INVERSE PROBLEMS FOR SECOND ORDER PARABOLIC AND HYPERBOLIC EQUATIONS

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
and that any and all required by the final
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To my family

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Abstract

Inverse problems for partial differential equation arise when solving many problems of mathematical physics and engineering. An inverse problem is a problem where a source term or some of coefficients of a partial differential equation modeling the appropriated process is unknown. To find the unknown solution and the unknown coefficient or resource term of such a problem, additional conditions will need to be provided. Such conditions may, for example, include partial information of the unknown fields (e.g., temperature) resulting from sensor (experimental) data at distinct points in the domain and time. In this work, our aim is to study the problem of existence and uniqueness of various inverse problems for second order parabolic and hyperbolic equations. We firstly consider the determination of the initial temperature distribution of heat equations from the final data. We study the problem of identification of the unknown source function of inhomogeneous heat equation. Then we study inverse problems of identification of coefficients in heat equation and wave equation.

Özet

Matematiksel fizik ve mühendislik sorularının birçoğunu çözerken kısmi diferansiyel denklemler için ters problemler ortaya çıkar. Bir ters problem kısmi difersansiyel denklemin kaynak teriminin veya bazı katsayılarının bilinmediği bir problemdir. Böyle bir problemin bilinmeyen çözümünü ve bilinmeyen katsayısını veya kaynak terimini bulmak için ek koşullar sağlanması gerekir. Örneğin bu koşullar tanım bölgesi ve zaman içinde farklı noktalarda sensör (deneysel) veriden çıkan bilinmeyen alanların kısmi bilgilerini içerebilir (örneğin, sıcaklık). Bu çalışmada amacımız ikinci dereceden parabolik ve hiperbolik denklemler için çeşitli ters problemlerin varlık ve teklik sorunlarını incelemektir. Öncelikle ısı denklemlerinde son veriden ilk sıcaklık dağılımının tanımlanması dikkate alınmaktadır. Sonra homojen olmayan ısı denkleminin bilinmeyen kaynak fonksiyonunun tanımlanması problemi incelenmektedir. Son olarak ısı denkleminde ve dalga denkleminde zamana bağlı katsayıların tanımlanmasının ters problemi çalışılmaktadır.

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Introduction

It is well known that second order parabolic and hyperbolic equations are modeling many dynamical processes in continuum mechanics and other fields of mathematical physics. Usually such mathematical models require certain state inputs in the form of initial and boundary data together with inputs such as coefficients or source terms which are related to the physical properties of the system. Proving existence and uniqueness of a solution for the associated problem constitutes solving the direct problem. Solving the direct problem permits the computation of various system outputs of physical interest. On the other hand, when some of the required inputs are not known we may instead be able to determine the missing input from outputs that are measured by formulating and solving an appropriate inverse problem (or an identification problem).. In particular, when the missing input is a coefficient in the partial differential equation, the problem is called a coefficient identification problem and when the source term is missing it is called a source identification problem (see [1],[2],[5]). We point out that the problem of identifying a linear source in parabolic and hyperbolic equations is very important and widely studied in the literature on inverse problems for PDEs.

In this work, our main goal is to study the questions of existence and uniqueness of various inverse problems for parabolic and hyperbolic equations.

In Chapter 1, we give a brief reminder for some common mathematical tools that we will use in the subsequent chapters. These include some calculus facts, various inequalities, functional analysis.

In Chapter 2, we study three inverse problems for the one-dimensional heat equation by reviewing the following book [2]. The aim of the first section is to determine the initial temperature distribution of a solution from certain additional information. The aim of the second section is to determine the unknown source function $F = F(x,t)$ in the heat conduction equation

$$u_t = u_{xx} + F(x,t)$$

from overspecified data.

In the last section, we consider the problem of determining the thermal diffusivity $a(t)$ of a heat equation, that is changing with time by reviewing the following book [1]. A physical example of such a problem arises from heat conduction in a material that is undergoing radioactive decay or damage. The thermal conductivity varies with the degree of decay, which can be related to time. The equation of heat conduction in such a material has the form

$$u_t = a(t)u_{xx}$$

where $a(t) > 0$ is the time-dependent thermal diffusivity coefficient.

In Chapter 3, reviewing the paper [4], we study the problem of an inverse problem for a second order hyperbolic equation. We consider the problem of determining the unknown coefficient $a(t)$ of a wave equation which has the form

$$u_{tt} = u_{xx} + a(t)u + F(x,t).$$

Chapter 1

Preliminaries

This section is a very brief reminder of some mathematical tools for reading the main chapters more comfortable. We only include the tools which we will need in our analysis in the main sections. Most results are given without proof since the proofs can be found in many sources. We may only give the proofs of results which have particular interest in our analysis.

1.1 Calculus

Theorem 1.1.1. ***(Fubini's Theorem)*** *If $f(x,y)$ is continuous on the rectangular region $R : a \leq x \leq b,\ c \leq y \leq d$, then the equality*

$$\int\int_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

holds.

Definition 1.1.1. The Fourier transform of an integrable function $f(x)$ is

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx.$$

Theorem 1.1.2. *Let f and g be two functions in $L_2(-\infty,\infty)$. We define their convolution to be*

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi,$$

and its Fourier transform is given by

$$\mathcal{F}[f * g](\xi) = \sqrt{2\pi}\hat{f}(\xi)\hat{g}(\xi).$$

Definition 1.1.2. The Fourier series of a periodic function $f(x)$ with period 2π is defined as the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{1.1.1}$$

where the coefficients $a_n,\ b_n$ are defined as

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx,\ \ n = 1, 2, \ldots, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx,\ \ n = 1, 2, \ldots. \end{aligned}$$

Note: The sine series defined by

$$\sum_{n=1}^{\infty} b_n \sin nx$$

and the cosine series defined by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

are special instances of Fourier series.

Theorem 1.1.3. ***(Parseval's Identity)*** *For* $f \in L_2[-\pi, \pi]$ *with Fourier series* (1.1.1), *we have*

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(t)|^2 dt = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2).$$

1.2 Functional Analysis

Definition 1.2.1. A sequence $\{x_n\}$ in a metric space X is said to converge if there is a point $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.

Definition 1.2.2. A sequence $\{x_n\}$ in a metric space X is sad to be a Cauchy sequence if for every $\varepsilon > 0$ there is a positive integer N such that for all $n, m \geq N$ we have $d(x_n, x_m) < \varepsilon$.

Definition 1.2.3. A metric space in which every Cauchy sequence converges is said to be complete.

Definition 1.2.4. Let f_n be a sequence of functions defined on a set E.
We say that f_n is pointwise bounded on E is the sequence $\{f_n(x)\}$ is bounded for every $x \in E$, that is, if there exits a finite-valued function ϕ defined on E such that

$$|f_n(x)| < \phi(x) \quad (x \in E,\ n = 1, 2, 3, \ldots).$$

We say that $\{f_n\}$ is uniformly bounded on E if there exits a positive number M such that

$$|f_n(x)| < M \quad (x \in E,\ n = 1, 2, 3, \ldots).$$

Theorem 1.2.1. *If $\{f_n\}$ is a pointwise bounded sequence of functions on a countable set E, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.*

Definition 1.2.5. A family $\mathfrak{F}$ of complex functions f defined on a set E in a metric space (X, d) is said to be equicontinuous on E if for every $\epsilon > 0$, there exits a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

whenever $d(x, y) < \delta,\ x \in E,\ y \in E$, and $f \in \mathfrak{F}$.

Theorem 1.2.2. ***(Ascoli-Arzela)*** *If $\{f_n\}$ is a uniformly bounded and equicontinuous sequence of functions defined on a compact set K, then $\{f_n\}$ contains a uniformly convergent subsequence.*

Theorem 1.2.3. ***(Dini's Theorem)*** *Let K be a compact space. Let $f : K \to \mathbb{R}$ be a continuous function and $f_n : K \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of continuous functions. If $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to f and if*

$$f_{n+1}(x) \geq f_n(x) \quad \text{for all } x \in K \text{ and all } n \in \mathbb{N}$$

then $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f.

Proposition 1.2.4. *Let X be a complete metric space, and let $Y \subseteq X$ be a closed subset of X. Then Y is complete.*

Definition 1.2.6. $C[a,b]$ is the Banach space of all continuous functions on $[a,b]$ equipped with the norm

$$\|u\|_{\mathcal{C}[a,b]} = \max_{x\in[a,b]} |u(x)|.$$

Definition 1.2.7. The linear space $L_p[a,b],\ p \geq 1$ of all functions continuous on $[a,b]$ is a normed space with the norm

$$\|u\|_{L_p[a,b]} = \left(\int_a^b |u(x)|^p \right)^{1/p}.$$

Definition 1.2.8. We denote by $\mathcal{B}_2^\alpha$ the set of functions of the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin kx,\ 0 < x < \pi,\ 0 < t < T,$$

where $u_k(t),\ k = 1, 2, 3 \ldots$ are continuous on $[0,T]$ and satisfy

$$\sum_{k=1}^{\infty} \left[k^\alpha \max_{0\leq t\leq T} |u_k(t)| \right]^2 < \infty,\ \alpha \geq 0.$$

In $\mathcal{B}_2^\alpha$, we define the norm

$$\|u\|_{\mathcal{B}_2^\alpha(Q)} = \left(\sum_{k=1}^{\infty} \left[k^\alpha \max_{0\leq t\leq T} |u_k(t)| \right]^2 \right)^{1/2} \tag{1.2.1}$$

where $\alpha \geq 0$ and $Q = [0,\pi] \times [0,T]$.

Lemma 1.2.5. *$\mathcal{B}_2^\alpha(Q)$ is a Banach space with respect to the given norm* (1.2.1).

Proof. Let $u^{(n)} \in \mathcal{B}_2^\alpha(Q)$ be a Cauchy sequence. Then, given $\varepsilon > 0$, there is $K > 0$ such that for all $n, m \geq K$

$$\left\| u_k^{(n)} - u_k^{(m)} \right\|_{\mathcal{C}[0,T]}^2 = \left[\max_{0\leq t\leq T} |u_k^{(n)}(t) - u_k^{(m)}(t)| \right]^2$$
$$\leq \sum_{k=1}^{\infty} \left[k^\alpha \max_{0\leq t\leq T} |u_k^{(n)}(t) - u_k^{(m)}(t)| \right]^2 = \left\| u^{(n)} - u^{(m)} \right\|_{\mathcal{B}_2^\alpha(Q)}^2 < \frac{\varepsilon}{2}.$$

Hence for every $k \geq 1$, the sequence $\{u_k^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}[0,T]$ and since $\mathcal{C}[0,T]$ is complete, the sequence $\{u_k^{(n)}\}_{n\in\mathbb{N}}$ converges to some u_k.

Let us show that

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin kx$$

is the limit in $\mathcal{B}_2^\alpha(Q)$ of the sequence $\{u^{(n)}\}_{n\in\mathbb{N}}$. To see this we first show that $u \in \mathcal{B}_2^\alpha(Q)$. Since $\{u^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}_2^\alpha(Q)$, we have

$$\sum_{k=1}^{N} \left[k^\alpha \max_{0\leq t\leq T} |u_k^{(n)}(t)| \right]^2 \leq \sum_{k=1}^{N} \left[k^\alpha \max_{0\leq t\leq T} |u_k^{(n)}(t) - u_k^{(K)}(t)| + k^\alpha \max_{0\leq t\leq T} |u_k^{(K)}(t)| \right]^2$$
$$\leq 2 \sum_{k=1}^{N} \left[k^\alpha \max_{0\leq t\leq T} |u_k^{(n)}(t) - u_k^{(K)}(t)| \right]^2 + 2 \sum_{k=1}^{N} \left[k^\alpha \max_{0\leq t\leq T} |u_k^{(K)}(t)| \right]^2$$
$$\leq 2 \left\| u^{(n)} - u^{(K)} \right\|_{\mathcal{B}_2^\alpha(Q)}^2 + 2 \left\| u^{(K)} \right\|_{\mathcal{B}_2^\alpha(Q)} \leq \varepsilon + 2 \left\| u^{(K)} \right\|_{\mathcal{B}_2^\alpha(Q)}$$

for every $N \geq 1$. Fixing N and taking limit as $n \to \infty$ we get

$$\sum_{k=1}^{N} \left[k^\alpha \max_{0\leq t\leq T} |u_k(t)| \right]^2 \leq \varepsilon + 2 \left\| u^{(K)} \right\|_{\mathcal{B}_2^\alpha(Q)}$$

and taking limit as $N \to \infty$ we get

$$\sum_{k=1}^{\infty} \left[k^\alpha \max_{0\leq t\leq T} |u_k(t)| \right]^2 \leq \varepsilon + 2 \left\| u^{(K)} \right\|_{\mathcal{B}_2^\alpha(Q)} < \infty.$$

So, $u \in \mathcal{B}_2^\alpha(Q)$. Next, we show that $\left\| u^{(n)} - u \right\|_{\mathcal{B}_2^\alpha(Q)} \to 0$ as $n \to \infty$. Since $\{u^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}_2^\alpha(Q)$, for every $N \geq 1$ we have

$$\sum_{k=1}^{N} \left[k^\alpha \max_{0\leq t\leq T} |u_k^{(n)}(t) - u_k^{(m)}(t)| \right]^2 \leq \left\| u^{(n)} - u^{(m)} \right\|_{\mathcal{B}_2^\alpha(Q)}^2 < \frac{\varepsilon}{2}.$$

With $n > K$ and N fixed, we let $m \to \infty$ to find that

$$\sum_{k=1}^{N} \left[k^{\alpha} \max_{0 \le t \le T} |u_k^{(n)}(t) - u_k(t)| \right]^2 \le \frac{\varepsilon}{2}.$$

Since this is true for every N,

$$\left\| u^{(n)} - u \right\|^2_{\mathcal{B}_2^{\alpha}(Q)} = \sum_{k=1}^{\infty} \left[k^{\alpha} \max_{0 \le t \le T} |u_k^{(n)}(t) - u_k(t)| \right]^2 \le \frac{\varepsilon}{2}$$

for $n > K$. Hence $u^{(n)} \to u$ and since $u \in \mathcal{B}_2^{\alpha}(Q)$, the space $\mathcal{B}_2^{\alpha}(Q)$ is complete normed space. □

Definition 1.2.9. Let (X, d) be a metric space. A mapping $T : X \to X$ is a contraction mapping, or contraction, if there exists a constant $c \in (0, 1)$ such that

$$d(T(x), T(y)) \le c d(x, y)$$

for all $x, y \in X$.

Theorem 1.2.6. ***(Contraction Mapping)*** *If $T : X \to X$ is a contraction mapping on a complete metric space (X, d), then the equation*

$$T(x) = x$$

has a unique solution $x \in X$. Such a solution is said to be a fixed point of T.

