GENERALIZED PSEUDOSPECTRA IN CONNECTION WITH MULTIPLE EIGENVALUES

by

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
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To my family

ABSTRACT

Wilkinson studied the distance from a square matrix with distinct eigenvalues to the set of defective matrices in 1960s due to its connection with the sensitivity of eigenvalues. Malyshev derived a singular value optimization characterization for the distance. Recently, Alam and Bora established that the distance to defectiveness from a matrix corresponds to the smallest ϵ such that two components of the ϵ-pseudospectrum of the matrix coalesce. Our main aim is to generalize this relation between the distance to defectiveness and the pseudospectra. First we attempt to relate the algebraic characterization of Malyshev and geometric characterization of Alam and Bora. Then we focus on the main theme of this thesis, the distance to the set of matrices with a multiple eigenvalue of prescribed algebraic multiplicity, which we call generalized Wilkinson distance, and its geometric characterization in terms of pseudospectra. We introduce the generalized pseudospectrum as the set comprised of eigenvalues of prescribed multiplicity of all matrices within a given neighborhood. As a generalization of the work of Alam and Bora, we derive an upper bound for the generalized Wilkinson distance in terms of the coalescence of components of the generalized pseudospectra.

ÖZETÇE

Wilkinson çoklu özdeğere sahip olmayan bir matristen çoklu özdeğere sahip matrisler kümesine uzaklığı, uzaklığın özdeğerlerin duyarlılığı ile ilintili olması yüzünden, 1960'ların sonunda çalıştı. Malyshev uzaklık için bir tekil değeri karakterizasyonu geliştirdi. Yakın zamanda Alam ve Bora uzaklığın ϵ yaklaşık spektrumunun bileşenlerinin birbirine değdiği en ufak ϵ değerine karşılık geldiğini kanıtladı. Ana amacımız uzaklık ile yaklaşık spektrum arasındaki bu ilintiyi genellemek. Önce Malyshev'in cebirsel karakterizasyonu ile Alam ve Bora'nin geometrik karakterizasyonunu ilişkilendirmeye çalışıyoruz. Sonra bu tezin ana teması üzerine yoğunlaşıyoruz, genelleşmiş Wilkinson uzaklığı diye adlandırdığımız, verilen bir çokluk değerine sahip bir özdeğeri olan matrisler kümesine uzaklık ve bu uzaklığın yaklaşık spektrum cinsinden geometrik karakterizasyonu. Genelleşmiş yaklaşık spektrumu, verilen bir komşulukta bulunan bütün matrislerin verilen çokluk değerine sahip özdeğerler kümesi olarak tanımlıyoruz. Alam ve Bora'nın çalışmasının bir genellemesi olarak, genelleşmiş Wilkinson uzaklığı için genelleşmiş yaklaşık spektrumunun bileşenleri cinsinden bir üst sınır çıkarımı sunuyoruz.

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INTRODUCTION

In this thesis, we consider the distance from a matrix to the set of matrices with multiple eigenvalues, which we call the Wilkinson distance. The connection of this distance with the ϵ-pseudospectrum of the matrix, the set comprised of the eigenvalues of all matrices within an ϵ-neighborhood of the matrix, has been a debated topic after a conjecture by Demmel [5]. It has been recently established [2] that the distance is the smallest ϵ such that at least two components of the ϵ-pseudospectrum of the matrix coalesce. Both the Wilkinson distance and the ϵ-pseudospectrum have been of interest to the numerical analysts due to their relation with the sensitivity of the eigenvalues. Our main concern in this thesis is to generalize the discovered connection between these two quantities, in particular we attempt to answer the following question: "is it possible to deduce a nearest matrix with a multiple eigenvalue of prescribed algebraic multiplicity from a given matrix based on the ϵ-pseudospectrum or a generalization of the ϵ-pseudospectrum?" In this chapter, we give a background to help us understand the notions in the thesis and a brief history of the problem of finding the Wilkinson distance.

1.1 Background

1.1.1 ϵ-Pseudospectra

For given $A \in \mathbb{C}^{n\times n}$ and $\epsilon > 0$, we define the ϵ-pseudospectrum of A as [17]

$$\Lambda_\epsilon(A) := \{z \in \mathbb{C} \mid \exists \Delta A \in \mathbb{C}^{n\times n} \text{ with } \|\Delta A\|_2 \leq \epsilon \text{ s.t. } z \text{ is an eigenvalue of } A + \Delta A\},$$

that is the set of all complex numbers which can be induced as eigenvalues by perturbations ΔA with $\|\Delta A\|_2 \leq \epsilon$. Here and elsewhere $\|\cdot\|_2$ denotes the matrix 2-norm (or ℓ_2 norm) induced by the Euclidean norm on $\mathbb{C}^n$. Note that for $\epsilon = 0$, this set is the spectrum of A, denoted by $\Lambda(A)$, which is the set of eigenvalues of A.

We know that if λ is an eigenvalue of A, $\text{rank}(A - \lambda I) \leq n - 1$. So we can also define $\Lambda_\epsilon(A)$ as follows:

$$\Lambda_\epsilon(A) := \{z \in \mathbb{C} \mid \exists \Delta A \in \mathbb{C}^{n\times n} \text{ s.t. } \|\Delta A\|_2 \leq \epsilon \text{ and } \text{rank}(A + \Delta A - zI) \leq n - 1\}.$$

It follows that a matrix closest to A with respect to the 2-norm and Frobenius norm with λ as an eigenvalue is located at a distance equal to the smallest singular value of $A - \lambda I$ due to Eckart-Young Theorem (see Theorem 1.1.3), i.e,

$$\min\{\|\Delta A\|_2 \mid \Delta A \in \mathbb{C}^{n\times n} \text{ s.t. } \text{rank}(A + \Delta A - \lambda I) \leq n - 1\} = \sigma_n(A - \lambda I)$$

where σ_n denotes the smallest singular value of $A - \lambda I$. (Singular values are defined and discussed in the next subsection.) So for the 2-norm and Frobenius norm, it is straightforward to deduce that

$$\Lambda_\epsilon(A) = \{z \in \mathbb{C} \mid \sigma_n(A - zI) \leq \epsilon\}. \tag{1.1.1}$$

Now to understand the connection between the Wilkinson distance, $W(A)$, and the ϵ-pseudospectrum of A, for a given eigenvalue λ of A, we define

$$\Delta_\lambda = \{z \in \mathbb{C} \mid \exists \Delta A \text{ with } \|\Delta A\|_2 \leq \epsilon \text{ s.t. } \tilde{\lambda}_{\Delta A}(1) = z\} \tag{1.1.2}$$

where $\tilde{\lambda}_{\Delta A} : [0, 1] \to \mathbb{C}$ given by $\tilde{\lambda}_{\Delta A}(t) := \lambda(A + t\Delta A)$ is any continuous curve with $\lambda(A + t\Delta A)$ denoting an eigenvalue of $A + t\Delta A$ such that $\lambda(A) = \lambda$. Here Δ_λ is called a component of $\Lambda_\epsilon(A)$. For sufficiently small value of ϵ, $\Lambda_\epsilon(A)$ has n disjoint components. As ϵ grows gradually, these components of $\Lambda_\epsilon(A)$ coalesce.

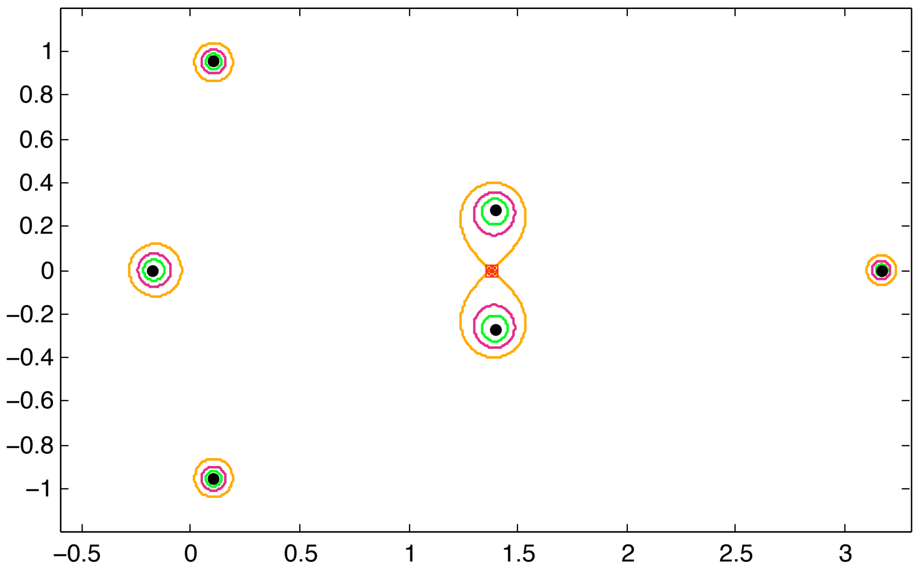

Figure 1.1: The ϵ-pseudospectra of the 6×6 Dramadah matrix D

Example: Consider a 6×6 Dramadah matrix

$$D = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

with six distinct eigenvalues. In Figure 1.1, the set Λ_ϵ for various ϵ are plotted for the 6×6 matrix D. As you can see, for smaller ϵ, $\Lambda_\epsilon(D)$ has 6 disjoint components and for $\epsilon = 0.0664$, two components coalesce at the red square.

Now we define an important quantity associated with the pseudospectra, which later will be shown to be equal to the Wilkinson distance,

$$\begin{aligned} C(A) &= \inf\{\epsilon \mid \text{at least two components of } \Lambda_\epsilon(A) \text{ coalesce}\} \\ &= \inf\{\epsilon \mid \# \text{ of disjoint components of } \Lambda_\epsilon(A) \leq n-1\}. \end{aligned}$$

In Figure 1.1, the Wilkinson distance $\epsilon = 0.0664$ is the smallest ϵ such that two components of $\Lambda_\epsilon(A)$ coalesce. Furthermore, the coalescence point $z_* = 1.3778$ marked with the red square is the multiple eigenvalue of a nearest matrix. Later, we will prove that these observations hold for any matrix.

1.1.2 Singular Value Decomposition

We reserve this subsection for singular values, emphasizing their geometric meaning, and presenting their connections with the distance to the set of matrices with prescribed rank. We conclude the subsection by summarizing some of the analytical properties of singular values.

Definition 1.1.1. *For a matrix $A \in \mathbb{C}^{m \times n}$, a nonnegative scalar σ is called **a singular value** of A, if there exist unit vectors $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^n$ such that*

$$Av = \sigma u \;\; \text{and} \;\; u^* A = \sigma v^*. \tag{1.1.3}$$

*The vectors u, v are called a **left singular vector** and a **right singular vector** associated with σ, respectively.*

Definition 1.1.2. *A **singular value decomposition** (SVD) of a matrix $A \in \mathbb{C}^{m \times n}$ is of the form*

$$A = U \Sigma V^*$$

where $U \in \mathbb{C}^{m \times m}, V \in \mathbb{C}^{n \times n}$ are unitary and $\Sigma \in \mathbb{R}^{m \times n}$ is diagonal.

Every matrix A has a singular value decomposition [16]. For simplicity let us suppose that the factor Σ in a singular value decomposition has distinct entries along its diagonal. Then the singular value decomposition becomes unique up to unit complex scalings on the columns U and V [16]. Furthermore, It is apparent from the singular value decomposition that

$$AV = U\Sigma \quad \text{and} \quad U^* A = \Sigma V^*$$

leading to $Av_j = \sigma_j u_j$ and $u_j^* A = \sigma_j v_j^*$ where u_j and v_j are the jth columns of U and V, σ_j is the (j, j)-entry of Σ. Thus the diagonal entries of Σ are the singular values of A. The corresponding left and right singular vectors are given by the corresponding columns U and

V. These observations hold in the more general setting, when the diagonal entries of Σ repeat, with the exception that the uniqueness of the factors U and V in the singular value decomposition becomes more intricate [7].

The geometric interpretation of an SVD is that the image of the unit sphere in the n-dimensional space under an $m \times n$ matrix is an ellipsoid in the m-dimensional space. For simplicity consider a matrix $A \in \mathbb{R}^{2\times 2}$ with an SVD

$$A = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix} = \sigma_1 u_1 v_1^* + \sigma_2 u_2 v_2^*.$$

Let S be the unit circle defined as

$$S = \{\alpha_1 v_1 + \alpha_2 v_2 \in \mathbb{R}^{2\times 2} \ \mid \ \alpha_1, \alpha_2 \in \mathbb{R} \ \text{ s.t. } \ \alpha_1^2 + \alpha_2^2 = 1\}.$$

Then the image of S under the matrix A is

$$\begin{aligned} AS &= \{A(\alpha_1 v_1 + \alpha_2 v_2) \in \mathbb{R}^{2\times 2} \ \mid \ \alpha_1, \alpha_2 \in \mathbb{R} \ \text{ s.t. } \ \alpha_1^2 + \alpha_2^2 = 1\} \\ &= \{\alpha_1 A v_1 + \alpha_2 A v_2 \in \mathbb{R}^{2\times 2} \ \mid \ \alpha_1, \alpha_2 \in \mathbb{R} \ \text{ s.t. } \ \alpha_1^2 + \alpha_2^2 = 1\} \\ &= \{\alpha_1 \sigma_1 u_1 + \alpha_2 \sigma_2 u_2 \in \mathbb{R}^{2\times 2} \ \mid \ \alpha_1, \alpha_2 \in \mathbb{R} \ \text{ s.t. } \ \alpha_1^2 + \alpha_2^2 = 1\}. \end{aligned}$$

Letting $\beta_1 = \alpha_1\sigma_1$ and $\beta_2 = \alpha_2\sigma_2$ we have

$$AS = \{\beta_1 u_1 + \beta_2 u_2 \in \mathbb{R}^{2\times 2} \ \mid \ \beta_1, \beta_2 \in \mathbb{R} \ \text{ s.t. } \ \frac{\beta_1^2}{\sigma_1^2} + \frac{\beta_2^2}{\sigma_2^2} = 1\}$$

is an ellipse with semi-axes $\sigma_1 u_1, \sigma_2 u_2$.

