-state, the market can not have any trivial equilibrium.*

Proof. Suppose, to get a contradiction, that the market has a trivial no-trade equilibrium. Clearly, executing a trade is the only way in which an agent can receive a positive surplus, therefore, the concept of a no-trade equilibrium by itself implies $v_b = v_s = 0$. Obviously, since there is no positive gain in staying in the market, yet there exists a sure negative cost, no one finds it profitable to stay in the market after their first period. This leaves us with a markets that contains only the "new-borns" each period, i.e. $l_b = l_s = 1$ in the equilibrium. Now, each buyer (seller) is born into the market, and since $r = 1$, finds a partner as soon as she is born. With probability β $(1 - \beta)$, she is designated as the proposer.

Clearly, the fact $v_b = v_s = 0$ implies that any price offer above that yields a nonzero share to the partner would be accepted. Therefore, the current equilibrium strategy σ_b for any buyer in the market is as follows:

- In the first period, the agent will inevitably enter the market. For the remaining periods, do not enter the market.
- In any period, if the agent enters the market, finds a match and is designated as the proposer, offers 0 as the price.
- The agent accepts any price offer in $[0, 1)$, and rejects offer 1.

One could also derive the strategy profile σ_s for any seller in the market; it will be defined similarly.

Here is an alternative strategy profile σ'_b that yields a strictly higher payoff for the seller:

- Enter the market in the first period. For the remaining periods, do not enter the market.
- In any period, if a match is found and the seller is designated as the proposer, offer $\epsilon > 0$ as the price.
- The agent accepts any price offer in $[0, 1)$, and rejects offer 1.

It is easy to check that, given σ_s, this strategy yields an expected payoff of $\beta\epsilon > 0$ each period, hence it is a profitable deviation, contradicting the choice of σ_b as the optimal strategy profile. □

Appendix D

Proof of Proposition 6.4

The proof of this proposition is composed of several steps. We begin by showing that the limit point of this equilibrium, e^*, admits perfect assortative matching. Because $e = (l, m, v) \to e^* = (l^*, m^*, v^*)$, we can then deduce that m satisfies the desired properties in the limit. The final thing to be demonstrated is that the convergence of m to m^* does not occur asymptotically, but rather it is completed before Δ reaches to 0. This is a relatively easier argument that basically follows from finiteness of types, and we'll provide it after the limit analysis. The limit analysis can, on the other hand, be considered as a restatement of Becker (1973) for a setup with finite types.

Lemma D.1. *Take any search economy $\mathcal{S}$, consider its frictionless counterpart $\mathcal{S}^*$ (which occurs in the limit when $\Delta = 0$). The equilibrium $e^* \in \mathcal{E}^*(\mathcal{S}^*)$ has the feature that $m_{ij} > 0$ if and only if $i = j$.*

Proof. For the economy in the limit, the three critical conditions, which are efficient bargaining, constant surplus and steady-state conditions, can be expressed as follows:

- If $v_b + v_s < f_{bs}$, then $m_{bs} = m_{sb} = 1$ and if $v_b + v_s > f_{bs}$, then $m_{bs} = m_{sb} = 0$.

- The valuations satisfy:

$$n(r)\beta \sum\nolimits_{S} p_s m_{bs}(f_{bs} - v_b - v_s) = 0 \text{ for all } b \in B,$$
$$\frac{n(r)}{r}(1-\beta) \sum\nolimits_{B} p_b m_{sb}(f_{bs} - v_b - v_s) = 0 \text{ for all } s \in S,$$

- The measures satisfy:

$$l_b n(r) \sum_{S} p_s m_{bs} \leq 1 \text{ for all } b \in B,$$
$$l_s \frac{n(r)}{r} \sum_{B} p_b m_{bs} \leq 1 \text{ for all } s \in S,$$

  where the inequality holds with equality for i with $v_i > 0$.