Theorem 1.2.7. *If X is a complete metric space and $f : X \to X$ is a mapping such that some iterate $f^N : X \to X$ is a contraction, then f has a unique fixed point.*

Proof. By contraction mapping theorem, f^N has a unique fixed point. Call it a, so $f^N(a) = a$. To show a is the only possible fixed point of f, observe that a fixed of f is a fixed point of f^N, thus must be a. To show a really is a fixed point of f, we note that $f(a) = f(f^N(a)) = f^N(f(a))$, so $f(a)$ is a fixed point of f^N. Therefore $f(a)$ and a are both fixed points of f^N. Since f^N has a unique fixed point, $f(a) = a$. □

Proposition 1.2.8. *Suppose that $K(x,y)$ is continuous on $[0,1]\times[0,1]$. Then the Volterra integral equation of the 2^{nd} kind*

$$\phi(x) + \int_0^x K(x,y)\phi(y)dy = f(x) \tag{1.2.2}$$

has a unique solution $\phi(x) \in \mathcal{C}[0,1]$ for any $f(x) \in \mathcal{C}[0,1]$.

Proof. Define the operator $T : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ by

$$T\phi = f(x) - \int_0^x K(x,y)\phi(y)dy$$

If $T\phi$ has a fixed point, such a fixed point must be a solution of (1.2.2). To show that such a fixed point exists we will show that T^n, for some n, will be a contraction operator. By theorem 1.2.7, T will then have a unique fixed point.

Define

$$Kf(x) = \int_0^x K(x,y)f(y)dy.$$

Then

$$\begin{aligned} K^2 f(x) &= K\left[\int_0^x K(x,y)f(y)dy\right] \\ &= \int_0^x K(x,z)\int_0^z K(z,y)f(y)dydz \\ &= \int_0^x \left[\int_y^x K(x,y)K(z,y)dz\right] f(y)dy \end{aligned}$$

It follows that K^2 is an integral operator, whose kernel is given by

$$\int_y^x K(x,y)K(z,y)dz$$

More generally it is easy to show that

$$K^n f(x) = \int_0^x K_n(x,y)f(y)dy$$

where $K_n(x,y)$ can be defined recursively by

$$\begin{aligned} K_n(x,y) &= \int_x^y K(x,z)K_{n-1}(z,y)dz, \quad n = 2,3,\ldots \\ K_1(x,y) &= K(x,y) \end{aligned}$$

Let us consider the operator

$$T\phi = f - K\phi$$

It is clear that

$$\begin{aligned} T^2\phi &= T[f - K\phi] = f - Kf + K^2\phi \\ &\vdots \\ T^n\phi &= f - Kf + K^2 f - K^3 f + \cdots (-1)^{n-1}K^{n-1}f + (-1)^n K^n\phi \end{aligned}$$

so that

$$\begin{aligned} \|T^n\phi_1 - T^n\phi_2\|_{\mathcal{C}[0,1]} &= \|K^n\phi_1 - K^n\phi_2\|_{\mathcal{C}[0,1]} \\ &= \left\| \int_0^x K_n(x,y)\,(\phi_1(y) - \phi_2(y))\,dy \right\|_{\mathcal{C}[0,1]} \end{aligned}$$

The kernel $K(x,y)$ is continuous on $[0,1]\times[0,1]$, and therefore uniformly bounded, say $|K(x,y)| \leq M$. Then, one can show by induction that

$$|K_n(x,y)| \leq \frac{M^n(x-y)^{n-1}}{(n-1)!},\ \ 0 \leq y \leq x$$

For $n = 1$, the above is obviously valid. If it is true for n, then

$$\begin{aligned} |K_{n+1}(x,y)| &\leq \int_y^x |K(x,y)||K_n(z,y)|dz \\ &\leq \frac{M^{n+1}(x-y)^n}{n!} \end{aligned}$$

We have, therefore,

$$\begin{aligned} \|T^n\phi_1 - T^n\phi_2\|_{\mathcal{C}[0,1]} &\leq \frac{M^n}{(n-1)!} \left\| \int_0^x (\phi_1(y) - \phi_2(y))dy \right\|_{\mathcal{C}[0,1]} \\ &\leq \frac{M^n}{(n-1)!} \|\phi_1(y) - \phi_2(y)\|_{\mathcal{C}[0,1]} \end{aligned}$$

For n sufficiently large

$$\frac{M^n}{(n-1)!} < 1$$

so that T^n is a contraction operator. Since $\mathcal{C}[0,1]$ is complete, T has a unique fixed point. □

Theorem 1.2.9. ***(Lebesgue Dominated-Convergence Theorem)*** *Let* $\{f_n\}$ *denote a sequence of integrable functions on* $[a, b]$ *such that* $f(x) = \lim_{b\to\infty} f_n(x)$. *Suppose that there exits a positive-valued integrable function* g *such that* $|f_n(x)| \leq g(x)$ *for all* $x \in [a, b]$ *and all* $n = 1, 2, \ldots$. *Then the limit function* $f(x)$ *is integrable and*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx.$$

1.3 Inequalities

Lemma 1.3.1. ***(Gronwall)*** *Let x and $\Psi \geq 0$ be real-valued continuous functions defined on $[a,b]$, and C be a constant. We suppose that on $[a,b]$ we have the inequality*

$$x(t) \leq C + \int_a^t \Psi(\tau)x(\tau)d\tau.$$

Then

$$x(t) \leq Ce^{\int_a^t \Psi(\tau)d\tau}.$$

Lemma 1.3.2. *Let $\varphi(t)$ be real-valued continuous function satisfying*

$$0 \leq \varphi(t) \leq C \int_0^t \frac{\varphi(t)}{(t-\tau)^{1/2}} d\tau, \ 0 \leq t \leq T, \tag{1.3.1}$$

where $C \geq 0$. Then

$$\varphi(t) = 0.$$

Proof. Let $z > t$. Multiplying both sides of (1.3.1) by $1/(z-t)^{1/2}$ and integrating both sides from 0 to z, we obtain

$$\int_0^z \frac{\varphi(t)}{(z-t)^{1/2}} \leq C \int_0^z \int_0^t \frac{\varphi(\tau)}{(z-t)^{1/2}(t-\tau)^{1/2}} d\tau.dt$$

If we interchange the order of integration on the left, we obtain

$$\int_0^z \frac{\varphi(t)}{(z-t)^{1/2}} dt \leq C \int_0^z \left[\int_\tau^z \frac{dt}{(z-t)^{1/2}(t-\tau)^{1/2}} \right] \varphi(\tau)d\tau. \tag{1.3.2}$$

Letting $t = \tau + (z-\tau)u$ we see that

$$\begin{aligned} \int_\tau^z \frac{dt}{(z-t)^{1/2}(t-\tau)^{1/2}} &= \int_0^1 \frac{(z-\tau)du}{(z-\tau)^{1/2}(1-u)^{1/2}(z-\tau)^{1/2}u^{1/2}} \\ &= \int_0^1 \frac{du}{(1-u)^{1/2}u^{1/2}}. \end{aligned}$$

Letting $u = v^2$ we see that

$$\int_0^1 \frac{du}{(1-u)^{1/2}u^{1/2}} = \int_0^1 \frac{2vdv}{\sqrt{1-v^2}v} = 2\int_0^1 \frac{dv}{\sqrt{1-v^2}} = 2\arcsin v\Big|_0^1 = \pi.$$

Then, from (1.3.2), we have

$$\int_0^z \frac{\varphi(t)}{(z-t)^{1/2}}dt \le C\pi \int_0^z \varphi(\tau)d\tau. \tag{1.3.3}$$

By using the inequality (1.3.1), left-hand side of (1.3.3) is greater than or equal to $\varphi(z)/C$. Therefore, we have

$$0 \le \varphi(z) \le \pi C^2 \int_0^z \varphi(\tau)d\tau,\ 0 \le z \le T.$$

Gronwall's lemma yields the result. □

Lemma 1.3.3. *Let y be a nonnegative continuous function and*

$$y(t) \le C_1 + C_2 \int_0^t y^2(\tau)d\tau,\ 0 \le t \le T$$

where C_1 and C_2 are some positive numbers and

$$C_1C_2T < 1.$$

Then

$$y(t) \le \frac{C_1}{1 - C_1C_2T}.$$

Proof. Set

$$v(t) = C_1 + C_2 \int_0^t y^2(\tau)d\tau.$$

Then, we have

$$v' = C_2y^2,\ v(0) = C_1,\ \text{and } y \le v.$$

From the last equality and inequality, we obtain

$$v' = C_2y^2 \le C_2v^2,$$

or

$$v'v^{-2} \le C_2.$$

Integrating the last inequality from 0 to t, we obtain

$$\int_0^t v'v^{-2}d\tau \leq \int_0^t C_2 d\tau. \qquad (1.3.4)$$

As $v'v^{-2} = [-v^{-1}]'$,

$$\int_0^t v'v^{-2}d\tau = \int_0^t [-v^{-1}]'d\tau = v^{-1}(0) - v^{-1}(t).$$

Then, from (1.3.4), we have

$$v^{-1}(0) - v^{-1}(t) \leq C_2 t \leq C_2 T.$$

By using the initial point $v(0) = C_1$, we obtain

$$v(t) \leq \frac{C_1}{1 - C_1 C_2 T}, \ 0 \leq t \leq T.$$

As $y \leq v$,

$$y(t) \leq \frac{C_1}{1 - C_1 C_2 T}, \ 0 \leq t \leq T.$$

□

Theorem 1.3.4. ***(Hölder's Inequality)*** *Let $p > 1$, $1/p + 1/q = 1$, $f(x)$ and $g(x)$ be continuous real-valued functions on $[a, b]$. Then, the Hölder's inequality for integrals states that*

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} \left(\int_a^b |g(x)|^q dx\right)^{1/q}.$$

Similarly, Hölder's inequality for sums states that

$$\sum_{k=1}^n |a_k b_k| \leq \left(\sum_{k=1}^n |a_k|^p\right)^{1/p} \left(\sum_{k=1}^n |b_k|^q\right)^{1/q}.$$

Corollary 1.3.5.

$$\left(\sum_{i=1}^n b_i\right)^2 \leq n \sum_{i=1}^n b_i^2$$

Proof. By using the Hölder's inequality for sums, we get

$$\left(\sum_{i=1}^n b_i\right)^2 = \left(\sum_{i=1}^n 1 \cdot b_i\right)^2 \leq \left[\left(\sum_{i=1}^n 1^2\right)^{1/2} \left(\sum_{i=1}^n b_i^2\right)^{1/2}\right]^2 = n \sum_{i=1}^n b_i^2$$

□

Chapter 2

Inverse Problem for Parabolic Equations

2.1 Determination of the Initial Temperature Distribution

2.1.1 Heat Equation on the Real Line

It is well-known that the Cauchy problem for the heat equation

$$u_t = a^2 u_{xx}, \ -\infty < x < \infty, \ t > 0; \tag{2.1.1}$$

$$u(x,0) = \varphi(x), \ -\infty < x < \infty. \tag{2.1.2}$$

is well-posed problem and the solution of this problem has the form

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi a^2 t}} e^{\frac{-(x-\xi)^2}{4a^2 t}} \varphi(\xi) d\xi. \tag{2.1.3}$$

Our aim in this section is to consider the inverse problem for the heat equation, i.e., find the function $\varphi(x)$ in (2.1.2) provided $u(x,T) = g(x), \ T > 0$ is given. That is we are going to solve the problem of finding solution of (2.1.1) at $t = 0$ under the condition

$$u(x,T) = g(x), \ -\infty < x < \infty. \tag{2.1.4}$$

Setting in (2.1.3) $t = T$ and using (2.1.4), we obtain

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi a^2 T}} e^{\frac{-(x-\xi)^2}{4a^2 T}} \varphi(\xi) d\xi = g(x), \ -\infty < x < \infty. \tag{2.1.5}$$

Firstly, we shall consider the existence of solution of equation (2.1.5). We have an integral equation

$$\int_{-\infty}^{\infty} K(x-\xi)\varphi(\xi)d\xi = g(x), \ -\infty < x < \infty, \tag{2.1.6}$$

where

$$K(x) = \frac{1}{\sqrt{4\pi a^2 T}} e^{\frac{-x^2}{4a^2T}}.$$

If the function $g(x) \in L_2(-\infty, \infty)$, then we can take the Fourier transform of both sides of the equation (2.1.6). Then we obtain

$$\sqrt{2\pi}\mathcal{F}[K]\mathcal{F}[\varphi] = \mathcal{F}[g],$$

and

$$\mathcal{F}[\varphi] = \mathcal{F}[g]e^{a^2\xi^2T}.$$

If the right side of the equation (2.1.1) is in $L_2(-\infty, \infty)$, we finally obtain

$$\varphi = \mathcal{F}^{-1}\left(\mathcal{F}[g]e^{a^2\xi^2T}\right).$$

Now, we shall prove that equation (2.1.5) has a unique solution in the space $L_2[-\infty, \infty]$. Assume that it has two different solution φ_1, φ_2 in $L_2[-\infty, \infty]$. Then, we have

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi a^2 T}} e^{\frac{-(x-y)^2}{4a^2T}} \varphi_i(y)dy = g(x), \ i = 1, 2.$$

Subtracting these two equations from each other, we obtain

$$\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4a^2T}} \phi(y)dy = 0, \ -\infty < x < \infty, \tag{2.1.7}$$

where $\phi = \varphi_1 - \varphi_2$. To establish the uniqueness, we need to show that the homogeneous equation (2.1.7) has only the trivial solution. Differentiating (2.1.7) with respect to x, we obtain

$$\int_{-\infty}^{\infty} (x-y)e^{\frac{-(x-y)^2}{4a^2T}} \phi(y)dy = 0, \ -\infty < x < \infty. \tag{2.1.8}$$

Multiplying (2.1.7) by x and using equation (2.1.8), we obtain

$$\int_{-\infty}^{\infty} ye^{\frac{-(x-y)^2}{4a^2T}}\phi(y)dy = 0, \quad -\infty < x < \infty. \tag{2.1.9}$$

Differentiating (2.1.9) with respect to x, we obtain

$$\int_{-\infty}^{\infty} y(x-y)e^{\frac{-(x-y)^2}{4a^2T}}\phi(y)dy = 0, \quad -\infty < x < \infty. \tag{2.1.10}$$

Multiplying (2.1.9) by x and using equation (2.1.10), we obtain

$$\int_{-\infty}^{\infty} y^2e^{\frac{-(x-y)^2}{4a^2T}}\phi(y)dy = 0, \quad -\infty < x < \infty.$$

Continuing this process, we obtain for all $n = 1, 2, \cdots$

$$\int_{-\infty}^{\infty} y^ne^{\frac{-(x-y)^2}{4a^2T}}\phi(y)dy = 0, \quad -\infty < x < \infty$$

Consider

$$g(z) = \int_{-\infty}^{\infty} e^{\left(iyz-\frac{(x-y)^2}{4a^2T}\right)}\phi(y)dy = 0, \quad -\infty < x < \infty$$

is defined and analytic on the complex plane since for each $R > 0$ the function $|\phi(y)|e^{\left(R|y|-\frac{(x-y)^2}{4a^2T}\right)}$ is integrable over $\mathbb{R}$. Then, from the equation (2.1.1), we obtain

$$g(0) = 0 \text{ and } g^{(n)}(0) = 0, \ n = 1, 2, \ldots.$$

So, $g \equiv 0$. Then an integrable function $\phi(y)e^{-\frac{(x-y)^2}{4a^2T}}$ has zero Fourier transform. Therefore, $\phi = 0$ a.e. on $L_2[-\infty, \infty]$, i.e., $\varphi_1(x) = \varphi_2(x)$. Thus, uniqueness of solution for equation (2.1.5) is established.