Let $A \in \mathbb{C}^{m\times n}$ be a rank k matrix with an SVD of the form $A = U\Sigma V^*$ where Σ=diag$(\sigma_1, \sigma_2, \ldots, \sigma_k, 0, 0, \ldots, 0)$. Then A can be decomposed into a sum of k rank one matrices:

$$A = \sum_{j=1}^{k} \sigma_j u_j v_j^*.$$

Now, for any $0 \le r \le k$, consider the sum of $k - r$ rank one matrices

$$\Delta A_* = \sum_{j=r+1}^{k} -\sigma_j u_j v_j^*.$$

Then we have $\text{rank}(A + \Delta A_*) = r$, furthermore $\|\Delta A_*\|_2 = \sigma_{r+1}(A)$. Here and throughout the text $\sigma_j(A)$ denotes the jth largest singular value of A. The following theorem is a fundamental result concerning the singular values of A with numerous applications in fields such as data compression, signal processing and statistics.

Theorem 1.1.3. *(Eckart-Young) Let* $A = U\Sigma V^* = U\text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k, 0, 0, \ldots, 0)V^* \in \mathbb{C}^{m\times n}$ *be a singular value decomposition. Then for any integer* r *with* $0 \leq r \leq k$,

$$\min\{\|\Delta A\|_2 \;\; | \;\; \text{rank}(A + \Delta A) \leq r\} = \|\Delta A_*\| = \sigma_{r+1}(A).$$

In particular, if A is a full rank $n \times n$ matrix, then for $r = n - 1$ the distance above reduces to the distance to the singularity given by $\sigma_n(A)$.

Analyticity of Singular Values

Throughout this thesis, analyticity of a singular value function plays an important role. A characterization of singular values is that they are the nonnegative eigenvalues of the matrix

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$$

. Indeed, let $A = U\Sigma V^*$ be a singular value decomposition. Then,

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & U\Sigma V^* \\ V\Sigma^* U^* & 0 \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} 0 & \Sigma \\ \Sigma^* & 0 \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & V^* \end{bmatrix}$$
$$= Q \begin{bmatrix} 0 & \Sigma \\ \Sigma^* & 0 \end{bmatrix} Q^*$$

where $Q = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}$, we see that the matrix $\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & \Sigma \\ \Sigma^* & 0 \end{bmatrix}$ and thus they have the same eigenvalues. The eigenvalues of the latter are the square roots of the diagonal elements of $\Sigma^*\Sigma$. This characterization and the following theorem shows the existence of analytic singular value decomposition for an arbitrary analytic matrix-valued function.

Theorem 1.1.4. *(Rellich [14]) Let $A(t) : \mathbb{R} \to \mathbb{C}^{n\times n}$ be an analytic Hermition matrix-valued function. Then $A(t)$ has the decomposition*

$$A(t) = Q(t)\Lambda(t)Q(t)^*$$

where $Q(t) : \mathbb{R} \to \mathbb{C}^{n\times n}$ is a unitary analytic matrix-valued function and $\Lambda(t) : \mathbb{R} \to \mathbb{C}^{n\times n}$ is a diagonal analytic matrix-valued function.

In [1], by applying the theorem above to the matrix-valued function $\begin{bmatrix} 0 & A(t) \\ A(t)^* & 0 \end{bmatrix}$, the following result is deduced.

Theorem 1.1.5. *Consider an analytic matrix-valued function $A(t) : \mathbb{R} \to \mathbb{C}^{n\times n}$. There exists a decomposition*

$$A(t) = U(t)\Sigma(t)V(t)^*$$

where $U(t), V(t) : \mathbb{R} \to \mathbb{C}^{n\times n}$ are unitary analytic matrix-valued functions and $\Sigma(t) : \mathbb{R} \to \mathbb{C}^{n\times n}$ is diagonal (with possibly negative entries on diagonal) and analytic at all t.

The diagonal entries of $\Sigma(t)$ above can be viewed as unsorted and signed (possibly negative) singular values of $A(t)$. Thus it follows from the theorem above that $\sigma_i(t)$ $-$ ith largest singular value of $A(t)-$ is analytic provided that it is simple and non-zero. Furthermore, for simple and non-zero $\sigma_i(t)$, denoting the corresponding pair of left and right singular vectors satisfying (1.1.3) with $u_i(t), v_i(t)$, it can be shown that [11]

$$\frac{d\sigma_i(t)}{dt} = \Re\left(u_i(t)^* \frac{dA(t)}{dt} v_i(t)\right). \tag{1.1.4}$$

1.1.3 Sylvester Equation

In this thesis, the dimensions of the solution spaces of certain Slyvester equations play key roles. Here, we will briefly discuss a Slyvester equation. Also we will define the Kronecker product and vectorization operator that help us to improve our understanding of the solution space.

Definition 1.1.6. *The Sylvester equation is*

$$AX - XB = 0 \tag{1.1.5}$$

where $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$ *are given square matrices and* $X \in \mathbb{C}^{m\times n}$ *is an unknown rectangular matrix. The associated Sylvester operator is* $S : \mathbb{C}^{m\times n} \to \mathbb{C}^{m\times n}$ *defined by*

$$S(X) = AX - XB.$$

Note that the solution set of the Sylvester equation is the same as the Kernel(S). The notions of Kronecker product and vec (vectorization) operator are helpful in identifying the solution set of the Sylvester equation.

Definition 1.1.7. *Let* A *be an* $m \times n$ *matrix and* B *be a* $p \times q$ *matrix. Then the Kronecker product* $A \otimes B$ *of* A *and* B *is the* $mp \times nq$ *block matrix defined as*

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}.$$

Two important properties of the Kronecker product [8] that we will refer are as follows:

(i) $(A \otimes B)(C \otimes D) = AC \otimes BD$

(ii) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

Definition 1.1.8. *The linear operator vec*: $\mathbb{C}^{m\times n} \to \mathbb{C}^{mn}$ *converts a matrix into a column vector, i.e., for any* $A \in \mathbb{C}^{m\times n}$,

$$vec(A) = [a_{11},\ \dots,\ a_{m1}, a_{12},\ \dots,\ a_{m2},\ \dots\ , a_{1n},\ \dots,\ a_{mn}]^T \in \mathbb{C}^{mn}.$$

The vectorization is frequently used in association with the Kronecker product to express the matrix associated with the linear map $X \mapsto AXC$ explicitly, in particular,

$$\mathrm{vec}(AXC) = (C^T \otimes A)\mathrm{vec}(X) \tag{1.1.6}$$

for matrices A, X, C (real or complex) of dimension $k \times l$, $l \times m$ and $m \times n$ respectively.

Now using the Kronecker product and vec operator, we can express the Sylvester equation (1.1.5) as a linear equation of the form

$$(I_m \otimes A - B^T \otimes I_n)\text{vec}(X) = 0. \tag{1.1.7}$$

Let $A = PJP^{-1}$ and $B^T = QKQ^{-1}$ be the Jordan canonical forms of A and B^T. Then by properties **(i)**, **(ii)**

$$\begin{aligned} I_m \otimes A - B^T \otimes I_n &= I_m \otimes (PJP^{-1}) - (QKQ^{-1}) \otimes I_n \\ &= (Q \otimes P)(I_m \otimes J)(Q^{-1} \otimes P^{-1}) - (Q \otimes P)(K \otimes I_n)(Q^{-1} \otimes P^{-1}) \\ &= (Q \otimes P)(I_m \otimes J - K \otimes I_n)(Q^{-1} \otimes P^{-1}). \end{aligned}$$

If we take $\text{vec}(\tilde{X}) = (Q^{-1} \otimes P^{-1})\text{vec}(X)$, then equation (1.1.7) is equivalent to

$$(I_m \otimes J - K \otimes I_n)\text{vec}(\tilde{X}) = 0.$$

Let λ_i's and μ_j's be the eigenvalues of A and B respectively. Then $I_m \otimes J - K \otimes I_n$ is an upper triangular matrix with diagonal elements $\lambda_i - \mu_j$. So if there exists i and j such that $\lambda_i = \mu_j$, then the system (1.1.5) has a non-trivial solution. Indeed the system has a non-trivial solution if and only if A and B have a common eigenvalues.

The following theorem concerns the dimension of the solution space for the system (1.1.5) and the proof of the theorem can be found in [6].

Theorem 1.1.9. *Let $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ and suppose that $\mu_1, \ldots, \mu_l$ are common eigenvalues of A and B. Then the dimension of the solution space for the Sylvester equation $AX - XB = 0$ is*

$$\dim\{X \in \mathbb{C}^{m\times n} \; : \; AX - XB = 0\} = \sum_{j=1}^{l} \sum_{i=1}^{l_j} \sum_{q=1}^{\tilde{l}_j} \min(c_{j,i}, p_{j,q})$$

where $c_{j,1}, \ldots, c_{j,l_j}$ and $p_{j,1}, \ldots, p_{j,\tilde{l}_j}$ are the sizes of the Jordan blocks of A and B associated with μ_j respectively.

1.1.4 The Conditioning of Eigenvalues

Let $f : X \to Y$ be a problem from a normed input vector space X to a normed output vector space Y. The **condition number** of f at $x \in X$, denoted by κ, measures how much the output can change for a small change in the input, formally defined as

$$\kappa = \lim_{\delta \to 0^+} \sup_{\|\delta x\| \le \delta} \frac{\|f(x+\delta x) - f(x)\|}{\|\delta x\|}.$$

If the condition number of f is large, the problem is called **ill-conditioned**, otherwise it is called **well-conditioned**.

Let $A \in \mathbb{C}^{n\times n}$ be a matrix and $\lambda \in \mathbb{C}$ be an eigenvalue of A with the corresponding eigenvector $v \in \mathbb{C}^n$. The problem we consider here is how a small perturbation in the matrix A changes the eigenvalue λ. For a small perturbation δA, is the eigenvalue of $A+\delta A$ close to λ or is it not? And the condition number of λ is a measure of sensitivity of λ to these perturbations. A small condition number implies an insensitive eigenvalue and such an eigenvalue is called well-conditioned. A sensitive eigenvalue has large condition number and is called ill-conditioned. In this section, we will give the condition number of a simple eigenvalue, i.e., an eigenvalue with algebraic multiplicity one.

Now assume that λ is a simple eigenvalue of $A \in \mathbb{C}^{n\times n}$ with unit right and left eigenvectors $x, y \in \mathbb{C}^n$, respectively. Also assume that the input matrix space is equipped with the 2-norm. For $\epsilon > 0$, consider the matrix

$$A(\epsilon) \;=\; A + \epsilon \Delta A$$

with $\|\Delta A\|_2 = 1$. Let $\lambda(\epsilon)$ be the eigenvalue of $A(\epsilon)$ with the associated right eigenvector $x(\epsilon)$:

$$(A + \epsilon\Delta A)x(\epsilon) = \lambda(\epsilon)x(\epsilon). \tag{1.1.8}$$

Note that due to the simplicity assumption on λ the eigenvalue $\lambda(\epsilon)$ is differentiable at $\epsilon = 0$, and the condition number of $\lambda(\epsilon)$ at $\epsilon = 0$ reduces to $\left|\frac{d\lambda}{d\epsilon}(0)\right|$. Thus differentiating the equation (1.1.8) w.r.t. ϵ and setting $\epsilon = 0$ yield

$$A\frac{dx(0)}{d\epsilon} + \Delta A \cdot x = \frac{d\lambda(0)}{d\epsilon}x + \lambda\frac{dx}{d\epsilon}(0).$$

Multiplying both sides by y^* from left gives

$$\frac{d\lambda(0)}{d\epsilon} = \frac{y^* \Delta A x}{y^* x}.$$

Using submultiplicative property (i.e., $\|CD\| \leq \|C\|\|D\|$) of the 2-norm, we get

$$\left| \frac{d\lambda(0)}{d\epsilon} \right| \leq \frac{1}{|y^* x|}.$$

Indeed, there exists a $\Delta A = yx^*$ so that the above inequality becomes an equality. Thus, if these two vectors x and y are almost orthogonal, the eigenvalue λ turns out to be ill-conditioned. So the quantity

$$\kappa = \frac{1}{|y^* x|}$$

can be taken as the condition number of a simple eigenvalue λ. All of the above arguments can be summarized as a theorem:

Theorem 1.1.10. *Let $A \in \mathbb{C}^{n \times n}$ and λ, x, y be a simple eigenvalue and its associated unit right and left eigenvectors,respectively. For $\epsilon > 0$, let $A(\epsilon) = A + \epsilon \Delta A$ with $\|\Delta A\|_2 = 1$. Then, if $\lambda(\epsilon)$ denote the differentiable eigenvalue of $A(\epsilon)$ such that $\lambda(0) = \lambda$, we have*

$$\left| \frac{d\lambda(0)}{d\epsilon} \right| \leq \frac{1}{|y^* x|}.$$

Note that if A is normal matrix, i.e., $AA^* = A^*A$, then the left and right eigenvectors y and x coincide, yielding $\kappa(\lambda) = \frac{1}{\|x\|} = 1$. So for normal matrices, the eigenvalue problem is always well-conditioned.

