EQUILIBRIUM BIDDING IN COMMON VALUE AUCTIONS WITH EX-POST INVESTMENT DECISIONS

by

Vyacheslav Arbuzov

A thesis submitted to the
Graduate School of Social Sciences and Humanities
in partial fulfillment for the
degree of
Master of Arts
in
Economics

May 2014

Department of Economics

Koç University

Acknowledgement

I would like to show my gratitude to all those who made it possible to complete this thesis. Foremost, I would like to express my sincere gratitude to my advisor Asst. Prof. Alp E. Atakan for introducing me to the topic and for all the support along the way. Without his outstanding guidance, enthusiasm, immense knowledge and his endless patience this thesis would not have been possible. I would also like to thank Assoc. Prof. Okan Yılankaya and Assoc. Prof. Oğuzhan Özbaş for being members of my thesis committee and for their valuable comments and suggestions.

Furthermore, I thank my family for their endless love and support. Their encouragement allowed me to pursue my graduate studies and to never give up. I also thank my colleagues who created an excellent and extremely productive atmosphere during my Master's Program.

Last, but not least, I am grateful to TÜBİTAK (through Asst. Prof. Alp E. Atakan's grant 112K023) and Vehbi Koç Foundation for their financial assistance throughout my graduate studies.

Abstract

I study a two-unit uniform-price auction with three participants who have unit demands. Before an auction, each participant receives a discrete signal that can take two (qualitatively different) values - l or h. After the auction, each winner needs to make an ex-post investment decision that will affect the value of the object. Hence, the values are determined jointly by the signal and ex-post investment decision. I characterize the necessary and sufficient conditions for the existence of monotone and non-monotone equilibria. I find that for some parameters, only non-monotone equilibria are possible. In addition, except for one class of monotone and non-monotone equilibria, the equilibrium behavior supposes that some agents choose to bid the same bid irrespective of the signal they receive. However, the players with qualitatively different signals cannot pool at the same bid in any (monotone or not) equilibrium of the model.

Keywords: Common value auctions, Monotonicity.

Contents

List of Figures

1 Introduction

Since one of the earliest reports in the fifth century B.C.E., auctions have been gaining more and more interest as one of the many alternatives to sell an object. Auctions are used to sell government land, rights for extracting natural resources and many other circumstances. Therefore, an analysis describing the existence or lack of auction's desirable features like efficiency is important. In particular, it is equally important to find out whether the price at auctions can effectively aggregate information each participant in the auction has and whether the price is a sufficient tool to correctly estimate the current state of the world (as it is in the competitive market structure). Previous research of large common value auctions (Pesendorfer and Swinkels 1997, Kremer 2002) shows that under particular assumptions information aggregation in auctions (be it second price private value auction or common value first price auction) could indeed occur. However, some researchers (Atakan and Ekmekci in press) demonstrate that with a slight modification of the setting and with minimal assumptions (which are practically justifiable) information aggregation fails to occur. The similarity of the settings in Atakan and Ekmekci (in press) and real life common value auctions draws a suspicion of inefficient resource allocation if the auction is used as a tool for recourse allocation. The problem bears extra attention because government resources are often sold using the auction mechanism.

Much of existing literature analyzes the properties of the auctions in large settings. However, some auctions, especially auctions aiming to allocate government resources, may have a small number of participants and even smaller number of objects that are available for sale. Typical examples of small auction settings are the sale of rights for oil extraction and the sale of spectrum rights. In both cases, economies of scale may require the government (or the possessor of resources) to limit the number of rights to a small number. In addition, some sectors just do not have enough participants to converge to the settings of a typical large common value auction. This was an issue in the auction of 3G licences in the UK[1]. Before the auction, there were four incumbent firms in the market; by the time of the auction, a total of thirteen firms decided to participate in the auction meaning that the government could attract only nine new firms despite providing some subsidies and privileges in establishing new infrastructure. The problem of failing to attract enough participants can originate in many other circumstances. For example, currently, there are twenty three different companies that operate in the oil extraction sector in Russian Federation. If the Russian government decides to sell its oil-drilling rights on the newly discovered petroleum deposits in Arctic, it is doubtful that the government could lure much of the new entrants and increase the number of participants to even 50 firms.

Besides the size of the auction, incompleteness of information available to the participants bears extra attention as a potential source of inefficiencies in the auctions. By introducing actions (or ex-post investment decisions), Atakan and Ekmekci (in press)

[1]For more information on the UK 3G rights auction refer to Binmore and Klemperer (2002)

demonstrate that an additional uncertainty may worsen information aggregation properties of large auctions. This result is especially important because many real life auctions require ex-post investment decisions. For example, the true value of the oil-drilling rights depends on what type of final good a possessor of the rights would want to produce. An optimal choice of final good, in turn, depends on future prices of various alternatives which are unknown by the time of acquisition of the rights.

A difference in equilibrium behavior in the settings presented by Atakan and Ekmekci (in press) and Pesendorfer and Swinkels (1997) demonstrates that additional uncertainty coming from the required ex-post decisions may lead to completely different equilibria of the auction. In particular, such uncertainty may allow participants to bid in a non-monotone fashion. Multiplicity of equilibria, in turn, may mean that some of the equilibria possess more favourable features and thus are more preferred than the standard equilibria. Hence, a full characterization of the optimal bidding in auctions with ex-post investment decisions possesses a potential to answer these questions.

In this study, I analyze a two-unit uniform-price auction with ex-post investment decisions. There are three participants who have unit demands. Before the auction, each participant receives a private discrete signal. Each winner of the object is required to take an action that will determine the true value of the object. I find that under mild assumptions that are reasonably justifiable a monotone equilibrium exists for some model parameters. In addition, I demonstrate that an existence of monotone equilibrium immediately implies the existence of non-monotone equilibria, but the reverse does not necessarily hold: only non-monotone equilibria are possible for some model parameters. Except for one case, pooling behavior is common in all equilibria (monotone or not). However, I find that the bidders with qualitatively different signals cannot pool at the same bid.

2 Literature Review

Much of the existing literature on common value auctions analyze conditions for the equilibrium existence and the equilibrium characterization in large auctions where the participants have unit demands and non-decreasing value function. Wilson (1977) first demonstrates that in large common value auctions with one object for sale the ending price converges in probability to the true value of the object. Milgrom (1979) extends this result to the auctions with arbitrary number of objects available for sale. Both results rely on the assumption that some bidders have arbitrarily precise signals about the true state of the world. While Wilson (1977) and Milgrom (1979) provide the asymptotic properties of common value auctions, Milgrom and Weber (1982) characterize the equilibrium bidding strategies in second-price sealed bid and English auctions with n participants and only one object sold. The main result is that monotone (pure strategy) equilibrium exists. This result is further extended by Pesendorfer and Swinkels (1997) to the k-unit uniform-price and first price auctions. Moreover, Pesendorfer and Swinkels find that the bidding function in both auctions is unique and monotone. But the most important result is that, under several assumptions in the k-unit uniform-price auction, the ending price of the object can fully aggregate the valuable information even if the bidders do not have arbitrarily precise signals. This result indicates that an auction has the desirable feature of predicting, or, rather, describing the current state of the world (as is in the case of perfect competition). Later, Kremer (2002) shows that different types of common value auctions, such as second-price, first-price, or English auctions can aggregate information about the current state of the economy. The properties of large common value auctions is further analyzed in different settings[2].

An important question to ask is how the results of the above research respond to slight modifications of the standard settings that are utilized in almost all the above papers. Atakan and Ekmekci (in press) demonstrate that a slight modification of the settings used in Pesendorfer and Swinkels (1997) (addition of ex-post investment decisions) could lead to a failure of information aggregation even in large sample common value auctions. Hence, an introduction of an additional uncertainty that modifies the behavior of the value function implies a violation of the results on information aggregation in large auctions.

Other researchers analyze the validity of the results drawn by Pesendorfer and Swinkels (1997), Milgrom and Weber (1982) and Kremer (2002) when the standard settings on signals and auction price formation do not hold. Riley (1988) draws inference on the efficiency of the sealed bid auctions in the setting where the price is a weighted average of all the losing bids. He shows that such a price formation improves efficiency of the auctions. In contrast, Goeree and Offerman (2003) characterize the equilibrium bidding behavior in the auctions where objects have both private and common component. They conclude that a greater uncertainty about the common value reduces efficiency of an auction.

While there exists extensive literature on large common value auctions of various types, the literature on small auction settings is not so broad[3]. Several papers on small auctions include Banerjee (2005) and Wang (1991). Banerjee (2005) analyzes a two-player first-price auction with asymmetric bidder information. He characterizes the equilibrium bidding strategies in the case where one player is more informed than the other. The inference on the efficiency of extra public information is not as straightforward as in the standard settings that are present in Milgrom and Weber (1982): it is uncertain if revealing public information would increase efficiency of the auction when the bidders are

[2]For the results on Double auctions, see Cripps and Swinkels (2006). Hong and Shum (2004) analyze the changes in the speed of convergence in k-unit uniform-price auctions under various assumptions. In contrast, Jackson and Kremer (2007) draw an inference on the informational inefficiency of large discriminatory auctions. De Castro and Karney (2012) provide an extensive summary of the results on equilibrium existence and its characterization under different auction settings and assumptions.

[3]I should note, though, that a few papers that analyze the asymptotic properties of large auctions first summarize the equilibrium behavior in small samples. Pesendorfer and Swinkels (1997) and Atakan and Ekmekci (in press) are some examples of such papers.

asymmetrically informed.

Similar to Banerjee (2005), Wang (1991) analyzes a first-price common value auction. However, Wang assumes discrete signals and extends the results to the large sample case. The important results are that the players with different signals use mixed strategies with non-overlapping support. In addition, as the number of bidders increase, the information aggregation can happen only partially: the auction price still fails to perfectly inform the true state of the world.

A yet greater limitation exists on the literature of non-monotone equilibria of the games that my analysis is closely related to. In most cases, the settings directly imply non-existence of non-monotone equilibria or the authors impose several assumptions to avoid complexities in characterization of non-monotone equilibria. For example, the multidimensionality of the values that Goeree and Offerman (2003) impose allows for non-monotone bidding strategy because of non-monotonicity of the summary statistic ordering. To avoid such complication, the authors impose an assumption and thus restrict their analysis to the monotone equilibria. Yet, such tactic is not employed by all the researchers. For example, in their analysis on the existence of monotone equilibria in asymmetric first-price auctions, Reny and Zamir (2004) demonstrate that a non-monotone bidding strategy may be the unique equilibrium bidding strategy if the players have multidimensional signals. Similarly, McAdams (2007) analyzes a simple model of small uniform-price auctions that exhibits only non-monotone equilibria. The author provides two examples of simple small auctions (where players have multi-unit demands) where the equilibrium bidding strategy of at least one player is non-monotone.

3 Model

Consider a sealed-bid uniform-price common value auction. There are three agents who have unit demands and who compete over two objects. There are two states of the world drawn from a discrete set $\Omega \equiv \{R, L\}$ with a generic element given as ω. A prior probability that the state is R is $\pi \in [0,1]$. Prior to the auction, each agent receives a signal that is drawn from a set $S \equiv [0,1]$. I call the signal "high" if $s > \frac{1}{2}$ and "low" otherwise. The probability of receiving a "high" signal (i.e. $s > \frac{1}{2}$) when the state is R is p (i.e. $\Pr(h|R) = p$) whereas the probability of receiving the same signal in state L is q. Hence, in state R the signals are distributed by a function $f(s|R) = 2(1-p)$ if $s \leq \frac{1}{2}$ and $f(s|R) = 2p$ otherwise. The density function conditional on state L is defined symmetrically. Each agent observes only her own signal. In the following assumption, I impose a condition on the distribution of the signals.

Assumption 1 (MLRP)**.** *The distribution of signals satisfies MLRP:*

$$\frac{p}{q} > \frac{1-p}{1-q} \Rightarrow p > q$$

Suppose that an agent i believes that the state is R with some probability z. Then, I say that the likelihood ratio of state R relative to state L is defined as $\rho \equiv \frac{z}{1-z}$. In a similar fashion, I define the likelihood ratio of an agent receiving a signal s, denoted as $\rho(s)$, as:

$$\rho(s) = \frac{\Pr(R)f(s|R)}{\Pr(L)f(s|L)}$$

In the remaining of the paper, I will make all the calculations of expected profits and payoffs using likelihood ratios instead of directly using probabilities because this approach simplifies the calculations.

After receiving a signal, each agent submits a bid $b_i \in [0, \infty)$. The agent wins an object if she submits at least the second highest bid. In case a tie occurs, a tie-breaking rule assigns an object to tied agents with equal probability. Each winner pays a price that is equal to the highest losing bid.

Conditional on winning an object, each winner chooses an action a from a discrete set A. For simplicity purposes, I assume that the set A consists of two alternatives - l and r. I construct the model such that the valuation of the object is both state- and action-dependent; i.e. the valuation of the object is given as $v(a, \omega)$. Action r is better in state R whereas action l gives greater payoff when the state is L. Without loss of generality, I assume that $v(r, R) \geq v(l, L)$ and make the following assumption on the shape of valuation function:

Assumption 2 (VAL)**.** *The valuation of the object satisfies the following relation:*

$$v(l, L) > v(r, L) = v(l, R) = 0$$

As the agent chooses her action after seeing her own signal, own bid and the price of the auction, the action strategy is a map $a : S \times [0, \infty) \times [0, \infty) \to A$.