A careful inspection of efficient bargaining and constant surplus conditions show that the inequality $v_b + v_s \geq f_{bs}$ must hold for each b, s pair. [1] Furthermore, again by the constant surplus condition, we have the following property: *if* $m_{bs} > 0$, *then* $v_b + v_s = f_{bs}$ *for any* b, s *pair.*

We will construct an inductive argument to show that $m_{ij} = 0$ for each $i \neq j$. We begin with the *highest* type, which is type n, where $n = |B| = |S|$. Our aim is to show that $m_{ni} = 0$ for each $i < n$. Suppose, to get a contradiction, that this is not the case, i.e. $m_{ni} > 0$ for some i. By the observation made in the previous paragraph, this means: $v_{b_n} + v_{s_i} = f_{ni}$.

Now, we will use the steady state conditions of agents to find a similar equality for b_n. For the moment, assume that the steady state conditions of b_n and s_n hold with

[1] This heavily depends on the assumption that both p_b and p_s are bounded away from zero. Note, however, the equilibrium we will construct in the end will have this property, thus the analysis indeed doesn't have any flows.

equality (we provide a proof of this assumption later on.) Then, we must have,

$$l_{b_n} n(r)(p_{s_1} m_{n1} + \ldots + p_{s_i} m_{ni} + \ldots + p_{s_n} m_{nn}) = 1$$
$$l_{s_n} \frac{n(r)}{r}(p_{b_1} m_{1n} + \ldots + p_{s_n} m_{nn}) = 1$$

Realize that since $m_{ni} > 0$ and since all probabilities are bounded away from zero, $l_{b_n} n(r) p_{s_i} m_{ni} > 0$ and thus $l_{b_n} n(r) p_{s_n} m_{nn} = l_{s_n} \frac{n(r)}{r} p_{b_n} m_{nn} < 1$. Therefore, for the second equality to hold, we must have s_n trading with some agent other than b_n, i.e. there exists some j s.t. $m_{jn} > 0$. Therefore, we must have $v_{b_j} + v_{s_n} = f_{jn}$.

Since the inequality $v_b + v_s \geq f_{bs}$ must hold for each pair, it needs to hold for pairs (b_n, s_n) and (b_j, s_i) in particular. Combining, we must have,

$$f_{nn} + f_{ji} \leq (v_{b_n} + v_{s_n}) + ((v_{b_j} + v_{s_i})) \quad \text{(D.1)}$$
$$= (v_{b_n} + v_{s_i}) + ((v_{b_j} + v_{s_n})) \quad \text{(D.2)}$$
$$= (f_{ni}) + (f_{jn}) \quad \text{(D.3)}$$
$$< f_{nn} + f_{ji} \quad \text{(D.4)}$$

where D.3 follows when we substitute the equalities derived above, and D.4 follows from strict supermodularity of the surplus function. We obtain $f_{nn} + f_{ji} < f_{nn} + f_{ji}$, a clear contradiction.

Conducting the symmetric analysis for the seller, it can be shown that $m_{ni} = m_{in} = 0$ for each $i \neq n$. One can then continue to apply the conductive argument for the *lower* types (beginning by type $n-1$), and reach the conclusion that $m_{ij} = 0$ for each $i \neq j$.

One thing that remains for this argument to be completed is to provide a discussion about the about an assumption that we made in the process, namely, the assumption of steady-state conditions holding with equality for b_n and s_n. First, realize that the assumption of b_n's condition binding does not serve any purpose for the sake of the analysis, and can be relaxed (nevertheless, the assumption is still necessary for the symmetric analysis, and needs to be defended in a way that is substantially the same as we do for the assumption of s_n's condition binding.) Here, we provide a proof of the observation that s_n's steady state condition holds with equality. Suppose, to get a contradiction, that it does not bind, which in turn implies that $v_{s_n} = 0$. Since the inequality $v_{b_i} + v_{s_n} \geq f_{in}$ for each b_i, we conclude that $v_{b_i} \geq f_{in}$ for each i. There exists two possibilities:

1. If $f_{in} > 0$ for each $i \neq n$, the analysis is simple. We obtain $v_{b_i} > 0$ for each i; therefore, the steady-state condition binds for each type of buyer. This means that each period, at total of $n = |B|$ measure of buyers enter to the market and prefer to stay until they conduct some trade. Consequently, for the market to remain in the steady state, a necessary condition is that a total of n buyers should conduct trade and leave each period. Accordingly, we must have a total measure n of sellers conducting the trade and leave each period. This enforces the steady state condition for each seller to bind, and the condition for s_n to bind in particular.