2.1.2 Heat Equation in a Finite Interval

Consider the following boundary-value problem

$$u_t = a^2 u_{xx},\ 0 < x < \pi,\ 0 < t < T; \tag{2.1.11}$$

$$u(0,t) = u(\pi,t) = 0,\ t \le t \le T; \tag{2.1.12}$$

$$u(x,0) = \varphi(x),\ 0 \le x \le \pi. \tag{2.1.13}$$

It is well-known that the solution of this problem has the form

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{\pi} \int_0^{\pi} \varphi(\xi)\sin(n\xi)d\xi e^{-n^2a^2t}\sin(nx). \tag{2.1.14}$$

Our aim in this section is to consider the inverse problem for the heat equation, i.e., find the function $\varphi(x)$ in (2.1.13) provided $u(x,T) = g(x)$ is given. That is we are going to solve the problem of finding solution of (2.1.11) at $t = 0$ under the condition

$$u(x,T) = g(x),\ 0 \le x \le \pi. \tag{2.1.15}$$

Setting in (2.1.14) $t = T$ and taking into account (2.1.15), we obtain

$$\sum_{n=1}^{\infty} \frac{2}{\pi} \int_0^{\pi} \varphi(\xi)\sin(n\xi)d\xi e^{-n^2a^2T}\sin(nx) = g(x),\ 0 \le x \le \pi. \tag{2.1.16}$$

Thus, the inverse problem is reduced to equation (2.1.16) for the unknown function $\varphi(x)$.

Firstly, we shall show that the equation (2.1.16) has a unique solution in the space $L_2[0,\pi]$. Assume that it has two different solutions φ_1, φ_2 in $L_2[0,\pi]$. Then, we have

$$\sum_{n=1}^{\infty} \frac{2}{\pi} \int_0^{\pi} \varphi_i(\xi)\sin(n\xi)d\xi e^{-n^2a^2T}\sin(nx) = g(x),\ i = 1,2.$$

Subtracting these two equations from each other, we obtain

$$\sum_{n=1}^{\infty} \frac{2}{\pi} \int_0^{\pi} \phi(\xi)\sin(n\xi)d\xi e^{-n^2a^2T}\sin(nx) = 0,\ 0 \le x \le \pi, \tag{2.1.17}$$

where $\phi = \varphi_1 - \varphi_2$. As the system of functions $\{\sqrt{\frac{2}{\pi}}\sin(nx)\}_{n\geq 1}$ is orthonormal in $L_2[0,\pi]$, and

$$e^{-n^2a^2T} \neq 0 \text{ for } n = 1, 2, \cdots,$$

then, multiplying (2.1.17) by $\sin(kx)$ and integrating it from 0 to π, we obtain

$$\int_0^\pi \phi(\xi)\sin(k\xi)d\xi = 0, \;\; k = 1, 2, \cdots.$$

Therefore, it follows that $\phi(x) = 0$, i.e., $\varphi_1(x) = \varphi_2(x)$ on $L_2[0,\pi]$. Thus, uniqueness of solution of the integral equation (2.1.16) is established.

Now we shall consider the problem of existence of solution of the integral equation (2.1.16). Let the equation (2.1.16) with $g(x) \in L_2[0,\pi]$ have a solution $\varphi(x) \in L_2[0,\pi]$. As the system of functions $\{\sqrt{\frac{2}{\pi}}\sin(nx)\}_{n\geq 1}$ is orthonormal in $L_2[0,\pi]$, multiplying (2.1.16) by $\frac{2}{\pi}\sin(kx)$ and integrating form 0 to π, we obtain, for $k = 1, 2, \cdots$,

$$\frac{2}{\pi}\int_0^\pi \varphi(\xi)\sin(k\xi)d\xi e^{-k^2a^2T} = \frac{2}{\pi}\int_0^\pi g(\xi)\sin(k\xi)d\xi. \tag{2.1.18}$$

Let

$$\varphi_k = \frac{2}{\pi}\int_0^\pi \varphi(\xi)\sin(k\xi)d\xi, \;\; g_k = \frac{2}{\pi}\int_0^\pi g(\xi)\sin(k\xi)d\xi$$

denote the Fourier coefficients of the functions $\varphi(x)$ and $g(x)$, respectively. Then, by using the equation (2.1.18), we have

$$\varphi_k = g_k e^{k^2a^2T}, \;\; k = 1, 2, \cdots. \tag{2.1.19}$$

By using the Parseval's identity and the equation (2.1.19), we have

$$\|\varphi\|^2_{L_2[0,\pi]} = \frac{\pi}{2}\sum_{n=1}^{\infty}\varphi_n^2 = \frac{\pi}{2}\sum_{n=1}^{\infty} g_n^2 e^{2n^2a^2T}. \tag{2.1.20}$$

Thus, in order that (2.1.16) has a solution in $L_2[0,\pi]$, it is necessary that the function $g(x) \in L_2[0,\pi]$ provides a converging series in the right-hand side of (2.1.20). As the terms of this series have the fast growing multiplier

$$e^{2n^2a^2T},$$

then the converging of the series imposes a heavy restriction on the decrease of the Fourier coefficients g_n of the function $g(x)$. As an example, we consider the function

$$\overline{g}(x) = \sum_{n=1}^{\infty} e^{-n} \sin(nx), \ \ 0 \leq x \leq \pi.$$

Firstly, we will show that $\overline{g} \in L_2[0,\pi]$. It is clear that

$$\begin{aligned}\int_0^\pi \overline{g}^2(x)dx &= \int_0^\pi \left(\sum_{n=1}^{\infty} e^{-n}\sin(nx)\right)^2 dx \\ &\leq \int_0^\pi \left(\sum_{n=1}^{\infty} \left[\frac{1}{e}\right]^n\right)^2 dx = \int_0^\pi \left(\frac{1}{1-\frac{1}{e}} - 1\right)^2 dx \\ &= \int_0^\pi \left(\frac{1}{1-e}\right)^2 dx < \infty.\end{aligned}$$

Therefore, $\overline{g} \in L_2[0,\pi]$. As the system of functions $\{\sqrt{\frac{2}{\pi}}\sin(nx)\}_{n\geq 1}$ is orthonormal, then the Fourier coefficients of $\overline{g}(x)$ are

$$\overline{g}_k = \frac{2}{\pi}\int_0^\pi \sum_{n=1}^{\infty} e^{-n}\sin(nx)\sin(kx)dx = e^{-k}, \ \ k = 1, 2, \cdots .$$

Thus, by using the equation (2.1.20), we obtain

$$\begin{aligned}\|\varphi\|^2_{L_2[0,\pi]} &= \frac{\pi}{2}\sum_{n=1}^{\infty} \overline{g}_n^2 e^{2n^2a^2T} \\ &= \sum_{n=1}^{\infty} e^{2n^2a^2T-2n}.\end{aligned} \tag{2.1.21}$$

But, the series on the right-hand side of (2.1.21) diverges, since

$$e^{2n^2a^2T-2n} \longrightarrow \infty \ \ \text{as} \ \ n \to \infty.$$

Therefore, the equation (2.1.16) has no solution for the function $\overline{g}(x)$.

Now we prove the uniqueness theorem by using the energy method. Suppose that $u_1(x,t)$ is a classical solution of the problem (2.1.11)-(2.1.13). Assume that the problem has not a unique solution, i.e. there exits another solution $u_2(x,t)$ of the problem (2.1.11)-(2.1.13).

Let us consider the function $u(x,t) = u_1(x,t) - u_2(x,t)$. Since the equation (2.1.11) is a linear homogeneous equation, the function $u(x,t)$ is also the solution of the problem (2.1.11)-(2.1.13).

Let us consider the following function

$$\psi(t) = \int_0^\pi u^2(x,t)dx. \tag{2.1.22}$$

If $\psi(t) = 0$ for all $t \in [0,T]$, then the solution of the problem is unique. If not, there exits an interval $(t_1,t_2) \in [0,T]$ such that $\psi(t) > 0,\ \forall t \in [t_1,t_2)$ and $\psi(t_2) = 0$.
Differentiating the function $\psi(t)$, we obtain

$$\psi'(t) = 2\int_0^\pi u(x,t)u_t(x,t)dx. \tag{2.1.23}$$

By using the equation (2.1.11), we obtain

$$\psi'(t) = 2a^2\int_0^\pi u(x,t)u_{xx}(x,t)dx = -2a^2\int_0^\pi u_x^2(x,t)dx.$$

From this equality, we obtain

$$\begin{aligned}\psi''(t) &= -4a^2\int_0^\pi u_x(x,t)u_{xt}(x,t)dx \\ &= 4a^2\int_0^\pi u_t(x,t)u_{xx}(x,t)dx = 4\int_0^\pi u_t^2(x,t)dx\end{aligned} \tag{2.1.24}$$

Now, we consider the function $h(t) = \ln(\psi(t))$. By using the equalities (2.1.23),(2.1.24) twice differentiating the function $h(t)$, we obtain

$$\begin{aligned}h''(t) &= \frac{d}{dt}\frac{\psi'(t)}{\psi(t)} = \frac{1}{\psi^2(t)}\left[\psi''(t)\psi(t) - \big(\psi'(t)\big)^2\right] \\ &= \frac{1}{\psi^2(t)}\left[4\int_0^\pi \big(u_t(x,t)\big)^2dx\int_0^\pi \big(u(x,t)\big)^2dx - 4\left(\int_0^\pi u(x,t)u_t(x,t)dx\right)^2\right].\end{aligned}$$

By using the Cauchy-Schwartz inequality, we obtain that

$$h''(t) \geq 0 \text{ for all } t \in [t_1,t_2).$$

So, the function $h(t)$ is a convex function on the interval $[t_1, t_2)$. Therefore, for each $t \in [t_1, t_2)$ and for each $\tau \in (0, 1)$ we have

$$h((1-\tau)t_1 + \tau t) \leq (1-\tau)h(t_1) + \tau h(t).$$

That is

$$\ln \psi((1-\tau)t_1 + \tau t) \leq (1-\tau)\ln \psi(t_1) + \tau \ln \psi(t).$$

or

$$\psi((1-\tau)t_1 + \tau t) \leq [\psi(t_1)]^{1-\tau}[\psi(t)]^{\tau}.$$

Passing to the limit as $t \to t_2^-$ we get

$$\psi((1-\tau)t_1 + \tau t_2) \leq [\psi(t_1)]^{1-\tau}[\psi(t_2)]^{\tau}$$

Since $\psi(t_2) = 0$, the last inequality implies that $\psi(t) = 0$ for each $t \in [t_1, t_2]$. This contradiction shows that solution of the problem is unique.

2.2 Determination of the Unknown Source Function $F(x,t)$

2.2.1 Inhomogeneous Heat Equation in a Finite Interval

Now, we consider the following inhomogeneous heat conductivity equation

$$u_t = a^2 u_{xx} + f(x)g(t),\ 0 < x < \pi,\ 0 < t < T; \tag{2.2.1}$$

$$u_x(0,t) = u_x(\pi,t) = 0,\ 0 \le t \le T; \tag{2.2.2}$$

$$u(x,0) = 0,\ 0 \le x \le \pi. \tag{2.2.3}$$

The solution of this problem may be obtained by

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t)\cos(nx). \tag{2.2.4}$$

It is clear that this series satisfies the boundary conditions (2.2.2). Setting the series (2.2.4) into the equation (2.2.1), we obtain

$$\sum_{n=0}^{\infty} T_n'(t)\cos(nx) = -a^2\sum_{n=0}^{\infty} n^2 T_n(t)\cos(nx) + f(x)g(t),$$

or

$$\sum_{n=0}^{\infty}[T_n'(t) + a^2n^2T_n(t)]\cos(nx) = f(x)g(t). \tag{2.2.5}$$

Expanding $f(x)$, we have

$$f(x) = \frac{f_0}{2} + \sum_{n=1}^{\infty} f_n\cos(nx), \tag{2.2.6}$$

where

$$f_n = \frac{2}{\pi}\int_0^{\pi} f(\xi)\cos(n\xi)d\xi,\ n = 0, 1, \cdots$$

From (2.2.5) and (2.2.6), we obtain

$$\left[T_0'(t) - \frac{f_0 g(t)}{2}\right] + \sum_{n=1}^{\infty}[T_n'(t) + a^2n^2T_n(t) - f_n g(t)]\cos(nx) = 0.$$

This equality holds if and only if

$$T_0'(t) = \frac{f_0 g(t)}{2} \tag{2.2.7}$$

and

$$T_n'(t) + a^2n^2T_n(t) = f_n g(t),\ n = 1, 2, \dots. \tag{2.2.8}$$

Taking into account the initial condition (2.2.3), we obtain

$$u(x, 0) = \sum_{n=0}^{\infty} T_n(0)\cos(nx) = 0. \tag{2.2.9}$$

It follows then

$$T_n(0) = 0\ , n = 0, 1, \dots. \tag{2.2.10}$$

Integrating (2.2.7) from 0 to t and using the initial condition (2.2.10), we obtain

$$T_0(t) = \int_0^t \frac{f_0 g(\tau)}{2} d\tau = \frac{1}{\pi}\int_0^t \int_0^\pi f(\xi)g(\tau)d\xi d\tau = \frac{1}{\pi}\int_0^\pi f(\xi)d\xi \int_0^t g(\tau)d\tau.$$

Multiplying (2.2.8) by $e^{a^2n^2t}$, we have

$$\left(T_n(t)e^{a^2n^2t}\right)' = e^{a^2n^2t} f_n g(t),\ n = 1, 2, \dots.$$

Integrating from 0 to t and using the initial condition (2.2.9), we obtain

$$T_n(t)e^{a^2n^2t} = \int_0^t e^{a^2n^2\tau} f_n g(\tau)d\tau,\ n = 1, 2, \dots,$$

or

$$\begin{aligned} T_n(t) &= \int_0^t e^{-a^2n^2(t-\tau)} f_n g(\tau)d\tau = \frac{2}{\pi}\int_0^t e^{-a^2n^2(t-\tau)} \int_0^\pi f(\xi)g(\tau)\cos(n\xi)d\xi d\tau \\ &= \frac{2}{\pi}\int_0^\pi f(\xi)\cos(n\xi)d\xi \int_0^t g(\tau)e^{-a^2n^2(t-\tau)}d\tau,\ n = 1, 2, \dots. \end{aligned}$$

Therefore, the solution of the problem (2.2.1)-(2.2.3) is as follows

$$\begin{aligned} u(x, t) &= \frac{1}{\pi}\int_0^\pi f(\xi)d\xi \int_0^t g(\tau)d\tau \\ &\quad + \sum_{n=1}^{\infty} \frac{2}{\pi}\int_0^\pi f(\xi)\cos(n\xi)d\xi \int_0^t g(\tau)e^{-a^2n^2(t-\tau)}d\tau \cos(nx). \end{aligned} \tag{2.2.11}$$

Our aim in this section is to find the function $g(t)$ in (2.2.1) provided $u(x_0, t) = h(t),\ 0 \le x_0 \le \pi$ is given. That is we are going to solve the problem of finding the source term $g(t)$

of (2.2.1) under the condition

$$h(t) = u(x_0, t),\ 0 \le t \le T \tag{2.2.12}$$

where $x_0 \in [0, \pi]$.