Using the beliefs of an agent denoted as ρ, I define the value of an object for this agent as a function $u : [0, \infty) \to \mathbb{R}$ that is given as:

$$u(\rho) = \max_{a \in \{l,r\}} \left\{ \frac{1}{\rho + 1} v(a, L) + \frac{\rho}{\rho + 1} v(a, R) \right\}$$

A map a that gives the optimal choice of action for the individual will maximize the expected payoff of the agent conditional on her beliefs ρ. Note that $u(0) = v(l, L)$ while $\lim_{\rho \to +\infty} u(\rho) = v(r, R)$. The assumption that $v(l, L) \leq v(r, R)$ and the fact that $u(\cdot)$ is continuous in ρ implies that exists a unique $\rho^* \in [0, \infty)$ such that it solves the following equation:

$$\frac{1}{\rho + 1} v(l, L) = \frac{\rho}{\rho + 1} v(r, R)$$

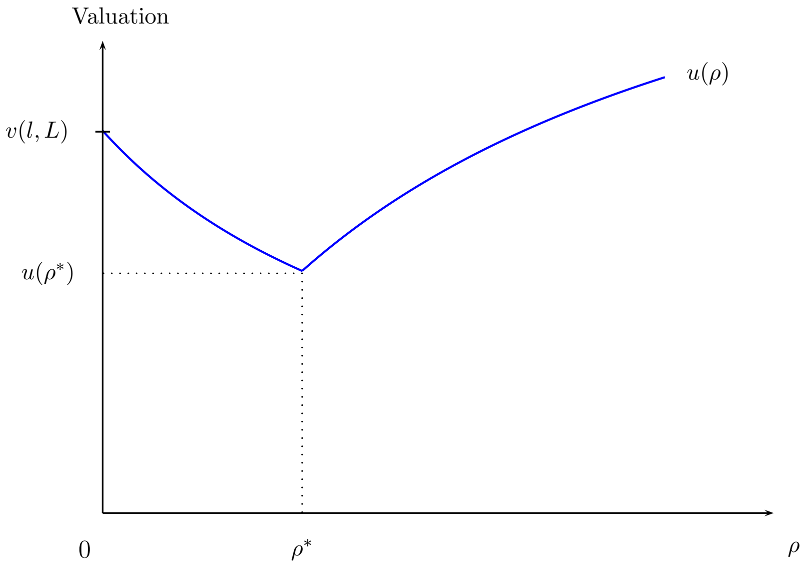

Figure 1: This figure depicts the behaviour of the value function across different beliefs. The value function is not monotone in signals and is strictly decreasing on the $[0, \rho^*]$ region; it is strictly increasing otherwise.

It is easy to see that $\rho^* = \frac{v(l,L)}{v(r,R)}$. The cutoff ρ^* is the belief that makes an agent indifferent between action r and l and thus determines the conditions for the optimal choice of action. Throughout my analysis, I will utilize the fact that the function $u(\cdot)$ is strictly decreasing on the interval $[0, \rho^*]$ and strictly increasing on $[\rho^*, \infty)$. Figure 1 depicts the shape of the value function.

An equilibrium of the model consists of the bidding strategy H_i and an action strategy a_i. A bidding strategy H_i is a measure on $S \times [0, \infty)$. An equilibrium bidding strategy is pure if there exists a function $b : S \to \mathbb{R}_+$ such that $H(\{s, b(s)\}_{s \in [0,1]}) = 1$. Description of the bidding strategy with the action strategy for each player will comprise the equilibrium of the described model.

4 Results

4.1 Monotone Equilibria

4.1.1 Behavior of Monotone Equilibria

This section summarizes the main results on the equilibrium existence and the characterization of the bidding function in different equilibria. My focus on the symmetric equilibria allows me to consider the strategy of one person. Without loss of generality, consider player 1 who receives a signal $s \in [0, 1]$. Let Y_2 denote the second highest signal of the remaining two players.

Before presenting the main results, I define one class of bidding strategies and equilibria that will be extensively used in the remaining of the paper.

Definition 1. *A bidding strategy, denoted by $b^{PS}(s)$, is said to be* **Pesendorfer-Swinkels**

(PS) bidding strategy *if it is given as:*

$$b(s) = E(v|s_1 = s, Y_2 = s)$$

Definition 2. *An equilibrium is said to be* **monotone** *if the equilibrium bidding strategy $b(s)$ is non-decreasing in signals. It is* **strictly monotone** *equilibrium if $b(s)$ is strictly increasing in s.*

With these two definitions, I can now present the first result on the general shape of any monotone equilibrium of the model.

Proposition 1. *Suppose that assumptions MLRP and VAL hold. Then, in any monotone equilibrium, there is a cutoff signal $0 \leq s^c \leq \frac{1}{2}$ such that the equilibrium bidding strategy is*

$$b(s) = \begin{cases} b^p < b^{PS}(s^c) & \forall s \in [0, s^c] \\ b^{PS}(s) & \forall s \in [s^c, 1] \end{cases}$$

Proof. See Appendix □

The following corollary characterizes the equilibrium behavior of the players with "high" signal.

Corollary 1. *In any monotone equilibrium, $b^{PS}(s) = \frac{\pi p^3}{\pi p^3 + (1-\pi)q^3} v(r, R)\ \forall s \in (\frac{1}{2}, 1]$*

Proof. See Appendix □

Notice that corollary 1 implies that a strictly monotone equilibrium is not possible under our settings. Even if the pooling does not occur around any signal $s \leq \frac{1}{2}$, the bidding strategy is constant in the above portion of the bidding strategy. That is, in any monotone equilibrium, the players with high signal bid the same bid. However, this kind of bidding is not a "proper" pooling behavior because the high signal players still behave based on the information coming from their private signal. This phenomenon is not general and is a result of the model's setting: the fact that the price is equal to the highest losing bid implies that whenever a player with high signal wins at his own bid, he is certain that the other two players received high signal. Though not general for any object-to-bidder ratio $\frac{k}{n}$, it is nevertheless holding for any k and n combination that satisfies $n = k + 1$.

An important result that proposition 1 demonstrates is that the pooling region cannot contain qualitatively different signals; that is, in any monotone equilibrium, the players with high signals will never want to pool if the players with low signals pool. In fact, the result is more general: the players with (qualitatively) different signals will never pool to the same bid in *any* equilibrium, be it monotone or not. The reason for such a result is that we can never find the pooling bid that is consistent with the individual rationality of the bidders with qualitatively different signals at the same time. An intuitive explanation for the incompatibility of the pooling behavior is as follows: if the players with qualitatively different signals pool to the same bid, then winning at the pooling price does not reveal much information because it may contain information favouring state R or state L or both. This information is not strong enough to offset the information that comes from players' private signals. In contrast, if only bidders with qualitatively same signals decide to pool, winning at the pooling bid reveals a strong signal towards only one state of the world thus making the pooling behavior optimal. The formal proof of generalization of this result to any non-monotone equilibrium will be presented as the proof of Corollary 3.

4.1.2 Characterization of Monotone Equilibria

Once I know the general behavior of the monotone bidding equilibrium, I can characterize the conditions under which such a monotone equilibrium exists. To do so, suppose that there exists a cutoff signal $0 \leq s^c \leq \frac{1}{2}$ such that the players with signal $s \leq s^c$ choose the pooling bid b^p and player with signals $s > s^c$ bid according to the PS bidding strategy $b^{PS}(s)$.

The three critical cases that affects the existence of monotone equilibria are: winning at the own bid, winning at the pooling bid (after bidding something higher), and winning after bidding the pooling bid. The beliefs conditional on these three cases will determine if the pooling behavior is compatible with the players' individual rationality. Hence, to characterize the conditions for the equilibrium existence, I need to define the players' beliefs on the state of the world when these three cases occur.

Consider player 1 who received signal s. Suppose that the player wins and the ending price is equal to his own bid. This can happen only if the second highest bid of the remaining two bids was equal to his bid. That is, the player with third highest signal received the same signal as player 1. Since we have a total of three players, this, in turn, implies that the other player receive a signal higher than s. Hence the underlying beliefs of the first player are:

$$\rho(s) = \frac{\pi(f(s|R))^2(1 - F(s|R))}{(1 - \pi)(f(s|L))^2(1 - F(s|L))}$$

Utilizing the definition of the c.d.f. $F(\cdot)$, I can rewrite the beliefs as:

$$\rho(s) = \begin{cases} \frac{\pi(1-p)^2(1-2(1-p)s)}{(1-\pi)(1-q)^2(1-2(1-q)s)} & \text{if } s \leq \frac{1}{2} \\ \frac{\pi p^3}{(1-\pi)q^3} & \text{if } s > \frac{1}{2} \end{cases}$$

Since the beliefs do not depend on signal when $s > \frac{1}{2}$ each player with high signal will choose to bid the same bid. This result was already demonstrated in corollary 1.

I can similarly define the beliefs of a person conditional on winning an object by bidding the pooling bid. Let $s^c \leq \frac{1}{2}$ be the cutoff signal such that the bidders with signals $s \leq s^c$ pool at b^p. Then, conditional on bidding b^p and winning the object, beliefs of the bidder with signal s are:

$$\begin{aligned}\rho^{pool}(s, s^c) &= \frac{\pi f(s|R)(\frac{2}{3}(F(s^c|R))^2 + F(s^c|R)(1 - F(s^c|R))}{(1-\pi)f(s|L)(\frac{2}{3}(F(s^c|L))^2 + F(s^c|L)(1 - F(s^c|L))} \\ &= \frac{\pi f(s|R)F(s^c|R)(3 - F(s^c|R))}{(1-\pi)f(s|L)F(s^c|L)(3 - F(s^c|L))}\end{aligned}$$

Bidding anything above the pooling bid and winning at the pooling price yields the beliefs of:

$$\begin{aligned}\rho(s, s^c) &= \frac{\pi f(s|R)((F(s^c|R))^2 + 2F(s^c|R)(1 - F(s^c|R))}{(1-\pi)f(s|L)((F(s^c|L))^2 + 2F(s^c|L)(1 - F(s^c|L))} \\ &= \frac{\pi f(s|R)F(s^c|R)(2 - F(s^c|R))}{(1-\pi)f(s|L)F(s^c|L)(2 - F(s^c|L))}\end{aligned}$$

By proposition 1, the cutoff signal must satisfy $s^c \leq \frac{1}{2}$. Hence, $F(s^c|R) = 2(1-p)s^c$. I can thus write the two beliefs as:

$$\begin{aligned}\rho^{pool}(s, s^c) &= \frac{\pi f(s|R)(1-p)(3 - 2(1-p)s^c)}{(1-\pi)f(s|L)(1-q)(3 - 2(1-q)s^c)} \\ \rho(s, s^c) &= \frac{\pi f(s|R)(1-p)(2 - 2(1-p)s^c)}{(1-\pi)f(s|L)(1-q)(2 - 2(1-q)s^c)}\end{aligned}$$

Several features of the belief functions should be noted. First, the beliefs $\rho^{pool}(s \leq \frac{1}{2}, 0)$ and $\rho(s \leq \frac{1}{2}, 0)$ coincide with the beliefs of a person with signal 0 conditional on winning the object at his own bid (i.e. $\rho^{pool}(s \leq \frac{1}{2}, 0) = \rho(s \leq \frac{1}{2}, 0) = \rho(0)$). Second, $\rho^{pool}(s, s^c) < \rho(s, s^c)$ for any $s^c > 0$. Last, both belief functions are increasing in s^c. These three observations will be helpful in characterizing the conditions for the existence of the monotone equilibrium of the model.

The following proposition characterizes the conditions under which a *unique* monotone equilibrium is possible.

Proposition 2. *Suppose that assumptions MLRP and VAL hold. If* $\rho(0) = \frac{\pi(1-p)^2}{(1-\pi)(1-q)^2} \geq \rho^*$ *also holds, then there exists a unique monotone equilibrium where the players with signals* $s \in [0, 1]$ *use the PS bidding strategy.*

Proof. See Appendix □

Condition $\rho(0) = \frac{\pi(1-p)^2}{(1-\pi)(1-q)^2} \geq \rho^*$ implies that every player, irrespective of his signal, believes that R is more likely. Hence, the model is reduced to the model without actions. Therefore, all the results presented by Pesendorfer and Swinkels (1997) come though in my case, as well. In particular, none of the players will want to bid anything lower than their expected value conditional on their bid being pivotal. The bidding strategy is therefore strictly increasing in the region $[0, \frac{1}{2}]$ because the beliefs $\rho(s)$ are increasing there. Note, however, that the bidding strategy is still only weakly monotone: the players with high signals still bid the same value (as was proposed in corollary 1). Yet, this phenomenon occurs only because of the fact that the price of the auction is equal to the lowest bid in the auction.

When $\rho(0) \geq \rho^*$ is not satisfied, the equilibrium bidding strategy cannot be strictly increasing around 0. To see this, notice that $\rho(0) < \rho^*$ implies that the value function is decreasing in signal in the neighbourhood of zero. Individual rationality of players implies that the bid cannot exceed the expected value conditional on being pivotal (as otherwise the profits would be negative). A strictly increasing bidding function would then imply that $b(0) < u(\rho(0))$. Hence, the player with signal 0 always wants to deviate to avoid losing at his own bid. Thus, some players with (quantitatively) different signals must pool in any monotone equilibrium.

