ON ROSENTHAL'S ℓ^1-THEOREM

by

Burçin Güneş

A Thesis Submitted to the
Graduate School of Sciences and Engineering
in Partial Fulfillment of the Requirements for
the Degree of Master of Science

in
Mathematics
Koç University
October 2013

Koç University
Graduate School of Sciences and Engineering

This is to certify that I have examined this copy of a master's thesis
by
Burçin Güneş

and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
examining committee have been made.

Thesis Committee Members:

Prof. Ali Ülger (Advisor)..........................
Prof. Varga Kalantarov ....
Assoc. Prof. Selçuk Demir ....

Date: October 2013

Abstract

Let X be a Banach space and $(x_n)_n$ be a bounded sequence in X. The sequence $(x_n)_n$ is said to be weakly Cauchy if, for each $f \in X^*$, the sequence $(f(x_n))_n$ converges. The sequence $(x_n)_n$ is said to be equivalent to the unit vector basis of ℓ^1 if there is a constant $C > 0$ such that, for any constants $c_1, \dots, c_n$ one has

$$\sum_{k=1}^{n} |c_k| \leq C \left\| \sum_{k=1}^{n} c_k x_k \right\|.$$

In 1974, Haskell P. Rosenthal [10] proved that a Banach space X does not contain an isomorphic copy of ℓ^1 if and only if every bounded sequence $(x_n)_n$ in X has a weakly Cauchy subsequence. In this thesis, we give combinatorial and topological proofs of this theorem and examine some of its equivalences. Then we present some applications of it.

Keywords: Baire-1 functions, Rosenthal's ℓ^1-theorem, weakly compact operators, Limited sets, Grothendieck property.

ÖZET

X bir Banach uzayı ve $(x_n)_n$ bu uzaydan sınırlı bir dizi olsun. Eğer her $f \in X^*$ için $(f(x_n))_n$ dizisi yakınsıyorsa $(x_n)_n$ dizisine zayıf Cauchy denir. Rastgele $c_1, \ldots, c_n$ sabitleri için

$$\sum_{k=1}^{n} |c_k| \leq C \left\| \sum_{k=1}^{n} c_k x_k \right\|$$

eşitsizliğini sağlayan bir $C > 0$ sabiti bulabiliyorsak $(x_n)_n$ dizisine ℓ^1 uzayının birim taban vektörlerine denk denir.

1974 yılında Haskell P. Rosenthal [10] bir Banach uzayının ℓ^1 'in eş yapısal bir kopyasını içermemesi ile o uzaydaki her sınırlı dizinin zayıf Cauchy bir alt dizisinin olmasının denk olduğunu kanıtladı. Bu tezde bu teoremin kombinatoryel ve topolojik kanıtlarını verecek ve bazı denkliklerini inceleyeceğiz. Daha sonra bazı uygulamalarını yapacağız.

Acknowledgement

Without the support, patience and guidance of the following people, this thesis would not have been completed. I owe my deepest gratitude to them.
Firstly, I would like to thank my supervisor Prof. Ali Ülger for his patience, guidance and criticism.
I would like to thank Haydar Göral for his unending patience, support and valuable comments during typing my thesis.
Finally, I would also like to extend my deepest gratitude to my parents: My mother Zühal Güneş and my father Ömer Güneş, whose love and guidance are with me in whatever I pursue. I also would like to thank my sister Burcu for always being my best friend and encouraging me to the better.

LIST OF SYMBOLS/ABBREVIATIONS

$B(x, \varepsilon)$	The open ball centered at x with radius ε.
$B'(x, \varepsilon)$	The closed ball centered at x with radius ε.
B_X	The closed unit ball of X.
X^*	The continuous dual of the Banach space X.
$[x_i]_{i=1}^n$	The subspace generated by the elements $x_1, \ldots, x_n$.
ℓ^1	The space of absolutely convergent sequences.
c_0	The space of all sequences converging to 0.
$B(X)$	The space of all bounded scalar functions on a set X.
$C(X)$	The real valued continuous functions on a topological space X.
$\{f < \alpha\}$	The preimage of $(-\infty, \alpha)$ under the function f.
$\{f > \beta\}$	The preimage of (β, ∞) under the function f.
$B_1(X)$	The set of all Baire-1 functions on a topological space X.
T^*	The adjoint operator of bounded linear operator T.

Contents

Chapter 1

BAIRE CATEGORY THEOREM

1.1 Baire Category Theorem and its various forms

In this section we present several forms of the Baire Category Theorem.

Let (X, d) be a metric space.

Definition 1.1.1. *A set $E \subseteq X$ is said to be nowhere dense if its closure $\overline{E}$ has an empty interior. If E can be written as a countable union of nowhere dense sets, we say that E is of the first category in X. If E is not of the first category we say that E is of the second category in X.*

Therefore a subset of X is either of the first category or of the second category.

Example 1.1.2.

1- Proper subspaces of $\mathbb{R}^n$ are nowhere dense.

2- $\mathbb{Q}$ is not nowhere dense in $\mathbb{R}$.

3- $\mathbb{Q} = \bigcup_{k=1}^{\infty} \frac{1}{k}\mathbb{Z}$ is of first category.

Lemma 1.1.3. *Suppose that (X, d) is a complete metric space and $(O_n)_n$ is a sequence of open dense subsets of X. Then $\bigcap_{n=1}^{\infty} O_n$ is also dense.*

Proof. It suffices to show that, for all $x \in X$ and $\varepsilon > 0$, we have

$$B(x, \varepsilon) \cap \bigcap_{n=1}^{\infty} O_n \neq \emptyset.$$

Let $x \in X$, $\varepsilon > 0$. Take $x_0 = x$ and $\varepsilon_0 = \frac{\varepsilon}{2}$. Since O_1 is open dense in X, $E_1 = B(x_0, \varepsilon_0) \cap O_1 \neq \emptyset$ is open. Let $x_1 \in E_1$ be such that $B'(x_1, \varepsilon_1) \subseteq E_1$ where $\varepsilon_1 \in (0, \varepsilon_0)$. Similarly, $E_2 = B(x_1, \varepsilon_1/2) \cap O_2 \neq \emptyset$ is open. Let $x_2 \in E_2$ be such that $B'(x_2, \varepsilon_2) \subseteq E_2$ where $\varepsilon_2 \in (0, \varepsilon_1/2)$.

In general, let $E_n = B(x_{n-1}, \varepsilon_{n-1}/2) \cap O_n \neq \emptyset$ and $x_n \in E_n$ be such that $B'(x_n, \varepsilon_n) \subseteq E_n$ where $\varepsilon_n \in (0, \varepsilon_{n-1}/2)$. So we have a sequence $(x_n)_n$ with the following properties:

i) For each $n \in \mathbb{N}$, for all $m > n$, $x_m \in B(x_n, \varepsilon_n)$ because

$$B(x_0, \varepsilon_0) \supseteq B'(x_1, \varepsilon_1) \supseteq B(x_1, \varepsilon_1) \supseteq \ldots \supseteq B'(x_n, \varepsilon_n) \supseteq B(x_n, \varepsilon_n) \supseteq \ldots$$

ii) Let $m > n$, then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + \ldots + d(x_{m-1}, x_m) \\ &\leq \varepsilon_n + \varepsilon_{n+1} + \ldots + \varepsilon_{m-1} \\ &\leq \frac{\varepsilon_0}{2^n} + \ldots + \frac{\varepsilon_0}{2^{m-1}} \\ &= \frac{\varepsilon_0}{2^n}\left(1 + \frac{1}{2} + \ldots + \frac{1}{2^{m-n-1}}\right) \leq \frac{\varepsilon_0}{2^{n-1}}. \end{aligned}$$

Hence the sequence $(x_n)_n$ is Cauchy.

Since (X, d) is complete, there is a $y \in X$ such that $x_n \to y$. Moreover, by i) we see that $y \in B(x_n, \varepsilon_n)$ for all $n \in \mathbb{N}$. Then $y \in \bigcap_{n=1}^{\infty} E_n \subseteq \bigcap_{n=1}^{\infty} O_n$. Thus $y \in B(x, \varepsilon) \cap \bigcap_{n=1}^{\infty} O_n \neq \emptyset$. □

Theorem 1.1.4. *Let (X, d) be a complete metric space. If $(F_n)_n$ is a sequence of closed sets with $X = \bigcup_{n=1}^{\infty} F_n$, then the set $\bigcup_{n=1}^{\infty} \mathring{F}_n$ is dense in X.*

Proof. We consider the boundary ∂F_n of each F_n. Since it is closed and has empty interior, the complement $X \backslash \partial F_n$ is an open dense set. Therefore, by Lemma 1.1.3, $\bigcap_{n=1}^{\infty} X \backslash \partial F_n$ is dense in X. Next, we show that $\bigcap_{n=1}^{\infty} (X \backslash \partial F_n) \subseteq \bigcup_{n=1}^{\infty} \mathring{F}_n$. Then, clearly, $\bigcup_{n=1}^{\infty} \mathring{F}_n$ is dense in X. Let $x \in \bigcap_{n=1}^{\infty} (X \backslash \partial F_n)$ be any point. Then $x \notin \partial F_n$ for all $n \in \mathbb{N}$

and $x \in F_m \subseteq \overline{F_m}$ for some $m \in \mathbb{N}$. Hence $x \in \mathring{\overline{F_m}}$. Thus we have the desired result. □

Theorem 1.1.5. *A complete metric space (X, d) is of the second category in X.*

Proof. Suppose $X = \bigcup_{n=1}^{\infty} A_n$, where A_n's are nowhere dense in X. Then $X = \bigcup_{n=1}^{\infty} \overline{A_n}$ too. Since $\mathring{\overline{A_n}} = \emptyset$ for all $n \in \mathbb{N}$, $X = X \backslash \mathring{\overline{A_n}} = \overline{(X \backslash \overline{A_n})}$ for all $n \in \mathbb{N}$. Hence open sets $X \backslash \overline{A_n}$ are dense in X. Therefore, by Lemma 1.1.3, $\bigcap_{n=1}^{\infty} (X \backslash \overline{A_n}) \neq \emptyset$. But this gives a contradiction as

$$\emptyset \neq \bigcap_{n=1}^{\infty} (X \backslash \overline{A_n}) = X \backslash (\bigcup_{n=1}^{\infty} \overline{A_n}) = \emptyset.$$

□

All these results Lemma 1.1.3, Theorem 1.1.4, Theorem 1.1.5 are known as "Baire Category Theorems" and the "category" in the name is due to Theorem 1.1.5.

Some Consequences

1) There is no function $f : \mathbb{R} \to \mathbb{R}$ which is continuous only on $\mathbb{Q}$.

   Let C_f be the set of points at which f is continuous. Then

$$C_f = \bigcap_{n=0}^{\infty} \bigcup \{U \subseteq \mathbb{R} : U \text{ is open and } diam f(U) < \frac{1}{n}\}.$$

   Thus the set C_f is a G_δ-set. But we know that $\mathbb{Q}$ is not a G_δ-set in $\mathbb{R}$. Therefore there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ which is continuous only on $\mathbb{Q}$.

2) A Banach space X is either of finite dimension or of uncountable dimension.

   Let $(e_n)_n$ be an algebraic basis for X. Let $M_n = \langle e_0, e_1, \ldots, e_n \rangle$. Then $\dim M_n = n + 1$. Thus M_n is closed. Since every $x \in X$ is a finite linear combination of some e_i's, we have $X = \bigcup_{n=0}^{\infty} M_n$. Hence, by Theorem 1.1.5, $\mathring{M_n} \neq \emptyset$ for some $n \in \mathbb{N}$ which is impossible as proper subspaces are nowhere dense.