Setting in (2.2.11) $x = x_0$, we obtain

$$\frac{1}{\pi}\int_0^\pi f(\xi)d\xi \int_0^t g(\tau)d\tau$$
$$+\sum_{n=1}^{\infty}\frac{2}{\pi}\int_0^\pi f(\xi)\cos(n\xi)d\xi \int_0^t g(\tau)e^{-a^2n^2(t-\tau)}d\tau\cos(nx_0) = h(t)\ 0 \le t \le T.$$

Changing the summation and integration, we obtain the Volterra integral equation of first kind for the function $g(t)$

$$\int_0^t K(t,\tau)g(\tau)d\tau = h(t),\ 0 \le t \le T \tag{2.2.13}$$

with the kernel

$$K(t,\tau) = \frac{1}{\pi}\int_0^\pi f(\xi)d\xi + \sum_{n=1}^{\infty}\frac{2}{\pi}\int_0^\pi f(\xi)\cos(n\xi)d\xi e^{-a^2n^2(t-\tau)}d\tau\cos(nx_0). \tag{2.2.14}$$

Now we consider the problem of existence and uniqueness of the solution of (2.2.13) in the space $\mathcal{C}[0, T]$.

Theorem 2.2.1. *Let* $f(x) \in \mathcal{C}^4[0, \pi]$ *and* $f'(0) = f'(\pi) = 0$. *If* $f(x_0) \neq 0$ *and* $h(t) \in \mathcal{C}^1[0, T]$, $h(0) = 0$, *then* (2.2.13) *has a unique solution* $g(t) \in \mathcal{C}[0, T]$.

Proof. By using the conditions of theorem, we have

$$\begin{aligned}\left|\int_0^\pi f(\xi)\cos(n\xi)d\xi\right| &= \left|\int_0^\pi f(\xi)\Big(\frac{1}{n}\sin(n\xi)\Big)' d\xi\right| \\ &= \left|\frac{1}{n}f(\xi)\sin(n\xi)\Big|_0^\pi - \frac{1}{n}\int_0^\pi f'(\xi)\sin(n\xi)d\xi\right| \\ &= \left|\frac{1}{n^2}\int_0^\pi f'(\xi)\Big(\cos(n\xi)\Big)' d\xi\right|\end{aligned}$$

$$
\begin{aligned}
&= \left| \frac{1}{n^2} f'(\xi)\cos(n\xi)\Big|_0^\pi - \frac{1}{n^2}\int_0^\pi f''(\xi)\cos(n\xi)d\xi \right| \\
&= \left| \frac{1}{n^3}\int_0^\pi f''(\xi)\Big(\sin(n\xi)\Big)' d\xi \right| \\
&= \left| \frac{1}{n^3} f''(\xi)\sin(n\xi)\Big|_0^\pi - \frac{1}{n^3}\int_0^\pi f'''(\xi)\sin(n\xi)d\xi \right| \\
&= \left| \frac{1}{n^4}\int_0^\pi f'''(\xi)\Big(\cos(n\xi)\Big)' d\xi \right| \\
&= \left| \frac{1}{n^4} f'''(\xi)\cos(n\xi)\Big|_0^\pi - \frac{1}{n^4}\int_0^\pi f''''(\xi)\cos(n\xi)d\xi \right| \\
&\le \frac{1}{n^4}\left(|f'''(\pi)| + |f'''(0)| + \int_0^\pi |f''''(\xi)| d\xi \right) \\
&\le \frac{1}{n^4}\left(|f'''(\pi)| + |f'''(0)| + C_1\pi \right) = \frac{C}{n^4}, \ n = 1, 2, \cdots ,
\end{aligned}
$$

where $C = |f'''(\pi)| + |f'''(0)| + C_1\pi > 0$.

By using the last inequality, we obtain for $0 \le \tau \le t \le T$,

$$\left| \frac{2}{\pi}\int_0^\pi f(\xi)\cos(n\xi)d\xi e^{-a^2n^2(t-\tau)}d\tau\cos(nx_0) \right| \le \frac{2C/\pi}{n^4},$$

and

$$\left| \frac{-2n^2a^2}{\pi}\int_0^\pi f(\xi)\cos(n\xi)d\xi e^{-a^2n^2(t-\tau)}d\tau\cos(nx_0) \right| \le \frac{2a^2C/\pi}{n^2}.$$

Since

$$\sum_{n=1}^\infty \frac{1}{n^p}$$

is convergent for $p > 1$, by using the comparison test, we see that the series

$$K(t,\tau) = \sum_{n=1}^\infty \frac{2}{\pi}\int_0^\pi f(\xi)\cos(n\xi)d\xi e^{-a^2n^2(t-\tau)}d\tau\cos(nx_0),$$

and

$$K_t(t,\tau) = \sum_{n=1}^\infty \frac{-2n^2a^2}{\pi}\int_0^\pi f(\xi)\cos(n\xi)d\xi e^{-a^2n^2(t-\tau)}d\tau\cos(nx_0)$$

are convergent for $0 \le \tau \le t \le T$. Therefore, $K(t,\tau)$ and $K_t(t,\tau)$ are continuous functions on $0 \le \tau \le t \le T$.

Differentiating (2.2.13) with respect to t, we obtain

$$K(t,t)g(t) + \int_0^t K_t(t,\tau)g(\tau)d\tau = h'(t),\ 0 \le t \le T. \tag{2.2.15}$$

Setting $t = T$ into the kernel (2.2.14), we obtain

$$K(t,t) = \frac{1}{\pi}\int_0^\pi f(\xi)d\xi + \sum_{n=1}^{\infty} \frac{2}{\pi}\int_0^\pi f(\xi)\cos(n\xi)d\xi d\tau \cos(nx_0). \tag{2.2.16}$$

The right-hand side of the equation (2.2.16) is the Fourier cosine series of $f(x)$ at the point $x = x_0$. Therefore, $K(t,t) = f(x_0) \neq 0$.

Dividing both sides of the equation (2.2.15) with $K(t,t)$, we obtain

$$g(t) + \int_0^t \frac{K_t(t,\tau)}{K(t,t)} g(\tau)d\tau = \frac{h'(t)}{K(t,t)},\ 0 \le t \le T. \tag{2.2.17}$$

As

$$\frac{K_t(t,\tau)}{K(t,t)}, \quad \text{and} \quad \frac{h'(t)}{K(t,t)}$$

are continuous for $0 \le \tau \le t \le T$, the equation (2.2.17) is a Volterra integral equation of the second kind with a continuous kernel and the right-hand side. By Theorem 1.2.8, the integral equation (2.2.17) has a unique solution $g(t) \in C[0,\pi]$. □

2.3 Determination of an Unknown Time-Dependent Diffusivity $a(t)$

2.3.1 Heat Equation on the Real Line

We consider the following initial-value problem

$$u_t = a(t)u_{xx}, \quad -\infty < x < \infty, \ t > 0; \tag{2.3.1}$$

$$u(x,0) = \begin{cases} 1, & -b \le x \le b; \\ 0, & |x| > b. \end{cases}, b > 0. \tag{2.3.2}$$

where $a(t) > 0$ is the time-dependent thermal diffusivity.

We shall determine the function $a(t) > 0$ and u from the interior temperature measurement

$$u(0,t) = h(t), \ t \ge 0. \tag{2.3.3}$$

The solution of the problem (2.3.1)-(2.3.2) for arbitrary $a(t) > 0$ can be found by using the Fourier transform in the form

$$u(x,t) = \frac{1}{\sqrt{4\pi \int_0^t a(y)dy}} \int_{-b}^{b} \exp\left\{ \frac{-(x-\xi)^2}{4\int_0^t a(y)dy} \right\} d\xi. \tag{2.3.4}$$

To determine the function $a(t) > 0$, we will use the additional condition (2.3.3).

Setting $x = 0$ in (2.3.4) and taking into account (2.3.3), we obtain

$$h(t) = \frac{1}{\sqrt{4\pi \int_0^t a(y)dy}} \int_{-b}^{b} \exp\left\{ \frac{-\xi^2}{4\int_0^t a(y)dy} \right\} d\xi.$$

Defining

$$F(\eta) = \frac{1}{\sqrt{4\pi\eta}} \int_{-b}^{b} e^{\frac{-\xi^2}{4\eta}} d\xi, \tag{2.3.5}$$

we can easily see that

$$F\left(\int_0^t a(y)dy \right) = h(t). \tag{2.3.6}$$

Setting $t = 0$ in (2.3.6), we obtain from the initial condition (2.3.2),

$$F(0) = h(0) = u(0,0) = 1.$$

By using the fact that e^{-x^2} is an even function and substituting the variable $\psi = \frac{\xi}{2\sqrt{\eta}}$ to the integral in (2.3.5), we obtain

$$\begin{aligned} F(\eta) &= \frac{1}{\sqrt{4\pi\eta}} \int_{-b}^{b} e^{\frac{-\xi^2}{4\eta}} d\xi = \frac{1}{\sqrt{\pi\eta}} \int_{0}^{b} e^{\frac{-\xi^2}{4\eta}} d\xi \\ &= \frac{2}{\sqrt{\pi}} \int_{0}^{b/2\sqrt{\eta}} e^{-\psi^2} d\psi. \end{aligned}$$

Then, we have $\lim_{\eta\to\infty} F(\eta) = 0$, and by using the Fundamental Theorem of Calculus,

$$F'(\eta) = \frac{-b}{2\sqrt{\pi}} e^{\frac{-b^2}{4\eta}} \eta^{-3/2} < 0, \ \eta > 0. \tag{2.3.7}$$

Consequently, $F : (0,\infty) \longrightarrow (0,1)$ is a strictly-decreasing function and then the inverse of F exists, say G. If $0 < h(t) < 1, \ t > 0$, then from (2.3.6),

$$\int_0^t a(y)dy = G(h(t)), \tag{2.3.8}$$

and

$$\begin{aligned} a(t) = G'(h(t))h'(t) &= \frac{F'(G(h(t)))}{F'(G(h(t)))} G'(h(t))h'(t) \\ &= \frac{[F(G(h(t)))]'}{F'(G(h(t)))} = \frac{h'(t)}{F'(G(h(t)))}, \ t > 0. \end{aligned} \tag{2.3.9}$$

From (2.3.8) and (2.3.9), we must require that h' is continuous, $h' < 0$, and $\lim_{t\to\infty} h(t) = 0$. From these requirements on h it follows that $a(t)$ is positive, integrable, and continuous for $t > 0$. For $a(t)$ to be continuous at $t = 0$ with $a(0) > 0$, we must have

$$\lim_{t\to 0^+} \frac{h'(t)}{F'(G(h(t)))} > 0.$$

Collecting all of the requirements, we obtain the result in the following theorem.

Theorem 2.3.1. *If h is continuously differentiable for $t > 0$, $h'(t) < 0$ for $t > 0$, and* $\lim_{t\to\infty} h(t) = 0$, *and if*

$$\lim_{t\to 0^+} \frac{h'(t)}{F'(G(h(t)))} > 0.$$

where F is defined by the equation (2.3.5) *and G is the inverse of F, then it follows that $a(t)$ defined by* (2.3.9) *and u defined by* (2.3.4) *constitute the unique solution to* (2.3.1)-(2.3.3).

2.3.2 Heat Equation in a Finite Segment

We consider the boundary-value problem

$$u_t = a(t)u_{xx},\ 0 < x < \pi,\ 0 < t; \tag{2.3.10}$$

$$u(0,t) = u(\pi,t) = 0,\ 0 \le t; \tag{2.3.11}$$

$$u(x,0) = \varphi(x),\ 0 \le x \le \pi. \tag{2.3.12}$$

where $a(t) > 0$ is the time-dependent thermal diffusivity.

We shall determine the function a and u from the interior temperature measurement

$$u(x_0,t) = h(t),\ t \ge 0 \tag{2.3.13}$$

where $x_0 \in (0,\pi)$.

The solution of the problem (2.3.10)-(2.3.12) for arbitrary $a(t) > 0$ can be found by using the separation of variables in the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 \int_0^t a(y)dy} \sin(nx) \tag{2.3.14}$$

where

$$A_n = \frac{2}{\pi}\int_0^{\pi} \varphi(\xi)\sin(n\xi)d\xi,\ n = 1,2,\ldots.$$

To determine the function $a(t)$, we will use the additional condition (2.3.13).

Setting $x = x_0$ in (2.3.14) and taking into account (2.3.13), we obtain

$$h(t) = \sum_{n=1}^{\infty} A_n e^{-n^2 \int_0^t a(y)dy} \sin(nx_0),\ t > 0.$$

The function

$$v(x,\eta) = \sum_{n=1}^{\infty} A_n e^{-n^2\eta} \sin(nx), \tag{2.3.15}$$

where

$$A_n = \frac{2}{\pi}\int_0^{\pi} \varphi(\xi)\sin(n\xi)d\xi,\ n = 1,2,\cdots$$

satisfies the problem

$$v_\eta = v_{xx},\ 0 < x < \pi,\ \eta > 0, \tag{2.3.16}$$

$$v(0,\eta) = v(\pi,\eta) = 0,\ \eta \geq 0, \tag{2.3.17}$$

$$v(x,0) = \varphi(x),\ 0 \leq x \leq \pi. \tag{2.3.18}$$

Suppose that φ is nonnegative continuous on $0 \leq x \leq \pi$, $\varphi(x_0) > 0$, and φ is twice continuously differentiable on $0 < x < \pi$ such that $\varphi'' < 0$ and φ'' is bounded for $0 < x < \pi$. Firstly, we have

$$v(x_0,0) = \varphi(x_0) > 0 \tag{2.3.19}$$

and, from (2.3.15),

$$\lim_{\eta\to\infty} v(x_0,\eta) = 0. \tag{2.3.20}$$

Next, we will show that $v_\eta(x_0,\eta) < 0$ for $\eta \geq 0$.