2. Alternatively, if $f_{jn} = 0$ for some particular j, things get a little complicated, yet still manageable. Following the logic provided in the previous paragraph, now, suppose b_j's steady state condition does not bind, and thus $v_{b_j} = 0$. Now, consider the agent s_i (the particular type who is assumed to be trading with b_n, such that $m_{ni} > 0$ and $v_{b_n} + v_{s_i} = f_{ni}$.) Since we must have $v_{b_j} + v_{s_i} \geq f_{ji}$, and since $v_{b_j} = 0$, we have $v_{s_i} \geq f_{ji}$. Furthermore, since we must have $v_{b_n} + v_{s_n} \geq f_{nn}$, and since $v_{s_n} = 0$, we have $v_{b_n} \geq f_{nn}$. Combining, we obtain:

$$v_{b_n} + v_{s_i} \geq f_{nn} + f_{ji} \tag{D.5}$$

$$> f_{ni} + f_{jn} \tag{D.6}$$

$$= (v_{b_n} + v_{s_i}) + 0 \tag{D.7}$$

where D.6 follows from strict supermodularity, and D.7 follows from the equalities derived just above. We obtain $v_{b_n} + v_{s_i} > v_{b_n} + v_{s_i}$, a contradiction. Therefore, we must have v_{s_n}'s steady state condition to bind in any case.

This wraps up the discussion that there is no *cross-trade* in the equilibrium. To show that the limit economy admits perfect assortative matching, one final thing that remains to be shown is that $m_{ii} > 0$ for each i in any $e^* \in \mathcal{E}^*(\mathcal{S}^*)$. This is rather easy to see, since assuming the contrary ($m_{ii} = 0$ for some i) would mean that b_i and s_i does not conduct any trade with any types in the market, and leave the market with zero surplus. One can always develop an alternative strategy (as done in the proof of Proposition C.1) to show that this strategy is not optimal. An easier way to see this would be to realize that zero surplus for both types would mean $v_{b_i} = v_{s_i} = 0$, which would contradict the inequality $v_{b_i} + v_{s_i} \geq f_{ii} > 0$. We conclude that $m_{ij} > 0$ if and only if $i = j$ in the economy at the limit. □

Now, having shown that the economy in the limit admits this critical feature, it remains to show that this is as well the case for a search economy with sufficiently low frictions.The following Lemma will be useful in demonstrating this:

Lemma D.2. *Take any search economy $\mathcal{S}$. In the equilibrium $e \in \mathcal{E}(\mathcal{S})$, for $i \neq j$, we have $m_{ij} = 0$ for sufficiently low frictions.*

Proof. Take any pair b_i and s_j such that $i \neq j$. By the previous lemma, we know that $m_{ij} = 0$ in the limit. There are two possible cases. If $v_{b_i} + v_{s_j} > f_{ij}$ in the limit, the argument is much easier. This is because $f_{ij} - \delta_{b_i} v_{b_i} - \delta_{s_j} v_{s_j} \to f_{ij} - v_{b_i} - v_{s_j} < 0$ as $\Delta \to 0$. Obviously, since every function we work on is continuous, there exists an $\epsilon > 0$ such that $f_{ij} - e^{-r_{b_i}\Delta} v_{b_i} - e^{-r_{s_j}\Delta} v_{s_j} < 0$ for $\Delta < \epsilon$.

If, on the other hand, $v_{b_i} + v_{s_j} = f_{ij}$ in the limit, the analysis gets more complicated. We do know that $m_{ij} = 0$ in the limit, yet we also need to show that this is the case for the search economy with sufficiently low frictions as well. Suppose the contrary, i.e. assume that $\lim_{\Delta\to 0} m_{ij} = 0$ but $m_{ij} > 0$ for each $\Delta > 0$. We will obtain a contradiction with this case.

