A JOURNEY AROUND MORSE HOMOLOGIES

(Modern Approach)

by

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This is to certify that I have examined this copy of a master's thesis by

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and that any and all revisions required by the final

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Abstract

This thesis is a workout of the lecture notes of M. Hutchings on Morse Homology. Our main goal is to study Morse Theory on finite dimensional manifolds by adapting the ideas of Floer Theory on infinite dimensional manifolds. In the first part of the thesis, we give all necessary definitions from Morse Theory such as Morse functions, moduli spaces and Morse homology. We prove that Morse homology is isomorphic to singular homology. We also study some applications of Morse homology to algebraic topology. Throughout the second part, we investigate Morse-Bott functions, moduli spaces and Morse-Bott homology by giving several examples. Finally, we make use of spectral sequences to simplify the computations of Morse-Bott homology.

ÖZET

Bu tezde M. Hutchings' in Morse homoloji ders notlarının detaylı çalışması yapılmıştır. Sonlu boyutlu manifoldlar üzerindeki Morse teorisi sonsuz boyutlu manifoldlar üzerinde çalışılan Floer teorisinin fikirleri adapte edilerek çalışılmıştır. Tezin ilk kısmında Morse fonksiyonları, moduli uzaylar ve Morse homoloji gibi Morse teorinin gerekli tüm tanımları yapılmıştır. Morse homolojinin singuler homolojiye izomorfik olduğunu söyleyen Morse Homoloji Teoremi ispatlanmıştır. Ayrıca Morse homolojinin cebirsel topolojiye bazı uygulamaları gösterilmiştir. Tezin ikinci kısmında ise Morse-Bott fonksiyonları, moduli uzaylar ve Morse-Bott homolojisine geçilmiş ve örnekleri verilmiştir. Son olarak, spektral diziler kullanılarak Morse-Bott homolojinin hesapları kolaylaştırılmıştır.

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INTRODUCTION

Morse Theory originated at the beginning of the 20th century, via the work of Marston Morse. The goal is to understand the topology of a manifold by studying a critical point of a suitable real-valued function on it. After then, theory was developed by Thom, Smale, Bott and Milnor and others.

In the classical Morse theory, given a finite dimensional manifold, we are interested in the critical points of Morse functions which are real valued functions whose critical points are nondegenerate and use them to understand the manifold. Currently, Morse theory is studied on the infinite dimensional manifolds. In this thesis, we focus on the finite dimensional manifolds, yet the ideas and techniques can be easily generalized to infinite dimensional manifolds.

In the first chapter, we introduce the basic definitions concerning manifold theory. In the second chapter, for a Morse function f and a generic metric g on a closed manifold M, we define a moduli space $\mathscr{M}(p,q)$ as the space of flow lines from a critical point p to another critical point q. A flow line $\gamma : \mathbb{R} \to M$ is a solution of the differential equation

$$\frac{\mathrm{d}\gamma}{\mathrm{d}s} = -gradf(\gamma(s)).$$

where *gradf* denote the gradient. (f, g) is called a Morse-Smale pair if the descending manifold of p and the ascending manifold of q intersect transversely. Here, the descending manifold of p and the ascending manifold of q are the set of all points in M that flow to p and q in backward and forward time, respectively. We mention that given any C^k Morse function f and a generic metric g, (f, g) is Morse-Smale. In the case that (f, g) is Morse-Smale, $\mathscr{M}(p,q)$ is a smooth manifold of dimension $i - j - 1$ where i and j are the indices of p and q respectively and $i > j$. Moreover, when the indices of p and q differ by one, zero dimensional manifold $\mathscr{M}(p,q)$ is compact, hence finite. A generalization of compactness results provides a necessary tool Morse complex who is denoted by $C_*^{Morse}(f, g)$. So, we can construct the Morse homology denoted by $H_*^{Morse}(f, g)$.

In the third chapter, we prove that the Morse homology is independent of the choice of a Morse function and a metric, i.e given two different Morse-Smale pairs (f_1, g_1) and (f_2, g_2) we show that

$$H_*^{Morse}(f_1, g_1) \cong H_*^{Morse}(f_2, g_2).$$

This proof is an adaptation of the analogous result in the infinite dimensional case which is studied extensively in [2].

In the fourth chapter, we prove the isomorphism between Morse homology and singular homology which is an alternative way to conclude that the Morse homology depends only on the smooth structure of the manifold M. This isomorphism can not be generalized in an obvious way to infinite dimensional manifolds.

In the fifth chapter, we study the proofs of some well-known theorems from algebraic topology such as Morse inequalities, Poincare duality and Kunneth formula by using Morse homology.

The last chapter is on the Morse-Bott Theory which studies functions whose critical points are not necessarily isolated, but form critical submanifolds. Such functions are called Morse-Bott functions. To define the chain complex in the Morse-Bott case we use moduli spaces of flow lines between critical submanifolds with simplicial complex of the critical submanifolds. This complicated process produces many chain complexes associated with a single Morse-Bott function f. There are also alternative ways of defining a Morse-Bott complex. For example, [6] and [7] include three different approaches together with the proof of the equivalence of these definitions. The fact that Morse-Bott homology is independent of the choice of a Morse-Bott function and a generic metric implies that it is isomorphic to singular homology of the manifold as the constant function is obviously Morse-Bott. We give an alternative definition of the Morse-Bott homology using spectral sequences and present some examples to clearify the definition.

We hope that it will be useful for those want to learn Morse theory with some basic knowledge on differentiable manifolds. There is no original result in this thesis. It is based on [15].

LIST OF SYMBOLS/ABBREVIATIONS

M	m dimensional smooth manifold
TM	tangent bundle of M
T^*M	cotangent bundle of M
$gradf$	the gradient vector field of f
$\mathrm{d}f_p$	differential of f at p
$H(f,p)$	Hessian of f at p
$\nabla_X(Y)$	the covariant derivative of Y in the direction of X
$Crit(f)$	the set of all critical points of f
$Crit_i(f)$	the set of all critical points of f of index i
$ind(p)$	the index of a critical point, the number of negative eigenvalues of the Hessian
$\mathscr{W}(p,q)$	the set of flow lines of the negative gradient vector field, $-gradf$ which converge to p and q
$\mathscr{M}(p,q)$	the quotient space of $\mathscr{W}(p,q)$ under the action of $\mathbb{R}$
$\mathscr{T}(M)$	the set of all sections of TM
$L^p_k(U)$	the completion of $C^\infty(U)$ with respect to the norm $\Vert.\Vert_{k,p}$
$L^p_{k,loc}$	locally L^p_k maps
$\nabla\psi$	the vertical differential of ψ
$\#\mathscr{M}(p,q)$	the total sum of the elements in $\mathscr{M}(p,q)$
$C_i^{Morse}(f,g)$	the free $\mathbb{Z}_2$-module generated by the elements of $Crit_i(f)$
$\mathbb{Z}_2$	the quotient ring of the ring of integers modulo the ideal of even numbers
$H_*^{Morse}(f,g)$	the Morse homology of the chain complex $C_*^{Morse}(f,g)$
$\Omega^m_c(M)$	the space of compactly supported smooth m-forms on M
$i_-(S)$	the dimension of the negative normal bundle N_pS^-
$i_+(S)$	$i_-(S)+dim(S)$
$[i_-(S), i_+(S)]$	the index of the critical submanifold S
$\mathscr{M}(S_j,S_k)$	the set of unparametrised flow lines between the critical submanifolds S_j and S_k
e_+	the endpoint map from $\mathscr{M}(S_j,S_k)$ to S_j sending $\gamma\mapsto lim_{s\to\infty}\gamma(s)$
e_-	the endpoint map from $\mathscr{M}(S_j,S_k)$ to S_k sending $\gamma\mapsto lim_{s\to-\infty}\gamma(s)$
$A\times_C B$	the fiber product of A and B
$\mathcal{O}$	the orientation sheaf
$C_*(S,\mathcal{O})$	the space of singular chains with coefficients in $\mathcal{O}$
C_*^{Bott}	the chain complex of a Morse-Bott function defined as $\bigoplus_S C_{k-i_-(S)}(S,\mathcal{O})$
$H_*^{Bott}(f,g)$	the Morse-Bott homology of the chain complex C_*^{Bott}

Contents

1 Preliminaries

In this chapter, we will present some basic definitions to understand the text. One can find so many books and lecture notes concerning manifolds. We will follow only the reference [18] through the text.

Definition 1.0.1. A manifold M of dimension n is a second countable, Hausdorff topological space for which every point has a neighborhood U that is homeomorphic to an open subset $\widetilde{U} \subset \mathbb{R}^n$. In particular, it is a topological space locally homeomorphic to $\mathbb{R}^n$. The pair (U, φ) where $\varphi : U \to \widetilde{U}$ is a homeomorphism is called a chart or a local coordinate system.

A manifold is called smooth if for any two distinct charts (U, φ) and (V, ψ) with $U \cap V \neq \emptyset$, whenever $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is a C^∞ function.

Definition 1.0.2. A function $f : M \to \mathbb{R}$ is called smooth if for all $p \in M$, there exists a smooth chart (U, φ) such that $p \in U$ and $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is smooth. $C^\infty(M)$ is defined to be the set of all smooth functions $f : M \to \mathbb{R}$. A function $F : M \to N$ is said to be smooth if for all $p \in M$, there exist some smooth charts (U, φ) and (V, ψ) such that $p \in U$, $q \in V$, $F(U) \subset V$ and $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$.

Definition 1.0.3. A map $X_p : C^\infty(M) \to \mathbb{R}$ is said to be a tangent vector at p if for every $f, g \in C^\infty(M)$ and $a, b \in \mathbb{R}$, it satisfies

i. $X_p(af + bg) = aX_p(f) + bX_p(g)$

ii. $X_p(fg) = f(p)X_p(g) + g(p)X_p(f)$

Tangent vectors at a point $p \in M$ form a vector space and it is called the tangent space at p denoted by T_pM. As a vector space, its basis is $\{ \left.\frac{\partial}{\partial x_1}\right|_p, ..., \left.\frac{\partial}{\partial x_n}\right|_p \}$ where $x_1, ..., x_n$ are any local coordinate functions $x_i : U \to \mathbb{R}$ such that $x_i(p) = \pi_i \circ \varphi(p)$. Here, $\left.\frac{\partial}{\partial x_1}\right|_p$ is defined by

$$\left.\frac{\partial}{\partial x_i}\right|_p f = \left.\frac{\partial}{\partial x_i}\right|_{\varphi(p)} (f \circ \varphi^{-1})$$

for any chart (U, φ) and a smooth function $f : U \to \mathbb{R}$. The disjoint union of tangent spaces at every point of the manifold is called the tangent bundle of M and denoted by TM. We write an element of this disjoint union as an ordered pair (p, X) with $p \in M$ and $X \in T_pM$. We define a projection map $\pi : TM \to M$, $\pi(p, X) = p$.

Lemma 1.0.4. *([18]) For any smooth n-manifold M, the tangent bundle TM has a natural topology and smooth structure that makes it into a $2n$-dimensional smooth manifold. With this structure π is smooth.*

Indeed, since M is second countable, we can choose a countable collection of smooth charts (U_i, ϕ_i) for M such that $\bigcup_i U_i = M$. Given (U_i, ϕ_i), let $(x^1, ..., x^n)$ denote the coordinate functions of ϕ_i and define $\widetilde{\phi_i} : \pi^{-1}(U_i) \to \mathbb{R}^{2n}$ by

$$\widetilde{\phi_i}(p, \sum_{i=1}^{n} v_i \left.\frac{\partial}{\partial x_i}\right|_p) = (x^1(p), ..., x^n(p), v^1, ..., v^n).$$

Then, we define the topology on TM by taking all sets of the form $(\widetilde{\phi_i})^{-1}(V)$ where $V \subset \mathbb{R}^{2n}$ is open, as a basis. In other words, we take the topology on TM which makes all $\widetilde{\phi_i}$' s continouos.

The dual of the tangent space T_pM is denoted by T_p^*M and called the cotangent space at p. Then, the covectors $(\lambda^1|_p, ..., \lambda^n|_p)$, defined by

$$\lambda^i(\left.\frac{\partial}{\partial x_j}\right|_p) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

are the dual basis for T_p^*M and any covector $w \in T_p^*M$ can be represented as $\sum_{i=1}^n w_i \lambda^i|_p$, where $w_i = w(\left.\frac{\partial}{\partial x_i}\right|_p)$. Given (U_i, ϕ_i), let $(x^1, ..., x^n)$ denote the coordinate functions of ϕ_i and define $\widetilde{\phi_i} : \pi^{-1}(U_i) \to \mathbb{R}^{2n}$ by

$$\widetilde{\phi_i}(p, \sum_{i=1}^{n} w_i \lambda^i|_p) = (x^1(p), ..., x^n(p), w^1, ..., w^n).$$

The disjoint union of cotangent spaces at every point of the manifold is called the cotangent bundle of M and denoted by T^*M. The topology on T^*M can be defined similarly depicted as above.

Definition 1.0.5. A vector field $X : M \to TM$ is a map that assigns to each point $p \in M$ a tangent vector at that point. In fact, it is a section of TM such that $\pi \circ X(p) = p$ for all $p \in M$. X is called smooth if $X : M \to TM$ is a smooth map. The set of all sections of TM is denoted by $\mathscr{T}(M)$.

Using the previous definition, we can define $X_p = \sum_{i=1}^n X_i(p) \left.\frac{\partial}{\partial x_i}\right|_p$, where $X_i : U \to \mathbb{R}$, $p \in U$, called the compenent functions of X in the given chart.

Lemma 1.0.6. *Let M be a manifold and $X : M \to TM$ be a vector field. If $(U, (x^i))$ is a smooth coordinate chart on M, then X is smooth if and only if its component functions are smooth with respect to these local coordinates.*

Let f be a smooth real-valued function on a smooth manifold M, $f : M \to \mathbb{R}$, We define a covector field $\mathrm{d}f$, the differential of f, by

$$\mathrm{d}f_p(X_p) = X_p f$$

for $X_p \in T_pM$. With local expression

$$\mathrm{d}f_p = \frac{\partial f}{\partial x_1}(p)\,\mathrm{d}x_1|_p + ... + \frac{\partial f}{\partial x_n}(p)\,\mathrm{d}x_n|_p$$

where $(x_1, ..., x_n)$ are local coordinates around p.

Definition 1.0.7. A Riemannian metric on a smooth manifold M is a smooth 2-tensor field g that satisfies the following conditions at each point of M:

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

i. Symmetry:
$\forall X_p, Y_p \in T_pM,\ g(X_p, Y_p) = g(Y_p, X_p)$

ii. Linearity:
$\forall a, b \in \mathbb{R}, X_p, Y_p, Z_p \in T_pM,\ g(aX_p + bY_p, Z_p) = ag(X_p, Z_p) + bg(Y_p, Z_p)$

iii. Positive definiteness:
$\forall X_p \in T_pM,\ g(X_p, X_p) \geq 0$

iv. Nondegeneracy:
$g(X_p, Y_p) = 0\ \forall Y_p \in T_pM$ if and only if $X_p = 0$

which defines an inner product on T_pM. By setting

$$\tilde{g}(X_p)(Y_p) = g(X_p, Y_p)$$

we get an isomorphism

$$\tilde{g} : T_pM \to T_p^*M$$

such that $X_p \mapsto \tilde{g}(X_p) = g(X_p, \cdot)$.

With this isomorphism, we define a vector field called the gradient of f and denoted by $gradf$ such that

$$gradf = \tilde{g}^{-1}(\mathrm{d}f_p).$$

We see that for any vector field X, it satisfies

$$g(gradf, X) = \mathrm{d}f(X) = Xf$$

or equivalently,

$$g(gradf, \cdot) = \mathrm{d}f.$$

With the Euclidean metric on $\mathbb{R}^n$

$$gradf = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$$

otherwise

$$gradf = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(g_{ij} \frac{\partial f}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

Definition 1.0.8. Let X be a smooth vector field on M. An integral curve is a smooth curve $\psi : \mathbb{R} \to M$ such that

$$\dot{\psi}(t) = X_{\psi(t)}$$

for all $t \in \mathbb{R}$.

Example: Consider the following vector field on $\mathbb{R}^2$: $X = y\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$. We will find the integral curve $\psi(t)$ with $\psi(0) = (a, b)$ for $b = 0$ and $b \neq 0$ seperately and find the values of t where $\psi(t)$ is defined. Let us write $\psi : \mathbb{R} \to \mathbb{R}^2$ where $\psi(t) = (x(t), y(t))$ and $\psi^{'}(t) = X_{\psi(t)}$. So,

$$x^{'} \left.\frac{\partial}{\partial x}\right|_{\psi(t)} + y^{'} \left.\frac{\partial}{\partial y}\right|_{\psi(t)} = y \left.\frac{\partial}{\partial x}\right|_{\psi(t)} + y^2 \left.\frac{\partial}{\partial y}\right|_{\psi(t)}.$$

We obtain two first order differential equations $x^{'} = \frac{\mathrm{d}x}{\mathrm{d}t} = y$ and $y^{'} = \frac{\mathrm{d}y}{\mathrm{d}t} = y^2$. If $y \neq 0$, we can solve the second seperable equation by integrating both sides, $y^{-2}\,\mathrm{d}y = \mathrm{d}t$. So, $y(t) = \frac{-1}{t+c}$. Again, if $y \neq 0$, we can solve the first seperable equation $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{-1}{t+c}$. So, $x(t) = ln\left|\frac{d}{t+c}\right|$. Note that if $y = 0$, $\frac{\mathrm{d}y}{\mathrm{d}t} = y^2$ is also satisfied and in this case, $x(t) = c$.

So, when $(a, b) = (a, 0)$, $\psi(t) = (a, 0)$ so that $\psi(0) = (a, 0)$. If $b \neq 0$,

$$\psi(t) = \left(ln \left| \frac{e^a}{b^t - 1} \right|, \frac{-b}{b^t - 1} \right)$$

so that $\psi(0) = (a, b)$. Moreover, if $\psi(0) = (a, b)$ with $b = 0$, $\psi(t) = (a, 0)$ so that ψ is defined for all $t \in \mathbb{R}$. If $\psi(0) = (a, b)$ with $b \neq 0$, $\psi(t) = \left(ln \left| \frac{e^a}{b^t - 1} \right|, \frac{-b}{b^t - 1} \right)$ so that ψ is defined for all $t \in \mathbb{R} - \left\{ \frac{1}{b} \right\}$.

2 Morse Homology

2.1 Morse Functions

Let M be a smooth manifold and $f : M \to \mathbb{R}$ be a smooth function.

Definition 2.1.1. A point $p \in M$ is called a critical point for f if $\mathrm{d}f_p = 0$. A point which is not critical is called a regular point. Critical values are points in the image of critical points and regular values are points in $\mathbb{R}$ with no critical point in the pre-image under f.

Definition 2.1.2. The Hessian of f at a critical point p, $H(f,p) : T_pM \to T_p^*M$ is defined by $H(f,p)(v) = \nabla_v(\mathrm{d}f)$ where ∇ is any connection on TM, $v \in T_pM$.

A connection, [19], in TM is a map

$$\nabla : \mathscr{T}(M) \times \mathscr{T}(M) \to \mathscr{T}(M)$$

defined by $\nabla(X,Y) = \nabla_X Y$ which satisfies the following conditions for any function $f, g \in C^\infty(M)$ and $a, b \in \mathbb{R}$:

i. $\nabla_X Y$ is linear over $C^\infty(M)$, i.e. $\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$

ii. $\nabla_X Y$ is linear over $\mathbb{R}$, i.e. $\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2$

iii. ∇ satisfies the Leibniz rule, i.e. $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$

Observe that the definition of the Hessian does not depend on the choice of the connection: The fact that ∇ is a connection means that for any function $f \in C^\infty(M)$

$$\nabla_{fv}(\mathrm{d}f) = f\nabla_v(\mathrm{d}f)$$

and

$$\nabla_v(f\,\mathrm{d}f) = f\nabla_v(\mathrm{d}f) + v(f)\cdot \mathrm{d}f$$

Note that ∇ is linear over $C^\infty(M)$ in v, but not in $\mathrm{d}f$. Also, $v(f)$ does not depend on ∇. Indeed, if ∇^1 and ∇^2 are two different connections on TM, the difference of these connections is linear in both v and $\mathrm{d}f$. Hence, the tensor $T(v, \mathrm{d}f) = \nabla^1_v(\mathrm{d}f) - \nabla^2_v(\mathrm{d}f)$ satisfies $T(v,0) = 0$ because T is a linear operator in $\mathrm{d}f$ for every fixed v. Since $\mathrm{d}f$ vanishes at a critical point p, we get $(\nabla^1 - \nabla^2)_v(\mathrm{d}f) = 0$ at p. Consequently, $\nabla^1 = \nabla^2$.

By using the Riemannian metric, the Hessian can be defined as a symmetric bilinear map [5], $H(f,p) : T_pM \times T_pM \to \mathbb{R}$ given by $H(f,p)(v,w) = \tilde{v}(\tilde{w}(f))_p = \tilde{v}_p(\tilde{w}(f))$, where $\tilde{v}$ and $\tilde{w}$ are smooth vector fields such that $\tilde{v}_p = v$ and $\tilde{w}_p = w$. Hessian is bilinear; it is also symmetric:

$$H(f,p)(v,w) - H(f,p)(w,v) = \tilde{v}_p(\tilde{w}(f)) - \tilde{w}_p(\tilde{v}(f)) = [\tilde{v},\tilde{w}]_p(f) = \mathrm{d}f_p([\tilde{v},\tilde{w}]) = 0$$

Hence, Hessian is symmetric bilinear map on T_pM because $\mathrm{d}f_p = 0$ for critical point p. Also, $H(f,p)$ is independent of the choice of the extensions of v and w since $\tilde{v}_p(\tilde{w}(f)) = v(\tilde{w}(f))$ and $\tilde{w}_p(\tilde{v}(f)) = w(\tilde{v}(f))$. If $x_1, ..., x_n$ are local coordinates for M near p and $v = \sum_{i=1}^n v_i(p)\frac{\partial}{\partial x_i}|_p$, $w = \sum_{i=1}^n w_i(p)\frac{\partial}{\partial x_i}|_p$, we can take $\tilde{w} = \sum_{i=1}^n w_i \frac{\partial}{\partial x_i}$. Then

$$H(f,p)(v,w) = v(\tilde{w}(f) = v\left(\sum_{i=1}^n w_i \frac{\partial f}{\partial x_i}\right) = \sum_{i=1}^n \sum_{j=1}^n a_j b_i \frac{\partial^2 f}{\partial x_i \partial x_j}$$

So, $H(f,p)$ is locally represented in terms of the basis $\left\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right\}$ by the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{n\times n}$.

Definition 2.1.3. Let p be a critical point of $f : M \to \mathbb{R}$.

i. The critical point p is called non-degenerate if the Hessian does not have zero as an eigenvalue.

ii. The index of a non-degenerate critical point, $ind(p)$, is defined to be the number of negative eigenvalues of the Hessian.

Lemma 2.1.4. *[5] Non-degenerate critical points are isolated.*

Proof. Consider a chart $\varphi : U \to \mathbb{R}^n$, with U as an open neighborhood of a non-degenerate critical point p, and $\varphi(p) = 0$. Define the map $g : \varphi(U) \to \mathbb{R}^n$ given by

$$g(x) = \left(\frac{\partial(f\circ\varphi^{-1})}{\partial x_1}(x), ..., \frac{\partial(f\circ\varphi^{-1})}{\partial x_n}(x)\right)$$

Then $g(0) = 0$ and since p is non-degenerate, dg is nonsingular . By the Inverse Function Theorem [18], g is a diffeomorphism of some open neighborhood V of p; in particular it is injective. This implies that $g(x) \neq 0$ for $0 \neq x \in V$. So, x can not be a critical point for f. ■

Definition 2.1.5. We say that a real valued function f on a manifold M is Morse if all of its critical points are non-degenerate.

From now on, f denotes a Morse function.

2.2 The gradient flow

Let g be a metric on M, and let $-gradf$ be the negative gradient of f with respect to g. Since gradient vector field is complete (i.e, its flow curves exist for all time), $-gradf$ generates a one-parameter group of diffeomorphisms $\psi_s : M \to M$ by sending $p \in M$ to the point obtained by following the integral curve starting at p for time s, i.e ψ_s is a diffeomorphism for all $s \in \mathbb{R}$ such that $\psi_{s+t} = \psi_s \circ \psi_t$, $\psi_0 = id$ and for any point $p \in M$ we have

$$\frac{d\psi_s}{dt}(p) = -gradf(\psi_s(p))$$

The fact that $\psi_s(p)$ is an integral curve for a fixed point $p \in M$ implies that any two different flows can not cross: if the flows intersect at any point y, then the tangent vectors at y must be the same, but this means that these two flows coincide.