Pooling in the bidding function requires extra conditions on the parameters of the model so that this bidding function is the equilibrium bidding function. Hence, to find the necessary and sufficient conditions, I need to determine the lowest value of the pooling bid b^p that is compatible with the pooling behavior. Note that, unless $u(\rho^{pool}(s \leq \frac{1}{2}, s^c)) \geq u(\rho(s \leq \frac{1}{2}, s^c))$ for some $s^c \leq \frac{1}{2}$, the pooling behavior is not sustainable. This is so because violation of the above inequality implies that expected payoff is higher if the player decides to bid above b^p in addition to higher chances of winning the object at the pooling price. Our condition that $\rho(0) < \rho^*$, however, implies that there exists such cutoff signal s^c. This result stems from two facts that $\rho^{pool}(s \leq \frac{1}{2}, 0) = \rho(s \leq \frac{1}{2}, 0) = \rho(0) < \rho^*$ and $\rho^{pool}(s \leq \frac{1}{2}, s^c) < \rho(s \leq \frac{1}{2}, s^c)$ for any $s^c \in B_\epsilon(0)$. Hence, $u(\rho^{pool}(s \leq \frac{1}{2}, s^c)) \geq u(\rho(s \leq \frac{1}{2}, s^c))$ is satisfied for any cutoff signal s^c in the neighbourhood of 0.

The fact that $u(\rho^{pool}(s \leq \frac{1}{2}, s^c)) \geq u(\rho(s \leq \frac{1}{2}, s^c))$ is satisfied is not, however, sufficient for the existence of the equilibrium. The bidding function can fail in equilibrium if the pooling bid b^p that is compatible with the pooling behavior of the players with low signal is not low enough to satisfy $b^p \leq b^{PS}(s)$. This can only occur when $\rho(s \leq \frac{1}{2}, s^c) < \rho^*$ for all cutoff signals $s^c \leq \frac{1}{2}$.

If $\rho(s \leq \frac{1}{2}, \frac{1}{2}) < \rho^*$, then $u(\rho^{pool}(s \leq \frac{1}{2}, s^c)) \geq u(\rho(s \leq \frac{1}{2}, s^c))$ is automatically satisfied for all cutoff signals that are less than $\frac{1}{2}$. Therefore, given any cutoff signal s^c, I can calculate a corresponding lower bound for the pooling bid that is compatible with the pooling behavior of the players with low signals. Let $b^p(s^c)$ be such bid. It can be written as:

$$b^p(s^c) = \frac{1}{\tilde{\rho}(s^c) + 1} v(l, L) \qquad \text{where} \quad \tilde{\rho}(s^c) = \frac{\pi(1-p)^2(3 - 4(1-p)s^c)}{(1-\pi)(1-q)^2(3 - 4(1-q)s^c)}$$

A method of calculation of this lower bound is described in the proof of proposition 1 that is given in appendix. For the cutoff signals s^c that satisfy $\rho(s \leq \frac{1}{2}, s^c) < \rho^*$, this

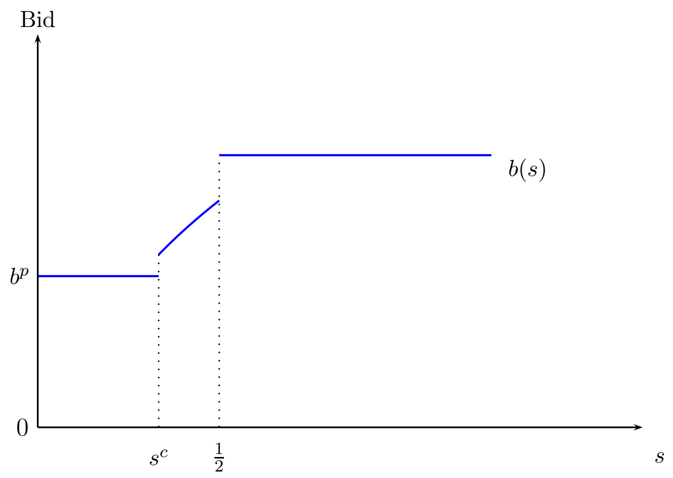

Figure 2: A typical (weakly) monotone equilibrium bidding function where pooling occurs. Bidders with signals below a cutoff s^c bid a pooling bid b^p, and those with signals above s^c use the PS bidding strategy.

lower bound is the pooling bid that makes the bidders with low signal indifferent between pooling and bidding anything above b^p and below $b^{PS}(s)$. The following proposition characterizes conditions for the existence of monotone equilibria.

Proposition 3. *Let assumptions MLRP and VAL hold.*

(a) If $u(\rho(s > \frac{1}{2}, \frac{1}{2})) < b^p(\frac{1}{2})$, then there is no monotone equilibrium of the model.

(b) If $u(\rho(s > \frac{1}{2}, \frac{1}{2})) \geq b^p(\frac{1}{2}) \geq u(\rho(s = \frac{1}{2}))$, then there exists a class of monotone equilibria where all players with signal $s \leq \frac{1}{2}$ choose the pooling bid b^p whereas players with signal $s > \frac{1}{2}$ bid $b^{PS}(s) > b^p$.

(c) If $u(\rho(\frac{1}{2})) > b^p(\frac{1}{2})$ and $\rho(0) < \rho^*$, then there exists a class of monotone equilibria where players bid according to the following strategy:

$$b(s) = \begin{cases} b^p < b(s^c) & \forall s \leq s^c < \frac{1}{2} \\ b^{PS}(s) & \forall s > s^c \end{cases}$$

Figure 2 depicts the shape of a typical bidding strategy with the pooling behavior. The shape of the bidding strategy determines the optimal choice of action conditional on winning at various prices. The players who bid anything higher than b^p and win the object at any price $p > b^p$ choose action r. Similarly, the bidders who pool and win the object choose action l conditional on winning the object. This kind of choice of action is quite intuitive because winning after pooling reveals an informative signal towards state L. The optimal choice of action after bidding any bid $b > b^p$ and winning at the pooling price is not so straightforward and depends on the parameters of the model. The multiplicity of optimal choice of action in such a case stems from the fact that winning at the pooling price reveals information in favour of state L, but the information may be so small that it fails to offset the information coming from the player's own signal. Hence a possibility of both actions to be optimal.

Another peculiar point is, if the cutoff signal s^c is strictly less than $\frac{1}{2}$, all bidders with low signal must be indifferent between pooling and bidding something above. This result stems from the fact that, though being quantitatively different, all signals that are lower than $\frac{1}{2}$ reveal exactly the same information about the true state of the world. Thus, if pooling is more profitable for the bidders with signal $s \leq s^c$, discreteness of the signals implies that pooling is more profitable for the bidders with signal $s \in (s^c, \frac{1}{2}]$, as well.

4.2 Non-monotone Equilibria

I now characterize all purely non-monotone equilibria - equilibria where the bidding strategy cannot be converted to a monotone bidding strategy by re-ordering of the signals. I first check if the monotone equilibrium with pooling can be converted to non-monotone by shifting the pooling region away from zero signal. I then move to characterizing the equilibrium where high signal bidders pool.

4.2.1 Non-monotone Equilibria with Pooling: Pooling by Low Signal Players

Suppose that players with signal $s \in [0, s^*)$ use a bidding strategy $b(s)$ that decreases as s increases. In addition, suppose players with signal $[s^*, \frac{1}{2}]$ bid the pooling bid b^p such that $b^p < b(s)$ for all $s < \underline{s}$ and that players with signal $s > \frac{1}{2}$ choose $b^{PS}(s) > b(0)$.

Like in the monotone equilibrium case, the bidders' beliefs conditional on winning at certain prices will affect the bidding behavior in equilibrium. The three critical beliefs are the beliefs conditional on winning at own bid, and the the beliefs conditional on winning at the pooling price.

Consider a bidder who bid above the pooling bid b^p. He wins the object at his own bid if one of the remaining bidders bid the same bid and the other bids something above. Hence, the beliefs can be written as:

$$\rho_N(s) = \begin{cases} \frac{\pi(1-p)^2(p+2(1-p)s)}{(1-\pi)(1-q)^2(q+2(1-q)s)} & s \in [0, \bar{s}) \\ \frac{\pi p^3}{(1-\pi)q^3} & s > \frac{1}{2} \end{cases}$$

Similarly, if the bidder with signal s decides to pool, his beliefs conditional on winning become:

$$\rho_N^{pool}(s, s^*) = \frac{\pi f(s|R)(1-p)(2+p+2(1-p)s^*)}{(1-\pi)f(s|L)(1-q)(2+q+2(1-q)s^*)}$$

Last, winning at the pooling bid after bidding any bid $b > b^p$ yields the beliefs of:

$$\rho_N(s, s^*) = \frac{\pi f(s|R)(1-p)(1+p+2(1-p)s^*)}{(1-\pi)f(s|L)(1-q)(1+q+2(1-q)s^*)}$$

Observe that $\rho_N(s)$ decreases in s under MLRP assumption. Hence, a (non-monotone) bidding strategy that is decreasing on $[0, s^*)$ is not possible unless $\rho_N(s) > \rho^*$. Also, note that the beliefs $\rho_N(s, s^*)$ and $\rho_N^{pool}(s, s^*)$ decrease as the cutoff signal s^* increases (i.e. as the pooling region widens), and that $\rho_N(s, s^*) \geq \rho_N^{pool}(s, s^*)$ for all $s^* \leq \frac{1}{2}$ with the inequality holding strictly if $s^* < \frac{1}{2}$.

Like in monotone equilibrium case, a player with low signal is willing to pool only if $u(\rho_N^{pool}(s \leq \frac{1}{2}, s^*)) \geq u(\rho_N(s \leq \frac{1}{2}, s^*))$ holds for some $s^* \in [0, \frac{1}{2}]$. This inequality, in turn, holds only if $\rho_N^{pool}(s \leq \frac{1}{2}, \frac{1}{2}) < \rho^*$.

Suppose that $\rho_N^{pool}(s \leq \frac{1}{2}, \frac{1}{2}) < \rho^*$ holds. Then, the behavior of the likelihood ratios $\rho_N^{pool}(\cdot)$ and $\rho_N(\cdot)$ implies that I can always find the cutoff signal s^* and a corresponding pooling bid $b_N^p(s^*)$ such that any player with signal $s \in [s^*, \frac{1}{2}]$ is indifferent between pooling and deviating slightly above the pooling bid. Such pooling bid takes the form

$$b_N^p(s^*) = \frac{1}{\hat{\rho}(s^*) + 1} v(l, L) \qquad \text{where} \qquad \hat{\rho}(s^*) = \frac{\pi(1-p)^2(1+2p+4(1-p)s^*)}{(1-\pi)(1-q)^2(1+2q+4(1-q)s^*)}$$

The method of calculation the lower bound is exactly same as the method used in the monotone bidding strategy case.

Even though some players with low signal prefer to pool, the non-monotone bidding strategy of interest may still fail to be an equilibrium bidding strategy. This can happen when the pooling bid is too high to be less than the bid by a person with signal s^*. Since the lower bound $b^p(s^*)$ is increasing in s^* and the value $u(\rho_N(\cdot))$ takes its highest value around zero, the relation between these two functions around 0 will determine the conditions that are necessary for the equilibrium existence. That is, to have a purely non-monotone equilibrium, at least the bidders with signals around zero should bid something higher than the pooling bid. The next proposition summarizes the results.

Proposition 4. *Suppose that assumptions MLRP and VAL hold. If* $\rho_N^{pool}(s \leq \frac{1}{2}, \frac{1}{2}) < \rho^*$ *and* $u(\rho_N(0)) > b_N^p(0)$ *also hold, then there exists a cutoff signals* $0 < s^* < \frac{1}{2}$ *such that the equilibrium bidding strategy is:*

$$b(s) = \begin{cases} b^p & \forall s \in [s^*, \frac{1}{2}] \\ u(\rho_N(s)) & \textit{otherwise} \end{cases}$$

Notice that the condition that $\rho_N^{pool}(s \leq \frac{1}{2}, \frac{1}{2}) < \rho^*$ corresponds to the condition $\rho(0) < \rho^*$ and that $u(\rho_N(0)) > b_N^p(0)$ correspond to the monotone equilibrium condition $u(\rho(\frac{1}{2})) > b^p(\frac{1}{2})$. In fact, there is a linear relation between the cutoff signals s^c and s^* that can be represented as $s^c = \frac{1}{2} - s^*$. This implies that without changing our conditions, we can shift the pooling region from 0 to $\frac{1}{2}$. Since the conditions for the existence did not change, the optimal choice of action also stays unchanged.

The following corollary demonstrates that such a non-monotone equilibrium is not unique: if we preserve the width of the pooling region, then the pooling region can be shifted to any place provided that the upper bound of the pooling region does not exceed $\frac{1}{2}$.

Corollary 2. *Let assumptions MLRP and VAL hold. Assume also that* $u(\rho_N(0)) > b_N^p(0)$. *Then, there exists an interval* $[\underline{s}, \bar{s}]$ *with* $\underline{s} > 0$ *and* $\bar{s} \leq \frac{1}{2}$ *such that, in equilibrium, players with signal* $s \in [\underline{s}, \bar{s}]$ *choose* b^p*, and others use a non-monotone bidding strategy* $b(s)$ *such that* $b(s) > b^p$ *for all* $s \notin [\underline{s}, \bar{s}]$

Proof. See Appendix. □

In a typical equilibrium where the bidders with low signal pool, winning at the pooling bid must bear a strong signals towards state L. This is indeed the case as all the bidders who win after bidding b^p choose action l. In addition, winning at any other price is an informative signal towards state R. Hence, the optimality of action r conditional on winning at any price $p > b^p$. Winning at the pooling price after bidding any bid above b^p still reveals information favouring state L, but the information hidden in price may be too weak to offset the information that is coming from the private signal. The high signal is too informative of the state R and hence the bidders with high signal choose action r conditional on winning at the pooling price. In contrast, the bidders with low signal may find both action r or l to be optimal conditional on bidding $b > b^p$ and winning at the pooling price.