1.2 History

Given $A \in \mathbb{C}^{nxn}$ with simple eigenvalues, consider the quantity

$$W(A) = \inf\{\|\Delta A\|_2 \mid A + \Delta A \text{ is defective}\}.$$

This quantity is called "the Wilkinson Distance". By a defective matrix, we mean that it does not have n linearly independent eigenvectors, i.e, it is not diagonalizable. Such a matrix has a Jordan block of size at least 2 in its Jordan canonical form. An eigenvalue corresponding to such a Jordan block is called defective, as its algebraic multiplicity is greater than its geometric multiplicity. By a multiple eigenvalue we mean that its algebraic multiplicity is greater than one. A multiple eigenvalue does not have to be defective. But the distance to the set of matrices with a defective eigenvalue is the same as the distance to the set of matrices with a multiple eigenvalue, since there exist arbitrarily small perturbations to a matrix with a non-defective multiple eigenvalue making the eigenvalue defective. Thus $W(A)$ can alternatively be defined as

$$\begin{aligned} W(A) &= \inf\{\|\Delta A\|_2 \mid \exists\lambda \text{ s.t. } A+\Delta A \text{ has } \lambda \text{ as a defective eigenvalue}\}. \\ &= \inf\{\|\Delta A\|_2 \mid \exists\lambda \text{ s.t. } A+\Delta A \text{ has } \lambda \text{ as a multiple eigenvalue}\}. \end{aligned}$$

The interest in the distance $W(A)$ goes back to 1960s. In his book [18], Wilkinson defined the condition number of a simple eigenvalue λ as $\text{cond}(\lambda) = \frac{1}{|y^*x|}$, where y, x are unit left and right eigenvectors corresponding to λ, i.e, $Ax = \lambda x$ and $y^*A = \lambda y^*$. Note that if λ is a defective eigenvalue, then there exists a pair of left and right eigenvectors y and x associated with it such that $y^*x = 0$. Wilkinson stated that such an eigenvalue is regarded as ill-conditioned. But he also observed that even if the eigenvalues are well-seperated from each other, thus not defective, they can still be very ill-conditioned. To illustrate this, Wilkinson considered the following matrices:

$$A = \begin{bmatrix} 20 & 20 & & & \\ & 19 & 20 & & \\ & & \ddots & & \\ & & & 2 & 20 \\ & & & & 1 \end{bmatrix} \quad \text{and} \quad A(\epsilon) = \begin{bmatrix} 20 & 20 & & & \\ & 19 & 20 & & \\ & & \ddots & & \\ & & & 2 & 20 \\ \epsilon & & & & 1 \end{bmatrix}.$$

He calculated that for $\epsilon = 7.8 \cdot 10^{-14}$, the matrix $A(\epsilon)$ has $\lambda = 10.5$ as a defective multiple eigenvalue. Based on these type of observations, he suspected that any matrix with an ill-conditioned eigenvalue - possibly away from the other eigenvalues - might be close to a defective matrix.

In [15], Axel Ruhe proved Wilkinson's conjecture, that is even if a matrix A with an ill-conditioned eigenvalue has well-seperated eigenvalues, it is close to a one having multiple eigenvalues. He gave the estimate

$$W(A) \leq \frac{n}{4} \max_{\substack{i,j \\ i \neq j}} |\lambda_i - \lambda_j| \tan\theta$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, the angle θ is such that $\sin\theta \leq \min_k |y_k^* x_k|^{\frac{1}{n-1}}$ for $k = 1, \ldots, n$, and y_k, x_k are unit left, right eigenvectors associated with λ_k.

After Ruhe has shown that Wilkinson's conjecture is true, Wilkinson made a detailed study of the distance $W(A)$ in [19,20]. He found a perturbation matrix that makes a simple eigenvalue λ a double eigenvalue keeping one of the left eigenvector x or the right eigenvector y fixed. This yields a bound sharper than that of Ruhe. Wilkinson's perturbation is given by

$$\Delta A = \frac{(\mathrm{cond}(\lambda)y - x)x^*(A - \lambda I)}{\left\| y - \frac{x}{\mathrm{cond}(\lambda)} \right\|^2},$$

and satisfies

$$W(A) \leq \|\Delta A\| \leq \frac{\|A\|_2}{\sqrt{\mathrm{cond}(\lambda)^2 - 1}},$$

where $\mathrm{cond}(\lambda) = \frac{1}{|y^* x|}$.

In his Ph.D. thesis [5], Demmel introduced the quantities: $diss(\sigma_1, \sigma_2, region)$ and $diss(\sigma_1, \sigma_2, path)$, where σ_1, σ_2 form a partition of A's spectrum into disjoint subsets. The former quantity is defined to be the smallest ϵ such that the associated pseudospectral components $\sigma_1(\epsilon), \sigma_2(\epsilon)$ containing σ_1, σ_2, respectively coalesce. To be formal, each of $\sigma_i(\epsilon)$ for $i = 1, 2$, is the union of the components $\Delta_{\lambda_{i_k}}$ containing the eigenvalue λ_{i_k} for $k = 1, \ldots, \ell$ where $\{\lambda_{i_1}, \ldots, \lambda_{i_\ell}\} = \sigma_i$ and $\Delta_{\lambda_{i_k}}$ is defined as in equation (1.1.2). The second quantity is defined as the norm of the smallest perturbation ΔA to A that makes an eigenvalue $\lambda_1 \in \sigma_1$ coalesce with $\lambda_2 \in \sigma_2$ to cause a double eigenvalue, that is there exist two continuous eigenvalue functions $\lambda_1(\Delta), \lambda_2(\Delta)$ corresponding to two of the eigenvalues of $A + \Delta$ such that $\lambda_1(\Delta A) = \lambda_2(\Delta A)$ whereas $\lambda_1(0) = \lambda_1 \in \sigma_1$, $\lambda_2(0) = \lambda_2 \in \sigma_2$.

If we take the minimum of these quantities over all partitions, $diss(\sigma_1, \sigma_2, path)$ corresponds to $W(A)$ and $diss(\sigma_1, \sigma_2, region)$ corresponds to $C(A)$. Demmel observed that for all norms $W(A) \geq C(A)$, since for any perturbation ΔA of norm $W(A)$ such that $A + \Delta A$ has a multiple eigenvalue the following holds: there exist two continuous functions

$\lambda_1, \lambda_2 : [0,1] \to \mathbb{C}$ satisfying

(i) $\lambda_1(t), \lambda_2(t)$ are eigenvalues of $A + t\Delta A$ such that $\lambda_1(t) \in \sigma_1(\epsilon)$ and $\lambda_2(t) \in \sigma_2(\epsilon)$ for all $t \in [0,1]$ and $\epsilon = W(A)$

(ii) $\lambda_1(1) = \lambda_2(1)$.

He also indicated it is an interesting open question as to whether the equality $W(A) = C(A)$ holds in the case of the 2-norm.

Wilkinson originally considered the distance to a nearest defective matrix from the viewpoint of its relation with the sensitivity of eigenvalues. But later in [21], like Demmel, Wilkinson discussed the notion of pseudospectra under the name "fundamental domain", denoted by $D(\epsilon)$. For any norm, he defined $D(\epsilon)$ as the set of complex numbers satisfying $\left\|(A - zI)^{-1}\right\|^{-1} \leq \epsilon$. With respect to the 2-norm, since

$$\left\|(A - zI)^{-1}\right\|_2 = \sigma_1\left((A - zI)^{-1}\right) = 1/\sigma_n(A - zI),$$

this identifies all z which can be induced from A as eigenvalues by perturbations ΔA with $\|\Delta A\|_2 \leq \epsilon$. Thus this is nothing but the ϵ-pseudospectrum of A. He observed that when ϵ is sufficiently small and $A \in \mathbb{C}^{n\times n}$ has distinct eigenvalues, $D(\epsilon)$ consists of n isolated domains, each containing one of the eigenvalues of A. In [21], his basic interest was to find the smallest value of ϵ for which two of these domains coalesces . His observation was that if $\tilde{z}$ is a point of coalescence of two components of $D(\epsilon)$, then there exist perturbations $\Delta A_1, \Delta A_2$ such that

(i) $\|\Delta A_1\| = \|\Delta A_2\| = \epsilon$

(ii) $\tilde{z}$ is an eigenvalue of $A + \Delta A_1$ and $A + \Delta A_2$.

Note that $W(A) = C(A)$ implies that two eigenvalues must travel to the same coalescence point under the same perturbation. This initiated a further investigation into the relation between $W(A)$ and $C(A)$. Wilkinson shed light into this relation by showing $W(A) > C(A)$ on certain examples for the ∞-norm and also hinted that the equality might hold for the 2-norm.

At the end of 1990s, Malyshev made an important contribution to the problem of characterizing $W(A)$. He derived the characterization

$$W(A) = \inf_{z \in \mathbb{C}} \sup_{\gamma \geq 0} \sigma_{2n-1}\left(\begin{bmatrix} A - zI & \gamma I \\ 0 & A - zI \end{bmatrix}\right)$$

when $W(A)$ is defined in terms of the 2-norm [12]. He did not even attempt to relate $W(A)$ with the pseudospectra. His characterization was purely algebraic. Inspired by the work of Malyshev, this yielded derivations of the singular value characterizations for the distance from a linear matrix pencil of the form $L : \mathbb{C} \to \mathbb{C}^{n\times n},\ L(\lambda) = A_0 + \lambda A_1$ to a nearest matrix pencil in 2-norm that has specified eigenvalues, and the distance in 2-norm from a matrix polynomial $P : \mathbb{C} \to \mathbb{C}^{n\times n},\ P(\lambda) = \lambda_{j=0}^{m} \lambda^j A_j$ to a nearest polynomial with specified eigenvalues [9, 10].

Finally, in [2] Alam and Bora affirmatively proved that the equality $W(A) = C(A)$ holds with respect to the 2-norm. A brief summary of their approach is as follows. For a matrix $A \in \mathbb{C}^{n\times n}$ with n distinct eigenvalues, recall that the ϵ-pseudospectrum of A with respect to the 2-norm can be characterized as

$$\Lambda_\epsilon(A) = \{z \in \mathbb{C} \ \mid \ \sigma_n(A - zI) \leq \epsilon\}.$$

For sufficiently small ϵ, $\Lambda_\epsilon(A)$ has n connected components. Let z be any boundary point of $\Lambda_\epsilon(A)$, i.e, $\sigma_n(A - zI) = \epsilon$, and u, v be a consistent pair of unit left and right singular vectors associated with $\sigma_n(A - zI)$. Then the perturbation $\Delta A = -\epsilon uv^*$ achieves the task of making z an eigenvalue of $A + \Delta A$ with left and right eigenvectors u and v, respectively. Indeed,

$$(A + \Delta A - zI)v = (A - zI)v + \Delta Av = \epsilon u - \epsilon uv^* v = \epsilon u - \epsilon u = 0,$$
$$u^*(A + \Delta A - zI) = u^*(A - zI) + u^*\Delta A = \epsilon v^* - \epsilon u^* uv^* = \epsilon v^* - \epsilon v^* = 0.$$

Now, let $\tilde{z}$ be a coalescence point of two distinct components of $\Lambda_{\tilde{\epsilon}}(A)$, where $\tilde{\epsilon} = \sigma_n(A - \tilde{z}I) = C(A)$. The remarkable observation by Alam and Bora is that if the multiplicty of $\sigma_n(A - \tilde{z}I)$ is one, then $\tilde{z}$ is a differentiable saddle point of the singular value function $\sigma_n(A - zI)$. The differentiability of $\sigma_n(A - zI)$ at $\tilde{z}$, and the formula (1.1.4) for the derivatives of singular value functions yield $\tilde{u}^* \tilde{v} = 0$, where $\tilde{u}, \tilde{v}$ are left and right singular vectors associated with $\sigma_n(A - \tilde{z}I)$. It follows that the perturbation $\Delta A = -\tilde{\epsilon}\tilde{u}\tilde{v}^*$ makes $\tilde{z}$ an eigenvalue of $A + \Delta A$ with a pair of orthogonal left and right eigenvectors, namely $\tilde{u}$ and $\tilde{v}$, respectively. This means that $\tilde{z}$ is a defective multiple eigenvalue (follows immediately from Jordan canonical form). Moreover, we have $\|\Delta A\|_2 = \tilde{\epsilon}$. This implies that $W(A) \leq C(A)$. If $\sigma_n(A - \tilde{z}I)$ has multiplicity two or greater (thus $\sigma_n(A - zI)$ is not differentiable at $\tilde{z}$) with left singular vectors $u_1,\ u_2$ and right singular vectors $v_1,\ v_2$ associated with it, then the perturbation $\Delta A = -\tilde{\epsilon}\begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix}^*$ makes $\tilde{z}$ an eigenvalue of $A + \Delta A$ with geometric multiplicity at least 2, and note that again $\|\Delta A\|_2 = \tilde{\epsilon}$. So in both cases we have $W(A) \leq C(A)$.