Definition 2.2.1. Let p be a critical point.

i. The descending manifold of p is defined to be:

$$\mathscr{D}(p) = \left\{ x \in M : \lim_{s \to -\infty} \psi_s(x) = p \right\}$$

ii. The ascending manifold of p is defined to be:

$$\mathscr{A}(p) = \left\{ x \in M : \lim_{s \to \infty} \psi_s(x) = p \right\}$$

$\mathscr{D}(p)$ and $\mathscr{A}(p)$ are the set of all points in M that flow to critical points in backward and forward time respectively. The descending manifold is sometimes called the unstable manifold and the ascending manifold is sometimes called the stable manifold. We can define stable and unstable sets for any dynamical system, however they may not always form submanifolds.

Proposition 2.2.2. *If p is a non-degenerate critical point, then $\mathscr{D}(p)$ and $\mathscr{A}(p)$ are embedded discs in M with dimensions $ind(p)$ and $dim(M) - ind(p)$, respectively. Further, $T_p\mathscr{D} \subset T_pM$ is the negative eigenspace of $H(f,p)$ and $T_p\mathscr{A} \subset T_pM$ is the positive eigenspace of $H(f,p)$.*

For the proof see [13] or [5].

Definition 2.2.3. A pair (f, g) is said to be Morse-Smale if $f : M \to \mathbb{R}$ is a Morse function and for all $p, q \in Crit(f)$, the descending manifold $\mathscr{D}(p)$ is transverse to the ascending manifold $\mathscr{A}(q)$, where $Crit(f)$ is the set of all critical points of f.

Remark 2.2.4. Let M be a smooth manifold. Two embedded submanifolds $S_1, S_2 \subset M$ are said to be transverse if for each $p \in S_1 \cap S_2$, the tangent spaces T_pS_1 and T_pS_2 together span T_pM, i.e $T_pM = T_pS_1 + T_pS_2$ for all $p \in S_1 \cap S_2$.

Proposition 2.2.5. *If two submanifolds S_1 and S_2 of M are transversal, then $S_1 \cap S_2$ is a submanifold of M and $dim(S_1 \cap S_2) = dim(S_1) + dim(S_2) - dim(M)$.*

2.3 The Moduli Spaces

Given a pair of critical points $p, q \in M$, we consider the set of flow lines of negative gradient vector field $-gradf$ which converge to p and q in forward and backward time, respectively.

$$\mathscr{W}(p,q) = \left\{ \gamma : \mathbb{R} \to M : \frac{d\gamma}{ds} = -gradf(\gamma(s)), \lim_{s\to-\infty} \gamma(s) = p, \lim_{s\to\infty} \gamma(s) = q \right\}.$$

There is a one-to-one correspondence between $\mathscr{D}(p) \cap \mathscr{A}(q)$ and $\mathscr{W}(p,q)$ as sets because given any point x in the intersection of descending and ascending manifolds of p and q, respectively, there is a flow line $\gamma_s(0) = x$. In fact all flow lines passing through that point are equivalent up to parametrization. Also, $\mathbb{R}$ acts on $\mathscr{W}(p,q)$, $\mathbb{R} \times \mathscr{W}(p,q) \longrightarrow \mathscr{W}(p,q)$ such that

$$(t, \gamma)(s) = \gamma(s+t).$$

The quotient

$$\mathscr{M}(p,q) = \mathscr{W}(p,q)/\mathbb{R}$$

is the moduli space of flow lines by this action. Under this action, the elements of $\mathscr{M}(p,q)$ are unparametrised flow lines. We now show that $\mathscr{M}(p,q)$ admits a unique smooth structure of dimension $ind(p) - ind(q) - 1$.

Definition 2.3.1. A Lie group is a smooth manifold G that is also a group such that the multiplication map $m : G \times G \to G$, $(g, h) \mapsto gh$ and the inversion map $i : G \to G$, $g \mapsto g^{-1}$ are both smooth.

Theorem 2.3.2. *(**Quotient Manifold Theorem**, [18]) Suppose a Lie group G acts smoothly, freely, and properly on a smooth manifold M. Then the quotient space M/G is a topological manifold of dimension $dimM - dimG$, and it has a unique smooth structure with the property that the quotient map $\pi : M \to M/G$ is a smooth submersion.*

Since, there is a natural Lie group structure on $\mathbb{R}$ under addition, it is enough to prove that the action defined above is smooth, free and proper. The action is free: given any $t \in \mathbb{R}$ with the property that $t \cdot \gamma = \gamma$, $t \cdot \gamma(s) = \gamma(s)$. This implies $\gamma(s + t) = \gamma(s)$. So, $t = 0$. Moreover, the action is proper:

Proposition 2.3.3. *([18], Proposition 9.13) A smooth action of a Lie group G on M is proper if and only if the following condition is satisfied: If p_i is a convergent sequence in M and g_i is a sequence in G such that $\{g_i \cdot p_i\}$ converges, then there is a convergent subsequence of $\{g_i\}$.*

In the sense of this proposition, we will show that the $\mathbb{R}$-action given by $t \cdot \gamma(s) = \gamma(s+t)$ is proper: Let $\{\gamma_n\}$ and $\{t_n\}$ be sequences of $\mathscr{W}(p, q)$ and $\mathbb{R}$, such that $\gamma_n \to \gamma \in \mathscr{W}(p, q)$ and $t_n \cdot \gamma_n \to \alpha \in \mathscr{W}(p, q)$. Our aim is to show that $\{t_n\}$ is included in a compact set $K \subset \mathbb{R}$. Suppose for the contrary that $\{t_n\}$ is not contained in a compact subset of $\mathbb{R}$. Then, it can not be bounded, so there exists a subsequence $\{t_{n_k}\}$ such that $t_{n_k} \to \infty$. Since $\gamma_n(.) \to \gamma(.) \in \mathscr{W}(p, q)$ uniformly, given any ϵ

$$d(\gamma_n, \gamma) < \epsilon.$$

On the other hand, $t_{n_k} \cdot \gamma \to q$ and $t_{n_k} \cdot \gamma_n \to \alpha \in \mathscr{W}(p, q)$. So, $\alpha = q$.

Proposition 2.3.4. *Let $p \neq q$ be two critical points of a Morse-Smale pair (f, g). If $\mathscr{W}(p, q)$ is non-empty, it is a smooth manifold of dimension $ind(p) - ind(q)$.*

Proof. This follows from Proposition (2.2.5), because Morse-Smale property of (f, g) assures the transversality of $\mathscr{D}(p)$ and $\mathscr{A}(q)$. Also $dim(\mathscr{W}(p, q)) = ind(p) + ind(M) - ind(q) - ind(M) = ind(p) - ind(q)$. ∎

Consequently, theorem 2.3.2 and proposition 2.3.3 guarantees that $\mathscr{M}(p, q)$ is a smooth manifold with dimension $ind(p) - ind(q) - 1$.

2.4 Transversality

In Proposition 2.3.3, we have shown that $\mathscr{W}(p,q)$ is a manifold of dimension $ind(p)-ind(q)$ by using Proposition 2.2.5. As a consequence, we observe that $\mathscr{M}(p,q)$ is actually a manifold of dimension $ind(p)-ind(q)-1$. Now, we will give a sufficient condition for a pair (f,g) to be Morse-Smale.

Theorem 2.4.1. *Let M be a closed smooth manifold, let k be a positive integer, and let $f: M \to \mathbb{R}$ be a C^{k+1} Morse function on M. Then for a generic C^k metric on X, the pair (f,g) is Morse-Smale.*

Definition 2.4.2. A property defined for elements of a topological space X is said to be generic if it is satisfied by a subset of objects in X which contains a countable intersection of open dense sets.

For the proof of theorem 2.4.1, we are going to state some definitions and results in functional analysis. More details are described in [1].

Definition 2.4.3. For an open set $U \subset \mathbb{R}^n$, $L^p_k(U)$ is the completion of $C^\infty(U)$ with respect to the norm $\|.\|_{k,p}$

$$\|u\|_{k,p} = \sum_{|I|\leq k} \left(\int_U |\partial^I u|^p \, \mathrm{d}x \right)^{1/p}$$

where $I=(i_1,i_2,...,i_n)$, $\partial^I = (\partial_1)^{i_1}.....(\partial_n)^{i_n}$, $\partial_j = \dfrac{\partial}{\partial x_j}$ and $|I| = i_1+i_2+....+i_n$.

We can also define L^p_k for a compact manifold M. $L^p_k(M)$ is the completion of $C^\infty(M)$ with respect to the norm defined below. Since M is compact, we pick a finite cover by charts

$$\varphi_i : U_i \subset M \to V_i \subset \mathbb{R}^n.$$

For $u: M \to \mathbb{R}^n$, define $\|u\|_{k,p} = \sum_i \|u \circ \varphi_i^{-1}\|_{L^p_k(V_i)}$. We sometimes use the notation L^p_k instead of $L^p_k(M)$.

Note that we construct completions by eliminating the Cauchy sequences which do not converge to any point. Changing the choice of chart, we get an equivalent norm because M is compact. For all choice of p, L^p_k is a Banach space and it is a Hilbert space if $p=2$.

Remark 2.4.4. $L^p_{k,loc}$ means the set of locally L^p_k maps, i.e it is the completion of smooth compactly supported functions C^∞_c with respect to the topology that u_n converges to u if and only if u_n converges to u on any compact set $C \subset\subset U$ means that C is open in U and $C \subset \overline{C} \subset U$.

Theorem 2.4.5. *(**Sobolev embedding theorem**, [1]) Let M be an n-dimensional manifold.*
If $k > k^{'}$ and $k - \frac{n}{p} \geq k^{'} - \frac{n}{p^{'}}$, then there is an embedding

$$L_k^p \hookrightarrow L_{k^{'}}^{p^{'}}.$$

If $k - \frac{n}{p} > l$, then there is a continuous embedding

$$L_k^p(M) \hookrightarrow C^l(M).$$

So, on a 1-manifold, $L_1^2 \subset C^0$.

Corollary 2.4.6. *Theorem holds for $L_{k,loc}^p$, i.e if $k > k^{'}$ and $k - \frac{n}{p} \geq k^{'} - \frac{n}{p^{'}}$, then there is an embedding*

$$L_{k,loc}^p \hookrightarrow L_{k^{'},loc}^{p^{'}}.$$

If $k - \frac{n}{p} > l$, then there is a continuous embedding

$$L_{k,loc}^p(M) \hookrightarrow C^l(M).$$

Proof. Let $u \in L_{k,loc}^p(M)$. So, the restriction of u on a compact set C, $u|_C$, is in $L_k^p(M)$. By theorem for L_k^p, $u|_C$ is in $L_{k^{'}}^{p^{'}}(M)$, and $u|_C$ is in $C^l(M)$ for $k - \frac{n}{p} > l$ and $k - \frac{n}{p} \geq k^{'} - \frac{n}{p^{'}}$. So, $u \in L_{k^{'},loc}^{p^{'}}$ and $u \in C^l(M)$. ■

If V and W are Banach spaces, then a function $f : V \to W$ is differentiable at $p \in V$ if there exists a bounded linear map $\mathrm{d}f_p : V \to W$ such that

$$\lim_{v \to 0} \frac{\|f(p+v) - f(p) - \mathrm{d}f_p(v)\|}{\|v\|} = 0.$$

If such a $\mathrm{d}f_p$ exists, then it will be unique. If f is differentiable for all p, then we obtain $\mathrm{d}f : V \to Hom(V, W)$, and we can talk about the differentiability of f. So, we can define infinite dimensional manifold structure for infinite dimensional Banach spaces.

Definition 2.4.7. A (smooth) Banach manifold M is a Hausdorff, second countable topological space with a cover U_α and a collection of charts $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$, where $\varphi_\alpha(U_\alpha)$

is open in some Banach space X_α. Given any two charts (U, φ) and (V, ϕ), we have that $\phi \circ \varphi^{-1} : \varphi(U \cap V) \to \phi(U \cap V)$ and $\varphi \circ \phi^{-1} : \phi(U \cap V) \to \varphi(U \cap V)$ are C^∞ mappings between open sets. If chart spaces X_α' s are equal to a fixed Banach space X, then we say that M is a Banach manifold modelled on X.

Example. Let M and N be closed manifolds. Then $C^k(N, M) = \left\{f : N \to M | f \in C^k\right\}$ is a C^∞ Banach manifold, [10].

We can also develop many notions of the finite dimensional manifolds to Banach manifolds such as Inverse Function Theorem, Implicit Function Theorem and Transversality definitions. Sard's Theorem can not be used directly as in the finite dimensional case, but it also works under some specific conditions.

Theorem 2.4.8. *Let $f : M \to N$ be a C^k map between Banach manifolds. If $q \in N$ is a regular value, then $f^{-1}(q)$ is a C^k submanifold of M with $T_p f^{-1}(q) = ker(\mathrm{d}f_p)$ for all $p \in f^{-1}(q)$.*

Definition 2.4.9. Let V and W be Banach spaces. A continuous linear map $F : V \to W$ is Fredholm if the following conditions are satisfied:

i. the kernel of F is finite dimensional, $dimKer(F) < \infty$.

ii. the image of F has finite codimension in W, $dimCoker(F) < \infty$.

iii. image of F is closed in W.

If F is Fredholm, then the index of F is defined to be,

$$ind(F) = dimKer(F) - dimCoker(F).$$

Example: For $0 < p < \infty$, ℓ^p is the subspace of the set of all real sequences $(x_n)_{n \in \mathbb{N}}$, satisfying

$$\sum_n |x_n|^p < \infty.$$

Consider the right and left shift operators R and L defined respectively by $R(x_1, x_2, ...) = (0, x_1, x_2, ...)$ and $L(x_1, x_2, ...) = (x_2, x_3, ...)$. These are Fredholm operators. Note that $kerR = 0$, $cokerR \cong \mathbb{R}$ and $kerL \cong \mathbb{R}$, $cokerL = 0$.

Remark 2.4.10. Consider the set of Fredholm operators $\mathscr{F}(V, W)$ with the norm topology, then $ind : \mathscr{F}(V, W) \to \mathbb{Z}$ is locally constant because of discreteness of $\mathbb{Z}$.

Definition 2.4.11. A map $F : M \to N$ between Banach manifolds is a Fredholm map of index k if

$$\mathrm{d}F_p : T_pM \to T_{F(p)}N$$

is a linear Fredholm operator of index k for all $p \in M$.

Now, we are ready to state Sard-Smale theorem for Banach manifolds.

Theorem 2.4.12. *Let M and N be seperable Banach manifolds. If $F : M \to N$ is a C^k Fredholm map and $k > max(0, ind(F))$, then a generic $q \in N$ is a regular value of F, i.e. $\mathrm{d}F_p$ is onto for all $p \in F^{-1}(q)$, so by Theorem* 2.4.8*, $F^{-1}(q)$ is naturally a submanifold of dimension $ind(F)$.*

Proof. See [25]. ■

Let M and E be Banach manifolds. A smooth surjective map $\pi : E \to M$ is said to be a Banach bundle, [20], if there is an open cover $\{U_\alpha\}$ of M and smooth maps $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times W_\alpha$, where W_α is a Banach space such that for any two maps φ_α, φ_β the map $U_\alpha \cap U_\beta \to Lin(W_\alpha, W_\beta)$, defined by $p \mapsto (\varphi_\beta \circ \varphi_\alpha^{-1})p$, is smooth. Here $Lin(X, Y)$ denotes the space of all continuous linear maps from a topological vector space X to a topological vector space Y. Also $E_p = \pi^{-1}(p)$ is a Banach space. We will write the elements of E as pairs $(p, e) \in M \times E$ where $\pi(e) = p$. The tangent space of E at $(p, 0)$ naturally splits into two parts

$$T_{(p,0)}E = T_pM \oplus E_p.$$

For $p \in M$, let $\pi_p : T_{(p,0)}E \to E_p$ be the projection onto the second factor. If $s : M \to E$ is a section, then we define a new function which is a composition of two functions defined as above, $\pi_p \circ \mathrm{d}s_p : T_pM \to E_p$. From now on, we will denote the composition as ∇s_p, i.e. $\nabla s_p = \pi_p \circ \mathrm{d}s_p$. Now we state and prove a theorem which is very useful for the proof of Theorem 2.4.1.

Theorem 2.4.13. *Let M and N be seperable Banach manifolds, $E \to M \times N$ a Banach space bundle, and $s : M \times N \to E$ a smooth section. Suppose that for all $(p, q) \in s^{-1}(0)$, the following hold:*

i. The differential $\nabla s_{(p,q)} : T_{(p,q)}(M \times N) \to E_{(p,q)}$ is surjective.

ii. The restricted differential $\nabla s_{(p,q)} : T_qN \to E_{(p,q)}$ is Fredholm of index l.

Then, for a generic $q \in M$, the set $\{q \in N | s(p,q) = 0\}$ is an l-dimensional submanifold of N. Moreover, $\nabla s_{(p,q)}$ is surjective on the tangent space to N.

Proof. By the first condition on s, we can say that $\nabla s_{(p,q)}$ is not zero, so (p,q) is a regular point, the image of this point is 0 and the implicit function theorem implies that $s^{-1}(0)$ is a Banach submanifold of $M \times N$. Let $\pi : s^{-1}(0) \to M$ be the projection onto the first factor. We claim that for each $(p,q) \in s^{-1}(0)$, the projection $\mathrm{d}\pi : T_{(p,q)}s^{-1}(0) \to T_pM$ is Fredholm. To prove that this claim, we first observe that the kernel of $\mathrm{d}\pi$ and the kernel of $\nabla s_{(p,q)}$ are isomorphic, i.e.

$$ker(\mathrm{d}\pi : T_{(p,q)}s^{-1}(0) \to T_pM) \cong ker(\nabla s_{(p,q)} : T_qN \to E_{(p,q)}).$$

Since $\mathrm{d}\pi(v,w) = v$,

$$\begin{aligned} ker(\mathrm{d}\pi) &= \big\{(0,w) \in T_{(p,q)}(M \times N) | \nabla s_{(p,q)}(w) = 0\big\} \\ &\cong \big\{w \in T_qN | \nabla s_{(p,q)}(w) = 0\big\} \\ &= ker(\nabla s_{(p,q)}). \end{aligned}$$

So, the dimension of $ker(\mathrm{d}\pi)$ is finite. Secondly, we observe that the cokernel of $\mathrm{d}\pi$ and the cokernel of $\nabla s_{(p,q)}$ are isomorphic, i.e.

$$coker(\mathrm{d}\pi : T_{(p,q)}s^{-1}(0) \to T_pM) \cong coker(\nabla s_{(p,q)} : T_qN \to E_{(p,q)})$$

that is, we will show that

$$T_pM/im(\mathrm{d}\pi) \cong E_{(p,q)}/im(\nabla s_{(p,q)}).$$

The map $\varphi : T_pM \xrightarrow{i} T_{(p,q)}(M \times N) \xrightarrow{\nabla s_{(p,q)}} E_{(p,q)} \xrightarrow{\tilde{\pi}} E_{(p,q)}/im(\nabla s_{(p,q)})$ defined by $v \mapsto (v,0)$ is a linear map. It is onto because of the first condition on the theorem. Now, it is enough to show that $ker(\varphi) = \mathrm{d}\pi(T_{(p,q)}s^{-1}(0))$: image of any element in $ker(\varphi)$ is also in $im(\nabla s_{(p,q)})$, on the other hand, we know that

$$\mathrm{d}\pi(T_{(p,q)}s^{-1}(0)) = \{v \in T_pM | \nabla s(v,0) \in \nabla s(T_qN)\}$$

So, the equality desired above is satisfied. Therefore, by the first isomorphism theorem, the finite dimensional cokernel property follows from the discussion above. Finally, it remains to be proven that the image of $\mathrm{d}\pi$ is closed. This follows from the fact that $\nabla s(T_qN)$ is closed and the inverse image of a closed set under the continuous function is closed. Note that $im(\mathrm{d}\pi)$ is the pre-image of $im(\nabla_{(p,q)}s)$ under the continuous map $v \mapsto \nabla s(v, 0)$.

By theorem 2.4.13, Sard-Smale theorem, and claim, a generic $p \in M$ is a regular value of $\pi : s^{-1}(0) \to M$. This implies that for such a p, the set $\pi^{-1}(p) = \{q \in N | s(p, q) = 0\}$ is a submanifold of N by the implicit function theorem. Again by Sadr-Smale theorem, we have that dimension of $\pi^{-1}(p)$ is the index of $ind(\mathrm{d}\pi)$ and $dimCoker(\mathrm{d}\pi) = 0$. Since $dimCoker(\mathrm{d}\pi) = dimCoker(\nabla s_{(p,q)})$ and $ker(\mathrm{d}\pi) = ker(\nabla s_{(p,q)})$, dimension of $\{q \in N | s(p, q) = 0\}$ is equal to index of $\nabla s_{(p,q)}$ which equals to l. Moreover, since $dimCoker(\nabla s_{(p,q)}) = 0$, for each (p, q) is in this submanifold, the restricted differential $\nabla s_{(p,q)} : T_qN \to E_{(p,q)}$ is surjective. ■

Proof of Theorem 2.4.1: We start proving theorem 2.4.1 by fixing distinct critical points $p \neq q \in Crit(f)$. Let Y be the space of all C^k-metrics on M. Note that Y is smooth Banach manifold: see, [11]. Let Z be the space of locally L^2_1 maps $\gamma : \mathbb{R} \to M$, i.e. $\gamma \in L^2_{1,loc}(\mathbb{R}, M)$ such that:

- $\lim_{s\to-\infty} \gamma(s) = p$ and for any sufficiently negative R such that $\gamma(-\infty, R]$ is contained in a coordinate chart around p, we have $\gamma|_{(-\infty,R]} \in L^2_1((-\infty, R], T_pM)$.
- $\lim_{s\to\infty} \gamma(s) = q$ and for any sufficiently positive R such that $\gamma[R, \infty)$ is contained in a coordinate chart around q, we have $\gamma|_{[R,\infty)} \in L^2_1([R, \infty), T_pM)$.

By Theorem 2.4.5 and its corollory, $L^2_{1,loc}(\mathbb{R}, M) \subset C^0_{loc}(\mathbb{R}, M)$, so $\gamma \in Z$ is continuous, and requiring convergence to p, q while s is going to $\mp\infty$ makes sense. Note that Z is a smooth Banach manifold modelled on $L^2_1(\mathbb{R}, \mathbb{R}^n)$ with $T_\gamma Z = L^2_1(\gamma^*TM)$.

Recall that the pullback bundle of C^k bundle E over M with projection $\pi : E \to M$ is a new C^k bundle f^*E over N with projection map $\hat{\pi} : f^*E \to N$, where $f : N \to M$ is a C^k map between smooth manifolds N and M. If $\alpha : \pi^{-1}(U) \to U \times E_{f(x)}$ is a trivialization of E, then

$$\hat{\alpha} : \hat{\pi}^{-1}(f^{-1}(U)) \to f^{-1}(U) \times E_x$$

is a trivialization of f^*E. The fibre of f^*E over a point $x \in N$ is the fiber of E over $f(x) \in M$.

Let $E \to Z$ be the Banach space bundle whose fiber bundle over $\gamma \in Z$ is $E_\gamma :=$

$L^2(\gamma^*TM)$, the space of L^2-sections of γ^*TM, whose fibre is $(\gamma^*TM)_s = T_{\gamma(s)}M$ over $s \in \mathbb{R}$. We can then define the section $\psi_g : Z \to E$ by $\gamma \mapsto \gamma^{'}(s) + gradf(\gamma(s))$. We now extend the section ψ_g to a section $\psi : Y \times Z \to E^*$, E^* is the pullback of the bundle E to $Y \times Z$ via projection to Z, as follows

$$\psi(g, \gamma)(s) = \gamma^{'}(s) + gradf(\gamma(s)).$$

First, we will show that ψ is a well-defined section: the derivative $\dfrac{\mathrm{d}}{\mathrm{d}s} : L^2_1 \to L^2$ is well-defined, so the issue comes from *gradf*. Since *gradf* is C^k, it is L^2 on compact sets, so we just need to check that $gradf(\gamma)$ is L^2 near the ends. Locally, near the critical point p, we have an estimate $|gradf(x)| \leq c|x|$ (This inequality comes from the Taylor expansion because of the equality $gradf(p) = 0$). Hence, $|gradf(\gamma)| \leq c|\gamma|$, so $gradf(\gamma)$ is in L^2 near ends since $\gamma \in L^2$.

If $\psi(g, \gamma) = 0$, then γ is a C^{k+1} negative gradient flow line of f from p to q with respect to g:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}s} = \gamma^{'} = -gradf(\gamma)$$

is continuous since γ is continuous. So, γ is in C^1. This implies that $gradf(\gamma)$ is C^1. So, γ is C^2. From this way we conclude that γ is C^{k+1}.