Like in the case of monotone equilibria, the pooling region can be shifted only in such a way that it does not contain qualitatively different signals. That is, I still cannot make the bidder with signal $s > \frac{1}{2}$ pool at b^p if some bidder with signal $s \leq \frac{1}{2}$ chooses to bid it. The next corollary summarizes the results.

Corollary 3. *In any non-monotone equilibrium with the pooling interval* $[\underline{s}, \bar{s}]$*, if* $\underline{s} < \frac{1}{2}$*, then* $\bar{s}$ *cannot exceed* $\frac{1}{2}$.

Proof. See Appendix. □

Figure 3 depicts the shape of a typical non-monotone equilibrium bidding function when low signal players decide to pool.

The result of Corollary 3 is quite intuitive. Notice that a bidder who bids b^p (say, bidder 1) can win the object only when at least one more bidder chose b^p. If the pooling region contains only low signals, then bidder 1 is certain that the other bidder who bid b^p received low signal. Knowing the quality of the other bidder's signal reveals strong enough information to offset (or reinforce) the information that is coming from the private signal. In contrast, when the pooling region contains qualitatively different signals, the bidder who bid b^p is not totally certain about the quality of the other pooler's signal: bidder 1 only knows that the signal of other pooler is high with some probability that is strictly less than 1. Hence, the information that is coming from winning after bidding b^p is not strong enough for the bidders with qualitatively different signals to disregard their own private signals and pool at some bid b^p.

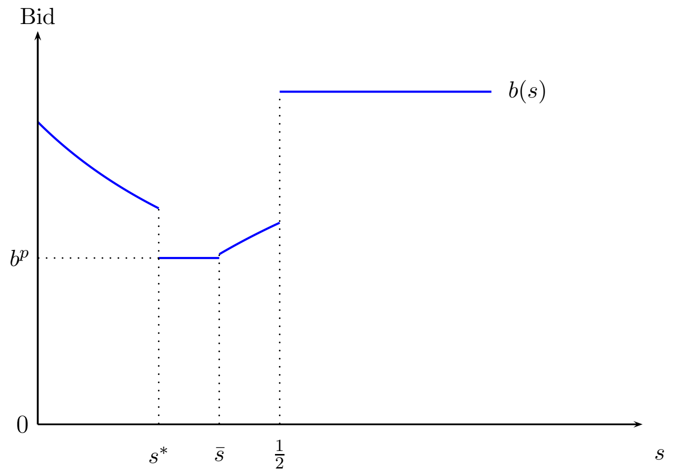

Figure 3: A typical non-monotone equilibrium bidding function with low signal players pooling.

4.2.2 Non-monotone Equilibria with Pooling: Pooling by High Signal Players

Since qualitatively different signals cannot be in the pooling region at the same time, the only type of the non-monotone bidding strategy left to analyze is the bidding strategy where only players with high signals pool.

Consider a non-monotone bidding strategy that takes the following form: the players with signals $s \in (\frac{1}{2}, \bar{s}]$ bid b^p whereas the others bid $b(s) > b^p$; $b(s)$ is increasing on $(\bar{s}, 1]$ and is non-increasing on $[0, \frac{1}{2}]$ and satisfies $b(\frac{1}{2}) \geq b(1)$. Figure 4 depicts the bidding strategy.

Let $\rho_h^N(s)$ denote the beliefs of a player with signal s conditional on winning at his own price $b(s) > b^p$. These beliefs can be written as:

$$\rho_h^N(s) = \begin{cases} \frac{\pi(f(s|R))^2 F(s|R)}{(1-\pi)(f(s|L))^2 F(s|L)} & \text{if } s \leq \frac{1}{2} \\ \frac{\pi(f(s|R))^2(1-F(s|R)+F(\frac{1}{2}|R))}{(1-\pi)(f(s|L))^2(1-F(s|L)+F(\frac{1}{2}|L))} & \text{if } s > \bar{s} \end{cases}$$

Using the definition of $F(\cdot)$, I can re-write the beliefs as:

$$\rho_h^N(s) = \begin{cases} \frac{\pi(1-p)^3}{(1-\pi)(1-q)^3} & \text{if } s \leq \frac{1}{2} \\ \frac{\pi p^2(1-p(2s-1))}{(1-\pi)q^2(1-q(2s-1))} & \text{if } s > \bar{s} \end{cases}$$

Similarly, let $\rho_h^{pool}(s, \bar{s})$ denote the beliefs of winning the object after bidding the pooling bid and $\rho_h(s, \bar{s})$ denote the beliefs of winning at the pooling price after deviating slightly above the pooling bid. I can write these beliefs as:

$$\rho_h^{pool}(s, \bar{s}) = \frac{\pi f(s|R)p(3 - p(2\bar{s} - 1))}{(1 - \pi)f(s|L)q(3 - q(2\bar{s} - 1))}$$

$$\rho_h(s, \bar{s}) = \frac{\pi f(s|R)p(2 - p(2\bar{s} - 1))}{(1 - \pi)f(s|L)q(2 - q(2\bar{s} - 1))}$$

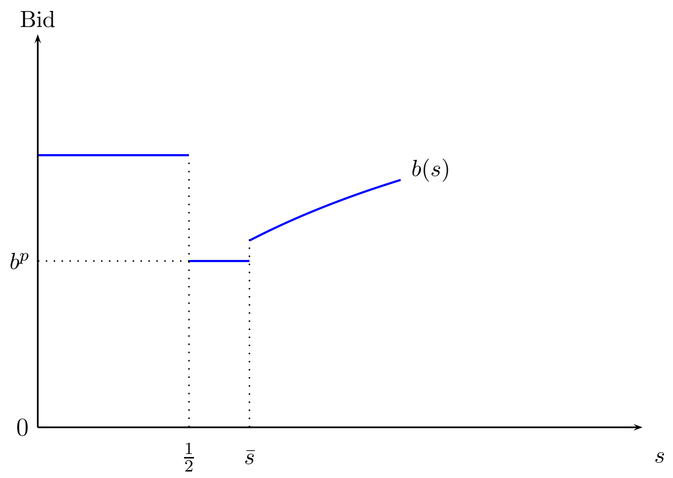

Figure 4: A typical non-monotone equilibrium bidding function with high signal players pooling.

MLRP assumption implies that $\rho_h(s, \bar{s}) < \rho_h^{pool}(s, \bar{s})$ for any cutoff signal $\bar{s} > \frac{1}{2}$ and that both beliefs decrease as the cutoff signal $\bar{s}$ increases. This implies that an equilibrium is not possible unless $\rho_h^{pool}(s > \frac{1}{2}, \bar{s}) > \rho^*$. MLRP also implies that $\rho_h^N(\bar{s}) < \rho_h(s > \frac{1}{2}, \bar{s})$. Note also that $\rho_h^N(s)$ decreases in s and hence, $\rho_h^N(\bar{s}) < \rho^*$ must hold in equilibrium to preserve non-monotonicity of the bidding strategy.

Since $\rho_h^{pool}(s > \frac{1}{2}, \bar{s})$ is decreasing in the cutoff signal $\bar{s}$, the belief's value in the neighbourhood of $\frac{1}{2}$ will determine if the pooling behavior in the non-monotone equilibrium is possible. Since the function is continuous in $\bar{s}$, its limit as $\bar{s}$ converges to $\frac{1}{2}$ is well defined. Let $\rho^+(\frac{1}{2})$ denote the function's limit point. It is easy to demonstrate that this limit point is

$$\rho^+(\frac{1}{2}) = \frac{\pi p^2}{(1-\pi)q^2}$$

The pooling behavior is not sustainable in equilibrium unless $\rho^+(\frac{1}{2}) > \rho^*$. If $\rho^+(\frac{1}{2}) > \rho^*$, then $\rho_h^{pool}(s > \frac{1}{2}, \bar{s}) > \rho^*$ for any $\bar{s} \in B_\epsilon(\frac{1}{2})$. The same holds for $\rho_h(s > \frac{1}{2}, \bar{s})$ because these beliefs converge to the same limit point as $\bar{s}$ approaches $\frac{1}{2}$. Hence, MLRP implies that $u(\rho_h(s > \frac{1}{2}, \bar{s})) < u(\rho_h^{pool}(s > \frac{1}{2}, \bar{s}))$ for any $\bar{s} \in B_\epsilon(\frac{1}{2})$. This, in turn, implies that, given any cutoff signal $\bar{s}$ in the neighbourhood of $\frac{1}{2}$, there exist a corresponding pooling bid $b^p(\bar{s})$ that makes the players with signal $s \in (\frac{1}{2}, \bar{s}]$ indifferent between pooling and deviating slightly above. This pooling bid can be written as:

$$b_h^p(\bar{s}) = \frac{\bar{\rho}(\bar{s})}{\bar{\rho}(\bar{s})+1} v(r, R) \text{ where } \bar{\rho}(\bar{s}) = \frac{\pi p^2(3-2p(2\bar{s}-1))}{(1-\pi)q^2(3-2q(2\bar{s}-1))}$$

This pooling bid is a lower bound for the pooling bid for any $\bar{s}$ such that $\rho_h(s > \frac{1}{2}, \bar{s}) > \rho^*$. To preserve pure non-monotonicity of the bidding strategy, the cutoff signal $\bar{s}$ must be strictly less than 1 (as otherwise we could re-order the signals and end up with the bidding strategy defined in part (b) of Proposition 3). The cutoff signal is less than one only if the bidders with signals 1 find it optimal to bid some bid $b(1) > b^p$. Hence the behavior of the bidder with signal 1 will determine if purely non-monotone equilibrium is possible. The following proposition summarizes the results.

Proposition 5. *Let assumptions MLRP and VAL hold. Assume also that* $u(\rho_h^N(1)) > b_h^p(1)$ *and* $\rho^+(\frac{1}{2}) > \rho^*$*. Then, there exists a cutoff signal* $\bar{s} \in (\frac{1}{2}, 1)$ *such that the equilibrium bidding strategy is:*

$$b(s) = \begin{cases} b^p & s \in (\frac{1}{2}, \bar{s}] \\ u(\rho_h^N(s)) \geq b^p & \textit{otherwise} \end{cases}$$

Proof. See Appendix. □

Condition $\rho^+(\frac{1}{2}) > \rho^*$ guarantees that the pooling behavior occurs in equilibrium whereas condition $u(\rho_h^N(1)) > b_h^p(1)$ guarantees that not all bidders with high signal pool: some bidders with high signal choose to bid above b^p in equilibrium. Intuitively, the winning after pooling reveals a strong signal in favour of state R (because only bidders with high signals pool and high signal is more probable in R); our condition $u(\rho_h^N(1)) > b_h^p(1)$ thus puts a limit on how informative pooling can be. If the pooling reveals a very strong information in favour of state R, then all the bidders with high signal will want to pool. Our condition $u(\rho_h^N(1)) > b_h^p(1)$ thus guarantees that some bidders with high signal find their signal more informative than the pooling bid.

Since the beliefs of the non-poolers must be less than ρ^* in equilibrium, each player who wins the object at any price other than the pooling bid must choose action l after winning the object. The players who pool choose action r conditional on winning the object. The optimal choice of action after bidding some bid $b > b^p$ and winning at the pooling price depends on the values of the cutoff signal $\bar{s}$. If the equilibrium cutoff signal is such that $\rho_h(s, \bar{s}) < \rho^*$, then the player chooses action l; otherwise, she chooses action r. Note that both cases are possible for different model parameters (i.e. π,p, q, and ρ^* combination) that satisfy the necessary conditions described in proposition 5.

The two types of the non-monotone equilibria are purely non-monotone in the sense that the equilibrium bidding strategies cannot be converted to the monotone strategies by re-ordering the players' signals. Also, the non-monotone equilibrium that is described in proposition 5 is an image of the bidding strategy given in proposition 4. To be more precise, if I flip the bidding strategy described in proposition 4 with respect to the $s = \frac{1}{2}$ point, I will get a function that behaves symmetrically as the bidding strategy in proposition 5. This implies that the results similar to those in corollary 2 are also present in the case when high signal players pool. The following corollary summarizes the results.

Corollary 4. *Let assumptions MLRP and VAL hold. Assume also that* $u(\rho_h^N(1)) > b_h^p(1)$ *and* $\rho^+(\frac{1}{2}) > \rho^*$*. Then, there exists an interval* $[\underline{s}, \bar{s}]$ *with* $\underline{s} > \frac{1}{2}$ *such that, in equilibrium, players with signal* $s \in [\underline{s}, \bar{s}]$ *choose* b^p*, and others use a non-monotone bidding strategy* $b(s)$ *such that* $b(s) > b^p$ *for all* $s \notin [\underline{s}, \bar{s}]$.

The proof of this corollary is similar to the proof of corollary 2 and stems from the fact that all beliefs stay the same if the width of the pooling region is preserved.