In [20], Wilkinson also discussed the distance to a nearest matrix that has an eigenvalue of prescribed albegraic multiplicity. The distance is generalized for any $r \geq 2$ and a given matrix $A \in \mathbb{C}^{n \times n}$, as

$$W_r(A) = \inf\{\|\Delta A\|_2 \mid (A+\Delta A) \text{ has an eigenvalue with algebraic multiplicity} \geq r\}. \quad (1.2.1)$$

Wilkinson argued that $W_r(A)$ can be considerably greater than $W(A)$ for $r \geq 3$ by giving numerical examples. Later Mengi derived the singular value characterization [13]

$$W_r(A) = \inf_{z \in \mathbb{C}} \left(\overline{f}_r(z) := \sup_{\gamma \in \mathbb{C}^{r(r-1)/2}} f_r(z, \gamma) \right) \quad (1.2.2)$$

where

$$f_r(z,\gamma) := \sigma_{nr-r+1}\left(\mathcal{A}(z,\gamma)\right) \quad \text{with} \quad \mathcal{A}(z,\gamma) := \begin{bmatrix} A - zI & \gamma_{1,2}I & & & \gamma_{1,r}I \\ 0 & A - zI & & & \gamma_{2,r}I \\ & & \ddots & & \\ & & & A - zI & \gamma_{r-1,r}I \\ 0 & & & 0 & A - zI \end{bmatrix}$$

and $\gamma := \begin{bmatrix} \gamma_{1,2} & \dots & \gamma_{r-1,r} \end{bmatrix}^T$. He conjectured the generalizations of the arguments regarding the relations between $W(A)$ and $C(A)$. He defined, for $r \geq 2$, the ϵ-pseudospectrum of order $r-1$ as

$$\Lambda_{\epsilon,r-1}(A) := \{z \in \mathbb{C} \mid \exists \Delta A \text{ s.t. } \|\Delta A\|_2 \le \epsilon \text{ and } \operatorname{rank}(A+\Delta A - zI)^{r-1} \le n-r+1\}. \tag{1.2.3}$$

Note that the condition $\operatorname{rank}(A+\Delta A-zI)^{r-1} \le n-r+1$ is equivalent to $(A+\Delta A)$ having z as an eigenvalue of algebraic multiplicity $r-1$. He claimed that $W_r(A)$ and $\Lambda_{\epsilon,r-1}(A)$ are related. We will discuss this in the third chapter.

1.3 Problem Definition, Contributions and Outline

In the previous section, we reported on how the various authors treated the problem of characterizing the distance to a nearest defective matrix $W(A)$. With respect to the 2-norm, we gave the singular value characterization of $W(A)$ and using the pseudospectral approach, we concluded that $W(A)$ is the smallest ϵ such that two components of $\Lambda_\epsilon(A)$ coalesce. Now for a given matrix $A \in \mathbb{C}^{n\times n}$ and $r \in [2,n]$, consider the distance $W_r(A)$ as defined in (1.2.1). The problem we consider here is the connection between $W_r(A)$ and the ϵ-pseudospectrum of A. We might be tempted to think

$$W_r(A) = \inf\{\epsilon \mid \textit{number of components of } \Lambda_\epsilon(A) \le n-r+1\}.$$

But $\Lambda_\epsilon(A)$ turns out to be irrelevant to $W_r(A)$ for $r \ge 2$. Consider the 4×4 smoke matrix with

$$S = \begin{bmatrix} i & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -i & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

four distinct eigenvalues. In Figure 1.2, the outermost curve is the boundary of $\Lambda_\epsilon(S)$ for $\epsilon = W_3(S) = 0.6672$. All components coalesce for this ϵ value. The blue asteriks representing the eigenvalue of a nearest matrix with multiplicity three is strictly inside $\Lambda_\epsilon(S)$ for $\epsilon = W_3(S)$ and not on the boundary.

The generalized ϵ-pseudospectrum $\Lambda_{\epsilon,r-1}(A)$ of order $r-1$ of A defined as in (1.2.3) turns out to be more relevant to $W_r(A)$. The distance $W_r(A)$ has the singular value characteriza-

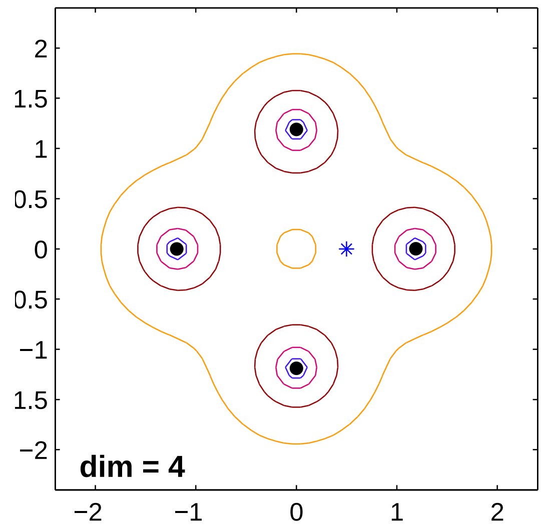

Figure 1.2: The ϵ-pseudospectra of the 4×4 smoke matrix S

tion (1.2.2) as discussed in the previous section, where the inner problem $\overline{f}_r(z)$ corresponds to the distance to a nearest matrix with z as an eigenvalue of algebraic multiplicity at least r. This leads us to

$$\Lambda_{\epsilon,r-1}(A) = \left\{ z \in \mathbb{C} \;\middle|\; \overline{f}_{r-1}(z) \leq \epsilon \right\},$$

which can be considered as a generalization of the singular value characterization (1.1.1) for the ordinary ϵ-pseudospectrum of A. We conjecture that

$$W_r(A) = C_{r-1}(A) := \inf\{\epsilon \mid \textit{two components of } \Lambda_{\epsilon,r-1}(A) \textit{ coalesce}\}.$$

Alternatively we could pose $C_{r-1}(A)$ as

$$C_{r-1}(A) = \inf\{\epsilon \mid \exists z_* \text{ s.t. } \overline{f}_{r-1}(z_*) = \epsilon \text{ and } z_* \text{ is a saddle point}\}.$$

The proof of the inequality $W_r(A) \geq C_{r-1}(A)$ appears to be problematic (unlike the case $r = 2$). For this one needs to develop an understanding of the components of $\Lambda_{\epsilon,r}(A)$. The set $\Lambda_\epsilon(A)$ has a component around each one of the eigenvalues of A. A natural question is what is the generalization of this fact for $\Lambda_{\epsilon,r}(A)$, that is what is located at the center of each component of $\Lambda_{\epsilon,r}(A)$? In this direction we claim the following.

Conjecture 1. *If z_* is a local minimum of of $\overline{f}_r(z)$, then z_* is a saddle point of $\overline{f}_{r-1}(z)$.*

According to the conjecture each component of $\Lambda_{\epsilon,r}(A)$ forms around a saddle point of $\overline{f}_{r-1}(z)$. For instance for $r = 2$ each component forms around a saddle point of $f(z) = \sigma_n(A - zI)$.

Suppose that z_* is a global minimizer of $\overline{f}_r(z)$ such that $W_r(A) = \overline{f}_r(z_*) = \epsilon$. Then according to Conjecture 1, z_* is a saddle point of $\overline{f}_{r-1}(z)$. Consequently, as $C_{r-1}(A)$ is the smallest saddle point of $\overline{f}_{r-1}(z)$, we have

$$C_{r-1}(A) \leq \overline{f}_{r-1}(z_*) \leq \overline{f}_r(z_*) = W_r(A).$$

1.3.1 Contributions

Now consider Conjecture 1 for the case $r = 2$, in which case the conjecture takes the form if z_* is a local minimum of of $\overline{f}_2(z) = \sup_{\gamma \geq 0} \sigma_{2n-1}\left(\begin{bmatrix} A - zI & \gamma I \\ 0 & A - zI \end{bmatrix}\right)$, then z_* is a saddle point of $\overline{f}_1(z) = \sigma_n(A - zI)$. The first contribution of this thesis is that we will prove that Conjecture 1 typically holds for the case $r = 2$ and when the supremum is attained at $\gamma = 0$. It reveals a connection between the purely algebraic characterization of Malyshev and geometric characterization of Alam and Bora.

The second contribution is the generalization of the work of Alam and Bora for $\Lambda_{r-1}(A)$. This establishes the inequality $W_r(A) \leq C_{r-1}(A)$. In particular, suppose that z_* is a coalescence point of two components of $\Lambda_{\epsilon,r-1}(A)$ satisying $\epsilon = \overline{f}_{r-1}(z_*) = C_{r-1}(A)$ and $\overline{f}_{r-1}(z)$ is differentiable at z_*. Then we construct a perturbation ΔA_* with $\|\Delta A_*\|_2 = C_{r-1}(A)$ such that z_* is an eigenvalue of $A + \Delta A_*$ with algebraic multiplicity at least r. This implies $W_r(A) \leq C_{r-1}(A)$. We prove this under mild multiplicity and linear independence assumptions.

1.3.2 Outline

This thesis is organized as follows. In Chapter 2, we prove Conjecture 1 for $r = 2$, and for a z such that the supremum of $f_2(z, \gamma)$ over all γ is attained at $\gamma = 0$. This attainment property at $\gamma = 0$ is true for each global minimizer z_* of $\overline{f}_2(z)$, and we believe it remains to be true for each local minimizer as well. This attainment property turns out be equivalent to the property that the left and right singular vectors u and v of $\sigma_n(A - z_* I)$ are orthogonal (Lemma 2.0.4), which typically implies that z_* is a saddle point of $f(z) = \sigma_n(A - zI)$ (Theorem 2.0.8). In Chapter 3, we give the details of the proof of the inequality $W_{r+1}(A) \leq C_r(A)$. Under mild multiplicity and linear independence assumptions, first we construct a perturbation of norm $C_r(A)$ that achieves the task of making a coalescence point z_* an eigenvalue with algebraic multiplicity at least r (Theorem 3.0.9). Second, we exploit the orthogonality relations among block components of left and right singular vectors of $\overline{f}_r(z_*)$ (Theorem 3.0.10). Then using the orthogonality results we conclude that z_* is indeed an eigenvalue of the perturbed matrix with algebraic multiplicity $r + 1$ (Theorem 3.0.12), which establishes the inequality. In Conclusion, we briefly list some open problems based on the observations from the previous chapters, and a possible outline towards a solution of Conjecture 1.

Chapter 2

CENTERS OF COMPONENTS OF SECOND ORDER PSEUDOSPECTRA

In the first chapter, we mentioned that many authors have studied the problem of characterizing the distance from an $n \times n$ matrix A to the set of defective matrices. There were several approaches to this question. Wilkinson and Ruhe have worked on the relationship between this distance and the conditioning of eigenvalues [15, 19, 20]. Later Demmel and Wilkinson have approached the problem by considering the coalescence of the components of the pseudospectrum [5, 21]. Malyshev contributed to the problem by deriving a singular value characterization for the distance [12]. Later we discussed how Alam&Bora proved the equality $W(A) = C(A)$ using the pseudospectral approach [2].

In this chapter, we will attempt to relate the results due to Malyshev, and Alam&Bora. The former was a purely algebraic characterization of the distance to matrices with multiple eigenvalues (or equivalently the distance to the set of defective matrices), while the latter was a geometric characterization. We are able to partly solve the following conjecture.

Conjecture 2. *Let* $g(z) = \sup_{\gamma \geq 0} \sigma_{2n-1}\left(\begin{bmatrix} A - zI & \gamma I \\ 0 & A - zI \end{bmatrix}\right)$. *Then if* $z_* \notin \Lambda(A)$ *is a local minimizer of* $g(z)$ *where* $\sigma_n(A - z_*)$ *is simple, then* z_* *is a saddle point or a local maximizer of the function* $f(z) = \sigma_n(A - zI)$.

Based on numerical evidence and intuition from maximum modulus principle, we believe that the local maximizers of $f(z) = \sigma_n(A - zI)$ can occur only at a point $\tilde{z}$ such that $\sigma_n(A - \tilde{z}I)$ is not simple. But a proof of this is open at the moment. A significant consequence of Conjecture 2 is that each component of $\Lambda_{\epsilon,2}(A)$ is centered around a saddle point of $f(z) = \sigma_n(A - zI)$ where two components of $\Lambda_\epsilon(A)$ coalesce or otherwise around a local maximizer of $f(z) = \sigma_n(A - zI)$. These remarks are illustrated in Figure 2.1 on a 6×6 matrix; the components of $\Lambda_\epsilon(A)$ are centered around the eigenvalue of A (which are the only local minimizer of $f(z) = \sigma_n(A - zI)$; see Corollary 2.0.7 below), while the components

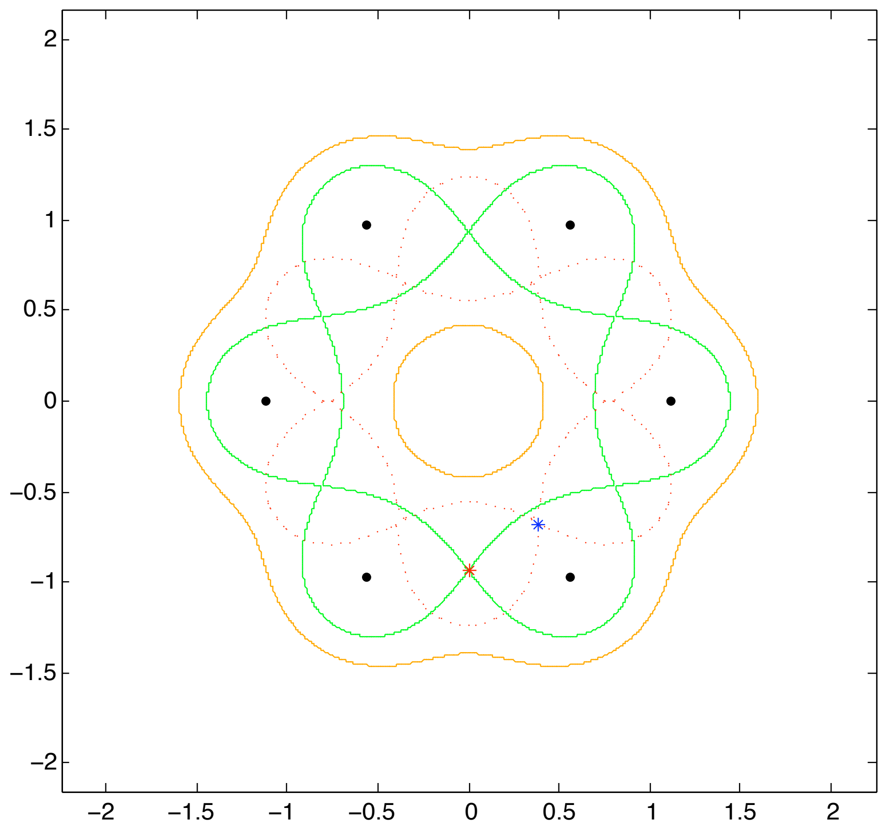

Figure 2.1: The pseudospectra of a 6×6 smoke matrix A is illustrated. The components of $\Lambda_\epsilon(A)$ for $\epsilon = W(A)$ (whose boundaries are represented by solid green curves) are centered around the eigenvalues of A marked with black dots. On the other hand, the components of $\Lambda_{\epsilon,2}(A)$ (whose boundaries are represented by dotted red curves) are centered around the coalescence points of the components of $\Lambda_\epsilon(A)$.

of $\Lambda_{\epsilon,2}(A)$ are centered around the coalescence points of the components of $\Lambda_\epsilon(A)$ (which are the global minimizers of $g(z)$).