Set $D_T = (0,\pi) \times (0,T]$ and $B_T = \overline{D_T} - D_T$. Define the auxiliary function

$$w(x,\eta) = v_\eta(x,\eta) + \varepsilon x^2 = v_{xx}(x,\eta) + \varepsilon x^2,$$

where ε is a positive number. Then w assumes its maximum on B_T. Otherwise, there would exist a point $(x_1,\eta_1) \in D_T$ s.t.

$$w(x_1,\eta_1) = \max_{D_T \cup B_T} w.$$

Hence, at $(x_1,\eta_1) \in D_T$,

$$w_{xx} - w_\eta \leq 0 \tag{2.3.21}$$

since $w_\eta(x_1,\eta_1) \geq 0$ and $w_{xx}(x_1,\eta_1) \leq 0$.

But

$$w_{xx} - w_\eta = (v_{\eta xx} + 2\varepsilon) - v_{xx\eta} = 2\varepsilon > 0$$

contradicts to (2.3.21). Then w has its maximum in B_T. Since

$$w \leq \max_{B_T} w,$$

and

$$v_\eta(x_0,\eta) < v_\eta(x_0,\eta) + \varepsilon x_0^2 = w(x_0,\eta)$$

where $x_0 \in (0,\pi), \eta > 0$, we have

$$v_\eta(x_0,\eta) < w(x_0,\eta) \le \max_{B_T} w \le \max_{B_T} v_\eta + \varepsilon \max_{B_T} x^2.$$

Since ε can be chosen arbitrarily,

$$v_\eta(x_0,\eta) < \max_{B_T} v_\eta = \max_{B_T} v_{xx} = \max_{0<x<\pi}\{0, \varphi''(x)\} = 0.$$

Also, we have

$$v_\eta(x_0,0) = v_{xx}(x_0,0) = \varphi''(x_0) < 0.$$

Thus, we have

$$v_\eta(x_0,\eta) < 0,\ \ 0 \le \eta. \tag{2.3.22}$$

Defining

$$F(\eta) = v(x_0,\eta).$$

we can easily see that

$$F\left(\int_0^t a(y)dy\right) = h(t). \tag{2.3.23}$$

Setting $t = 0$, we have $F(0) = h(0)$. Then, by using the results (2.3.19),(2.3.20), and (2.3.22), we have

$$F(0) = h(0) = \varphi(x_0),\ \ \lim_{\eta\to\infty} F(\eta) = 0,\ \ F'(\eta) < 0,\ \ \eta \ge 0.$$

Then, $F : [0,\infty) \longrightarrow (0,\varphi(x_0)]$ is strictly-decreasing function and then the inverse of F exists, say G. If $0 < h(t) \le \varphi(x_0),\ t \ge 0$, then from (2.3.23), we have

$$\int_0^t a(y)dy = G(h(t)),\ \ t \ge 0 \tag{2.3.24}$$

and

$$\begin{aligned} a(t) = G'(h(t))h'(t) &= \frac{F'(G(h(t)))}{F'(G(h(t)))}G'(h(t))h'(t) \\ &= \frac{[F(G(h(t)))]'}{F'(G(h(t)))} = \frac{h'(t)}{F'(G(h(t)))}, \ t \geq 0. \end{aligned} \tag{2.3.25}$$

Note that $F'(G(h(t))) < 0, \ t \geq 0$. From (2.3.24) and (2.3.25), we must require that h' is continuous, $h' < 0$, and $\lim_{t\to\infty} h(t) = 0$. From these requirements on h it follows that $a(t)$ is positive, integrable, and continuous for $t \geq 0$.

Collecting all of the requirements, we obtain the result in the following theorem.

Theorem 2.3.2. *If φ is nonnegative, twice continuously differentiable with bounded $\varphi'' < 0$, and $\varphi(x_0) > 0, \ 0 < x_0 < \pi$, and if h is continuously differentiable for $t \geq 0$, $h' < 0$, and $\lim_{t\to\infty} h(t) = 0$, then $a(t)$ given by* (2.3.25) *and u given by* (2.3.14) *constitute the unique solution to problem* (2.3.10)-(2.3.13).

2.3.3 Heat Equation on the Half-Line

Now, we consider in this section the determination of a positive continuous function $a(t)$ defined on the interval $0 \leq t < T$ and a function $u = u(x,t)$ defined on $0 \leq x < \infty,\ 0 \leq t < T$, such that the pair (a, u) satisfies

$$u_t = a(t)u_{xx},\ 0 < x < \infty,\ 0 < t < T; \tag{2.3.26}$$

$$u(x,0) = 0,\ 0 \leq x < \infty; \tag{2.3.27}$$

$$u(0,t) = \psi(t),\ 0 \leq t < T. \tag{2.3.28}$$

where $\psi(t)$ is given function defined on $0 \leq t < T$.

We shall determine the function $a(t)$ defined on the interval $0 \leq t < T$ and u defined on $0 \leq x < \infty,\ 0 \leq t < T$ from the boundary-flux measurement

$$-a(t)u_x(0,t) = g(t),\ 0 < t < T \tag{2.3.29}$$

where g is given function defined on $0 < t < T$.

Definition 2.3.1. Let $Q = [0,\infty) \times [0,T)$. A pair of functions $\{a(t), u(x,t)\}$ is called a solution of (2.3.26)-(2.3.29) if

1. $u \in \mathcal{C}^{2,1}(Q)$, and positive function $a \in \mathcal{C}\big([0,T)\big)$
2. (2.3.27)-(2.3.29) is satisfied in the usual sense.

The first step in this section is based upon the representation of solutions of the heat equation to which equation (2.3.26) can be reduced via the transformation

$$\theta(t) = \int_0^t a(y)dy, \quad 0 \leq t \leq T.$$

Since

$$\theta'(t) = a(t) > 0, \quad 0 \leq t \leq T, \tag{2.3.30}$$

the continuous function $\theta(t)$ is invertible, i.e., there exits a function φ such that

$$\varphi(\theta(t)) = t, \quad 0 \le t \le T, \tag{2.3.31}$$

$$\theta(\varphi(\tau)) = \tau, \quad 0 \le \tau \le \theta(T). \tag{2.3.32}$$

Let

$$\eta = \theta(t),\ 0 \le t \le T.$$

Differentiating the both sides of (2.3.31) with respect to t, we have

$$1 = \frac{d}{dt}\varphi(\theta(t)) = \frac{d}{dt}\varphi(\eta) = \varphi'(\eta)\frac{d\eta}{dt} = \varphi'(\eta)\theta'(t)$$

or, by using the equation (2.3.30), we have

$$\varphi'(\eta) = \frac{1}{\theta'(t)} = \frac{1}{\theta'\big(\varphi(\theta(t))\big)} = \frac{1}{\theta'\big(\varphi(\eta)\big)} = \frac{1}{a(\varphi(\eta))}, \quad 0 \le \eta \le \theta(T). \tag{2.3.33}$$

Let

$$U(x,\eta) = u\big(x, \underbrace{\varphi(\eta)}_{t}\big). \tag{2.3.34}$$

Differentiating the equation (2.3.34) with respect to η and using the result (2.3.33), we have

$$\begin{aligned} U_\eta(x,\eta) &= u_t(x,\varphi(\eta))\varphi'(\eta) = u_t(x,\varphi(\eta))\frac{1}{a(\varphi(t))} \\ &= u_{xx}(x,\varphi(\eta)) = U_{xx}(x,\eta). \end{aligned}$$

Consequently, to obtain the representation for $u(x,t)$, we substitute $\eta = \theta(t)$ into the representation for $U(x,\eta)$.

So, the problem (2.3.26)-(2.3.28) is reduced to

$$U_\eta = U_{xx},\ 0 < x < \infty,\ 0 < \eta < \theta(T); \tag{2.3.35}$$

$$u(x,0) = 0,\ 0 \le x < \infty; \tag{2.3.36}$$

$$U(0,\eta) = \psi(\varphi(\eta)),\ 0 \le \eta \le \theta(T). \tag{2.3.37}$$

The solution of the problem (2.3.35)-(2.3.37) can be found by using the Laplace transform in the form

$$U(x,\eta) = \int_0^{\eta} \frac{x}{\sqrt{4\pi(\eta - y)^3}} e^{-x^2/4(\eta - y)} \psi(\varphi(y))dy.$$

Substituting in $\eta = \theta(t)$, we obtain

$$u(x,t) = \int_0^{\theta(t)} \frac{x}{\sqrt{4\pi(\theta(t) - y)^3}} e^{\dfrac{-x^2}{4(\theta(t) - y)}} \psi(\varphi(y))dy.$$

If we make the substitution $y = \theta(\tau)$, we obtain $dy = \theta'(\tau)d\tau = a(\tau)d\tau$ and then

$$\begin{aligned} u(x,t) &= \int_0^{t} \frac{x}{\sqrt{4\pi(\theta(t) - \theta(\tau))^3}} e^{\dfrac{-x^2}{4(\theta(t) - \theta(\tau))}} \psi(\varphi(\theta(\tau)))a(\tau)d\tau \\ &= \frac{1}{2\sqrt{\pi}} \int_0^{t} \frac{x}{\left(\int_\tau^t a(y)dy\right)^{3/2}} e^{\dfrac{-x^2}{4\int_\tau^t a(y)dy}} \psi(\tau)a(\tau)d\tau. \end{aligned} \tag{2.3.38}$$

Now, we shall determine the function $a(t)$ defined on the interval $0 \leq t < T$ from the boundary-flux measurement (2.3.29). Differentiating the equality (2.3.38) with respect to x and setting $x = 0$, we obtain

$$u_x(0,t) = \frac{1}{2\sqrt{\pi}} \int_0^{t} \frac{\psi(\tau)a(\tau)}{\left(\int_\tau^t a(y)dy\right)^{3/2}} d\tau. \tag{2.3.39}$$

Under the assumption that ψ is continuously differentiable, integrating by parts in (2.3.39) and using the initial condition $\psi(0) = u(0,0) = 0$, we obtain

$$\begin{aligned} \int_0^{t} \frac{\psi(\tau)a(\tau)}{\left(\int_\tau^t a(y)dy\right)^{3/2}} d\tau &= \int_0^{t} 2\psi(\tau) \left(\frac{1}{\left(\int_\tau^t a(y)dy\right)^{1/2}} \right)' d\tau \\ &= \left(2\psi(\tau) \frac{1}{\left(\int_\tau^t a(y)dy\right)^{1/2}} \right|_{\tau=0}^{\tau=t} - \int_0^{t} \frac{2\psi'(\tau)}{\left(\int_\tau^t a(y)dy\right)^{1/2}} d\tau \\ &= -\int_0^{t} \frac{2\psi'(\tau)}{\left(\int_\tau^t a(y)dy\right)^{1/2}} d\tau. \end{aligned}$$

By using this result, we can write the equation (2.3.39) in the form

$$u_x(0,t) = \frac{-1}{\sqrt{\pi}} \int_0^{t} \frac{\psi'(\tau)}{\left(\int_\tau^t a(y)dy\right)^{1/2}} d\tau. \tag{2.3.40}$$

From the equation (2.3.29) and (2.3.40), we get the nonlinear integral equation

$$a(t) = \frac{-g(t)}{u_x(0,t)} = \frac{\sqrt{\pi}g(t)}{\displaystyle\int_0^t \left[\frac{\psi'(\tau)}{\left(\int_\tau^t a(y)dy\right)^{1/2}}\right] d\tau}, \ 0 < t < T. \tag{2.3.41}$$

Then, the existence of a unique solution to the problem (2.3.26)-(2.3.29) is equivalent to the existence of a unique solution to the integral equation (2.3.41).

Define

$$\mathcal{F}a(t) = \frac{\sqrt{\pi}g(t)}{\displaystyle\int_0^t \left[\frac{\psi'(\tau)}{\left(\int_\tau^t a(y)dy\right)^{1/2}}\right] d\tau}, \ 0 < t < T \tag{2.3.42}$$

Now, the existence of a unique solution to the integral equation (2.3.41) is equivalent to the existence of a unique fixed point of the operator $\mathcal{F}$.

Assumption

We shall assume that

1. ψ is continuously differentiable on every compact subset of $0 \leq t < T$;
2. $\psi' > 0, \ 0 < t < T$;
3. g is continuous for $0 \leq t < T$, and positive for $0 < t < T$;
4. The function

   $$h(t) = \frac{\sqrt{\pi}g(t)}{\displaystyle\int_0^t \left[\frac{\psi'(\tau)}{(t-\tau)^{1/2}}\right] d\tau}, \ 0 < t < T,$$

   satisfies

   $$\lim_{t \to 0^+} h(t) = h_0 > 0.$$

Definition 2.3.2. For any function $\varphi(t)$ defined for $0 \leq t < T$, Let

$$s(\varphi, t) = \sup_{0<y<t} \varphi(y), \ i(\varphi, t) = \inf_{0<y<t} \varphi(y).$$

Lemma 2.3.3. *The function $\mathcal{F}a(t)$ satisfies*

$$\sqrt{i(a,t)}i(h,t) \le \mathcal{F}a(t) \le \sqrt{s(a,t)}s(h,t),\ \ 0 < t < T.$$

Proof. Since $g > 0$ and $\psi' > 0$,

$$\begin{aligned}\mathcal{F}a(t) =& \frac{\sqrt{\pi}g(t)}{\displaystyle\int_0^t \left[\frac{\psi'(\tau)}{\left(\int_\tau^t a(y)dy\right)^{1/2}}\right] d\tau} \le \frac{\sqrt{\pi}g(t)}{\displaystyle\int_0^t \left[\frac{\psi'(\tau)}{\left(\int_\tau^t \sup\limits_{0<y<t} a(y)dy\right)^{1/2}}\right] d\tau}\\ =& \frac{\sqrt{\pi}g(t)}{\displaystyle\int_0^t \left[\frac{\psi'(\tau)}{\big((t-\tau)s(a,t)\big)^{1/2}}\right] d\tau} = \sqrt{s(a,t)}h(t) \le \sqrt{s(a,t)}s(h,t)\end{aligned}$$

Likewise,

$$\begin{aligned}\mathcal{F}a(t) =& \frac{\sqrt{\pi}g(t)}{\displaystyle\int_0^t \left[\frac{\psi'(\tau)}{\left(\int_\tau^t a(y)dy\right)^{1/2}}\right] d\tau} \ge \frac{\sqrt{\pi}g(t)}{\displaystyle\int_0^t \left[\frac{\psi'(\tau)}{\left(\int_\tau^t \inf\limits_{0<y<t} a(y)dy\right)^{1/2}}\right] d\tau}\\ =& \frac{\sqrt{\pi}g(t)}{\displaystyle\int_0^t \left[\frac{\psi'(\tau)}{\big((t-\tau)i(a,t)\big)^{1/2}}\right] d\tau} = \sqrt{i(a,t)}h(t) \ge \sqrt{i(a,t)}i(h,t)\end{aligned}$$

