# STABILITY ANALYSIS OF FITZHUGH-NAGUMO EQUATIONS

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# Table of Contents

# Abstract


FitzHugh-Nagumo system is generally used to model some biological phenomena and electrical circuits. This system is obtained by reduction of the Hodgkin-Huxley system which is also widely used but harder to analyze.

In this work, our aim is to study the existence and uniqueness of solutions to reaction-diffusion equations and some stability properties of FitzHugh-Nagumo equations.

We firstly consider the problem of local and global existence and uniqueness of the solution to the reaction-diffusion equation.

Then, we study the problem of stabilization of solutions of FitzHugh-Nagumo system on a bounded domain. We show that by applying a feedback controller on a subdomain, the system can be exponentially stabilized.

Finally, considering again a FitzHugh-Nagumo system, we study the continuous dependence of solutions to this system on the diffusivity coefficient.

# Özet

FitzHugh-Nagumo diferensiyal denklem sistemi, bazı biyolojik olayları ve elektrik devrelerini modellemek için kullanılır. Bu sistem, yine bazı biyolojik olayları modellemek için çokça kullanılan fakat daha karmaşık olan Hodgkin-Huxley sisteminden iki değişkenin indirgenmesiyle elde edilir.

Bu çalışmadaki amacımız, reaksiyon-yayılma denklemlerinin çözümlerinin varlığını-tekliğini ve FitzHugh-Nagumo denklemlerinin bazı kararlılık özelliklerini incelemektir.

Öncelikle, reaksiyon-yayılma denklemlerinin çözümlerinin yerel ve küresel varlık-teklik problemini inceleyeceğiz.

Sonrasında, sınırlı tanım kümesinde tanımlı olan FitzHugh-Nagumo sisteminin çözümlerinin kararlılaştırılması problemini inceleyeceğiz.

Son olarak, FitzHugh-Nagumo sisteminin çözümlerinin yayılma katsayısına sürekli bağımlılığını inceleyeceğiz.

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# Notation

| | |
|---|---|
| $D(f)$ | The domain of $f$ |
| $R(f)$ | The range of $f$ |
| $X'$ | The dual space of $X$ |
| $\hookrightarrow$ | continuous embedding |
| $\subset\subset$ | compact embedding |
| $\rightharpoonup$ | weak convergence |
| $C(\Omega)$ | continuous functions on $\Omega$ |
| $C^k(\Omega)$ | k times continuously differentiable functions on $\Omega$ |
| $B$ | Banach space of all bounded, continuous functons $u(x,t)$ on $\mathbb{R}$ (t fixed) |
| $\lVert u(.,t)\rVert_B$ | $\sup_{x\in\mathbb{R}\lvert u(x,t)\rvert}$ (t fixed) |
| $C([0,T];B)$ | The space of all continuous functions defined on $[0,T]$ that have values in B |
| $\lVert u\rVert$ | $\sup_{t\in[0,T]}\lVert u(.,t)\rVert_B$ |
| $\Delta u$ | $\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}$ |
| $\nabla u$ | $(\partial_{x_1}u,...,\partial_{x_n}u)$ |
| $\lvert\Omega\rvert$ | The Lebesgue measure of $\Omega$ |

# Introduction

We will basically consider some stability properties of FitzHugh-Nagumo type of differential equations.

The problem of the existence and uniqueness of solutions to FitzHugh-Nagumo model and the stability analysis of this model have been studied by various approaches. The existence-uniqueness of solutions to nonlinear reaction-diffusion equations and stability properties (such as asymptotic stability or global asymptotic stability) of solutions are studied in [10], [7], [11]. Jackson proved the existence and uniqueness of solution to the initial boundary value problem for FitzHugh-Nagumo system by using the energy method in [3]. Some researchers focused on the long-time behaviour of the system. The asymptotic behaviour of the system defined on a bounded domain is studied in [9]. On the other hand, some researchers focused on the travelling wave solutions and the stability of these solutions in [4], [18]. There is also some work on the structural stability of the solution to the problem. This kind of stability is defined as the continuous dependence of solutions on the diffusivity coefficient. The problem of continuous dependence of solutions to the semi-linear reaction-diffusion system is studied, in [2].

FitzHugh-Nagumo equation is a system of reaction-diffusion equations which has the form

$$u_t - Du_{xx} = f(u)$$

where $D$ is called the diffusivity coefficient, $f(u)$ is called the reaction term and $u(x,t)$ is the unknown function.

In physiology, an action potential is a short-lasting event in which the electrical membrane potential of a cell rapidly rises and falls, following a consistent trajectory. Action potentials occur in several types of animal cells, called excitable cells, which include neurons, muscle cells, and endocrine cells, as well as in some plant cells.

Consider the following system of differential equaitons

$$\begin{aligned}
C_m \frac{\partial V}{\partial t} &= \frac{a}{2R_i}\frac{\partial^2 V}{\partial x^2} + \bar{g}_K n^4 \left(V_K - V\right) + \bar{g}_{Na} m^3 h \left(V_{Na} - V\right) + g_l \left(V_l - V\right) + I(x,t), \\
\frac{\partial m}{\partial t} &= \alpha_m(V)(1-m) - \beta_m(V)m, \\
\frac{\partial n}{\partial t} &= \alpha_n(V)(1-n) - \beta_n(V)n, \\
\frac{\partial h}{\partial t} &= \alpha_h(V)(1-h) - \beta_h(V)h.
\end{aligned}$$

This system was proposed to describe the evolution in time $t > 0$ and space $0 < x < L$ of the depolarization $V(x,t) = V_m(x,t) - V_R$, where $V_m(x,t)$ is the actual membrane potential and $V_R$ is (assumed to be constant) the resting potantial. The quantities $C_m$, $\bar{g}_K$, $\bar{g}_{Na}$, $g_l$, and $I(x,t)$ are respectively the membrane capacitance, maximal potassium conductance, maximal sodium conductance and applied current density for unit area. $R_i$ is the intracellular resistivity and $\alpha$ is the fiber radius, $n(x,t)$, $m(x,t)$, and $h(x,t)$ are the dimensionless potasium activation, sodium activation and sodium inactivation variables.

This model is called Hodgkin-Huxley model and it describes how action potentials in neurons are initiated and propagated. Although it is difficult to analyze the Hodgkin-Huxley model, it provides a clear, biological and mechanistic model for cardiac action potentials. This model was investigated by Alan Lloyd Hodgkin and Andrew Fielding Huxley. They carried out an elegant series of electrophysiological experiments on the squid giant axon in the late 1940s and early 1950s. The squid giant axon is notable for its extraordinarily large diameter which allowed Hodgkin and Huxley to insert the electrodes of the voltage clamp apparatus into the lumen of the axon. This ability combined with the system's simplicity was crucial for the success of their study of action potentials.

In a series of five articles published in 1952, these investigators (together with Bernard Katz) unveiled the key properties of the ionic conductances underlying the nerve action potential. For this achievement, Hodgkin and Huxley were awarded the 1963 Nobel Prize in Physiology and Medicine (shared with John Eccles, for his work on potentials and conductances at motoneuron synapses).

The following system of partial differential equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_u^2 \Delta u + f(u) - \sigma u, \\ \tau \frac{\partial v}{\partial t} &= d_v^2 \Delta v + u - v, \end{aligned}$$

where $f(u) = \lambda u - u^3 - \kappa$, describes how an action potential travels through a nerve. Here $d_u$, $d_v$, $\tau$, $\sigma$ and $\lambda$ are positive constants.

This model is called FitzHugh-Nagumo model and was obtained by reduction of the Hodgkin-Huxley model. This reduction is from four variables to two variables. Basically, the FitzHugh-Nagumo model extracts the essential behaviour of the

Hodgkin-Huxley fast-slow phase plane and presents it in a simplified form.

The motivation for the FitzHugh-Nagumo model was to isolate the mathematical properties of excitation and propagation from the electrochemical properties of sodium and potasium ion flow. There are two variables in the model. One of them is a voltage-like variable having cubic nonlinearity that allows regenerative self-excitation via positive feedback. The other variable is a recovery variable having a linear dynamics that provides a slower negative feedback.

FitzHugh modified van der Pol model to explain the basic properties of excitability as exhibited by the more complex Hodgkin-Huxley equations. In the original papers of FitzHugh, the FitzHugh-Nagumo model was called Bonhoeffer-van der Pol oscillator (named after Karl Friedrich Bonhoeffer and Balthasar van der Pol), since it contains the van der Pol oscillator as a special case.

The FitzHugh-Nagumo model can be derived from a simplified model of the cell membrane. Here the cell (or membrane patch) consists of three components, a capacitor representing the membrane capacitance, a nonlinear current-voltage device for the fast current. In 1962, an equivalent circuit model suggested by Jin-Ichi Nagumo, Suguru Arimoto and Shuji Yoshizawa.

The FitzHugh-Nagumo equations are used to model electrical waves of the heart or cortisol secretion, which is controlled by the hypothalamic pituitary adrenal axis.

In this thesis, our aim is to study the problem of stabilization of solutions and continuous dependence of solutions of FitzHugh-Nagumo equations on the diffusivity coefficient. The thesis consists of four chapters. First chapter is the preliminaries. In the second chapter, following [8], we study local and global existence and uniqueness of the solution to the reaction-diffusion equation. Then in the third chapter, following

[19], we study the problem of stabilization of solutions to FitzHugh-Nagumo system on a bounded domain. Finally, in the last chapter, following [17], we study the problem of structural stability of solutions to initial boundary value problem for the FitzHugh-Nagumo system.

# Chapter 1

# Preliminaries

In this chapter, we will give some definitions, inequalities, theorems and concepts that we will use in the following chapters.

**Definition 1.1.** *A metric space is an ordered pair* $(M, d)$ *where* $M$ *is a set and* $d$ *is a metric on* $M$*. That is, a function* $d : M \times M \to \mathbb{R}$ *such that for any* $x, y, z \in M$*, the following holds:*

*(a)* $d(x,y) \geq 0$,

*(b)* $d(x,y)=0$ *if and only if* $x=y$,

*(c)* $d(x,y)=d(y,x)$,

*(d)* $d(x,z) \leq d(x, y) + d(y, z)$.

**Definition 1.2.** *A metric space* $M$ *is called complete if every Cauchy sequence in* $M$ *has a limit that is also in* $M$*.*

**Definition 1.3.** *A vector space over* $\mathbb{F}$ *is a non-empty set* $\boldsymbol{V}$ *together with two functions, one from* $\boldsymbol{V} \times \boldsymbol{V}$ *to* $\boldsymbol{V}$ *and the other from* $\mathbb{F} \times \boldsymbol{V}$ *to* $\boldsymbol{V}$*, denoted by* $x+y$ *and* $\alpha x$ *respectively, for all* $x, y \in \boldsymbol{V}$ *and* $\alpha \in \mathbb{F}$*, such that, for any* $\alpha, \beta \in \mathbb{F}$ *and any* $x, y, z \in \boldsymbol{V}$*,* *(a)* $x+y=y+x, \qquad x+(y+z)=(x+y)+z$*;*

*(b) there exists a unique* $0 \in \boldsymbol{V}$ *(independent of x) such that* $x+0=x$*;*

*(c) there exists a unique* $-x \in \boldsymbol{V}$ *such that* $x+(-x)=0$*;*

*(d)* $1x=x, \qquad \alpha(\beta x)=(\alpha\beta)x$*;*

*(e)* $\alpha(x+y)=\alpha x+\alpha y, \qquad (\alpha+\beta)x=\alpha x+\beta x$*.*

*If* $\mathbb{F}=\mathbb{R}$ *(respectively,* $F=\mathbb{C}$*) then* V *is a real (respectively, complex) vector space. Elements of* $\mathbb{F}$ *are called scalars, while elements of* $\boldsymbol{V}$ *are called vectors. The operation* $x+y$ *is called vector addition, while the operation* $\alpha x$ *is called scalar multiplication.*

**Definition 1.4.** *Let* $\boldsymbol{V}$ *be a vector space over* $\mathbb{F}$*. A norm on* $\boldsymbol{V}$ *is a function* $\|\cdot\| : \boldsymbol{V} \to \mathbb{R}$ *such that for all* $x, y \in \boldsymbol{V}$ *and* $\alpha \in F$*,*

*(a)* $\|x\| \geq 0$*;*

*(b)* $\|x\| = 0$ *if and only if* $x = 0$*;*

*(c)* $\|\alpha x\| = |\alpha|\,\|x\|$*;*

*(d)* $\|x+y\| \leq \|x\| + \|y\|$*.*

**Definition 1.5.** *A vector space* $\boldsymbol{V}$ *on which there is a norm is called a normed vector space or just a normed space.*