We claim now that the hypothesis of Theorem 2.4.13 are satisfied: If $\psi(g, \gamma) = 0$, then

$$\nabla\psi(\dot{g}, \dot{\gamma}) = \nabla_{\gamma'}\dot{\gamma} - \nabla_{\dot{\gamma}}(gradf) - (gradf)^{'}.$$

For the computation of this formula, see [15].

We start to verify the statements of Theorem 2.4.13. First, we will show that $\nabla\psi$ is surjective. Assume $\nabla\psi$ is not onto. If we can show that the image of $\nabla\psi$ is closed then there is a nonzero $w \in L^2(\gamma^*TM)$ such that

$$\int \langle \nabla\psi(\dot{g}, \dot{\gamma}), w \rangle \ \mathrm{d}s = 0$$

for all $(\dot{g}, \dot{\gamma}) \in T_{(g,\gamma)}(Y \times Z)$. By choosing $\dot{\gamma}$ is zero, we get

$$\int \left\langle (gradf)^{'}, w \right\rangle \ \mathrm{d}s = 0$$

for all $\dot{g} \in T_gY$.

We now assume that the restricted map $\nabla\psi_g$ is Fredholm (we will show it later). We use this assumption to show that $\nabla\psi$ is closed: since $im(\nabla\psi_g) \subset im(\nabla\psi)$, $im(\nabla\psi) = im(\nabla\psi_g) \oplus (C \cap im(\nabla\psi))$ where C is the complement of $im(\nabla\psi_g)$ in $E_{(g,\gamma)}$ and it is finite dimensional. Then, $C \cap im(\nabla\psi)$ is also finite dimensional. So, $im(\nabla\psi)$ is closed because of closedness of $im(\nabla\psi_g))$.

Fix $s_0 \in \mathbb{R}$. For any vector w in the fibre of γ^*TM over s_0, we claim that there is a $\dot{g} \in T_gY$ such that $(gradf)' = w$: since $\mathrm{d}f = g(gradf, .)$, locally $gradf = g^{-1}(\mathrm{d}f_p)$. So

$$\begin{aligned}
(gradf)'\dot{g} &= (\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (g + t\dot{g})^{-1})\,\mathrm{d}f_p \\
&= (\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} [g \cdot (1 + tg^{-1}\dot{g})]^{-1})\,\mathrm{d}f_p \\
&= (\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (1 + tg^{-1}\dot{g})^{-1} \cdot g^{-1})\,\mathrm{d}f_p \\
&= (\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (1 - tg^{-1}\dot{g} + t^2(g^{-1}\dot{g})^2 - \cdots)) \cdot g^{-1}\,\mathrm{d}f_p \\
&= -g^{-1}\dot{g}g^{-1}\,\mathrm{d}f_p
\end{aligned}$$

By taking $\dot{g} = gSg$, where S is the symmetric matrix, $(gradf)'\dot{g}$ will be symmetric and since it is arbitrary at s_0, we pick $\dot{g}$ so that $(gradf)' = w$.

By our assumption, w is nonzero so we can fix an s_0 such that $w(s_0) \neq 0$. So, we can choose $\dot{g}$ such that $(gradf)'(s_0) = w(s_0)$. So, $\langle (gradf)', w\rangle$ is greater than 0. By multiplying $\dot{g}$ by a bump function $\beta : \mathbb{R} \to \mathbb{R}$, 0 away from $\gamma(s_0)$ and 1 at $\gamma(s_0)$, $\tilde{\dot{g}}(s) = \beta(s)\dot{g}(s)$. Using the C^k-smoothness of w,

$$\int \Big\langle (gradf)', w \Big\rangle\, \mathrm{d}s > 0$$

a contradiction, so $\nabla\psi_g$ must be onto.

Secondly, we now show that the restricted differential $\nabla\psi_{(g,\gamma)} : T_\gamma Z \to E_{(g,\gamma)}$ defined by $\dot{\gamma} \mapsto \nabla_{\gamma'}\dot{\gamma} + \nabla_{\dot{\gamma}}(gradf)$ is Fredholm. If we travialize γ^*TM using parallel translation with respect to ∇ on TM in the following way, see [19] : Let $e_1, e_2, ..., e_m$ be the basis of γ^*TM with $\nabla_{\gamma'}e_i = 0$ and $\dot{\gamma} = (V^1, V^2, ..., V^m)^T = \sum V^i e_i$. Then $\nabla_{\gamma'}\dot{\gamma} = \sum \partial_s V^i e_i$ and

$\nabla_{\dot{\gamma}}(gradf) = \sum A_s V^j e_i$, so in this trivialization, we have

$$\nabla\psi_{(g,\gamma)} : L^2_1(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n)$$

defined by $\dot{\gamma} \mapsto (\partial_s - A_s)\dot{\gamma}$ where $A_s : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $A_s(w) = \nabla_w(gradf(\gamma(s))$, in fact A_s is the covariant derivative from $T_{\gamma(s)}M$ to $T_{\gamma(s)}M$. From the definition of Hessian, we observe that $\lim_{s\to-\infty} A_s = -H(f,p)$ and $\lim_{s\to\infty} A_s = -H(f,q)$. Since these are self-adjoint and invertible, theorem stated below applies to prove the Fredholm property.

Theorem 2.4.14. *Let $\mathscr{H}$ be Hilbert space and let $\{A_s | s \in \mathbb{R}\}$ be a continuous family of operators on $\mathscr{H}$. We assume that A_s converges in the norm topology to invertible self-adjoint operators $A^{\pm}$ as $s \to \pm\infty$. Then the operator*

$$\partial_s - A_s : L^2_1(\mathbb{R}, \mathscr{H}) \to L^2(\mathbb{R}, \mathscr{H})$$

is Fredholm, and $ind(\partial_s - A_s) = -SF\{A_s\}$.

The term $SF\{A_s\}$ is called the spectral flow of the family of operators A_s. It equals the number of eigenvalues of A_s which pass from negative to positive as s goes from $-\infty$ to ∞ minus the number of eigenvalues of A_s which pass from positive to negative as s goes from $-\infty$ to ∞. So,

$$\begin{aligned} SF\{A_s\} &= ind(H(f,p)) - ind(H(f,q)) \\ &= ind(p) - ind(q) \end{aligned}$$

We prove the theorem for a finite dimensional Hilbert space $\mathscr{H}$. Since the finite dimensional Hilbert spaces are isomorphic to $\mathbb{R}^n$, we use $\mathbb{R}^n$ instead of $\mathscr{H}$. Before proving this theorem we state a powerful lemma, [28].

Lemma 2.4.15. *Suppose that A, B and C are Banach spaces, $L : A \to B$ is a bounded linear operator, and $K : A \to C$ is a compact linear operator. If $||a||_A \leq c.(||La||_B + ||Ka||_Z)$ for all $a \in A$ and c is a constant, then L has a closed range and a finite dimensional kernel.*

In our case, $A = L^2_1(\mathbb{R}, \mathbb{R}^n)$, $B = L^2(\mathbb{R}, \mathbb{R}^n)$, $C = L^2([-S,S], \mathbb{R}^n)$, and the maps $L = \partial_s - A_s$ and K is the restriction. K is a compact and bounded linear operator because of Sobolev embedding theorem. For the inequality in lemma, see [2]. So, $L = \partial_s - A_s$ has closed range and finite dimensional kernel. Now, our aim is to show that the cokernel of $L = \partial_s - A_s$ is also finite dimensional, so $\partial_s - A_s$ will be Fredholm. Before showing this we will identify kernel of $\partial_s - A_s$ and determine its dimension to compute the index of the operator $\partial_s - A_s$.

For each $h \in \mathbb{R}^n$, by the fundamental theorem of ODE's, there exists a unique differentiable function $f_h : \mathbb{R} \to \mathbb{R}^n$ solving the equation

$$(\partial_s - A_s)f_h(s) = 0$$

$$f_h(0) = h$$

In fact, since $\mathbb{R}^n$ is not compact, the existence theorem for ODE's gives us the short-time solution defined for $s \in (-\delta, \delta)$ for some $\delta > 0$. But in our case the short-time solution can be continued for all time because we have a uniform upper bound for the eigenvalues of A. This function f_h may or may not be in L^2_1. Accordingly, we define

$$\mathscr{H}^+ = \left\{ h \in \mathbb{R}^n : \lim_{s\to\infty} f_h(s) = 0 \right\}$$

and

$$\mathscr{H}^- = \left\{ h \in \mathbb{R}^n : \lim_{s\to-\infty} f_h(s) = 0 \right\}$$

Lemma 2.4.16. *The map*

$$\Psi : \mathscr{H}^+ \cap \mathscr{H}^- \to Ker(\partial_s - A_s)$$

defined by $h \mapsto f_h$ is an isomoprhism.

Proof. First, we need to show that $f_h \in L^2_1$: $(\partial_s - A_s)f_h = 0$ implies that f_h is in C^1. So, either $f_h(s) \to \infty$ as $s \to \infty$ or $f_h(s) \to 0$ exponentially fast as $s \to \infty$. When $f_h(s) \to \infty$, $f_h \notin L^2$, so not in L^2_1. When $f_h(s) \to 0$ exponentially fast, $f_h \in L^2$. So, $\partial_s f_h = A_s f_h \in L^2$. This implies that $f_h \in L^2_1$.

Ψ is one to one: Suppose $f_h = 0$. This implies that $f_h(s) = 0$ for all $s \in \mathbb{R}$. So, $f_h(0) = 0$ as well, and it is equal to h by the initial condition. So, $h = 0$ and $Ker = \{0\}$.

Ψ is onto: Let $f \in ker(\partial_s - A_s) \subset L_1^2$. This implies that f goes to 0 as $s \to \infty$. Hence, $f = f_h$ for $h = f(0)$. ■

Moreover, if $E^-(A^+)$ denotes the negative eigenspace of A^+ and $E^+(A^-)$ denotes the positive eigenspace of A^-, then we have isomorphisms

$$\mathscr{H}^+ \to E^-(A^+)$$

defined by $h \mapsto |h| \lim_{s\to\infty} \dfrac{f_h(s)}{|f_h(s)|}$.

Similary,

$$\mathscr{H}^- \to E^+(A^-)$$

Thus, we get the equality $\dim(\mathscr{H}^+)= \dim(E^-(A^+))$ and $\dim(\mathscr{H}^-)= \dim(E^+(A^-))$.

Now we examine the cokernel of $\partial_s - A_s$. To do this, we first study some details in functional analysis.

Let $L : A \to B$ be a bounded linear operator, where A is a Banach space and B is a Hilbert space.

Fact 1: If $im(L)$ is closed, then $cokerL \cong (imL)^\perp = \{b \in B | \langle La, b\rangle = 0\}$ for all $a \in A$: In general, if $V \subset B$ is a closed subspace, then $B = V \oplus V^\perp$, so $V^\perp \cong B/V$.

Definition 2.4.17. The formal adjoint $L^* : A^* \to B^*$ of $L : A \to B$, where A and B are Hilbert space, is defined by

$$\int_M \langle La, b\rangle_B = \int_M \langle a, L^*b\rangle_A$$

for all $a \in A$ and $b \in B$.

Fact 2: $(imL)^\perp \cong kerL^*$: $b \perp imL \Leftrightarrow \langle La, b\rangle = 0 = \langle a, L^*b\rangle \, \forall a \in A \Leftrightarrow L^*b = 0$

Given the fact that the image of $\partial_s - A_s$ is closed and the Fact 1, we have the following isomorphism

$$coker(\partial_s - A_s) \cong im(\partial_s - A_s)^\perp$$

The formal adjoint of $\partial_s - A_s$ exists [2] and equals to $-\partial_s - A_s^*$: since $(L_1^2)^* = L_1^2$ and

$(L^2)^* = L^2$, the formal adjoint is defined from L_1^2 to L^2. So, we have

$$\begin{aligned}\int_M \langle(\partial_s - A_s)f, g\rangle\, ds &= \int_M \langle f, (\partial_s - A_s)^* g\rangle\, ds \\ &= \int_M \langle f, \partial_s^* g\rangle\, ds - \int_M \langle f, A_s^* g\rangle\, ds \\ &= \int_M \langle f, (-\partial_s - A_s^* g\rangle\, ds\end{aligned}$$

Then, the cokernel of $\partial_s - A_s$ is just the kernel of its adjoint; $coker(\partial_s - A_s) = ker(\partial_s + A_s^*)$

Lemma 2.4.18. *The map*

$$ker(\partial_s + A_s^*) \to (\mathscr{H}^+)^\perp \cap (\mathscr{H}^-)^\perp$$

defined by $\tilde{f} \mapsto \tilde{f}(0)$ *is an isomorphism.*

Proof. The map is well-defined: Suppose $\tilde{f} \in ker(\partial_s + A_s^*)$ and let $h \in \mathscr{H}^\pm$. So, we have

$$\begin{aligned}\partial_s \left\langle \tilde{f}, f_h \right\rangle &= \left\langle \partial_s \tilde{f}, f_h \right\rangle + \left\langle \tilde{f}, \partial_s f_h \right\rangle \\ &= \left\langle -A_s^* \tilde{f}, f_h \right\rangle + \left\langle \tilde{f}, A_s f_h \right\rangle \\ &= -\left\langle \tilde{f}, A_s f_h \right\rangle + \left\langle \tilde{f}, A_s f_h \right\rangle \\ &= 0\end{aligned}$$

Since $\tilde{f}$ is in $ker(\partial_s + A^*)$, $\lim_{s\to\pm\infty} \tilde{f}(s) = 0$. Also, since $h \in \mathscr{H}^\pm$, we have $\lim_{s\to\pm\infty} f_h(s) = 0$, so $\lim_{s\to\pm\infty} \left\langle \tilde{f}(s), f_h(s) \right\rangle = 0$. That is $\left\langle \tilde{f}, f_h \right\rangle = 0$ for all $s \in \mathbb{R}$. By setting $s = 0$, we get $\left\langle \tilde{f}(0), f_h(0) \right\rangle = \left\langle \tilde{f}(0), h \right\rangle = 0$. Hence, $\tilde{f}(0) \in (\mathscr{H}^\pm)^\perp$. Since the differential equation $(\partial_s + A_s^*)(\tilde{f}) = 0$ has a unique solution, $\tilde{f}(0) = \tilde{g}(0)$, $\tilde{f} = \tilde{g}$. Let $h' \in (\mathscr{H}^+)^\perp \cap (\mathscr{H}^-)^\perp$. So, $\langle h', h\rangle = 0$ for a given $h \in \mathscr{H}^\pm$. And, $\langle h', f_h(0)\rangle = 0 = \left\langle \tilde{f}(0), f_h(0) \right\rangle$. Hence, $h' = \tilde{f}(0)$. ∎

Therefore,

$$\begin{aligned} ind(\partial_s - A_s) &= dim(\mathscr{H}^+ \cap \mathscr{H}^-) - dim((\mathscr{H}^+)^\perp \cap (\mathscr{H}^-)^\perp) \\ &= dim(\mathscr{H}^+ \cap \mathscr{H}^-) - (n - dim(\mathscr{H}^+ \cup \mathscr{H}^-)) \\ &= dim(\mathscr{H}^+) + dim(\mathscr{H}^-) - n \\ &= dim(E^-(A^+)) + dim(E^+(A^-)) - n \\ &= n - dim(E^+(A^+)) + dim(E^+(A^-)) - n \\ &= -SFA_s \end{aligned}$$

By the theorem 2.4.14, we conclude that for a generic g, the restricted map $\nabla\psi_(g,\gamma)$ is surjective for all flow line γ. Now, we will show that this onto map implies the Morse-Smale transversality condition. We observe that if γ is a flow line from p to q, then for the trivialization of γ^*TM above,

$$\mathscr{H}^+ = T_{\gamma(0)}\mathscr{D}(p)$$

and

$$\mathscr{H}^- = T_{\gamma(0)}\mathscr{A}(p)$$

Because $\nabla\psi_{(g,\gamma)}$ is surjective, $\partial_s - A_s$ is surjective, as well. This implies $coker(\partial_s - A_s) = 0$. Hence, by Lemma 2.4.18, $(\mathscr{H}^+)^\perp \cap (\mathscr{H}^-)^\perp = 0$. From this, we get $\mathscr{H}^+ \oplus \mathscr{H}^- = \mathbb{R}$. By the above observation, $T_{\gamma(0)}\mathscr{D}(p) \oplus T_{\gamma(0)}\mathscr{A}(q) = T_{\gamma(0)}M$. So, $\mathscr{D}(p)$ and $\mathscr{A}(q)$ intersect transversely at $\gamma(0)$.

2.5 Compactness and Gluing Theorems

When $ind(p) - ind(q) = 1$, the moduli space $\mathscr{M}(p,q)$ has dimension zero. We would like to count the points in $\mathscr{M}(p,q)$. For this, it is enough to know $\mathscr{M}(p,q)$ is compact. In many cases, $\mathscr{M}(p,q)$ may not be compact. So, we now review a compactification process and a theorem that is more powerful than we want. To follow the details of theorems presented in this section , [13],[27],[2] are excellent references.

To compactify a topological space X, we identify which sequences in X can fail to converge, then we add the "limit points", ∂X, to our space and declare the new open sets to ensure those new points are indeed limit points in $X \cup \partial X$. Note that the induced topology on X inherited from $X \cup \partial X$ coincides with the original topology for X.

Topology of moduli space $\mathscr{M}(p,q) = \mathscr{W}(p,q)/\mathbb{R}$: Recall that

$$\mathscr{W}(p,q) \subset U \subset L^2_{1,loc}(\mathbb{R}, M)$$

and $\mathscr{W}(p,q)$ is a submanifold of U, so the topology of $\mathscr{W}(p,q)$ is induced topology from U. Recall the quotient map $\pi : \mathscr{W}(p,q) \to \mathscr{W}(p,q)/\mathbb{R} = \mathscr{M}(p,q)$ by $u \mapsto [u]$. The $\mathbb{R}$-action is given by shifting the s by a constant

$$v = [u] = [u(. + constant)]$$

We call such $u = \tilde{v} \in \mathscr{W}(p,q)$ a lift of $v \in \mathscr{M}(p,q)$. The quotient topology is: $V \subset \mathscr{M}(p,q)$ is an open subset iff $\pi^{-1}(V) \subset \mathscr{W}(p,q)$ is open.

Definition 2.5.1. A sequence (v_n) in $\mathscr{M}(p,q)$ converges to $v \in \mathscr{M}(p,q)$ if for any lifts $\tilde{v}_n$ of v_n and $\tilde{v}$ of v, there are shifts $t_n \in \mathbb{R}$ such that $t_n \cdot \tilde{v}_n \to \tilde{v}$. More precisely, $[u_n] \to [u]$ in $\mathscr{M}(p,q)$ if and only if $u_n(. + t_n) \to u(.)$ in $\mathscr{W}(p,q)$.

Useful Lemmas of negative gradient flows:

Let $\gamma : \mathbb{R} \to M$ be a solution of the equation $\dfrac{\mathrm{d}\gamma}{\mathrm{d}s} = -gradf(\gamma(s))$.

Lemma 2.5.2. *If γ is nonconstant and $s_2 > s_1$, then $f(\gamma(s_1)) > f(\gamma(s_2))$, this means that f decreasess along γ.*

Proof. The fundamental theorem of calculus applied to the composition $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ yields

$$\begin{aligned} f(\gamma(s_1)) - f(\gamma(s_2)) = -\int_{s_1}^{s_2} \frac{\mathrm{d}}{\mathrm{d}s} f(\gamma(s)) &= -\int_{s_1}^{s_2} \mathrm{d}f_{\gamma(s)}(\frac{\mathrm{d}\gamma}{\mathrm{d}s})\,\mathrm{d}s \\ &= -\int_{s_1}^{s_2} \mathrm{d}f_{\gamma(s)}(-gradf(\gamma(s)))\,\mathrm{d}s \\ &= -\int_{s_1}^{s_2} g(gradf(\gamma(s)), -gradf(\gamma(s)))\,\mathrm{d}s \\ &= \int_{s_1}^{s_2} \|\frac{\mathrm{d}\gamma}{\mathrm{d}s}\|^2\,\mathrm{d}s = \|\frac{\mathrm{d}\gamma}{\mathrm{d}s}\|^2(s_2 - s_1) > 0 \end{aligned}$$

∎

The behavior at the ends of a flow line is described by the following:

Lemma 2.5.3. *If* $\lim_{s\to-\infty}\gamma(s)=p$ *or* $\lim_{s\to\infty}\gamma(s)=p$*, then* $p\in Crit(f)$*.*

Proof. We suppose that $\lim_{s\to\infty}\gamma(s)=p$, the other case is similar. By the previous lemma, we have

$$\lim_{s\to\infty}\int_0^s \|\frac{\mathrm{d}\gamma}{\mathrm{d}s}\|^2\,\mathrm{d}s=\lim_{s\to\infty}(f(\gamma(0))-f(\gamma(s)))=f(\gamma(0))-f(p)<\infty$$

This implies that $\|\frac{\mathrm{d}\gamma}{\mathrm{d}s}\|\to 0$ as $s\to\infty$. Hence,

$$\|gradf(p)\|=\|gradf(\lim_{s\to\infty}\gamma(s))\|=\lim_{s\to\infty}\|gradf(\gamma(s))\|=\lim_{s\to\infty}\|\frac{\mathrm{d}\gamma}{\mathrm{d}s}\|=0$$

So, p is a critical point. ■

Definition 2.5.4. (C^0_{loc}**-convergence**) Let M be a closed manifold, $f:M\to\mathbb{R}$ a Morse function. $\gamma_n\to\gamma$ in C^0_{loc} means $\gamma_n|_K\to\gamma|_K$ in C^0, where K is any compact set in M.

Lemma 2.5.5. *Let* (γ_n) *be a sequence of maps from* $\mathbb{R}$ *to* X *satisfying the equation* $\frac{\mathrm{d}\gamma_n}{\mathrm{d}s}=-gradf(\gamma_n(s))$*. Then there exists a subsequence* (γ_{n_j}) *such that* $\gamma_{n_j}\to\gamma$ *in* C^0_{loc}*, where* γ *is also satisfies* $\frac{\mathrm{d}\gamma}{\mathrm{d}s}=-gradf(\gamma(s))$*.*

Remark 2.5.6. Suppose $\frac{\mathrm{d}\gamma_n}{\mathrm{d}s}=-gradf(\gamma_n(s))$ and $\frac{\mathrm{d}\gamma}{\mathrm{d}s}=-gradf(\gamma)$. Then $\gamma_n\to\gamma$ in C^0_{loc} if and only if $\gamma_n\to\gamma$ in C^k_{loc} for all k: If $\gamma_n\to\gamma$ in C^0_{loc}, then $-gradf(\gamma_n)\to-gradf(\gamma)$ in C^0_{loc}, this implies $\gamma_n\to\gamma$ in C^1_{loc}, and by continuing in this way, we conclude $\gamma_n\to\gamma$ in C^k_{loc} for all k.

Now, we state an important theorem which says that C^k_{loc}-convergence in $\mathscr{W}(p,q)$ implies convergence in usual sense. The details of the proof are so technical and can be found in [27].

Theorem 2.5.7. *If a sequence* (γ_n) *in* $\mathscr{W}(p,q)$ *converges to* γ *in* $\mathscr{W}(p,q)$ *in the* C^k_{loc} *sense, then* $\gamma_n\to\gamma$ *with respect to the usual topology of* $\mathscr{W}(p,q)$*.*

Now we will talk about the compactness in the following theorem, [13].

Theorem 2.5.8. *Let* p *and* q *be critical points of* f*. Then for any sequence* $(u_n(t))_{n\in\mathbb{N}}$ *in* $\mathscr{W}(p,q)$*, after selection of a subsequence, there exist critical points* $p=r_0,r_1,r_2,...,r_{k+1}=q$*, flow lines* $v_i\in\mathscr{W}(p_i,p_{i+1})$ *and* $t_{n,i}\in\mathbb{R}$ *where* $i=0,...,k$ *and* $n\in\mathbb{N}$ *such that the flow lines* $u_n(t+t_{n,i})$ *converge to* v_i *as* n *tends to* ∞

Let $(\hat{u}_n) \subset \mathscr{M}(p,q)$. Consider a lifting of this sequence $(u_n) \subset \mathscr{W}(p,q)$. With the above theorem, we define the "limit" of the sequence $\hat{u}_n$ to be the broken flow lines :

$$(\hat{v}_0, \hat{v}_1, ..., \hat{v}_k) \in \mathscr{M}(p, r_1) \times \mathscr{M}(r_1, r_2) \times ... \times \mathscr{M}(r_k, q)$$

To compactify $\mathscr{M}(p,q)$, we will add the all broken flow lines, $\partial\mathscr{M}(p,q)$, to the space $\mathscr{M}(p,q)$. Hence,

$$\overline{\mathscr{M}(p,q)} = \mathscr{M}(p,q) \cup \partial\mathscr{M}(p,q).$$

Proof. By lemma 2.5.5, there exists a convergent subsequence (u_{n_k}) such that

$$u_{n_k} \overset{C^k_{loc}}{\to} v$$

v does not have to be in $\mathscr{W}(p,q)$, but the limit points $\lim_{s\to\infty} v(s) = q^{'}$ and $\lim_{s\to-\infty} v(s) = p^{'}$ satisfy the following inequality:

$$f(q) \leq f(\lim_{s\to\infty} v(s)) \leq f(\lim_{s\to-\infty} v(s)) \leq f(p).$$

If $p = p^{'}$ and $q = q^{'}$, then by theorem 2.5.7, u_{n_k} converges to v on topology on $\mathscr{W}(p,q)$ So, we are done.