Suppose, as in the proof of previous Lemma, that $v_{b_j} > 0$ and $v_{s_i} > 0$ in the limit (we'll handle the alternative cases later on.) Then, for sufficiently low frictions, we must have the steady-state conditions of these two types to be holding with equality. This, on the other hand, implies that both b_j and s_i must be trading with some other agents at any nonnegative Δ (because $l_{b_j} n(r) p_{s_j} m_{jj} = l_{s_j} \frac{n(r)}{r} p_{b_j} m_{jj} < 1$, and similarly for s_i.) Say, these agents are s_k and b_m respectively, i.e. we must have $m_{jk} > 0$ and $m_{mi} > 0$. But then, following the same reasoning, b_k and s_m must be trading with some other as well, and the process continues until we have a cycle. Eventually, we must end up with an ordered cyclic set of indices $(i, j, k, \ldots, m, i)$ such that $m_{ij} > 0, m_{jk} > 0, \ldots, m_{mi} > 0$.

It is indeed quite straightforward to obtain a contradiction once one has such a cycle. For simplicity, we'll present the case with only three indices, but the general argument is similar. Suppose the cycle is of the form (i, j, k, i), i.e. suppose we have $m_{ij} > 0, m_{jk} > 0, m_{ki} > 0$ for each $\Delta > 0$. It is easy to observe that this implies:

$$v_{b_i} + v_{s_j} = f_{ij}$$
$$v_{b_j} + v_{s_k} = f_{jk}$$
$$v_{b_k} + v_{s_i} = f_{ki}$$

in the limit (because otherwise, if any of these inequalities is strict, we would need to have $m_{bs} = 0$ for these particular types for sufficiently small frictions.) Adding the three equalities, we obtain:

$$(v_{b_i} + v_{s_j}) + (v_{b_j} + v_{s_k}) + (v_{b_k} + v_{s_i}) = f_{ij} + f_{jk} + f_{ki} \tag{D.8}$$
$$(v_{b_i} + v_{s_i}) + (v_{b_j} + v_{s_j}) + (v_{b_k} + v_{s_k}) = f_{ij} + f_{jk} + f_{ki} \tag{D.9}$$
$$f_{ii} + f_{jj} + f_{kk} = f_{ij} + f_{jk} + f_{ki} \tag{D.10}$$

where D.10 follows from the fact that $m_{ii} > 0, m_{jj} > 0, m_{kk} > 0$ in the limit (by Lemma D.1.) Now, without loss of generality, assume that $\min\{i, j, k\} = k$. Strict supermodularity of surplus function then implies: $f_{ji} + f_{kk} > f_{jk} + f_{ki}$. Adding f_{ij} to both sides, we obtain: $f_{ij} + f_{ji} + f_{kk} > f_{ij} + f_{jk} + f_{ki}$. By strict supermodularity, we also have $f_{ii} + f_{jj} > f_{ij} + f_{ji}$, and adding f_{kk} to both sides, we obtain: $f_{ii} + f_{jj} + f_{kk} > f_{ij} + f_{ji} + f_{kk}$. Combining both inequalities, we obtain: $f_{ii} + f_{jj} + f_{kk} > f_{ij} + f_{jk} + f_{ki}$, a contradiction to D.10.

The only thing that remains, therefore, is handling the alternative cases to the case assumed above (i.e. $v_{b_j} > 0$ and $v_{s_i} > 0$.) First, assume that $v_{b_j} = v_{s_i} = 0$ in the limit. This has two implications: first, we need to have $v_{b_i} = f_{ii}$ and $v_{s_j} = f_{jj}$ in the limit. Second, since the inequality $v_{b_j} + v_{s_i} \geq f_{ji}$ must hold in the limit, we must have $f_{ji} = 0$. By the first implication, $v_{b_i} + v_{s_j} = f_{ii} + f_{jj}$ and by strict supermodularity, $f_{ii} + f_{jj} > f_{ij} + f_{ji}$, thus $v_{b_i} + v_{s_j} > f_{ij} + f_{ji}$. Using the second implication, this gives: $v_{b_i} + v_{s_j} > f_{ij}$. But, by the efficient bargaining condition, this implies that $m_{ij} = 0$ in the limit, a contradiction to our base assumption.