**Remark 1.1.** *Every normed space is a metric space.*

**Definition 1.6.** *A complete normed vector space is called a Banach space.*

**Definition 1.7.** *Let* $(M, d)$ *be a metric space. Then the map* $\phi : M \to M$ *is a contraction mapping of* $(M, d)$ *if for some real number* $0 \leq k < 1$*, called the constant of contraction, we have*

$$d(\phi(x), \phi(y)) \leq kd(x, y), \qquad \forall x, y \in M.$$

**Theorem 1.1.** *(Banach Fixed-Point Theorem) If $\phi : X \to X$ is a contraction mapping on a Banach space $X$, then $\phi$ has precisely one fixed point. That is, there exists a unique $u \in X$ such that*

$$\phi(u) = u.$$

**Remark 1.2.** *In some cases, the map $\phi$ may not be a contraction mapping on the entire Banach space, but rather only on a closed ball in the space, and the Fixed-Point Theorem remains valid on closed balls of a Banach space.*

**Definition 1.8.** *Let $H_1$ and $H_2$ be two normed vector spaces, with norms $\|.\|_1$ and $\|.\|_2$, respectively. If*

*(a) $H_1 \subset H_2$,*

*(b) there exists $C \in \mathbb{R}^+$ such that*

$$\|u\|_{H_2} \leq C \|u\|_{H_1} \qquad \forall u \in H_1,$$

*then $H_1$ is said to be continuously embedded in $H_2$ and denoted by $H_1 \hookrightarrow H_2$.*

**Definition 1.9.** *If*

*(a) $H_1 \hookrightarrow H_2$,*

*(b) for any sequence $\{u_n\}_{n\in\mathbb{N}}$ bounded in $H_1$, there is a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that $u_{n_k} \to u$ in $H_2$,*

*then $H_1$ is said to be compactly embedded in $H_2$ and denoted by $H_1 \subset\subset H_2$.*

**Definition 1.10.** *A linear operator $T$ is an operator such that*

*(i) the domain $D(T)$ of $T$ is a vector space and the range $R(T)$ lies in a vector space over the same field,*

*(ii) for all* $x, y \in D(T)$*, and each scalar* $\alpha$*,*

$$\begin{aligned} T(x+y) &= T(x)+T(y), \\ T(\alpha x) &= \alpha T(x). \end{aligned}$$

**Definition 1.11.** *Let* $X$ *and* $Y$ *be normed spaces and* $T : D(T) \to Y$ *a linear operator, where* $D(T) \subset X$*. The operator* $T$ *is said to be bounded if there is a positive number* $c$ *such that for all* $x \in D(T)$*,*

$$\|Tx\| \leq c\,\|x\|\,.$$

**Definition 1.12.** *A bounded linear functional* $f$ *is a bounded linear operator with range in the scalar field of the normed space* $X$ *in which the domain* $D(f)$ *lies. Thus there exists a positive number* $c$ *such that for all* $x \in D(f)$*,*

$$|f(x)| \leq c\,\|x\|\,.$$

*Furthermore, the norm of* $f$ *is*

$$\|f\| = \sup_{\substack{x \in D(f) \\ x \neq 0}} \frac{|f(x)|}{\|x\|}$$

*or*

$$\|f\| = \sup_{\substack{x \in D(f) \\ \|x\|=1}} |f(x)|.$$

**Definition 1.13.** *Let* $X$ *be a normed space. Then the set of all bounded linear functionals on* $X$ *constitutes a normed space with the norm defined by*

$$\|f\| = \sup_{\substack{x\in X\\ x\neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x\in X\\ \|x\|=1}} |f(x)|$$

*which is called the dual space of $X$ and is denoted by $X'$.*

**Definition 1.14.** *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a normed space $X$ is said to be weakly convergent if there is an element $x \in X$ such that for every $f \in X'$,*

$$\lim_{n\to\infty} f(x_n) = f(x).$$

*This is written as $x_n \rightharpoonup x$. The element $x$ is called the weak limit of $\{x_n\}_{n\in\mathbb{N}}$, and we say that $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to $x$.*

**Lemma 1.2.** *Let $\{x_n\}_{n\in\mathbb{N}^+}$ be a weakly convergent sequence in a normed space $X$ with weak limit $x$. Then we have*

$$\liminf_{n\to\infty} \|x_n\| \geq \|x\|.$$

*Proof.* Let $f \in X'$ be an arbitrary but fixed. We know that $|f(x)| \leq \|f\|\,\|x\|$ for any $f \in X'$ and $x \in X$. We also have by assumption that $\{f(x_n)\}_{n\in\mathbb{N}^+}$ is a convergent sequence. So, consider that

$$|f(x)| = \lim_{n\to\infty} |f(x_n)| \leq \liminf_{n\to\infty} \|f\|\,\|x_n\| = \|f\| \liminf_{n\to\infty} \|x_n\|.$$

Take the supremum over all $f \in X'$ with $\|f\| \leq 1$ to get

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

**Proposition 1.3.** *If $\{x_n\}_{n\in\mathbb{N}^+}$ is a bounded sequence in a Hilbert space, then it has a weakly convergent subsequence (see [14], p.155).*

**Definition 1.15.** *The linear space of all functions $f$ integrable over the domain $G$ is denoted by $L^1(G)$. That is,*

$$L^1(G) = \left\{ f : \int_G |f(x)|\, dx < \infty \right\}.$$

This space equipped with the norm

$$\|f\|_{L^1(G)} := \int_G |f(x)|\, dx$$

is a Banach space.

**Definition 1.16.** *The linear space of functions $f$ such that $|f|^p \in L^1(G)$ is denoted by $L^p(G)$. That is,*

$$L^p(G) = \left\{ f : \int_G |f(x)|^p\, dx < \infty \right\}.$$

This space equipped with the norm

$$\|f\|_{L^p(G)} := \left( \int_G |f(x)|^p\, dx \right)^{1/p}$$

is a Banach space.

**Theorem 1.4.** *(Weierstrass M-Test) Let $f_n$ be defined on a set $S$ and let $M_n \geq 0$ such that $\sum_{n=1}^{\infty} M_n < \infty$. If $|f_n| \leq M_n$ for all $n \in \mathbb{N}^+$ and $x \in S$, then $\sum_{n=1}^{\infty} f_n$ converges absolutely and uniformly on $S$.*

**Definition 1.17.** *Let $f$ be a function. The support of $f : \mathbb{R} \to \mathbb{R}$ is defined as follows*

$$supp(f) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}.$$

**Definition 1.18.** *Let $G$ be an open set in $\mathbb{R}^n$ and $f : G \to \mathbb{C}$ be a Lebesgue measurable function. If for all compact domains $K$ of $G$,*

$$\int_K |f(x)|\, dx < \infty$$

*then $f$ is called locally integrable.*

**Definition 1.19.** *The space $C_0^\infty(G)$ consists of functions $f$ which are defined on $G$, infinitely differentiable and have compact support within $G$.*

**Definition 1.20.** *Given $\mathbb{R}^n$, define a multi-index $\alpha$ as an ordered collection of non-negative integers $\alpha = (\alpha_1, ..., \alpha_n)$, such that its length is given by $|\alpha| = \sum_{i=1}^n a_i$.*

**Remark 1.3.** *If $f$ is an $m$-times differentiable function, then for any $\alpha$ with $|\alpha| \leq m$, the derivative can be expressed as*

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$$

**Definition 1.21.** *Let $\alpha = (\alpha_1, ..., \alpha_n)$ be a given multi-index, and $f$ and $g$ be locally integrable functions over $G$. Assume that for all $\beta \in C_0^\infty(G)$ the following integral identity is satisfied*

$$\int_G f(x) D^\alpha \beta(x) dx = (-1)^{|\alpha|} \int_G g(x)\beta(x) dx.$$

*Then the function $g$ is called the weak $\alpha$-th derivative of $f$ on the region $G$.*

**Definition 1.22.** *The Sobolev space $H^1(G)$ is an inner product space of all functions $f \in L^2(G)$ that have all first order weak derivatives $f_{x_i}$, $i = 1, ..., n$ belonging to $L^2(G)$. The inner product in $H^1(G)$ is defined by*

$$(f, g)_{H^1(G)} = \int_G \left[ f(x)g(x) + \sum_{j=1}^n f_{x_j}(x) g_{x_j}(x) \right] dx.$$

Thus, the norm on $H^1(G)$ is given by

$$\|f\|_{H^1(G)} = \left( \int_G \left[ |f(x)|^2 + |\nabla f(x)|^2 \right] dx \right)^{1/2}.$$

**Remark 1.4.** *The space $H^1(G)$ is complete. That is, this space is a Hilbert space.*

**Remark 1.5.** *The space $H_0^1(G)$ is defined as the completion of $C_0^\infty(G)$ in the sense of the norm $H^1(G)$. The norm on this space is defined as follows*

$$\|u\|_{H_0^1(G)} = \|\nabla u\|_{L^2(G)}.$$

**Definition 1.23.** *The Sobolev space $H^m(G)$, $m = 1, 2, 3, ...$ is a separable Hilbert space of all functions $f \in L^2(G)$ that have all $m^{th}$-order weak derivatives belonging to $L^2(G)$ with the inner product*

$$(f, g)_{H^m(G)} = \int_G \sum_{|\alpha| \le m} D^\alpha f(x) D^\alpha g(x) dx.$$

**Inequality 1.5.** *(Cauchy-Schwartz Inequality) Let $f$ and $g$ be square-integrable functions defined on a domain $\Omega$. Then we have*

$$\left| \int_\Omega f(x) g(x) dx \right|^2 \le \int_\Omega |f(x)|^2 dx \cdot \int_\Omega |g(x)|^2 dx$$

**Inequality 1.6.** *(Young's Inequality) Let $1 < p, q < \infty$ with $1/p + 1/q = 1$ and $a$, $b$ be nonnegative real numbers and $\epsilon > 0$, then we have*

$$ab \le \frac{\epsilon}{p} a^p + \frac{1}{q \epsilon^{1/(p-1)}} b^q.$$

**Inequality 1.7.** *(Jensen's Inequality) Let $\varphi$ be a convex function on $\mathbb{R}$ and $f$ be an integrable function on [0,1]. Then*

$$\varphi \left[ \int_0^1 f(t) dt \right] \le \int_0^1 \varphi(f(t)) dt.$$

**Inequality 1.8.** *(Interpolation Inequality) For all $u \in H^2(\Omega) \cap H_0^1(\Omega)$, we have*

$$\|\nabla u\|^2 \le \|u\| \, \|\Delta u\|.$$

*Proof.* Since $C_0^\infty(\Omega)$ is dense in $H^2(\Omega) \cap H_0^1(\Omega)$, it is enough to prove the inequality for functions in $C_0^\infty(\Omega)$.

Firstly, note that

$$\nabla(u(x)\nabla u(x)) = |\nabla u(x)|^2 + u(x)\Delta u(x). \tag{1.1}$$

Integrating both sides of (1.1), we write

$$\int_G \nabla(u(x)\nabla u(x))dx = \int_G |\nabla u(x)|^2 dx + \int_G u(x)\Delta u(x)dx. \tag{1.2}$$

By Divergence Theorem applied to the left-hand side of (1.2), we get that

$$\int_G \nabla(u(x)\nabla u(x))dx = \int_{\partial G} (u(x)\nabla u(x)) \cdot \overrightarrow{n}\, dx,$$

where $\overrightarrow{n}$ is the outward unit normal field of $\partial G$. Since $u \in H_0^1(G)$, we see that $\int_{\partial G}(u(x)\nabla u(x)) \cdot \overrightarrow{n}\, dx = 0$. So, we obtain

$$\int_G |\nabla u(x)|^2 dx = -\int_G u(x)\Delta u(x)dx \tag{1.3}$$

By Cauchy-Schwartz Inequality, the right-hand side of (1.3) can be estimated as follows

$$\begin{aligned} -\int_G u(x)\Delta u(x)dx &\leq \int_G |u(x)\Delta u(x)|dx \\ &\leq \left(\int_G |u(x)|^2 dx\right)^{1/2} \left(\int_G |\Delta u(x)|^2 dx\right)^{1/2} \\ &= \|u\| \, \|\Delta u\| \, . \end{aligned}$$

Then combining the last inequality with (1.3), we get $\|\nabla u\|^2 \leq \|u\| \, \|\Delta u\|$ which is the desired result. □

**Inequality 1.9.** *(Sobolev Inequality) Let $G \subset \mathbb{R}^n$ with $n \leq 3$ and $u \in H^2(G) \cap H_0^1(G)$ be a function defined on $G$. Then we have the following inequality*

$$\max_{x \in G} |u(x)| \leq C \|\Delta u\|$$

*for some positive constant $C$ depending on $G$(see [13], p.423,470).*

**Inequality 1.10.** *Let $u_1, v_1 \in \mathbb{R}^n$. Then for any $q \geq 2$, there exists a positive constant $d_1$ such that*

$$d_1 |u_1 - u_2|^q \leq \left\langle |u_1|^{q-2} u_1 - |u_2|^{q-2} u_2, u_1 - u_2 \right\rangle$$

*Proof.* Define $J(q) := \langle |u_1|^{q-2} u_1 - |u_2|^{q-2} u_2, u_1 - u_2 \rangle$.