1.2 Baire-1 functions and continuity of them

Let (X, d) be a metric space and $f_n : X \to \mathbb{R}$ be a sequence of continuous functions on a subset E. Suppose that $(f_n)_n$ converges uniformly to a function f on E. Then, as is well-known, f is also continuous on E. However, if $(f_n)_n$ converges pointwise to f, f need not be continuous. In this section, we consider functions which are pointwise limits of continuous functions and study continuity of them.

Definition 1.2.1. *Let $f : X \to \mathbb{R}$ be a function. We say that f is a Baire-1 function if there exists a sequence of continuous functions $f_n : X \to \mathbb{R}$ such that $f_n \to f$ pointwise on X.*

Example 1.2.2. *Let $X = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be derivative of a function $g : \mathbb{R} \to \mathbb{R}$. For $x \in \mathbb{R}$, put $f_n(x) = \frac{g(x+1/n)-g(x)}{1/n}$. Each $f_n : \mathbb{R} \to \mathbb{R}$ is continuous and for all $x \in \mathbb{R}$, $f(x) = \lim\limits_{n\to\infty} f_n(x)$. So f is a Baire-1 function.*

Remarks. 1) Baire-1 functions are closed under addition, multiplication, scalar multiplication and taking quotients by nowhere vanishing denominators.
2) If $g : X \to \mathbb{R}$ is a bounded Baire-1 function, say by K, then the sequence of continuous functions $(g_n)_n$ converging pointwise to g can be chosen so that g_n's are bounded also by K.

Theorem 1.2.3. *The limit of a uniformly convergent sequence of Baire-1 functions is also Baire-1.*

Proof. Let f be the uniform limit of the Baire-1 functions $(f_n)_n$. First, note that by passing to a subsequence of f_n, if necessary, we may suppose that f_n's are given so that $|f_n(x) - f(x)| < 2^{-n}$ for each $x \in X$ and $n \in \mathbb{N}$. Therefore, for each $x \in X$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} |f_{n+1}(x) - f_n(x)| &\leq |f_{n+1}(x) - f(x)| + |f(x) - f_n(x)| \\ &< 2^{-(n+1)} + 2^{-n} < 2 \cdot 2^{-n}. \end{aligned}$$

So, for each $x \in X$, the sum $\sum_{n=1}^{\infty} |f_{n+1}(x) - f_n(x)|$ makes sense. As each f_n is a Baire-1 function, so is $f_{n+1} - f_n$. Hence there is a sequence $(g_{n,m})_m$ of continuous functions with $\lim_{m\to\infty} g_{n,m}(x) = f_{n+1}(x) - f_n(x)$ for all $x \in X$. Moreover, as in Remark 2, these $(g_{n,m})_m$'s can be chosen such that each $|g_{n,m}(x)| \leq 2 \cdot 2^{-n}$ for all $x \in X$. Now, for $m \geq 1$, put $g_m := g_{1,m} + \ldots + g_{m,m}$. Each g_m is continuous.
We show that at each $x \in X$, $\lim_{m\to\infty} g_m(x)$ exists and

$$\lim_{m\to\infty} g_m(x) = \sum_{n=1}^{\infty} (f_{n+1}(x) - f_n(x)).$$

To this end, let $\varepsilon > 0$ be given and $N \in \mathbb{N}$ be chosen so that $4 \cdot 2^{-N} < \frac{\varepsilon}{3}$. Then, for each x, we have

$$\sum_{n=N+1}^{\infty} |f_{n+1}(x) - f_n(x)| < \frac{\varepsilon}{3}.$$

So,

$$\left| \sum_{n=N+1}^{\infty} (f_{n+1}(x) - f_n(x)) \right| < \frac{\varepsilon}{3}.$$

Hence given $x \in X$, there is an $M > N$ such that for all $n, m \geq M$ we get

$$|f_{n+1}(x) - f_n(x) - g_{n,m}(x)| < \frac{\varepsilon}{3N}.$$

Therefore,

$$\begin{aligned}
\left| \sum_{n=1}^{\infty} (f_{n+1}(x) - f_n(x)) - g_m(x) \right| &= \left| \sum_{n=1}^{\infty} (f_{n+1}(x) - f_n(x)) - \sum_{n=1}^{m} g_{n,m}(x) \right| \\
&\leq \left| \sum_{n=1}^{\infty} (f_{n+1}(x) - f_n(x)) - \sum_{n=1}^{N} (f_{n+1}(x) - f_n(x)) \right| \\
&+ \left| \sum_{n=1}^{N} (f_{n+1}(x) - f_n(x) - g_{n,m}(x)) \right| \\
&+ \sum_{n=N+1}^{m} |g_{n,m}(x)| < \varepsilon.
\end{aligned}$$

Hence $\lim_{m\to\infty} g_m = \sum_{n=1}^{\infty}(f_{n+1} - f_n)$ is a Baire-1 function. So

$$f(x) = \lim_{n\to\infty} f_n(x) = f_1(x) + \sum_{n=1}^{\infty}(f_{n+1} - f_n)(x),$$

which is a sum of two Baire-1 functions, is also a Baire-1 function.

□

We now study the continuity of Baire-1 functions.

Theorem 1.2.4. *Let (X, d) be a complete metric space. If $f : X \to \mathbb{R}$ is a Baire-1 function then C_f is dense in X.*

Proof. Let, for $n \in \mathbb{N}$ and $k \in \mathbb{N}\setminus\{0\}$, $A_n(k) = \bigcup_{p\in\mathbb{N}}\{x \in X : |f_{n+p}(x) - f(x)| \leq \frac{1}{k}\}$. We first show that the points of continuity of f coincide with "the points of uniform convergence". i.e.

$$C_f = \bigcap_{k=1}^{\infty}\bigcup_{n\in\mathbb{N}} \mathring{A}_n(k).$$

Let $K = \bigcap_{k=1}^{\infty}\bigcup_{n\in\mathbb{N}} \mathring{A}_n(k)$ and $x_0 \in K$ be any point. We need to show that f is continuous at x_0. Let $\varepsilon > 0$ be given. Then there is a $k \in \mathbb{N}$ such that $k\varepsilon > 3$. Then $x_0 \in \bigcup_{n\in\mathbb{N}} \mathring{A}_n(k)$. Hence $x_0 \in \mathring{A}_{n_0}(k)$ for some $n_0 \in \mathbb{N}$. Thus there is an $\eta_k > 0$ such that $B(x_0, \eta_k) \subseteq \mathring{A}_{n_0}(k)$. So, for all $x \in B(x_0, \eta_k)$, $|f_{n_0+p}(x) - f(x)| \leq \frac{1}{k}$ for some $p \in \mathbb{N}$. In particular, $|f_{n_0+p}(x_0) - f(x_0)| \leq \frac{1}{k}$. Now, since $f_n(x_0) \to f(x_0)$ there is $p \in \mathbb{N}$ such that $|f_{n_0+p}(x_0) - f(x_0)| \leq \frac{1}{k}$. As f_{n_0+p} is continuous at x_0, there is a $\eta < \eta_k$ such that for all $x \in B(x_0, \eta)$ $|f_{n_0+p}(x) - f_{n_0+p}(x_0)| < \frac{1}{k}$. Then, for $x \in B(x_0, \eta)$,

$$\begin{aligned}|f(x) - f(x_0)| &\leq |f(x) - f_{n_0+p}(x)| + |f_{n_0+p}(x) - f_{n_0+p}(x_0)| + |f_{n_0+p}(x_0) - f(x_0)| \\ &< \frac{3}{k} < \varepsilon.\end{aligned}$$

Hence, $K \subseteq C_f$.

Conversely, let $x_0 \in C_f$ and let $k \geq 1$. Since $f_n(x_0) \to f(x_0)$, for some $n \in \mathbb{N}$, $|f_n(x_0) - f(x_0)| \leq \frac{1}{2k}$. Now, the function $g_n = f_n - f$ is continuous at x_0 and $|g_n(x_0)| \leq \frac{1}{2k}$. So there is $\eta > 0$ such that for all $x \in B(x_0, \eta)$, $|g_n(x_0)| \leq \frac{1}{k}$. Hence $B(x_0, \eta) \subseteq A_n(k)$. So $x_0 \in \mathring{A}_n(k) \subseteq \bigcup\limits_{n\in\mathbb{N}} \mathring{A}_n(k)$. Since k is arbitrary, we have $x_0 \in \bigcap\limits_{k=1}^{\infty} \bigcup\limits_{n\in\mathbb{N}} \mathring{A}_n(k)$. Therefore, $K = C_f$.

Next, for $k \geq 1$, $n \in \mathbb{N}$, let

$$B_n(k) = \{x \in X : \sup_{p\in\mathbb{N}} |f_{n+p}(x) - f(x)| \leq \frac{1}{k}\}.$$

Note that $B_n(k) \subseteq A_n(k)$ for all $k \geq 1$ and $n \in \mathbb{N}$. Also, since f_n's are continuous, $B_n(k) = \bigcap\limits_{p\in\mathbb{N}} \{x \in X : |f_{n+p}(x) - f(x)| \leq \frac{1}{k}\}$ is closed. Moreover, since for each $x \in X$, $f_n(x)$ converges, it is Cauchy. So $x \in B_n(k)$ for some $n \in \mathbb{N}$. Hence $X = \bigcup\limits_{n\in\mathbb{N}} B_n(k)$. Then, by Theorem 1.1.4, $\bigcup\limits_{n\in\mathbb{N}} \mathring{B}_n(k)$ is dense in X. Also, as $\bigcup\limits_{n\in\mathbb{N}} \mathring{B}_n(k) \subseteq \bigcup\limits_{n\in\mathbb{N}} \mathring{A}_n(k)$, we have $O_k = \bigcup\limits_{n\in\mathbb{N}} \mathring{A}_n(k)$ is open and dense in X. Therefore, the set of points of continuity $C_f = \bigcap\limits_{k\in\mathbb{N}} O_k$ is a dense set by Lemma 1.1.3.

□

1.3 Baire's Great Theorem

For a given function $f : X \to \mathbb{R}$, deciding whether it is Baire-1 or not may not be easy. In this section, we try to find necessary and sufficient conditions for f to be a Baire-1 function.

Lemma 1.3.1. *For every Baire-1 function $f : X \to \mathbb{R}$ and every open subset U of $\mathbb{R}$, the preimage $f^{-1}(U)$ is an F_σ-set.*

Proof. Let $f : X \to \mathbb{R}$ be a Baire-1 function and $f_n : X \to \mathbb{R}$ be a sequence of continuous functions with $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$. It sufficies to show that, for every rational number q, the sets

$$\{x \in X : f(x) < q\} \qquad \text{and} \qquad \{x \in X : f(x) \geq q\}$$

are F_σ. As each f_n is continuous on X and

$$\{x \in X : f(x) < q\} = \bigcup_{\substack{q \in \mathbb{Q} \\ p < q}} \bigcup_{m=1}^{\infty} \bigcap_{n \geq m} \{x \in X : f_n(x) \leq p\}$$

and

$$\{x \in X : f(x) \geq q\} = \bigcup_{\substack{q \in \mathbb{Q} \\ p > q}} \bigcup_{m=1}^{\infty} \bigcap_{n \geq m} \{x \in X : f_n(x) \geq p\},$$

the sets $\{x \in X : f(x) < q\}$ and $\{x \in X : f(x) \geq q\}$ are F_σ-sets.

□

Lemma 1.3.2. *If $A \subseteq X$ is both F_σ and G_δ then χ_A is a Baire-1 function.*

Proof. Let $A = \bigcup_{n \geq 1} A_n$ and $X \setminus A = \bigcup_{n \geq 1} B_n$ where A_n's and B_n's are closed and disjoint. Moreover, we can assume that the sequences $(A_n)_n$ and $(B_n)_n$ are increasing. By Urysohn's Lemma, for each $n \in \mathbb{N}$, there is a continuous function $f_n : X \to [0,1]$ such that $f_{n \restriction A_n} = 1$ and $f_{n \restriction B_n} = 0$. Then $\chi_A = \lim_{n \in \mathbb{N}} f_n$, so that χ_A is a Baire-1 function.