□

Lemma 2.3.4. *If $a(t)$ is a solution of the nonlinear integral equation* (2.3.41)*, then*

$$i(h,t)^2 \le a(t) \le s(h,t)^2,\ \ 0 \le t < T.$$

Proof. By Lemma 2.3.3,

$$a(t) = \mathcal{F}a(t) \le \sqrt{s(a,t)}s(h,t).$$

whence it follows that

$$s(a,t) \le \sqrt{s(a,t)}s(h,t).$$

and

$$\sqrt{s(a,t)} \le s(h,t). \tag{2.3.43}$$

Likewise,

$$a(t) \geq \sqrt{i(a,t)}i(h,t).$$

whence it follows that

$$i(a,t) \geq \sqrt{i(a,t)}i(h,t).$$

and

$$\sqrt{i(a,t)} \geq i(h,t). \tag{2.3.44}$$

Combining (2.3.43) and (2.3.44),we have

$$i(h,t) \leq \sqrt{i(a,t)} \leq \sqrt{a(t)} \leq \sqrt{s(a,t)} \leq s(h,t) \tag{2.3.45}$$

Therefore, the result follows by taking the square of each term in (2.3.45). □

We now restrict our attention to the class of functions defined as

$$\mathcal{G} = \{a \in C\big([0,T)\big) \mid i(h,t)^2 \leq a(t) \leq s(h,t)^2\} \tag{2.3.46}$$

Lemma 2.3.5. *$\mathcal{F}$ maps $\mathcal{G}$ into $\mathcal{G}$*

Proof. Let $a(t)$ be in $\mathcal{G}$. From the Lemma 2.3.3, we have

$$\mathcal{F}a(t) \leq \sqrt{s(a,t)}s(h,t)$$

But, since $a(t) \in \mathcal{G}$,

$$\sqrt{s(a,t)} \leq \sqrt{s(h,t)^2}.$$

Then

$$\mathcal{F}a(t) \leq \sqrt{s(a,t)}s(h,t) \leq s(h,t)^2.$$

Likewise,

$$\mathcal{F}a(t) \geq \sqrt{i(a,t)}s(h,t)$$

But, since $a(t) \in \mathcal{G}$,

$$\sqrt{i(a,t)} \geq \sqrt{i(h,t)^2}.$$

Then

$$\mathcal{F}a(t) \geq \sqrt{i(a,t)}i(h,t) \geq i(h,t)^2.$$

Therefore the result follows. □

Lemma 2.3.6. *If a_1 and a_2 are $\mathcal{G}$ and $a_1 \leq a_2$, then $\mathcal{F}a_1 \leq \mathcal{F}a_2$.*

Proof. From the definition (2.3.42) of $\mathcal{F}$, we obtain

$$\frac{\sqrt{\pi}g(t)}{\mathcal{F}a_2} = \int_0^t \frac{\psi'(\tau)}{\left(\int_\tau^t a_2(y)dy\right)^{1/2}} d\tau \leq \int_0^t \frac{\psi'(\tau)}{\left(\int_\tau^t a_1(y)dy\right)^{1/2}} d\tau = \frac{\sqrt{\pi}g(t)}{\mathcal{F}a_1}.$$

Thus

$$\mathcal{F}a_1(t) \leq \mathcal{F}a_2.$$

□

Lemma 2.3.7. *The image $\mathcal{F}\mathcal{G}$ is an equicontinuous, uniformly bounded family of functions.*

Proof. Since $i(h,t)^2 \leq \mathcal{F}a(t) \leq s(h,t)^2$ and

$$\lim_{t\to 0^+} i(h,t) = \lim_{t\to 0^+} s(h,t) = \lim_{t\to 0^+} h(t) = h_0,$$

it follows that the family $\mathcal{F}\mathcal{G}$ is equicontinuous at $t = 0$. Now, we will consider $\mathcal{F}\mathcal{G}$ at $t,\ 0 < t < T$.

Let t_0 be fixed such that $t < t_0 < \min(2t, T)$ and let $\delta > 0$ satisfy $t < t + \delta < t_0$. Set

$$\Delta(a,\delta) = \int_0^{t+\delta} \frac{\psi'(\tau)}{\left(\int_\tau^{t+\delta} a(y)dy\right)^{1/2}} d\tau - \int_0^t \frac{\psi'(\tau)}{\left(\int_\tau^t a(y)dy\right)^{1/2}} d\tau$$

Then,

$$
\begin{aligned}
|\Delta(a,\delta)| &= \Bigg| \int_t^{t+\delta} \frac{\psi'(\tau)}{\left(\int_\tau^{t+\delta} a(y)dy\right)^{1/2}} d\tau \\
&+ \int_0^t \frac{\psi'(\tau)}{\left(\int_\tau^{t+\delta} a(y)dy\right)^{1/2}} d\tau - \int_0^t \frac{\psi'(\tau)}{\left(\int_\tau^{t} a(y)dy\right)^{1/2}} d\tau \Bigg| \\
&\leq \left| \int_t^{t+\delta} \frac{\psi'(\tau)}{\left(\int_\tau^{t+\delta} a(y)dy\right)^{1/2}} d\tau \right| \\
&+ \left| \int_0^t \frac{\psi'(\tau)}{\left(\int_\tau^{t+\delta} a(y)dy\right)^{1/2}} d\tau - \int_0^t \frac{\psi'(\tau)}{\left(\int_\tau^{t} a(y)dy\right)^{1/2}} d\tau \right| \\
&= \left| \int_t^{t+\delta} \frac{\psi'(\tau)}{\left(\int_\tau^{t+\delta} a(y)dy\right)^{1/2}} d\tau \right| \\
&+ \left| \int_0^t \frac{(-1)\psi'(\tau)}{\left(\int_\tau^{t} a(y)dy\right)^{1/2} - \left(\int_\tau^{t+\delta} a(y)dy\right)^{1/2}} d\tau \right| \\
&= I_1 + I_2
\end{aligned}
$$

where

$$
I_1 = \int_t^{t+\delta} \frac{\psi'(\tau)}{\left(\int_\tau^{t+\delta} a(y)dy\right)^{1/2}} d\tau
$$

and

$$
I_2 = \int_0^t \frac{\psi'(\tau)}{\left(\int_\tau^{t} a(y)dy\right)^{1/2} - \left(\int_\tau^{t+\delta} a(y)dy\right)^{1/2}} d\tau
$$

By using the inequality $i(h,t) \leq a(t),\ 0 \leq t < T$ obtained from the Lemma 2.3.4, we have

$$\begin{aligned}
I_1 &= \int_t^{t+\delta} \frac{\psi'(\tau)}{\left(\int_\tau^{t+\delta} a(y)dy\right)^{1/2}} d\tau \leq \int_t^{t+\delta} \frac{\psi'(\tau)}{\left(\int_\tau^{t+\delta} \inf\limits_{0<y<t_0} a(y)dy\right)^{1/2}} d\tau \\
&\leq \int_t^{t+\delta} \frac{\psi'(\tau)}{\left(\int_\tau^{t+\delta} i(h,t_0)^2 dy\right)^{1/2}} d\tau = \frac{1}{i(h,t_0)} \int_t^{t+\delta} \frac{\psi'(\tau)}{(t+\delta-\tau)^{1/2}} d\tau \\
&\leq \frac{1}{i(h,t_0)} \left(\sup_{t\leq\tau\leq t_0} \psi'(\tau)\right) \int_t^{t+\delta} \frac{1}{(t+\delta-\tau)^{1/2}} d\tau \\
&= \frac{1}{i(h,t_0)} \left(\sup_{t\leq\tau\leq t_0} \psi'(\tau)\right) \left(-2(t+\delta-\tau)^{1/2} \Big|_{\tau=t}^{\tau=t+\delta} \right. = C_1 \delta^{1/2}
\end{aligned} \tag{2.3.47}$$

where

$$C_1 = \frac{2}{i(h,t_0)} \left(\sup_{t\leq\tau\leq t_0} \psi'(\tau)\right)$$

which is positive and finite by our assumptions and independent of $a(t)$.

For $b_i > 0$, $i = 1, 2$, it follows that

$$\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}} = \frac{\sqrt{b_2} - \sqrt{b_1}}{\sqrt{b_1}\sqrt{b_2}} = \frac{b_2 - b_1}{\sqrt{b_1}\sqrt{b_2}(\sqrt{b_1} + \sqrt{b_2})}. \tag{2.3.48}$$

By using this result, we obtain

$$\begin{aligned}
I_2 &= \int_0^t \frac{\psi'(\tau)}{\left(\int_\tau^t a(y)dy\right)^{1/2} - \left(\int_\tau^{t+\delta} a(y)dy\right)^{1/2}} d\tau \\
&= \int_0^t \frac{\left(\int_t^{t+\delta} a(y)dy\right)\psi'(\tau)d\tau}{\left(\int_\tau^t a(y)dy\right)^{1/2}\left(\int_\tau^{t+\delta} a(y)dy\right)^{1/2}\left[\left(\int_\tau^t a(y)dy\right)^{1/2} + \left(\int_\tau^{t+\delta} a(y)dy\right)^{1/2}\right]} \\
&\leq \frac{\left(\sup\limits_{0<y<t_0} a(y)\right)}{\left(\inf\limits_{0<y<t_0} a(y)\right)^{3/2}} \int_0^t \frac{\left(\int_t^{t+\delta} dy\right)\psi'(\tau)d\tau}{\left(\int_\tau^t dy\right)^{1/2}\left(\int_\tau^{t+\delta} dy\right)^{1/2}\left[\left(\int_\tau^t dy\right)^{1/2} + \left(\int_\tau^{t+\delta} dy\right)^{1/2}\right]} \\
&\leq \frac{s(h,t_0)^2}{i(h,t_0)^3} \int_0^t \frac{\delta}{(t-\tau)^{1/2}(t+\delta-\tau)^{1/2}\left[(t-\tau)^{1/2} + (t+\delta-\tau)^{1/2}\right]} \psi'(\tau)d\tau \\
&= \frac{s(h,t_0)^2}{i(h,t_0)^3} \int_0^t \left[\frac{1}{(t-\tau)^{1/2}} - \frac{1}{(t+\delta-\tau)^{1/2}}\right] \psi'(\tau)d\tau
\end{aligned} \tag{2.3.49}$$

We now let η, $0 < \eta < \frac{1}{2}t$. Then, from the inequality (2.3.49), we have

$$\begin{aligned}
I_2 &\le C_2 \int_0^t \left[\frac{1}{(t-\tau)^{1/2}} - \frac{1}{(t+\delta-\tau)^{1/2}}\right] \psi'(\tau) d\tau \\
&= C_2 \int_0^\eta \left[\frac{1}{(t-\tau)^{1/2}} - \frac{1}{(t+\delta-\tau)^{1/2}}\right] \psi'(\tau) d\tau \\
&+ C_2 \int_\eta^t \left[\frac{1}{(t-\tau)^{1/2}} - \frac{1}{(t+\delta-\tau)^{1/2}}\right] \psi'(\tau) d\tau \\
&\le C_2 \int_0^\eta \frac{\psi'(\tau)}{(t-\tau)^{1/2}} d\tau + C_2 \sup_{\eta \le \tau \le t_0} \psi'(\tau) \int_0^t \left[\frac{1}{(t-\tau)^{1/2}} - \frac{1}{(t+\delta-\tau)^{1/2}}\right] d\tau \\
&= C_2 \left[\left(\psi(\tau) \frac{1}{(t-\tau)^{1/2}} \Big|_{\tau=0}^{\tau=\eta} - \int_0^\eta \frac{\psi(\tau)}{2(t-\tau)^{3/2}} d\tau \right] \right. \\
&\qquad\qquad + C_2 \sup_{\eta \le \tau \le t_0} \psi'(\tau) \Big(-2(t-\tau)^{1/2} + 2(t+\delta-\tau)^{1/2} \Big|_{\tau=0}^{\tau=t} \\
&\le C_2 \psi(\eta) \frac{1}{(t-\eta)^{1/2}} + C_2 \sup_{\eta \le \tau \le t_0} \psi'(\tau) \left[2\delta^{1/2} + 2t^{1/2} - 2(t+\delta)^{1/2} \right] \\
&\le C_2 \psi(\eta) \eta^{-1/2} + 2C_2 \delta^{1/2} \sup_{\eta \le \tau \le t_0} \psi'(\tau)
\end{aligned}$$

where

$$C_2 = \frac{s(h,t_0)^2}{i(h,t_0)^3}$$

Since ψ is continuous and $\psi(0) = 0$, we can select η sufficiently small so that

$$C_2 \psi(\eta) \eta^{-1/2} < 2^{-1}\varepsilon, \ \varepsilon > 0.$$

Fixing η, we then can select δ sufficiently small so that

$$2C_2 \delta^{1/2} \sup_{\eta \le \tau \le t_0} \psi'(\tau) < 2^{-1}\varepsilon \ \varepsilon > 0.$$

Consequently, for each $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ independent of a such that

$$I_2 < \varepsilon \tag{2.3.50}$$

for all $0 < \delta < \delta_\varepsilon$.

Combining (2.3.47) and (2.3.50), it follows that $\Delta(a, \delta)$ tends to zero uniformly with respect to $a \in \mathcal{G}$ as δ tends to zero from above. By a similar argument, $\Delta(a, \delta)$ tends to zero uniformly with respect to $a \in \mathcal{G}$ as δ tends to zero from below.

Thus, the functions

$$\frac{\sqrt{\pi}g(t)}{\mathcal{F}a(t)} = \int_0^t \frac{\psi'(\tau)}{\left(\int_\tau^t a(y)dy\right)^{1/2}} d\tau$$

for $a \in \mathcal{G}$ are equicontinuous for $0 \le t < T$. As g is continuous for $0 \le t < T$, it follows that the functions $\mathcal{F}a(t)$ for $a \in \mathcal{G}$ are equicontinuous. The uniformly boundedness follows from results of Lemma 2.3.5 □

Now, we will consider the existence of a fixed point of the operator $\mathcal{F}$.

Let $a_0(t) = i(h,t)^2,\ 0 \le t < T$. Then a_0 is in $\mathcal{G}$. Since $\mathcal{F}$ maps $\mathcal{G}$ into $\mathcal{G}$, $\mathcal{F}a_0 \in \mathcal{G}$, which implies that $\mathcal{F}a_0(t) \ge i(h,t)^2 = a_0(t)$. As $\mathcal{F}a_0(t) \ge a_0(t)$, by the Lemma 2.3.6, $\mathcal{F}^2 a_0(t) \ge \mathcal{F}a_0(t)$, and by induction the sequence $\mathcal{F}^n a_0(t)$ is a monotone increasing sequence of functions on $0 \le t < T$. As $\mathcal{F}^n a_0(t) \in \mathcal{G},\ n = 1, 2, 3, \ldots$, they are bounded above by $s(h,t)^2$. Hence,

$$\lim_{n\to\infty} \mathcal{F}^n a_0(t)$$

exists for $0 \le t < T$, say $\widetilde{a}(t)$. But, from Lemma (2.3.7), $\mathcal{F}^n a_0(t),\ n = 1, 2, 3 \ldots$, are equicontinuous and uniformly bounded. From the Ascoli-Arzela Theorem, there exists a uniformly convergent subsequence on each compact subset of $0 \le t < T$. This, together with the monotonicity of the sequence, implies that the entire sequence $\mathcal{F}^n a_0,\ n = 1, 2, 3, \ldots$, converges uniformly to $\widetilde{a}(t)$ on each compact subset of $0 \le t < T$.