Then, by Fundamental Theorem of Calculus, we can write

$$J(q) = \left\langle \int_0^1 \left\{ \frac{d}{ds} \left[ |su_1 + (1-s)u_2|^{q-2} \left(su_1 + (1-s)u_2\right) \right] \right\} ds, u_1 - u_2 \right\rangle$$

Since, for any differentiable function $f$, $\frac{d}{ds}|f(s)| = \frac{\langle f(s), f'(s) \rangle}{|f(s)|}$, we get

$$\begin{aligned} J(q) &= \left\langle \int_0^1 |su_1 + (1-s)u_2|^{q-2} (u_1 - u_2) ds, u_1 - u_2 \right\rangle \\ &+ (q-2) \int_0^1 \left[ |su_1 + (1-s)u_2|^{q-4} \langle su_1 + (1-s)u_2, u_1 - u_2 \rangle^2 ds \right. \end{aligned}$$

Since $(q-2) \int_0^1 \left[ |su_1 + (1-s)u_2|^{q-4} \langle su_1 + (1-s)u_2, u_1 - u_2 \rangle^2 ds \geq 0 \right.$, we have $J(q) \geq |u_1 - u_2|^2 \int_0^1 |su_1 + (1-s)u_2|^{q-2} ds$.

Now, consider the following cases:

If $|u_1| \geq |u_1 - u_2|$, then

$$\begin{aligned} |su_1 + (1-s)u_2| &= |u_1 - (1-s)(u_1 - u_2)| \\ &\geq ||u_1| - (1-s)|u_1 - u_2|| \\ &= ||u_1| - |u_1 - u_2| + s|u_1 - u_2|| \\ &\geq s|u_1 - u_2| \end{aligned}$$

and by definition of the inner product, we obtain

$$\begin{aligned} J(q) &\geq |u_1 - u_2|^2 \int_0^1 s^{q-2} |u_1 - u_2|^{q-2} ds \\ &= |u_1 - u_2|^q \int_0^1 s^{q-2} ds \\ &= \frac{1}{q-1} |u_1 - u_2|^q. \end{aligned} \tag{1.4}$$

If $|u_1| < |u_1 - u_2|$, then

$$\begin{aligned} |su_1 + (1-s)u_2| &= |u_1 - (1-s)(u_1 - u_2)| \\ &\leq |u_1| + (1-s)|u_1 - u_2| \\ &\leq |u_1 - u_2| + (1-s)|u_1 - u_2| \\ &= (2-s)|u_1 - u_2|. \end{aligned}$$

Since $0 \leq s \leq 1$, we get

$$\frac{|su_1 + (1-s)u_2|^2}{4} \leq \frac{|su_1 + (1-s)u_2|^2}{(2-s)^2} \leq |u_1 - u_2|^2.$$

So,

$$J(q) \geq \frac{1}{4} \int_0^1 (|su_1 + (1-s)u_2|^q ds. \tag{1.5}$$

Since $q \geq 2$, by Jensen's Inequality, from (1.5) we obtain

$$\begin{aligned} J(q) &\geq \frac{1}{4}\left(\int_0^1 |su_1+(1-s)u_2|^2 ds\right)^{q/2} \\ &= \frac{1}{4.3^{q/2}}\left(|u_1|^2+\langle u_1,u_2\rangle+|u_2|^2\right)^{q/2} \\ &\geq \frac{1}{4.12^{q/2}}|u_1-u_2|^q. \end{aligned} \tag{1.6}$$

From (1.4) and (1.6) with $d_1 := \min\left\{\frac{1}{q-1}, \frac{1}{4.12^{q/2}}\right\}$, we deduce

$$d_1|u_1-u_2|^q \leq \left\langle |u_1|^{q-2}u_1 - |u_2|^{q-2}u_2, u_1-u_2\right\rangle$$

□

**Definition 1.24.** *Let $X$ and $Y$ be two non-empty subsets of a metric space $(M,d)$. We define their Hausdorff distance $d_H(X,Y)$ by*

$$d_H(X,Y) = \max\left\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)\right\}.$$

**Theorem 1.11.** *(Divergence Theorem) Let $R\subset\mathbb{R}^n$ be a region in space with smooth boundary $\partial R$. Let $F$ be a vector field whose components have first order continouus partial derivatives and $\overrightarrow{n}$ be the outward unit normal field of the boundary $\partial R$. Then,*

$$\int_R \left(\nabla\cdot\overrightarrow{F}\right)dV = \int_{\partial R}(\overrightarrow{F}\cdot\overrightarrow{n})dS.$$

**Lemma 1.12.** *(Gronwall's Lemma) Let $I$ be the interval $[a,\infty)$, $[a,b]$ or $[a,b)$ in $\mathbb{R}$. Let $\alpha$, $\beta$ and $z$ be real valued functions defined on $I$. Assume that $\beta$ and $z$ are continuous and the negative part of $\alpha$ is integrable on every closed and bounded subinterval of $I$.*

*(a) If $\beta$ is nonnegative and $z$ satisfies the integral inequality*

$$z(t) \leq \alpha(t) + \int_a^t \beta(s)z(s)ds, \qquad \forall t\in I, \tag{1.7}$$

*then*

$$z(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s)e^{\int_s^t \beta(r)dr}ds, \qquad t \in I \tag{1.8}$$

*(b) If, in addition the function $\alpha$ is nondecreasing, then*

$$z(t) \leq \alpha(t)e^{\int_a^t \beta(s)ds}, \qquad t \in I.$$

*Proof.* (a)Let us define the following auxiliary function

$$v(s) = exp\left(-\int_a^s \beta(r)dr\right)\int_a^s \beta(r)(r)dr, \qquad s \in I. \tag{1.9}$$

By using the condition (1.7), we obtain

$$\begin{aligned} v'(s) &= \left(z(s) - \int_a^s \beta(r)z(r\beta)dr\right)\beta(s)exp\left(-\int_a^s \beta(r)dr\right) && (1.10)\\ &\leq \alpha(s)\beta(s)exp\left(-\int_a^s \beta(r)dr\right), \qquad s \in I. && (1.11)\end{aligned}$$

Since $v(a) = 0$, integrating (1.10) from $a$ to $t$, we obtain

$$v(t) \leq \int_a^t \alpha(s)\beta(s)exp\left(-\int_a^s \beta(r)dr\right). \tag{1.12}$$

It follows from (1.9) that

$$\int_a^t \beta(s)z(s)ds = exp\left(\int_a^t \beta(r)dr\right)v(t). \tag{1.13}$$

From (1.12) and (1.13), we obtain

$$\begin{aligned} \int_a^t \beta(s)z(s)ds &\leq \int_a^t \alpha(s)\beta(s)exp\left(\int_a^t \beta(r)dr - \int_a^s \beta(r)dr\right)ds && (1.14)\\ &= \int_a^t \alpha(s)\beta(s)exp\left(\int_s^t \beta(r)dr\right). && (1.15)\end{aligned}$$

Finally, by using (1.14) and the inequality (1.7), we get the desired inequality.

(b) If the function $\alpha$ is nondecreasing, we have $\alpha(s) \leq \alpha(t)$, and then using the Fundamental Theorem of Calculus, we obtain

$$\begin{aligned} z(t) &\leq \alpha(t) + \left(-\alpha(t)exp\left(\int_s^t \beta(r)dr\right)\right)\Big|_{s=a}^{s=t} \\ &= \alpha(t)exp\left(\int_a^t \beta(r)dr\right), \qquad t \in I. \end{aligned}$$

□

**Theorem 1.13.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. All of the eigenvalues of the following problem are positive.*

$$-\Delta v = \lambda v, \qquad x \in \Omega,$$
$$v = 0, \qquad x \in \partial\Omega.$$

*Proof.* Assume that $v$ be an eigenfunction corresponding to the eigenvalue $\lambda$. Then by Divergence Theorem, we have

$$\begin{aligned} \lambda \int_\Omega v^2(x)dx &= -\int_\Omega (\Delta v(x))v(x)dx \\ &= \int_\Omega |\nabla v(x)|^2 dx - \int_{\partial\Omega} v\frac{\partial v}{\partial \nu} dS(x) \\ &= \int_\Omega |\nabla v(x)|^2 dx \\ &\geq 0, \end{aligned}$$

where $\nu$ is the outward unit normal vector.

We claim that $\|\nabla v\|_{L^2(\Omega)} > 0$. If $\|\nabla v\|_{L^2(\Omega)} = 0$, then we get $\nabla v = (0, 0, \cdots, 0)$ which means that $v$ is constant on $\Omega$. But, by assumption, $v = 0$ on $\partial\Omega$. Therefore, if $v$ is constant on $\Omega$ and $v = 0$ on $\partial\Omega$, then $v \equiv 0$. However 0 cannot be an eigenfunction. So, we obtain

$$\lambda \int_\Omega v^2(x)dx > 0.$$

This shows $\lambda > 0$. □

Note that we can normalize eigenfunctions by the condition

$$\int_G u^2(x)dx = 1.$$

**Remark 1.6.** *The smallest eigenvalue of the operator $\Delta$ under the homogeneous Dirichlet boundary condition is given as*

$$\lambda_1(\Omega) = \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u(x)|^2 dx}{\int_\Omega u^2(x)dx},$$

*and this infimum is achieved by the corresponding eigenfunction $u_1$ (see [15]).*

# Chapter 2

# Existence of Solutions to Reaction-Diffusion Equation

In this chapter, we consider the problem of existence and uniqueness of a solution to the nonlinear initial value problem

$$\begin{aligned} &u_t - Du_{xx} = f(u), \qquad x \in \mathbb{R}, \quad t > 0 \\ &u(x,0) = u_0(x), \qquad x \in \mathbb{R}. \end{aligned} \tag{2.1}$$

Some problems in the form (2.1) with different $f(u)$ and $u_0(x)$, may have a solution blowing up in a finite time or may not have a unique solution. So, we need to impose some conditions on $f(u)$.

The best way to exhibit the existence of a solution to a problem is of course writing down a formula for the solution. This is possible for some certain linear problems. Consider the following linear, nonhomogeneous diffusion problem

$$\begin{aligned} &u_t - Du_{xx} = g(x,t), \qquad x \in \mathbb{R}, \quad t > 0, \\ &u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \end{aligned} \tag{2.2}$$

where $g$ and $u_0$ are continuous bounded functions. By using Fourier transform, we can derive the solution to this problem by

$$u(x,t) = \int_{\mathbb{R}} K(x-y,t)u_0(y)\ dy\ +\ \int_0^t \int_{\mathbb{R}} K(x-y,t-s)g(y,s)\ dyds, \qquad (2.3)$$

where $K(x,t)$ is the diffusion kernel given by

$$K(x,t) = \left(\frac{1}{4\pi Dt}\right)^{1/2} \exp\ \left(\frac{-x^2}{4Dt}\right). \qquad (2.4)$$

But, for nonlinear problems, it is usually impossible to proceed in this manner, and in such kind of situations, alternative methods must be found in order to prove the existence of solution to a problem rather than describing it by a formula. Fixed point iteration is one of these kind of methods. The basic idea is to produce a sequence, through iteration of a certain map, that converges to the solution of the problem, thus showing the existence.

We now consider the question of existence of a solution to the nonlinear initial value problem (2.1). We can do this by using the solution (2.3) to the linear, non-homogeneous problem (2.2). Suppose that $f$ and $u_0$ are continuous and bounded functions on $\mathbb{R}$. In this case, instead of explicit solution (2.3) for the linear equation, we reduce the problem to the following nonlinear integral equation for $u(x,t)$.

$$u(x,t) = \int_{\mathbb{R}} K(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}} K(x-y,t-s)f(u(y,s))dyds \qquad (2.5)$$

It is easy to see that $u = u(x,t)$ is a solution of (2.1) if and only if $u = u(x,t)$ is a solution of (2.5). The equation (2.5) can be written in the form

$$u = \Phi(u),$$

where $\Phi$ is a nonlinear integral operator defined on the set of bounded continuous functions by

$$\Phi(u)(x,t) \equiv \int_{\mathbb{R}} K(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}} K(x-y,t-s)f(u(y,s))dyds.$$

So, we can define the fixed point iteration by

$$u_{n+1} = \Phi(u_n),$$

or

$$\begin{aligned} u_{n+1}(x,t) = & \int_{\mathbb{R}} K(x-y,t)u_0(y)dy \\ & + \int_0^t \int_{\mathbb{R}} K(x-y,t-s)f(u_n(y,s))dyds, \qquad n = 0,1,2,... \end{aligned} \tag{2.6}$$

with

$$u_0(x,t) = \int_{\mathbb{R}} K(x-y,t)u_0(y)dy \tag{2.7}$$

Now, we will prove the existence theorem for the initial value problem (2.1) under suitable assumptions on the nonlinear term $f(\cdot)$. In the proof of the following theorem, we use the fact that the diffusion kernel $K$ is strictly positive and that

$$\int_{\mathbb{R}} K(x-y,t-s)\ dx = 1, \qquad for\ all\ y\ and\ all\ s < t. \tag{2.8}$$

**Theorem 2.1.** *Consider the initial value problem (2.1) where $u_0(x)$ is a bounded continuous function on $\mathbb{R}$ and where $f$ is a bounded continuous function on $\mathbb{R}$ that satisfies the global Lipschitz condition*

$$|f(u) - f(v)| \leq k|u - v|, \qquad \textit{for all } u, v \in \mathbb{R} \tag{2.9}$$

*where $k$ is a positive constant independent of $u$ and $v$. Then for any $T > 0$, there exists a unique, bounded solution $u(x,t)$ of (2.1) for $x \in \mathbb{R}$ and $0 \leq t \leq T$.*

*Proof.* We will show that the sequence defined by (2.6) and (2.7) converges uniformly on $\mathbb{R} \times [0,T]$ to a function that is a solution of (2.1).