If without loss of generality $p \neq p^{'}$, then we can assume that $f(p^{'}) < f(p)$, we choose a regular value $a \in \mathbb{R}$ such that $f(p^{'}) < a < f(p)$ and $t_{n,i}$ with $f(u_n(t_{n,i})) = a$. Again, we apply the lemma 2.5.5 to get

$$u_n(t + t_{n,i}) \overset{C^k_{loc}}{\to} w$$

where $w : \mathbb{R} \to M$. w need not be in $\mathscr{W}(p,p^{'})$, but the limit points $\lim_{s\to\infty} w(s) = q^{''}$ and $\lim_{s\to-\infty} w(s) = p^{''}$ satisfy the following inequality:

$$f(q) \leq f(q^{''}) \leq f(p^{''}) \leq f(p).$$

Also, we have $f(p^{'}) \leq f(q^{''})$: we first show that as $n \to \infty$, the shifts $t_{n,i}$ are not bounded from below. Assume for the contrary that there exists $M \in \mathbb{R}$ such that $t_{n,i} > M$ for all $n \in \mathbb{N}$. Since f decreases along u, we get

$$f(u_n(t_{n,i})) < f(u_n(M)).$$

But $f(u_n(t_{n,i})) = a$ and $\lim_{n\to\infty} f(u_n(M)) = f(v(M)) < f(p^{'})$. But we have choosen that $a > f(p^{'})$. This gives a contradiction.

Now suppose $f(q^{''}) < f(p^{'})$. Then there exist an $\epsilon > 0$, and $s_0 \in \mathbb{R}$ such that

$$f(w(s_0)) = f(p^{'}) - 4\epsilon$$

and we can also choose an $s_1 \in \mathbb{R}$ such that

$$f(v(s_1)) = f(p^{'}) - \epsilon.$$

Since $\lim_{n\to\infty} f(u_n(s_0 + t_{n,i})) = f(w(s_0))$ and $\lim_{n\to\infty} f(u_n(s_1)) = f(v(s_1))$, there is an $N \in \mathbb{N}$ such that for all $n > N$,

$$f(u_n(s_0 + t_{n,i})) - f(w(s_0)) < \epsilon.$$

This gives us the following inequality, $f(u_n(s_0 + t_{n,i})) < f(p^{'}) - 3\epsilon$. Also,

$$f(u_n(s_1)) - f(v(s_1)) < \epsilon$$

gives us the inequality $f(u_n(s_1)) > f(p^{'}) - 2\epsilon$. Hence we get

$$f(u_n(s_0 + t_{n,i})) < f(u_n(s_1))$$

By lemma 2.5.2, we have $s_0 + t_{n,i} > s_1$ and $t_{n,i} > s_1 - s_0$. This means that $t_{n,i}$ are bounded from below and this gives a contradiction.

If $f(q^{''}) = f(p^{'})$, then $w \in \mathscr{W}(p, p^{'})$. So, if $f(q^{''}) < f(p^{'})$, then we can proceed this process at finitely many step, because the critical points are finite on M. ■

2.6 Gluing

The second part of the compactification process is the gluing theorem. An important question for compactification process is that every broken flow line arises as a limit point. The following theorem says that the broken flow lines $(\hat{v}_0, \hat{v}_1, ..., \hat{v}_k)$ glued together with a gluing parameter ρ mapped into $\mathscr{M}(p, q)$ and conversely any sequence of flow lines converging to broken flow line lies in the range of gluing map $\widehat{\#}$ which is precisely defined in the theorem 2.6.1. For simlicity, we will state the theorem for broken flow lines with only one break. After this theorem, compactification process will finish theorem 2.6.3.

Theorem 2.6.1. *Given a compact set of simply broken flow lines* $K \subset \mathscr{W}(p,r) \times \mathscr{W}(r,q)$, *there is a lower bound* $\rho_K \geq 0$ *and a smooth map*

$$\# : K \times [\rho_K, \infty) \to \mathscr{W}(p,q)$$

where $(u, v, \rho) \mapsto u\#_\rho v$ *satisfying: The map* $\#_\rho : K \hookrightarrow \mathscr{W}(p,q)$ *is an embedding for each gluing parameter* $\rho \geq \rho_K$. *Moreover, given a compact set* $\widehat{K} \subset \mathscr{M}(p,r) \times \mathscr{M}(r,q)$ *of unparametrized flow lines,* $\#$ *induces a smooth embedding*

$$\widehat{\#} : \widehat{K} \times [\rho_{\widehat{K}}, \infty) \hookrightarrow \mathscr{M}(p,q)$$

such that we obtain C^∞_{loc} *convergence toward the simply broken flow line*

$$\widehat{\widehat{u}\#_\rho \widehat{v}} \overset{C^\infty_{loc}}{\to} (\widehat{u}, \widehat{v})$$

as ρ *tends to infinity. Conversely, if* $\widehat{w_n} \overset{C^\infty_{loc}}{\to} (\widehat{u}, \widehat{v})$, *then* $\widehat{w_n} \in \mathscr{M}(p,q)$ *for sufficiently large* n.

The proof of this theorem is so technical and really we do not need it to construct Morse homology. We can give an excellent reference for details of the proof of this theorem, [27].

We conclude that all broken flow lines appear as limit points, so if $ind(p) = ind(q) + 2$, then $\overline{\mathscr{M}(p,q)}$ becomes one dimensional compact manifold with boundary.

Definition 2.6.2. ([18]) A manifold with corners M of dimension n is a second countable, Hausdorff topological space such that every point $x \in M$ has an open neighborhood U_α and homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^{n-k} \times [0,\infty)^k$ for some $0 \leq k \leq n$ such that the transition functions $\phi_\beta \circ \phi_\alpha^{-1}$ are smooth. For $0 \leq k \leq n$, we define the codimension k-stratum of M to be the set M_k of points $x \in M$ with a chart $\phi : U \to \mathbb{R}^{n-k} \times [0,\infty)^k$ such that at least one the last k coordinates of $\phi(x)$ is zero. Note that M_0 is just the interior of M, and if $M_2 = M_3 = ... = M_n = \emptyset$, then M is just a manifold with boundary and $M_1 = \partial M$.

Now we are ready to finish the compactification process by the help of the compactness and gluing theorems. By the three facts which states that 1) every broken flow line appears as a boundary (COMPACTNESS), 2) we can glue them and 3) embed those broken flow lines smoothly into the unparametrized moduli spaces (GLUING), we can generate a manifold with corner. More precisely,

Theorem 2.6.3. *If M is closed (compact without boundary) and (f, g) is Morse- Smale, then for any two critical points p, q the moduli space $\mathscr{M}(p, q)$ has a natural compactification to a smooth manifold with corners $\overline{\mathscr{M}(p, q)}$ whose codimension k stratum is*

$$\overline{\mathscr{M}(p, q)}_k = \bigcup_{r_1, ..., r_k \in Crit(f)} \mathscr{M}(p, r_1) \times \mathscr{M}(r_1, r_2) \times ... \times \mathscr{M}(r_{k-1}, r_k) \times \mathscr{M}(r_k, q)$$

with $p, r_1, ..., r_k, q$ are all different. In particular, for the case $k = 1$, as oriented manifolds, the boundary of $\overline{\mathscr{M}(p, q)}$

$$\partial \overline{\mathscr{M}(p, q)} = \bigcup_{r \in Crit(f)} (-1)^{ind(p)+ind(r)+1} \mathscr{M}(p, r) \times \mathscr{M}(r, q).$$

Results of the Theorem:

• If $ind(p) \leq ind(q)$, then $\mathscr{M}(p, q) = \emptyset$.

• If $ind(p) = i$ and $ind(q) = i-1$, then $\mathscr{M}(p, q)$ is a compact zero dimensional manifold. Hence, we can count its elements because compact discrete sets have to be finite. The total sum of the elements will be denoted by $\#\mathscr{M}(p, q)$.

• If $ind(q) = i - 2$, then $\overline{\mathscr{M}(p, q)}$ is compact one-dimensional manifold with boundary

$$\partial \overline{\mathscr{M}(p, q)} = \bigcup_{r \in Crit(f)} \mathscr{M}(p, r) \times \mathscr{M}(r, q).$$

We now define the Morse complex $(C_*^{Morse}(f, g), \partial^{Morse})$ as follows.

2.7 The Morse chain complex

Let $f : M \to \mathbb{R}$ be a Morse function and $Crit_i(f)$ denote the set of critical points of f with index i. The chain group $C_k^{Morse}(f, g)$ is the free $\mathbb{Z}_2$-module generated by the elements in $Crit_k(f)$. We can also define this complex with $\mathbb{Z}$-coeffcients if we define orientations on the moduli spaces $\mathscr{M}(p, q)$. It is more easy to work with $\mathbb{Z}_2$ for now.

The differential $\partial^{Morse} : C_k^{Morse}(f, g) \to C_{k-1}^{Morse}(f, g)$ counts the negative gradient flow lines between critical points. If $p \in Crit_k(f)$, then

$$\partial^{Morse}(p) = \sum_{q \in Crit_{k-1}(f)} \#\mathscr{M}(p, q) \cdot q$$

Remark 2.7.1. The sum is well-defined because of the second result of theorem 2.6.3.

Lemma 2.7.2. $(\partial^{Morse})^2 = 0$.

Proof. If $p \in Crit_k(f)$, then

$$\begin{aligned}(\partial^{Morse})^2(p) &= \partial^{Morse}(\sum_{r\in Crit_{k-1}(f)} \#\mathscr{M}(p,r)\cdot r) \\ &= \sum_{r\in Crit_{k-1}(f)} \#\mathscr{M}(p,r)(\sum_{q\in Crit_{k-2}(f)} \#\mathscr{M}(r,q)\cdot q) \\ &= \sum_{r\in Crit_{k-1}(f)}\sum_{q\in Crit_{k-2}(f)} \#\mathscr{M}(p,r)\#\mathscr{M}(r,q)\cdot q \\ &= \sum_{r\in Crit_{k-1}(f), q\in Crit_{k-2}(f)} \#(\mathscr{M}(p,r)\times\mathscr{M}(r,q))\cdot q \\ &= \#\partial\overline{\mathscr{M}(p,q)}.\end{aligned}$$

Observe that $\overline{\mathscr{M}(p,q)}$ is a compact one dimensional manifold with boundary. So, $\overline{\mathscr{M}(p,q)}$ is the disjoint union of finitely many circles and closed intervals. Hence, $\#\overline{\mathscr{M}(p,q)}$ is even, so 0 modulo 2. ■

The homology of the complex $(C_*^{Morse}(f,g), \partial^{Morse})$ is called the Morse homology,

$$H_*^{Morse}(f,g) = \frac{ker(\partial^{Morse} : C_*^{Morse}(f,g) \to C_{*-1}^{Morse}(f,g))}{im(\partial^{Morse} : C_{*+1}^{Morse}(f,g) \to C_*^{Morse}(f,g))}.$$

It is defined for a generic pair (f,g). Later, we will prove that the Morse homology is independent of the choice of a generic pair (f,g).

Example (Sphere): Let $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$ be the 2-sphere, and define $f : \mathbb{S}^2 \to \mathbb{R}$ as $f(x,y,z) = z$. Then f is the Morse function; in fact f has two critical points: the north pole $(0,0,1)$ and the south pole $(0,0,-1)$, each of index $2,0$ respectively. So the chain complex is as follows

$$0 \to \langle N\rangle \xrightarrow{\partial_2} 0 \xrightarrow{\partial_1} \langle S\rangle \xrightarrow{0} 0$$

We observe that all boundary operators are 0 because there are no critical points of index 1, so no gradient flow lines between successive critical points. Hence the homology of 2-sphere with coefficient in $\mathbb{Z}$ is $H_n^{Morse}(f,g)=\begin{cases}\mathbb{Z}_2, & \text{if } n = 0,2\\ 0, & otherwise\end{cases}$

Example (A deformed sphere): Consider $\mathbb{S}^2$ and the Morse function $f : \mathbb{S}^2 \to \mathbb{R}$ with two critical points r, s of index 2, one critical point q of index 1 and one critical point p of index 0. Then for any metric g, the pair (f, g) is Morse-Smale. We thus get the following Morse complex

$$0 \to \langle r, s\rangle \xrightarrow{\partial_2} \langle q\rangle \xrightarrow{\partial_1} \langle p\rangle \xrightarrow{0} 0$$

and all we need to calculate the homology groups is to find ∂ and for that we have to study the flow lines joining the critical points of the succesive index. Let us begin with the critical points of index 2 and 1. For suitable orientation choices, the sign associated with a gradient flow line is $+1$ or -1. So, $\partial_2(r) = \mp q$ and $\partial_2(s) = \pm q$. Hence, $\partial_2(r + s) = 0$ or $\partial_2(r - s) = 0$ since the gradient flow is downward and in both cases $ker\partial_2 = \mathbb{Z}$ and the homology group $H_2^{Morse}(f, g) = \mathbb{Z}$. Since $\partial_2(r) = q$, we see that $H_1^{Morse}(f, g) = 0$ because the generator of $\langle q\rangle$ is in the image of ∂_2. Since $\partial_1(q) = \partial_1^2(r) = 0$, we see that p is not in the image of ∂_1, and hence $H_0^{Morse}(f, g)$ is generated by p. Thus, $H_n^{Morse}(f, g) = \begin{cases} \mathbb{Z}, & \text{if } n\text{= 0,2} \\ 0, & otherwise \end{cases}$

Example (Real Projective Plane): Consider the set of all lines through the origin in Euclidean space $\mathbb{R}^3$. This is the projective space $\mathbb{R}P^2$ and has a two dimensional smooth manifold structure. There are three charts that cover projective plane. $\mathbb{R}P^2$ is in fact homemorphic to the quotient space $\mathbb{S}^2/\sim$, where the equivalence relation is $x \sim -x$. Let $(\lambda_1, ..., \lambda_n)$ be an increasing sequence of positive real numbers and consider the function

$$f(x_1, ..., x_{n+1}) = \sum_{i=1}^{n+1} \lambda_i x_i^2$$

For the first chart on $\mathbb{R}P^2$ is $\varphi_1(x_1, x_2, x_3) = (\frac{x_2}{x_1}, \frac{x_3}{x_1})$ and its inverse is $\varphi_1^{-1}(u, v) = (\sqrt{1 - u^2 - v^2}, u, v)$. So the function is $f \circ \varphi_1^{-1}(u, v) = u^2 + 2v^2 + 1$.
For the second chart on $\mathbb{R}P^2$ is $\varphi_2(x_1, x_2, x_3) = (\frac{x_1}{x_2}, \frac{x_3}{x_2})$ and its inverse is $\varphi_2^{-1}(u, v) = (u, \sqrt{1 - u^2 - v^2}, v)$. So the function is $f \circ \varphi_2^{-1}(u, v) = -u^2 + v^2 + 2$.
For the third chart on $\mathbb{R}P^2$ is $\varphi_3(x_1, x_2, x_3) = (\frac{x_1}{x_3}, \frac{x_2}{x_3})$ and its inverse is $\varphi_3^{-1}(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$. So the function is $f \circ \varphi_3^{-1}(u, v) = -2u^2 - v^2 + 3$.
Hence the derivative of each function $f \circ \varphi_i^{-1}$ is $2u\,\mathrm{d}u + 4v\,\mathrm{d}v$, $-2u\,\mathrm{d}u + 2v\,\mathrm{d}v$ and $-4u\,\mathrm{d}u - 2v\,\mathrm{d}v$,respectively. So the critical points are $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ each of index $0, 1, 2$ respectively.

The vectors that span the tangent space to $\mathbb{S}^2$ at point (u, v) are $(1, 0, \frac{-u}{\sqrt{1-u^2-v^2}})$ and $(0, 1, \frac{-v}{\sqrt{1-u^2-v^2}})$. So, the inner product of them in $\mathbb{R}^3$ is

$$\left\langle (1, 0, \frac{-u}{\sqrt{1-u^2-v^2}}), (1, 0, \frac{-u}{\sqrt{1-u^2-v^2}}) \right\rangle = \frac{1-v^2}{1-u^2-v^2}$$

$$\left\langle (1, 0, \frac{-u}{\sqrt{1-u^2-v^2}}), (0, 1, \frac{-v}{\sqrt{1-u^2-v^2}}) \right\rangle = \frac{uv}{1-u^2-v^2}$$

$$\left\langle (0, 1, \frac{-v}{\sqrt{1-u^2-v^2}}), (0, 1, \frac{-v}{\sqrt{1-u^2-v^2}}) \right\rangle = \frac{1-u^2}{1-u^2-v^2}$$

and the induced metric on $\mathbb{S}^2$ is the matrix

$$g = \begin{bmatrix} \frac{1-v^2}{1-u^2-v^2} & \frac{uv}{1-u^2-v^2} \\ \frac{uv}{1-u^2-v^2} & \frac{1-u^2}{1-u^2-v^2} \end{bmatrix}$$

and its inverse matrix is

$$g^{-1} = \begin{bmatrix} 1-u^2 & -uv \\ -uv & 1-v^2 \end{bmatrix}$$

So, the gradient vector field is

$$-grad(f \circ \varphi_1^{-1}) = -g^{-1}(\mathrm{d}(f \circ \varphi_1^{-1})) = \begin{bmatrix} u^2-1 & uv \\ uv & v^2-1 \end{bmatrix} \begin{bmatrix} 2u \\ 4v \end{bmatrix}$$

$$-grad(f \circ \varphi_2^{-1}) = -g^{-1}(\mathrm{d}(f \circ \varphi_2^{-1})) = \begin{bmatrix} u^2-1 & uv \\ uv & v^2-1 \end{bmatrix} \begin{bmatrix} -2u \\ 2v \end{bmatrix}$$

$$-grad(f \circ \varphi_3^{-1}) = -g^{-1}(\mathrm{d}(f \circ \varphi_3^{-1})) = \begin{bmatrix} u^2-1 & uv \\ uv & v^2-1 \end{bmatrix} \begin{bmatrix} -4u \\ -2v \end{bmatrix}$$

The gradient flow lines can be found by solving following differential equation systems for each chart φ_i. Using φ_1;

$$\frac{\mathrm{d}\gamma_1(t)}{\mathrm{d}t} = 2\gamma_1(t)(\gamma_1^2(t)-1) + 4\gamma_1(t)\gamma_2^2(t), \frac{\mathrm{d}\gamma_2(t)}{\mathrm{d}t} = 2\gamma_1^2(t)\gamma_2(t) + 4\gamma_2(t)(\gamma_2^2(t)-1)$$

Using φ_2,

$$\frac{\mathrm{d}\gamma_1(t)}{\mathrm{d}t} = -2\gamma_1(t)(\gamma_1^2(t)-1) + 2\gamma_1(t)\gamma_2^2(t), \frac{\mathrm{d}\gamma_2(t)}{\mathrm{d}t} = -2\gamma_1^2(t)\gamma_2(t) + 2\gamma_2(t)(\gamma_2^2(t)-1)$$

Using φ_3,

$$\frac{\mathrm{d}\gamma_1(t)}{\mathrm{d}t} = -4\gamma_1(t)(\gamma_1^2(t)-1) - 2\gamma_1(t)\gamma_2^2(t), \frac{\mathrm{d}\gamma_2(t)}{\mathrm{d}t} = -4\gamma_1^2(t)\gamma_2(t) - 2\gamma_2(t)(\gamma_2^2(t)-1)$$

Hence, we can conclude that there are two gradient flow lines between the critical points of index 2 and 1, and between the critical points of index 1 and 0. Since $2 \equiv 0(mod2)$, the differential operator ∂_2^{Morse} and ∂_1^{Morse} are 0. Therefore, the homology groups with coefficients in $\mathbb{Z}_2$,

$$0 \to \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to 0$$

which gives us the following homology groups, $H_n^{Morse}(f,g) = \begin{cases} \mathbb{Z}_2, & \text{if } n\text{= 0,1,2} \\ 0, & otherwise \end{cases}$

3 Invariance of Morse Homology

We will show that $H_*^{Morse}(f, g)$ does not depend on f and g directly. This means that given another Morse function f_1 and metric g_1 satisfying Morse-Smale condition,

$$H_*^{Morse}(f, g) \cong H_*^{Morse}(f_1, g_1)$$

There are lots of different techniques to show invariance of Morse homology for finite dimensional manifolds, for example we will prove that Morse homology is isomorphic to singular homology in the next chapter. This implies that Morse homology depends only the structure of given manifold. But, the point that we explain now will provide a very useful method to get an invariant for Morse homology of infinite dimensional manifolds. There might not be another homology to compare with the Floer homology.

3.1 Continuation Maps

Let (f_0, g_0) and (f_1, g_1) be two Morse-Smale pairs associated with Morse complexes (C_*^0, ∂^0) and (C_*^1, ∂^1). Let

$$\Gamma = \{(f_t, g_t) : t \in [0, 1]\}$$

be a path of pairs from (f_0, g_0) to (f_1, g_1).

Remark 3.1.1. Note that the pairs (f_t, g_t) do not have to be Morse-Smale for all t. For example, take a path on $\mathbb{R}$ with the usual metric, $f_t : \mathbb{R} \to \mathbb{R}$ such that $f_t(x) = x^3 - 3(t - \frac{1}{2})x$, $t \in [0, 1]$. f_1 has two critical points at $x = \mp\sqrt{\frac{1}{2}}$ and it is a Morse function, f_0 has no critical point so it is also a Morse function. But, $f_{1/2}$ is not a Morse function because it has only one critical point at $x = 0$ and it is degenerate.

We define the continuation map $\Phi_\Gamma : C_*^0 \to C_*^1$ as follows: Define a vector field V on $[0, 1] \times X$ by

$$V := (1 - t)t(1 + t)\frac{\partial}{\partial t} + V_t$$

where V_t denote the negative gradient vector field of $f_t : X \to \mathbb{R}$ with respect to the metric g_t. Now, we can define the critical points, flow lines, ascending and descending manifolds. In particular, the first coordinate of V is the gradient of the function $\mathbb{R} \to \mathbb{R}$ defined by $t \mapsto \frac{1}{4}(t + 1)^2(t - 1)^2$. This function has a critical point of index 1 at $t = 0$ and a critical point of index 0 at $t = 1$ with no critical point between them. Thus, the critical points of index i of V are precisely

$$crit_i(V) = \{0\} \times crit_{i-1}(f_0) \cup \{1\} \times crit_i(f_1)$$

We say that the family Γ is admissible if the ascending and descending manifolds of the critical points of V intersect transversely. If (f_0, g_0) and (f_1, g_1) are Morse-Smale, then the generic homotopy Γ between them is admissible, but the converse is not true in general, i.e, for an admissible Γ, there might be some t such that (f_t, g_t) is not Morse- Smale. As an example, $f_t : \mathbb{R} \to \mathbb{R}$ such that $f_t(x) = x^3 - 3(t - \frac{1}{2})x$, $t \in [0, 1]$ can be given. So, for the rest section we assume that Γ is admissible.