□

Lemma 1.3.3. *If $f : X \to \mathbb{R}$ is such that $f^{-1}(O)$ is an F_σ-set for every open set $O \subseteq \mathbb{R}$ then f is a Baire-1 function.*

Proof. Without loss of generality we may assume that $f : X \to (0,1)$ as $\mathbb{R}$ and $(0,1)$ are homeomorphic. We fix $n \geq 1$ and define the sets

$$A_k := \{x \in X : \tfrac{k}{n} < f(x)\} \qquad \text{and} \qquad B_k := \{x \in X : f(x) < \tfrac{k+1}{n}\}$$

for $k \in \{0, 1, \ldots, n-1\}$. Then $X = A_k \cup B_k$ for $k \in \{0, 1, \ldots, n-1\}$. All sets A_k and B_k are F_σ by assumption. So, for fixed k, there are closed sets $F_{k,l}$ and $F'_{k,l}$ such that $A_k = \bigcup_{l=1}^{\infty} F_{k,l}$ and $B_k = \bigcup_{l=1}^{\infty} F'_{k,l}$. Then $g_k := \sum_{l=1}^{\infty} 2^{-l} \chi_{F_{k,l}}$ is a

Baire-1 function since each $\chi_{F_{k,l}}$ is Baire-1. Similarly, $g'_k := \sum_{l=1}^{\infty} 2^{-l}\chi_{F'_{k,l}}$ is a Baire-1 function. Clearly, $\{g_k > 0\} = A_k$ and $\{g'_k > 0\} = B_k$. Therefore, $f_k := \frac{g_k}{g_k+g'_k}$ is also a Baire-1 function which satisfies $f_k = 0$ on $X\setminus A_k$, $f_k = 1$ on $X\setminus B_k$ and $0 < f_k < 1$ elsewhere. Then $\frac{1}{n}(f_1 + \ldots + f_n)$ converges uniformly to f. Let $x \in X$. Then there is a k_0 such that $\frac{k_0}{n} < f(x) < \frac{k_0+1}{n}$. Therefore $f_i(x) = 1$ for $i \in \{1, \ldots, k_0 - 1\}$ and $f_i(x) = 0$ for $i \in \{k_0, \ldots, n\}$. Then $\left|\frac{1}{n}(f_1(x) + \ldots + f_n(x)) - f(x)\right| < \frac{1}{n}$. Hence f is itself Baire-1 by Theorem 1.2.3.

□

Proposition 1.3.4. *Let (X, d) be a metric space and $f : X \to \mathbb{R}$ be such that for each nonempty closed subset F of X, the restriction function $f_{\restriction F}$ of f is continuous at least at one point $x_0 \in F$. Then f is a Baire-1 function.*

Proof. For $C \subseteq X$, define

$$osc(f, C) := sup\{d(f(x), f(y)) : x, y \in C\}.$$

For $n \in \mathbb{N}$, let C_n be the class of subsets C with $osc(f, C) < 2^{-n}$. First note that if F is a closed subset of X and a_F is a point of continuity of the restriction function $f_{\restriction F}$, then for each $n \in \mathbb{N}$ there is a neighbourhood $U_{a_F,n}$ of a_F such that $F \cap U_{a_F,n} \neq \emptyset$ and $F \cap U_{a_F,n}$ belongs to C_n.

Now, we fix an $n \in \mathbb{N}$ and define a family of strictly decreasing closed sets of X, for any ordinal by transfinite induction.

i. Let $Z_0 = X$.

ii. If $\alpha = \beta + 1$ and $Z_\beta \neq \emptyset$, put $Z_\alpha = Z_\beta \setminus U_{a_{Z_\beta},n}$ where $U_{a_{Z_\beta},n}$ is an open neighbourhood of the point a_{Z_β} of continuity of $f_{\restriction Z_\beta}$ with $Z_\beta \cap U_{a_{Z_\beta},n} \neq \emptyset$ and $Z_\beta \cap U_{a_{Z_\beta},n}$ belongs to C_n.

iii. If α is a limit ordinal, put $Z_\alpha = \bigcap_{\beta<\alpha} Z_\beta$.

Then there is an ordinal $\gamma_n \leq card(X)$ such that $Z_{\gamma_n} = \emptyset$ and $Z_\alpha \neq \emptyset$ for all $\alpha < \gamma_n$. Hence we have found a family of closed sets $(Z_\alpha)_{\alpha \in [0,\gamma_n]}$ of X with nonempty difference sets $D_\alpha = Z_\alpha \setminus Z_{\alpha+1}$ belongs to C_n for each $\alpha \in [0, \gamma_n]$.
Let $x_\alpha \in D_\alpha$ and consider $g_n : X \to \mathbb{R}$ such that for every $\alpha \in [0, \gamma_n]$ and $x \in D_\alpha$, $g_n(x) = f(x_\alpha)$. Then each g_n is Baire-1 and f is the uniform limit of $(g_n)_n$. Hence f is also a Baire-1 function by Theorem 1.2.3.

□

Theorem 1.3.5 (Baire's Great Theorem)**.** *Let (X, d) be a complete metric space and $f : X \to \mathbb{R}$ be a given function. Then f is Baire-1 if and only if for each nonempty closed subset F of X, the restriction function $f_{\restriction_F}$ of f is continuous at least at one point $x_0 \in F$.*

Proof. Suppose first that for each nonempty closed subset F of X, the restriction function $f_{\restriction_F}$ of f is continuous at least at one point $x_0 \in F$. Then f is Baire-1 by Proposition 1.3.4.

Conversely, suppose f is Baire-1 function. Let F be a nonempty closed subset of X. Then F itself can be seen as a complete metric space. Then $f_{\restriction_F}$ is still a Baire-1 function. Hence, by Theorem 1.2.4, $C_{f_{\restriction_F}}$ is dense in F. So F contains a point of continuity of the restriction function $f_{\restriction_F}$.

□

Chapter 2

BASIC SEQUENCES IN BANACH SPACES

2.1 Schauder Basis

A sequence $(x_n)_{n\geq 1}$ in a Banach space X is called a *Schauder basis* (or *basis)* for X if for each $x \in X$ there is a unique sequence $(\alpha_n)_{n\geq 1}$ of scalars such that

$$x = \lim_{n\to\infty} \sum_{k=1}^{n} \alpha_k x_k.$$

A sequence $(x_n)_{n\geq 1}$ which is a Schauder basis of its closed linear span is called a *basic sequence.*

Let $(X, \|\cdot\|)$ be a Banach space with a Schauder basis $(x_k)_{k\geq 1}$. As for each $x \in X$ we have a unique sequence of scalars, the Schauder basis consists of linearly independent vectors. Moreover, we can identify each element of X with the corresponding unique sequence of scalars $(\alpha_k)_{k\geq 1}$. Indeed, let S be the set of sequences of scalars $(s_k)_{k\geq 1}$ such that $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} s_k x_k$ exists in X. Clearly, S is a vector space with the coordinatewise addition and scalar multiplication. Let $(s_k)_{k\geq 1} \in S$. Also, since $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} s_k x_k$ exists, the limit of the real sequence $\left(\left\| \sum\limits_{k=1}^{n} s_k x_k \right\|\right)_{n\geq 1}$ exists. Hence $\left(\left\| \sum\limits_{k=1}^{n} s_k x_k \right\|\right)_{n\geq 1}$ is bounded. Thus we can talk about the supremum of this sequence. For $(s_n)_{n\geq 1} \in S$, we define

$$|||(s_n)_{n\geq 1}||| := \sup_{n\geq 1} \left\| \sum_{k=1}^{n} s_k x_k \right\|.$$

This definition makes S into a Banach space: Let $(y_p)_p = ((s_{p,i})_i)_p$ be a Cauchy sequence in S. Since

$$|s_{p,i} - s_{q,i}|\|x_i\| \leq 2 \sup_{n\geq 1} \left\| \sum_{i=1}^{n} (s_{p,i} - s_{q,i}) x_i \right\| = 2|||y_p - y_q|||$$

$(s_{p,i})_p$ converges for each i. For $i \in \mathbb{N}$, let $s_i := \lim_{p\to\infty} s_{p,i}$. We show that $(s_i)_i$ is in S. To this end, let $\varepsilon > 0$ be a given number. Since $(y_p)_p$ is Cauchy there is an $r \in \mathbb{N}$ such that for $p \geq r$ we have

$$|||y_p - y_r||| < \varepsilon.$$

Therefore, $p \geq r$, $\left\| \sum_{i=1}^{n} (s_{p,i} - s_{r,i}) x_i \right\| < \varepsilon$. Also, as $y_r = (s_{r,i}) \in S$, there is an $N_1 \in \mathbb{N}$ such that whenever $m \geq n \geq N_1$

$$\left\| \sum_{i=n}^{m} s_{r,i} x_i \right\| < \varepsilon.$$

Hence we get, for $m \geq n \geq N_1$,

$$\left\| \sum_{i=n}^{m} s_i x_i \right\| < 3\varepsilon.$$

This means that $\left(\sum_{i=1}^{m} s_i x_i \right)_n$ is Cauchy, so convergent. Hence $(s_i)_i = \lim_{p\to\infty} y_p \in S$. Now, S is a Banach space and

$$\left\| \lim_{n\to\infty} \sum_{k=1}^{n} s_k x_k \right\| \leq |||(s_k)_k|||.$$

Therefore, by Open Mapping Theorem, S and X are isomorphic via the injective norm-decreasing map

$$B : (S, |||\cdot|||) \to (X, \|\cdot\|)$$

given by $B((s_k)_k) = \sum_{k=1}^{\infty} s_k x_k$.

Then each of the *coefficient functionals* $x_k^* : \sum_n \alpha_n x_n \mapsto \alpha_k$ is continuous as

$$|\alpha_k|\|x_k\| \leq 2\|B^{-1}\| \Big\| \sum_n \alpha_n x_n \Big\|.$$

Hence, the projections $P_n : X \to X$, defined by $P_n(\sum_{i=1}^{\infty} a_i x_i) = \sum_{i=1}^{n} a_i x_i$ are bounded linear operators and for any $x \in X$, we have $x = \lim_{n\to\infty} P_n x$. Thus, $\sup_n \|P_n\| < \infty$ by *Banach-Steinhaus Theorem* (the number $\sup_n \|P_n\|$ is called the basis constant of $(x_n)_n$).