Firstly, note that by Lipschitz condition (2.9) we have

$$|f(u_0(x,t)) - f(0)| \leq k|u_0(x,t)|. \tag{2.10}$$

From (2.10), we deduce

$$|f(u_0(x,t))| \leq |f(0)| + k|u_0(x,t)| \leq (1+k)m \tag{2.11}$$

$$m = \max\left\{ f(0), \sup_{0 \leq t \leq T} |u_0(x,t)| \right\}. \tag{2.12}$$

Here, let us use the notation

$$M_n(t) = sup \left\{ |u_n(x,s) - u_{n-1}(x,s)| \ : \ x \in \mathbb{R}, \ s \leq t \right\}, \qquad t \leq T.$$

Now, by (2.6), (2.11) and (2.12) we get

$$
\begin{aligned}
|u_1(x,t)-u_0(x,t)| &\leq \int_0^t \int_R K(x-y,t-s)|f(u_0(y,s))|\ dyds \\
&\leq \int_0^t (1+k)m \int_R K(x-y,t-s)\ dyds \\
&= \int_0^t (1+k)m\ ds \\
&= (1+k)mt.
\end{aligned}
$$

Denote $M := (1+k)m$ and take the supremum over $\mathbb{R}\times[0,t]$ to get

$$
M_1(t) \leq Mt, \qquad 0 \leq t \leq T. \tag{2.13}
$$

Now, let us obtain a bound for $|u_{n+1}-u_n|$. Again by using (2.6) and (2.8), we obtain

$$
\begin{aligned}
|u_{n+1}(x,t)-u_n(x,t)| &\leq \int_0^t \int_R K(x-y,t-s)|f(u_n(y,s))-f(u_{n-1}(y,s))|\ dyds \\
&\leq \int_0^t \int_R K(x-y,t-s)k|u_n(y,s)-u_{n-1}(y,s)|\ dyds \\
&\leq \int_0^t kM_n(s) \int_R K(x-y,t-s)\ dyds \\
&= k\int_0^t M_n(s)\ ds.
\end{aligned}
$$

Then taking the supremum,

$$
M_{n+1}(t) \leq k\int_0^t M_n(s)\ ds, \qquad 0 \leq t \leq T, \quad n=1,2,3,... \tag{2.14}
$$

On the other hand, consider that

$$
M_2(t) \leq k\int_0^t M_1(s)\ ds \leq k\int_0^t Ms\ ds = \frac{kMt^2}{2.1} = \frac{M}{k}\frac{(kt)^2}{2!},
$$

$$M_3(t) \leq k \int_0^t M_2(s)\ ds \leq k \int_0^t \frac{kMs^2}{2}\ ds = \frac{k^2Mt^3}{3.2.1} = \frac{M}{k}\frac{(kt)^3}{3!}$$

and continuing this procedure, we get

$$M_n(t) \leq \frac{M}{k}\frac{(kt)^n}{n!}, \qquad 0 \leq t \leq T. \tag{2.15}$$

Now, let us consider the series

$$u_0(x) + \sum_{n=1}^{\infty} \left(u_n(x,t) - u_{n-1}(x,t)\right), \tag{2.16}$$

where $u_0(x,t) \equiv u_0(x)$.

Since the $m^{th}$ partial sum of the series (2.16) is

$$S_m(x,t) = u_0(x) + \sum_{n=1}^{m} \left(u_n(x,t) - u_{n-1}(x,t)\right) = u_m(x,t),$$

the series (2.16) is convergent if and only if the sequence $\{u_m(x,t)\}_{n\in\mathbb{N}^+}$ is convergent and the sum of the series is the limit of the sequence $\{u_m(x,t)\}_{m\in\mathbb{N}^+}$. Let us show that the series (2.16) is uniformly convergent. In fact, we have

$$|u_n(x,t) - u_{n-1}(x,t)| \leq M_n(t) \leq \frac{M}{k}\frac{(kT)^n}{n!}$$

and since

$$\sum_{n=1}^{\infty} \frac{(kT)^n}{n!} = e^{kT} < \infty,$$

by using Weierstrass M-Test (1.4), we deduce that $u_n(x,t)$ converges uniformly on $\mathbb{R} \times [0,T]$ to some continuous, bounded function $u(x,t)$. Now, we can take the limit of both sides of (2.6) and using the uniform convergence we can pass the limit inside of the integral. This shows that the limiting function $u(x,t)$ satisfies the integral equation which means that u(x,t) is a solution to the initial value problem (2.1).

In order to show the uniqueness of solution, suppose that the problem has two different solutions $u$ and $v$ which are bounded, continuous functions satisfying the integral equation. Then we get

$$\begin{aligned} |u(x,t)-v(x,t)| &\leq \int_0^t \int_{\mathbb{R}} K(x-y,t-s)|f(u(y,s))-f(v(y,s))|dyds \\ &\leq \int_0^t \int_{\mathbb{R}} K(x-y,t-s)k|u(y,s)-v(y,s)|dyds. \end{aligned} \tag{2.17}$$

Let $M(t) = sup\{|u(x,s)-v(x,s)|, x \in \mathbb{R}, s \leq t\}$. Then from the inequality (2.17) we obtain

$$M(t) \leq k\int_0^t \int_{\mathbb{R}} K(x-y,t-s)M(s)dyds = k\int_0^t M(s)ds.$$

By Gronwall's Lemma, $M(t) \equiv 0$. Therefore, $u = v$ and the solution to the problem is unique. □

In the preceding theorem we assumed that the nonlinear term $f(u)$ satisfies the uniform Lipschitz condition (2.9). But in many models described by reaction-diffusion equations the nonlinear reaction terms are not satisfying the uniform Lipschitz condition. For example, reaction term for the Fisher equation has the form $f(u) = u(1-u)$. Since

$$|f(u)-f(v)| = |u(1-u)-v(1-v)| = |u-u^2-v+v^2| = |1-u-v||u-v|$$

and the right hand side cannot be bounded by $k|u-v|$ with some constant $k > 0$ for all $u$ and $v$. However, this nonlinearity satisfies the local Lipschitz condition. The nonlinear terms of the well-known Kolmogorov-Petrovsky-Piskunov equation and the FitzHugh-Nagumo equations are cubic polynomials, and they also satisfy just local Lipschitz condition. Thus, we seek to formulate an existence-uniqueness theorem by weakening the hypothesis in (2.1) to a local Lipschitz condition.

Now, we will formulate and prove the existence theorem for the initial value problem (2.1), where only a local Lipschitz condition is required. But, firstly let us introduce some notations. Consider a function $u = u(x,t)$. For each fixed $t$, we consider $u$ as a function of $x$, defined on $\mathbb{R}$. For the following formulation, we denote by $B$ the space $C_B(\mathbb{R})$ of all bounded, continuous functions $v(x)$ on $\mathbb{R}$, and let $\|v\|_B$ denote the norm of a function $v(x)$ in $B$, i.e.,

$$\|u(t)\|_B = \sup_{x\in\mathbb{R}}|u(x,t)| \qquad for\ t\ fixed. \tag{2.18}$$

Now, let $T > 0$ and let $C([0,T];B)$ be the set of all continuous functions defined on $0 \leq t \leq T$ with values in the Banach space $B$. That is, to each $t \in [0,T]$ we associate a bounded, continuous function $u(x,t)$ of $x \in \mathbb{R}$ ($t$ fixed) which is an element of the Banach space $B$. The set $C([0,T];B)$ whose elements will be denoted by $u$, is a Banach space with the norm

$$\|u\| = \sup_{t\in[0,T]} \|u(t)\|_B \,. \tag{2.19}$$

We also introduce the convolution operation

$$(K * u)(x,t) = \int_{\mathbb{R}} K(x-y,t)u(y,t)dy,$$

where $K(x,t)$ is the diffusion kernel. We call $K * u$ the *convolution* of $K$ with $u$, and $(K * u)(x,t)$ also belongs to $B$.

**Theorem 2.2.** *(Local Existence) Consider the initial value problem (2.1) where* $u_0 \in B$ *and* $f$ *satisfies the conditions*

*(i)* $f \in C^1(\mathbb{R})$.

*(ii)* $f(0) = 0$, *and for each fixed* $t$ *in* $[0, T]$, $f(u(x,t)) \in B$, *where* $u(x,t) \in B$.

*(iii) For any* $M > 0$ *there exists a constant* $k$, *depending only on* $M$, *such that*

$$\|f(u(t)) - f(v(t))\|_B \leq k \|u(t) - v(t)\|_B$$

*for all* $t \in [0, T]$ *and all* $u(x,t)$ *and* $v(x,t)$ *in* $B$ *with* $\|u(t)\|_B \leq M$ *and* $\|v(t)\|_B \leq M$.

*Then there exists* $t_0 > 0$, *where* $t_0$ *depends only on* $f$ *and* $\|u_0\|_B$, *such that the initial value problem (2.1) has a unique solution* $u = u(x,t)$ *in* $C([0, t_0]; B)$ *and* $\|u\| \leq 2\|u_0\|_B$.

*Proof.* We will define a closed subspace $P$ of the Banach space $C([0, t_0]; B)$ and show that the mapping

$$\Phi(u)(x,t) = \int_R K(x-y, t-s) u_0(y)\, dy \; + \int_0^t \int_R K(x-y, t-s) f(u(y,s))\, dyds$$

is a contraction mapping on $P$. Then we will apply the remark (1.2) to produce a solution to $u = \Phi(u)$, which is the solution of the initial value problem. Define

$$P = \left\{ u \in C([0, t_0]; B) : \|u(t) - (K * u_0)(t)\|_B \leq \|u_0\|_B \,, \; for \; 0 \leq t \leq t_0 \right\},$$

where $t_0 = 1/2k$. The set $P$ is closed and nonempty. Also, from the defining property of $P$, by using triangle inequality we obtain

$$\|u(t)\|_B - \|(K * u_0)(t)\|_B \leq \|u(t) - (K * u_0)(t)\|_B \leq \|u_0\|_B \,. \tag{2.20}$$

Then the fact

$$\|(K * u)(t)\|_B \leq \|u(t)\|_B$$

with (2.20) imply that

$$\|u(t)\|_B \leq 2 \|u_0\|_B$$

and then taking the supremum over $[0, t_0]$, we get

$$\|u\| \leq 2 \|u_0\|_B \,. \tag{2.21}$$

This proves the last statement of the theorem. Now, we have from $(iii)$, for any $0 \leq t \leq t_0$,

$$\|f(u(t)) - f(v(t))\|_B \leq k \|u(t) - v(t)\|_B \leq k \|u - v\| \,.$$

We know that $k$ depends only on the supremum norm of $u$ and $v$, and on $f$. So, clearly we can say that $t_0$ depends only on $f$ and the supremum norm of $u_0$, by the inequality (2.21).

Now, consider that

$$\begin{aligned}
\|\Phi(u)(t)-(K*u_0)(t)\|_B &= \left\|\int_0^t\int_{\mathbb{R}} K(x-y,t-s)f(u(y,s))dyds\right\|_B \\
&= \sup_{x\in\mathbb{R}}\left|\int_0^t\int_{\mathbb{R}} K(x-y,t-s)f(u(y,s))dyds\right| \\
&\le \sup_{x\in\mathbb{R}}\int_0^t\int_{\mathbb{R}} K(x-y,t-s)|f(u(y,s))|dyds \\
&\le \int_0^t \|f(u(s))\|_B\,ds \le \int_0^t \ k\,\|u(s)\|_B\,ds \\
&\le \int_0^t \ 2k\,\|u_0\|_B\,ds = 2kt\,\|u_0\|_B\,, \quad for\ all\ t\in[0,t_0] \\
&\le 2kt_0\,\|u_0\|_B = \|u_0\|_B\,,
\end{aligned}$$

since $t_0 = 1/2k$. That is, $\Phi$ maps $P$ into $P$.