4.3 Comparison of Equilibria

The existence of different classes of monotone and non-monotone equilibria raises questions about their possibility to co-exist under various model parameters. That is, which equilibria can survive at the same time (for the same parameters) is an important question to ask. Such an inference is the most straightforward if I compare the beliefs that come from the conditions of the existence of various equilibria. A comparison of $\rho(0)$, $\rho(s > \frac{1}{2}, s^c = \frac{1}{2})$, $\tilde{\rho}(s^c = \frac{1}{2})$ (coming from low signal player pooling case), $\rho^+(\frac{1}{2})$, $\rho_h^N(1)$ and $\bar{\rho}(\bar{s} = 1)$ (coming from conditions of high signal player pooling equilibria) will demonstrate the relation between different classes of equilibria. The relation between these beliefs can be summarized as follows:

$$\rho^+(\frac{1}{2}) > \bar{\rho}(\bar{s} = 1) > \rho_h^N(1) > \rho(s > \frac{1}{2}, s^c = \frac{1}{2}) > \tilde{\rho}(s^c = \frac{1}{2}) > \rho(0)$$

The sustainability of a particular equilibrium bidding strategy depends on where ρ^* enters this sequence of inequalities.

Suppose that $\rho(0) > \rho^*$. Then, there exists a monotone equilibrium with bidding function that is strictly increasing around 0. Note that $\rho(0) > \rho^*$ immediately implies that $\rho_h^N(1) > \rho^*$ and hence $u(\rho_h^N(1)) > b_h^p(1)$ is never satisfied. That is the pooling bid is too high to make the person with signal 1 be willing to bid anything above the pooling bid.

Now, suppose that there exists a monotone equilibrium where the players with low signal pool. Since $\rho(s > \frac{1}{2}, s^c = \frac{1}{2}) > \tilde{\rho}(s^c = \frac{1}{2})$, a necessary condition for such an equilibrium is that $\rho(s > \frac{1}{2}, s^c = \frac{1}{2}) > \rho^*$ (as otherwise we cannot make the high signal players bid anything above the pooling bid). But then, $\rho_h^N(1) > \rho^*$ also holds implying that $u(\rho_h^N(1)) < b_h^p(1)$ for any parameters that are compatible with monotone equilibrium. Therefore, the non-monotone equilibrium with high signal players pooling cannot exist if a monotone equilibrium exists. Similarly, the two classes of non-monotone equilibria (with high signal and low signal players pooling) cannot exist at the same time. This implies that for some (π, p, q, ρ^*) combination that satisfies MLRP and VAL assumptions the only equilibrium of the model is the non-monotone equilibrium.

5 Conclusion

In this thesis, I analyze a small common value auction where there are three participants and two objects to be sold. I analyze a discrete signal two-unit uniform-price auction where each participant receives either high or low signal. I find all the monotone and non-monotone equilibria of the model. In particular, I derive the necessary and sufficient conditions for an existence of a monotone equilibrium. In addition, I demonstrate that if a monotone equilibrium exists, then a non-monotone equilibrium should also exist. The non-monotone equilibria that I derive are of two types: the one where the players with low signal players pool and the other with high signal players pooling. I demonstrate that the two types of non-monotone equilibria cannot co-exist at the same time leading to the conclusion that the model can have only non-monotone equilibria. In addition, I demonstrate that bidders with qualitatively different signals cannot pool at the same bid in any monotone or non-monotone equilibrium. Such a result occurs because, once the bidders with different signals pool, the information that is coming from the pooling bid is diluted and hence is not precise enough to offset the information that is coming from the bidder's private signal. Another peculiar result is that an existence of any equilibrium of the model implies that there are infinitely many equilibria. However, many equilibria require exactly same conditions for their existence and therefore cannot lead to qualitatively different results in terms of features of the auction.

A Appendix

Proof of Proposition 1.

Existence of the cutoff signal for any pooling equilibrium

Fix a monotone equilibrium. Suppose that the equilibrium bidding strategy $b(s)$ is strictly increasing around $s = 0$. Then conditional on winning (or losing) at his own bid, the player with signal $s \in B_\epsilon(0)$ has the beliefs of the form:

$$\rho(s) = \frac{\pi(f(s|R))^2(1 - F(s|R))}{(1 - \pi)(f(s|L))^2(1 - F(s|L))}$$

Since we consider signals $s \in B_\epsilon(0)$, the beliefs can be written as:

$$\rho(s) = \frac{\pi(1 - p)^2(1 - 2(1 - p)s)}{(1 - \pi)(1 - q)^2(1 - 2(1 - q)s)}$$

Note that $\rho(s)$ is increasing in s and that $\rho(0) = \frac{\pi(1-p)^2}{(1-\pi)(1-q)^2}$. If $\rho(0) > \rho^*$, the value function $u(\rho(s))$ increases in s and all players choose the same action. Hence, the setting are exactly same as in Pesendorfer and Swinkels (1997). There exists a unique monotone equilibrium where types bid using the PS type bidding function. But then, there is a cutoff signal $s^c = 0$ that satisfies our strategy description.

If $\rho(0) < \rho^*$, then the equilibrium bidding strategy cannot be strictly increasing around 0. To see that, assume for a contradiction that a monotone bidding strategy $b(s)$ is increasing around 0. We know that the beliefs of a player with signal s conditional on winning at his own bid increase in s; the fact that $\rho(0) < \rho^*$ implies that the value function $u(\rho(s))$ is decreasing in signal around $s = 0$. That is $u(\rho(0)) > u(\rho(s))$ for any $s \in B_\epsilon(0)$. Also, individual rationality implies that $u(\rho(s|s \in B_\epsilon(0))) \geq b(s)$ must hold (as otherwise the player receives negative profit and has an incentive to deviate to 0). Hence, conditional on losing at his own bid, the player with signal 0 has an expected value of $u(\rho(0)) > b(0)$. So, the player with signal 0 has a profitable deviation. This implies that if $\rho(0) < \rho^*$ holds, the players with signals around 0 must submit the same bid in any monotone equilibrium. This, in turn, implies that there is a cutoff signal s^c such that players with signal $s \in [0, s^c]$ choose the same bid b^p.

The cutoff signal s^c must be bounded away from 1. To see this, assume for a contradiction that $s^c = 1$ in a monotone equilibrium. But then, winning the auction does not reveal any information and hence posterior beliefs stay unchanged. MLRP assumption then implies that $u(\rho(s > \frac{1}{2})) \neq u(\rho(s \leq \frac{1}{2}))$ and so, no matter what the bid is, at least one player would have a profitable deviation from that bid. A contradiction. Thus, $s^c < 1$ in any monotone equilibrium.

A condition that s^c implies that there are players with signal $s > s^c$ who choose a bid that is different from b^p. Consider this player with signal $s > s^c$. If he chooses any bid $b < b^{PS}(s)$, then conditional on winning at his own bid, his expected profit is positive. Hence, this player has a profitable deviation to avoid a tie. This contradicts to our assumption of equilibrium behavior. Similarly, if $b > b^{PS}(s)$, then the player has a negative profit if he wins when his bid is pivotal and hence has a profitable deviation. Therefore, the only bid that a player with signal $s > s^c$ can submit in any monotone equilibrium is $b^{PS}(s)$.

Cutoff signal cannot exceed $\frac{1}{2}$

We now show that s^c cannot exceed $\frac{1}{2}$. Assume for a contradiction that there exists $s^c > \frac{1}{2}$ such that in a monotone equilibrium the players with signals in $[0, s^c]$ pool at b^p and the players with signals in $(s^c, 1]$ use the PS bidding strategy.

Consider a person with signal $s \leq s^c$. Conditional on winning after bidding b^p, his beliefs are:

$$\begin{aligned}
\rho^{pool}(s, s^c) &= \frac{\pi f(s|R)(\frac{2}{3}(F(s^c|R))^2 + F(s^c|R)(1 - F(s^c|R))}{(1-\pi) f(s|L)(\frac{2}{3}(F(s^c|L))^2 + F(s^c|L)(1 - F(s^c|L))} \\
&= \frac{\pi f(s|R) F(s^c|R)(3 - F(s^c|R))}{(1-\pi) f(s|L) F(s^c|L)(3 - F(s^c|L))}
\end{aligned}$$

If he bids slightly above the pooling bid, his beliefs are:

$$\begin{aligned}
\rho(s, s^c) &= \frac{\pi f(s|R)((F(s^c|R))^2 + 2F(s^c|R)(1 - F(s^c|R))}{(1-\pi) f(s|L)((F(s^c|L))^2 + 2F(s^c|L)(1 - F(s^c|L))} \\
&= \frac{\pi f(s|R) F(s^c|R)(2 - F(s^c|R))}{(1-\pi) f(s|L) F(s^c|L)(2 - F(s^c|L))}
\end{aligned}$$

Our assumption that $s^c > \frac{1}{2}$ implies that $F(s^c|R) = 1 - 2p(1 - s^c)$. We can thus re-write the beliefs as:

$$\rho^{pool}(s,s^c) = \frac{\pi f(s|R)(1-2p(1-s^c))(2+2p(1-s^c))}{(1-\pi)f(s|L)(1-2q(1-s^c))(2+2q(1-s^c))}$$
$$\rho(s,s^c) = \frac{\pi f(s|R)(1-2p(1-s^c))(1+2p(1-s^c))}{(1-\pi)f(s|L)(1-2q(1-s^c))(1+2q(1-s^c))}$$

If $\rho^{pool}(s,s^c) > \rho^*$, then $u(\rho^{pool}(s,s^c)) < u(\rho(s,s^c))$ which violates our assumption of equilibrium existence. So, $\rho^{pool}(s,s^c) < \rho^*$ must hold in an equilibrium where $s^c > \frac{1}{2}$. Individual rationality of a player with signal s^c implies that in equilibrium, $b^p \leq u(\rho^{pool}(s,s^c))$ must be satisfied.

In addition, a player with signal $s \leq \frac{1}{2}$ must have no incentive to deviate from b^p. To derive the conditions for which a player with low signal has no incentive to deviate, define the profit from bidding b^p as

$$\begin{aligned}\Pi(b^p,s) &= \Pr(L|s)\Pr(win, \text{bid } b^p, p=b^p|L)(v(l,L)-b^p)\\ &\quad + \Pr(R|s)\Pr(win, \text{bid } b^p, p=b^p|R)(v(l,R)-b^p)\\ &= \frac{1}{3}\Big((1-\pi)(1-q)(1-2q(1-s^c))(2+2q(1-s^c))(v(l,L)-b^p)\\ &\quad - \pi(1-p)(1-2p(1-s^c))(2+2p(1-s^c))b^p\Big)\frac{1}{\pi(1-p)+(1-\pi)(1-q)}\end{aligned}$$

Because $\rho^{pool}(s>\frac{1}{2},s^c) < \rho^*$ in equilibrium and because $\rho(s \leq \frac{1}{2}, s^c) < \rho^{pool}(s > \frac{1}{2}, s^c)$ under MLRP assumption, any player with signal $s \leq \frac{1}{2}$ chooses action l conditional on winning at the pooling bid even if he bids anything above b^p. Hence, we can write the profit from deviating slightly above b^p as

$$\begin{aligned}\Pi(b^p+\epsilon,s) &= \Pr(L|s)\Pr(win, \text{bid } b^p+\epsilon, p=b^p|L)(v(l,L)-b^p)\\ &\quad + \Pr(R|s)\Pr(win, \text{bid } b^p+\epsilon, p=b^p|R)(v(l,R)-b^p)\\ &= \Big((1-\pi)(1-q)(1-2q(1-s^c))(1+2q(1-s^c))(v(l,L)-b^p)\\ &\quad - \pi(1-p)(1-2p(1-s^c))(1+2p(1-s^c))b^p\Big)\frac{1}{\pi(1-p)+(1-\pi)(1-q)}\end{aligned}$$

A condition that $\Pi(b^p,l) \geq \Pi(b^p+\epsilon,l)$ gives the lower bound for b^p that makes a person with signal $s \leq \frac{1}{2}$ not willing to deviate from b^p. The pooling bid must be at least as low as $b^p(s^c)$ which is set such that $\Pi(b^p+\epsilon,s) = \Pi(b^p,s)$ is satisfied and is given as:

$$b^p(s^c) = \frac{1}{\tilde{\rho}(s^c)+1}v(l,L) \text{ where } \tilde{\rho}(s^c) = \frac{\pi(1-p)(1-2p(1-s^c))(1+4p(1-s^c))}{(1-\pi)(1-q)(1-2q(1-s^c))(1+4q(1-s^c))}$$

But then MLRP implies that $b^p(s^c) > u(\rho^{pool}(s^c,s^c))$ which violates individual rationality of the player with signal s^c and hence violates our assumption of equilibrium behavior. So, $s^c \leq \frac{1}{2}$. □

Proof of Corollary 1.

From proposition 1, we know that $s^c \leq \frac{1}{2}$ in any monotone equilibrium. Hence, a player with signal $s > \frac{1}{2}$ must bid $b^{PS} > b^p$. The definition of the PS bidding strategy implies that $b^{PS}(s) = u(\rho(s))$ where $\rho(s)$ are the player's beliefs conditional on winning at his own bid. We can write the beliefs as:

$$\rho(s) = \frac{\pi(f(s|R))^2(1-F(s|R))}{(1-\pi)(f(s|L))^2(1-F(s|L))}$$

Since we focus on $s > \frac{1}{2}$, we know that $F(s|R) = 1 - 2p(1-s)$ and $f(s|R) = 2p$ for any $s > \frac{1}{2}$. Hence, the beliefs can be re-written as:

$$\rho(s) = \frac{\pi p^3(1-s)}{(1-\pi)q^3(1-s)} = \frac{\pi p^3}{(1-\pi)q^3}$$

These beliefs are independent of the signal s. Thus, the bid $b^{PS}(s)$ is same for any signal $s > \frac{1}{2}$. Last, the fact that $u(\rho(s))$ must be non-decreasing in signals and that $\rho(s > \frac{1}{2}) > \rho(s^c)$ implies that $\rho(s > \frac{1}{2}) > \rho^*$. Thus $b^{PS}(s) = \frac{\pi p^3}{\pi p^3 + (1-\pi)q^3} v(r, R)$. □

Proof of Proposition 2.