Now, let $m < n$ and $\sum_{k=1}^{\infty} a_k x_k \in X$. Then

$$\begin{aligned}\Big\| \sum_{k=1}^{m} a_k x_k \Big\| &= \Big\| P_m\Big(\sum_{k=1}^{\infty} a_k x_k \Big) \Big\| = \Big\| P_m P_n \Big(\sum_{k=1}^{\infty} a_k x_k \Big) \Big\| \\ &= \Big\| P_m\Big(\sum_{k=1}^{n} a_k x_k \Big) \Big\| \leq \|P_m\| \Big\| \sum_{k=1}^{n} a_k x_k \Big\| \\ &\leq \sup_n \|P_n\| \Big\| \sum_{k=1}^{n} a_k x_k \Big\|.\end{aligned}$$

Conversely, suppose that we have a sequence $(x_n)_n$ of nonzero vectors for which there is a $K > 0$ such that whenever $m < n$,

$$\Big\| \sum_{k=1}^{m} a_k x_k \Big\| \leq K \Big\| \sum_{k=1}^{n} a_k x_k \Big\| \tag{2.1.1}$$

holds. If a vector x has a representation of the form $\sum_{k=1}^{\infty} a_k x_k = \lim_{m\to\infty} \sum_{k=1}^{m} a_k x_k$, (2.1.1) ensures that the representation is unique. For instance, let $j, k \geq 1$. Then we have

$$|a_j|\|x_j\| = \|a_j x_j\| \le K\Big\|\sum_{i=j}^{j+k} a_i x_i\Big\|.$$

Hence if $\sum_{k=1}^{\infty} a_k x_k = \sum_{k=1}^{\infty} b_k x_k$ and $j \ge 1$ is the least index such that $a_j \ne b_j$ we get

$$\|a_j - b_j\| \le \frac{K}{\|x_j\|}\Big\|\sum_{i\ge j}(a_i - b_i)x_i\Big\|,$$

which forces $a_j = b_j$. Clearly, each element in $[x_n]$ is representable in such a form and (2.1.1) gives rise to projections from $[x_n]$ to itself that are bounded linear operators. Then each P_m has a bounded linear extension, still called P_m, projecting $[x_n : n \ge 1]$ onto $[x_n : 1 \le n \le m]$. This again gives the continuity of the coefficient functionals x_k^* defined on $span(x_n)$ and hence by Hahn-Banach theorem x_k^* has unique extensions to all of $[x_n : n \ge 1]$, given by $x_k^*(x)x_k = P_k(x) - P_{k-1}(x)$. Let $x \in [x_n : n \ge 1]$ and $\varepsilon > 0$ be given. Then there is a $\sigma \in [x_n : 1 \le n \le n_\varepsilon]$ for some n_ε so that $\|x - \sigma\| < \varepsilon$. Now, if $n \ge n_\varepsilon$, then

$$\begin{aligned}\|x - P_n(x)\| &\le \|x-\sigma\| + \|\sigma - P_n(\sigma)\| + \|P_n(\sigma) - P_n(x)\|\\ &= \|x-\sigma\| + \|\sigma-\sigma\| + \|P_n(\sigma) - P_n(x)\|\\ &< \varepsilon + \|P_n\|\varepsilon \le (1+K)\varepsilon.\end{aligned}$$

Therefore, $x = \lim_{n\to\infty} P_n(x) = \lim_{n\to\infty}\sum_{k=1}^{n} x_k^*(x)x_k.$

Hence we have the following theorem:

Theorem 2.1.1. *Let $(x_n)_n$ be a sequence of nonzero vectors in the Banach space X. Then in order that $(x_n)_n$ be a basic sequence it is necessary and sufficient that there be a finite constant $K > 0$ so that for any choice of scalars $(a_n)_n$ and any* *integers $m < n$ we have*

$$\Big\|\sum_{k=1}^{m} a_k x_k\Big\| \leq K\Big\|\sum_{k=1}^{n} a_k x_k\Big\|.$$

Lemma 2.1.2. *Let F be a finite dimensional subspace of the infinite dimensional Banach space X, and let $\varepsilon > 0$. Then there is an $x \in X$ such that $\|x\| = 1$ and*

$$\|y\| \leq (1+\varepsilon)\|y + \lambda x\|$$

for all $y \in F$ and all scalars λ.

Proof. Suppose $\varepsilon < 1$. Since F is finite dimensional, $S_F = \{y \in F : \|y\| = 1\}$ is compact. Hence there are $y_1, \dots, y_k \in F$ such that $S_F \subseteq B(y_1, \varepsilon/2) \cup \dots \cup B(y_k, \varepsilon/2)$. Let $y_i \in X^*$ be such that ${y_i}^*(y_i) = 1$. Then there is an $x \in X$ such that ${y_i}^*(x) = 0$ for all $i \in \{1, \dots, k\}$. Now, let $y \in S_F$ and λ be any scalar. Therefore, for some i, we have

$$\begin{aligned}\|y + \lambda x\| &\geq \|y_i + \lambda x\| - \|y - y_i\| \geq \|y_i + \lambda x\| - \varepsilon/2 \\ &\geq y_i^*(y_i + \lambda x) - \varepsilon/2 = 1 - \varepsilon/2 \\ &\geq \frac{1}{1+\varepsilon}\end{aligned}$$

Thus $\|y\| \leq (1+\varepsilon)\|y + \lambda x\|$ for all λ and $\|y\| = 1$. Hence the result follows as λ being arbitrary.

□

Corollary 2.1.3. *Every infinite dimensional Banach space contains a basic sequence.*

Proof. Let X be an infinite dimensional Banach space and $\varepsilon > 0$. Choose a sequence $(\varepsilon_n)_n$ of positive numbers such that $\prod_{n=1}^{\infty}(1+\varepsilon_n) \leq 1+\varepsilon$. Now, let $x_1 \in S_X$

and pick $x_2 \in S_X$ such that

$$\|y\| \leq (1+\varepsilon_1)\|y+\lambda x_2\|$$

for all $y \in [x_1]$ and scalars λ. Next, choose $x_3 \in S_X$ so that

$$\|y\| \leq (1+\varepsilon_2)\|y+\lambda x_3\|$$

for all $y \in [x_1, x_2]$ and scalars λ. Suppose we chose $x_1, \ldots, x_n$. Pick x_{n+1} so that

$$\|y\| \leq (1+\varepsilon_n)\|y+\lambda x_{n+1}\|$$

for all $y \in [x_1, \ldots, x_n]$ and scalars λ. Thus we have a sequence $(x_n)_n$ so that for any scalars $(a_n)_n$ and any integers $m < n$ we have

$$\left\|\sum_{k=1}^{m} a_k x_k\right\| \leq K\left\|\sum_{k=1}^{n} a_k x_k\right\|$$

where $K = \prod_{n=1}^{\infty}(1+\varepsilon_n) \leq 1+\varepsilon$. Hence, by Theorem 2.1.1, $(x_n)_n$ is a basic sequence. $\square$

Definition 2.1.4. *Let $(x_n)_n$ be a basis for X and $(y_n)_n$ be a basis for Y. We say that $(x_n)_n$ and $(y_n)_n$ are equivalent if the convergence of $\sum_{n=1}^{\infty} a_n x_n$ is equivalent to that of $\sum_{n=1}^{\infty} a_n y_n$.*

Theorem 2.1.5. *The bases $(x_n)_n$ and $(y_n)_n$ are equivalent if and only if there is an isomorphism between X and Y that carries each x_n to y_n.*

Proof. Recall that renorming X by taking any $x = \sum_n s_n x_n$ and defining

$$|||x||| = \sup_n \left\|\sum_{k=1}^{n} s_k x_k\right\|,$$

X can be seen as a monotone basis, i.e. $\left\| \sum_{k=1}^{m} s_k x_k \right\| \leq \left\| \sum_{k=1}^{m+n} s_k x_k \right\|$ for any $m, n \geq 1$. An isomorph of X in which $(x_n)_n$ is still a basis but is now a monotone basis. Now, look at the operator $T : X \to Y$ that takes $\sum_{k=1}^{\infty} s_k x_k$ to $\sum_{k=1}^{\infty} s_k y_k$. T is one-to-one and onto. T also has a closed graph. Therefore T is an isomorphism and takes x_n to y_n.

□

2.2 Unconditional Basis

Let $(x_n)_n$ be a sequence of vectors in a Banach space X. A series $\sum_{n=1}^{\infty} x_n$ is said to be unconditionally convergent if for every permutation σ of natural numbers $\sum_{n=1}^{\infty} x_{\sigma(n)}$ converges.

A basis $(x_n)_n$ of a Banach space X is said to be unconditional if for every $x \in X$, its expansion in terms of the basis $\sum_{n=1}^{\infty} a_n x_n$ converges unconditionally.

Proposition 2.2.1. *A basic sequence $(x_n)_n$ is unconditional if and only if any of the following conditions holds.*

(i) For every permutation σ of the natural numbers the sequence $(x_{\sigma(n)})_n$ is a basic sequence.

(ii) For every subset M of the natural numbers the convergence of $\sum_{n=1}^{\infty} a_n x_n$ implies the convergence of $\sum_{n \in M} a_n x_n$.

(iii) The convergence of $\sum_{n=1}^{\infty} a_n x_n$ implies the convergence of $\sum_{n \in M} b_n x_n$ whenever $|b_n| \leq |a_n|$ for all n.

2.3 Basic sequences equivalent to the unit basis of c_0

Let $(x_n)_n$ be a normalized basic sequence of c_0. Suppose there is a constant $K > 0$ such that

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \leq K \sup_{1 \leq i \leq n} |a_i| \tag{2.3.1}$$

for any n and any scalars $a_1, \ldots, a_n$. Then, clearly, $(x_n)_n$ is equivalent to the unit vector basis of c_0.

Conversely, if we are given a normalized basic sequence $(x_n)_n$ which is equivalent to the unit vector basis $(e_n)_n$ of c_0. Then there is an isomorphism $T : c_0 \to c_0$ such that $T(e_i) = x_i$ for all $i \in \mathbb{N}$. So for arbitrary $n \in \mathbb{N}$ and scalars $a_1, ..., a_n$, we have

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| = \left\| T\left(\sum_{i=1}^{n} a_i e_i \right) \right\| \leq \|T\| \left\| \sum_{i=1}^{n} a_i e_i \right\| = \|T\| \sup_{1 \leq i \leq n} |a_i|.$$

Therefore a normalized basic sequence $(x_n)_n$ in c_0 is equivalent to the unit vector basis of c_0 if and only if (2.3.1) holds.

We continue with a definition.

Definition 2.3.1. *A series $\sum_n x_n$ is said to be weakly unconditionally Cauchy (wuC) if, given any permutation σ of natural numbers, the $\left(\sum_{k=1}^{n} x_{\sigma(k)} \right)_n$ is a weakly convergent sequence. In other words, $\sum_n x_n$ is wuC if and only if for each $x^* \in X^*$, $\sum_n |x^*(x_n)| < \infty$.*

Theorem 2.3.2. *The following statements regarding a formal series $\sum_n x_n$ in a Banach space are equivalent:*

1. *$\sum_n x_n$ is wuC.*

2. *There is a* $C > 0$ *such that for any* $(t_n)_n \in \ell^\infty$

$$\sup_n \left\| \sum_{k=1}^{n} t_k x_k \right\| \leq C \sup_n |t_n|.$$

3. *For any* $(t_n)_n \in c_0$, $\sum_n t_n x_n$ *converges.*

4. *There is a* $C > 0$ *such that for any finite subset* Δ *of* $\mathbb{N}$ *and any signs* $\pm$, *we have* $\left\| \sum_{n\in\Delta} \pm x_n \right\| \leq C$.

Proof. Suppose 1 holds and define $T : X^* \to \ell^1$ by

$$Tx^* = x^*(x_n).$$

T is a well-defined linear map with a closed graph; therefore, T is bounded. From this we see that for any $(t_n)_n \in B_{\ell^\infty}$ and any $x^* \in B_{X^*}$,

$$\left| x^* \left(\sum_{k=1}^{n} t_k x_k \right) \right| = |(t_1, \ldots, t_n, 0, 0, \ldots) \cdot (Tx^*)| \leq \|T\|.$$

Part 2 follows from this.

If we suppose 2 holds and let $(t_n)_n \in c_0$, then keeping $m < n$ and letting both go to ∞, we have

$$\left\| \sum_{k=m}^{n} t_k x_k \right\| \leq C \sup_{m\leq k\leq n} |t_k| \to 0$$

from which 3 follows easily.

If 3 holds, then the operator $T : c_0 \to X$ defined by $T(t_n) = \sum_n t_n x_n$ cannot be far behind; part 3 assures us that T is well-defined. T is plainly linear and has a closed graph, so T is bounded. The values of T on B_{c_0} are bounded. In particular, vectors of the form $\sum_{n\in\Delta} \pm x_n$, where Δ ranges over the finite subsets Δ of $\mathbb{N}$ and we allow all the $\pm$'s available, are among the values of T on B_{c_0}, and that is statement 4.

Finally, if 4 is in effect, then for any $x^* \in B_{X^*}$ we have $x^* \sum_{n\in\Delta} \pm x_n = \sum_{n\in\Delta} \pm x^* x_n \leq \left\| \sum_{n\in\Delta} \pm x_n \right\| \leq C$ for any finite subset Δ of $\mathbb{N}$ and any choice of signs $\pm$. That $\sum_n |x^* x_n| < \infty$ follows directly from this and along with it we get part 1.