The rest of this section is devoted to a partial solution of Conjecture 2. Consider the singular value function $h(\gamma) = \sigma_{2n-1}\left(\begin{bmatrix} A - zI & \gamma I \\ 0 & A - zI \end{bmatrix}\right)$ for a fixed $z \in \mathbb{C}$, that is continuous and symmetric, i.e, $h(\gamma) = h(-\gamma)$. In [12], Malyshev has shown that $h(\gamma)$ possesses a number of remarkable properties helpful to enlighten Conjecture 2.

Lemma 2.0.1. *$h(\gamma) \to 0$ as $|\gamma| \to \infty$.*

Note that due to the continuity of $h(\gamma)$, Lemma 2.0.1 implies that if we take the supremum of $h(\gamma)$ over γ, then it must be attained at a finite value. In addition to Lemma 2.0.1, another important property shown by Malyshev is the following result.

Lemma 2.0.2. *Let* $h(\gamma) = \sigma_{2n-1}\left(\begin{bmatrix} A - zI & \gamma I \\ 0 & A - zI \end{bmatrix}\right) \not\equiv 0$. *Then either* $h(\gamma)$ *has no local extrema on* $(0, \infty)$ *or it has only one, which is a global maximum.*

Denoting a global minimizer of $g(z)$ with z_{**}, it can be easily verified that $g(z_{**})$ is attained at $\gamma = 0$. Indeed we have $g(z_{**}) = W(A) = \sigma_n(A - z_{**}I)$, where the first equality is due to Malyshev [12], and the second equality is due to Alam and Bora [2], which means $g(z_{**})$ is attained at $\gamma = 0$. We will prove Conjecture 2 for any z_* such that $g(z_*)$ is attained at $\gamma = 0$. Here is a brief outline of the proof.

(1) $g(z_*)$ is attained at $\gamma = 0$ if and only if

$$u^* v = 0 \tag{2.0.1}$$

where u, v are left and right singular vectors associated with $\sigma_n(A - z_*I)$.

(2) Equation (2.0.1) implies z_* is a saddle point of the function $\sigma_n(A - zI)$ or a local maximizer.

Step **(1)** is involved and relies on the following lemma for recognizing multiple eigenvalues. Here we benefit from the assumptions that $z_* \notin \Lambda(A)$ and $\sigma_n(A - z_*I)$ is simple, which ensures that $\sigma_n(A - zI)$ is differentiable at z_*.

Lemma 2.0.3. *Let* $A \in \mathbb{C}^{n \times n}$ *be a matrix with an eigenvalue* λ *with an associated pair of left and right eigenvectors* $u, v \in \mathbb{C}^n$ *such that* $u^* v = 0$. *Then the eigenvalue* λ *is multiple.*

Lemma 2.0.4. *Let* $z_* \notin \Lambda(A)$ *be a point such that* $\sigma_n(A - z_*I)$ *is simple. Then the following are equivalent:*

(i) $\sup_\gamma \ \sigma_{2n-1}\left(\begin{bmatrix} A - z_*I & \gamma I \\ 0 & A - z_*I \end{bmatrix}\right)$ *is attained at* $\gamma = 0$.

(ii) $u^* v = 0$, *where* u, v *are left and right singular vectors associated with* $f(z_*)$.

Proof. First suppose that $u^* v = 0$. If we set $\Delta A = -f(z_*)uv^*$, then we have

- $(A + \Delta A - z_* I)v = (A - z_* I)v + \Delta A v = f(z_*)u - f(z_*)u = 0$
- $u^*(A + \Delta A - z_* I) = u^*(A - z_*) + u^* \Delta A = f(z_*)v^* - f(z_*)v^* = 0$

That is u and v are left and right eigenvectors of $A + \Delta A$ corresponding to the eigenvalue z_*. Thus Lemma 2.0.3 implies that z_* is multiple. Moreover we have $\|\Delta A\| = f(z_*)$. This means that

$$\begin{aligned} \sup_{\gamma} \ \sigma_{2n-1}\left(\begin{bmatrix} A - z_* I & \gamma I \\ 0 & A - z_* I \end{bmatrix}\right) &= W(A, z_*) \\ &\leq f(z_*) \\ &= \sigma_n(A - z_* I) = \sigma_{2n-1}\left(\begin{bmatrix} A - z_* I & 0 \\ 0 & A - z_* I \end{bmatrix}\right). \end{aligned}$$

Thus the supremum is attained at $\gamma = 0$.

Conversely suppose that **(i)** holds. Consider, the analytic matrix function

$$F(\gamma) = \begin{bmatrix} A - z_* I & \gamma I \\ 0 & A - z_* I \end{bmatrix}.$$

for γ near 0. Let us focus on the analytic SVD of $F(\gamma)$ [1]

$$F(\gamma) = \tilde{U}(\gamma)\tilde{\Sigma}(\gamma)\tilde{V}(\gamma)^*$$

where $\tilde{U}(\gamma) = \begin{bmatrix} \tilde{u}_1(\gamma) & \tilde{u}_2(\gamma) & \ldots & \tilde{u}_{2n}(\gamma) \end{bmatrix}$, $\tilde{V}(\gamma) = \begin{bmatrix} \tilde{v}_1(\gamma) & \tilde{v}_2(\gamma) & \ldots & \tilde{v}_{2n}(\gamma) \end{bmatrix}$ are analytic unitary matrix functions and $\tilde{\Sigma}(\gamma) = \text{diag}(\tilde{\sigma}_1(\gamma), \tilde{\sigma}_2(\gamma), \ldots, \tilde{\sigma}_{2n}(\gamma))$ is an analytic diagonal matrix function. The assumption that $\sigma_n(A - z_* I)$ is simple implies

$$\tilde{\sigma}_{2n}(0) = \tilde{\sigma}_{2n-1}(0) = \sigma_n(A - z_* I) < \sigma_{n-1}(A - z_* I) = \tilde{\sigma}_{2n-2}(0) = \tilde{\sigma}_{2n-3}(0) \leq \tilde{\sigma}_j(0)$$

for each $j = 1, \ldots, 2n-4$. Due to the continuity of the singular values $\tilde{\sigma}_j(\gamma)$ for $j = 1, \ldots, 2n$ there exists a neighborhood Γ of 0 such that $\tilde{\sigma}_{2n}(\gamma) < \tilde{\sigma}_j(\gamma)$ and $\tilde{\sigma}_{2n-1}(\gamma) < \tilde{\sigma}_j(\gamma)$ for all $\gamma \in \Gamma$ and for each $j = 1, \ldots, 2n-2$. Let $\tilde{u}_{2n-1}(\gamma), \tilde{v}_{2n-1}(\gamma), \tilde{u}_{2n}(\gamma), \tilde{v}_{2n}(\gamma)$ be unit analytic consistent pair of left and right singular vectors of $F(\gamma)$ associated with $\tilde{\sigma}_{2n-1}(\gamma), \tilde{\sigma}_{2n}(\gamma)$.

As $\tilde{\sigma}_{2n-1}(0) = \tilde{\sigma}_{2n}(0) = \sigma_n(A - z_* I)$, we get that

$$\tilde{u}_{2n-1}(0) = \begin{bmatrix} k_1 u \\ l_1 u \end{bmatrix}, \tilde{u}_{2n}(0) = \begin{bmatrix} k_2 u \\ l_2 u \end{bmatrix}$$

for some k_1, k_2, l_1, l_2 such that $\begin{bmatrix} k_1 & k_2 \\ l_1 & l_2 \end{bmatrix}$ is unitary. Define the scalars α and β satisfying

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} k_1^* & l_1^* \\ k_2^* & l_2^* \end{bmatrix} \begin{bmatrix} 1 \\ v^* u \end{bmatrix} \frac{1}{\sqrt{1 + |v^* u|^2}}. \tag{2.0.2}$$

Notice that $|\alpha|^2 + |\beta|^2 = 1$

Now let us define two analytic vector functions $x(\gamma) = \alpha \tilde{v}_{2n-1}(\gamma) + \beta \tilde{v}_{2n}(\gamma)$ and $y(\gamma) = \alpha \tilde{u}_{2n-1}(\gamma) + \beta \tilde{u}_{2n}(\gamma)$. These vector functions satisfy the following properties:

(i) $x(\gamma)^* x(\gamma) = (\alpha^* \tilde{v}_{2n-1}(\gamma)^* + \beta^* \tilde{v}_{2n}(\gamma)^*)(\alpha \tilde{v}_{2n-1}(\gamma) + \beta \tilde{v}_{2n}(\gamma)) = |\alpha|^2 + |\beta|^2 = 1$ for all $\gamma \in \Gamma$

(ii) $F(\gamma)x(\gamma) = \alpha \tilde{\sigma}_{2n-1}(\gamma) \tilde{u}_{2n-1}(\gamma) + \beta \tilde{\sigma}_{2n}(\gamma) \tilde{u}_{2n}(\gamma)$ for all $\gamma \in \Gamma$

(iii) $\|F(\gamma)x(\gamma)\|^2 = |\alpha|^2 \tilde{\sigma}_{2n-1}(\gamma)^2 + |\beta|^2 \tilde{\sigma}_{2n}(\gamma)^2 \leq (|\alpha|^2 + |\beta|^2)\sigma_n(A - z_* I)^2 = \sigma_n(A - z_* I)^2$ for all $\gamma \in \Gamma$

(iv) $F(0)x(0) = \sigma_n(A - z_* I)y(0)$ and $F(0)^* y(0) = \sigma_n(A - z_* I)x(0)$

(v) $\|F(0)x(0)\| = \sigma_n(A - z_* I)$

Properties **(iii)** and **(v)** together imply that the analytic function $g(\gamma) = \|F(\gamma)x(\gamma)\|^2$ has a local maximum at $\gamma = 0$.

Differentiating $g(\gamma) = \|F(\gamma)x(\gamma)\|^2 = x(\gamma)^*F(\gamma)^*F(\gamma)x(\gamma)$ at $\gamma = 0$, we obtain

$$\frac{d(x(\gamma)^*F(\gamma)^*F(\gamma)x(\gamma))}{d\gamma}\bigg|_{\gamma=0} = \frac{dx(\gamma)^*}{d\gamma}\bigg|_{\gamma=0} F(0)^*F(0)x(0) + x(0)^*\frac{dF(\gamma)^*}{d\gamma}\bigg|_{\gamma=0} F(0)x(0)$$

$$+ x(0)^*F(0)^*\frac{dF(\gamma)}{d\gamma}\bigg|_{\gamma=0} x(0) + x(0)^*F(0)^*F(0)\frac{dx(\gamma)}{d\gamma}\bigg|_{\gamma=0}. \tag{2.0.3}$$

Using property **(iv)**, equation (2.0.3) becomes

$$\frac{d(x(\gamma)^*F(\gamma)^*F(\gamma)x(\gamma))}{d\gamma}\bigg|_{\gamma=0} = (\sigma_n(A - z_*I))^2\left(\frac{dx(\gamma)^*}{d\gamma}\bigg|_{\gamma=0} x(0) + x(0)^*\frac{dx(\gamma)}{d\gamma}\bigg|_{\gamma=0}\right)$$

$$+ \sigma_n(A - z_*I)\left(x(0)^*\frac{dF(\gamma)^*}{d\gamma}\bigg|_{\gamma=0} y(0) + y(0)^*\frac{dF(\gamma)}{d\gamma}\bigg|_{\gamma=0} x(0)\right). \tag{2.0.4}$$

Furthermore, as $x(\gamma)^*x(\gamma) = 1$,

$$\frac{dx(\gamma)^*x(\gamma)}{d\gamma}\bigg|_{\gamma=0} = \frac{dx(\gamma)^*}{d\gamma}\bigg|_{\gamma=0} x(0) + x(0)^*\frac{dx(\gamma)}{d\gamma}\bigg|_{\gamma=0} = 0. \tag{2.0.5}$$

From equation (2.0.2), the vector function $x(\gamma)$ at $\gamma = 0$ is given by

$$x(0) = \alpha\tilde{v}_{2n-1}(0) + \beta\tilde{v}_{2n}(0) = \alpha\begin{bmatrix} k_1 v \\ l_1 v \end{bmatrix} + \beta\begin{bmatrix} k_2 v \\ l_2 v \end{bmatrix} = \begin{bmatrix} (\alpha k_1 + \beta k_2)v \\ (\alpha l_1 + \beta l_2)v \end{bmatrix}$$

$$= \frac{1}{\sqrt{1 + |v^*u|^2}}\begin{bmatrix} v \\ (v^*u)v \end{bmatrix}.$$

Similarly $y(0) = \frac{1}{\sqrt{1+|v^*u|^2}}\begin{bmatrix} u \\ (v^*u)u \end{bmatrix}$. So we have

$$x(0)^*\frac{dF(\gamma)^*}{d\gamma}\bigg|_{\gamma=0} y(0) = y(0)^*\frac{dF(\gamma)}{d\gamma}\bigg|_{\gamma=0} x(0) = \frac{|v^*u|^2}{1 + |v^*u|^2}. \tag{2.0.6}$$

Employing equations (2.0.5) and (2.0.6) in (2.0.4), and employing the fact that $\gamma = 0$ is a local maximum, we have

$$\left.\frac{d(x(\gamma)^* F(\gamma)^* F(\gamma) x(\gamma))}{d\gamma}\right|_{\gamma=0} \;=\; 2\sigma_n(A - z_* I)\frac{|v^* u|^2}{1 + |v^* u|^2} \;=\; 0.$$

This implies $v^* u = 0$ as desired. □

Step **(2)** is a simple consequence of the fact that the only minimizers of $f(z) = \sigma_n(A - zI)$ are the eigenvalues of A, which follows from the maximum modulus principle [4].