Suppose that $\rho(0) \geq \rho^*$ is satisfied. Since $\rho^{pool}(s \leq \frac{1}{2}, 0) = \rho(0)$, we conclude that $\rho(s, s^c) > \rho^{pool}(s, s^c) > \rho^*$ for any $s^c > 0$. This, in turn, implies that $u(\rho(s, s^c)) > u(\rho^{pool}(s, s^c))$ for any $s^c > 0$. If condition $u(\rho(s, s^c)) > u(\rho^{pool}(s, s^c))$ holds, then by deviating from the pooling bid, a player increases his payoff (value minus price) and his chances of winning the object. That is, he increases his expected profit if bids slightly above the pooling bid. Hence, a player with signal $s < s^c$ will never want to pool. So, $s^c = 0$.

We now need to show that none of the players has a profitable deviation from the Pesendorfer-Swinkels bidding strategy. Consider a person with signal $s \leq \frac{1}{2}$. If he bids $b^{PS}(s)$, conditional on winning at his own bid, the beliefs of the player are:

$$\rho(s) = \frac{\pi(1-p)^2(1-2(1-p)s)}{(1-\pi)(1-q)^2(1-2(1-q)s)}$$

He receives a zero profit if wins when the price is equal his own bid. Similarly, winning at any price $p < b^{PS}(s)$ yields a profit of zero because of the signals being discrete (that is, because the signals $s \leq \frac{1}{2}$ reveal exactly same information). So, there is no incentive to deviate below $b^{PS}(s)$. If the player mimics any person with signal $\frac{1}{2} \geq s' > s$, he again has an expected profit of zero conditional on winning at his bid. This is so because the beliefs conditional on winning when the price is $b^{PS}(s')$ is:

$$\rho(s, p = b^{PS}(s')) = \frac{\pi(f(s'|R))^2(1-F(s'|R))}{(1-\pi)(f(s'|L))^2(1-F(s'|L))} = \frac{\pi(1-p)^2(1-2(1-p)s')}{(1-\pi)(1-q)^2(1-2(1-q)s')}$$

Note that $\rho(s, p = b^{PS}(s')) = \rho(s')$ and hence $u(\rho(s, p = b^{PS}(s'))) = u(\rho(s')) = b^{PS}(s')$.

Hence, deviation to any low signal player's bid does not increase profit.

If the player deviates to $b^{PS}(s > \frac{1}{2})$ (i.e. if mimics the player with high signal), then his beliefs conditional on winning at the price $b^{PS}(s > \frac{1}{2})$ are:

$$\rho(s, p = b^{PS}(s > \frac{1}{2})) = \frac{\pi p^2(1-p)}{(1-\pi)q^2(1-q)}$$

From corollary 1, we know that $b^{PS}(s > \frac{1}{2}) = u(\rho(s > \frac{1}{2}))$ where $\rho(s > \frac{1}{2}) = \frac{\pi p^3}{(1-\pi)q^3}$. But then, MLRP and assumption that $\rho(0) \geq \rho^*$ implies that $u(\rho(s, p = b^{PS}(s > \frac{1}{2}))) < b^{PS}(s > \frac{1}{2})$. Hence, a player with signal $s \leq \frac{1}{2}$ has no incentive to mimic a player with high signal.

Consider the player with signal $s > \frac{1}{2}$. He has no incentive to deviate above his bid because, by doing so, he only increases his chances to win an object at the price $p = b^{PS}(s > \frac{1}{2})$. Winning at the price $p = b^{PS}(s > \frac{1}{2})$ yields the profit of 0 and hence this deviation does not affect the expected profit of the player with signal $s > \frac{1}{2}$.

If he deviates to $b = b^{PS}(s')$ where $s' \leq \frac{1}{2}$ (i.e. if mimics any player with low signal), then conditional on losing (as well as winning) at this bid, his beliefs are

$$\rho^{dev}(s > \frac{1}{2}) = \frac{\pi p(1-p)(1-2(1-p)s')}{(1-\pi)q(1-q)(1-2(1-q)s')}$$

MLRP and the fact that $\rho(0) \geq \rho^*$ implies that $u(\rho^{dev}(s > \frac{1}{2})) > u(\rho(s')) = b^{PS}(s')$. So, the player with high signal has a profitable deviation from any $b \neq b^{PS}(s > \frac{1}{2})$.

Hence, if $\rho(0) \geq \rho^*$, there indeed is a unique monotone equilibrium. □

Proof of Proposition 3.

Proof of part (a)

Part (a) follows immediately: the pooling bid that makes the players with low signal to pool is too high to satisfy individual rationality of a person with high signal. Conditional on winning at the pooling price, the player with high signal has a value of $u(\rho(s > \frac{1}{2}, s^c))$. If $u(\rho(s > \frac{1}{2}, \frac{1}{2})) < b^p(\frac{1}{2})$ holds, then $u(\rho(s > \frac{1}{2}, s^c)) < b^p(s^c)$ holds for any $s^c < \frac{1}{2}$. The condition If $u(\rho(s > \frac{1}{2}, \frac{1}{2})) < b^p(\frac{1}{2})$ also implies that all the players with low signal choose the pooling bid. Therefore, the auction price is either $b^{PS}(s > \frac{1}{2})$ or b^p. Winning at the price $p = b^{PS}(s > \frac{1}{2})$ yields a payoff of zero. Since the player with high signal loses money at the pooling price, his expected profit is negative. So, the high signal bidder has a profitable deviation to zero.

Proof of part (b)

If $u(\rho(s > \frac{1}{2}, \frac{1}{2})) \geq b^p(\frac{1}{2}) \geq u(\rho(s = \frac{1}{2}))$ holds, then $\rho(0) < \rho^*$ must also hold. In addition, the only class of equilibria possible is the one where all players with low signal pool. This result stems from the fact that $b^p(\frac{1}{2}) \geq u(\rho(s = \frac{1}{2}))$ implies that the player with signal $s \leq \frac{1}{2}$ cannot bid anything greater than the pooling bid and get a non-negative profit conditional on winning at his own bid. Therefore, $s^c = \frac{1}{2}$ must hold in equilibrium. Also, $b^p(\frac{1}{2}) \geq u(\rho(s = \frac{1}{2}))$ can hold only if $\rho(s \leq \frac{1}{2}, \frac{1}{2}) < \rho^*$. But then, since $\rho^{pool}(s, s^c) < \rho(s, s^c)$ for any cutoff signal $s^c > 0$, we conclude that $u(\rho^{pool}(s \leq \frac{1}{2}, \frac{1}{2})) > u(\rho(s \leq \frac{1}{2}, \frac{1}{2}))$. Therefore, we can find a pooling bid b^p such that pooling is more profitable than deviating slightly above b^p. Since we are in $\rho(s \leq \frac{1}{2}, \frac{1}{2}) < \rho^*$ case, the lower bound that is compatible with the pooling behavior by all low signal players is by construction equal to $b^p(\frac{1}{2})$. Hence, for any pooling bid $b^p \in [b^p(\frac{1}{2}), u(\rho^{pool}(s \leq \frac{1}{2}, \frac{1}{2}))]$, the optimal strategy for the low signal players is to choose b^p.

Consider the bidder with signal $s > \frac{1}{2}$. Since $\rho(s > \frac{1}{2}, \frac{1}{2}) > \tilde{\rho}(\frac{1}{2})$, the condition $u(\rho(s > \frac{1}{2}, \frac{1}{2})) \geq b^p(\frac{1}{2})$ holds only if $\rho(s > \frac{1}{2}, \frac{1}{2}) > \rho^*$. Since $\rho(s > \frac{1}{2}, \frac{1}{2}) < \rho(s > \frac{1}{2})$ by MLRP assumption, we conclude that $b^{PS}(s > \frac{1}{2}) = u(\rho(s > \frac{1}{2})) > b^p(\frac{1}{2})$. Therefore, there are values for the pooling bid that are compatible with the PS bidding strategy that high signal players use in equilibrium. Also, $u(\rho(s > \frac{1}{2}, \frac{1}{2})) \geq b^p(\frac{1}{2})$ implies that we can find b^p such that the high signal players receive non-negative profits if they use the PS bidding strategy. In fact, for any b^p that satisfies $b^p \leq u(\rho(s > \frac{1}{2}, \frac{1}{2}))$, bidders with high signals do not want to deviate to zero.

We now demonstrate that the high signal bidder have no incentive to deviate to any bid outside $b^{PS}(\cdot)$. If the bidder deviates slightly above, then he only increases the odds of winning the object at the price $p = b^{PS}(s > \frac{1}{2})$. Since winning at this price yields a payoff of zero, such a deviation is not profitable. Similarly deviating to any bid $b > b^p$ does not increase the expected profit.

Suppose that the bidder with signal $s > \frac{1}{2}$ pools. Then, conditional on winning the object, his beliefs are $\rho^{pool}(s > \frac{1}{2}, \frac{1}{2})$. MLRP implies that $\rho^{pool}(s > \frac{1}{2}, \frac{1}{2}) > \tilde{\rho}(\frac{1}{2})$. This, in turn, implies that $u(\rho^{pool}(s > \frac{1}{2}, \frac{1}{2})) < u(\rho(s > \frac{1}{2}, \frac{1}{2}))$. Hence, the bidder with high signal always has an incentive to deviate from the pooling bid.

Therefore, if we choose the pooling bid b^p such that $b^p(\frac{1}{2}) \leq b^p \leq \min\{u(\rho(s > \frac{1}{2}, \frac{1}{2})), u(\rho^{pool}(s \leq \frac{1}{2}, \frac{1}{2}))\}$, then none of the players has an incentive to deviate from his bid. The fact that such b^p may take different values implies that the model may have infinitely many equilibria.

Proof of part (c)

Suppose that $u(\rho(s = \frac{1}{2})) > b^p(\frac{1}{2})$ holds. Since $\rho(s) > \tilde{\rho}(s^c)$ for any $\frac{1}{2} \geq s > s^c$, and since $\rho(0) = \tilde{\rho}(0) < \rho^*$, we can conclude that $u(\rho(s)) < b^p(s^c)$ for any $s = s^c \in B_\epsilon(0)$. This implies that there exists $\hat{s} < \frac{1}{2}$ such that $u(\rho(\hat{s})) = b^p(\hat{s})$. Notice that $\rho(\hat{s}) > \rho^*$ must hold for such equality to happen.

Set $s^c = \hat{s}$. Then, if we choose the pooling bid such that $b^p = b^p(\hat{s})$, the players with signals $s \leq s^c$ will be indifferent between bidding b^p and deviating slightly above.

Consider a person with signal $s \leq \frac{1}{2}$. If he wins at any price $b(s')$ such that $b^p < b(s') < b(s > \frac{1}{2})$, his beliefs, denoted as $\rho(s, p = b(s'))$, coincide with $\rho(s')$. This happens because of the discreteness of our signals. Therefore, $u(\rho(s, p = b(s'))) = b(s') = u(\rho(s'))$ meaning that the low signal player gets a profit of 0 if he wins at any price that is higher than the pooling price. This immediately implies the bidder with signal $s > s^c$ has no incentive to deviate to the other low signal player's bid. Also, notice that if any player with signal $s > s^c$ wins at the pooling price, his beliefs are equal to $\rho(s, s^c)$. Discreteness of the signals imply that $\rho(s, s^c) = \rho(s', s^c)$ for any $s, s' \leq \frac{1}{2}$. Thus, our method of calculation of the pooling bid $b^p(\hat{s})$ immediately implies that none of the players with signal $\frac{1}{2} \geq s > s^c$ have an incentive to pool. None of the bidders with low signal has an incentive to mimic his own type.

Mimicking any bidder with high signal is not profitable either. To see this, suppose that the bidder with signal $s \leq \frac{1}{2}$ bids $b^{PS}(s > \frac{1}{2})$. Conditional on winning at this bid, the player's beliefs become $\rho(s, p = b^{PS}(s > \frac{1}{2})) = \frac{\pi p^2(1-p)}{(1-\pi)q^2(1-q)} > \rho(\hat{s}) > \rho^*$. But then, $u(\rho(s, p = b^{PS}(s > \frac{1}{2}))) < b^{PS}(s > \frac{1}{2})$ meaning that the player loses money if deviates to $b^{PS}(s > \frac{1}{2})$. Hence, none of the players with low signal has an incentive to mimic the high signal player. Thus, the low signal players do not want to deviate from their bids.

Last, the players with signal $s > \frac{1}{2}$ have no incentive to deviate from their bids, either. Deviating slightly above $b^{PS}(s)$ leaves the expected profit unchanged; deviating to the pooling bid, on the other hand, makes the player worse off. This is so because $\rho(s > \frac{1}{2}, s^c) > \rho(s \leq \frac{1}{2}) > \rho^*$ for any (s, s^c) pair and because $\rho^{pool}(s > \frac{1}{2}, s^c) > \tilde{\rho}(s^c)$ for any $s^c < \frac{1}{2}$ that is consistent with monotonicity of the bidding function (i.e. for any s^c such that $u(\rho(s^c)) > b^p(s^c)$ is satisfied). So, no matter what the position of $\rho^{pool}(s > \frac{1}{2}, s^c)$ relative to ρ^* is, it is always the case that $u(\rho^{pool}(s > \frac{1}{2}, s^c)) < u(\rho(s > \frac{1}{2}, s^c))$. So, bidding above the pooling bid is profitable for any player with high signal.