Since $\mathcal{F}^n a_0(t)$ converges uniformly to $\widetilde{a}(t)$ on $0 \le t \le T_0 < T$, for each τ, $0 < \tau < t \le T_0$, we have

$$\lim_{n\to\infty} \int_\tau^t \mathcal{F}^n a_0(y)dy = \int_\tau^t \widetilde{a}(y)dy.$$

Then, we have

$$\lim_{n\to\infty} \frac{\psi'(\tau)}{\left(\int_\tau^t \mathcal{F}^n a_0(y)dy\right)^{1/2}} = \frac{\psi'(\tau)}{\left(\int_\tau^t \widetilde{a}(y)dy\right)^{1/2}}.$$

Since $\mathcal{F}^n a_0(t)$ is a monotone increasing sequence and $\lim_{n\to\infty} \mathcal{F}^n a_0(t) = \widetilde{a}(t)$, we have for $0 < t \leq T$,

$$a_0(t) \leq \mathcal{F}^n a_0(t) \leq \widetilde{a}(t),\ n = 1, 2, 3, \ldots$$

By using this result, we obtain

$$0 < \frac{\psi'(\tau)}{\left(\int_\tau^t \widetilde{a}(y)dy\right)^{1/2}} \leq \frac{\psi'(\tau)}{\left(\int_\tau^t \mathcal{F}^n a_0(y)dy\right)^{1/2}} \leq \frac{\psi'(\tau)}{\left(\int_\tau^t a_0(y)dy\right)^{1/2}}$$

Then, from the Lebesgue dominated-convergence theorem, we obtain

$$\lim_{n\to\infty} \int_0^t \frac{\psi'(\tau)}{\left(\int_\tau^t \mathcal{F}^n a_0(y)dy\right)^{1/2}} d\tau = \int_0^t \frac{\psi'(\tau)}{\left(\int_\tau^t \widetilde{a}(y)dy\right)^{1/2}} d\tau.$$

Then, we also have

$$\lim_{n\to\infty} \frac{\sqrt{\pi}g(t)}{\int_0^t \frac{\psi'(\tau)}{(\int_\tau^t \mathcal{F}^n a_0(y)dy)^{1/2}} d\tau} = \frac{\sqrt{\pi}g(t)}{\int_0^t \frac{\psi'(\tau)}{(\int_\tau^t \widetilde{a}(y)dy)^{1/2}} d\tau}.$$

By using the definiton 2.3.42 of $\mathcal{F}$, from the last equality, we obtain

$$\lim_{n\to\infty} \mathcal{F}\Big(\mathcal{F}^n a_0(t)\Big) = \mathcal{F}\widetilde{a}(t)$$

Since the left-hand side of last equation equals to $\widetilde{a}(t)$, we have

$$\widetilde{a}(t) = \mathcal{F}\widetilde{a}(t)$$

Thus, $\widetilde{a}(t) = \lim_{n\to\infty} \mathcal{F}^n a_0(t)$ is a fixed point of the operator $\mathcal{F}$.

Now, we will consider the uniqueness of the fixed point of the operator $\mathcal{F}$. Suppose that $a_1(t)$ and $a_2(t)$ are two different solutions of the nonlinear integral equation (2.3.41). Then, we see that

$$\frac{\sqrt{\pi}g(t)}{a_1(t)} - \frac{\sqrt{\pi}g(t)}{a_2(t)} = \int_0^t \left[\frac{1}{\left(\int_\tau^t a_1(y)dy\right)^{1/2}} - \frac{1}{\left(\int_\tau^t a_2(y)dy\right)^{1/2}} \right] \psi'(\tau)d\tau. \tag{2.3.51}$$

By using the equation (2.3.48), we can write (2.3.51) as

$$a_2(t) - a_1(t) = \frac{a_1(t)a_2(t)}{\sqrt{\pi}g(t)} \int_0^t \frac{\left(\int_\tau^t a_1(y)dy\right)^{-1/2} \left(\int_\tau^t a_2(y)dy\right)^{-1/2} \left(\int_\tau^t (a_2(y) - a_1(y))dy\right)}{\left[\left(\int_\tau^t a_1(y)dy\right)^{1/2} + \left(\int_\tau^t a_2(y)dy\right)^{1/2}\right]} \psi'(\tau)d\tau.$$

Employing the fact that $a_1(t)$ and $a_2(t)$ are in $\mathcal{G}$ defined by (2.3.46), we obtain

$$\begin{aligned}
|a_2(t) - a_1(t)| &\leq \frac{s(h,t)^4}{\sqrt{\pi}g(t)i(h,t)^3} \int_0^t \frac{\int_\tau^t |a_2(y) - a_1(y)|dy}{(\int_\tau^t dy)^{1/2}(\int_\tau^t dy)^{1/2}\left[(\int_\tau^t dy)^{1/2} + (\int_\tau^t dy)^{1/2}\right]} \psi'(\tau)d\tau \\
&\leq \frac{s(h,t)^4}{2\sqrt{\pi}g(t)i(h,t)^3} \int_0^t \frac{\int_\tau^t |a_2(y) - a_1(y)|dy}{(t-\tau)^{3/2}} \psi'(\tau)d\tau \\
&\leq \frac{s(h,t)^4 s(|a_1 - a_2|, t)}{2\sqrt{\pi}g(t)i(h,t)^3} \int_0^t \frac{\int_\tau^t dy}{(t-\tau)^{3/2}} \psi'(\tau)d\tau \\
&\leq \frac{s(h,t)^4 s(|a_1 - a_2|, t)}{2\sqrt{\pi}g(t)i(h,t)^3} \int_0^t \frac{\psi'(\tau)}{(t-\tau)^{1/2}} d\tau = \frac{s(h,t)^4 s(|a_1 - a_2|, t)}{2\sqrt{\pi}g(t)i(h,t)^3} \frac{\sqrt{\pi}g(t)}{h(t)} \\
&\leq \frac{s(h,t)^4}{2i(h,t)^4} s(|a_1 - a_2|, t)
\end{aligned} \tag{2.3.52}$$

Thus, we have

$$s(|a_2 - a_1|, t) \leq \frac{s(h,t)^4}{2i(h,t)^4} s(|a_1 - a_2|, t)$$

for $0 \leq t < T$. Since $\lim_{t \to 0^+} h(t) = h_0$, we have

$$\lim_{t \to 0^+} \frac{s(h,t)^4}{2i(h,t)^4} = \frac{1}{2}. \tag{2.3.53}$$

Since $h(t)$ is continuous, from the equation (2.3.53), for each $0 < \epsilon < \frac{1}{2}$, there exists a $t_0 > 0$ such that for all t, $0 \leq t \leq t_0$, we have

$$\left| \frac{s(h,t)^4}{2i(h,t)^4} - \frac{1}{2} \right| < \frac{1}{2} - \epsilon,$$

or

$$0 < \frac{s(h,t)^4}{2i(h,t)^4} < 1-\varepsilon, \quad 0<\varepsilon<\frac{1}{2}.$$

Thus, for $0 \le t \le t_0$, we have

$$s(|a_2-a_1|,t) \le (1-\varepsilon)s(|a_1-a_2|,t). \tag{2.3.54}$$

which implies that

$$a_1(t) \equiv a_2(t)$$

for $0 \le t \le t_0$.

Now, we will consider the inequality (2.3.52). Then, we have

$$\begin{aligned}|a_2(t)-a_1(t)| &\le \frac{s(h,t)^4}{2\sqrt{\pi}g(t)i(h,t)^3}\int_0^t \frac{\int_\tau^t |a_2(y)-a_1(y)|dy}{(t-\tau)^{3/2}}\psi'(\tau)d\tau, \quad t_0 \le t \le T_0 \\ &\le C\int_0^t (t-\tau)^{-3/2}\int_\tau^t |a_2(y)-a_1(y)|dyd\tau, \quad t_0 \le t \le T_0,\end{aligned} \tag{2.3.55}$$

where

$$C = \frac{s(h,T_0)^4 s(\psi',T_0)}{2\sqrt{\pi}i(h,T_0)^3 \inf\limits_{t_0\le t\le T_0} g(t)}.$$

Applying Fubini's Theorem,

$$\begin{aligned}\int_0^t (t-\tau)^{-3/2}\int_\tau^t |a_2(y)-a_1(y)|dyd\tau &= \int_0^t |a_2(y)-a_1(y)|\int_0^y (t-\tau)^{-3/2}d\tau dy \\ &= \int_0^t |a_2(y)-a_1(y)|\big[2(t-\tau)^{-1/2}\Big|_{\tau=0}^{\tau=y} dy \\ &= 2\int_0^t |a_2(y)-a_1(y)|\big[(t-y)^{-1/2}-t^{-1/2}\big]dy \\ &\le 2\int_0^t (t-y)^{-1/2}|a_2(y)-a_1(y)|dy\end{aligned} \tag{2.3.56}$$

Substituting (2.3.56) into (2.3.55), we obtain

$$|a_2(t) - a_1(t)| \leq 2C \int_0^t (t-y)^{-1/2} |a_2(y) - a_1(y)| dy, \quad t_0 \leq t \leq T_0. \tag{2.3.57}$$

As $a_1 \equiv a_2$ for $0 \leq t \leq t_0$, (2.3.57) holds for all t, $0 \leq t \leq T_0$. From this inequality it follows from Lemma (1.3.2) that $a_1(t) \equiv a_2(t)$ for $0 \leq t \leq T_0$. As T_0 is an arbitrary positive number less than T, we see that the solution to the nonlinear integral equation (2.3.41) is unique.

Theorem 2.3.8. *If ψ is continuously differentiable for $0 \leq t < T$, $\psi'(t) > 0,\ 0 < t < T$, g is continuous for $0 \leq t < T$, and positive for $0 < t < T$, and the function*

$$h(t) = \frac{\sqrt{\pi} g(t)}{\displaystyle\int_0^t \left[\frac{\psi'(\tau)}{(t-\tau)^{1/2}} \right] d\tau}, \ 0 < t < T,$$

satisfies

$$\lim_{t \to 0^+} h(t) = h_0 > 0,$$

then there is a unique solution to the nonlinear integral equation (2.3.41).

Chapter 3

Inverse Problem for Wave Equation

Now, we consider in this section the determination of a continuous function $a(t)$ defined on the interval $0 \le t \le T$ and a function $u = u(x,t)$ defined on $0 \le x \le \pi,\ 0 \le t \le T$, such that the pair (a, u) satisfies

$$u_{tt} = u_{xx} + a(t)u + F(x,t),\ 0 < x < \pi,\ 0 < t < T; \tag{3.0.1}$$

$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x),\ 0 \le x \le \pi; \tag{3.0.2}$$

$$u(0,t) = u(\pi,t) = 0,\ 0 \le t \le T; \tag{3.0.3}$$

where $\varphi(x),\ \psi(x),\ F(x,t)$ are given functions. We shall determine the functions $a(t)$ and $u(x,t)$ from the measurement

$$u_x(0,t) = g(t),\ 0 \le t \le T. \tag{3.0.4}$$

Definition 3.0.3. Let $Q = (0,\pi) \times (0,T)$. A pair of functions $\{a(t), u(x,t)\}$ is called a solution of (3.0.1)-(3.0.4) if

1. $u \in \mathcal{C}^2(\bar{Q})$, and $a \in \mathcal{C}[0,T]$
2. (3.0.2)-(3.0.4) is satisfied in the usual sense.

Now, we will assume that $\varphi(x),\ \psi(x),\ g(t),\ F(x,t)$ satisfy

Assumption

1. $\varphi(x) \in \mathcal{C}^3[0,\pi]$, $\varphi^{(4)}(x) \in L_2(0,\pi)$, and $\varphi(0) = \varphi(\pi) = \varphi''(0) = \varphi''(\pi) = 0$;

2. $\psi(x) \in \mathcal{C}^2[0,\pi]$, $\psi'''(x) \in L_2(0,\pi)$, and $\psi(0) = \psi(\pi) = \psi''(0) = \psi''(\pi) = 0$;

3. $g(t) \in \mathcal{C}^2[0,T]$, $g(t) \neq 0$, and $g(0) = \varphi'(0),\ g'(0) = \psi'(0)$;

4. $F(x,t) \in \mathcal{C}(\overline{Q})$, $F_{xx}(x,t) \in \mathcal{C}(\overline{Q})$, $F_{xxx}(x,t) \in L_2(Q)$, and $F(0,t) = F(\pi,t) = F_{xx}(0,t) = F_{xx}(\pi,t) = 0$ for all $t \in [0,T]$.

The solution of the problem (3.0.1)-(3.0.4) may be obtained by

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin kx \tag{3.0.5}$$

It is clear that this series satisfies the boundary conditions (3.0.3). Setting the series (3.0.5) into the equation (3.0.1), we obtain

$$\sum_{k=1}^{\infty} u_k''(t) \sin kx = -\sum_{k=1}^{\infty} k^2 u_k(t) \sin kx + \sum_{k=1}^{\infty} a(t) u_k(t) \sin kx + F(x,t) \tag{3.0.6}$$

Expanding $F(x,t)$, we have

$$F(x,t) = \sum_{k=1}^{\infty} F_k(t) \sin kx \tag{3.0.7}$$

where

$$F_k(t) = \frac{2}{\pi} \int_0^{\pi} F(\xi,t) \sin k\xi d\xi,\ \ k = 1,2,\ldots$$

From (3.0.6) and (3.0.7), we obtain

$$\sum_{k=1}^{\infty} \left[u_k''(t) + k^2 u_k(t) - a(t) u_k(t) - F_k(t) \right] \sin kx = 0$$

This equality holds if and only if

$$u_k''(t) + k^2 u_k(t) = a(t) u_k(t) + F_k(t),\ \ k = 1,2,\ldots \tag{3.0.8}$$

Taking into account the initial conditions (3.0.2) and using the variation of parameters method, we can easily see that the solution of the problem (3.0.8) is of the form

$$u_k(t) = \varphi_k \cos kt + \frac{\psi_k}{k} \sin kt + \int_0^t \frac{1}{k} F_k(\tau) \sin[k(t-\tau)]d\tau$$

$$+ \int_0^t \frac{1}{k} a(\tau) u_k(\tau) \sin[k(t-\tau)]d\tau. \tag{3.0.9}$$

Thus, the solution of the problem (3.0.1)-(3.0.4) is of the form

$$u(x,t) = \sum_{k=1}^{\infty} \varphi_k \cos kt \sin kx + \sum_{k=1}^{\infty} \frac{\psi_k}{k} \sin kt \sin kx$$

$$+ \sum_{k=1}^{\infty} \frac{2}{k\pi} \int_0^t \int_0^{\pi} F(\xi,\tau) \sin k\xi \sin[k(t-\tau)]d\xi d\tau \sin kx \tag{3.0.10}$$

$$+ \sum_{k=1}^{\infty} \frac{2}{k\pi} \int_0^t \int_0^{\pi} a(\tau) u(\xi,\tau) \sin k\xi \sin[k(t-\tau)]d\xi d\tau \sin kx.$$