Now, we will prove that $\Phi$ is a contraction mapping.

$$\begin{aligned}
\|\Phi(u)(t)-\Phi(v)(t)\|_B &= \sup_{x\in\mathbb{R}}|\Phi(u)(x,t)-\Phi(v)(x,t)| \\
&\le \sup_{x\in\mathbb{R}}\int_0^t\int_{\mathbb{R}} K(x-y,t-s)|f(u(y,s))-f(v(y,s))|dyds \\
&= \int_0^t\int_{\mathbb{R}} K(x-y,t-s)\,\|f(u(s))-f(v(s))\|_B\,dyds \\
&= \int_0^t \|f(u(s))-f(v(s))\|_B\,ds \\
&\le \int_0^t \ k\,\|u-v\|\,ds \\
&\le kt_0\,\|u-v\| \\
&= \frac{1}{2}\,\|u-v\|\,.
\end{aligned}$$

Taking the supremum over $t\in[0,T]$, we get

$$\|\Phi(u)-\Phi(v)\| \le \frac{1}{2}\,\|u-v\|\,.$$

This shows that $\Phi$ is a contraction mapping on the closed subset $P$ of the Banach space $C([0,T];B)$. By the remark (1.2), we deduce that there is a unique fixed point of the operator $\Phi$ in the ball $P$. This proves that there is a unique solution to the initial value problem (2.1) in $P$.

Finally, it remains to show that there are no solutions outside of the set $P$. This fact results from the following argument. If $u, v \in C([0,T];B)$ are two solutions of the initial value problem (2.1), then we obtain

$$|u(x,t) - v(x,t)| \leq \int_0^t \int_{\mathbb{R}} K(x-y,t-s)|f(u(y,s)) - f(v(y,s))| \, dyds.$$

Taking supremum on $x$ gives

$$212 \, \|u(t) - v(t)\|_B \leq \int_0^t \int_{\mathbb{R}} K(x-y,t-s) \, \|f(u(y,s)) - f(v(y,s))\|_B \, dyds \quad (2.22)$$

$$\leq k \int_0^t \|u(s) - v(s)\|_B \, ds. \quad (2.23)$$

Then, multiplying (**??**) by $e^{-kt}$, we get

$$\frac{d}{dt}\left[e^{-kt} \int_0^t \|u(s) - v(s)\|_B \, ds\right] \leq 0$$

which implies by Gronwall's Lemma $\|u(t) - v(t)\|_B = 0$ and we get $u \equiv v$. □

This theorem is only a local existence result, guaranteeing a solution for $0 \leq t \leq t_0$, for some $t_0$. Under certain conditions we may extend the solution to any finite time. We have the following result.

**Theorem 2.3.** *(Global Existence) Suppose that all conditions of the theorem (2.2) are satisfied. If in addition there exists a constant $C$ depending on $\sup_{x \in \mathbb{R}} |u_0(x)|$ such* *that*

$$\sup_{x\in\mathbb{R}} |u(x,t)| \leq C, \qquad \forall t \in [0,T].$$

*Then the solution of the problem (2.1) exists on* $[0,T]$*, and* $u(x,t) \in B$*. Here* $T$ *may be infinity, giving global existence.*

*Proof.* The local theorem guarentees a solution $u$ on $[0,t_0]$. Then we can apply the local theorem again with initial condition $u(x,t_0)$ to get a solution on $[t_0, 2t_0]$. Continuing in this manner we can obtain, after a finite number of steps, a solution on $[0,T]$. □

**Remark 2.1.** *The proof of the existence and uniqueness of solution to the system of reaction diffusion equations is similar.*

# Chapter 3

# The Stabilization of FitzHugh-Nagumo System with One Feedback Controller

In this chapter, we will investigate the internal feedback stabilization of a FitzHugh-Nagumo system on a bounded domain. We will show that the system, given below, can be stabilized exponentially by one feedback controller acting on a subdomain.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, $\omega \subset \Omega$ be an open nonempty subdomain of $\Omega$ with smooth boundary $\partial\omega$ such that $\bar{\omega} \subset \Omega$. We will consider the following system,

$$\begin{aligned}
&u_t - \Delta u - u(1-u)(u-a) + v = mw; \quad (x,t) \in \Omega \times \mathbb{R}^+, \\
&v_t - \sigma u + \beta v = 0; \quad (x,t) \in \Omega \times \mathbb{R}^+, \\
&u(x,t) = 0; \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, \\
&u(x,0) = u_0(x), v(x,0) = v_0(x); \quad x \in \Omega,
\end{aligned} \tag{3.1}$$

where $u$, $v$ are unknown functions, $w$ is the control input, $a$, $\sigma$, $\beta$ are positive constants, and $m$ is the characteristic function of the domain $\bar{\omega}$.

In order to investigate the stabilization of the system (3.1), we apply the feedback controller

$$w = -ku, \tag{3.2}$$

where $k > 0$. Then the system (3.1) becomes

$$\begin{aligned} & u_t - \Delta u - u(1-u)(u-a) + v = -kmu; \quad (x,t) \in \Omega \times \mathbb{R}^+, \\ & v_t - \sigma u + \beta v = 0; \quad (x,t) \in \Omega \times \mathbb{R}^+, \\ & u(x,t) = 0; \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ & u(x,0) = u_0(x), v(x,0) = v_0(x); \quad x \in \Omega. \end{aligned} \tag{3.3}$$

**Definition 3.1.** *If there exists a feedback controller $w = -ku$ for some $k > 0$ such that the corresponding solution $[u, v]$ of (3.3) satisfies the inequality*

$$\int_\Omega (u^2(x,t) + v^2(x,t))dx \le Me^{-\alpha t} \int_\Omega (u_0^2(x) + v_0^2(x))dx$$

*for any $t > 0$ and some constants $\alpha > 0$, $M > 0$, then we say that the FitzHugh-Nagumo system (3.1) can be stabilized via the feedback controller $w = -ku$.*

Now, let $A_{\Omega_\omega}$ be the Laplace operator with Dirichlet boundary condition defined on $\Omega_\omega = \Omega \setminus \bar{\omega}$. i.e.,

$$\begin{aligned} A_{\Omega_\omega} u = -\Delta u; \quad u \in D\left(A_{\Omega_\omega}\right), \\ D\left(A_{\Omega_\omega}\right) = H^2\left(\Omega_\omega\right) \cap \ H_0^1\left(\Omega_\omega\right). \end{aligned}$$

Let us denote the first eigenvalue of $A_{\Omega_\omega}$ by $\lambda^1\left(A_{\Omega_\omega}\right)$, and by remark (1.6), we write

$$\lambda^1\left(A_{\Omega_\omega}\right) = \inf\left\{\int_{\Omega_\omega} \left|\nabla u(x)\right|^2 dx \ : \ u \in H_0^1\left(\Omega_\omega\right), \|u\|_{L^2(\Omega_\omega)} = 1\right\}.$$

**Remark 3.1.** *Note that*

$$\lambda^1\left(A_{\Omega_\omega}\right) \to \infty \quad as \quad d_H\left(\partial\Omega, \partial\omega\right) \to \ 0$$

*where $d_H$ is the "Hausdorff distance".*

By remark (3.1), for any positive constant $a$, we can choose $\Omega_\omega$ "sufficiently thin" so that

$$\lambda^1\left(A_{\Omega_\omega}\right) - \frac{(a-1)^2}{4} > 0. \tag{3.4}$$

**Lemma 3.1.** *For any $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that for all $k > K(\epsilon)$, the following inequality holds*

$$\left(\lambda^1\left(A_{\Omega_\omega}\right) - \epsilon\right) \int_\Omega u^2(x)\ dx \leq \int_\Omega \left(|\nabla u(x)|^2 + kmu^2(x)\right) dx, \quad u \in H_0^1(\Omega). \tag{3.5}$$

*Proof.* Firstly, let us define the operator $A_k$ as follows

$$A_k u = -\Delta u + kmu; \quad u \in D(A_k),$$
$$D(A_k) = H^2(\Omega) \cap \ H_0^1(\Omega)$$

and let $\lambda^1\left(A_k\right)$ be the first eigenvalue of the operator $A_k$. That is,

$$\lambda^1\left(A_k\right) = \inf\left\{\int_\Omega \ \left(|\nabla u(x)|^2 + kmu(x)^2\right) dx : u \in H_0^1(\Omega), \|u\|_{L^2(\Omega)} = 1\right\}. \tag{3.6}$$

Then we obtain that

$$\begin{aligned}\lambda^1(A_k) &\leq \inf\left\{\int_{\Omega_\omega} \ \left(|\nabla u(x)|^2 + kmu^2(x)\right) dx : u \in H_0^1(\Omega_\omega), \|u\|_{L^2(\Omega_\omega)} = 1\right\} \\ &= \inf\left\{\int_{\Omega_\omega} |\nabla u(x)|^2\, dx : u \in H_0^1(\Omega_\omega), \|u\|_{L^2(\Omega_\omega)} = 1\right\} \\ &= \lambda^1\left(A_{\Omega_\omega}\right).\end{aligned} \tag{3.7}$$

Here, note that $\{\lambda^1(A_k)\}_{k\in\mathbb{N}^+}$ is an increasing sequence of real numbers.

On the other hand, let $\phi_k^1$ be the eigenfunction corresponding to $\lambda^1(A_k)$ for each $k \in \mathbb{N}^+$. Without loss of generality, we may assume that $\|\phi_k^1\|_{L^2(\Omega)} = 1$ for each $k \in \mathbb{N}^+$. Using remark (1.6) again, we write

$$\begin{aligned}
\lambda^1(A_k) &= \int_\Omega \left(\left|\nabla\phi_k^1(x)\right|^2 + km(\phi_k^1(x))^2\right)dx \\
&= \int_\Omega \left|\nabla\phi_k^1(x)\right|^2 dx + k\int_\omega (\phi_k^1(x))^2 dx \\
&= \left\|\nabla\phi_k^1\right\|_{L^2(\Omega)}^2 + k\left\|\phi_k^1\right\|_{L^2(\omega)}^2,
\end{aligned}$$

where $m = \chi_{\bar{\omega}}$

Then using (3.6) and (3.7), for any $k \in \mathbb{N}^+$ we get,

$$\left\|\nabla\phi_k^1\right\|_{L^2(\Omega)}^2 + k\left\|\phi_k^1\right\|_{L^2(\omega)}^2 \leq \lambda^1\left(A_{\Omega_\omega}\right). \tag{3.8}$$

We know that

$$\left\|\phi_k^1\right\|_{H_0^1(\Omega)} = \left\|\nabla\phi_k^1\right\|_{L^2(\Omega)}$$

and considering (3.8), we deduce that $\{\phi_k^1\}_{k\in N^+}$ is a bounded sequence in $H_0^1(\Omega)$. Hence, by proposition (1.3), $\{\phi_k^1\}_{k\in\mathbb{N}^+}$ has a weakly convergent subsequence (also denoted by $\{\phi_k^1\}_{k\in\mathbb{N}^+}$) which is also bounded and converging to $\phi^1$. That is,

$$\phi_k^1 \rightharpoonup \phi^1 \quad as \quad k \to \infty \quad in \quad H_0^1(\Omega)$$

Now, since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, by definition (1.9), this subsequence has a subsequence (also denoted by $\{\phi_k^1\}_{k\in\mathbb{N}^+}$) such that

$$\phi_k^1 \to \phi^1 \quad as \quad k \to \infty \quad in \quad L^2(\Omega)$$

and in particular,

$$\phi_k^1 \to \phi^1 \quad as \quad k \to \infty \quad in \quad L^2(\bar{\omega}). \tag{3.9}$$

Note that by lemma (1.2), we obtain that

$$\liminf_{n\to\infty} \left\|\phi_k^1\right\|_{H_0^1(\Omega)} \geq \left\|\phi^1\right\|_{H_0^1(\Omega)},$$

or equivalently, we write

$$\liminf_{n\to\infty} \left\|\nabla\phi_k^1\right\|_{L^2(\Omega)} \geq \left\|\nabla\phi^1\right\|_{L^2(\Omega)}. \tag{3.10}$$

On the other hand, since $\|\nabla\phi_k^1\|_{L^2(\Omega)}^2 + k\,\|\phi_k^1\|_{L^2(\omega)}^2 \geq 0$, for all $k \in \mathbb{N}^+$, considering (3.8), we see that $\lim_{k\to\infty} \|\phi_k^1\|_{L^2(\omega)} = 0$, and also considering (3.9) above, we deduce $\phi^1 = 0$ almost everywhere on $\bar{\omega}$. Since $\|\phi^1\|_{L^2(\Omega)} = 1$, we obtain that $\|\phi^1\|_{L^2(\Omega_\omega)} = 1$.