For critical points P and Q of V, we define $\mathscr{W}(P, Q)$ to be the moduli space of flow lines from P to Q and $\mathscr{M}(P, Q)$ the moduli space from P to Q, modulo the $\mathbb{R}$ action as usual. Since Γ is admisible, $\mathscr{M}(P, Q)$ is a $(ind(P) - ind(Q) - 1)$-dimensional manifold without boundary. If $ind(p) = i = ind(q) + 1$, then $ind((0, p)) = i + 1$ and $ind((1, q)) = i - 1$, then $\mathscr{M}((0, p), (1, q))$ is a one dimesional manifold and has a compactification to a compact oriented 1-manifold $\overline{\mathscr{M}((0, p), (1, q))}$ by adding broken flow lines passing through critical points of index i. So we have,

$$\partial\overline{\mathscr{M}((0, p), (1, q))} = \bigcup_{r' \in Crit_i(f_1)} \mathscr{M}((0, p), (1, r')) \times \mathscr{M}((1, r'), (1, q))$$
$$\cup \bigcup_{r \in Crit_{i-1}(f_0)} \mathscr{M}((0, p), (0, r)) \times \mathscr{M}((0, r), (1, q))$$

If $\mathscr{M}_0$ and $\mathscr{M}_1$ denote the moduli spaces for (f_0, g_0) and (f_1, g_1), then we have

$$\mathscr{M}((0, p), (0, r)) = \mathscr{M}_0(p, r)$$

$$\mathscr{M}((1, r'), (1, q)) = \mathscr{M}_1(r', q)$$

For $p \in crit_i(f_0)$ and $q \in crit_i(f_1)$ we have, $ind((0, p)) = i + 1$ and $ind((1, q)) = i$, hence $\mathscr{M}((0, p), (1, q))$ is a compact 0-dimensional manifold. We now define $\Phi_\Gamma : C_i^0 \to C_i^1$ on $p \in crit_i(f_0)$ by

$$\Phi_\Gamma(p) = \sum_{q \in crit_i(f_1)} \#\mathscr{M}((0, p), (1, q)) \cdot q$$

The sum is well-defined because $\mathscr{M}((0,p),(1,q))$ is 0-dimensional and compact.

Proposition 3.1.2. *Every compact* 1*-dimensional manifold is homeomorphic to a disjoint union of circles and closed intervals.*

Proof. See, [24] ∎

Corollary 3.1.3. *The number of boundary components of any compact* 1*-dimensional manifold is even.*

Lemma 3.1.4. Φ_Γ *is chain map, i.e,* $\Phi_\Gamma \circ \partial^0 = \partial^1 \circ \Phi_\Gamma$.

Proof. We will show that $(\Phi_\Gamma \circ \partial^0 - \partial^1 \circ \Phi_\Gamma)(p)$ is zero for all $p \in C_i^0$. For $p \in crit_i(f_0)$, we have

$$\begin{aligned}
\Phi_\Gamma \circ \partial^0(p) &= \Phi_\Gamma\Big(\sum_{r \in crit_{i-1}(f_0)} \#\mathscr{M}_0(p,r)\cdot r\Big) \\
&= \sum_{r \in crit_{i-1}(f_0)} \sum_{q \in crit_{i-1}(f_1)} \#\mathscr{M}((0,r),(1,q))\#\mathscr{M}_0(p,r)\cdot q \\
&= \sum_{r \in crit_{i-1}(f_0)} \sum_{q \in crit_{i-1}(f_1)} \#\mathscr{M}((0,r),(1,q))\#\mathscr{M}((0,p),(0,r))\cdot q
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\partial^1 \circ \Phi_\Gamma(p) &= \partial^1\Big(\sum_{r \in crit_i(f_1)} \#\mathscr{M}((0,p),(1,r))\cdot r\Big) \\
&= \sum_{r \in crit_i(f_1)} \sum_{q \in crit_{i-1}(f_1)} \#\mathscr{M}((0,p),(1,r))\#\mathscr{M}_1(r,q)\cdot q \\
&= \sum_{r \in crit_i(f_1)} \sum_{q \in crit_{i-1}(f_1)} \#\mathscr{M}((0,p),(1,r))\#\mathscr{M}((1,r),(1,q))\cdot q
\end{aligned}$$

Therefore, the coefficient of q in $(\Phi_\Gamma \circ \partial^0 - \partial^1 \circ \Phi_\Gamma)(p)$ is

$$\#\partial\overline{\mathscr{M}(0,p),(1,q))}$$

This is the number of boundary components in the compact 1-manifold $\overline{\mathscr{M}((0,p),(1,q))}$, hence is zero modulo 2 by Corollary 3.1.3. ∎

Thus Φ_Γ induces a map on Morse homology

$$(\Phi_\Gamma)_* : H^{Morse}(f_0, g_0) \to H^{Morse}(f_1, g_1)$$

3.2 Chain homotopies

Now, we consider two paths between pairs (f_0, g_0) and (f_1, g_1) and corresponding the continuation maps Φ_{Γ_1} and Φ_{Γ_2}, respectively. We regard the homotopy as a family of pairs $\Psi = \{(f_d, g_d) : d \in D\}$, where D is a digon (a 2-dimensional manifold with corners such that D has two edges e and $e^{'}$, and two vertices v and w), such that $\Psi|e = \Gamma_1$ and $\Psi|e^{'} = \Gamma_2$. This means that $(f_v, g_v) = (f_0, g_0)$ and $(f_w, g_w) = (f_1, g_1)$. Put a metric $\hat{g}$ on D such that the length of the edges are 1, and let $\hat{f} : D \to \mathbb{R}$ be a smooth function with an index 2 critical point at v and an index 0 critical point at w and no other critical points. Further assume the negative gradient vector field $\hat{V}$ of $\hat{f}$ with respect to $\hat{g}$ is tangent to the edges and is equal to $(t+1)t(t-1)$ there. So we define a vector field V on $D \times X$ by $V := \hat{V} + V_d$, where V_d is the negative gradient vector field of f_d with respect to g_d. We see that V restricted to the slice $e \times X$ and $e^{'} \times X$ coincide with the vector field on $[0, 1] \times X$ for Γ_1 and the vector field $[0, 1] \times X$ for Γ_2, respectively. So we can again define critical points, flow lines, ascending and descending manifolds of V. We still say Ψ is admissible if the ascending and descending manifolds of all critical points intersect transversely. The critical points of index i of V are

$$crit_i(V) = \{v\} \times crit_{i-2}(f_0) \cup \{w\} \times crit_i(f_1)$$

For critical points P and Q of V, we define $\mathscr{W}(P, Q)$ to be the moduli space of flow lines from P to Q and $\mathscr{M}(P, Q)$ the moduli space from P to Q, modulo the $\mathbb{R}$ action as usual. For any admissible family Ψ, $\mathscr{M}(P, Q)$ is a $(ind(P) - ind(Q) - 1)$-dimensional manifold without boundary. If $P = (v, p)$ and $Q = (w, q)$ with $ind(p) = i = ind(q)$, then $ind((v, p)) = i + 2$ and $ind((w, q)) = i$, then $\mathscr{M}((v, p), (w, q))$ is a one dimesional manifold and has a compactification to a 1-manifold $\overline{\mathscr{M}((v, p), (w, q))}$ with boundary by adding broken flow lines. The broken flow lines that pass through critical point of index $i+1$ are included in the boundary. But, $\overline{\mathscr{M}((v, p), (w, q))}$ also has boundary points coming from those flow lines which stay in the slice $e \times X$ and $e^{'} \times X$. These flow lines are in correspondence with moduli spaces $\mathscr{M}^{\Gamma_1}((v, p), (w, q))$ for Γ_1 and $\mathscr{M}^{\Gamma_2}((v, p), (w, q))$ for

Γ_2 which are defined previous section. Hence,

$$\begin{aligned}\partial\overline{\mathscr{M}((v,p),(w,q))} = \mathscr{M}^{\Gamma_1}((v,p),(w,q)) \cup \mathscr{M}^{\Gamma_2}((v,p),(w,q))\\ \cup \bigcup_{r\in Crit_{i-1}(f_0)} \mathscr{M}((v,p),(v,r)) \times \mathscr{M}((v,r),(w,q))\\ \cup \bigcup_{s\in Crit_{i+1}(f_1)} \mathscr{M}((v,p),(w,s)) \times \mathscr{M}((w,s),(w,q))\end{aligned}$$

Note that the flow lines in $D \times X$ that stay in the $v \times X$ slice correspond to the flows in X with respect to (f_0, g_0) and similarly flow lines for the $w \times X$ slice. So,

$$\mathscr{M}((v,p),(v,r)) = \mathscr{M}_0(p,r)$$

$$\mathscr{M}((w,s),(w,q)) = \mathscr{M}_1(s,q)$$

We now define $K : C^0_i \to C^1_{i+1}$ on $p \in crit_i(f_0)$ by

$$K(p) = \sum_{q\in crit_{i+1}(f_1)} \#\mathscr{M}((v,p),(w,q)) \cdot q$$

Lemma 3.2.1. *A generic homotopy between the admissible paths Γ_1 and Γ_2 induces a chain homotopy*

$$K : C^0_i \to C^1_{i+1}$$

such that $\partial^1 \circ K + K \circ \partial^0 = \Phi_{\Gamma_1} - \Phi_{\Gamma_2}$.

Proof. Let $p \in crit_i(f_0)$

$$\begin{aligned}\partial^1 \circ K(p) &= \partial^1\Big(\sum_{r\in crit_{i+1}(f_1)} \#\mathscr{M}((v,p),(w,r)) \cdot r\Big)\\ &= \sum_{r\in crit_{i+1}(f_1)} \sum_{q\in crit_i(f_1)} \#\mathscr{M}((v,p),(w,r))\#\mathscr{M}_1(r,q) \cdot q\\ &= \sum_{r\in crit_{i+1}(f_1)} \sum_{q\in crit_i(f_1)} \#\mathscr{M}((v,p),(w,r))\#\mathscr{M}((w,r),(w,q)) \cdot q\end{aligned}$$

On the other hand,

$$\begin{aligned}
K \circ \partial^0(p) &= K(\sum_{r \in crit_{i-1}(f_0)} \#\mathscr{M}_0(p,r) \cdot r) \\
&= \sum_{r \in crit_{i-1}(f_0)} \sum_{q \in crit_i(f_1)} \#\mathscr{M}((v,r),(w,q)) \#\mathscr{M}_0(p,r) \cdot q \\
&= \sum_{r \in crit_{i-1}(f_0)} \sum_{q \in crit_i(f_1)} \#\mathscr{M}((v,r),(w,q)) \#\mathscr{M}((v,p),(v,r)) \cdot q
\end{aligned}$$

By definition of Φ_Γ, we observe that $\Phi_{\Gamma_1}(p) = \sum_{q \in crit_i(f_1)} \#\mathscr{M}^{\Gamma_1}((0,p),(1,q)) \cdot q$ and $\Phi_{\Gamma_2}(p) = \sum_{q \in crit_i(f_1)} \#\mathscr{M}^{\Gamma_2}((0,p),(1,q)) \cdot q$. Also, we know when $\Psi|e$, it coincides with Γ_1 and when $\Psi|e^{'}$, coincides with Γ_2. Therefore, the coefficient of q in $(\partial^1 \circ K + K \circ \partial^0 + \Phi_{\Gamma_2} - \Phi_{\Gamma_1})(p)$ is

$$\#\partial\overline{\mathscr{M}((v,p),(w,q))}$$

But it is zero modulo 2, so is $(\partial^1 \circ K + K \circ \partial^0 + \Phi_{\Gamma_2} - \Phi_{\Gamma_1})(p) = 0$. ■

This Lemma shows that $(\Phi_\Gamma)_*$ depends only on the homotopy class of Γ. However, the space of metrics, the subspace of the space of all functions from $M \times M$ to $\mathbb{R}$, is contractible because for any two metrics g_1, g_2 and arbitrary $t \in [0,1]$, $(1-t)g_1 + tg_2$ is also a metric. So, all paths between any pairs are homotopic. This implies that $(\Phi_\Gamma)_*$ does not depend on Γ. We want to prove that $(\Phi_\Gamma)_*$ is an isomorphism. It is enough to show bijectivity of $(\Phi_\Gamma)_*$. To do this we will show that the induced map of composition of paths $\Gamma_2 * \Gamma_1$, where the end point of Γ_1 is the starting point of Γ_2, is chain homotopic to the composition of induced maps $(\Phi_{\Gamma_2})_* \circ (\Phi_{\Gamma_1})_*$.

Lemma 3.2.2. *$\Phi_{\Gamma_2 * \Gamma_1}$ is chain homotopic to $\Phi_{\Gamma_2} \circ \Phi_{\Gamma_1}$.*

Proof. We will use the same arguments as defined in Lemma 2.2.1. In this case we use a triangle instead of a digon, let T be a triangle, that is a 2-manifold with three vertices, u, v, w and three edges, e_{uv}, e_{vw}, e_{uw}. Let $\Omega = \{(f_d, g_d) : d \in T\}$ be a family of pairs such that $(f_u, g_u) = \Gamma_1(0)$, $(f_v, g_v) = \Gamma_1(1) = \Gamma_2(0)$ and $(f_w, g_w) = \Gamma_2(1)$, also $\Omega|e_{uv} = \Gamma_1$, $\Omega|e_{vw} = \Gamma_2$ and $\Omega|e_{uw} = \Gamma_2 * \Gamma_1$. Put a metric on T such that the length of all edges are one, and take a smooth function $f : T \to \mathbb{R}$ where its only critical points are u, v and w with $ind(u) = 2$, $ind(v) = 1$ and $ind(w) = 0$. So we define a vector field V_Ω on $T \times X$ by $V_\Omega := \hat{V} + V_d$, where V_d is the negative gradient vector field of f_d with respect to g_d. The critical points of index i of V_Ω are

$$crit_i(V_\Omega) = \{u\} \times crit_{i-2}(f_u) \cup \{v\} \times crit_{i-1}(f_v) \cup \{w\} \times crit_i(f_w).$$

If $ind(p) = i = ind(q)$, then $ind((u,p)) = i+2$ and $ind(w,q) = i$, hence $\mathscr{M}((u,p),(w,q))$ is a one dimensional manifold and has a natural compactification to a one dimensional manifold with boundary $\overline{\mathscr{M}((u,p),(w,q))}$. The boundary of this compact manifold has more complicated broken flow lines. The broken flow lines passing through one critical point of index $i+1$ are contained in the boundary. We also have boundary points coming from those flow lines stay in the $e_{uw} \times X$ slice. These flow lines correspond to $\mathscr{M}^{\Gamma_2 * \Gamma_1}((u,p),(w,q))$. Hence,

$$\begin{aligned}\partial\overline{\mathscr{M}((u,p),(w,q))} = {} & \mathscr{M}^{\Gamma_2*\Gamma_1}((u,p),(w,q)) \\ & \cup \bigcup_{r \in Crit_{i-1}(f_u)} \mathscr{M}((u,p),(u,r)) \times \mathscr{M}((u,r),(w,q)) \\ & \cup \bigcup_{r' \in Crit_{i+1}(f_w)} \mathscr{M}((u,p),(w,r')) \times \mathscr{M}((w,r'),(w,q)) \\ & \cup \bigcup_{s \in crit_i(f_v)} \mathscr{M}((u,p),(v,s)) \times \mathscr{M}((v,s),(w,q)).\end{aligned}$$

Note that the flow lines that stay in the $u \times X$ slice correspond to $\mathscr{M}^{f_u}(p,r)$, and similarly the flow lines that stay in the $w \times X$ slice correspond to $\mathscr{M}^{f_w}(r',q)$. Therefore,

$$\mathscr{M}((u,p),(u,r)) = \mathscr{M}^{f_u}(p,r)$$

$$\mathscr{M}((w,r'),(w,q)) = \mathscr{M}^{f_w}(r',q).$$

The flow lines from (u,p) to (v,s) stay in the $e|uv \times X$ slice and the flows from (v,s) to (w,q) stay in the slice $e|vw \times X$. So,

$$\mathscr{M}((u,p),(v,s)) = \mathscr{M}^{\Gamma_1}((u,p),(v,s))$$

$$\mathscr{M}((v,s),(w,q)) = \mathscr{M}^{\Gamma_2}((v,s),(w,q)).$$

Now, we define $H : C_i^u \to C_{i+1}^w$ by

$$H(p) = \sum_{q \in crit_{i+1}(f_w)} \#\mathscr{M}((u,p),(w,q)).q$$

By using the same technique in the proof of Lemma 3.2.1, we want to prove that

$$(\partial^w \circ H + H \circ \partial^u - \Phi_{\Gamma_2 * \Gamma_1} + \Phi_{\Gamma_2} \circ \Phi_{\Gamma_1})(p) = 0$$

for all $p \in crit_i(f_u)$. We can compute

$$\begin{aligned}
\partial^w \circ H(p) &= \partial^w(\sum_{r' \in crit_{i+1}(f_w)} \#\mathscr{M}((u,p),(w,r')) \cdot r') \\
&= \sum_{r' \in crit_{i+1}(f_w)} \sum_{q \in crit_i(f_w)} \#\mathscr{M}((u,p),(w,r'))\#\mathscr{M}^{f_w}(r',q) \cdot q \\
&= \sum_{r' \in crit_{i+1}(f_w)} \sum_{q \in crit_i(f_w)} \#\mathscr{M}((u,p),(w,r'))\#\mathscr{M}((w,r'),(w,q)) \cdot q
\end{aligned}$$

On the other hand,

$$\begin{aligned}
H \circ \partial^u(p) &= H(\sum_{r \in crit_{i-1}(f_u)} \#\mathscr{M}^{f_u}(p,r) \cdot r) \\
&= \sum_{r \in crit_{i-1}(f_u)} \sum_{q \in crit_i(f_w)} \#\mathscr{M}^{f_u}(p,r)\#\mathscr{M}((u,r),(w,q)) \cdot q \\
&= \sum_{r \in crit_{i-1}(f_u)} \sum_{q \in crit_i(f_w)} \#\mathscr{M}((u,p),(u,r))\#\mathscr{M}((u,r),(w,q)) \cdot q
\end{aligned}$$

$$\begin{aligned}
\Phi_{\Gamma_2} \circ \Phi_{\Gamma_1}(p) &= \Phi_{\Gamma_2}(\sum_{s \in crit_i(f_v)} \#\mathscr{M}^{\Gamma_1}((u,p),(v,s)) \cdot s) \\
&= \sum_{s \in crit_i(f_v)} \sum_{q \in crit_i(f_w)} \#\mathscr{M}^{\Gamma_1}((u,p),(v,s))\#\mathscr{M}^{\Gamma_2}((v,s),(w,q)) \cdot q \\
&= \sum_{s \in crit_i(f_v)} \sum_{q \in crit_i(f_w)} \#\mathscr{M}((u,p),(v,s))\#\mathscr{M}((v,s),(w,q)) \cdot q
\end{aligned}$$

Since $\Omega|e_{uw} = \Gamma_2 * \Gamma_1$, and $\Phi_{\Gamma_2 * \Gamma_1}(p) = \sum_{q \in crit_i(f_w)} \#\mathscr{M}^{\Gamma_2 * \Gamma_1}((u,p),(w,q)) \cdot q$, we conclude that the coefficient of q in $(\partial^w \circ H + H \circ \partial^u - \Phi_{\Gamma_2 * \Gamma_1} + \Phi_{\Gamma_2} \circ \Phi_{\Gamma_1})(p)$ is

$$\#\partial \overline{\mathscr{M}((u,p),(w,q))}$$

which is zero modulo 2. ■

Proposition 3.2.3. *If* $\Gamma = \{f_t, g_t\}$ *is a constant family with* (f_t, g_t) *Morse-Smale, then* Γ *is admissible and* $\Phi_\Gamma = id$.

Proof. Let V be the vector field on $[0,1] \times X$ induced by the constant path Γ. Since the vector field $(1-t)t(1+t)\dfrac{\partial}{\partial t}$ on $[0,1]$ is directly from 0 to 1, we have that the descending manifold of a critical point $(0,p)$ of V is $\mathscr{D}(0,p) = [0,1) \times \mathscr{D}(p)$ and the ascending manifold of a critical point $(1,q)$ is $\mathscr{A}(1,q) = (0,1] \times \mathscr{A}(q)$. These manifolds intersect transversely. Also, the descending manifold of $(1,q)$ is $\{1\} \times \mathscr{D}(q)$ and the ascending manifold of $(0,p)$ is $\{0\} \times \mathscr{A}(p)$. These manifolds which do not intersect at any point automatically intersect transversely. Therefore Γ is admissible.
For $p, q \in crit_i(f)$, a flow from $(0,p)$ to $(1,q)$ in $[0,1] \times X$ projects to a flow from p to q in the $\{0\} \times X$ slice. This implies that $p = q$ because there is no flow between critical points of equal index, unless it is the constant flow. Therefore, $\Phi_\Gamma(p) = p$. ■

In conclusion, we get the following results:

• Lemma 3.2.1 and lemma 3.2.2 implies that $(\Phi_{\Gamma_2})_* \circ (\Phi_{\Gamma_1})_* = (\Phi_{\Gamma_2 * \Gamma_1})_*$.

• Prop. 3.2.3 and lemma 3.2.1 implies that $(\Phi_\Gamma)_* = id$ for constant family of pairs Γ.

• If Γ_1 is an admissible path from (f_0, g_0) to (f_1, g_1), and Γ_2 is a path from (f_1, g_1) to (f_0, g_0), then $\Gamma_2 * \Gamma_1$ is a constant family of pairs, where $f_t = f_0$ and $g_t = g_0$ for all $t \in [0,1]$. So, $(\Phi_{\Gamma_2})_* \circ (\Phi_{\Gamma_1})_* = id$ and $(\Phi_{\Gamma_1})_*$ is injective. Also, $\Gamma_1 * \Gamma_2$ is a constant family of pairs, where $f_t = f_1$ and $g_t = g_1$ for all $t \in [0,1]$. So, $(\Phi_{\Gamma_1})_* \circ (\Phi_{\Gamma_2})_* = id$ and $(\Phi_{\Gamma_1})_*$ is surjective.

Hence, $(\Phi_{\Gamma_1})_* : H^{Morse}(f_0, g_0) \to H^{Morse}(f_1, g_1)$ is an isomorphism.

4 Isomorphism to Singular Homology

The goal of this section is to establish the connection between Morse homology and singular homology. As described at the previous section, our main aim is to show that Morse homology is independent of the choice of a Morse function f and a Riemannian metric g. By proving the isomorphism between Morse homology and singular homology, we observe that the homology depends only on the manifold structure.

Theorem 4.0.4. *Given a closed manifold M and a Morse-Smale pair (f, g) on M, the homology of the Morse-Smale complex $(C_*^{Morse}(f, g), \partial^{Morse})$ is isomorphic to the homology of the singular chain complex (C_*, ∂). That is;*

$$H_*^{Morse}(f, g) \cong H_*(M).$$

Before proving this theorem, we give some backgrounds related to singular homology.

Definition 4.0.5. Let $\Omega_c^m(M)$ be the space of compactly supported, smooth m-forms on M. An m-current T on M is a functional on $\Omega_c^m(M)$, $T : \Omega_c^m(M) \to \mathbb{R}$.

Integration determines an m-current $[M]$ for a compact, oriented m dimensional manifold M with boundary in the following way:

$$[M](w) = \int_M w.$$

This can be checked by the definition of integral that $[M]$ is a linear functional. Note that the topology of the space of m-currents comes from the *weak convergence* of currents, i.e a sequence T_k of currents converges to a current T if $T_k(w) \to T(w)$ for all $w \in \Omega_c^m(M)$.

By Stokes' Theorem, we automatically obtain the following equality:

$$[M]dw = \int_M dw = \int_{\partial M} w = [\partial M](w).$$

Now, we observe that the singular i-simplex $\sigma : \Delta^i \to M$ defines an i-current $[\sigma]$ by

$$[\sigma](w) = \int_{\Delta^i} \sigma^* w.$$

We define the chain groups $C_i(M)$ to be the free abelian groups generated by $[\sigma]$, where $\sigma : \Delta^i \to M$ is *generic*, i.e it is smooth and each face of σ is transverse to the ascending manifolds of all critical points of f. The differential $\partial : C_i(M) \to C_{i-1}(M)$ is defined as above

$$\partial[\sigma](w) = \int_{\Delta^i} \sigma^* dw$$

where $w \in \Omega_c^{i-1}(M)$. Note that $\partial^2 = 0$ because $d^2 = 0$. Considering the chain complex $(C_*(M), \partial_*)$, we have the homology of currents $H_*(C_*(M))$. This homology is canonically isomorphic to the singular homology groups $H_*(M)$ of M. This can be proved by showing the Eilenberg-Steenrod axioms for $H_*(C_*(M))$. We will skip this observation because the details do not help us to prove the isomorphism between Morse homology and singular homology.

To continue the isomorphism process, we now compactify the descending manifold $\mathscr{D}(p)$ for $p \in Crit(f)$. It has a natural compactification as the moduli space $\mathscr{M}(p,q)$ prensented above [section 2.6]. So, we have the following theorem.