Theorem 2.0.5 (Maximum Modulus Principle)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function on a connected open subset U of $\mathbb{C}$. If there is a point $z_0 \in U$ satisfying $|f(z_0)| \geq |f(z)|$ for all z in a neighborhood of z_0, then f is constant on U.*

Theorem 2.0.6. *Let $\mathcal{A}(z) : \mathbb{C} \to \mathbb{C}^{n\times n}$ be an analytic matrix function on an open subset U of $\mathbb{C}$. Then, either $\|\mathcal{A}(z)\|$ is constant on a neighborhood of some $\tilde{z} \in U$, or otherwise $\|\mathcal{A}(z)\|$ does not have any local maximizer in U.*

Proof. Suppose $\|\mathcal{A}(z)\|$ has a local maximizer $z_* \in U$. There exists unit vectors $u, v \in \mathbb{C}^n$ such that

$$|u^* \mathcal{A}(z_*) v| = \|\mathcal{A}(z_*)\| \geq \|\mathcal{A}(z)\| \geq |u^* \mathcal{A}(z) v|. \tag{2.0.7}$$

for all z in a neighborhood $\mathcal{N} \subset U$ of z_*. But $w(z) = u^* \mathcal{A}(z) v$ is analytic, and the above relations would imply that $|w(z_*)| \geq |w(z)|$ for all $z \in \mathcal{N}$. It follows from the maximum modulus principle that $w(z)$ is constant on U, which in turn would mean that $\|\mathcal{A}(z)\|$ is constant on $\mathcal{N}$ from (2.0.7). □

Corollary 2.0.7. *The eigenvalues of the matrix A are the only local minimizers of the function $f(z) = \sigma_n(A - zI)$.*

Proof. Noting that $\sigma_n(A - zI) = 1/\|(A - zI)^{-1}\|$, and applying Theorem 2.0.6 with $\mathcal{A}(z) = (A - zI)^{-1}$, we deduce that any $z \notin \Lambda(A)$ cannot be a local maximizer of $\|(A - zI)^{-1}\|$, and thus cannot be a local minimizer of $\sigma_n(A - zI)$. (Note that none of the analytic singular values of $(A - zI)$ is constant, since they all blow up as $z \to \infty$. Thus $\|A - zI\|$ cannot be constant either.) □

Corollary 2.0.8. *Let $z_* \notin \Lambda(A)$ be such that $\sigma_n(A - z_*I)$ is simple, and u, v be left and right singular vectors of $A - z_*I$ corresponding to $\sigma_n(A - z_*I)$ such that $u^*v = 0$. Then z_* is a saddle point of the function $f(z) = \sigma_n(A - zI)$ or a local maximizer.*

Proof. Since the function $\sigma_n(A - zI)$ is nonzero and simple at z_*, it is (real) analytic at z_* with respect to real and imaginary parts of z disjointly; indeed

$$\frac{d\sigma_n(A - zI)}{d\Re z}\Big|_{z=z_*} = -\Re(u^*v) = 0 \qquad \text{and} \qquad \frac{d\sigma_n(A - zI)}{d\Im z}\Big|_{z=z_*} = \Im(u^*v) = 0$$

i.e., z_* is a critical point of $\sigma_n(A - zI)$. Moreover, as only local minimizers of $\sigma_n(A - zI)$ are the eigenvalues of A, we deduce that z_* is either a saddle point or a local maximizer of $\sigma_n(A - zI)$. □

Chapter 3

COALESCENCE POINTS OF COMPONENTS OF HIGHER ORDER PSEUDOSPECTRA

In the first chapter, we discussed the connection between the Wilkinson distance $W(A)$, and the ϵ-pseudospectrum $\Lambda_\epsilon(A)$, and characterized as the smallest ϵ so that $\Lambda_\epsilon(A)$ has $n-1$ or fewer components. Thus, $W(A)$ is the smallest ϵ so that two components of $\Lambda_\epsilon(A)$ coalesce. Moreover, the point of coalescence $z_* \in \mathbb{C}$ for $\epsilon = W(A)$ is the multiple eigenvalue of a nearest matrix.

In this chapter we consider the following generalization of the Wilkinson distance for a given matrix $A \in \mathbb{C}^{n\times n}$ and an integer $r \in [2, n]$.

$$W_r(A) = \inf\{\|\Delta A\|_2 \mid (A + \Delta A) \text{ has an eigenvalue with algebraic multiplicity} \geq r\}.$$

Inspired by the work of Malyshev [12], for the special case $r = 2$, Mengi [13] established that

$$W_r(A) = \inf_{z\in\mathbb{C}} \left(\overline{f}_r(z) := \sup_{\gamma\in\mathbb{C}^{r(r-1)/2}} f_r(z,\gamma) \right)$$

where

$$f_r(z,\gamma) := \sigma_{nr-r+1}\left(\mathcal{A}(z,\gamma)\right) \quad \text{with} \quad \mathcal{A}(z,\gamma) := \begin{bmatrix} A - zI & \gamma_{1,2}I & & & \gamma_{1,r}I \\ 0 & A - zI & & & \gamma_{2,r}I \\ & & \ddots & & \\ & & & A - zI & \gamma_{r-1,r}I \\ 0 & & & 0 & A - zI \end{bmatrix}$$

and $\gamma := \begin{bmatrix} \gamma_{1,2} & \dots & \gamma_{r-1,r} \end{bmatrix}^T$. The value of $\overline{f}_r(z)$ for a fixed $z \in \mathbb{C}$ corresponds to the distance from A to the nearest matrix with z as an eigenvalue with algebraic multiplicity at least equal to r.

In the first chapter, we mentioned that $\Lambda_\epsilon(A)$ turns out to be irrelevant to $W_r(A)$ for

$r > 2$; we provided an example illustrating this. To be relevant, we defined the generalized ϵ-pseudospectrum of order r as

$$\begin{aligned}\Lambda_{\epsilon,r}(A) &:= \{z \in \mathbb{C} \mid \exists \Delta A \text{ s.t. } \|\Delta A\|_2 \leq \epsilon \text{ and } \operatorname{rank}(A + \Delta A - zI)^{r-1} \leq n - r + 1\} \\ &= \left\{z \in \mathbb{C} \mid \overline{f}_r(z) \leq \epsilon\right\}.\end{aligned}$$

We conjecture that

$$W_{r+1}(A) = C_r(A) := \inf\{\epsilon \mid \textit{two components of } \Lambda_{\epsilon,r}(A) \textit{ coalesce}\}.$$

The following example illustrates our conjecture for $r = 2$.

Example:

Consider the 6×6 Dramadah matrix

$$D = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

with six distinct eigenvalues. A plot of $\Lambda_{\epsilon,2}(D)$ for the Dramadah matrix is provided in Figure 3.1 for various ϵ. The inner-most curve represents the boundary of $\Lambda_{\epsilon,2}(D)$ for $\epsilon = W_3(D)$. Remarkably two components of $\Lambda_{\epsilon,2}(D)$ coalesce for $\epsilon = W_3(D)$ at $\lambda_{**} = 0.8413$ (marked with a red square in the figure), which is the eigenvalue with algebraic multiplicity three of a nearest matrix.

Remember that denoting a consistent pair of unit left and right singular vectors associated with $\sigma_n(A - zI) = \epsilon$ with u and v, respectively, the rank one perturbation $\Delta A = -\epsilon uv^*$ with $\|\Delta A\| = \epsilon$ makes z an eigenvalue of $A + \Delta A$ with u and v as the associated left and right eigenvectors. Moreover, a coalescence point z_* of $\Lambda_\epsilon(A)$ is a saddle point of $\sigma_n(A - zI)$. If $\sigma_n(A - zI)$ is differentiable at z_* (i.e., $\sigma_n(A - zI)$ is simple), we have $u^*v = 0$. It follows that z_* is an eigenvalue of $A + \Delta A$ with a pair of orthogonal left and right eigenvectors meaning z_* is indeed a defective multiple eigenvalue. This implies $W(A) \leq C(A)$, provided

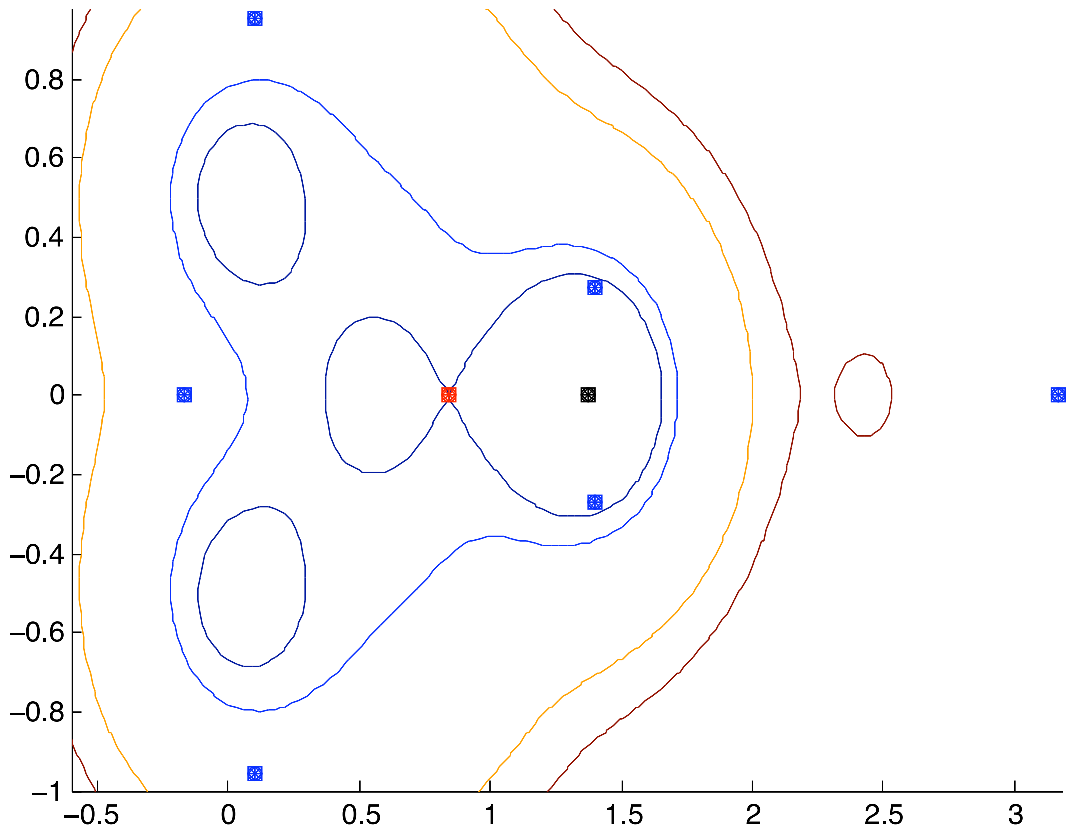

Figure 3.1: The ϵ-pseudospectra of the 6×6 Dramadah matrix of order two are shown for various ϵ. The blue squares represent the eigenvalues of the matrix, whereas the black and red squares correspond to the nearest multiple eigenvalue and the eigenvalue of algebraic multiplicity three, respectively, under smallest perturbation possible.

$\sigma_n(A - zI)$ is differentiable at the smallest saddle point z_*.

In this chapter, we generalize these observations for $\Lambda_{\epsilon,r}(A)$ under mild assumptions to prove the inequality $W_{r+1}(A) \leq C_r(A)$. In this respect note that, if we have a coalescence point z_* of two components of $\Lambda_{\epsilon,r}(A)$, then

(1) there exists a rank r perturbation ΔA with $\|\Delta A\|_2 = \epsilon$ such that z_* is an eigenvalue of $A + \Delta A$ with algebraic multiplicity r or greater,

(2) z_* is a saddle point of $\overline{f}_r(z)$ under linear independence and multiplicity assumptions stated below.