As the final alternative case, assume, without loss of generality, that $v_{s_i} = 0$ but $v_{b_j} > 0$. But then, for her steady-state condition b_j must be trading with someone else other than s_j. Say this agent is s_k, i.e. $m_{jk} > 0$. Again, this has to continue until either (i) we obtain a cycle, in which case one can obtain a contradiction similar to the one above, or (ii) we encounter some buyer with zero valuation. Suppose case (ii) occurs, and again for simplicity, assume that $v_{b_k} = 0$. Now we have:

$$m_{ij} > 0 \Rightarrow v_{b_i} + v_{s_j} = f_{ij},$$
$$m_{jk} > 0 \Rightarrow v_{b_j} + v_{s_k} = f_{jk},$$
$$v_{b_i} + v_{s_i} = f_{ii}, v_{b_j} + v_{s_j} = f_{jj}, v_{b_k} + v_{s_k} = f_{kk},$$
$$v_{s_i} = v_{b_k} = 0$$

in the limit. Combining all these equalities, one can easily get: $f_{ii} + f_{jj} + f_{kk} = f_{ij} + f_{jk}$. Note, however, as discussed above, strict supermodularity of the surplus function yields $f_{ii} + f_{jj} + f_{kk} > f_{ij} + f_{jk} + f_{ki}$, a contradiction. □

By the help of Lemma D.2, it is now straightforward to show that any search economy with sufficiently low frictions has the property that $m_{ij} = 0$ for any $i \neq j$ with sufficiently low frictions. In particular, by Lemma D.2, it follows that for each (b_i, s_j) pair with $i \neq j$, one can find a $\Delta_{ij} > 0$ such that $m_{ij} = 0$ when $\Delta < \Delta_{ij}$. Picking $\underline{\Delta} = \min\{(\Delta_{ij})_{(b_i,s_j)\in B\times S}\} > 0$, we have the property in Lemma D.1 carried into a search economy with low frictions.

The only thing that remains to complete the proof of Proposition 6.4 is, therefore, the demonstration that $m_{ii} = 1$ for each i. The following Lemma shows this.

Lemma D.3. *In the search equilibrium $e \in \mathcal{E}(\mathcal{S})$, $m_{ii} = 1$ for each i when $\Delta < \underline{\Delta}$.*

Proof. Realize that Lemma D.2 implies that the steady state condition of each agent simplifies to: $l_{b_i} n(r) p_{s_i} \leq 1$ for each $b_i \in B$, and similarly for each $s_i \in S$. We'll use this simplified version for the rest of analysis.

Suppose, to get a contradiction, that $m_{ii} < 1$ for some particular i. By the efficient bargaining condition, this implies: $f_{ii} - \delta v_{b_i} - \delta v_{s_i} \leq 0 \Rightarrow (v_{b_i} + v_{s_i}) \geq \frac{f_{ii}}{\delta} > 0$.[2] This, in turn, implies that $v_{b_i} > 0$ or $v_{s_i} > 0$. Either way, at least one of the steady-state conditions associated with these types must hold with equality, and since $l_{b_i} n(r) m_{ii} p_{s_i} = l_{s_i} \frac{n(r)}{r} m_{ii} p_{b_i}$, we can be sure that both steady state conditions hold with equality. Rearranging terms in these steady state conditions, one obtains $m_{ii} = \frac{1}{\frac{l_{b_i} l_{s_i}}{L} \frac{n(r)}{r}} < 1$. Now, this inequality implies that $l_{b_i} > 1$ or $l_{s_i} > 1$ (otherwise, if $l_{b_i} = l_{s_i} = 1$, plugging them into the inequality gives $\frac{L}{n(r)/r} < 1$. Realize that the left-hand side of the inequality corresponds to the total size of the longer side of the market. Taking into account that $l_i \geq 1$ for each i, any side of the market must ne at least of size n, so this is clearly impossible.) But then, by the constant surplus condition, at least one of the inequalities denoting expected payoffs of b_i

[2]For the sake of simplicity, we assume here that $\delta_{b_i} = \delta_{s_i} = \delta$. The proof also works with different patience levels.