Similarly, if the high signal player mimics the player with signal $\frac{1}{2} \geq s' > s^c$ (i.e. if bids $b(s')$), then his beliefs conditional on losing at his own bid are $\rho(s, p = b(s'))$. It is easy to show that MLRP implies that $\rho(s, p = b(s')) > \rho(s')$. Since $\rho(s) > \rho^*$ for all $s > s^c$, we conclude that $u(\rho(s, p = b(s'))) > u(\rho(s')) = p$. So, the player is better of if he bids above $b(s')$ and avoids losing at this price. Since this holds for an arbitrary s', this result holds for any $s' \leq \frac{1}{2}$. Hence, the high signal player has no incentive to deviate form his bid. □

Proof of Proposition 4.
Let assumptions MLRP and VAL and condition $u(\rho_N(0)) > b^p_N(0)$ hold. Note that $\rho_N(s^*) > \rho_N(s, s^*) > \rho^{pool}_N(s, s^*)$ for any $s \leq \frac{1}{2}$ and $s^* < \frac{1}{2}$. Since $\rho_N(\frac{1}{2}) = \rho_N(s, \frac{1}{2}) = \rho^{pool}_N(s, \frac{1}{2}) < \rho^*$, we can conclude that $u(\rho_N(s)) < b^p_N(s)$ for any $s \in B_\epsilon(\frac{1}{2})$. Continuity of the value function $u(\cdot)$ and the lower bound $b^p(\cdot)$ implies that the exists a signal $\hat{s} < \frac{1}{2}$ such that $u(\rho_N(\hat{s})) = b^p_N(\hat{s})$. This equality is possible only if $\rho_N(\hat{s}) > \rho^* > \hat{\rho}(\hat{s})$.

Let $s^* = \hat{s}$. If we set $b^p = b^p_N(\hat{s})$, then none of the players with signal $s \leq s^*$ has an incentive to deviate slightly above the pooling bid. Similarly, they have no incentive to mimic any player with low signal. The discreteness of the signals implies that if the bidder mimics any player with signal $\frac{1}{2} \geq s' > \hat{s}$ and wins at the price $p = b(s')$, his beliefs are same as the beliefs of the bidder with signal s'. That is $\rho(s, p = b(s')) = \rho_N(s')$. Hence, $u(\rho(s, p = b(s'))) = b(s')$. Similarly, since $\rho_N(s, \hat{s}) = \rho_N(s', \hat{s})$ for any $s, s' \leq \frac{1}{2}$, we conclude that the choice of b^p such that $b^p = b^p(\hat{s})$ implies that any player who pools has no incentive to deviate from the pooling bid. By the same logic, none of the bidders with signal $\frac{1}{2} \geq s > \hat{s}$ has an incentive to pool.

If the low signal bidder deviates to $b(s > \frac{1}{2})$ and wins at this bid, his beliefs are:

$$\rho(s, p = b(s > \frac{1}{2})) = \frac{\pi p^2(1-p)}{(1-\pi)q^2(1-p)}$$

MLRP assumption implies that $\rho_N(\hat{s}) < \rho(s, p = b(s > \frac{1}{2})) < \rho_N(s > \frac{1}{2})$. Since $\rho_N(\hat{s}) > \rho^*$, we conclude that $u(\rho(s, p = b(s > \frac{1}{2}))) < u(\rho_N(s > \frac{1}{2})) = b(s > \frac{1}{2})$. Therefore, the low signal bidder loses money if mimics any player with high signal. Hence none of the players with low signal has a profitable deviation form his bid.

The players with high signals have no incentive to deviate, either. To see this, note that deviating slightly above increases the chances of winning the object at the own price. Since winning at the own bid yields the payoff of zero, such deviation is not profitable. Also, the high signal players receive a positive payoff if the price is equal to the low signal player's bid. Hence, the high signal player will always want to bid above $b(s \leq \frac{1}{2})$ to avoid losing at such price. Similarly, winning at the pooling price yields a positive payoff. To see this, note that $\rho_N(s^*) < \rho_N^{pool}(s > \frac{1}{2}, s^*) < \rho_N(s > \frac{1}{2}, s^*)$ for any $s^* \in (0, \frac{1}{2}]$. We know that $\rho_N(\hat{s}) > \rho^*$ must hold in equilibrium. Hence, we get that $u(\rho_N(\hat{s})) < u(\rho_N^{pool}(s > \frac{1}{2}, \hat{s})) < u(\rho_N(s > \frac{1}{2}, \hat{s}))$ and non-monotonicity of the bidding strategy implies $u(\rho_N(s^*)) \geq b^p$. Hence, the player with signal $s > \frac{1}{2}$ always receives strictly positive profit conditional on winning at the pooling bid. Hence, no incentive to deviate to zero. The fact that $u(\rho_N^{pool}(s > \frac{1}{2}, \hat{s})) < u(\rho_N(s > \frac{1}{2}, \hat{s}))$, in turn, implies that pooling is not profitable when the cutoff signal is $\hat{s}$. Hence, the player with high signal has no incentive to deviate from his bid. □

Proof of Corollary 2.

The result follows from proposition 4. To see this, note that assumptions MLRP and VAL together with the condition $u(\rho_N(0)) > b_N^p(0)$ implies that there exists a non-monotone equilibrium where the pooling occurs in the $[s^*, \frac{1}{2}]$ interval.

Now suppose that there is pooling in the $[\underline{s}, \bar{s}]$ interval. In addition, suppose that the bidders with signals $s \in [0, \underline{s})$ use a decreasing bidding strategy whereas the bidders with signals $s \in (\bar{s}, \frac{1}{2}]$ use an increasing bidding function. Suppose also that $b(\frac{1}{2} \geq s > \bar{s}) \leq b(0)$. The high signal bidders again use the PS bidding strategy. The critical cases that affects the optimality of such bidding strategy are when: a) bidders win when the price is equal to their own bid; b) when they bid $b > b^p$ and win at the pooling price; and c) when the bidders pool and win. Let $\tilde{\rho}_N(s)$ denote the beliefs of a player with signal $s < \frac{1}{2}$ conditional on winning at own bid, $\tilde{\rho}_N^{pool}(s)$ denote the beliefs conditional on winning the auction after bidding b^p, and $\tilde{\rho}_N^{dev}(s)$ denote the beliefs conditional bidding any $b > b^p$ and winning the object at the pooling price. We can write these beliefs as:

$$\tilde{\rho}_N(s) = \begin{cases} \frac{\pi(1-p)^2(p+2(1-p)s)}{(1-\pi)(1-q)^2(p+2(1-q)s)} & \text{if } s < \underline{s} \\ \frac{\pi(1-p)^2(1-2(1-p)(s-\underline{s}))}{(1-\pi)(1-q)^2(1-2(1-q)(s-\underline{s}))} & \text{if } s \in (\bar{s}, \frac{1}{2}] \end{cases}$$

$$\tilde{\rho}_N^{pool}(s) = \frac{\pi(1-p)^2(3-2(1-p)(\bar{s}-\underline{s}))}{(1-\pi)(1-q)^2(3-2(1-q)(\bar{s}-\underline{s}))}$$

$$\tilde{\rho}_N^{dev}(s) = \frac{\pi(1-p)^2(2-2(1-p)(\bar{s}-\underline{s}))}{(1-\pi)(1-q)^2(2-2(1-q)(\bar{s}-\underline{s}))}$$

Notice that $\tilde{\rho}_N(s)$ is decreasing is s on the $[0, \underline{s}]$ region and is increasing on $[\bar{s}, \frac{1}{2}]$. Also, $\tilde{\rho}_N(s)$ is minimized when $s = \bar{s}$. Hence, the condition for such an equilibrium is that $\tilde{\rho}_N(\bar{s}) > \rho^*$.

We require that the pooling bid that is compatible with the pooling behavior is also compatible with the behavior of the players who bid above b^p. In particular, we require that $b(\bar{s}) \geq b^p$. But this condition always holds under the assumptions that we impose. This is so because $\tilde{\rho}_N^{pool}(s) = \rho_N^{pool}(s, s^*)$, $\tilde{\rho}_N^{dev}(s) = \rho_N(s, s^*)$, and $\tilde{\rho}_N(s|s > \bar{s}) = \rho_N(s)$ if $\frac{1}{2} - s^* = \bar{s} - \underline{s}$. That is, the beliefs coincide if the pooling interval stays the same. Hence, an existence of a non-monotone equilibrium with the pooling in the $[s^*, \frac{1}{2}]$ interval implies that there exists an equilibrium where the pooling occurs on the interval $[\underline{s}, \bar{s}]$ where $\underline{s} \neq s^*$ and $\bar{s} < \frac{1}{2}$. □

Proof of Corollary 3.

Suppose for a contradiction that there exists a non-monotone equilibrium where the players receiving both high and low signals pool to some bid b^*. In particular, suppose that the players with signals $s \in [\underline{s}, \bar{s}]$, where $\underline{s} < \frac{1}{2} < \bar{s}$, choose b^*. None of the players with signals from this region must have an incentive to deviate from b^*. To find the conditions for existence of the equilibrium, we define the beliefs of winning at the price b^*. Let $\rho^{pool}(s)$ denote the beliefs of a player with signal s conditional on winning an object after bidding b^*. These can be written as

$$\rho^{pool}(s) = \frac{\pi f(s|R)(1-2p(1-\bar{s})-2(1-p)\underline{s})(2+2p(1-\bar{s})+2(1-p)\underline{s})}{(1-\pi)f(s|L)(1-2q(1-\bar{s})-2(1-p)\underline{s})(2+2q(1-\bar{s})+2(1-q)\underline{s})}$$

Similarly, the beliefs of winning an object at the pooling price after bidding any $b > b^*$ can be written as:

$$\rho^{dev}(s) = \frac{\pi f(s|R)(1-2p(1-\bar{s})-2(1-p)\underline{s})(1+2p(1-\bar{s})+2(1-p)\underline{s})}{(1-\pi)f(s|L)(1-2q(1-\bar{s})-2(1-p)\underline{s})(1+2q(1-\bar{s})+2(1-q)\underline{s})}$$

The relation between these two beliefs depends on the value of $\bar{s}$ and $\underline{s}$. If $\underline{s} < 1 - \bar{s}$, then $\rho^{dev}(s) > \rho^{pool}(s)$. If $\underline{s} > 1 - \bar{s}$, then $\rho^{dev}(s) < \rho^{pool}(s)$. The beliefs are equal if $\underline{s} = 1 - \bar{s}$. We now consider all the three cases separately.

Suppose that $\underline{s} = 1 - \bar{s}$ hold. Then, $\rho^{pool}(s) = \rho^{dev}(s) = \frac{\pi f(s|R)}{(1-\pi)f(s|L)}$ and hence pooling reveals no new and valuable information. Hence, the pooling behavior is sustainable only if $b^p = u(\rho^{pool}(s \leq \frac{1}{2})) = u(\rho^{pool}(s > \frac{1}{2}))$. The condition $u(\rho^{pool}(s \leq \frac{1}{2})) = u(\rho^{pool}(s > \frac{1}{2}))$, in turn, is satisfied when $\rho^{pool}(s \leq \frac{1}{2}) < \rho^* < \rho^{pool}(s > \frac{1}{2})$. So, suppose that this is the case. Whether such a pooling region is sustainable in a non-monotone equilibrium will depend on what the players with the signals outside the pooling region bid.

Suppose first that at least two bidders with qualitatively different signals bid the same bid; i.e. suppose that $b(s) = b(s') > b^p$ for some $s \leq \frac{1}{2}$ and $s' > \frac{1}{2}$. Consider such a player with signal s. Conditional on winning at his own bid, the player's beliefs become:

$$\rho(s) = \frac{\pi f(s|R)}{(1-\pi) f(s|L)} \frac{\Pr(\text{win}, p = b(s)|R)}{\Pr(\text{win}, p = b(s)|L)}$$

Note that $\frac{\Pr(\text{win},p=b(s)|R)}{\Pr(\text{win},p=b(s)|L)} \neq 1$ must hold as otherwise, the beliefs are identical to $\rho^{pool}(s)$ and hence $b^p = b(s)$ violating our assumption that the players bid above b^p. Thus, the ratio can either be greater than 1 or less than 1. Suppose first that $\frac{\Pr(\text{win},p=b(s)|R)}{\Pr(\text{win},p=b(s)|L)} > 1$. Then, since winning at the player's own bid reveals limited information because both high signal player and low signal player bid this bid, $\frac{\Pr(\text{win},p=b(s)|R)}{\Pr(\text{win},p=b(s)|L)}$ cannot exceed $\frac{p}{q}$ (corresponding to the case when winning reveals information that favours state R the most). Hence, we have $\rho^{pool}(s > \frac{1}{2}) > \rho(s) > \rho^{pool}(s \leq \frac{1}{2})$ for any signal $s < \underline{s}$ implying that $u(\rho(s)) < b^p$. Hence, the player with low signal loses money if bids above the pooling bid. Hence, an incentive to deviate. Similarly, if $\frac{\Pr(\text{win},p=b(s)|R)}{\Pr(\text{win},p=b(s)|L)} < 1$, the players with high signals have profitable deviation. Therefore, the bidding functions cannot be overlapping for the players with qualitatively different signals.