Then by Divergence Theorem, consider that

$$\begin{aligned}
\left\|\nabla\phi^1\right\|_{L^2(\Omega)}^2 &= \int_{\Omega_\omega} \left|\nabla\phi^1(x)\right|^2 dx + \int_{\omega} \left|\nabla\phi^1(x)\right|^2 dx \\
&= \int_{\Omega_\omega} \left|\nabla\phi^1(x)\right|^2 dx - \int_{\omega} \phi^1(x)\Delta\phi^1(x)dx \\
&= \int_{\Omega_\omega} \left|\nabla\phi^1(x)\right|^2 dx \\
&= \left\|\nabla\phi^1\right\|_{L^2(\Omega_\omega)}^2.
\end{aligned}$$

Moreover, since we have $\lambda^1(A_k) \geq \|\nabla\phi_k^1\|_{L^2(\Omega)}^2$, for all $k \in \mathbb{N}^+$, considering (3.10), we obtain that

$$\begin{aligned}
\lim_{k\to\infty} \lambda^1(A_k) &\geq \liminf_{n\to\infty} \left\|\nabla \phi_k^1\right\|^2_{L^2(\Omega)} \\
&\geq \left\|\nabla \phi^1\right\|^2_{L^2(\Omega)} \\
&= \left\|\nabla \phi^1\right\|^2_{L^2(\Omega_\omega)} \\
&\geq \lambda^1\left(A_{\Omega_\omega}\right).
\end{aligned}$$

From (3.7) we have $\lim_{k\to\infty} \lambda^1(A_k) \leq \lambda^1\left(A_{\Omega_\omega}\right)$. Thus, we obtain that $\lim_{k\to\infty} \lambda^1(A_k) = \lambda^1\left(A_{\Omega_\omega}\right)$. By definition, given $\epsilon > 0$ there exists $K \in \mathbb{N}^+$ such that $\forall k \geq K$ we have $|\lambda^1(A_k) - \lambda^1\left(A_{\Omega_\omega}\right)| < \epsilon$. Then, we get

$$\lambda^1\left(A_{\Omega_\omega}\right) - \epsilon \leq \frac{\int_\Omega \left(|\nabla u(x)|^2 + kmu^2(x)\right)dx}{\int_\Omega u^2(x)dx}, \quad for\ all\ u \in H_0^1(\Omega) \ \ with\ u \neq 0,$$

$$\left(\lambda^1\left(A_{\Omega_\omega}\right) - \epsilon\right) \int_\Omega u^2(x)dx \leq \int_\Omega \left(|\nabla u(x)|^2 + kmu^2(x)\right) dx, \quad for\ all\ u \in H_0^1(\Omega).$$

**Theorem 3.2.** *There exists $K > 0$ such that if $k > K$, the FitzHugh-Nagumo system (3.1) can be stabilized via the feedback controller $w = -ku$. That is, for $u_0 \in L^2(\Omega)$, $v_0 \in L^2(\Omega)$, the solution $(u, v)$ of (3.3) satisfies*

$$\int_\Omega \left(u^2(x,t) + v^2(x,t)\right) dx \leq Me^{-\alpha t} \int_\Omega \left(u_0^2(x) + v_0^2(x)\right) dx$$

*for any $t > 0$ and some constants $\alpha > 0$, $M > 0$.*

*Proof.* Multiplying the first equation of (3.3) by $\sigma u$ and integrating over $\Omega \times (0,t)$,

$$\begin{aligned}
&\frac{\sigma}{2}\int_\Omega u^2(x,t)dx+\sigma\int_0^t\int_\Omega|\nabla u(x,s)|^2\,dxds+\sigma\int_0^t\int_\Omega u^4(x,s)dxds\\
&\quad-\sigma(a+1)\int_0^t\int_\Omega u^3(x,s)dxds+\sigma a\int_0^t\int_\Omega u^2(x,s)dxds\\
&\quad+\sigma\int_0^t\int_\Omega u(x,s)v(x,s)dxds=\frac{\sigma}{2}\int_0^t u_0^2(x)dx-\sigma k\int_0^t\int_\Omega mu^2(x,s)dxds.
\end{aligned}\tag{3.11}$$

Now, multipliying the second equation of (3.3) by $v$ and integrating over $\Omega\times(0,t)$,

$$\begin{aligned}
&\frac{1}{2}\int_\Omega v^2(x,t)dx-\sigma\int_0^t\int_\Omega u(x,s)v(x,s)dxds\\
&\qquad\qquad+\beta\int_0^t\int_\Omega v^2(x,s)dxds=\frac{1}{2}\int_\Omega v_0^2(x)dx.
\end{aligned}\tag{3.12}$$

Summing (3.11) and (3.12), we get

$$\begin{aligned}
&\frac{\sigma}{2}\int_\Omega u^2(x,t)dx+\frac{1}{2}\int_\Omega v^2(x,t)dx+\sigma\int_0^t\int_\Omega u^4(x,s)dxds\\
&\qquad=-\sigma\int_0^t\int_\Omega|\nabla u(x,s)|^2\,dxds-\sigma k\int_0^t\int_\Omega mu^2(x,s)dxds\\
&\qquad+\sigma(a+1)\int_0^t\int_\Omega u^3(x,s)dxds-\sigma a\int_0^t\int_\Omega u^2(x,s)dxds\\
&\qquad-\beta\int_0^t\int_\Omega v^2(x,s)dxds+\frac{\sigma}{2}\int_\Omega u_0^2(x)dx+\frac{1}{2}\int_\Omega v_0^2(x)dx.
\end{aligned}\tag{3.13}$$

Now, by using Young's Inequality (1.6) with $\epsilon=2$ and $p=q=2$, we obtain

$$\begin{aligned}
\sigma(a+1)\int_0^t\int_\Omega u^3(x,s)dxds&=\int_0^t\int_\Omega\sqrt{\sigma}u^2(x,s)\sqrt{\sigma}(a+1)u(x,s)dxds\\
&\le\sigma\int_0^t\int_\Omega u^4(x,s)dxds\\
&+\frac{\sigma(a+1)^2}{4}\int_0^t\int_\Omega u^2(x,s)dxds.
\end{aligned}\tag{3.14}$$

By using (3.13) and (3.14), we get

$$
\begin{aligned}
\frac{\sigma}{2}\int_\Omega u^2(x,t)dx + \frac{1}{2}\int_\Omega v^2(x,t)dx& \\
&\leq -\sigma \int_0^t \int_\Omega \left(|\nabla u(x,s)|^2 + kmu^2(x,s)\right)dxds \\
&+ \frac{\sigma(a-1)^2}{4}\int_0^t \int_\Omega u^2(x,s)dxds - \beta \int_0^t \int_\Omega v^2(x,s)dxds \\
&+ \frac{\sigma}{2}\int_\Omega u_0^2(x)dx + \frac{1}{2}\int_\Omega v_0^2(x)dx.
\end{aligned} \tag{3.15}
$$

By using (3.4), we can choose a small $\epsilon > 0$ such that $\lambda^1\left(A_{\Omega_\omega}\right) - \frac{(a-1)^2}{4} > \epsilon$. Let

$$
\delta := \sigma\left(\lambda^1\left(A_{\Omega_\omega}\right) - \frac{(a-1)^2}{4} - \epsilon\right) > 0.
$$

Combining (3.5) and (3.15), we have for any $k > K$,

$$
\begin{aligned}
\frac{\sigma}{2}\int_\Omega u^2(x,t)dx + \frac{1}{2}\int_\Omega v^2(x,t)dx &\leq -\sigma\left(\lambda^1\left(A_{\Omega_\omega}\right) - \epsilon\right)\int_0^t \int_\Omega u^2(x,s)dxds \\
&+ \frac{\sigma(a-1)^2}{4}\int_0^t \int_\Omega u^2(x,s)dxds \\
&- \beta\int_0^t \int_\Omega v^2(x,s)dxds + \frac{\sigma}{2}\int_\Omega u_0^2(x)dx + \frac{1}{2}\int_\Omega v_0^2(x)dx \\
&= -\delta\int_0^t \int_\Omega u^2(x,s)dxds - \beta\int_0^t \int_\Omega v^2(x,s)dxds \\
&+ \frac{\sigma}{2}\int_\Omega u_0^2(x)dx + \frac{1}{2}\int_\Omega v_0^2(x)dx.
\end{aligned}
$$

Let $c_1 = \min\left\{\frac{\sigma}{2}, \frac{1}{2}\right\}$, $c_2 = \max\left\{\frac{\sigma}{2}, \frac{1}{2}\right\}$, $\alpha = \frac{1}{c_1}\min\{\delta, \beta\}$, then we obtain

$$
c_1\int_\Omega (u^2(x,t)+v^2(x,t))dx \leq -\alpha c_1\int_0^t \int_\Omega (u^2(x,s)+v^2(x,s))dxds + c_2\int_\Omega (u_0^2(x)+v_0^2(x))dx.
$$

Dividing both sides by $c_1$, we get

$$\int_\Omega (u^2(x,t)+v^2(x,t))dx \le -\alpha \int_0^t \int_\Omega (u^2(x,s)+v^2(x,s))dxds + \frac{c_2}{c_1}\int_\Omega (u_0^2(x)+v_0^2(x))dx \quad (3.16)$$

Now, denoting $\int_\Omega (u^2(x,t)+v^2(x,t))dx$ by $Y(t)$ and $\frac{c_2}{c_1}$ by $\gamma$ we can write (3.16) as follows

$$Y(t) \le -\alpha \int_0^t Y(s)ds + \gamma Y(0). \quad (3.17)$$

Then multiplying both sides of (3.17) by $e^{\alpha t}$ and arranging we obtain

$$\int_0^t Y(s)ds \le int_0^t \gamma Y(0)e^{-\alpha s}ds.$$

Since $Y(s) \ge 0$ and $\gamma Y(0)e^{-\alpha s} \ge 0$ on $[0,t]$, we deduce $Y(t) \le \gamma Y(0)e^{-\alpha t}$ which is equivalent to

$$\int_\Omega (u^2(x,t)+v^2(x,t))dx \le \left[\frac{c_2}{c_1}\int_\Omega (u_0^2(x)+v_0^2(x))dx\right] e^{-\alpha t}.$$

Letting $M := \frac{c_2}{c_1}$, we obtain the desired stability result

$$\int_\Omega (u^2(x,t)+v^2(x,t))dx \le Me^{-\alpha t}\int_\Omega (u_0^2(x)+v_0^2(x))dx.$$

$\square$

# Chapter 4

# Structural Stability for FitzHugh-Nagumo Equation

In this chapter, we will consider an initial boundary value problem for a system of nonlinear parabolic equations that can be considered as a regularization of the FitzHugh-Nagumo model. For the system, given below, we will study the problem of continuous dependence of solutions to the probem on the diffusivity coefficient. Note that, this type of stability is called "structural stability".

Let $G \subset \mathbb{R}^n$ $(n \leq 4)$ be a bounded domain with sufficiently smooth boundary $\partial G$, and $u_0$, $v_0$ be given functions. Consider the system,

$$\begin{aligned} &u_t - \Delta u + g\,|u|^p\,u + cu^2 + au - v = 0, \quad x \in G,\ t > 0 \\ &v_t - k\Delta v + fv + bu = 0, \quad x \in G,\ t > 0 \\ &u(x,0) = u_0(x),\ v(x,0) = v_0(x), \quad x \in G \\ &u(x,t) = 0,\ v(x,t) = 0, \quad x \in \partial G,\ t > 0 \end{aligned} \tag{4.1}$$

where $a > 0$, $b > 0$, $f > 0$, $g > 0$, $p \geq 2$, and $c \in \mathbb{R}$ are given numbers.

Here, we assume that $[u, v]$ is the classical solution of the system (4.1).

Firstly, we derive a priori estimates for solutions of the problem (4.1) which are uniform with respect to $t \in \mathbb{R}^+$.