□

Corollary 2.3.3. *A basic sequence for which* $\inf_n \|x_n\| > 0$ *and* $\sum_n x_n$ *is wuC is equivalent to the unit vector basis of* c_0*.*

Proof. If $(x_n)_n$ is a basic sequence and $\sum_n t_n x_n$ is convergent, then $\left(\sum_{k=1}^{n} t_k x_k\right)$ is a Cauchy sequence. Therefore, letting n tend to infinity, the sequence

$$|t_n|\|x_n\| = \left\| \sum_{k=1}^{n} t_k x_k - \sum_{k=1}^{n-1} t_k x_k \right\|$$

tends to 0; from this and the restraint $\inf_n \|x_n\| > 0$, it follows that $(t_n)_n \in c_0$.
On the other hand, if $(x_n)_n$ is a basic sequence and $\sum_n x_n$ is wuC, then $\sum_n t_n x_n$ converges for each $(t_n)_n \in c_0$, thanks to previous theorem part 3.
Consequently, a basic sequence $(x_n)_n$ with $\inf_n \|x_n\| > 0$ and for which $\sum_n x_n$ is wuC is equivalent to the unit vector basis of c_0.

□

Theorem 2.3.4. *A Banach space* X *has a subspace isomorphic to* c_0 *if and only if there is a wuC series* $\sum_n x_n$ *in* X *such that* $\sum_n x_n$ *fails to converge.*

Proof. The "only if" part is trivial: we simply take $(x_n)_n$ as a basic sequence which is equivalent to the unit vector basis of c_0. To prove the "if" part let $(x_n)_n$ be such that $\sum_n |x^*(x_n)| < \infty$ for every $x^* \in X^*$ but $\sum_n x_n$ diverges. It follows from the uniform boundedness principle that there is a constant M so that $\sum_n |x^*(x_n)| \leq M\|x^*\|$ for every $x^* \in X^*$. Since $\sum_n x_n$ diverges there is an $\varepsilon > 0$ and the integers $p_1 < q_1 < p_2 < q_2 < \ldots$ so that $\left\| \sum_{n=p_k}^{q_k} x_n \right\| \geq \varepsilon$ for every k. For

$k = 1, 2, \dots$, put $y_k = \sum_{n=p_k}^{q_k} x_n$. Since $\sum_k |x^*(y_k)| < \infty$ for every $x^* \in X^*$ it follows that $y_k \xrightarrow{w} 0$ and $\inf_k \|y_k\| > 0$. By passing to a subsequence of $(y_k)_k$ if necessary we may assume that $(y_k)_k$ forms a basic sequence. Then, by Corollary 2.3.3, $(y_k)_k$ is equivalent to unit vector basis of c_0.

□

2.4 Basic sequences equivalent to the unit basis of ℓ^1

Let $(x_n)_n$ be a normalized basic sequence of ℓ^1. Suppose there is a constant $K > 0$ such that

$$\sum_{i=1}^{n} |a_i| \leq K \left\| \sum_{i=1}^{n} a_i x_i \right\| \tag{2.4.1}$$

for any n and any scalars $a_1, \dots, a_n$. Then, clearly, $(x_n)_n$ is equivalent to the unit vector basis of ℓ^1.

Conversely, if we are given a normalized basic sequence $(x_n)_n$ which is equivalent to the unit vector basis $(e_n)_n$ of ℓ^1. Then there is an isomorphism $T : \ell^1 \to \ell^1$ such that $T(x_i) = e_i$ for all $i \in \mathbb{N}$. So for arbitrary $n \in \mathbb{N}$ and scalars $a_1, ..., a_n$, we have

$$\sum_{i=1}^{n} |a_i| = \left\| T\Big(\sum_{i=1}^{n} a_i e_i \Big) \right\| \leq \|T\| \left\| \sum_{i=1}^{n} a_i e_i \right\| = \|T\| \left\| \sum_{i=1}^{n} a_i x_i \right\|.$$

Thus a normalized basic sequence $(x_n)_n$ in ℓ^1 is equivalent to the unit vector basis of ℓ^1 if and only if (2.4.1) holds.

Definition 2.4.1. *We say that a sequence $(A_n, B_n)_{n \in \mathbb{N}}$ of sets is independent if for every pair of disjoint finite nonempty subsets B, G of $\mathbb{N}$,*

$$\bigcap_{n \in B} A_n \cap \bigcap_{n \in G} B_n \neq \emptyset.$$

A (finite or infinite) sequence of real-valued functions $(f_n)_n$ on a set Ω is called *independent on a set $A \subseteq \Omega$ if there exist numbers $\alpha < \beta$ such that the sequence of pairs $(\{f_n < \alpha\} \cap A, \{f_n > \beta\} \cap A)_n$ is independent. If we want to specify α and β we say that $(f_n)_n$ is (α, β)-independent on A.*

Proposition 2.4.2. *If $(f_n)_n$ is a (uniformly) bounded independent sequence of functions on a set Ω, then $(f_n)_n$ is a sequence equivalent to the unit basis of ℓ^1.*

Proof. Let $\alpha < \beta$ be such that $(f_n)_n$ is (α, β)-independent. Since the sequence $(f_n)_n$ is bounded, it will be equivalent to the unit basis of ℓ^1 if we can show that for every finite sequence $\alpha_1, \ldots, \alpha_k$ we have

$$\left\| \sum_{i=1}^{k} \alpha_i f_i \right\| \geq \frac{1}{2}(\beta - \alpha) \sum_{i=1}^{k} |\alpha_i|. \tag{2.4.2}$$

We distinguish two cases:

CASE 1. $(\alpha + \beta) \sum_{i \leq k} \alpha_i \geq 0$.

Putting $P := \{i \leq k : \alpha_i \geq 0\}$ and $Q := \{i \leq k : \alpha_i < 0\}$ we then have by the (α, β)-independence of $(f_n)_n$ that

$$\bigcap_{i \in Q} \{f_i < \alpha\} \cap \bigcap_{i \in P} \{f_i > \beta\} \neq \emptyset.$$

For any t in this intersection,

$$\begin{aligned} \sum_{i=1}^{k} \alpha_i f_i(t) &\geq \beta \sum_{i \in P} \alpha_i + \alpha \sum_{i \in Q} \alpha_i \\ &= \frac{\alpha + \beta}{2} \sum_{i=1}^{k} \alpha_i + \frac{\beta - \alpha}{2} \sum_{i=1}^{k} |\alpha_i| \\ &\geq \frac{1}{2}(\beta - \alpha) \sum_{i=1}^{k} |\alpha_i|. \end{aligned}$$

CASE 2. $(\alpha + \beta) \sum_{i \leq k} \alpha_i < 0$.

If we replace the α_i by $-\alpha_i$ we are in the case 1 and it follows for some $s \in T$,

$$-\sum_{i=1}^{k} \alpha_i f_i(s) \geq \frac{1}{2}(\beta - \alpha) \sum_{i=1}^{k} |\alpha_i|.$$

So in either case we have (2.4.2). Hence $(f_n)_n$ is equivalent to the unit basis of ℓ^1.

□

Chapter 3

ROSENTHAL'S ℓ^1-THEOREM

In this section we present two proofs of Rosenthal's ℓ^1-Theorem. The first proof is combinatorial and the second is topological.

3.1 Combinatorial Proof

Definition 3.1.1. *Let S be a set, $(A_n, B_n)_{n\in\mathbb{N}}$ a sequence of pairs of subsets of S with $A_n \cap B_n = \emptyset$ for all $n \in \mathbb{N}$ and X a subset of S. We say that $(A_n, B_n)_{n\in\mathbb{N}}$ converges on X if for every point $x \in X$ we have either* $\lim_{n\to\infty} \chi_{A_n}(x) = 0$ *or* $\lim_{n\to\infty} \chi_{B_n}(x) = 0$.

Of course, every such sequence $(A_n, B_n)_{n\in\mathbb{N}}$ converges on the empty set and if $(A_n, B_n)_{n\in\mathbb{N}}$ converges on X then every subsequence of $(A_n, B_n)_{n\in\mathbb{N}}$ is also convergent on X. Moreover, if $(A_n, B_n)_{n\in\mathbb{N}}$ converges on subsets $X_1, ..., X_l$ of S then $(A_n, B_n)_{n\in\mathbb{N}}$ converges on $\bigcup_{i=0}^{l} X_i$.

Lemma 3.1.2. *Let $l \geq 1$, $(A_n, B_n)_{n\in\mathbb{N}}$ a sequence of pairs of subsets of a set S with $A_n \cap B_n = \emptyset$ for all $n \in \mathbb{N}$, $X_1, ..., X_l$ disjoint subsets of S. Suppose that for each $1 \leq i \leq l$, $(A_n, B_n)_{n\in\mathbb{N}}$ has no subsequence convergent on X_i. Then there exist a j and an infinite subset M of $\mathbb{N}$ so that for each i, $1 \leq i \leq l$, $(A_n, B_n)_{n\in M}$ still has no subsequence convergent on $X_i \cap A_j$ and also on $X_i \cap B_j$.*

Proof. The proof will be proceed by induction on l. For the case $l = 1$, we develop an algorithm to produce the desired j and M; and then we will make use of this algorithm in the inductive step.

Suppose $(A_n, B_n)_{n\in\mathbb{N}}$ are as in the lemma. i.e. the sequence $(A_n, B_n)_{n\in\mathbb{N}}$ has no subsequence convergent on X. Clearly, without loss of generalization we can assume that $S = X$.

To continue we need a definition.

Definition 3.1.3. *We say that j and M work for X if $(A_n, B_n)_{n\in\mathbb{N}}$ has no subsequence convergent on $X \cap A_j$ or on $X \cap B_j$. More generally, we say that j and M r-work if for every $1 \le i \le r$, $(A_n, B_n)_{n\in\mathbb{N}}$ has no subsequence convergent on $X_i \cap A_j$ or on $X_i \cap B_j$.*

The Basic Algorithm. Let $n_1 \in \mathbb{N}$ be any. If n_1 and $N_0 := \mathbb{N}$ do not work, let N_1 be arbitrary subset of N_0 such that $(A_n, B_n)_{n\in N_1}$ converges on A_{n_1} or B_{n_1}. Suppose $k > 1$ and $N_{k-1} \subseteq \mathbb{N}$ and $n_{k-1} \in \mathbb{N}$ are defined. Let $n_k \in N_{k-1}$ with $n_k > n_{k-1}$. If n_k and N_{k-1} do not work, let N_k be arbitrary subset of N_{k-1} such that $(A_n, B_n)_{n\in N_k}$ converges on A_{n_k} or B_{n_k}.

This process can only be continued only finitely many times. That is, as long as the n_j's and N_j's are chosen as above, there must exists a $k \ge 1$ such that n_k works for N_{k-1}.

Suppose the process continued infinitely many times. Then we have an increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ and decreasing sequence of subsets of natural numbers N_k with $n_k \in N_{k-1}$. Also, $(A_n, B_n)_{n\in N_k}$ converges on $\varepsilon_{n_k} A_{n_k}$ where $\varepsilon_{n_k} = \pm 1$ defined for all $k \in \mathbb{N}$ and

$$\varepsilon_{n_k} A_{n_k} = \begin{cases} A_{n_k} & \text{if } \varepsilon_{n_k} = 1, \\ B_{n_k} & \text{if } \varepsilon_{n_k} = -1. \end{cases}$$

Now, put $M = \{n_1, n_2, \ldots\}$. Therefore, for every k, $(A_n, B_n)_{n\in M}$ is a subsequence of $(A_n, B_n)_{n\in N_k}$. Hence $(A_n, B_n)_{n\in M}$ converges on $\bigcup_{k\ge 1} \varepsilon_{n_k} A_{n_k}$. By passing to an infinite subset of M if necessary, we may suppose that $(A_n, B_n)_{n\in M}$ converges on either on $\bigcup_{k\geq 1} A_{n_k}$ or on $\bigcup_{k\geq 1} B_{n_k}$. Without loss of generality, suppose we are in the former case. Since $(A_n, B_n)_{n\in M}$ does not converge on X there is an $x \in X$ so that both $\{n \in M : x \in A_n\}$ and $\{n \in M : x \in B_n\}$ are infinite. But then $x \in \bigcup_{k\geq 1} A_{n_k}$ and hence $(A_n, B_n)_{n\in M}$ cannot converge on $\bigcup_{k\geq 1} \varepsilon_{n_k} A_{n_k}$, a contradiction.