Theorem 4.0.6. *There is a natural compactification of $\mathscr{D}(p)$ of a critical point p to a smooth manifold with corners $\overline{\mathscr{D}(p)}$, whose codimension k stratum is*

$$\overline{\mathscr{D}(p)}_k = \bigcup_{q_1,\dots,q_k \in Crit(f)} \mathscr{M}(p,q_1) \times \mathscr{M}(q_1,q_2) \times \dots \times \mathscr{M}(q_{k-1},q_k) \times \mathscr{D}(q_k)$$

with $p, q_1, \dots, q_k$ are all different. In particular, for the case $k = 1$, as oriented manifolds, we have

$$\partial\overline{\mathscr{D}(p)} = \bigcup_{q \in Crit(f)} (-1)^{ind(p)+ind(q)+1} \mathscr{M}(p,q) \times \mathscr{D}(q).$$

Furthermore, we define an extended inclusion map $e : \overline{\mathscr{D}(p)} \to M$ from the inclusion $i : \mathscr{D}(p) \to M$ by setting $e := \pi_k : \overline{\mathscr{D}(p)}_k \to M$ with π_k is the projection to the last factor $\mathscr{D}(q_k) \subset M$.

Because $\overline{\mathscr{D}(p)}$ is homeomorphic to a closed ball of dimension $ind(p)$, the compact oriented manifold with corners $\overline{\mathscr{D}(p)}$ has a fundamental current $[\overline{\mathscr{D}(p)}] : \Omega_c^m(M) \to \mathbb{R}$. Together with the map $e : \overline{\mathscr{D}(p)} \to M$, we define the pushforward of $[\overline{\mathscr{D}(p)}]$ by the equation $e_*[\overline{\mathscr{D}(p)}](w) = [\overline{\mathscr{D}(p)}](e^*w)$. So the pushforward of $[\overline{\mathscr{D}(p)}]$ is also a linear functional from $\Omega_c^*(M)$ to $\mathbb{R}$ defined by

$$w \mapsto \int_{\overline{\mathscr{D}(p)}} e^* w.$$

Hence, $e_*[\overline{\mathscr{D}(p)}] \in C_*(M)$.

Remark 4.0.7. Let M and N be smooth manifolds, and $F : M \to N$ be the smooth map.

Then for a suitable current T, we have the following equality: $\partial(F_*T) = F_*\partial T$. This can be shown by applying a suitable form w both sides. That is, $\partial(F_*T)(w) = \partial T(F^*w) = F_*\partial T(w)$.

We now define a chain map $D : C_*^{Morse}(f, g) \to C_*(M)$ by

$$p \in Crit(f) \mapsto D(p) = e_*[\overline{\mathscr{D}(p)}].$$

Lemma 4.0.8. *D is a chain map: $\partial D = D\partial^{Morse}$.*

Proof. Let $p \in Crit_i(f)$. By Thm. 4.0.6, we have

$$\partial\overline{\mathscr{D}(p)} = \bigcup_{q\in Crit(f)} (-1)^{ind(p)+ind(q)+1}\mathscr{M}(p, q) \times \mathscr{D}(q).$$

Therefore,

$$\begin{aligned}
\partial D(p)(w) &= \partial e_*[\overline{\mathscr{D}(p)}](w) \\
&= e_*\partial[\overline{\mathscr{D}(p)}](w) \\
&= e_*[\partial\overline{\mathscr{D}(p)}](w) \\
&= e_*[\bigcup_{q\in Crit(f)} (-1)^{ind(p)+ind(q)+1}\overline{\mathscr{M}(p, q)} \times \overline{\mathscr{D}(q)}](w) \\
&= \int_{\bigcup_{q\in Crit(f)}(-1)^{ind(p)+ind(q)+1}\overline{\mathscr{M}(p,q)}\times\overline{\mathscr{D}(q)}} e^*w \\
&= \sum_{q\in Crit(f)} (-1)^{ind(p)+ind(q)+1} \int_{\overline{\mathscr{M}(p,q)}\times\overline{\mathscr{D}(q)}} e^*w \\
&= \sum_{q\in Crit(f)} (-1)^{ind(p)+ind(q)+1} e_*[\overline{\mathscr{M}(p, q)} \times \overline{\mathscr{D}(q)}](w).
\end{aligned}$$

Hence,

$$\partial D(p) = \sum_{q\in Crit(f)} (-1)^{ind(p)+ind(q)+1} e_*[\overline{\mathscr{M}(p, q)} \times \overline{\mathscr{D}(q)}] \in C_{i-1}(M).$$

If $ind(q) > i-1$, then $\mathscr{M}(p, q)$ is empty because $dim(\mathscr{M}(p, q)) = ind(p) - ind(q) - 1 < 0$ in that case. So, the right hand side becomes the current of dimension greater than $i - 1$. If $ind(q) < i-1$, then the right hand side becomes the current of dimension less than or equal to $i - 2$ because e maps $\overline{\mathscr{M}(p, q)} \times \overline{\mathscr{D}(q)}$ to the last factor $e_*[\overline{\mathscr{D}(q)}]$. Then, $ind(q) = i - 1$.

So, $\mathscr{M}(p,q)$ is the finite set of points because $\mathscr{M}(p,q)$ is 0-dimensional and compact. Therefore,

$$\begin{aligned} \partial D(p) &= \sum_{q \in Crit_{i-1}(f)} \#\mathscr{M}(p,q) \cdot e_*[\overline{\mathscr{D}(q)}] \\ &= D(\partial^{Morse}(p)). \end{aligned}$$

■

So, the chain map D between chain complexes $C_*^{Morse}(f,g)$ and $C_*(M)$ induces homomorphism between the homology groups of the two complexes:

$$D_* : H_*^{Morse}(f,g) \to H_*(M)$$

4.1 The left inverse chain map

For the inverse map,let σ be a generic i-simplex and q be a critical point. Consider the moduli space $\mathscr{M}(\sigma,q)$ of gradient flow lines from σ to q, i.e

$$\mathscr{M}(\sigma,q) = \left\{ \gamma : [0,\infty) \to M | \gamma(0) \in \sigma, \gamma'(s) = -gradf(\gamma(s)), \lim_{s\to\infty} \gamma(s) = q \right\}.$$

We again omit the orientation on the moduli space and use the $\mathbb{Z}/(2)$ coefficient to count the number of boundary components. We state a compactification theorem for the moduli space $\mathscr{M}(\sigma,q)$.

Theorem 4.1.1. *There is a natural compactification of $\mathscr{M}(\sigma,q)$ to a smooth manifold with corners $\overline{\mathscr{M}(\sigma,q)}$ whose codimension k stratum is*

$$\overline{\mathscr{M}(\sigma,q)}_k = \bigcup_{j=0}^{k} \bigcup_{p_1,\ldots,p_j \in Crit(f)} \mathscr{M}(\sigma_{k-j},p_1) \times \mathscr{M}(p_1,p_2) \times \ldots. \times \mathscr{M}(p_{j-1},p_j) \times \mathscr{M}(p_j,q)$$

where $p_1, \ldots., p_j, q$ are all distinct and σ_j denotes the codimension j stratum of σ. When $k=1$, as oriented manifolds, we have

$$\partial\overline{\mathscr{M}(\sigma,q)} = \mathscr{M}(\partial\sigma,q) \cup \bigcup_{p \in Crit(f)} (-1)^{i+ind(q)} \mathscr{M}(\sigma,p) \times \mathscr{M}(p,q)$$

We now define the map $A : C_*(M) \to C_*^{Morse}(f,g)$ by

$$A(\sigma) = \sum_{p \in Crit_i(f)} \#\mathscr{M}(\sigma, p) \cdot p$$

This map is well-defined because $dim(\mathscr{M}(\sigma,p)) = i - ind(p) = 0$ and $\mathscr{M}(\sigma,p)$ is compact. We will show that A is a chain map and it is inverse of D.

Lemma 4.1.2. *A is a chain map: $A\partial = \partial^{Morse}A$.*

Proof. We show that $(A\partial - \partial^{Morse}A)(\sigma) = 0$ for all $\sigma \in C_i(M)$. For $\sigma \in C_i(M)$ and $q \in Crit_{i-1}(f)$,

$$A\partial(\sigma) = A(\partial\sigma) = \sum_{q \in Crit_{i-1}(f)} \#\mathscr{M}(\partial\sigma, q) \cdot q.$$

On the other hand,

$$\begin{aligned}
\partial^{Morse}A(\sigma) &= \partial^{Morse}\Big(\sum_{p \in Crit_i(f)} \#\mathscr{M}(\sigma,p) \cdot p\Big) \\
&= \sum_{p \in Crit_i(f)} \#\mathscr{M}(\sigma,p) \cdot \partial^{Morse}(p) \\
&= \sum_{p \in Crit_i(f)} \sum_{q \in Crit_{i-1}(f)} \#\mathscr{M}(\sigma,p) \cdot \#\mathscr{M}(p,q) \cdot q
\end{aligned}$$

So, the coefficient of q is

$$\sum_{q \in Crit_{i-1}(f)} \#\mathscr{M}(\partial\sigma, q) - \sum_{p \in Crit_i(f)} \sum_{q \in Crit_{i-1}(f)} \#\mathscr{M}(\sigma,p) \cdot \#\mathscr{M}(p,q) = \#\partial\overline{\mathscr{M}(\sigma,q)}$$

Since $\mathscr{M}(\sigma,q)$ is a 1-dimensional compact manifold, $\#\partial\overline{\mathscr{M}(\sigma,q)}$ is zero modulo 2. ∎

Lemma 4.1.3. *$A \circ D = id : C_i^{Morse} \to C_i^{Morse}$.*

Proof. Let p be an index i critical point. Then

$$A \circ D(p) = A(D(p)) = \sum_{q \in Crit_i(f)} \#\mathscr{M}(D(p), q) \cdot q.$$

While $q = p$, $\mathscr{M}(D(p), p)$ contains only one point which is constant gradient flow line. If q is another critical point of index i, then $\mathscr{M}(D(p), q)$ is empty set because there is no gradient flow line from $D(p)$ to q except p. But $\mathscr{M}(p, q)$ is empty as well because of the Morse-Smale condition. Hence, $A \circ D(p) = p$. ■

Since A is also a chain map and the composition of D and A is identity, the induced map from D on the previous section becomes one-to-one and onto. Now, it is enough to show that the composition $D \circ A$ and id are homotopic. From this way, it will follow that $D \circ A$ and id give rise to the same function on the homology level.

4.2 The chain homotopy

Let σ be a generic simplex. The forward orbit $\mathscr{F}(\sigma)$ of σ is a set defined as follows:

$$\mathscr{F}(\sigma) = \{(s, x) : s \geq 0, x \in \sigma\}$$

together with a map $e : \mathscr{F}(\sigma) \to M$, $(s, x) \mapsto \varphi_s(\sigma(x))$.

Remark 4.2.1. Every dynamical system has an orbit at a point x, that is the sequence of states that starts given initial state $\{\varphi_s(x) : s \in \mathbb{R}\}$. The forward orbit is the subsequence $\{\varphi_s(x) : s \geq 0\}$. For each fixed x, $\varphi_s(x)$ defines a curve on M as s varies over $\mathbb{R}$, this is the orbit of x.

Theorem 4.2.2. *There is a natural compactification of $\mathscr{F}(\sigma)$ to a smooth manifold with corners $\overline{\mathscr{F}(\sigma)}$ whose codimension k stratum is*

$$\overline{\mathscr{F}(\sigma)}_k = \mathscr{F}(\sigma_k) \cup \bigcup_{j=1}^{k} \bigcup_{p_1,\ldots,p_j \in Crit(f)} \mathscr{M}(\sigma_{k-j}, p_1) \times \mathscr{M}(p_1, p_2) \times \ldots. \times \mathscr{M}(p_{j-1}, p_j) \times \mathscr{D}(p_j)$$

where $p_1, \ldots., p_j$ are all distinct and σ_j denotes the codimension j stratum of σ. When $k = 1$, as oriented manifolds, we have

$$\partial\overline{\mathscr{F}(\sigma)} = -\sigma \cup -\mathscr{F}(\partial\sigma) \cup \bigcup_{p \in Crit(f)} \mathscr{M}(\sigma, p) \times \mathscr{D}(p).$$

We define a map $F : C_i(M) \to C_{i+1}(M)$ by

$$F(\sigma) = e_*[\overline{\mathscr{F}(\sigma)}].$$

Lemma 4.2.3. *F is a chain homotopy between $D \circ A$ and identity on $C_i(M)$, that is*

$$\partial F + F\partial = D \circ A - id_{C_i(M)}.$$

Proof. Our aim is to show $(\partial F + F\partial - D \circ A + id_{C_i(M)})(\sigma) = 0$ for all $\sigma \in C_i(M)$. Let σ be an i-simplex and w be an i-form.

$$\begin{aligned}
\partial F(\sigma)(w) &= \partial e_*[\overline{\mathscr{F}(\sigma)}](w) \\
&= e_*\partial[\overline{\mathscr{F}(\sigma)}](w) \\
&= e_*[\partial\overline{\mathscr{F}(\sigma)}](w) \\
&= e_*[-\sigma \cup -\mathscr{F}(\partial\sigma) \cup \bigcup_{p \in Crit(f)} \mathscr{M}(\sigma,p) \times \mathscr{D}(p)](w) \\
&= \int_{-\sigma \cup -\overline{\mathscr{F}(\partial\sigma)} \cup \bigcup_{p \in Crit(f)} \overline{\mathscr{M}(\sigma,p)} \times \overline{\mathscr{D}(p)}} e^* w \\
&= e_*[-\sigma] - e_*[\overline{\mathscr{F}(\partial\sigma)}] + \sum_{p \in Crit(f)} e_*[\overline{\mathscr{M}(\sigma,p)} \times \overline{\mathscr{D}(p)}] \\
&= e_*[-\sigma] - e_*[\overline{\mathscr{F}(\partial\sigma)}] + \sum_{p \in Crit(f)} \#\mathscr{M}(\sigma,p) \cdot e_*[\overline{\mathscr{D}(p)}]
\end{aligned}$$

The last equality comes from the fact that $dim(\mathscr{M}(\sigma,p)) = i - ind(p) = 0$. Now, apply σ to the following function

$$\begin{aligned}
F\partial(\sigma) &= F(\partial\sigma) \\
&= e_*[\overline{\mathscr{F}(\partial\sigma)}]
\end{aligned}$$

$$\begin{aligned}
D \circ A(\sigma) &= D(\sum_{p \in Crit_i(f)} \#\mathscr{M}(\sigma,p) \cdot p) \\
&= \sum_{p \in Crit_i(f)} \#\mathscr{M}(\sigma,p) \cdot e_*[\overline{\mathscr{D}(p)}]
\end{aligned}$$

Hence, we have shown the equality $(\partial F + F\partial - D \circ A + id_{C_i(M)})(\sigma) = 0$. ∎

5 Applications of Morse Homology

We will use the isomorphism between Morse homology and singular homology to prove some theorems coming from algebraic topology such as Morse inequalities, Poincare duality and Kunneth formula for homology groups. For original proofs without using Morse Homology Theorem, one can check the reference [17].

5.1 Morse inequalities

Let b_k be the k-th Betti number of M, that is $b_k = dimH_k(M) = rankH_k(M)$ and c_k be the number of elements in the set of critical points of a Morse function f of index k, that is $c_k = dimC_k^{Morse}(f, g)$. We first observe that $c_k \geq b_k$ for all $k = 0, 1, ..., n$: we have a short exact sequence at each $k = 0, 1, ..., n$

$$0 \rightarrow ker\partial_k^{Morse} \rightarrow C_k^{Morse}(f, g) \rightarrow im\partial_k^{Morse} \rightarrow 0.$$

Hence, we get the following equality

$$dimC_k^{Morse}(f, g) = dim(im(\partial_k^{Morse}) + dim(ker(\partial_k^{Morse}))).$$

We also have a short exact sequence

$$0 \rightarrow im\partial_{k+1}^{Morse} \rightarrow ker\partial_k^{Morse} \rightarrow H_k^{Morse}(f, g) \rightarrow 0.$$

Hence, we get another equality

$$dimH_k^{Morse}(f, g) = dim(ker(\partial_k^{Morse})/im(\partial_{k+1}^{Morse})) = dim(ker(\partial_k^{Morse})) - dim(im(\partial_{k+1}^{Morse})).$$

By substituting the second equality into the first one and using the theorem 4.0.4, we get

$$c_k = dimC_k^{Morse}(f, g) = dim(im(\partial_k^{Morse})) + dim(im(\partial_{k+1}^{Morse})) + b_k.$$

Hence, $c_k \geq b_k$ for all $k = 0, 1, ..., n$. We now state a very strong theorem which is called the *Euler-Poincare theorem.*

Theorem 5.1.1. *([2]) Given any chain complex described as below*

$$0 \rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} .... \xrightarrow{\partial_1} C_0 \rightarrow 0$$

we have

$$\sum_{i=0}^{n}(-1)^i dimC_i = \sum_{i=0}^{n}(-1)^i dimH_i(C_*).$$

Proof. The equalities obtained from the exact sequences as above implies the equality

$$dim(ker(\partial_k)) = dimC_k - dim(im(\partial_k)) = dim(im(\partial_{k+1})) + dimH_k(C_*).$$

Therefore,

$$\sum_{i=0}^{n}(-1)^i(dimC_i - dim(im(\partial_i))) = \sum_{i=0}^{n}(-1)^i(dim(im(\partial_{i+1}))) + dimH_i(C_*).$$

Hence, we conclude the desired result

$$\sum_{i=0}^{n}(-1)^i dimC_i = \sum_{i=0}^{n}(-1)^i dimH_i(C_*).$$

■

Now we pass to Morse inequalities.

Theorem 5.1.2. *Let $f : M \to \mathbb{R}$ be a Morse function. Let c_i be the number of critical points of f of index i, and let $b_i = dimH_i(M)$ be the i-th Betti number of M. Then*

$$c_k - c_{k-1} + ... + (-1)^k c_0 \geq b_k - b_{k-1} + ... + (-1)^k b_0$$

and for $k = m$, m is the dimension of M, we get the equality:

$$\sum_{i=0}^{m} c_i = \sum_{i=0}^{m} b_i.$$

Proof. By Theorem 4.0.4, we know that $dimH_i^{Morse}(f,g) = dimH_i(M) = b_i$. Let $0 \leq k \leq m$, and the new chain complex (C_*, ∂_*) by defining $C_k = C_k^{Morse}(f,g)$ if $k \leq m$ and $C_k = 0$ if $k > m$:

$$0 \to C_k \to C_{k-1} \to ..... \to C_1 \to C_0 \to 0.$$

So by Euler-Poincare theorem we have

$$\sum_{i=0}^{k}(-1)^i dimC_i = \sum_{i=0}^{k}(-1)^i dimH_i(C_*).$$

Now we multiply both sides with $(-1)^k$ to get

$$(-1)^k \sum_{i=0}^{n} (-1)^i dim C_i = (-1)^k \sum_{i=0}^{n} (-1)^i dim H_i(C_*).$$

For the last term, we have $H_k(C_*) = ker\partial_k$. And we have $H_k^{Morse}(f,g) = ker\partial_k / im(\partial_{k+1})$ as well. Hence, we observe that $H_k^{Morse}(f,g)$ is the quotient of $H_k(C_*)$ and $b_k \leq dim H_k(C_*)$. So together with the equality above, we obtain the desired result. Also, if $k = m$, then $dim(im(\partial_k)) = 0$. Hence, inequality becomes equality. ■

5.2 Poincare duality

In this section, we will focus on the $\mathbb{Z}_2$ coefficients. We define the Morse cochain complex with $C^k_{Morse}(f,g)$ as cochain groups and the differential ∂^k_{Morse} by

$$\begin{aligned} \partial^k_{Morse} : C^k_{Morse}(f,g) &\longrightarrow C^{k+1}_{Morse}(f,g) \\ \phi &\longmapsto \partial^k_{Morse}(\phi) : C^{Morse}_{k+1}(f,g) \longrightarrow \mathbb{Z}_2 \\ & \qquad p \longmapsto \textstyle\sum_{q \in Crit_k(f)} \#\mathscr{M}(p,q) \cdot \phi(q) \end{aligned}$$

We observe that $\partial^{k+1}_{Morse} \circ \partial^k_{Morse} = 0$. This can be proved in the similar way to lemma 1.7.2. So $(C^*_{Morse}(f,g), \partial^*_{Morse})$ becomes the complex called the Morse cochain complex. Therefore, we can define the Morse cohomology:

$$H^k_{Morse}(f,g) = ker\partial^k_{Morse} / im\partial^{k-1}_{Morse}.$$

There is a very useful isomorphism between the cohomology induced from Morse homology and de Rham cohomology [26], $H^*_{Morse}(f,g) \cong H^*_{dR}(M,R)$ for any coefficient ring. Moreover, $H^*(M) \cong H^*_{dR}(M,R)$, the singular cohomology of a manifold and de Rham cohomology are isomorphic, [18]. So, $H^*_{Morse}(f,g) \cong H^*(M,R)$.

Theorem 5.2.1. *If M is oriented, then $H_{n-*}(M) \cong H^*(M, \mathbb{Z}_2)$.*

Proof. Let f be any Morse function on M. Then $-f$ is also a Morse function. The critical points of f and $-f$ are the same but the indexes are different for a critical point p. If the index of p with respect to f is k, then the index of p with respect to $-f$ is $n-k$. This is because the change of signs on the Hessians. Since M is closed, we know that there exist finitely many critical points of f. Noting that we use finitely many critical points, we get a canonical isomorphism $C_k^{Morse}(f,g) \cong C^k_{Morse}(f,g)$ defined by $p \mapsto \phi_p$ such that $\phi_p(q) = 1$ if $q = p$, $\phi_p(q) = 0$ if otherwise. Also we have $C_k^{Morse}(f,g) = C_{n-k}^{Morse}(-f,g)$. Therefore we get

$$C^k_{Morse}(f,g) \cong C_{n-k}^{Morse}(-f).$$

Now we will show that the isomorphism $\Psi : C_{n-k}^{Morse}(-f) \to C^k_{Morse}(f,g)$ is a chain map, i.e $\Psi \circ \partial^{Morse}(p) = \partial_{Morse} \circ \Psi(p)$, for all $p \in Crit_{n-k}(-f)$. First, let $p \in Crit_{n-k}(-f)$. So,

$$\begin{aligned} \Psi \circ \partial^{Morse}(p) &= \Psi\Big(\sum_{q \in Crit_{n-(k+1)}(-f)} \#\mathscr{M}(q,p) \cdot q\Big) \\ &= \sum_{q \in Crit_{n-(k+1)}(-f)} \#\mathscr{M}(q,p) \cdot \Psi(q) \end{aligned}$$

On the other hand, let $q \in Crit_{k+1}(f)$. So,

$$\partial_{Morse} \circ \Psi(p)(q) = \sum_{r \in Crit_k(f)} \#\mathscr{M}(q,r) \cdot \Psi(p)(r)$$

Observe that $\mathscr{M}(p,q)$ with respect to f is isomorphic to $\mathscr{M}(q,p)$ with $-f$ by the map $[\gamma(s)] \mapsto [\gamma(-s)]$. We obtained the desired equality $\Psi \circ \partial^{Morse} = \partial_{Morse} \circ \Psi$. Hence, we canonically obtain the isomorphism

$$\Psi_* : H_{n-*}^{Morse}(-f,g) \to H^*_{Morse}(f,g)$$

Since the Morse homology and the Morse cohomology are independent of the choice of a Morse function, we proved the Poincare duality. ■

5.3 Kunneth Formula

Again, in this section we will use the $\mathbb{Z}_2$ coefficients. We follow the way of [1].

Theorem 5.3.1. *([17])*
Let M and N be two closed manifolds. For any $k > 0$ and the coefficient ring $\mathbb{Z}_2$,

$$H_k(M \times N; \mathbb{Z}_2) \cong \bigoplus_{i+j=k} H_i(M;\mathbb{Z}_2) \otimes H_j(N;\mathbb{Z}_2).$$

It can be proved by using the CW-complex structure on $M \times N$ as presented in [17]. Yet, we will prove this by using the isomorphism between singular homology and Morse homology. Let $f_1 : M \to \mathbb{R}$ and $f_2 : N \to \mathbb{R}$ be two Morse functions on M and N, respectively. Let $-gradf_1$ and $-gradf_2$ be the negative gradients of f_1 and f_2 with respect to the metrics g_1 and g_2, respectively. In the sense of section 1.2, we can say that $-gradf_1$ generates a one-parameter group of diffeomorphism $\psi_s : M \to M$ and $-gradf_2$ generates a one-parameter group of diffeomorphism $\varphi_s : N \to N$. We now define the function $f_1 \oplus f_2 : M \times N \to \mathbb{R}$, $f_1 \oplus f_2(m,n) = f_1(m) + f_2(n)$. Because f_1 and f_2 are Morse, so is $f_1 + f_2$: Let p and q be nondegenerate critical points of f_1 and f_2, respectively. Our aim is to show that the Hessian of $f_1 \oplus f_2$ does not have zero as an eigenvalue at $p \times q$. For $X \times Y \in T_{p\times q}(M \times N) = T_pM \times T_qN$,

$$H(f_1 \oplus f_2, p \times q)(X \times Y) = \nabla_{X\times Y}(\mathrm{d}(f_1 \oplus f_2)) = \nabla_X(\mathrm{d}f_1) \oplus \nabla_Y(\mathrm{d}f_2).$$

Since p and q are nondegenerate, we have conclude that $p \times q$ is nondegenerate, too.