**Theorem 4.1.** *Suppose that $u_0, v_0 \in H_0^1(G)$. Then the following estimates hold true*

$$\begin{aligned} &\|\nabla u(t)\|, \|\nabla v(t)\|, \int_0^t \|\nabla u(s)\|^2 ds, k\int_0^t \|\nabla v(s)\|^2 ds, \\ &\int_0^t \|\Delta u(s)\|^2 ds, k\int_0^t \|\Delta v(s)\|^2 ds \le D, \qquad \forall t \in \mathbb{R}^+. \end{aligned} \tag{4.2}$$

*for some constant $D$.*

*Proof.* Let us multiply the first equation in (4.1) by $bu$ and then integrate the result over $G$,

$$\begin{aligned} \frac{b}{2}\int_G \frac{\partial}{\partial t}(u^2(x,t))dx - b\int_G u(x,t)\Delta u(x,t)dx + bg\int_G |u(x,t)|^{p+2}\, dx \\ + ab\int_G u^2(x,t)dx - b\int_G u(x,t)v(x,t)dx = -bc\int_G u^3(x,t)dx. \end{aligned}$$

From this, we get

$$\begin{aligned} \frac{d}{dt}\left[\frac{b}{2}\|u(t)\|^2\right] + b\|\nabla u(t)\|^2 + bg\int_G |u(x,t)|^{p+2}\, dx + ab\|u(t)\|^2 \\ - b\int_G u(x,t)v(x,t)dx = -bc\int_G u^3(x,t)dx. \end{aligned} \tag{4.3}$$

Multiply the second equation in (4.1) by $v$ and then integrate over $G$,

$$\frac{1}{2}\int_G \frac{\partial}{\partial t}v^2(x,t)dx - k\int_G v(x,t)\Delta v(x,t)dx + f\int_G v^2(x,t)dx + b\int_G u(x,t)v(x,t)dx = 0$$

Then, we obtain that

$$\frac{d}{dt}\left[\frac{1}{2}\|v(t)\|^2\right] + k\|\nabla v(t)\|^2 + f\|v(t)\|^2 + b\int_G u(x,t)v(x,t)dx = 0. \tag{4.4}$$

Summing (4.3) and (4.4), we obtain

$$\begin{aligned}\frac{d}{dt}\left[\frac{b}{2}\|u(t)\|^2+\frac{1}{2}\|v(t)\|^2\right]&+ab\|u(t)\|^2+f\|v(t)\|^2+b\|\nabla u(t)\|^2\\&+k\|\nabla v(t)\|^2+bg\int_G|u(x,t)|^{p+2}dx\\&=-bc\int_G u^3(x,t)dx\end{aligned}\tag{4.5}$$

By using Young's Inequality (1.6) for the right-hand side of (4.5) with $\epsilon=\frac{bg(p+2)}{6}$, we obtain

$$-bc\int_G u^3(x,t)dx\le\frac{bg}{2}\int_G|u(x,t)|^{p+2}dx+\frac{p-1}{p+2}\left(\frac{bg(p+2)}{6}\right)^{3/(1-p)}(b|c|)^{\frac{p+2}{p-1}}|G|.$$

Letting $C_0=\frac{p-1}{p+2}\left(\frac{bg(p+2)}{6}\right)^{3/(1-p)}(b|c|)^{\frac{p+2}{p-1}}|G|$, we write the above inequality in the form

$$-bc\int_G u^3(x,t)dx\le\frac{bg}{2}\int_G|u(x,t)|^{p+2}dx+C_0.\tag{4.6}$$

Combining (4.5) and (4.6), we get

$$\begin{aligned}\frac{d}{dt}\left[b\|u(t)\|^2+\|v(t)\|^2\right]&+2ab\|u(t)\|^2+2f\|v(t)\|^2+2b\|\nabla u(t)\|^2\\&+2k\|\nabla v(t)\|^2+bg\int_G|u(x,t)|^{p+2}dx\le 2C_0.\end{aligned}\tag{4.7}$$

Then, from (4.7), we obtain that

$$\frac{d}{dt}\left[b\|u(t)\|^2+\|v(t)\|^2\right]+2ab\|u(t)\|^2+2f\|v(t)\|^2\le 2C_0.$$

Here, letting $\nu_1=\min\{2a,2f\}$, we get

$$\frac{d}{dt}\left[b\left\|u(t)\right\|^2+\left\|v(t)\right\|^2\right]+\nu_1\left(b\left\|u(t)\right\|^2+\left\|v(t)\right\|^2\right)\leq 2C_0.$$

This implies that

$$b\left\|u(t)\right\|^2+\left\|v(t)\right\|^2\leq e^{-\nu_1 t}\left(b\left\|u_0\right\|^2+\left\|v_0\right\|^2-\frac{2C_0}{\nu_1}\right)+\frac{2C_0}{\nu_1}.$$

Since for $\nu_1>0$ and $t>0$, $e^{-\nu_1 t}<1$, we get the inequality

$$b\left\|u(t)\right\|^2+\left\|v(t)\right\|^2\leq C_1,\quad \forall t\in\mathbb{R}^+, \tag{4.8}$$

where $C_1=b\left\|u_0\right\|^2+\left\|v_0\right\|^2$.

Here, it is important that $C_0$ and $C_1$ do not depend on $k$. Now, integrating (4.7) with respect to $t$, we get

$$\begin{aligned}b\left\|u(t)\right\|^2&+\left\|v(t)\right\|^2-b\left\|u_0\right\|^2-\left\|v_0\right\|^2+2ab\int_0^t\left\|u(s)\right\|^2ds\\&+2f\int_0^t\left\|v(s)\right\|^2ds+2b\int_0^t\left\|\nabla u(s)\right\|^2ds+2k\int_0^t\left\|\nabla v(s)\right\|^2ds\\&+bg\int_0^t\int_G\left|u(x,s)\right|^{p+2}dxds\leq 2C_0t.\end{aligned}$$

Then since

$$b\left\|u(t)\right\|^2+\left\|v(t)\right\|^2+2ab\int_0^t\left\|u(s)\right\|^2ds+2d\int_0^t\left\|v(s)\right\|^2ds\geq 0,$$

by using (4.8) we obtain that

$$2b\int_0^t \|\nabla u(s)\|^2\, ds + 2k\int_0^t \|\nabla v(s)\|^2\, ds$$
$$+ bg\int_0^t\int_G |u(x,s)|^{p+2}\, dxds$$
$$\leq 2C_0t + C_1 \leq 2C_0T + C_1,$$

for all $t \in [0,T]$. So, we deduce that there exists $C_2(T)$ depending on $b, c, g, p, |G|$, initial data and $T$ such that

$$\int_0^t \|\nabla u(s)\|^2\, ds, \quad k\int_0^t \|\nabla v(s)\|^2\, ds,$$
$$\int_0^t\int_G |u(x,s)|^{p+2}\, dxds \leq C_2(T), \quad \forall t \in [0,T]. \tag{4.9}$$

Now, multiplying the first equation in (4.1) by $-b\Delta u$ and then integrating over $G$, we get

$$-b\int_G u_t(x,t)\Delta u(x,t)dx + b\int_G |\Delta u(x,t)|^2\, dx - bg\int_G |u(x,t)|^p\, u(x,t)\Delta u(x,t)dx$$
$$- bc\int_G u^2(x,t)\Delta u(x,t)dx - ab\int_G u(x,t)\Delta u(x,t)dx$$
$$+ b\int_G v(x,t)\Delta u(x,t)dx = 0.$$

From this equality, we obtain that

$$\frac{b}{2}\frac{d}{dt}\|\nabla u(t)\|^2 + b\|\Delta u(t)\|^2 + bg(p+1)\int_G |u(x,t)|^p\, |\nabla u(x,t)|^2\, dx$$
$$+ bc\int_G u(x,t)\, |\nabla u(x,t)|^2\, dx + ab\|\nabla u(t)\|^2 \tag{4.10}$$
$$+ b\int_G u(x,t)\Delta v(x,t)dx = 0$$

Then multiplying the second equation in (4.1) by $-\Delta v$ and integrating over $G$, we get that

$$-\int_G v_t(x,t)\Delta v(x,t)dx + k\int_G |\Delta v(x,t)|^2\,dx - f\int_G v(x,t)\Delta v(x,t)dx \\ - b\int_G u(x,t)\Delta v(x,t)dx = 0.$$

By using Divergence Theorem (1.11), we get that

$$\frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|^2 + k\|\Delta v(t)\|^2 + f\|\nabla v(t)\|^2 - b\int_G u(x,t)\Delta v(x,t)dx = 0. \tag{4.11}$$

Then, summing (4.10) and (4.11), we obtain that

$$\begin{aligned}&\frac{d}{dt}\left[\frac{b}{2}\|\nabla u(t)\|^2 + \frac{1}{2}\|\nabla v(t)\|^2\right] + ab\|\nabla u(t)\|^2 + f\|\nabla v(t)\|^2 + b\|\Delta u(t)\|^2\\ &+ k\|\Delta v(t)\|^2 + bg(p+1)\int_G |u(x,t)|^p\,|\nabla u(x,t)|^2\,dx\\ &= -bc\int_G u(x,t)\,|\nabla u(x,t)|^2\,dx.\end{aligned} \tag{4.12}$$

Now, let us use Young's Inequality (1.6) and Interpolation Inequality (1.8) for the right-hand side of (4.12) with $\epsilon = \frac{bgp(p+1)}{2}$,

$$
\begin{aligned}
-bc\int_G u(x,t)\left|\nabla u(x,t)\right|^2 dx &\le b\left|c\right|\int_G \left|u(x,t)\right|\left|\nabla u(x,t)\right|^2 dx \\
&= b\left|c\right|\int_G \left|u(x,t)\right|\left|\nabla u(x,t)\right|^{2/p}\left|\nabla u(x,t)\right|^{(2p-2)/p} dx \\
&\le \frac{bg(p+1)}{2}\int_G \left|u(x,t)\right|^p\left|\nabla u(x,t)\right|^2 dx \\
&+ \frac{p-1}{p}\left(\frac{bgp(p+1)}{2}\right)^{1/(1-p)}(b\left|c\right|)^{p/(p-1)}\int_G\left|\nabla u(x,t)\right|^2 dx \\
&= \frac{bg(p+1)}{2}\int_G \left|u(x,t)\right|^p\left|\nabla u(x,t)\right|^2 dx + C_3\left\|\nabla u(t)\right\|^2 \\
&\le \frac{bg(p+1)}{2}\int_G \left|u(x,t)\right|^p\left|\nabla u(x,t)\right|^2 dx + C_3\left\|u(t)\right\|\left\|\Delta u(t)\right\| \\
&\le \frac{bg(p+1)}{2}\int_G \left|u(x,t)\right|^p\left|\nabla u(x,t)\right|^2 dx + \frac{b}{2}\left\|\Delta u(t)\right\|^2 \\
&+ \frac{C_3}{2b}\left\|u(t)\right\|^2 \\
&= \frac{bg(p+1)}{2}\int_G \left|u(x,t)\right|^p\left|\nabla u(x,t)\right|^2 dx + \frac{b}{2}\left\|\Delta u(t)\right\|^2 \\
&+ C_4\left\|u(t)\right\|^2
\end{aligned}
$$

where $C_3 = \frac{p-1}{p}\left(\frac{bgp(p+1)}{2}\right)^{1/(1-p)}(b\left|c\right|)^{p/(p-1)}$ and $C_4 = \frac{C_3}{2b}$. Note that here we also used the inequality

$$
\left\|u(t)\right\|\left\|\Delta u(t)\right\| \le \epsilon\left\|u(t)\right\|^2 + \frac{1}{4\epsilon}\left\|\Delta u(t)\right\|^2,
$$

with $\epsilon = 1/2b$.

By using the above inequality in (4.12), we obtain that

$$
\begin{aligned}
\frac{d}{dt}\left[b\|\nabla u(t)\|^2+\|\nabla v(t)\|^2\right]&+2ab\|\nabla u(t)\|^2+2f\|\nabla v(t)\|^2+2b\|\Delta u(t)\|^2\\
&+2k\|\Delta v(t)\|^2+2bg(p+1)\int_G|u(x,t)|^p|\nabla u(x,t)|^2dx\\
&\le bg(p+1)\int_G|u(x,t)|^p|\nabla u(x,t)|^2dx+b\|\Delta u(t)\|^2\\
&+2C_4\|u(t)\|^2.
\end{aligned}
$$

From this inequality, we get

$$
\begin{aligned}
\frac{d}{dt}\left[b\|\nabla u(t)\|^2+\|\nabla v(t)\|^2\right]&+2ab\|\nabla u(t)\|^2+2f\|\nabla v(t)\|^2\\
&+b\|\Delta u(t)\|^2+2k\|\Delta v(t)\|^2\\
&+bg(p+1)\int_G|u(x,t)|^p|\nabla u(x,t)|^2dx \qquad (4.13)\\
&\le 2C_4\|u(t)\|^2\\
&\le C_5,
\end{aligned}
$$

where $C_5=\frac{2C_1C_4}{b}$. Now, integrating (4.13) with respect to t, we obtain that

$$
\begin{aligned}
b\|\nabla u(t)\|^2+\|\nabla v(t)\|^2-b\|\nabla u(0)\|^2-\|\nabla v(0)\|^2+2ab\int_0^t\|\nabla u(s)\|^2ds\\
+2f\int_0^t\|\nabla v(s)\|^2ds+b\int_0^t\|\Delta u(s)\|^2ds+2k\int_0^t\|\Delta v(s)\|^2ds\\
+bg(p+1)\int_0^t\int_G|u(x,s)|^p|\nabla u(x,s)|^2dxds\\
\le C_5t.
\end{aligned}
$$

From this inequality we obtain the following inequality

$$
\begin{aligned}
b\left\|\nabla u(t)\right|^2 + \left\|\nabla v(t)\right\|^2 + b\int_0^t \left\|\Delta u(s)\right\|^2 ds + 2k\int_0^t \left\|\Delta v(s)\right\|^2 ds \\
\le C_5 t + b\left\|\nabla u(0)\right\|^2 + \left\|\nabla v(0)\right\|^2 \\
\le C_5 T + b\left\|\nabla u(0)\right\|^2 + \left\|\nabla v(0)\right\|^2 ,
\end{aligned}
\tag{4.14}
$$

for all $t \in [0, T]$. So, we deduce that there exists a constant $C_6(T)$ depending on $a, b, c, d, g, |G|$, initial data and $T$ such that

$$
\begin{aligned}
\left\|\nabla u(t)\right\|^2, \ \left\|\nabla v(t)\right\|^2, \ \int_0^t \left\|\Delta u(s)\right\|^2 ds, \\
k\int_0^t \left\|\Delta v(s)\right\|^2 ds \ \le \ C_6(T), \quad \forall t \in [0, T].
\end{aligned}
\tag{4.15}
$$

□

So, we are ready to prove the following theorem on continuous dependence of solution to the problem (4.1) on the diffusivity coefficient k.