Thus, the process in the basic algorithm is finite. Therefore, for the case $l = 1$ of the lemma, we have a j and an infinite subset M of $\mathbb{N}$ so that $(A_n, B_n)_{n\in\mathbb{N}}$ has no subsequence convergent on $X \cap A_j$ and on $X \cap B_j$.

Next, for the induction hypothesis we assume that Lemma 3.1.2 is proved for the case $l = r$. Let X_i's and $(A_n, B_n)_{n\in\mathbb{N}}$ be given as in the assumptions for the case $l = r + 1$. Again, we shall say that j and M $r - work$ if for every $1 < i < r$, $(A_n, B_n)_n$ has no subsequence convergent on $X_i \cap A_j$ or on $X_i \cap B_j$. By the induction hypothesis, there are $n_1 \in \mathbb{N}$ and $N_1' \subseteq \mathbb{N}$ $r - work$. If n_1 and N_1' do not work for X_{r+1}, choose a subset N_1 of N_1' so that $(A_n, B_n)_{n\in N_1}$ converges on $A_{n_1} \cap X_{r+1}$ or on $B_{n_1} \cap X_{r+1}$. Suppose for $k > 1$ we have defined $N_{k-1} \subseteq \mathbb{N}$ and $n_{k-1} \in \mathbb{N}$. Since $(A_n, B_n)_{n\in N_{k-1}}$ is a subsequence of $(A_n, B_n)_{n\in\mathbb{N}}$, we may apply the induction hypothesis to choose an $n_k \in N_{k-1}$ with $n_k > n_{k-1}$ and $N_k' \subseteq of N_{k-1}$ so that n_k and N_k' $r - work$. Again, if n_k and N_{k-1}' do not work for X_{r+1} choose $N_k \subseteq N_k'$ so that $(A_n, B_n)_{n\in N_k}$ converges on $A_{n_k} \cap X_{r+1}$ or on $B_{n_k} \cap X_{r+1}$. Now, this process cannot be continued indefinitely, since n_k's and N_k's are constructed to satisfy the criteria of the *Basic Algorithm* and $(A_n, B_n)_{n\in\mathbb{N}}$ has no subsequence convergent on X_{r+1}. Thus, there must exists a $k > 1$ so that n_k and N_k' work. By construction, n_k and N_k' $r - work$, hence by definition, n_k and N_k' satisfy the conclusion of Lemma 3.1.2.

□

Theorem 3.1.4. *Let $(A_n, B_n)_{n\in\mathbb{N}}$ be a sequence of pairs of subsets of a set S with $A_n \cap B_n = \emptyset$ for all n, and suppose that $(A_n, B_n)_{n\in\mathbb{N}}$ has no convergent subsequence. Then there is an infinite subset $M \subseteq \mathbb{N}$ so that $(A_n, B_n)_{n\in M}$ is independent.*

Proof. We apply Lemma 3.1.2 for the case $l = 1$. Then there are $n_1 \in \mathbb{N}$ and $M_1 \subseteq \mathbb{N}$ so that $(A_n, B_n)_{n \in M_1}$ has no convergent subsequence on either A_{n_1} or B_{n_1}. Suppose we have further chosen $n_1 < \ldots < n_k$ and M_k so that on each of the disjoint 2^k sets $\bigcap_{j=1}^{k} \varepsilon_j A_{n_j}$, $(A_n, B_n)_{n \in M_k}$ has no convergent subsequence, where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ ranges over all 2^k choices of signs $\varepsilon_j = \pm 1$ all j. Now, applying Lemma 3.1.2 for the case $l = 2^k$, choose $n_{k+1} \in M_k$, $n_{k+1} > n_k$, and M_{k+1} a subset of M_k so that for each $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$, $(A_n, B_n)_{n \in M_{k+1}}$ has no convergent subsequence on $\bigcap_{j=1}^{k} \varepsilon_j A_{n_j} \cap A_{n_{k+1}}$ and on $\bigcap_{j=1}^{k} \varepsilon_j A_{n_j} \cap B_{n_{k+1}}$. So by induction we have $(n_j)_j$ and $(M_j)_j$. Then $M = \{n_1, n_2, \ldots\}$ is the desired subset.

□

Proposition 3.1.5. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of uniformly bounded real-valued functions defined on a set S and δ and r be real numbers with $\delta > 0$. Assume, putting $A_n = \{x \in S : f_n(x) > \delta + r\}$ and $B_n = \{x \in S : f_n(x) < r\}$ for all $n \in \mathbb{N}$, that $(A_n, B_n)_{n \in \mathbb{N}}$ is independent. Then $(f_n)_{n \in \mathbb{N}}$ is equivalent in the sup-norm to the usual ℓ^1-basis.*

Proof. By multiplying f_n's by an appropriate constant if necessary, we may assume that $\delta + r > 0$. Let $(c_i)_{i \in \mathbb{N}}$ be a sequence of scalars with only finitely many of c_i's non-zero with $\sum |c_i| = 1$. We will show that there is an $s \in S$ with

$$\left| \sum c_i f_i(s) \right| \geq \frac{\delta}{2}. \tag{3.1.1}$$

Hence $\left\| \sum c_i f_i \right\| \geq \frac{\delta}{2}$. Let $G = \{i \in \mathbb{N} : c_i > 0\}$ and $B = \{i \in \mathbb{N} : c_i < 0\}$. Then both G and B are finite and hence $\bigcap_{n \in B} A_n \cap \bigcap_{n \in G} B_n \neq \emptyset$ and $\bigcap_{n \in G} A_n \cap \bigcap_{n \in B} B_n \neq \emptyset$ as $(A_n, B_n)_{n \in \mathbb{N}}$ is independent. Let $x \in \bigcap_{n \in B} A_n \cap \bigcap_{n \in G} B_n$ and $y \in \bigcap_{n \in G} A_n \cap \bigcap_{n \in B} B_n$.

If we suppose first that $r \geq 0$ and $B' = \{i \in B : f_i(x) > 0\}$, then

$$\sum_{i \in B} c_i f_i(x) \geq \sum_{i \in B'} c_i f_i(x) > -r \sum_{i \in B'} |c_i| \geq \sum_{i \in B} |c_i|(-r). \tag{3.1.2}$$

Similarly,

$$\sum_{i \in G} c_i f_i(y) \geq \sum_{i \in G} |c_i|(-r). \tag{3.1.3}$$

By (3.1.2) and (3.1.3), we have

$$\sum c_i f_i(x) \geq \sum_{i \in G} |c_i|(\delta + r) + \sum_{i \in B} |c_i|(-r) \tag{3.1.4}$$

and

$$\sum c_i f_i(y) \geq \sum_{i \in B} |c_i|(\delta + r) + \sum_{i \in G} |c_i|(-r). \tag{3.1.5}$$

Similarly, (3.1.4) and (3.1.5) hold if $r < 0$. Now, the sum of right-hands of equations (3.1.4) and (3.1.5) is equal to δ. Hence maximum of the left-hand sides is at least $\frac{\delta}{2}$. Therefore x or y is the element s satisfying (3.1.1). Hence $(f_n)_n$ is equivalent, in the sup-norm to the usual ℓ^1-basis.

□

Lemma 3.1.6. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of uniformly bounded real-valued functions defined on a set S with no pointwise convergent subsequence on S. For each subset M of $\mathbb{N}$, let*

$$\delta(M) = \sup_{x \in S}(\limsup_M f_m(x) - \liminf_M f_m(x)).$$

Then there exists a subset Q of $\mathbb{N}$ so that for all subsets L of Q, $\delta(L) = \delta(Q)$.

Proof. Since $(f_n)_n$ has no pointwise convergent subsequence, we note that $\delta(M) > 0$ for all subsets M of $\mathbb{N}$. Also, for any subsets L and M of $\mathbb{N}$ with L almost contained in M, we have that $\delta(L) \leq \delta(M)$. Suppose the conclusion is false. Then there is a transfinite family $\{N_\alpha : \alpha < \omega_1\}$ of subsets of $\mathbb{N}$, indexed by the set of ordinals α less than the first uncountable ordinal ω_1, with the property that for all $\alpha < \beta < \omega_1$, N_β is almost contained in N_α and $\delta(N_\beta) \leq \delta(N_\alpha)$. But this is impossible because there is no transfinite strictly decreasing sequence of positive real numbers. Hence we have a subset Q of $\mathbb{N}$ such that for all subsets L of Q, $\delta(L) = \delta(Q)$.

□

Lemma 3.1.7. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of uniformly bounded real-valued functions defined on a set S with no pointwise convergent subsequence on S. Let Q be a subset of* $\mathbb{N}$ *satisfying the conclusion of Lemma* 3.1.6 *and put $\delta = \delta(Q)/2$. There exist a subset M' of Q and a rational number r so that for every subset L of M', there is an $x \in S$ satisfying*

$$\limsup_L f_i(x) > \delta + r \quad \text{and} \quad \liminf_L f_i(x) < r.$$

Proof. Suppose for a contradiction that the conclusion of the lemma does not hold. Let $r_1, r_2, \ldots$ be an enumeration of the rational numbers. Choose $L_1 \subset Q$ such that for all $x \in S$,

$$\limsup_L f_i(x) \leq \delta + r \quad \text{or} \quad \liminf_L f_i(x) \geq r. \tag{3.1.6}$$

for $L = L_1$ and $r = r_1$ Suppose for $k \geq 1$we have chosen the subset L_k of Q. Choose $L_{k+1} \subset L_k$ so that (3.1.6) holds for $L = L_{k+1}$ and $r = r_{k+1}$. Hence we have $L_1 \supset L_2 \supset L_3 \supset \ldots \supset L_k \supset \ldots$ by induction. Now by the standard diagonal procedure, choose an infinite set L with L almost contained in L_k for all $k \in \mathbb{N}$. Then (3.1.6) holds for all rational numbers r. Since L is almost contained in Q and Q satisfies the conclusion of Lemma 3.1.6, $\delta(L) = \delta(Q) = 2\delta$. Let $\varepsilon = \delta/2$.

By the very definition of $\delta(L)$, there is an $x \in S$ so that

$$\limsup_L f_i(x) - \liminf_L f_i(x) > \delta(L) - \varepsilon.$$

Let $a = \limsup_L f_i(x)$ and $b = \liminf_L f_i(x)$. Then, let r be a rational number such that $b + \delta/2 > r > b$. Thus

$$b < r < r + \delta = r - b + \delta + b < 2\delta - \varepsilon + b < a.$$

Hence $a > \delta + r$ and $b < r$, contradicting (3.1.6).

□

Theorem 3.1.8. *Let S be a set and $(f_n)_n$ a uniformly bounded sequence of real valued functions defined on S. Then $(f_n)_n$ has a subsequence $(f_{n_k})_k$ satisfying one of the following alternatives:*

(i) $(f_{n_k})_k$ converges pointwise on S.

(ii) $(f_{n_k})_k$ is equivalent to the usual ℓ^1-basis.