The critical points of $f_1 \oplus f_2$ are the points (p,q) such that $p \in Crit(f_1)$ and $q \in Crit(f_2)$. And also, the index of (p,q) is the sum of the index of p and the index of q. Suppose (p_1,q_1) be the critical point of index k, it will flow the critical point (p_2,q_2) of index $k-1$ in the following flow:

$$(\varphi \times \psi)_s(p,q) = (\varphi_s(p), \psi_s(q)).$$

It is generated by the negative gradient $(-gradf_1, -gradf_2)$. Given a pair of critical points (p_1,q_1) and (p_2,q_2) in $M \times N$, we consider the set of flow lines of negative gradient vector field $(-gradf_1, -gradf_2)$, in particular,

$$\mathscr{M}(p_1 \times q_1, p_2 \times q_2) \cong \mathscr{M}(p_1,p_2) \times \mathscr{M}(q_1,q_2).$$

Here, $p_1 \neq p_2$ and $q_1 \neq q_2$ so that $\mathscr{M}(p_1,p_2) \times \mathscr{M}(q_1,q_2)$ is not empty. The indices have to satisfy the following inequalities; $ind(p_1) \geq ind(p_2) + 1$ and $ind(q_1) \geq ind(q_2) + 1$ or $ind(p_1,q_1) \geq ind(p_2,q_2) + 2$. So, for the critical points with consecutive indices,

$$\mathscr{M}(p_1 \times q_1, p_2 \times q_2) = \begin{cases} \{p_1\} \times \mathscr{M}(q_1,q_2), & \text{if } p_1 = p_2 \\ \mathscr{M}(p_1,p_2) \times \{q_1\}, & \text{if } q_1 = q_2 \end{cases}$$

and

$$\#\mathscr{M}(p_1 \times q_1, p_2 \times q_2) = \begin{cases} \#\mathscr{M}(q_1, q_2), & \text{if } p_1 = p_2 \\ \#\mathscr{M}(p_1, p_2), & \text{if } q_1 = q_2 \\ 0, & \textit{otherwise} \end{cases}$$

Note that the map

$$\Phi : \bigoplus_{i+j=k} C_i(f_1) \otimes C_j(f_2) \to C_k(f_1 \oplus f_2)$$

where $\Phi(a_1 \otimes a_2) = (a_1, a_2)$ is an isomorphism. And we will show that it is a chain map so that it induces an isomorphism on homology groups.

Proposition 5.3.2. *Φ is a chain map, i.e $\Phi \circ (\partial_M^{Morse} \otimes 1 + 1 \otimes \partial_N^{Morse}) = \partial_{M,N}^{Morse} \circ \Phi$.*

Proof. Let p_1 be a critical point of f_1 of index i and q_1 be a critical point of f_2 of index j and p_2 be a critical point of f_1 of index $i-1$ and q_2 be a critical point of f_2 of index $j-1$.

$$\begin{aligned}
\Phi \circ (\partial_M^{Morse} \otimes 1 + 1 \otimes \partial_N^{Morse})(p_1 \otimes q_1) &= \Phi(\partial_M^{Morse}(p_1) \otimes q_1 + p_1 \otimes \partial_N^{Morse})(q_1)) \\
&= \Phi(\sum \#\mathscr{M}(p_1, p_2) \cdot p_2 \otimes q_1 + \sum \#\mathscr{M}(q_1, q_2) \cdot p_1 \otimes q_2) \\
&= \sum \#\mathscr{M}(p_1, p_2) \cdot \Phi(p_2 \otimes q_1) + \sum \#\mathscr{M}(q_1, q_2) \cdot \Phi(p_1 \otimes q_2) \\
&= \sum \#\mathscr{M}(p_1, p_2) \cdot (p_2, q_1) + \sum \#\mathscr{M}(q_1, q_2) \cdot (p_1, q_2)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\partial_{(M,N)}^{Morse} \circ \Phi(p_1 \otimes q_1) &= \partial_{(M,N)}^{Morse}(p_1, q_1) \\
&= \sum_{(p_2,q_2) \in Crit_{i+j-1}(f \oplus g)} \#\mathscr{M}((p_1, q_1), (p_2, q_2)) \cdot (p_2, q_2) \\
&= \sum_{p_2 \in Crit_{i-1}(f_1)} \#\mathscr{M}(p_1, p_2) \cdot (p_2, q_1) + \sum_{q_2 \in Crit_{j-1}(f_2)} \#\mathscr{M}(q_1, q_2) \cdot (p_1, q_2)
\end{aligned}$$

So, we have shown that Φ is a chain map. ■

Hence, Φ_* induces an isomorphism on Morse homology

$$\Phi_* : H^{Morse}(\bigoplus_{i+j=k} C_i(f_1) \otimes C_j(f_2), \partial_M^{Morse} \otimes 1 + 1 \otimes \partial_N^{Morse}) \to H^{Morse}(C_k(f_1 \oplus f_2), \partial_{(M,N)}^{Morse}).$$

6 Morse-Bott Theory

Definition 6.0.3. Let $f : M \to \mathbb{R}$ be a smooth function on m-dimensional manifold M. An n dimensional submanifold $S \subset M$ is said to be critical if all points in S are critical.

Definition 6.0.4. Let M be a finite dimensional closed manifold. A function $f : M \to \mathbb{R}$ is said to be Morse-Bott if the followings are satisfied:

i. $Crit(f)$, the set of all critical points of f, is the disjoint union of submanifolds $S_i \subset M$.

ii. If S is a critical submanifold, for all $p \in S$ the kernel of Hessian of f at p consists only T_pS.

We now explain how to extend the idea of Morse Theory to Morse-Bott Theory. Although the set of all Morse functions is dense in the space of continuous functions, being a Morse function is difficult because of nondegeneracy condition for all critical points. Since any Morse-Bott function can be perturbed to a Morse function and Morse-Bott functions enable us to study on critical points which are degenerate, we are interested in Morse-Bott theory. For more detailed reading, [3] is an excellent reference.

Definition 6.0.5. (Normal Bundle,[19]) Let M be a Riemannian manifold with Riemannian metric g and S be a submanifold of M. For a given $p \in S$, we say that $n \in T_pM$ is normal to S if $g(n, v) = 0$ for all $v \in T_pS$. The set of all such n, N_pS, is said to be the normal space to S at p. The set

$$NS := \coprod_{p \in S} N_pS$$

is called the normal bundle.

Remark 6.0.6. ([6]) Equivalent to the second condition, given any metric on M, the Hessian $H(f, p)$ induces an invertible self adjoint map on the normal bundle

$$H(f, p) : N_pS \to N_pS.$$

Examples

• Every Morse function on M is also a Morse-Bott function; the critical submanifolds are critical points.

• Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = -x^2 + y^2$. By taking partial derivatives with respect to x, y and z, we get $\dfrac{\mathrm{d}f}{\mathrm{d}x} = -2x$, $\dfrac{\mathrm{d}f}{\mathrm{d}y} = 2y$ and $\dfrac{\mathrm{d}f}{\mathrm{d}z} = 0$ respectively. So, the critical points are $(0, 0, z)$ for all z. Therefore, the critical submanifold is z-axis. Note that the kernel of the Hessian of f at any point z is the set of solutions of the matrix equation

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So, $kerH(f, (0, 0, z)) = \{(0, 0, z) : z \in \mathbb{R}\}$. Also, since z-axis is $\mathbb{R}$ and the tangent space of $\mathbb{R}$ is again $\mathbb{R}$, $kerH(f, (0, 0, z)) = T_{(0,0,z)}\mathbb{R}$. Hence, f is a Morse-Bott function.

• Let $f : \mathbb{S}^2 \to \mathbb{R}$ be defined by $(x, y, z) \mapsto z^2$. We parametrize $\mathbb{S}^2$ with six charts:

$$(x, y, \mp\sqrt{1 - x^2 - y^2}) \mapsto 1 - x^2 - y^2$$

$$(x, \mp\sqrt{1 - y^2 - z^2} \mapsto z^2$$

$$(\mp\sqrt{1 - y^2 - z^2}, y, z) \mapsto z^2$$

The first one gives two critical points $(0, 0, \mp 1)$ and other ones give the critical submanifold, *equator*. So, the critical submanifolds of f are $S_0 = \mathbb{S}^1$, $S_1 = \emptyset$ and $S_2 = \{N, S\}$.

We now define the index for critical submanifolds. By using metric on M, we decompose the normal bundle as

$$N_pS = N_pS^- \oplus N_pS^+$$

The index is regarded as an interval $[i_-(S), i_+(S)]$, where $i_-(S)$ is the dimension of the negative normal bundle N_pS^- and $i_+(S) = i_-(S) + dim(S)$. In the second example, the index of the critical submanifold *z-axis* is $[1, 2]$ because $N_p\mathbb{R} = \mathbb{R} \oplus \mathbb{R}$ and the dimension of $\mathbb{R}$ is 1.

Remark 6.0.7. Every Morse-Bott function can be perturbed to a Morse function: let $f : M \to \mathbb{R}$ be a Morse-Bott function and $f_i : S_i \to \mathbb{R}$ be Morse functions on critical submanifolds of M. First, extend f_i to the manifold M by multiplying with bump functions on the tubular neighborhoods, then define the function

$$h_\epsilon(x) = f(x) + \epsilon(\sum_i \tilde{f}_i(x))$$

for all $x \in M$ and $\epsilon > 0$. Here, $\tilde{f}_i : M \to \mathbb{R}$ is the extension of f_i. The set of critical points of f_ϵ, $Crit(f_\epsilon)$, is the union of the set of critical points of f_i' s. So, f_ϵ becomes a Morse function. Let $p \in Crit(f_i)$, then the index of p of f_ϵ is

$$ind(p) + i_-(S_i).$$

As an example of the perturbation of a Morse-Bott function, consider $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) = -x^2 + y^2$. It is shown above that f is a Morse-Bott function. It has only one critical submanifold which is $\mathbb{R}$. Now, let $g : \mathbb{R} \to \mathbb{R}$ defined by $g(z) = -z^2$. This is a Morse function and it has one critical point 0 of index 1. First we extend it to $\mathbb{R}^3$ and define a new function $h_\epsilon(x, y, z) = -x^2 + y^2 - \epsilon z^2$. The critical point of h_ϵ is just $(0, 0, 0)$ of index 2. Also, the Hessian does not have 0 as an eigenvalue. So, h_ϵ is a Morse function.

6.1 Morse-Bott Homology

Fix a Morse-Bott function f on M. Let g be a metric and $-gradf$ be a negative gradient vector field of f. If S_j, S_k are two critical submanifolds, we define the set of flow lines as follows:

$$\mathscr{W}(S_j, S_k) = \left\{ \gamma : \mathbb{R} \to M | \gamma^{'}(s) = -gradf(\gamma(s)), \lim_{s \to -\infty} \gamma(s) \in S_j, \lim_{s \to \infty} \gamma(s) \in S_k \right\}.$$

As $\mathbb{R}$ acts on $\mathscr{W}(S_j, S_k)$ by precomposition with translation, we get the set of unparametrized flow lines beginning in S_j and ending in S_k

$$\mathscr{M}(S_j, S_k) = \mathscr{W}(S_j, S_k)/\mathbb{R}.$$

Remark 6.1.1. The descending and ascending manifolds of a critical submanifold S_j can be defined analogous to the case of nondegenerate critical points.

i. The descending manifold of S_j is defined to be:

$$\mathscr{D}(S_j) = \left\{ x \in M : \lim_{s \to -\infty} \psi_s(x) \in S_j \right\}$$

ii. The ascending manifold of S_j is defined to be:

$$\mathscr{A}(S_j) = \left\{ x \in M : \lim_{s \to \infty} \psi_s(x) \in S_j \right\}$$

For any generic metric g, the descending manifold of S_j and the ascending manifold of S_k will intersect transversely so that $\mathscr{W}(S_j, S_k)$ is a manifold.

Remark 6.1.2. The dimension of the descending manifold of S_j is $n_j + i_-(S_j)$ and the dimension of the ascending manifold is $m - (n_k + i_-(S_k))$, where n_j and n_k are dimensions of S_j and S_k, respectively. Then,

$$\begin{aligned} dim\mathscr{W}(S_j, S_k) &= n_j + i_-(S_j) + (n_k + (m - (n_k + i_-(S_k)))) - m \\ &= n_j + i_-(S_j) - i_-(S_k) \\ &= i_+(S_j) - i_-(S_k) \end{aligned}$$

So, $dim\mathscr{M}(S_j, S_k) = i_+(S_j) - i_-(S_k) - 1$.

We now focus on the case of Morse-Bott complex. There are natural endpoint maps

$$e_+ : \mathscr{M}(S_j, S_k) \to S_j$$

and

$$e_- : \mathscr{M}(S_j, S_k) \to S_k$$

sending $\gamma \mapsto \lim_{s \to -\infty} \gamma(s)$ and $\gamma \mapsto \lim_{s \to -\infty} \gamma(s)$, respectively.

Proposition 6.1.3. *If A, B and C are smooth manifolds and the maps $f : A \to C$ and $g : B \to C$ are transverse to each other, then*

$$A \times_C B = \{(a, b) \in A \times B : f(a) = g(b)\}$$

is a smooth manifold of dimension $dim(A) + dim(B) - dim(C)$.

Proof. Consider the map $f \times g : A \times B \to C \times C$ defined by $(a, b) \mapsto (f(a), g(b))$. Observe that $(f \times g)^{-1}(\Delta_C) = A \times_C B$, where $\Delta_C = \{(c, c) : c \in C\}$. Since f and g are transverse to each other, $f \times g$ and Δ_C intersect transversally, as well. By [14], we conclude that $(f \times g)^{-1}(\Delta_C)$ is a smooth submanifold of $A \times B$. ∎

Lemma 6.1.4. *([12]) For a generic metric, the maps $e_+ : \mathscr{M}(S_j, S_k) \to S_j$ and $e_- : \mathscr{M}(S_j, S_k) \to S_k$ are transverse to each other.*

Lemma 6.1.5. *For a generic metric, the space of gradient lines $\mathscr{M}(S_j, S_k)$ can be compactified to a manifold with corner $\overline{\mathscr{M}(S_j, S_k)}$ and the codimension l stratum is*

$$\overline{\mathscr{M}(S_j, S_k)_l} = \bigcup_{S_{i_1},...,S_{i_l}} \mathscr{M}(S_j, S_{i_1}) \times_{S_{i_1}} \mathscr{M}(S_{i_1}, S_{i_2}) \times_{S_{i_2}} ... \times_{S_{i_l}} \mathscr{M}(S_{i_l}, S_k)$$

where $S_{i_1}, ..., S_{i_n}$ are different critical submanifolds. In particular, for $l = 1$, the boundary of $\overline{\mathscr{M}(S_j, S_k)}$

$$\partial\overline{\mathscr{M}(S_j, S_k)} = \bigcup_{S_{i_1}} \mathscr{M}(S_j, S_{i_1}) \times_{S_{i_1}} \mathscr{M}(S_{i_1}, S_k)$$

Like in Lemma 6.1.4, for a generic metric, the maps are transverse to each other

$$e_-^{i_1,...,i_m} : \mathscr{M}(S_j, S_{i_1}) \times_{S_{i_1}} ... \times_{S_{i_{m-1}}} \mathscr{M}(S_{i_{m-1}}, S_{i_m}) \to S_{i_m}$$

$$e_+^{i_m,...,i_n} : \mathscr{M}(S_{i_m}, S_{i_{m+1}}) \times_{S_{i_{m+1}}} ... \times_{S_{i_n}} \mathscr{M}(S_{i_n}, S_k) \to S_{i_m}.$$

Note that the moduli space does not have to be orientable even if S_1 and S_2 are orientable. As an example, let S_1 and S_2 are circles and the moduli space is a Klein bottle. However, we can orient $\mathscr{M}(S_1, S_2)$ locally. Let σ be a generic simplex in S_1, and define

$$\overline{\mathscr{M}(\sigma, S_2)} := \sigma \times_{S_1} \overline{\mathscr{M}(S_1, S_2)}.$$

We observe that σ and e_- are transversal, so $\overline{\mathscr{M}(\sigma, S_2)}$ is a manifold, see [7].

We give a local orientation to $\mathscr{M}(\sigma, S_2)$. If $\gamma \in \mathscr{M}(\sigma, S_2)$ is a gradient flow line from p_1 to p_2, then orintations of σ, $\mathscr{D}(p_1)$ and $\mathscr{D}(p_2)$ determine a local orientation of $\mathscr{M}(\sigma, S_2)$ in such a way that

$$T_{p_1}\sigma \oplus T_{p_1}\mathscr{D}(p_1) \cong T_\gamma\mathscr{M}(\sigma, S_2) \oplus T\gamma \oplus T_{p_2}\mathscr{D}(p_2)$$

where $p_1 \in \sigma$.

Let us introduce the orientation sheaf $\mathcal{O}$ to define the chain groups of Morse-Bott complex. Before doing this, we recall the definitions of presheaf and sheaf which provide informations from local data to global data. For more detail, see [9].

Definition 6.1.6. (**Presheaf**) Let X be a topological space. We say that $\mathcal{F}$ is a presheaf of groups over X if the following conditions are satisfied

i. For each open set $U \subset X$, there is a group denoted by $\mathcal{F}(U)$

ii. For every inclusion $V \subset U$, there is a map $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ with the properties that $\mathcal{F}(\emptyset) = 0$, $\rho_U^U = id$ for any open set $U \subset X$ and for open sets $W \subset V \subset U$ the composition $\rho_W^V \circ \rho_V^U = \rho_W^U$.

The elements of a presheaf $\mathcal{F}(U)$ are sections on U and $\rho_V^U(f)$ denoted by $f|_U$.

Definition 6.1.7. (**Sheaf**) Let X be a topological space. A presheaf $\mathcal{F}$ is called a sheaf if for any open set $U \subset X$ and for any open cover $\{U_i\}_{i\in I}$ of U, the followings are satisfied

i. Locality:
If $f \in \mathcal{F}(U)$ such that $f|_{U_i} = 0$ for all i, then $f = 0$.

ii. Gluing:
Given a collection of sections $\{f_i\}_{i\in I}$ such that $\rho^{U_i}_{U_i\cap U_j}(f_i) = \rho^{U_j}_{U_i\cap U_j}(f_j)$, then there exists a section $f \in \mathcal{F}(U)$ such that $\rho^U_{U_i}(f) = f_i$ for all $i \in I$. This means that if $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all i, j, then there is an $f \in \mathcal{F}(U)$ with $f|_{U_i} = f_i$ for all i.

Note that this definition can be generalized to all algebraic objects such as sets, rings, modules, etc.

Remark 6.1.8. A presheaf does not have to be a sheaf: Let A be an Abelian group. We can define a presheaf $\tilde{A}$ by $\tilde{A}(U) = A$ for all U. The restriction maps ρ are the identity maps on A. This is called the *constant presheaf* on X. It is obviously a presheaf. But it may not be a sheaf. For example, let $X = U_1 \cup U_2$ such that $U_1 \cap U_2 = \emptyset$, then if $a \in \tilde{A}(U_1)$ and $b \in \tilde{A}(U_2)$, a and b are distinct in A, then a and b do not glue an element in $\tilde{A}(X)$.

Definition 6.1.9. (**Constant sheaf**) Let X be a topological space and A be an Abelian group with a topological structure, discrete topology. We define the *constant sheaf* by $\mathcal{F}(U) = C(U, A)$, the group of continuous functions from U to A. The restriction maps are the function restriction.

Definition 6.1.10. (**Stalk**) Let $\mathcal{F}$ be a presheaf on a topological space X. Let $p \in X$. We define the stalk of $\mathcal{F}$ at p

$$\mathcal{F}_p = \lim_{\substack{\rightarrow \\ p\in U}} \mathcal{F}(U)$$

where U is the neighborhood of p. The direct limit is over for each neighborhood of p.

We have seen that not all presheafs are sheafs. But we can obtain a sheaf by applying a process which is called the *sheafification*. More precisely, let $\mathcal{F}$ be a presheaf. We define a new sheaf $\tilde{\mathcal{F}}$ in the following way: Let $\tilde{\mathcal{F}}(U)$ be the set of all functions f from U to disjoint union on stalks $\mathcal{F}_p$

$$f : U \to \bigcup_{p \in U} \mathcal{F}_p$$

that satisfy the following properties

i. $f(p) \in \mathcal{F}_p$

ii. For al $p \in U$, there is an open neighborhood $V \subset U$ of p and a section $s \in \mathcal{F}(V)$ such that for all $q \in V$ we have $f(q) = s_q$.

Lemma 6.1.11. *Let $\mathcal{F}$ be a presheaf and $\tilde{\mathcal{F}}$ its sheafification. The stalks of $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are the same.*

Let M be an n-dimensional smooth manifold. Then the orientation sheaf $\mathcal{O}$ is the sheafification of the presheaf $U \mapsto H_n(M, M - U, \mathbb{Z})$. It is always a locally constant sheaf [9].

Remark 6.1.12. Because of the above lemma 6.1.11, the stalk of $\mathcal{O}$ is the stalk of the presheaf $U \mapsto H_n(M, M - U, \mathbb{Z})$. Since $H_n(M, M - U, \mathbb{Z}) \cong \mathbb{Z}$, the stalk of $\mathcal{O}$ is $\mathbb{Z}$, [15]: Note that for all $p \in M$, $H_q(M, M - \{p\}) = \mathbb{Z}$ if $q = n$, and 0 if $q \neq n$, let U be an open neighborhood of p, then $\overline{M - U} = M - U \subset M - \{p\} = Int(M - \{p\}$. So, by excision theorem

$$\begin{aligned} H_q(M, M - \{p\}) &\cong H_q(U, U - \{p\}) \\ &\cong H_q(\mathbb{R}^n, \mathbb{R}^n - \{p\} \\ &\cong \tilde{H}_{q-1}(\mathbb{R}^n - \{p\}) \\ &\cong \tilde{H}_{q-1}(\mathbb{S}^{n-1}) \end{aligned}$$

the last isomorphism comes from the long exact sequence theorem and $\tilde{H}_{q-1}(\mathbb{S}^{n-1}) \cong \mathbb{Z}$ if $q = n$, and 0 if $q \neq n$. Also, given an open neighborhood W of p, there exists an open neighborhood U of p such that $U \subset W$ and $H_*(M, M - U) \cong H_*(M, M - \{p\})$ for all $p \in U$.

Remark 6.1.13. The orientation sheaf $\mathcal{O}$ is trivial iff M is orientable, [9]. Trivial orientation sheaf means that $\mathcal{O}$ is constant sheaf.

Now, we can use all these informations to the critical submanifold S. There is an orientation sheaf $\mathcal{O}$ on S whose stalk is defined by

$$\mathcal{O}_p := H_{i_-(S)-1}(\mathscr{D}(p)/p) \cong \mathbb{Z}.$$

The orientation on $\mathscr{D}(p)$ determines such an isomorphism and the orientation on $\mathscr{A}(p)$ determines the opposite orientation. Let $C_*^{sing}(S,\mathcal{O})$ be the space of singular chains with coefficients in $\mathcal{O}$,

$$C_*^{sing}(S,\mathcal{O}) := \mathbb{Z}[(\sigma,o) : \sigma \subset S, o \in \Gamma(Im(\sigma),\mathcal{O})]/\sim$$

where σ is simplex in S, $\Gamma(Im(\sigma),\mathcal{O})$ is the set of sections and the relation $\sim$ is defined by $(\sigma,-o) \sim -(\sigma,o)$. To define the Morse-Bott chain complex, we choose the generic simplecies which means that σ is smooth and each face of σ is transverse to e_+ of all moduli spaces of flow lines between critical submanifolds and all iterated fiber products thereof. We define the resulting chain complex as $C_*(S,\mathcal{O}) \subset C_*^{sing}(S,\mathcal{O})$ and the differential defined in a standard way. Thus, the k^{th} Morse-Bott chain group is defined as follows

$$C_k^{Bott} := \bigoplus_S C_{k-i_-(S)}(S,\mathcal{O}).$$

If $\sigma \in C_*(S,\mathcal{O})$ is a generic simplex and for $S^{'} \neq S$, then we have the following composition of maps

$$\overline{\mathscr{M}(\sigma,S')} \xrightarrow{i} \overline{\mathscr{M}(S,S')} \xrightarrow{e_-} S^{'}$$

gives us the following well defined current

$$(i \circ e_-)[\overline{\mathscr{M}(\sigma,S')}] \in C_*(S^{'},\mathcal{O}).$$

We always think σ with an orientation data o, we will write σ instead of (σ,o) and we use the notation e_- instead of $(i \circ e_-)$ for simplicity. Moreover, if $dim(\sigma) = k - i_-(S)$, then

$$\begin{aligned} dim(\mathscr{M}(\sigma,S^{'})) &= dim(\sigma) + dim(\overline{\mathscr{M}(S,S')}) - dim(S) \\ &= k - i_-(S) + i_+(S) - i_-(S^{'}) - 1 - dim(S) \\ &= k - 1 - i_-(S^{'}). \end{aligned}$$

We now define the differential $D : C_k^{Bott} \to C_{k-1}^{Bott}$ as follows:

$$D(\sigma) := \partial\sigma + \sum_{S' \neq S} e_-[\overline{\mathscr{M}(\sigma, S')}].$$

Lemma 6.1.14. $D^2 = 0$.