**Theorem 4.2.** *Suppose that* $[u_i, v_i]$, $i = 1, 2$ *are strong solutions of the problem (4.1), that is;*

$$
\begin{aligned}
&\partial_t u_i - \Delta u_i + \left|u_i\right|^p u_i + c u_i^2 + a u_i - v_i = 0, \quad x \in G, \ t > 0, \\
&\partial_t v_i - k_i \Delta v_i + f v_i + b u_i = 0, \quad x \in G, \ t > 0, \\
&u_i(x, 0) = u_0(x), v_i(x, 0) = v_0(x), \quad x \in G, \\
&u_i(x, t) = v_i(x, t) = 0, \quad x \in \partial G, \ t > 0.
\end{aligned}
\tag{4.16}
$$

*Then the following a priori estimate with* $\tilde{k} = k_1 - k_2$ *holds true*

$$
\left\|u_1(t) - u_2(t)\right\| \le D_1(T)\sqrt{\tilde{k}} e^{D_2 t}, \quad \forall t \in [0, T], \tag{4.17}
$$

*for some constants* $D_1(T)$ *and* $D_2$.

*Proof.* It is clear that the pair of functions $[w, z] = [u_1 - u_2, v_1 - v_2]$ is a solution of the problem

$$\begin{aligned}
&w_t - \Delta w + |u_1|^p \, u_1 - |u_2|^p \, u_2 + c(u_1 + u_2)w + aw - z = 0, \quad x \in G, \ t > 0, \\
&z_t - k_1 \Delta z + fz + bw = \tilde{k}\Delta v_2, \quad x \in G, \ t > 0, \\
&w(x, 0) = 0, z(x, 0) = 0, \quad x \in G, \\
&w(t, x) = z(t, x) = 0, \quad x \in \partial G, \ t > 0.
\end{aligned} \tag{4.18}$$

Multiplying the first equation in (4.18) by $bw$ and then integrating over $G$, we get the relation

$$\begin{aligned}
&\frac{b}{2}\frac{d}{dt} \|w(t)\|^2 + b \|\nabla w(t)\|^2 \\
&+ b \int_G w(x,t) \left( |u_1(x,t)|^p u(x,t)_1 - |u(x,t)_2|^p u(x,t)_2 \right) dx \\
&+ bc \int_G w^2(x,t) \left( u_1(x,t) + u_2(x,t) \right) dx + ab \|w(t)\|^2 \\
&- b \int_G w(x,t) z(x,t) dx = 0.
\end{aligned} \tag{4.19}$$

Now, multiply the second equation in (4.18) by $z$ and integrate over $G$ to get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \|z(t)\|^2 + k_1 \|\nabla z(t)\|^2 + f \|z(t)\|^2 + b \int_G w(x,t) z(x,t) dx \\
= \tilde{k} \int_G z(x,t) \Delta v_2 d(x,t) x.
\end{aligned} \tag{4.20}$$

Summing (4.19) and (4.20), we obtain that

$$\frac{d}{dt}\left[\frac{b}{2}\|w(t)\|^2+\frac{1}{2}\|z(t)\|^2\right]+f\|z(t)\|^2+b\|\nabla w(t)\|^2+k_1\|\nabla z(t)\|^2$$
$$+ab\|w(t)\|^2+b\int_G w(x,t)\left(|u_1(x,t)|^p u_1(x,t)\right.$$
$$\left.-|u_2(x,t)|^p u_2(x,t)dx\right. \tag{4.21}$$
$$=\tilde{k}\int_G z(x,t)\Delta v_2(x,t)dx$$
$$-bc\int_G (u_1(x,t)+u_2(x,t))w^2(x,t)dx.$$

Combining the inequality (1.10) with (4.21), we obtain that

$$\frac{d}{dt}\left[\frac{b}{2}\|w(t)\|^2+\frac{1}{2}\|z(t)\|^2\right]+f\|z(t)\|^2+b\|\nabla w(t)\|^2$$
$$+k_1\|\nabla z(t)\|^2+ab\|w(t)\|^2+bd_1\int_G |w(x,t)|^{p+2}\,dx\le$$
$$\tilde{k}\int_G z(x,t)\Delta v_2(x,t)dx-bc\int_G (u_1(x,t)+u_2(x,t))w^2(x,t)dx.$$

Then clearly, we get that

$$\frac{d}{dt}\left[\frac{b}{2}\|w(t)\|^2+\frac{1}{2}\|z(t)\|^2\right]\le\tilde{k}\int_G z(x,t)\Delta v_2(x,t)dx$$
$$-bc\int_G (u_1(x,t)+u_2(x,t))w^2(x,t)dx. \tag{4.22}$$

For the first term on the right-hand side of (4.22), using Young's Inequality (1.6) with $\epsilon=f$, and $p=q=2$ we obtain the inequality

$$\left|\tilde{k}\int_G \Delta v_2(x,t)z(x,t)dx\right|\le\int_G |\tilde{k}\Delta v_2(x,t)z(x,t)|dx$$
$$\le\frac{f}{2}\int_G |z(x,t)|^2\,dx+\frac{\tilde{k}^2}{2f}\int_G |\Delta v_2(x,t)|^2\,dx \tag{4.23}$$
$$=\frac{f}{2}\|z(t)\|^2+\frac{\tilde{k}^2}{2f}\|\Delta v_2(t)\|^2.$$

For the second term on the right-hand side of (4.22), we have

$$\begin{aligned}\left|\int_G \left(u_1(x,t)+u_2(x,t)\right) w^2(x,t)dx\right| &\leq \int \left(|u_1(x,t)|+|u_2(x,t)|\right) w^2(x,t)dx \\ &\leq \max_{x\in G}\left(|u_1(x,t)|+|u_2(x,t)|\right)\int_G w^2(x,t)dx \\ &\leq \left(\max_{x\in G}|u_1(x,t)|+\max_{x\in G}|u_2(x,t)|\right)\|w(t)\|^2 .\end{aligned}$$

Then, using the Sobolev Inequality (1.9), we obtain

$$\left|\int_G \left(u_1(x,t)+u_2(x,t)\right) w^2(x,t)dx\right| \leq \left(r_1\|\Delta u_1(t)\|+r_2\|\Delta u_2(t)\|\right)\|w(t)\|^2 .$$

Letting $r=\max\{r_1,r_2\}$, we get

$$\left|\int_G \left(u_1(x,t)+u_2(x,t)\right) w^2(x,t)dx\right| \leq r\left(\|\Delta u_1(t)\|+\|\Delta u_2(t)\|\right)\|w(t)\|^2 . \qquad (4.24)$$

Now, using (4.23) and (4.24) in (4.22) we get that

$$\begin{aligned}\frac{d}{dt}\left[\frac{b}{2}\|w(t)\|^2+\frac{1}{2}\|z(t)\|^2\right] &\leq \frac{f}{2}\|z(t)\|^2+\frac{\tilde{k}^2}{2f}\|\Delta v_2(t)\|^2 \\ &+ b|c|r\left(\|\Delta u_1(t)\|+\|\Delta u_2(t)\|\right)\|w(t)\|^2 \\ &\leq \left[|c|r\left(\|\Delta u_1(t)\|^2+\|\Delta u_2(t)\|^2\right)+\frac{f}{2}\right]\left(b\|w(t)\|^2+\|z(t)\|^2\right) \\ &+ \frac{\tilde{k}^2}{2f}\|\Delta v_2(t)\|^2 .\end{aligned}$$

From this inequality, we get that

$$\begin{aligned}\frac{d}{dt}\left[b\|w(t)\|^2+\|z(t)\|^2\right] &\leq \left[2|c|r\left(\|\Delta u_1(t)\|^2+\|\Delta u_2(t)\|^2\right)+f\right]\left(b\|w(t)\|^2+\|z(t)\|^2\right) \\ &+ \frac{\tilde{k}^2}{f}\|\Delta v_2(t)\|^2 .\end{aligned}$$

Letting

$$y(t) := b\left\|w(t)\right\|^2 + \left\|z(t)\right\|^2$$

, $\beta(t) := 2|c|r\left(\left\|\Delta u_1(t)\right\|^2 + \left\|\Delta u_2(t)\right\|^2\right) + f$,and $\alpha(t) := \frac{\tilde{k}^2}{d}\left\|\Delta v_2(t)\right\|^2$, we can write the last inequality as

$$y'(t) \leq \alpha(t) + \beta(t)y(t).$$

Observe that $y(0) = 0$. Integrating it over $[0, t]$, we get

$$y(t) \leq \int_0^t \alpha(s)ds + \int_0^t \beta(s)y(s)ds.$$

Due to Gronwall's Lemma, we have

$$y(t) \leq \left(\int_0^t \alpha(s)ds\right) e^{\int_0^t \beta(s)ds}.$$

That is,

$$y(t) \leq \left(\frac{\tilde{k}^2}{f}\int_0^t \left\|\Delta v_2(t)\right\|^2 ds\right) e^{\int_0^t \left[2|c|r\left(\left\|\Delta u_1(s)\right\|^2 + \left\|\Delta u_2(s)\right\|^2\right) + f\right]ds}. \tag{4.25}$$

Remember that we have from (4.15) that $\int_0^t \left\|\Delta u(s)\right\|^2 ds \leq C_6(T)$, $\forall t \in [0, T]$ and so,

$$\begin{aligned}
&\int_0^t [2|c|r\left(\left\|\Delta u_1(s)\right\|^2 + \left\|\Delta u_2(s)\right\|^2 + f\right]ds \\
&\qquad = 2|c|r\left(\int_0^t \left\|\Delta u_1(s)\right\|^2 ds + \int_0^t \left\|\Delta u_2(s)\right\|^2 ds\right) + ft \\
&\qquad \leq 4|c|rC_6(T) + ft.
\end{aligned}$$

Using (4.15) again and employing the above inequalities in (4.25), we get

$$b\,\|w(t)\|^2 + \|z(t)\|^2 \leq \frac{C_6(T)\tilde{k}}{f} e^{C_7}$$

where $C_7 := 4|c|rC_6(T) + ft$. Then, we also have $b\,\|w(t)\|^2 \leq \frac{C_6(T)\tilde{k}}{f} e^{C_7}$, and since $C_7$ depends on $t$ linearly, we deduce that there exist constatnts $D_1(T)$ and $D_2$ such that

$$\|w(t)\| \leq D_1(T)\sqrt{\tilde{k}} e^{D_2 t}, \qquad \forall t \in [0, T].$$

Hence, the estimate (4.17) is satisfied. □

**Corollary 4.3.** *Since the constant $D_1(T)$ in (4.17) does not depend on diffusivity coefficients $k_1$ and $k_2$, it follows that on each finite interval $[0, T]$ a solution $[u, v]$ to the problem (4.1) tends as $k \to 0^+$ to the solution $[\tilde{u}, \tilde{v}]$ of the initial boundary value problem for the system*

$$\begin{aligned}
&\tilde{u}_t - \Delta\tilde{u} + g\,|\tilde{u}|^p\,\tilde{u} + c\tilde{u}^2 + a\tilde{u} - \tilde{v} = 0, \quad x \in G,\ t > 0\\
&\tilde{v}_t + f\tilde{v} + b\tilde{u} = 0, \quad x \in G,\ t > 0\\
&\tilde{u}(x, 0) = u_0(x),\ \tilde{v}(x, 0) = v_0(x), \quad x \in G\\
&\tilde{u}(x, t) = 0, \quad x \in \partial G,\ t > 0.
\end{aligned} \tag{4.26}$$

This shows that we can approximate solution to the system (4.26) by solutions of the system

$$\begin{aligned}
&u_t - \Delta u + g\,|u|^p\,u + cu^2 + au - v = 0, \quad x \in G,\ t > 0\\
&v_t - k\Delta v + fv + bu = 0, \quad x \in G,\ t > 0.
\end{aligned}$$

# Conclusion

In this thesis, our main aim is to study some stability properties of Fitzhugh-Nagumo Equations.

We know that a system of FitzHugh-Nagumo Equations is special type of a system of reaction-diffusion equations. So, firstly we proved the existence and uniqueness of solutions of a reaction-diffusion equation and noted that the result for the system of reacrion-diffusion equations is similar.

After that,we studied some stability properties of FitzHugh-Nagumo model. We showed that the solutions of a FitzHugh-Nagumo model can be stabilized by applying a feedback controller on a bounded subdomain. For another FitzHugh-Nagumo model, we showed that the solutions of this model are continuously dependening on the diffusivity coefficient.

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