Proof. Suppose $(f_n)_n$ has no pointwise convergent subsequence. Let M', δ and r be chosen as in Lemma 3.1.7, and for each $n \in M'$, let $A_n = \{x : f_n(x) > \delta + r\}$ and $B_n = \{x : f_n(x) < r\}$. Then taking $L = M'$, we have that $(A_n, B_n)_{n \in M'}$ does not have any convergent subsequence on S. Hence, by Theorem 3.1.4, there is an infinite subset $M \subseteq M'$ with $(A_n, B_n)_{n \in M}$ is independent. Therefore, $(f_n)_n$ is a sequence satisfying assumptions of Proposition 3.1.5, thus $(f_n)_n$ is equivalent to the usual basis of ℓ^1.

□

If $(b_n)_n$ is a bounded sequence in a Banach space B, we let S denote the closed unit ball of B^* and then define $f_n(s) = s(b_n)$ for all $s \in S$ and n. Therefore we have the following theorem :

Theorem 3.1.9 (Rosenthal's ℓ^1-Theorem)**.** *Let $(f_n)_n$ be a bounded sequence in a real Banach space B. Then $(f_n)_n$ has a subsequence $(f_{n_k})_k$ satisfying one of the following two mutually exclusive alternatives:*

(i) $(f_{n_k})_k$ is a weak Cauchy subsequence.

(ii) $(f_{n_k})_k$ is equivalent to the usual ℓ^1-basis.

Corollary 3.1.10. *Let B be a (real or complex) Banach space. Then B does not contain an isomorphic copy of ℓ^1 if and only if every bounded sequence $(x_n)_n$ in B has a weakly Cauchy subsequence.*

3.2 Baire Category Proof

Let T be a compact space and $Z \subseteq C(T)$ bounded set of continuous functions.

Definition 3.2.1. *A nonempty closed set $L \subseteq T$ is called topologically critical (t-critical) for Z if there exist numbers $\alpha < \beta$ such that for all $k, l \in \mathbb{N}$, the intersection*

$$\left(\bigcup_{f \in Z} \{f < \alpha\}^k \times \{f > \beta\}^l \right) \cap L^{k+l} \tag{3.2.1}$$

is dense in L^{k+l}.
Z is called topologically stable (t-stable) if no t-critical sets exist.

Proposition 3.2.2. *If Z is not t-stable, then Z contains an independent sequence.*

Proof. Let $L \subseteq T$ be a t-critical set for Z and let $\alpha < \beta$ be such that (3.2.1) is satisfied. The key to the inductive proof below is the following reformulation of (3.2.1):

For every $n \in \mathbb{N}$ and for every n-tuple $U_1, U_2, \dots, U_n$ of nonempty open subsets of L there exists an $f \in Z$ that on each U_i $(i = 1, \dots, n)$ attains values $< \alpha$ and values $> \beta$.

Suppose we are given $n \in \mathbb{N}$ and $U_1, U_2, \ldots, U_n$ we clearly have

$$U_1 \times \ldots \times U_n \times U_1 \times \ldots \times U_n \cap \left(\bigcup_{f \in Z} \{f < \alpha\}^n \times \{f > \beta\}^n \right) \neq \emptyset.$$

The construction of the independent sequence is now easy. For $n = 1$, take $U_1 = L$. Then by there is an $f_1 \in Z$ such that $U_1 \cap \{f_1 < \alpha\} \neq \emptyset$ and $U_1 \cap \{f_1 > \beta\} \neq \emptyset$. Suppose $f_1, \ldots, f_n$ have been selected so that $(f_i)_{i=1}^n$ is (α, β)-independent on L (Remember that a (finite or infinite) sequence of functions $(f_n)_n$ on a set Ω is called (α, β)-independent on a set $A \subseteq \Omega$ if the sequence of pairs $(\{f_n < \alpha\} \cap A, \{f_n > \beta\} \cap A)_n$ is independent). To choose f_{n+1} we apply above reformulation to the 2^n-tuple of nonempty open subsets $U_P \cap L$, where

$$U_P := \left(\bigcap_{k \in P} \{f_k < \alpha\} \right) \cap \left(\bigcap_{k \notin P} \{f_k > \beta\} \right)$$

for every nonempty $P \subseteq \{1, \ldots n\}$. Observe that $U_P \cap L \neq \emptyset$ by the induction hypothesis. Let $f_{n+1} \in Z$ be as in the reformulation of (3.2.1) for these $U_P \cap L$. Then both $\{f_{n+1} < \alpha\}$ and $\{f_{n+1} > \beta\}$ meet each $U_P \cap L$, i.e. $(f_i)_{i=1}^{n+1}$ is (α, β)-independent on L. This completes the induction and the proof.

□

Proposition 3.2.3. *If T is compact and $Z \subseteq C(T)$ is not t-stable, then there exists a Radon measure μ on T such that $L^1(\mu)$is isometric to $L^1 := L^1[0,1]$ and so that Z is not totally bounded in $L^1(\mu)$.*

Proof. By Proposition 3.2.2, Z contains a sequence $(f_n)_n$ which is (α, β)-independent for some $\alpha < \beta$. Observe that the sets $\{f_n < \alpha\} \cup \{f_n > \beta\}$ $(n = 1, 2, \ldots)$ satisfy the finite intersection property. Hence $K := \bigcap_{n=1}^{\infty} \{f_n < \alpha\} \cup \{f_n > \beta\} \neq \emptyset$ and compact. We now define a map $h : K \to \{0,1\}^{\mathbb{N}}$ with components h_n by

$$h_n(t) := \begin{cases} 0 & \text{if } f_n(t) \leq \alpha, \\ 1 & \text{if } f_n(t) \geq \beta. \end{cases}$$

Since h_n is continuous and (by the independence of $(f_n)_n$) $h(K)$ is dense in $\{0,1\}^{\mathbb{N}}$, h is a surjection. Letting ν be the product measure $(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)^{\mathbb{N}}$ on $\{0,1\}^{\mathbb{N}}$ we know from Prop B.1 in [5] that there is a Radon probability μ on K such that $h\mu = \nu$ and with the additional property that $L^1(\mu) \cong L^1(\nu) \cong L^1$. Since for $m \neq n$

$$K \cap \{f_n \leq \alpha\} \cap \{f_m \geq \beta\} = h^{-1}\{(\varepsilon_k)_k \in \{0,1\}^{\mathbb{N}} : \varepsilon_n = 0 \text{ and } \varepsilon_m = 1\},$$

we have $\mu(\{f_n \leq \alpha\} \cap \{f_m \geq \beta\}) = \frac{1}{4}$. It immediately follows that $\|f_m - f_n\| \geq \frac{1}{4}(\beta - \alpha)$, so that $(f_n)_n$ is not totally bounded in $L^1(\mu)$. Neither in Z.

□

We now give a characterization of Baire-1 functions in terms of stability.

Lemma 3.2.4. *The following are equivalent for a function f on T:*

(i) $f \in B_1(X)$.

(ii) For every nonempty closed subset L of T and for all numbers $\alpha < \beta$ the sets $L \cap \{h < \alpha\}$ and $L \cap \{h > \beta\}$ are not both dense in L.

Proof. (i) $\Rightarrow$ (ii): If there is a closed subset $L \subseteq T$ and numbers $\alpha < \beta$ such that $\overline{L \cap \{h < \alpha\}} = \overline{L \cap \{h > \beta\}} = L$ then $f_{\restriction F}$ has no point of continuity. Thus f cannot be in $B_1(T)$ by Theorem 1.3.5.

(ii) $\Rightarrow$ (i): Let $L \subseteq T$ and let $((\alpha_n, \beta_n))$ be an enumeration of all rational numbers (α, β) with $\alpha < \beta$. For each $n \in \mathbb{N}$, put

$$A_n := L \cap f < \alpha_n \qquad \text{and} \qquad B_n := L \cap f > \beta_n.$$

Consider the sets $L_n := \overline{A_n} \cap \overline{B_n}$. Each L_n is nowhere dense in L. Therefore, by Baire Category Theorem, $F := \bigcap_{n \in \mathbb{N}} (L \setminus L_n)$ is dense in L. Moreover, $f_{\restriction F}$ is continuous at every point in $\bigcap_{n\in\mathbb{N}}(L \setminus L_n)$. Hence $f \in B_1(X)$.

□

Lemma 3.2.5. *If Z is t-stable, then Z is relatively τ_p-compact in $B_1(T)$.*

Proof. Suppose for a contradiction that h is in the τ_p-closure of Z, but $h \notin B_1(T)$. Then,by Lemma 3.2.4, there are a closed subset $L \subseteq T$ and numbers $\alpha < \beta$ such that $\overline{L \cap \{h < \alpha\}} = \overline{L \cap \{h > \beta\}} = L$. But this implies that for all $k, l \in \mathbb{N}$,

$$\bigcup_{f\in Z}(\{f < \alpha\}^k \times \{f > \beta\}^l) \cap L^{k+l},$$

is dense in L^{k+l}, contradicting the stability of Z.

□

Proposition 3.2.6. *Let T be compact and let $Z \subseteq C(T)$ be t-stable. Then every sequence $(f_n)_n$ in Z has a pointwise convergent subsequence.*

Proof. Let $(f_n)_n$ be a sequence in Z. Consider the map $F : T \to \mathbb{R}^{\mathbb{N}}$ defined by $F(t) := (f_n(t))_n$ for $t \in T$ and put $S := F(T)$. Then S is compact. It suffices to show that every τ_p-cluster point of sequence $(e_n)_n$ of coordinate functions on S is the τ_p-limit of a subsequence of $(f_n)_n$, since $f_n = e_n \circ F$. So all we have to do is show that $(e_n)_n$ is t-stable. For a contradiction suppose that $L \subseteq S$ is t-critical (hence compact) for $(e_n)_n$ and let $\alpha < \beta$ be as in (3.2.1). By an application of Zorn's Lemma there is a minimal compact $M \subseteq T$ with $F(M) = L$, i.e. such that $M' \subset M$, compact $F(M') \subset L$. We claim that M is t-critical for $(f_n)_n$, contradicting the fact that $(f_n)_n$ is t-stable. Indeed, for any k-tuple of nonempty open sets $U_1, \dots U_k \subseteq M$ we have by minimality of M that each $F(U_i)$ contains a nonempty open subset $V_i \subset L$ $(i = 1, \dots k)$. Since we are assuming that L is t-critical for $(e_n)_n$, some $(e_n)_n$ takes values $< \alpha$ and $> \beta$ on each V_i. This implies that the corresponding $f_n = e_n \circ F$ takes values $< \alpha$ and $> \beta$ on each U_i.

Because $U_1, \ldots U_k$ were arbitrary we have now proved that M is t-critical for $(f_n)_n$, a contradiction.

□

Let X be a Banach space and T be the closed unit ball of X^*, equipped with its weak* topology. Then T is compact by Alaoglu's theorem. We identify X^{**} with a subspace of $\mathbb{R}^T$. Notice that under this identification the weak* topology of X^{**} corresponds to the topology of pointwise convergence on T. Next, we see the elements of X as continuous functions on X^*. Then by restricting the functions on X we get $Z := B_X \subseteq C(T)$.

Suppose first that every sequence in Z has a pointwise convergent subsequence. Note that by Riesz Representation Theorem, bounded sequences in $C(T)$ are τ_p-Cauchy if and only if they are weakly Cauchy. Then since the unit vectors in ℓ^1 have no weakly Cauchy subsequence, Z cannot contain a sequence equivalent to ℓ^1.

Conversely, suppose that Z has a subsequence with no pointwise convergent subsequence. Therefore Z cannot be t-stable by Proposition 3.2.6. Hence Z contains an independent sequence by Proposition 3.2.2. Then Z has a sequence equivalent to the unit basis of ℓ^1 by Proposition 2.4.2.

Thus we have again reached the following conclusion:

Theorem 3.2.7. *Let X be a Banach space. Then X does not contain an isomorphic copy of ℓ^1 if and only if every bounded sequence $(x_n)_n$ in X has a weakly Cauchy subsequence.*

3.3 Some equivalences

Theorem 3.3.1. *Let B be a separable Banach space. The following are equivalent:*

(1) B contains no subspace isomorphic to ℓ^1.