Proof.

$$\begin{aligned}
D(D\sigma) &= D(\partial\sigma + \sum_{S' \neq S} e_-[\overline{\mathscr{M}(\sigma, S')}]) \\
&= D(\partial\sigma) + D(\sum_{S' \neq S} e_-[\overline{\mathscr{M}(\sigma, S')}])) \\
&= \partial^2\sigma + \sum_{S' \neq S} e_-[\overline{\mathscr{M}(\partial\sigma, S')}] + \partial(\sum_{S' \neq S} e_-[\overline{\mathscr{M}(\sigma, S')}]) + \sum_{S'' \neq S'} e_-[\overline{\mathscr{M}(\sum_{S' \neq S} e_-[\overline{\mathscr{M}(\sigma, S')}], S'')}] \\
&= \sum_{S' \neq S} e_-[\partial\sigma \times_S \overline{\mathscr{M}(S, S')}] + \partial(\sum_{S' \neq S} e_-[\sigma \times_S \overline{\mathscr{M}(S, S')}]) \\
&+ \sum_{S'' \neq S'} e_-[\sum_{S' \neq S} e_-[\overline{\mathscr{M}(\sigma, S')}] \times_{S'} \overline{\mathscr{M}(S', S'')}] \\
&= \sum_{S' \neq S} e_-[\sigma \times_S \partial\overline{\mathscr{M}(S, S')}] + \sum_{S'' \neq S'} e_-[\sum_{S' \neq S} e_-[\sigma \times_S \overline{\mathscr{M}(S, S')}] \times_{S'} \overline{\mathscr{M}(S', S'')}]
\end{aligned}$$

The last equality comes from the fact that, see [12]

$$\partial(\sigma \times_S \overline{\mathscr{M}(S, S')}) = \partial\sigma \times_S \overline{\mathscr{M}(S, S')} \bigcup \sigma \times_S \partial\overline{\mathscr{M}(S, S')}.$$

Then, we get the equality from the compactness property of moduli sapace $\mathscr{M}(S, S^{'})$:

$$\begin{aligned}
\sum_{S' \neq S} e_-[\sigma \times_S \partial\overline{\mathscr{M}(S, S')}] &= \sum_{S' \neq S} e_-[\sigma \times_S \bigcup_{S''} \mathscr{M}(S, S^{''}) \times_{S''} \mathscr{M}(S^{''}, S^{'})] \\
&= \sum_{S' \neq S} \sum_{S \neq S''} e_-[\sigma \times_S \mathscr{M}(S, S^{''}) \times_{S''} \mathscr{M}(S^{''}, S^{'})]
\end{aligned}$$

Hence,

$$D^2\sigma = 0.$$

■

The homology of the chain complex (C_*^{Bott}, D) is called the Morse-Bott homology $H_*^{Bott}(f, g)$.

Examples

• Let $f : M \to \mathbb{R}$ be the 0 map. Since all points in M are critical points, $Crit(f) = M$. So, f is a Morse-Bott function. The only critical submanifold of M is itself. Then $S = M$ and $C_*^{Bott} = C_*(M)$. In this case, the all homology group are the same as all singular homology groups of M. Hence, $H_*^{Bott}(f, g) = H_*(M)$.

• Consider the Morse-Bott function $f : \mathbb{S}^2 \to \mathbb{R}$ such that $f(x, y, z) = z^2$. We have shown that the critical submanifolds are *equatorial* $S_0 = \mathbb{S}^1$, and the set of critical points $S_2 = \{N, S\}$ with the corrosponding index 0 and 2, respectively. So the Morse-Bott chain groups are the followings:

$$C_0^{Bott} = C_0(S_0, \mathcal{O})$$

$$C_1^{Bott} = C_1(S_0, \mathcal{O})$$

$$C_2^{Bott} = C_2(S_0, \mathcal{O}) \oplus C_0(S_2, \mathcal{O}) = C_0(S_2, \mathcal{O})$$

Since S_0 is $\mathbb{S}^1$ and the orientation sheaf is trivial, $C_0^{Bott} = C_0(\mathbb{S}^1)$. Moreover, the chain groups for $k \geq 3$ are all empty set. Then, the chain complex is as follows

$$0 \to C_0(S_2, \mathcal{O}) \overset{D_2}{\to} C_1(\mathbb{S}^1) \overset{D_1}{\to} C_0(\mathbb{S}^1) \overset{0}{\to} 0.$$

In this chain complex, $D_1(\sigma) = \partial\sigma$, $D_2(S) = e_-[\mathscr{M}(S, S_0)] = [S_0]$ and $D_2(N) = e_-[\mathscr{M}(N, S_0)] = [S_0]$. Then, $ker(D_2) =< S + N >= \mathbb{Z}_2$, $im(D_2) = \mathbb{Z}_2$ and $ker(D_1) = \mathbb{Z}_2$. Therefore, the Morse-Bott homology groups of $\mathbb{S}^2$ are

$$H_n^{Bott}(f, g) = \begin{cases} \mathbb{Z}_2, & \text{if } n\text{= 0,2} \\ 0, & otherwise \end{cases}$$

• Let $f : T^2 \to \mathbb{R}$ be a Morse-Bott function on the torus with two critical submanifolds, S_0 and S_1, circle of minima and circle of maxima, respectively. The circle of minima is of index 0 and the circle of maxima is of index 1. The chain groups are the followings;

$$C_0^{Bott} = C_0(S_0, \mathcal{O})$$

$$C_1^{Bott} = C_1(S_0, \mathcal{O}) \oplus C_0(S_1, \mathcal{O})$$

$$C_2^{Bott} = C_1(S_1, \mathcal{O})$$

so, the Morse- Bott chain complex is

$$0 \to C_1(S_1, \mathcal{O}) \stackrel{D_2}{\to} C_1(S_0, \mathcal{O}) \oplus C_0(S_1, \mathcal{O}) \stackrel{D_1}{\to} C_0(S_0, \mathcal{O}) \stackrel{0}{\to} 0.$$

Since the circle is orientable, the orientation sheaf is trivial. Also, all simplices in the critical submanifolds are generic. If we choose a symmetric metric on torus, for each point in S_1 there are two flow lines to the same point in S_0. So, the differential $D_2(\sigma_1) = \partial\sigma_1$ and $D_1(\sigma_0, .) = \partial\sigma_0$. Then $ker(D_1) =< [S_0], . >= \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $im(D_1) = 0$, $ker(D_2) =< [S_1] >= \mathbb{Z}_2$ and $im(D_2) = 0$. Therefore, the Morse-Bott homology of T^2 is

$$H_n^{Bott}(f, g) = \begin{cases} \mathbb{Z}_2, & \text{if } n\text{= 0,2} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & \text{if } n = 1 \\ 0, & otherwise \end{cases}$$

• In this example, we will do surgery on a horizontal circle of the previous example: First, having cut out the tubular neighborhood of the horizontal circle, we are left with a cylinder $\mathbb{S}^1 \times \mathbb{D}^1$. We glue $\mathbb{D}^2 \times \partial\mathbb{D}^1 = \mathbb{D}^2 \sqcup \mathbb{D}^2$ back in the cylinder. The resulting manifold will be homeomorphic to the sphere.

We now put a Morse-Bott function on $\mathbb{S}^2$ with a circle S_0 of minima, a circle S_1 of maxima, an isolated minimum m_0 and an isolated maksimum m_2 as critical submanifolds. The indices are 0, 1, 0 and 2, respectively. The chain groups are

$$C_0^{Bott} = C_0(S_0, \mathcal{O}) \oplus C_0(m_0, \mathcal{O})$$

$$C_1^{Bott} = C_1(S_0, \mathcal{O}) \oplus C_0(S_1, \mathcal{O})$$

$$C_2^{Bott} = C_1(S_1, \mathcal{O}) \oplus C_0(m_2, \mathcal{O})$$

Since the critical submanifolds are orientable, all orientation sheaves are trivial. So the Morse-Bott chain complex is

$$0 \to C_1(S_1, \mathcal{O}) \oplus C_0(m_2, \mathcal{O}) \stackrel{D_2}{\to} C_1(S_0, \mathcal{O}) \oplus C_0(S_1, \mathcal{O}) \stackrel{D_1}{\to} C_0(S_0, \mathcal{O}) \oplus C_0(m_0, \mathcal{O}) \to 0$$

Up to orientation, $D_1(S_0) = 0$ and $D_1(p) = m_0 + q$ where $p \in S_1$ and $q = \phi(p)$, $\phi : S_1 \to S_0$ is a diffeomorphism. Furthermore, $D_2(m_2) = [S_0]$ and $D_2(\sigma) = \partial\sigma + e_-[\mathscr{M}(\sigma, S_0)] = [S_0]$. Since there are two flow lines at the same point on S_0, $D_2 = 0$. Therefore, the Morse-Bott homology is depicted below

$$H_n^{Bott}(f, g) = \begin{cases} \mathbb{Z}_2, & \text{if } n = 0,2 \\ 0, & otherwise \end{cases}$$

As easily seen in the examples, the Morse-Bott homology groups computed from the complex (C_*^{Bott}, D) are independent of the choice of a Morse-Bott function and a generic metric. This is analogous to the case of Morse homology.

Theorem 6.1.15. *Given two different Morse-Bott functions f_0 and f_1 with generic metrics g_0 and g_1, there is a canonical isomorphism*

$$H_*^{Bott}(f_0, g_0) \cong H_*^{Bott}(f_1, g_1).$$

We already know from the previous examples that $H_*^{Bott}(f_1, g_1) \cong H_*(X)$ for $f_1 = 0$ and any metric g_1. Then the theorem implies that for a Morse-Bott function f_0 and a metric g_0, there is a canonical isomorphism

$$H_*^{Bott}(f_0, g_0) \cong H_*(X).$$

In conclusion, Morse-Bott homology depends only on the manifold structure. The proof of the theorem can also be shown by an alternative technique, see [3].

6.2 The Morse-Bott spectral sequences

Let C_* be a chain complex, and let $A_* \subset C_*$ be a subcomplex. The short exact sequence of chain complexes

$$0 \to A_* \hookrightarrow C_* \to C_*/A_* \to 0$$

leads to a long exact sequence in homology, [17]:

$$\to ... \to H_{q+1}(C_*, A_*) \to H_q(A_*) \to H_q(C_*) \to H_q(C_*, A_*) \to H_{q-1}(A_*) \to ....$$

A filtred chain complex is a chain complex with a filtration F_pC_i of each C_i such that $\partial(F_pC_i) \subset F_pC_{i-1}$. When one has a filtration of a chain complex C_*, there is an increasing sequence of subcomplexes $F_pC_* \subset F_{p+1}C_*$ such that $C_* = \cup_p F_pC_*$, [22].

Let G_pC_* be the subquotient complex $G_pC_* = F_pC_*/F_{p-1}C_*$. We get a short exact sequence

$$0 \to F_{p-1}C_* \hookrightarrow F_pC_* \to G_pC_* \to 0.$$

Then there is a long exact sequence in the homology for each p,

$$\to ... \to H_{q+1}(G_pC_*) \to H_q(F_{p-1}C_*) \to H_q(F_pC_*) \to H_q(G_pC_*) \to H_{q-1}(F_{p-1}C_*) \to ....$$

Let us denote the graded module by $E^0_{p,q} = G_pC_{p+q} = F_pC_{p+q}/F_{p-1}C_{p+q}$. The boundary map on the chain complex C_* induces the zeroth differential, which we denote by

$$\mathrm{d}^0_{p,q} : E^0_{p,q} \to E^0_{p,q-1}.$$

Since $\mathrm{d}^0_{p,q} \circ \mathrm{d}^0_{p,q+1} = 0$, we let

$$E^1_{p,q} = H_{p+q}(G_pC_*).$$

Now, we define the first differential map $\mathrm{d}^1_{p,q} : E^1_{p,q} \to E^1_{p-1,q}$ as follows: A homology class $\alpha \in E^1_{p,q}$ can be represented by a chain $x \in F_pC_{p+q}$ such that $\partial x \in F_{p-1}C_{p+q-1}$. We define $\mathrm{d}^1_{p,q}(\alpha) = [\partial x]$. Since $\partial^2 = 0$, then $\mathrm{d}^1_{p,q}$ is well defined and $\mathrm{d}^1_{p,q} \circ \mathrm{d}^1_{p+1,q} = 0$, we let the homology

$$E^2_{p,q} = ker(\mathrm{d}^1_{p,q} : E^1_{p,q} \to E_{p-1,q})/Im(\mathrm{d}^1_{p+1,q} : E^1_{p+1,q} \to E^1_{p,q}).$$

All in all, a spectral sequence is a collection of R-modules $E^r_{p,q}$ and morphisms $\mathrm{d}^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ satisfies $\mathrm{d}^r_{p,q} \circ \mathrm{d}^r_{p+r,q-r+1} = 0$, so that $E^{r+1}_{p,q}$ is the homology of the chain complex $(E^r, \mathrm{d}^r_{p,q})$

$$E^{r+1}_{p,q} = \frac{ker(E^r_{p,q} \to E^r_{p-r,q+r-1})}{Im(E^r_{p+r,q-r+1} \to E^r_{p-r,q+r-1})}.$$

A spectral sequence is called convergent if for every p, q there exists $r_{p,q}$ such that for all $r \geq r_{p,q}$, the differentials $\mathrm{d}^r_{p,q}$ and $\mathrm{d}^r_{p+r,q-r+1}$ are zero. When $\left\{E^r_{p,q}, \mathrm{d}^r_{p,q}\right\}_r$ is convergent, its limit is denoted by $E^\infty_{p,q}$, [22].

Proposition 6.2.1. *([16], [8]) Let (F_pC_*, ∂) be a filtred complex. Then, there is a spectral* *sequence* $\left\{E^r_{p,q},\, \mathrm{d}^r_{p,q}\right\}$*, defined for* $r \geq 0$*, with*

$$E^1_{p,q} = H_{p+q}(G_p C_*).$$

If the filtration of C_i *is bounded for each* i*, then the spectral sequence converges to*

$$E^\infty_{p,q} = G_p H_{p+q}(C_*).$$

Thus, $E^\infty_{*,*}$ *determines the homology* $H_*(C_*)$ *up to extensions. In particular,*

$$H_i(C_*) \cong \bigoplus_{p+q=i} E^\infty_{p,q}.$$

The pair (f,g) is weakly self-indexing if $\mathscr{M}(S,S') = \emptyset$ whenever $i_-(S) < i_-(S')$. In this case, i_- defines a filtration on the complex (C^{Bott}_*, D)

$$F_i C^{Bott}_* = \bigoplus_{i_-(S)\leq i} C_{*-i_-(S)}(S,\mathcal{O}).$$

According to the proposition stated above, we obtain a spectral sequence converging to the Morse-Bott homology with the $E^1_{p,q}$ term of the spectral sequence as follows:

$$\begin{aligned}
E^1_{p,q} &= H_{p+q}(F_p(C^{Bott}_*)/F_{p-1}(C^{Bott}_*)) \\
&= H_{p+q}(\bigoplus_{i_-(S)\leq p} C_*(S,\mathcal{O})[i_-(S)]/\bigoplus_{i_-(S)\leq p-1} C_*(S,\mathcal{O})[i_-(S)]) \\
&= H_{p+q}(\bigoplus_{i_-(S)=p} C_*(S,\mathcal{O})[i_-(S)]) \\
&= H_q(\bigoplus_{i_-(S)=p} C_*(S,\mathcal{O})) \\
&= \bigoplus_{i_-(S)=p} H_q(C_*(S,\mathcal{O})) \\
&= \bigoplus_{i_-(S)=p} H_q(S,\mathcal{O}).
\end{aligned}$$

Hence, the $E^1_{p,q}$ term of the spectral squence is:

$$\vdots$$

$$\begin{array}{ccccccccc}
\bigoplus_{i_-(S)=0} H_3(S,\mathcal{O}) & \xleftarrow{\mathrm{d}^1_{1,3}} & \bigoplus_{i_-(S)=1} H_3(S,\mathcal{O}) & \xleftarrow{\mathrm{d}^1_{2,3}} & \bigoplus_{i_-(S)=2} H_3(S,\mathcal{O}) & \xleftarrow{\mathrm{d}^1_{3,3}} & \bigoplus_{i_-(S)=3} H_3(S,\mathcal{O}) & & \cdots \\
\bigoplus_{i_-(S)=0} H_2(S,\mathcal{O}) & \xleftarrow{\mathrm{d}^1_{1,2}} & \bigoplus_{i_-(S)=1} H_2(S,\mathcal{O}) & \xleftarrow{\mathrm{d}^1_{2,2}} & \bigoplus_{i_-(S)=2} H_2(S,\mathcal{O}) & \xleftarrow{\mathrm{d}^1_{3,2}} & \bigoplus_{i_-(S)=3} H_2(S,\mathcal{O}) & & \cdots \\
\bigoplus_{i_-(S)=0} H_1(S,\mathcal{O}) & \xleftarrow{\mathrm{d}^1_{1,1}} & \bigoplus_{i_-(S)=1} H_1(S,\mathcal{O}) & \xleftarrow{\mathrm{d}^1_{2,1}} & \bigoplus_{i_-(S)=2} H_1(S,\mathcal{O}) & \xleftarrow{\mathrm{d}^1_{3,1}} & \bigoplus_{i_-(S)=3} H_1(S,\mathcal{O}) & & \cdots \\
\bigoplus_{i_-(S)=0} H_0(S,\mathcal{O}) & \xleftarrow{\mathrm{d}^1_{1,0}} & \bigoplus_{i_-(S)=1} H_0(S,\mathcal{O}) & \xleftarrow{\mathrm{d}^1_{2,0}} & \bigoplus_{i_-(S)=2} H_0(S,\mathcal{O}) & \xleftarrow{\mathrm{d}^1_{3,0}} & \bigoplus_{i_-(S)=3} H_0(S,\mathcal{O}) & & \cdots
\end{array}$$

where the differential $\mathrm{d}^1_{p,q}$ is defined

$$\mathrm{d}^1_{p,q} : E^1_{p,q} \to E^1_{p-1,q}$$

as follows: Let $\alpha \in H_q(S,\mathcal{O})$, represent it with a cycle C. For each S' with $i_-(S') = p-1$, up to orientation we obtain $\mathrm{d}^1_{p,q}(\alpha) = \sum_{i_(S')=p-1} \pm[e_-(C \times_S \mathscr{M}(S,S'))]$. The map is well-defined because the dimension of the fiber product $C \times_S \mathscr{M}(S,S')$ is 1. The higher differentials are similar with $\mathrm{d}^1_{p,q}$ when there are no broken flow lines involved. Below, we present some examples of Morse-Bott functions to calculate their Morse-Bott homology in the sense of spectral sequences.

Examples

• Take the Morse-Bott function on the torus which has two critical submanifolds, S_1 and S_0, the indices of submanifolds are 1 and 0, respectively. We calculated the Morse-Bott homology of torus on previous pages. We realized that the Morse-Bott homology of torus is the same as its singular homology. Now, by showing all terms of the spectral sequence, we will calculate the homology by using Prop 5.2.1. The $E^1_{p,q}$ term of the spectral sequence

$$\vdots$$

$$\begin{array}{ccccccc}
0 & \xleftarrow{\mathrm{d}^1_{1,2}} & 0 & \xleftarrow{\mathrm{d}^1_{2,2}} & 0 & \xleftarrow{\mathrm{d}^1_{3,2}} & \cdots \\
\mathbb{Z} & \xleftarrow{\mathrm{d}^1_{1,1}} & \mathbb{Z} & \xleftarrow{\mathrm{d}^1_{2,1}} & 0 & \xleftarrow{\mathrm{d}^1_{3,1}} & \cdots \\
\mathbb{Z} & \xleftarrow{\mathrm{d}^1_{1,0}} & \mathbb{Z} & \xleftarrow{\mathrm{d}^1_{2,0}} & 0 & \xleftarrow{\mathrm{d}^1_{3,0}} & \cdots
\end{array}$$

Observe that $\mathrm{d}^1_{1,0}$ and $\mathrm{d}^1_{1,1}$ are not zero. So, we pass to $E^2_{p,q}$. To do this, we have to find all terms of $E^2_{p,q}$. The kernel of the differential $\mathrm{d}^1_{1,0}$ is the set $\{\alpha \in H_0(S_1) : \mathrm{d}^1_{p,q}(\alpha) = 0\}$. So, $e_-[C \times_{S_0} \mathscr{M}(S_1, S_0)] = 0$. Since the moduli space $\mathscr{M}(S_1, S_0)$ is one dimensional manifold, the kernel is the set of points., i.e $\mathbb{Z}$. From the same reason, the kernel of the differential $\mathrm{d}^1_{1,1}$ is $\mathbb{Z}$. Hence, the $E^2_{p,q}$ term of the spectral sequence looks like

$$\begin{matrix} 0 & 0 & 0 \\ \mathbb{Z} & \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} & 0 \end{matrix}$$

Then all differentials $\mathrm{d}^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+1}$ are 0. Also, for every p and q, the differentials $\mathrm{d}^r_{p,q}$ and $\mathrm{d}^r_{p+r,q-r+1}$ are zero for $r \geq 2$. So, $E^\infty_{p,q}$ is as above. Hence, we get the homology groups as follows:

$$H_0^{Bott}(f, g) = E^2_{0,0} = \mathbb{Z}$$

$$H_1^{Bott}(f, g) = E^2_{1,0} \oplus E^2_{0,1} = \mathbb{Z} \oplus \mathbb{Z}$$

$$H_2^{Bott}(f, g) = E^2_{2,0} \oplus E^2_{0,2} \oplus E^2_{1,1} = \mathbb{Z}$$

• Let $f : \mathbb{S}^2 \to \mathbb{R}$ such that $f(x, y, z) = z^2$. The critical submanifolds are the equatorial $\mathbb{S}^1$, north pole N and south pole S. Their indices are 0, 2 and 2, respectively. The $E^1_{p,q}$ term of the spectral sequence is

$$\begin{array}{c} \vdots \\ 0 \xleftarrow{\mathrm{d}^1_{1,2}} 0 \xleftarrow{\mathrm{d}^1_{2,2}} 0 \xleftarrow{\mathrm{d}^1_{3,2}} \qquad \cdots \\ \mathbb{Z} \xleftarrow{\mathrm{d}^1_{1,1}} 0 \xleftarrow{\mathrm{d}^1_{2,1}} 0 \xleftarrow{\mathrm{d}^1_{3,1}} 0 \qquad \cdots \\ \mathbb{Z} \xleftarrow{\mathrm{d}^1_{1,0}} 0 \xleftarrow{\mathrm{d}^1_{2,0}} \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{\mathrm{d}^1_{3,0}} 0 \quad \cdots \end{array}$$

All differentials on the $E^1_{p,q}$ are zero but not all on $E^2_{p,q}$. For example, $\mathrm{d}^2_{2,0} : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ is not zero since $\mathrm{d}^2_{2,0}(N, S) = e_-[\mathscr{M}(N, \mathbb{S}^1)] = \pm[\mathbb{S}^1]$. So we pass to $E^3_{p,q}$. The kernel of the differential $\mathrm{d}^2_{2,0}$ is $\mathbb{Z}$. Hence, the $E^3_{p,q}$ term of the spectral sequence looks like

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathbb{Z} & 0 & \mathbb{Z} \end{matrix}$$

The differentials $\mathrm{d}^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ and $\mathrm{d}^r_{p+r,q-r+1} : E^r_{p+r,q-r+1} \to E^r_{p,q}$ are zero for $r \geq 3$. So, the $E^\infty_{p,q}$ term of is as above. The Morse- Bott Homology groups is as follows:

$$H_0^{Bott}(f,g) = E^3_{0,0} = \mathbb{Z}$$

$$H_1^{Bott}(f,g) = E^3_{1,0} \oplus E^3_{0,1} = 0$$

$$H_2^{Bott}(f,g) = E^3_{2,0} \oplus E^3_{0,2} \oplus E^3_{1,1} = \mathbb{Z}$$

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