and s_i must hold with equality. Suppose, without loss of generality, that it is b_i. Therefore, we must have:

$$-c_{b_i} + n(r) \sum\nolimits_S p_s m_{is}(f_{is} - \delta v_{b_i} - \delta v_s)) = (1-\delta) v_{b_i}$$

We have already shown that $m_{ij} = 0$ for each $i \neq j$, and $1 - \delta v_{b_i} - \delta v_{s_i} \leq 0$. Substituting gives: $-c_{b_i} = (1-\delta)v_{b_i}$, which, by the individual rationality condition, indicates a contradiction. □

Lemmas D.2 and D.3 do simultaneously imply that, if $\Delta < \underline{\Delta}$, the economy admits perfect assortative matching, thus the proof of Proposition 6.4 follows.

Appendix E

Proof of Proposition 6.5

The argument will follow from the observation that steady-state measures of each corresponding type (l_{b_i} and l_{s_i} for each i) must remain comparable to each other. The following Lemma is therefore crucial.

Lemma E.1. *In any search equilibrium $\mathcal{E}(\mathcal{S})$, the steady-state measures or corresponding types remain comparable to each other, i.e. the ratio $\frac{l_{b_i}}{l_{s_i}}$ is bounded from above (away from infinity) and from below (away from zero).*

Proof. Assume, without loss of generality, that $l_{b_i} \geq l_{s_i}$ for a particular i. We will derive an upper bound for $\frac{l_{b_i}}{l_{s_i}}$. If $l_{b_i} = 1$, then everything is already under control. Therefore, we can assume that $l_{b_i} > 1$. The constant surplus condition, then, yields two equations:

$$-c_{b_i} + n(r)\beta p_{s_i}(f_{ii} - \delta_{b_i} v_{b_i} - \delta_{s_i} v_{s_i}) = (1 - \delta_{b_i}) v_{b_i}$$
$$-c_{s_i} + \frac{n(r)}{r}(1 - \beta) p_{b_i}(f_{ii} - \delta_{b_i} v_{b_i} - \delta_{s_i} v_{s_i}) \leq (1 - \delta_{s_i}) v_{s_i}$$

Rearrangement of two equations and utilization of the facts $p_{b_i} = \frac{l_{b_i}}{L}$, $p_{s_i} = \frac{l_{s_i}}{Lr}$ gives the upper bound:

$$\frac{l_{b_i}}{l_{s_i}} \leq \frac{\frac{v_{s_i}}{1-\beta} + \frac{c_{s_i}/(1-\beta)}{1-\delta_{s_i}}}{\frac{v_{b_i}}{\beta} + \frac{c_{b_i}/\beta}{1-\delta_{b_i}}} \frac{1 - \delta_{s_i}}{1 - \delta_{b_i}}$$

It is trivial to see that this upper bound is finite. It depends on the values of v_{b_i} and v_{s_i}, but can not converge towards infinity even in the worst case. This is because, as frictions disappear, we have:

$$\lim_{\Delta\to 0}\frac{c_i}{1-\delta_i}=\lim_{\Delta\to 0}\frac{\Delta\kappa_i}{1-e^{-r_i\Delta}}=\lim_{\Delta\to 0}\frac{\kappa_i}{r_i}=\frac{\kappa_i}{r_i}>0$$

where the second equality follows from L'Hôpital's Rule. This observation shows that the denominator never vanishes, hence $\frac{l_{b_i}}{l_{s_i}}$ remains finite.

If we employ some stronger assumptions, such as a perfectly symmetric market structure ($\frac{c_{b_i}}{c_{s_i}}=\frac{\beta}{1-\beta}$ and $\delta_{b_i}=\delta_{s_i}=\delta_i$ for each i), value of the upper bound converges to $\frac{\frac{v_{s_i}}{1-\beta}+\theta}{\frac{v_{b_i}}{\beta}+\theta}$, which is bounded above by $\frac{\frac{f_{ii}}{1-\beta}+\theta}{\theta}$ (where $\theta=\lim_{\Delta\to 0}\frac{c_{b_i}/\beta}{1-\delta_i}=\frac{c_{s_i}/(1-\beta)}{1-\delta_i}$).