(2) Every bounded sequence in B has a weak Cauchy subsequence.

(3) B^* is weak* sequentially dense in B^{**}.

(4) The cardinality of B^{**} equals the cardinality of B.

(5) Every bounded sequence in B^{**} has a weak* convergent subsequence.

(6) Every bounded subset of B is weakly sequentially dense in its weak closure.

(7) Every bounded subset of B^{**} is weak* sequentially dense in its weak* closure.

(8) Every bounded weak* closed convex subset of B^* is the norm closed convex hull of the set of its extreme points.

Theorem 3.3.2. *Let X be a Banach space. Then the following are equivalent:*

(1) X contains no subspace isomorphic to ℓ^1.

(2) Every bounded sequence in X has a weak Cauchy subsequence.

(3) X^* contains no subspace isomorphic to $L^1 := L^1[0,1]$.

(4) X^* contains no subspace isomorphic to $C[0,1]^*$.

Chapter 4

SOME APPLICATIONS OF ROSENTHAL'S ℓ^1-THEOREM

In this section we give some applications of Rosenthal's ℓ^1-theorem.

4.1 Weakly compact operators

Let X and Y be two Banach spaces on $\mathbb{R}$.

Definition 4.1.1. *A subset A in X is called conditionally weakly compact, if every sequence in A admits a weak Cauchy subsequence.*

A linear map $T : X \to Y$ from the Banach space X into Y is called weakly compact/Rosenthal, if it maps the closed unit ball of X onto a relatively weakly compact/a conditionally weakly compact set in Y.

Clearly every weakly compact operator is Rosenthal and Rosenthal's ℓ^1-theorem implies that $T : X \to Y$ is Rosenthal if and only if T is not an isomorphism on any copy of ℓ^1 in X.

By Eberlien-Šmulian Theorem, $T : X \to Y$ is weakly compact if and only if for every bounded sequence $(x_n)_n$ of X the sequence $(T(x_n))_n$ has a weakly convergent subsequence in Y.

Linear combinations of weakly compact operators are weakly compact. The composition of a weakly compact operator and a bounded linear operator when possible is weakly compact. Moreover, the limit in the operator norm of a sequence of weakly compact linear operators is a weakly compact linear operator if it exists.

4.2 Schur property

Definition 4.2.1. *We say that a Banach space X has the Schur property if weakly convergent sequences in X are norm convergent.*

Theorem 4.2.2 (Schur's lemma)**.** *ℓ^1 has Schur property.*

Proposition 4.2.3. *A Banach space X with the Schur property is weakly sequentially complete.*

Proof. Let X be a Banach space with Schur property. Suppose $(x_n)_n$ is weakly Cauchy sequence in X. Take two increasing sequence $(n_k)_k$ and $(m_k)_k$ of natural numbers. Then $(x_{n_k} - x_{m_k})_k$ is weakly null. Since X has Schur property $(x_{n_k} - x_{m_k})_k$ is norm null. Thus $(x_n)_n$ is norm Cauchy. Then it is norm convergent and hence weakly convergent. Hence X is weakly sequentially complete.

□

Proposition 4.2.4. *Let X be a weakly sequentially complete Banach space. Then either X is reflexive or X contains a subspace isomorphic to ℓ^1.*

Proof. Suppose X is not reflexive. Then X contains a bounded sequence $(x_n)_n$ that has no weakly convergent subsequence. Hence, by Theorem 3.1.9, $(x_n)_n$ has either a weakly Cauchy subsequence or a subsequence equivalent to the usual basis of ℓ^1. But if the first case is true, say $(x_{n_k})_k$ is such a subsequence, we would have found a weakly convergent subsequence of $(x_{n_k})_k$ since X is weakly sequentially complete, contradicting our assumption. Hence the latter case is true. i.e. X contains a subspace isomorphic to ℓ^1.

□

Definition 4.2.5. *A Banach space X is said to have the Dunford-Pettis property if every weakly compact operator into a Banach space Y transforms weakly compact sets in X into norm compact sets in Y.*

Proposition 4.2.6. *The dual X^* of a Banach space X has the Schur property if and only if X has the Dunford-Pettis property and does not contain ℓ^1.*

Corollary 4.2.7. *If both X and X^* have the Schur property then X is finite dimensional.*

Theorem 4.2.8. *Let X be a Banach space not containing ℓ^1. Then every bounded linear operator into a Banach space Y which carries weak Cauchy sequences to norm Cauchy sequences is compact.*

Proof. Let $T : X \to Y$ be such an operator and $(x_n)_n$ be a bounded sequence in X. Since X does not contain ℓ^1, by Rosenthal's ℓ^1-Theorem, $(x_n)_n$ has a weakly Cauchy (hence convergent) subsequence $(x_{n_k})_k$. Then $(T(x_{n_k}))_k$ is norm convergent. Therefore for every bounded sequence $(x_n)_n$ in X the sequence $(T(x_n))_n$ has a norm convergent subsequence. Hence T is compact.

□

4.3 Limited sets

Let X be a Banach space.

Definition 4.3.1. *A subset B of X is said to be limited if every weak* null sequence $(x_n^*)_n$ in X^* converges uniformly on B, that is*

$$\lim_n \sup_{x \in B} x_n^*(x) = 0.$$

For example, every relatively compact subset of X is limited by uniform boundedness principle. Also, limited sets are bounded.

Theorem 4.3.2. *Let X be a Banach space not containing ℓ^1. Then every limited subset of X^* is relatively compact.*

Proof. Let K be a limited subset of X^*. We define an operator $T : X \to B(K)$ by putting $T(x)(x^*) := x^*(x)$. T is a bounded linear operator that sends weak Cauchy sequences to norm Cauchy sequences. Hence, by Theorem 4.2.8, T is compact. Also, the adjoint operator T^* of T is compact. Now, let $x^* \in K$. We define $F_{x^*} : B(K)^* \to \mathbb{R}$ by $F_{x^*}(f) = f(x^*)$ for any $f \in B(K)$. This implies that $T^*(F_{x^*})(x^*) = x^*(x)$ and hence $T^*(F_{x^*}) = x^*$ for any $x^* \in K$. Since $K = \{T^*(F_{x^*}) : x^* \in K\} \subseteq T^*(B_{B(K)^*})$ we have that K is relatively compact.

□

4.4 Grothendieck property

Definition 4.4.1. *A Banach space X is called a Grothendieck space whenever weak* convergence and weak convergence of sequences coincide in the dual space X^*. A Banach space is said to be a Grothendieck space if it has the Grothendieck property.*

For example, reflexive spaces have Grothendieck property. So does ℓ^∞.

Proposition 4.4.2. *Let X be a Banach space. Then X contains a quotient isomorphic to c_0 if and only if X^* contains a weak* null sequence equivalent to the unit basis of ℓ^1.*

Proof. Note that there is a bijection between linear continuous maps $T : X \to c_0$ and weak* null sequences $(x_n^*)_n$. We have $T(x) = (x_n^*(x))_n$ for all $x \in X$ and $T^*(\alpha) = \sum_n \alpha_n x_n^*$ for all $\alpha = (\alpha_n)_n \in \ell^1$.

($\Rightarrow$): Assume that $T : X \to c_0$ is a quotient map. Then T^* is an isomorphism into and hence (x_n^*) is equivalent to the unit basis of ℓ^1.

($\Leftarrow$): Take $T(x) = (x_n^*(x))_n$. Since (x_n^*) is equivalent to the unit basis of ℓ^1

there is a constant $K > 0$ such that $\sum_{i=1}^{n} |\alpha_i| \leq K \left\| \sum_{i=1}^{n} \alpha_i x_i^* \right\|$ for any n and any scalars $\alpha_1, \ldots, \alpha_n$. Thus we have that $\sum_{i=1}^{\infty} |\alpha_i| \leq K \left\| \sum_{i=1}^{\infty} \alpha_i x_i^* \right\| = \|T^*(\alpha)\|$ for all $\alpha = (\alpha_n)_n \in \ell^1$ which implies T^* is an isomorphism into. Therefore, the range of T is dense and closed and we deduce that T is onto.

□

We now give a characterization for Banach spaces having Grothendieck property. To this end we need:

Claim 4.4.3. *Let X be a Banach space with Grothendieck property and Y be a separable Banach space then every bounded linear operator $T : X \to Y$ is weakly compact.*

Proof. Let $T : X \to Y$ be a bounded linear operator. We show that the adjoint operator $T^* : Y^* \to X^*$ is weakly compact. First note that B_{Y*} is weak* compact by Alaoglu's Theorem and as Y is separable B_{Y^*} is also weak* metrizable. Now, let $(y_n^*)_n$ be a sequence in B_{Y^*}. Then we can find a weak*, hence weak, convergent subsequence $(y_{n_k}^*)_n$ of $(y_n^*)_n$. Moreover, T^* is weakly continuous since T is bounded linear. Thus $(T^*(y_{n_k}^*))_n$ converges weakly. Therefore T^* is weakly compact. Hence T itself is weakly compact.

□

Now, we see that a separable quotient of a Grothendieck space is reflexive.

Let X/M be a separable quotient of a Grothendieck space X. Consider the quotient map $\pi : X \to X/M$. It is bounded and linear. Therefore it is weakly compact by above claim. Hence $\pi(X) = X/M$ is closed, which means X/M is reflexive.

Theorem 4.4.4. *For any Banach space X the following are equivalent:*

(1) X has the Grothendieck property.

(2) X^* is weakly sequentially complete and no quotient of X is isomorphic to c_0.

Proof. (1) $\Rightarrow$ (2) : Let X be a Grothendieck space and $(x_n^*)_n$ a weakly Cauchy sequence in X^*. Then it is weak* convergent. Since X is a Grothendieck space, $(x_n^*)_n$ converges weakly. Hence X^* is weakly sequentially complete. Clearly, no quotient of X is isomorphic to c_0 by above remark.

(2) $\Rightarrow$ (1) : Suppose for a contradiction that X does not have the Grothendieck property. So there is a weak* null sequence $(x_n^*)_n$ in X^* which is not weakly null. Then as X^* is weakly sequentially complete $(x_n^*)_n$ has no weakly Cauchy subsequence. Then X^* contains a sequence equivalent to the unit vector basis of ℓ^1 and we can find a subsequence $(x_{n_k^*})_n$ which is equivalent to the unit basis of ℓ^1. But this is not possible by Proposition 4.4.2. Hence X is a Grothendieck space.

□

Bibliography

[1] F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, Springer, New York, 2005.

[2] J. B. Conway, *A Course in Functional Analysis*, Springer-Verlag, New York, 1990.

[3] R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Longman Scientific & Technical Ed., England, 1993.

[4] J. Diestel, *Sequences and Series in Banach Spaces*, Springer-Verlag, New York, 1984.

[5] D. van Dulst, *Characterizations of Banach spaces not containing* l^1, Centrum voor Wiskunde en Informatica, Amsterdam, 1989.

[6] N. Dunford, J. T. Schwartz, *Linear Operators Part I*, Wiley - Interscience Publication, New York, Chichester, Brisbane, Toronto, Singapore, 1988.

[7] F. Hausdorff, *Set Theory*, American Mathematical Society, Providence, 2005.

[8] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I and II*, Springer-Verlag, New York, 1996.

[9] T. J. Morrison, *Functional Analysis: An Introduction to Banach Space Theory*, Wiley, New York, 2001.

[10] H. P. Rosenthal, *A characterization of Banach spaces containing* l^1, Proc. Nat. Acad. Sci. U.S.A., 71 (1974), 2411-2413, Providence, 1974.

[11] H. P. Rosenthal, *Point-wise compact subsets of the first Baire class*, Amer. J. Math., 99 (1977), 362-378.

[12] A. Ülger, *Real Analysis*, Lecture Notes, 2006.