Assumptions

Let z_* be a point of coalescence of two components of $\Lambda_{\epsilon,r}(A)$ and $U, V \in \mathbb{C}^{n \times r}$ be such that $u = \text{vec}(U)$ and $v = \text{vec}(V)$ consist of a pair of unit left and right singular vectors associated with the singular value

$$\overline{f}_r(z_*) = \sigma_{nr-r+1}\left(\mathcal{A}(z_*, \gamma_*)\right) = \sup_{\gamma}\ \sigma_{nr-r+1}\left(\mathcal{A}(z_*, \gamma)\right).$$

Note that $\sup_\gamma \sigma_{nr-r+1}\left(\mathcal{A}(z_*, \gamma)\right)$ is guaranteed to be attained, since $\sigma_{nr-r+1}\left(\mathcal{A}(z_*, \gamma)\right)$ decays to zero as any of the components of γ goes to infinity in modulus [10]. Throughout this chapter the arguments hold under the following mild assumptions.

- **(linear independence assumption)** $\text{rank}(U) = r$ (equivalently $\text{rank}(V) = r$).
- **(multiplicity assumption)** Multiplicity of $\overline{f}_r(z_*) = \sigma_{nr-r+1}\left(\mathcal{A}(z_*, \gamma_*)\right)$ is one.

Note that under the multiplicity assumption, we have the equality $U^*U = V^*V$ [13]. The next theorem not only shows the existence of a rank r perturbation matrix ΔA that induce z_* as an eigenvalue of $A + \Delta A$ with algebraic multiplicity r or greater but also identifies the left and right generalized eigenspaces of $A + \Delta A$ associated with z_*.

Theorem 3.0.9. *Suppose that the linear independence and multiplicity assumptions hold. Then the perturbation* $\Delta A = -\overline{f}_r(z_*)UV^+$ *satisfies the following:*

(i) $\|\Delta A\|_2 = \overline{f}_r(z_*)$,

(ii) *z_* is an eigenvalue of $A + \Delta A$ with algebraic multiplicity at least r,*

(iii) *$(A + \Delta A - z_* I)^r v_j = 0$ for $j = 1, \ldots, r$ where v_j is the jth column of V, and*

(iv) *$u_j^*(A + \Delta A - z_* I)^r = 0$ for $j = 1, \ldots, r$ where u_j is the jth column of U.*

Above V^+ denotes the Monroe-Penrose pseudoinverse of V.

Proof. Suppose that U, V satisfy the equation $U^*U = V^*V$. Then

$$\begin{aligned}\left\|UV^+\right\|_2 = & \ \sup_{x\in\mathbb{C}^n, x\neq 0} \frac{\left\|UV^+x\right\|_2}{\|x\|_2} = \sup_{x\in\mathbb{C}^n, x\neq 0} \frac{\sqrt{x^*(V^+)^*U^*UV^+x}}{\|x\|_2} \\ = & \ \sup_{x\in\mathbb{C}^n, x\neq 0} \frac{\sqrt{x^*(V^+)^*V^*VV^+x}}{\|x\|_2} = \sup_{x\in\mathbb{C}^n, x\neq 0} \frac{\left\|VV^+x\right\|_2}{\|x\|_2} = \left\|VV^+\right\|_2.\end{aligned}$$

Note that VV^+ is an orthogonal projector onto $\mathrm{Col}(V)$ and so $\|VV^+\| = 1$. Thus we have $\|UV^+\| = 1$ yielding

$$\|\Delta A\|_2 = \left\|-\overline{f}_r(z_*)UV^+\right\|_2 = \overline{f}_r(z_*)\left\|UV^+\right\|_2 = \overline{f}_r(z_*).$$

Next we will establish **(iii)**. We have, as $\overline{f}_r(z_*)$ is a singular value of $\mathcal{A}(z_*, \gamma_*)$ with associated left singular vector $u = \mathrm{vec}(U)$ and right singular vector $v = \mathrm{vec}(V)$,

$$\mathcal{A}(z_*, \gamma_*)v = \overline{f}_r(z_*)u \quad \text{and} \quad \mathcal{A}(z_*, \gamma_*)^*u = \overline{f}_r(z_*)v. \tag{3.0.1}$$

Noting that $\mathcal{A}(z_*, \gamma_*) = I \otimes A - C(z_*, \gamma_*)^T \otimes I$, and using equation (1.1.6), we can write the first equation as a matrix equation of the form

$$AV - VC(z_*, \gamma_*) = \overline{f}_r(z_*)U$$

where

$$C(z_*, \gamma_*) = \begin{bmatrix} z_* & 0 & & & 0 \\ -\gamma_{1,2} & z_* & & & \\ & & \ddots & & \\ & & & z_* & \\ -\gamma_{1,r} & & & -\gamma_{r-1,r} & z_* \end{bmatrix}.$$

Due to the linear independence assumption, i.e, $\text{rank}(V) = r$, we have $V^+V = I$. This yields

$$AV - VC(z_*, \gamma_*) = \overline{f}_r(z_*)UV^+V = -\Delta AV \Longrightarrow (A + \Delta A)V - VC(z_*, \gamma_*) = 0.$$

Using again vec operator, we obtain

$$\begin{bmatrix} A+\Delta A - z_*I & \gamma_{1,2}I & & & \gamma_{1,r}I \\ 0 & A+\Delta A - z_*I & & & \gamma_{2,r}I \\ & & \ddots & & \\ & & & A+\Delta A - z_*I & \gamma_{r-1,r}I \\ 0 & & & 0 & A+\Delta A - z_*I \end{bmatrix} v = 0. \quad (3.0.2)$$

An implication of the last equation is

$$(A + \Delta A - z_*I)^r v_j = 0, \qquad j = 1, \ldots, r \qquad (3.0.3)$$

where v_j denotes the jth column of V. Indeed, equation (3.0.3) can be shown using induction. For the base case $j = r$, we obtain, from the last block row of equation (3.0.2),

$$(A + \Delta A - z_*I)v_r = 0.$$

Now as the inductive hypothesis assume that for $l = j + 1, \ldots, r$

$$(A + \Delta A - z_*I)^{r-l+1}v_l = 0.$$

From the jth block row of (3.0.1), we have

$$\begin{aligned}
(A + \Delta A - z_* I)v_j + \sum_{l=j+1}^{r} \gamma_{l,j} v_l = 0 \Longrightarrow \\
(A + \Delta A - z_* I)^{r-j+1} v_j + \sum_{l=j+1}^{r} \gamma_{l,j} (A + \Delta A - z_* I)^{r-j} v_l = 0 \Longrightarrow \\
(A + \Delta A - z_* I)^{r-j+1} v_j + \sum_{l=j+1}^{r} \gamma_{l,j} (A + \Delta A - z_* I)^{l-j-1} (A + \Delta A - z_* I)^{r-l+1} v_l = 0 \Longrightarrow \\
(A + \Delta A - z_* I)^{r-j+1} v_j = 0.
\end{aligned}$$

as desired. Above we used the inductive hypothesis in the second to last equality. This completes the proof of **(iii)**.

Similarly, by writing the second equation in (3.0.1) in matrix form we have

$$A^* U - U C^*(z_*, \gamma) = \overline{f}_r(z_*) V,$$

or equivalently

$$\left(A - \overline{f}_r(z_*)(V U^+)^*\right)^* U - U C^*(z_*, \gamma) = 0.$$

Due to the property $U^* U = V^* V$ we have $-\overline{f}_r(z_*)(V U^+)^* = -\overline{f}_r(z_*) U V^+ = \Delta A$. By expressing the last matrix equation in the vector form we deduce

$$\begin{bmatrix}
A + \Delta A - z_* I & \gamma_{1,2} I & & & \gamma_{1,r} I \\
0 & A + \Delta A - z_* I & & & \gamma_{2,r} I \\
 & & \ddots & & \\
 & & & A + \Delta A - z_* I & \gamma_{r-1,r} I \\
0 & & & 0 & A + \Delta A - z_* I
\end{bmatrix}^* u = 0,$$

which yields, by a similar reasoning above,

$$u_j^* (A + \Delta A - z_* I)^r = 0, \qquad j = 1, \ldots, r \tag{3.0.4}$$

completing the proof of **(iv)**. Finally, equation (3.0.3) implies that

$$\dim \text{Null}(A + \Delta A - z_* I)^r \geq r,$$

or equivalently $\text{rank}(A + \Delta A - z_*)^r \leq n - r$ proving **(ii)**. □

Remember that for any matrix function $\mathcal{F} : \mathbb{R} \to \mathbb{C}^{n \times n}$ depending on a parameter analytically each singular value σ_j is analytic at a given point as long as it is non-zero and its multiplicity is one. In this case the derivative is given by

$$\frac{d\sigma_j(\tilde{\omega})}{d\omega} = \Re \left(\mathcal{U}_j^* \, \frac{d\mathcal{F}(\tilde{\omega})}{d\omega} \, \mathcal{V}_j \right) \tag{3.0.5}$$

where $\mathcal{U}_j, \mathcal{V}_j$ denote a consistent pair of unit left and right singular vectors associated with $\sigma_j(\tilde{\omega})$.

Theorem 3.0.10. *Under the linear independence and multiplicity assumptions, we have*

$$u_i^* v_j = 0 \qquad i = 1, \ldots, r, \quad j = i, \ldots, r.$$

Proof. We view $f_r(z_*, \gamma)$ as a mapping $\mathbb{R}^{r(r-1)} \to \mathbb{R}$ by decomposing each complex parameter γ_{ij} contained in γ into its real and imaginary parts $\Re\gamma_{ij}$ and $\Im\gamma_{ij}$. Due to the multiplicity assumption, $f_r(z_*, \gamma)$ is analytic with respect to the real and imaginary parts at γ_*. By an application of the formula (3.0.5) we obtain

$$\frac{\partial f_r(z_*, \gamma_*)}{\partial \Re\gamma_{ij}} = \Re(u_i^* v_j) \quad \text{and} \quad \frac{\partial f_r(z_*, \gamma_*)}{\partial \Im\gamma_{ij}} = -\Im(u_i^* v_j).$$

Since γ_* is a local maximizer of $f_r(z_*, \gamma)$, all of these partial derivatives must vanish yielding

$$u_i^* v_j = 0 \qquad i = 1, \ldots, r, \;\; j = i+1, \ldots, r.$$

A consequence of the multiplicity assumption is the differentiability, indeed analyticity, of $\overline{f}_r(z)$ at z_* [3, Proposition 4.12]. Its derivatives are given by

$$\frac{\partial \overline{f}_r(z_*)}{\partial \Re z} = \Re \left(u^* \frac{\partial \mathcal{A}(z_*, \gamma_*)}{\partial \Re z} v \right) = -\Re(u^* v)$$

and

$$\frac{\partial \overline{f}_r(z_*)}{\partial \Im z} = \Re\left(u^* \frac{\partial \mathcal{A}(z_*, \gamma_*)}{\partial \Im z} v\right) = \Im(u^* v).$$

Since z_* is a saddle point of $\overline{f}_r(z)$, the derivatives above yield

$$u^* v = \sum_{j=1}^{r} u_j^* v_j = 0. \tag{3.0.6}$$

Now, for $j = 1, \ldots, r$, define the functions

$$\overline{f}_r^{(j)}(z) := \sup_{\gamma} \ \sigma_{nr-r+1}\left(\mathcal{A}^{(j)}(z, \gamma)\right)$$

with

$$\mathcal{A}^{(j)}(z,\gamma) := \begin{bmatrix} A - z_* I & \gamma_{1,2} I & & & & & \gamma_{1,r} I \\ 0 & A - z_* I & & & & & \gamma_{2,r} I \\ & & \ddots & & & & \\ & & & \underbrace{A - zI}_{(j,j)\text{ block}} & & & \\ & & & & \ddots & & \\ & & & & & A - z_* I & \gamma_{r-1,r} I \\ 0 & & & & & 0 & A - z_* I \end{bmatrix}.$$