Let α be the upper bound defined above. If $\alpha\geq 1$, then this effectively works as an upper bound on $\frac{l_{b_i}}{l_{s_i}}$. If $0<\alpha<1$, then, we must have a case where $l_{b_i}<l_{s_i}$. It can also be shown that $\frac{l_{s_i}}{l_{b_i}}$ is bounded above by $\frac{1}{\alpha}$ in this case.

One final remark is that, if we have a case where $l_{b_i}>1$, $l_{s_i}>1$, then the upper bound binds, in the sense that we must have $\frac{l_{b_i}}{l_{s_i}}=\alpha$. In this case, we have a well-defined ratio of steady-state measures. □

Another thing that will be helpful in the analysis is to realize that the steady state conditions of corresponding types can be simplified further, and can even be unified. Begin by observing that $l_{b_i}n(r)p_{s_i}=l_{s_i}\frac{n(r)}{r}p_{b_i}$ for any i; furthermore, plugging in the values of p_{b_i} and p_{s_i}, one can conclude that the steady state conditions indeed correspond to:

$$\frac{l_{b_i}l_{s_i}}{L_l}\leq 1$$

(where L_l denotes the total measure of the longer side of the market) and the inequality holds with equality for i if $v_{b_i}>0$ or $v_{s_i}>0$. Based on the revised version of this steady state condition, and based on the comparability result obtained in

Lemma E.1, we now argue that the market size cannot grow unboundedly. The proof of Proposition 6.5, which depends heavily on Lemma E.1, is as follows:

Proof. **(of Proposition 6.5)** Suppose, to get a contradiction, that L grows infinitely, and let $\lambda \to \infty$ denote the order of convergence in which this part grows (i.e. $\frac{L}{\lambda}$ remains finite and nonzero.) Since the set of available types is finite, there must be at least one type whose steady state measure grows in the order of λ, i.e. there exists an i such that $l_{b_i} \to \infty$ grows with λ. By Lemma E.1, then, l_{s_i} must also grow unboundedly in the order of λ. Note, however, that by the steady state condition for i, we must have $\frac{l_{b_i} l_{s_i}}{L_l} \leq 1$, so L_l must be growing in the order of λ^2. Obviously, L grows in the order of λ, so we conclude that the sellers are at the longer side of the market, i.e. $L_l = Lr$. But then, by the same argument, there must be some j such that $l_{s_j} \to \infty$ in the order of λ^2, and, by Lemma E.1 again, l_{s_j} must be growing in the order of λ^2. This implies that L grows at least in the order of λ^2, a contradiction. □

Appendix F

Proof of Proposition 6.7

The eminent method to prove this Proposition is to show that $\hat{q}$ satisfies the constraints of the primal problem and v satisfies the constraints of the dual, and finally to show that the values of the primal and dual objective functions are equal, i.e. $\sum_s v_s + \sum_b v_b = \sum_{B\times S} \hat{q}_{bs} f_{bs}$.

The first thing to notice is that the parameters which are likely to change as $\Delta \to 0$, i.e. the tuple $(p_b, p_s, \hat{q}_{bs}, v_b, v_s)\forall(b,s) \in B \times S$, indeed reside in a compact set. This is because for each b, s, $(p_b, p_s, \hat{q}_{bs}) \in [0,1]^3$ and $(v_b, v_s) \in [0, f_{bs}]^2$. Furthermore, realize that none of the conditions that define (l, m, v) include any discontinuity at all. Therefore, we can work with the values in the limit, and confidently assume that the conditions of search equilibrium are also satisfied for $\Delta = 0$.

The fact that $(\hat{q}) \in \mathbb{R}_+^{|B||S|}$ and $(v) \in \mathbb{R}_+^{|B|+|S|}$ are obvious, by construction.To show that (3.3) holds for $\hat{q}_b$, use the steady-state condition, which says: $Lp_b \sum_S n(r)p_s m_{bs} \leq 1\forall b$, which, by definition of $\hat{q}_{bs}$, translates into: $\sum_{s\in S} \hat{q}_{bs} \leq 1\forall b$. Similarly, (3.4) can be obtained via the symmetric steady-state condition.