Now, suppose that the players with qualitatively different signals do not bid the same bids. Suppose first that the high signal players always bid above any player with low signal. This implies that the high signal players use the PS bidding strategy. Since we are looking for non-monotone bidding strategy, all the players with $s < \underline{s}$ must bid strictly above the pooling bid b^p. Suppose that such players utilize a strictly decreasing function. Then, conditional on winning at the own bid, a player with signal s has the beliefs

$$\rho(s) = \frac{\pi (f(s|R))^2 (F(s|R) + 1 - F(\frac{1}{2}|R))}{(1-\pi)(f(s|L))^2 (F(s|L) + 1 - F(\frac{1}{2}|L))} = \frac{\pi(1-p)^2(p + 2(1-p)s)}{(1-\pi)(1-q)^2(q + 2(1-q)s)}$$

These beliefs are decreasing in s. Hence $\rho(s) > \rho^*$ in equilibrium. But this violates our assumption that $\rho^{pool}(s \leq \frac{1}{2}) < \rho^* < \rho^{pool}(s > \frac{1}{2})$. So, no such equilibrium. Suppose then that the low signal players use an increasing bidding strategy. Then, the beliefs are of the following form:

$$\rho(s) = \frac{\pi (f(s|R))^2 (1 - F(\bar{s}|R) + F(s|R))}{(1-\pi)(f(s|L))^2 (1 - F(\bar{s}|L) + F(s|L))} = \frac{\pi(1-p)^2(p + 2(1-p)(\bar{s} - s)}{(1-\pi)(1-q)^2(q + 2(1-q)(\bar{s} - s)}$$

The beliefs are again increasing in s implying that $\rho(s) > \rho^*$ for any $s < \underline{s}$. This again violates our assumption that $\rho^{pool}(s \leq \frac{1}{2}) < \rho^* < \rho^{pool}(s > \frac{1}{2})$. The similar calculations yield the result that no equilibrium is possible when the low signal players bid above the players with high signals. We thus conclude that no equilibrium is possible when $\underline{s} = 1-\bar{s}$.

Suppose that $\underline{s} < 1 - \bar{s}$ holds. Then, we know that $\rho^{dev}(s) > \rho^{pool}(s)$ also holds. For the players with signal $s > \frac{1}{2}$ to be willing to pool, $u(\rho^{pool}(s > \frac{1}{2})) \geq u(\rho^{dev}(s > \frac{1}{2}))$ must hold (as otherwise, deviating slightly above b^* is always profitable). This, in turn can old only if $\rho^{pool}(s > \frac{1}{2}) < \rho^*$.

MLRP assumption implies that $\rho^{pool}(s > \frac{1}{2}) > \rho^{dev}(s \leq \frac{1}{2}) > \rho^{pool}(s \leq \frac{1}{2})$ irrespective of $\bar{s}$ and $\underline{s}$. This implies that a player with low signal chooses action l after winning the object at the pooling price (irrespective of his own bid). So, for the pooling behavior to be optimal for the players with low signal, the pooling bid must satisfy

$$b^* \geq \frac{1}{\tilde{\rho}+1} v(l, L) \qquad \text{where}$$

$$\tilde{\rho} = \frac{\pi(1-p)(1-2p(1-\bar{s})-2(1-p)\underline{s})(1+4p(1-\bar{s})+4(1-p)\underline{s})}{(1-\pi)(1-q)(1-2q(1-\bar{s})-2(1-p)\underline{s})(1+4q(1-\bar{s})+4(1-q)\underline{s})}$$

Using simple math, we can demonstrate that MLRP assumption implies $\tilde{\rho} < \rho^{pool}(s > \frac{1}{2})$. Since $\rho^{pool}(s > \frac{1}{2}) < \rho^*$ must hold in equilibrium, we get that $u(\rho^{pool}(s > \frac{1}{2})) < b^*$. So, if we choose the pooling bid that makes the player with "low" signal want to bid b^*, the player with "high" signal has a profitable deviation (and vice versa). So, there cannot be a non-monotone equilibrium where $\underline{s} < 1 - \bar{s}$ and $\bar{s} > \frac{1}{2}$.

Now, suppose that $\underline{s} > 1-\bar{s}$ holds. Then, $\rho^{dev}(s) < \rho^{pool}(s)$. An equilibrium may exist only if $\rho^{pool}(s \leq \frac{1}{2}) > \rho^*$. This is so because otherwise $u(\rho^{dev}(s \leq \frac{1}{2})) > u(\rho^{pool}(s \leq \frac{1}{2}))$.

Assume that $\rho^{pool}(s \leq \frac{1}{2}) > \rho^*$. MLRP implies that $\rho^{pool}(s > \frac{1}{2}) > \rho^{dev}(s > \frac{1}{2}) > \rho^*$. The position of the beliefs relative to ρ^* implies that the player with high signal always chooses action r conditional on winning at the pooling price. Hence the lower bound of the pooling bid b^p that is consistent with the pooling behavior of the players with high signal takes the form:

$$\hat{b} = \frac{\hat{\rho}}{\hat{\rho}+1} v(r, R) \qquad \text{where}$$

$$\hat{\rho} = \frac{\pi p(1-2p(1-\bar{s})-2(1-p)\underline{s})(1+4p(1-\bar{s})+4(1-p)\underline{s})}{(1-\pi)q(1-2q(1-\bar{s})-2(1-p)\underline{s})(1+4q(1-\bar{s})+4(1-q)\underline{s})}$$

MLRP again implies that $\hat{\rho} > \rho^{pool}(s \leq \frac{1}{2})$ and such a lower bound does not satisfy individual rationality of the players who received low signal and chose the pooling bid. That is, no matter what the pooling bid b^* is, at least one of the players has a profitable deviation from the pooling bid.

Hence, the players with (qualitatively) different signals cannot pool to the same bid.

□

Proof of Proposition 5.

Notice that the functions $\rho_h^N(\cdot)$ and $\bar{\rho}(\cdot)$ converge to $\rho^+(\frac{1}{2})$ as the cutoff signal $\bar{s}$ approaches $\frac{1}{2}$. The fact that $\rho^+(\frac{1}{2}) > \rho^*$ then implies that $\rho_h^N(\bar{s}) > \rho^*$ for any $\bar{s} \in B_\epsilon(\frac{1}{2})$.

We know that MLRP implies $\rho_h^N(\bar{s}) < \bar{\rho}(\bar{s})$ for any $\bar{s} > \frac{1}{2}$. Hence, $u(\rho_h^N(\bar{s})) < b_h^p(\bar{s})$ when $\bar{s} \in B_\epsilon(\frac{1}{2})$. Continuity of both functions and our assumption that $u(\rho_h^N(1)) > b_h^p(1)$ then implies that there exists a signal $\hat{s} \in (\frac{1}{2}, 1)$ such that $u(\rho_h^N(\hat{s})) = b_h^p(\hat{s})$.

Let $\bar{s} = \hat{s}$. Since the condition that $u(\rho_h^N(\hat{s})) = b_h^p(\hat{s})$ can hold only when $\rho_h^N(\hat{s}) < \rho^*$ and $\bar{\rho}(\hat{s}) > \rho^*$, the lower bound $b_h^p(\bar{s})$ is by construction the true value of the pooling bid that makes the player with signal $s \in (\frac{1}{2}, \bar{s}]$ indifferent between pooling and deviating slightly above. Also, note that $\rho_h(s, \bar{s})$ is same across any signal $s > \frac{1}{2}$. So, if we demonstrate that winning at any price greater than b^p yields a payoff of zero, we would prove that none of the players with high signal has a profitable deviation from their bid.

The shape of the bidding strategy implies that a player with high signal can only win if the price is equal to some bid of a player with high signal. So, conditional on winning at any price $p = b(s') > b^p$, the beliefs of the player are:

$$\rho_h(s, p = b(s')) = \frac{\pi p^2(1 - p(2s' - 1))}{(1 - \pi)q^2(1 - q(2s' - 1))}$$

These beliefs are identical to $\rho_h^N(s')$ and hence $u(\rho_h(s, p = b(s'))) = u(\rho_h^N(s')) = b(s')$. Since this holds for an arbitrary signal $s' > \frac{1}{2}$, winning at any price that equals to the bid of the player with high signal yields a payoff of 0 for all high signal players. Hence, the players with signals $s > \bar{s}$ have no incentive to pool and the players with signals $(\frac{1}{2}, \bar{s}]$ have no incentive to bid above.

None of the high signal players has an incentive to mimic a player with low signal, either. To see this, suppose that a player with signal $s > \bar{s}$ bids $b(s' < \frac{1}{2})$. Conditional on winning at this bid, his beliefs are:

$$\rho_h(s, p = b(s')) = \frac{\pi p(1 - p)^2}{(1 - \pi)q(1 - q)^2}$$

MLRP implies that $\rho_h^N(s) > \rho_h(s, p = b(s')) > \rho_h^N(s')$. We also know that $\rho_h^N(s) < \rho^*$ holds in equilibrium. Hence, $u(\rho_h(s, p = b(s'))) < b(s') = u(\rho_h^N(s'))$. The player with high signal loses money if he mimics any player with low signal. Same holds for those who pool. Hence, none of the players with high signal wants to deviate from their bid.

We are left to demonstrate that the low signal player have no incentive to deviate, either. First, note that $\rho_h(s, p = b(s')) < \rho_h^N(s')$ for any $s \leq \frac{1}{2}$ and $s' > \bar{s}$. Hence $u(\rho_h(s, p = b(s'))) > u(\rho_h^N(s'))$ implying that the low signal player would always want to bid above $b(s')$ to avoid losing at this bid. So, low signal player does not want to mimic non-pooling player with high signal.

The last step is to show that none of the players with low signal (i.e. with $s \leq \frac{1}{2}$) has an incentive to pool. To show this, it would suffice to demonstrate that $u(\rho_h(s \leq \frac{1}{2})) > u(\rho_h^{pool}(s \leq \frac{1}{2}))$. Notice that MLRP implies that $\rho_h^{pool}(s \leq \frac{1}{2}, \bar{s}) < \rho_h^N(1)$ for any cutoff signal $\bar{s} > \frac{1}{2}$. But then, since $\rho_h^N(\hat{s})$ must hold in any equilibrium with the bidding strategy of interest, $\rho_h^N(1) < \rho^*$ must also hold. We conclude that $\rho_h^{pool}(s \leq \frac{1}{2}, \hat{s}) < \rho^*$ also holds. Since MLRP implies that $\rho_h^{pool}(s, \bar{s}) > \rho_h(s, \bar{s})$ for any $\bar{s} > \frac{1}{2}$ and any $s \in [0, 1]$, we conclude that $u(\rho_h^{pool}(s \leq \frac{1}{2}, \hat{s})) < u(\rho_h(s \leq \frac{1}{2}, \hat{s}))$. Therefore, it is always profitable for the player with low signal to deviate from the pooling bid. Hence, the low signal player has no incentive to bid anything outside his bid. Thus, the bidding strategy characterized above is indeed an equilibrium bidding strategy of the model. □

B References

Atakan, A. E., and Ekmekci, M. (in press). Auctions, actions, and the failure of information aggregation. *American Economic Review*.

Banerjee, P. (2005). Common value auctions with asymmetric bidder information. *Economics Letters*, *88*(1), 47–53.

Binmore, K., and Klemperer, P. (2002). The biggest auction ever: the sale of the british 3g telecom licences. *The Economic Journal*, *112*(478), C74–C96.

Cripps, M. W., and Swinkels, J. M. (2006). Efficiency of large double auctions. *Econometrica*, *74*(1), 47–92.

De Castro, L. I., and Karney, D. H. (2012). Equilibria existence and characterization in auctions: achievements and open questions. *Journal of Economic Surveys*, *26*(5), 911–932.

Goeree, J. K., and Offerman, T. (2003). Competitive bidding in auctions with private and common values*. *The Economic Journal*, *113*(489), 598–613.

Hong, H., and Shum, M. (2004). Rates of information aggregation in common value auctions. *Journal of Economic Theory*, *116*(1), 1–40.

Jackson, M. O., and Kremer, I. (2007). On the informational inefficiency of discriminatory price auctions. *Journal of Economic Theory*, *132*(1), 507–517.

Kremer, I. (2002). Information aggregation in common value auctions. *Econometrica*, *70*(4), 1675–1682.

McAdams, D. (2007). On the failure of monotonicity in uniform-price auctions. *Journal of Economic Theory*, *137*(1), 729–732.

Milgrom, P. R. (1979). A convergence theorem for competitive bidding with differential information. *Econometrica: Journal of the Econometric Society*, 679–688.

Milgrom, P. R., and Weber, R. J. (1982). A theory of auctions and competitive bidding. *Econometrica: Journal of the Econometric Society*, *50*, 1089–1122.

Pesendorfer, W., and Swinkels, J. M. (1997). The loser's curse and information aggregation in common value auctions. *Econometrica: Journal of the Econometric Society*, *65*, 1247–1281.

Reny, P. J., and Zamir, S. (2004). On the existence of pure strategy monotone equilibria in asymmetric first-price auctions. *Econometrica*, *72*(4), 1105–1125.

Riley, J. G. (1988). Ex post information in auctions. *The Review of Economic Studies*, *55*(3), 409–429.

Wang, R. (1991). Common-value auctions with discrete private information. *Journal of Economic Theory*, *54*(2), 429–447.

Wilson, R. (1977). A bidding model of perfect competition. *The Review of Economic Studies*, 511–518.