Note that under the multiplicity and linear independence assumptions the functions $\overline{f}_r^{(j)}$ for $j = 1, \ldots, r$ correspond to the same distance function. To see this, let $C^{(j)}(z, \gamma)$ be the same as $C(z_*, \gamma)$ except its entry at position (j, j) is z. By applying the vec operator to both sides of

$$(A + \Delta A)X - XC^{(j)}(z, \gamma) = 0 \tag{3.0.7}$$

and exploiting the Kronecker product identity (1.1.6), we obtain

$$\mathcal{A}^{(j)}(\Delta A, z, \gamma)x =$$

$$\begin{bmatrix} A+\Delta A - z_* I & \gamma_{1,2} I & & & & & \gamma_{1,r} I \\ 0 & A+\Delta A - z_* I & & & & & \gamma_{2,r} I \\ & & \ddots & & & & \\ & & & \underbrace{A+\Delta A - zI}_{(j,j)\text{ block}} & & & \\ & & & & \ddots & & \\ & & & & & A+\Delta A - z_* I & \gamma_{r-1,r} I \\ 0 & & & & & 0 & A+\Delta A - z_* I \end{bmatrix} x = 0,$$

where $x = \text{vec}(X)$. Thus the solution space of the Slyvester equation (3.0.7) and null space of $\mathcal{A}^{(j)}(\Delta A, z, \gamma)$ have the same dimension for all γ. We can indeed define $\overline{f}_r^{(j)}(z)$ as [13]

$$\begin{aligned} \overline{f}_r^{(j)}(z) = &\inf \left\{ \|\Delta A\|_2 \mid \text{rank}(\mathcal{A}^{(j)}(\Delta A, z, \gamma)) \le nr - r + 1 \;\; \forall \gamma \in \mathcal{G}^{(j)}(z) \right\} \\ = &\inf \left\{ \|\Delta A\|_2 \mid \dim \text{Null}(\mathcal{A}^{(j)}(\Delta A, z, \gamma)) \ge r \;\; \forall \gamma \in \mathcal{G}^{(j)}(z) \right\} \\ = &\inf \left\{ \|\Delta A\|_2 \mid \dim \text{Kernel}\left(X \mapsto (A+\Delta A)X - XC^{(j)}(z,\gamma) \right) \ge r \;\; \forall \gamma \in \mathcal{G}^{(j)}(z) \right\} \end{aligned}$$

where $\mathcal{G}^{(j)}(z)$ denotes the set of (generic) γ values such that $C^{(j)}(z, \gamma)$ has full Jordan blocks. Now for each $\gamma \in \mathcal{G}^{(j)}(z) \cap \mathcal{G}^{(k)}(z)$, the matrices $C^{(j)}(z, \gamma)$ and $C^{(k)}(z, \gamma)$ have the same eigenvalues and the same Jordan canonical form, so Theorem 1.1.9 implies

$$\dim \text{Kernel}\left(X \mapsto (A+\Delta A)X - XC^{(j)}(z,\gamma) \right) = \dim \text{Kernel}\left(X \mapsto (A+\Delta A)X - XC^{(k)}(z,\gamma) \right)$$

for all ΔA. Thus, $\overline{f}_r^{(j)}(z) = \overline{f}_r^{(k)}(z)$. Again the multiplicity assumption ensures the differentiability of the functions $\overline{f}_r^{(j)}(z)$ at z_*. Now applications of the formula (3.0.5) yield

$$\frac{\partial \overline{f}_r^{(j)}(z)}{\partial \Re z} = -\Re(u_j^* v_j) = -\Re(u_k^* v_k) = \frac{\partial \overline{f}_r^{(k)}(z)}{\partial \Re z}$$

and

$$\frac{\partial \overline{f}_r^{(j)}(z)}{\partial \Im z} = \Im(u_j^* v_j) = \Im(u_k^* v_k) = \frac{\partial \overline{f}_r^{(k)}(z)}{\partial \Im z},$$

consequently $u_j^* v_j = u_k^* v_k$ for all j, k. By combining this with (3.0.6) we conclude with

$$u_j^* v_j = 0, \qquad j = 1, \ldots, r.$$

completing proof. □

We have shown that the algebraic multiplicity of a saddle point z_* of $\overline{f}_r(z)$ as an eigenvalue of $A + \Delta A$ is at least r (Theorem 3.0.9). Further can be inferred about the algebraic multiplicity of z_* by exploiting the orthogonality relations deduced in the previous theorem among the block components of u and v, which happen to compromise bases for the generalized eigenspaces of $A + \Delta A$ associated with z_* by Theorem 3.0.9.

Lemma 3.0.11. *Let $A \in \mathbb{C}^{n \times n}$ and z be an eigenvalue of A such that*

$$\text{rank}(A - zI)^r \leq n - r$$

(i.e., the algebraic multiplicity of z is at least r). Suppose also that $\{\mathcal{V}_1, \ldots, \mathcal{V}_r\}$ is a linearly independent set in $\mathbb{C}^n$ and $\mathcal{U}_1 \in \mathbb{C}^n$ and satisfy

(1) $(A - zI)^r \mathcal{V}_j = 0 \quad j = 1, \ldots, r$

(2) $\mathcal{U}_1^* (A - zI)^r = 0$, *and*

(3) $\mathcal{U}_1^* \mathcal{V}_j = 0 \quad j = 1, \ldots, r.$

Then A has z as an eigenvalue of algebraic multiplicity at least $r + 1$.

Proof. Let $A = PJP^{-1}$ be the Jordan canonical form such that

$$J = \left[\begin{array}{c|c} J_z & \\ \hline & \tilde{J} \end{array} \right]$$

where J_z is of size $r \times r$, and consist of the Jordan blocks associated with the eigenvalue z. Note that $\mathcal{U}_1, \mathcal{V}_1, \ldots, \mathcal{V}_r$ are the generalized left eigenvector and right eigenvectors associated with z, respectively. Also, note that

$$(A - zI)^r = P(J - zI)^r P^{-1} \quad \text{with} \quad (J - zI)^r = \left[\begin{array}{c|c} 0 & \\ \hline & (\tilde{J} - zI)^r \end{array}\right].$$

From **(2)**,

$$\mathcal{U}_1^*(A - zI)^r = 0 \Longrightarrow \mathcal{U}_1^* P P^{-1}(A - zI)^r P = 0 \Longrightarrow \mathcal{U}_1^* P(J - zI)^r = 0. \qquad (3.0.8)$$

Let $x^* = \mathcal{U}_1^* P$. Then from **(3)** we deduce that for $j = 1, \ldots, r$,

$$\mathcal{U}_1^* \mathcal{V}_j = 0 \Longrightarrow \mathcal{U}_1^* P P^{-1} \mathcal{V}_j = 0 \Longrightarrow x^* e_j = 0$$

yielding that the first r entries of the vector x is 0, i.e, $x^* = [\ 0 \quad \tilde{x}^*\ ]$, where $\tilde{x} \in \mathbb{C}^{n-r}$. Using equation (3.0.8), we get

$$[\ 0 \quad \tilde{x}^*\ ](J - zI)^r = [\ 0 \quad \tilde{x}^*\ ] \left[\begin{array}{c|c} 0 & \\ \hline & (\tilde{J} - zI)^r \end{array}\right] = 0$$

yielding $\tilde{x}^*(\tilde{J} - zI)^r = 0$. Thus z is an eigenvalue of $\tilde{J}$ showing that z is an eigenvalue of J of algebraic multiplicity at least $r + 1$. The same is, therefore, true for A. □

Applying the previous lemma to the eigenvalue z_* of $A + \Delta A$, and parts **(iii)**-**(iv)** of Theorem 3.0.9 we deduce the following.

Theorem 3.0.12. *The scalar z_* is an eigenvalue of $A + \Delta A$ with algebraic multiplicity at least $r + 1$.*

The significance of the theorem above to establish the connection between $\mathcal{C}_r(A)$ and $\mathcal{W}_{r+1}(A)$ is as follows. Suppose that the function $\overline{f}_r(z)$ is differentiable at the point of coalescence z_* of the components of $\Lambda_{\epsilon,r}(A)$ for $\epsilon = \mathcal{C}_r(A)$. (The differentiability of $\overline{f}_r(z)$ at z_* is ensured for instance by the multiplicity assumption.) Then the argument above establishes the existence of a perturbation ΔA with norm $\|\Delta A\|_2 = \mathcal{C}_r(A) = \overline{f}_r(z_*)$ such that z_* is an eigenvalue of $A + \Delta A$ with algebraic multiplicity at least $r + 1$. Consequently, $\mathcal{W}_{r+1}(A) \leq \mathcal{C}_r(A)$.

Chapter 4

CONCLUSION

We studied the Wilkinson distance of a matrix A, which is the 2-norm of the smallest perturbation ΔA so that $A + \Delta A$ has a multiple eigenvalue. Malyshev derived a singular value optimization characterization of the distance. Alam&Bora showed that the distance is the smallest ϵ such that the number of connected components of the ϵ-pseudospectrum of A is at most $n - 1$. Chapter 2 builds a connection between the works done by Malyshev and Alam&Bora. Chapter 3 is dedicated to generalize the observations of Alam&Bora.

In the second chapter, we partly prove Conjecture 1 for $r = 2$, suggesting how the conjecture should be approached for an arbitrary r. We prove that the attainment of the supremum $\overline{f}_2(z_*) = \sup_\gamma f_2(z, \gamma)$ at $\gamma = 0$ (where $\overline{f}_r(z)$ and $f_r(z, \gamma)$ are as defined in (1.2.2)) is equivalent to the orthogonality of left and right singular vectors of $\sigma_n(A - z_* I)$. This usually means that z_* is a saddle point of the function $f(z) = \sigma_n(A - zI)$. Thus, at the centers of the components of the second order pseudospectra $\Lambda_{\epsilon,2}(A)$ there usually lie coalescence points of the components of the ordinary pseudospectra $\Lambda_\epsilon(A)$.

In the third chapter, we establish the inequality $W_{r+1}(A) \leq C_r(A)$ for $r \geq 2$ under the multiplicity and linear independence assumptions. We first provide a perturbation that induces an eigenvalue with algebraic multiplicity r at a coalescence point z_* of two components of a generalized pseudospectrum of order r. We next demonstrate that the block components of the left and right singular vectors corresponding to $\overline{f}_r(z_*)$ are orthogonal, which implies that z_* is indeed an eigenvalue of algebraic multiplicity at least $r + 1$ of the perturbed matrix. This yields the desired inequality $W_{r+1}(A) \leq C_r(A)$.

4.1 Open Problems

4.1.1 Conjecture 1 for $r \geq 2$

In section 1.3, we stated that our main problem is to show that the equality $W_{r+1}(A) = C_r(A)$ holds for $r \geq 2$. In Chapter 3, we proved the inequality $W_{r+1}(A) \leq C_r(A)$. A proof of the other direction, $W_{r+1}(A) \geq C_r(A)$, is an open problem. The solution of Conjecture 1 would lead us to the other direction, and this may be achieved by following the steps below:

(1) $\overline{f}_r(z)$ has a local minimum at $z_* \Longrightarrow \overline{f}_r(z_*)$ is attained at $\gamma_{1r} = \gamma_{2r} = \cdots = \gamma_{r-1,r} = 0$.

(2) $\overline{f}_r(z_*)$ is attained at $\gamma_{1r} = \gamma_{2r} = \cdots = \gamma_{r-1,r} = 0$ if and only if

$$u_1^* v_1 = u_1^* v_2 = \cdots = u_1^* v_r = 0 \tag{4.1.1}$$

where $u_1, v_1, \ldots, v_r$ are columns of U, V and $u = \text{vec}(U)$, $v = \text{vec}(V)$ consist of a pair of unit left and right singular vectors associated with the singular value $\overline{f}_r(z_*)$.

(3) Equation (4.1.1) implies that z_* is a saddle point of the function $\overline{f}_{r-1}(z)$.

Note that the steps are the generalization of the ones for the case $r = 2$.

4.1.2 Multiplicity and Linear Independence Assumptions

The results in the third chapter are proved under the multiplicity and linear independence assumptions. Numerical experiments suggest that our results may hold even if these assumptions are not satisfied. Here we provide an example for which the multiplicity and linear independence assumptions are not satisfied. Consider the diagonal matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

In Figure 4.1, the boundaries of $\Lambda_\epsilon(A)$ and $\Lambda_{\epsilon,2}(A)$ are plotted for $\epsilon = W_3(A) = 0.7070$. Remarkably two components of $\Lambda_{\epsilon,2}(A)$ coalesce at $z_* = 2$, which is the eigenvalue with algebraic multiplicity three of a nearest matrix. For this example we have $C_2(A) = W_3(A)$ confirming our conjecture. For the coalescence point $z_* = 2$, we observe that $\overline{f}_3(z_*)$ is attained at $\gamma_* = [\ 0.0926 - 0.0526i \quad -0.6023 - 0.3396i \quad 0.0890 - 0.0534i\ ]^T$, and that the multiplicity of $\overline{f}_3(z_*)$ at this optimal γ is 3, indeed $\sigma_5\left(\mathcal{A}(z_*, \gamma_*)\right) = \sigma_6\left(\mathcal{A}(z_*, \gamma_*)\right) = \sigma_7\left(\mathcal{A}(z_*, \gamma_*)\right) = 0.7070$. Hence the multiplicity condition at the optimal γ is violated. Moreover all three pairs of singular vectors corresponding to 0.7070 violate the linear independence assumption.

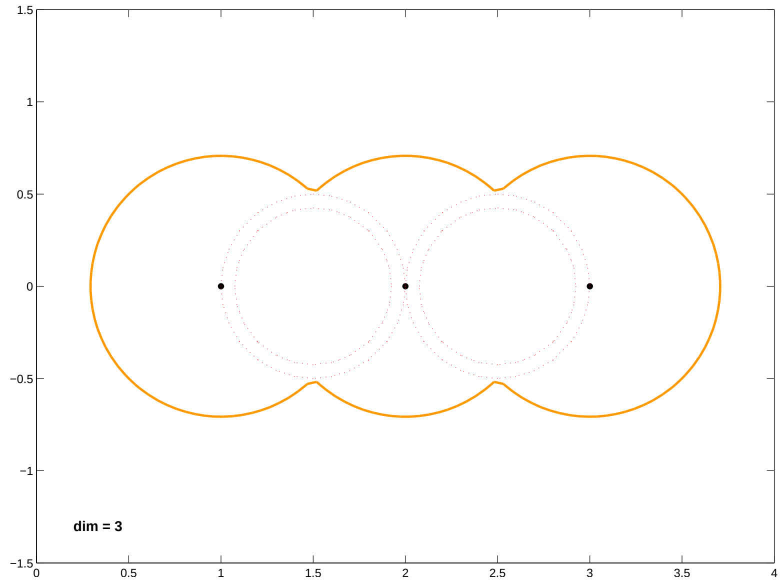

Figure 4.1: The sets $\Lambda_\epsilon(A)$ (*solid curve*) and $\Lambda_{\epsilon,2}(A)$ (*dotted curve*) for $\epsilon = W_3(A) = 0.7070$ are illustrated for a 3×3 diagonal matrix A.

But our main result in the third chapter is still true. So the inequality $W_{r+1}(A) \leq C_r(A)$ may hold regardless of these assumptions.

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