To show that inequality (3.5) is satisfied by elements of v: Take any two types $(b,s) \in B \times S$, and suppose, to get a contradiction, that $f_{bs} - v_b - v_s > 0$. Note that by the efficient bargaining condition, this implies: $m_{bs} = 1$. Now, use the constant surplus condition for one member of the pair (say, for the b). We have:

$$n(r)\beta p_s(f_{bs} - v_b - v_s) \leq 0$$

Where, by Proposition 6.6, we know that $p_s > 0$, and thus the left hand-side is strictly positive, a contradiction.

The final thing that remains to be shown, therefore, is that the primal and dual values are equal. To show this, we need the help of following Lemma (whose proof is provided in the Appendix.)

Lemma F.1. *In any competitive equilibrium* $e^* = (q, v)$*,* $\sum_s v_s + \sum_b v_b = \sum_{B\times S} q_{bs} f_{bs}$ *if and only if* $v_{b_i} + v_{s_i} = f_{ii}$ *for each* i*.*

Proof. We first show that q maintains a specific form in any competitive equilibrium. Our claim is that q_{ij} takes the form of the Kronecker's delta function, i.e.

$$q_{ij} = \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j; \\ 1, & \text{if } i = j. \end{cases}$$

We receive support from several sources in proving this claim. The observation that q_{bs} takes integer values hinges on the fact that the assignment problem of Shapley and Shubik (1972) is a special case of a linear program: an integer program. An earlier treatment of this observation (which Shapley and Shubik (1972) uses as well) is available in Dantzig (1963). A more recent treatment involves showing that such an assignment problem's constraint matrix is totally unimodular (TUM), and the example of such an approach can be found in Vohra (2005).

Given $q_{bs} \in \{0, 1\}$ for each (b, s), to show that it is equal to Kronecker's delta function, one can refer to the proof in Becker (1973). The following originally belongs to Becker (1973), and is adapted into our model.

We need to show that the maximizing sum occurs when the corresponding types are matched with each other, i.e.

$$\sum_{j=1}^{n} f_{j,i_j} < \sum_{i=1}^{n} f_{i,i}$$

for all permutations $(i_1, i_2, \ldots, i_n) \neq (1, 2, \ldots, n)$.

Suppose, to get a contradiction, that the maximizing sum occurs for a permutation $(i_1, i_2, \ldots, i_n)$ that violates the rule $i_1 < i_2 < \ldots < i_n$. Then there is a particular j with the property $i_j > i_{j+1}$. But then, by the strict supermodularity of surplus function, we have:

$$f_{j,i_j} + f_{j+1,i_{j+1}} < f_{j,i_{j+1}} + f_{j+1,i_j}$$

which contradicts the optimality of $(i_1, i_2, \ldots, i_n)$.

Now, based on this argument, we can revise the statement in Lemma F.1 as: "$\sum_{i=1}^{n} v_{s_i} + \sum_{i=1}^{n} v_{b_i} = \sum_{i=1}^{n} f_{ii}$ if and only if $v_{b_i} + v_{s_i} = f_{ii}$ for each i." The *if* part of the statement is obvious, and follows directly. For the *only if* part, suppose $v_{b_i} + v_{s_i} > f_{ii}$ for some particular i. Then, for the equality to hold, it must be the case that $v_{b_j} + v_{s_j} < f_{jj}$ for some j, which contradicts that e^* satisfies the constraint in the dual problem. □

Lemma F.1 simplifies the effort required tremendously, since it's almost self-evident that the condition $v_{b_i} + v_{s_i} = f_{ii}$ is satisfied in the limit. This is because $v_{b_i} + v_{s_i} < f_{ii}$ would contradict to the constant surplus condition, and if $v_{b_i} + v_{s_i} > f_{ii}$, there would be no trading opportunities for types i, which leave them with $v_{b_i} = v_{s_i} = 0$, another contradiction. This completes the proof of Proposition 6.7.

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