Taking into account the equation (3.0.4), we get the system of integral equation

$$\sum_{k=1}^{\infty} \varphi_k k \cos kt + \sum_{k=1}^{\infty} \psi_k \sin kt + \sum_{k=1}^{\infty} \frac{2}{\pi} \int_0^t \int_0^{\pi} F(\xi,\tau) \sin k\xi \sin[k(t-\tau)]d\xi d\tau$$

$$+ \sum_{k=1}^{\infty} \frac{2}{\pi} \int_0^t \int_0^{\pi} a(\tau) u(\xi,\tau) \sin k\xi \sin[k(t-\tau)]d\xi d\tau = g(t). \tag{3.0.11}$$

Differentiating (3.0.11) with respect to t, we obtain

$$-\sum_{k=1}^{\infty} \varphi_k k^2 \sin kt + \sum_{k=1}^{\infty} \psi_k k \cos kt + \sum_{k=1}^{\infty} \frac{2k}{\pi} \int_0^t \int_0^{\pi} F(\xi,\tau) \sin k\xi \cos[k(t-\tau)]d\xi d\tau$$

$$+ \sum_{k=1}^{\infty} \frac{2k}{\pi} \int_0^t \int_0^{\pi} a(\tau) u(\xi,\tau) \sin k\xi \cos[k(t-\tau)]d\xi d\tau = g'(t). \tag{3.0.12}$$

Again, differentiating the equation (3.0.12) with respect to t, we obtain

$$-\sum_{k=1}^{\infty}\varphi_k k^3\cos kt-\sum_{k=1}^{\infty}\psi_k k^2\sin kt+\sum_{k=1}^{\infty}\frac{2k}{\pi}\int_0^{\pi}F(\xi,t)\sin k\xi d\xi$$

$$-\sum_{k=1}^{\infty}\frac{2k^2}{\pi}\int_0^t\int_0^{\pi}F(\xi,\tau)\sin k\xi\sin[k(t-\tau)]d\xi d\tau+a(t)\sum_{k=1}^{\infty}\frac{2k}{\pi}\int_0^{\pi}u(\xi,t)\sin k\xi d\xi \quad (3.0.13)$$

$$-\sum_{k=1}^{\infty}\frac{2k^2}{\pi}\int_0^t\int_0^{\pi}a(\tau)u(\xi,\tau)\sin k\xi\sin[k(t-\tau)]d\xi d\tau=g''(t).$$

As

$$u(x,t)=\sum_{k=1}^{\infty}\frac{2}{\pi}\int_0^{\pi}u(\xi,t)\sin k\xi d\xi\sin kx,$$

then we have

$$g(t)=u_x(0,t)=\sum_{k=1}^{\infty}\frac{2k}{\pi}\int_0^{\pi}u(\xi,t)\sin k\xi d\xi. \quad (3.0.14)$$

Similarly, from the equation (3.0.7), we obtain

$$F_x(0,t)=\sum_{k=1}^{\infty}\frac{2k}{\pi}\int_0^{\pi}F(\xi,t)\sin k\xi d\xi. \quad (3.0.15)$$

By using the equations (3.0.14) and (3.0.15), we rewrite the equation (3.0.13) to obtain

$$a(t)=\frac{1}{g(t)}\sum_{k=1}^{\infty}\varphi_k k^3\cos kt+\frac{1}{g(t)}\sum_{k=1}^{\infty}\psi_k k^2\sin kt+\frac{g''(t)}{g(t)}-\frac{F_x(0,t)}{g(t)}$$

$$+\frac{1}{g(t)}\sum_{k=1}^{\infty}\frac{2k^2}{\pi}\int_0^t\int_0^{\pi}F(\xi,\tau)\sin k\xi\sin[k(t-\tau)]d\xi d\tau \quad (3.0.16)$$

$$+\frac{1}{g(t)}\sum_{k=1}^{\infty}\frac{2k^2}{\pi}\int_0^t\int_0^{\pi}a(\tau)u(\xi,\tau)\sin k\xi\sin[k(t-\tau)]d\xi d\tau.$$

Theorem 3.0.9. *Under the assumption 1) - 4), suppose that* T_1 *and* T_2 *are some positive numbers that satisfy the conditions*

$$C_1\left(\frac{\pi^2}{3}+\frac{\pi^2}{2H_*^2}\right)T_1^2<1$$

and

$$C_1+\left(\frac{\pi^2}{3}+\frac{\pi^2}{2H_*^2}\right)M^4T_2^2\le M^2$$

where

$$M=\left\{\frac{C_1}{1-C_1\left(\frac{\pi^2}{3}+\frac{\pi^2}{2H_*^2}\right)T_1^2}\right\}^{\frac{1}{2}}, \tag{3.0.17}$$

$$\begin{aligned}C_1=&\frac{8}{\pi}\|\varphi'''\|^2_{L_2(0,\pi)}+\frac{8}{\pi}\|\psi''\|^2_{L_2(0,\pi)}+\frac{8T}{\pi}\left\|\frac{\partial^2 F}{\partial x^2}\right\|^2_{L_2(Q)}+\frac{6}{H_*^2}\|g''\|^2_{\mathcal{C}[0,T]}\\&+\frac{6}{H_*^2}\|F_x(0,\cdot)\|^2_{\mathcal{C}[0,T]}+\frac{2\pi}{H_*^2}\left\|\varphi^{(4)}\right\|^2_{L_2(0,\pi)}+\frac{2\pi}{H_*^2}\left\|\psi^{(3)}\right\|^2_{L_2(0,\pi)}\\&+\frac{2\pi T}{H_*^2}\left\|\frac{\partial^3 F}{\partial x^3}\right\|^2_{L_2(Q)},\end{aligned}$$

$$H_*=\min_{0\le t\le T}|g(t)|.$$

Then, the problem (3.0.1)-(3.0.4) *has a unique solution with* $T\le\min\{T_1,T_2\}$.

Now, we write (3.0.10) and (3.0.16) in the form

$$v=\phi[v] \tag{3.0.18}$$

where

$$v=\{u,a\},\qquad \phi=\{\phi_1(u,a),\phi_2(u,a)\}, \tag{3.0.19}$$

and $\phi_i(u,a),\ i=1,2$ are defined by the right hand side of (3.0.10) and (3.0.16), respectively.

Let $E=\mathcal{B}_2^3(Q)\times\mathcal{C}[0,T]$ be defined with the norm

$$\|v\|_E=\left(\|u\|^2_{\mathcal{B}_2^3(Q)}+\|a\|^2_{\mathcal{C}[0,T]}\right)^{\frac{1}{2}} \tag{3.0.20}$$

Corollary 3.0.10. *E is a Banach space with respect to the given norm* (3.0.20).

Proof. Let $\{v_n\}_{n\in\mathbb{N}} \in E$ be a Cauchy sequence. Then, given $\varepsilon > 0$, there is $K > 0$ such that for all $n, m \geq K$ we have

$$\|v_n - v_m\|_E = \left(\|u_n - u_m\|^2_{\mathcal{B}^3_2(Q)} + \|a_n - a_m\|^2_{\mathcal{C}[0,T]}\right)^{\frac{1}{2}} < \varepsilon$$

Then, it is easy to see that for all $n, m \geq K$

$$\|u_n - u_m\| < \varepsilon \quad \text{and} \quad \|a_n - a_m\| < \varepsilon.$$

Then, $\{u_n\}_{n\in\mathbb{N}}$ and $\{a_n\}_{n\in\mathbb{N}}$ are also Cauchy sequences. Since $\mathcal{B}^3_2(Q)$ and $C[0,T]$ are complete, sequences u_n and a_n converge to some $u \in \mathcal{B}^3_2(Q)$ and $a \in \mathcal{C}[0,T]$, respectively. So there is $K_1 > 0$ such that for all $n \geq K_1$ we have

$$\|u_n - u\|_{\mathcal{B}^3_2(Q)} < \frac{\varepsilon}{\sqrt{2}},$$

and there is $K_2 > 0$ such that for all $n \geq K_2$ we have

$$\|a_n - a\|_{C[0,T]} < \frac{\varepsilon}{\sqrt{2}}.$$

Set $v = \{u, a\}$. Then, for all $n \geq \max K_1, K_2$ we have

$$\|v_n - v\|_E < \varepsilon$$

Hence, v_n converges to $v \in E$. It shows that E is a complete metric space. □

Now, let us show that solutions of (3.0.18) are bounded in E.

Lemma 3.0.11. *If v is the solution of* (3.0.18)*, then*

$$\|v\|_E \leq M \tag{3.0.21}$$

where M is defined by (3.0.17).

Proof. From the equation (3.0.9) and by using the Hölder's Inequality, we obtain

$$\begin{aligned} k^3 \max_{0\leq\tau\leq t} |u_k(\tau)| &\leq k^3|\varphi_k| + k^2|\psi_k| \\ &+ \frac{2}{\pi}\left[\int_0^t \left(k^2 \int_0^\pi F(\xi,\tau)\sin k\xi d\xi\right)^2 d\tau\right]^{\frac{1}{2}} \left[\int_0^t \sin^2[k(t-\tau)]d\tau\right]^{\frac{1}{2}} \\ &+ \left[\int_0^t \left(k^2 a(\tau) u_k(\tau)\right)^2 d\tau\right]^{\frac{1}{2}} \left[\int_0^t \sin^2[k(t-\tau)]d\tau\right]^{\frac{1}{2}}. \end{aligned} \tag{3.0.22}$$

Note that

$$\int_0^t \sin^2[k(t-\tau)]d\tau \le \int_0^t d\tau = t \le T. \tag{3.0.23}$$

From the equation (3.0.22), by using the inequality $\left(\sum_{i=1}^n b_i\right)^2 \le n\sum_{i=1}^n b_i^2$, we obtain

$$\begin{aligned}\sum_{k=1}^\infty \left[k^3 \max_{0\le\tau\le t}|u_k(\tau)|\right]^2 &\le 4\sum_{k=1}^\infty \left(k^3\varphi_k\right)^2 + 4\sum_{k=1}^\infty \left(k^2\psi_k\right)^2 \\ &+ \frac{16T}{\pi^2}\sum_{k=1}^\infty \int_0^T \left(k^2\int_0^\pi F(\xi,\tau)\sin k\xi d\xi\right)^2 d\tau \\ &+ 4T\sum_{k=1}^\infty \int_0^t \left(k^2 a(\tau)u_k(\tau)\right)^2 d\tau.\end{aligned} \tag{3.0.24}$$

By using the simple inequality $2ab \le a^2+2ab+b^2 = (a+b)^2$, we have

$$\begin{aligned}4T\sum_{k=1}^\infty \int_0^t \left(k^2 a(\tau)u_k(\tau)\right)^2 d\tau &\le 4T\int_0^t a^2(\tau)\sum_{k=1}^\infty\left[k^3\max_{0\le\eta\le\tau}|u_k(\eta)|\right]^2\frac{1}{k^2}d\tau \\ &\le 4T\int_0^t a^2(\tau)\sum_{k=1}^\infty\left[k^3\max_{0\le\eta\le\tau}|u_k(\eta)|\right]^2\sum_{k=1}^\infty\frac{1}{k^2}d\tau \\ &\le 2T\int_0^t\left(\left[\max_{0\le\eta\le\tau}|a(\eta)|\right]^2+\sum_{k=1}^\infty\left[k^3\max_{0\le\eta\le\tau}|u_k(\eta)|\right]^2\right)^2\sum_{k=1}^\infty\frac{1}{k^2}d\tau.\end{aligned} \tag{3.0.25}$$

Since

$$\sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{6},$$

from the equation (3.0.24), we have

$$\begin{aligned}\sum_{k=1}^\infty \left[k^3 \max_{0\le\tau\le t}|u_k(\tau)|\right]^2 &\le 4\sum_{k=1}^\infty \left(k^3\varphi_k\right)^2 + 4\sum_{k=1}^\infty \left(k^2\psi_k\right)^2 \\ &+ \frac{16T}{\pi^2}\sum_{k=1}^\infty \int_0^T \left(k^2\int_0^\pi F(\xi,\tau)\sin k\xi d\xi\right)^2 d\tau \\ &+ \frac{T\pi^2}{3}\int_0^t\left(\left[\max_{0\le\eta\le\tau}|a(\eta)|\right]^2+\sum_{k=1}^\infty\left[k^3\max_{0\le\eta\le\tau}|u_k(\eta)|\right]^2\right)^2 d\tau.\end{aligned}$$

Note that

$$\begin{aligned}\|\varphi'''\|^2_{L_2(0,\pi)} &= \int_0^\pi [\varphi'''(x)]^2 dx = \int_0^\pi \left[\frac{d^3}{dx^3}\left(\sum_{k=1}^\infty \varphi_k \sin kx\right)\right]^2 dx \\ &= \int_0^\pi \left(\sum_{k=1}^\infty k^3\varphi_k \cos kx\right)^2 dx = \sum_{k=1}^\infty k^6 \varphi_k^2 \int_0^\pi \cos^2 kx dx \\ &= \frac{\pi}{2}\sum_{k=1}^\infty k^6\varphi_k^2 = \frac{\pi}{2}\sum_{k=1}^\infty \left(k^3\varphi_k\right)^2,\end{aligned} \tag{3.0.26}$$

$$\begin{aligned}\|\psi''\|^2_{L_2(0,\pi)} &= \int_0^\pi [\psi''(x)]^2 dx = \int_0^\pi \left[\frac{d^2}{dx^2}\left(\sum_{k=1}^\infty \psi_k \sin kx\right)\right]^2 dx \\ &= \int_0^\pi \left(\sum_{k=1}^\infty k^2\psi_k \sin kx\right)^2 dx = \sum_{k=1}^\infty k^4 \psi_k^2 \int_0^\pi \sin^2 kx dx \\ &= \frac{\pi}{2}\sum_{k=1}^\infty k^4\varphi_k^2 = \frac{\pi}{2}\sum_{k=1}^\infty \left(k^2\varphi_k\right)^2,\end{aligned}$$

and

$$\begin{aligned}\left\|\frac{\partial^2 F}{\partial x^2}\right\|^2_{L_2(Q)} &= \int_0^T\int_0^\pi \left[\frac{\partial^2 F(x,\tau)}{\partial x^2}\right]^2 dxd\tau \\ &= \int_0^T\int_0^\pi \left[\frac{\partial^2}{\partial x^2}\left(\sum_{k=1}^\infty F_k(\tau)\sin kx\right)\right]^2 dxd\tau \\ &= \int_0^T\int_0^\pi \left(\sum_{k=1}^\infty k^2 F_k(\tau)\sin kx\right)^2 dxd\tau \\ &= \int_0^T \sum_{k=1}^\infty k^4 F_k^2(\tau)\int_0^\pi \sin^2 kxdxd\tau \\ &= \frac{\pi}{2}\sum_{k=1}^\infty \int_0^T k^4F_k^2(\tau)d\tau = \frac{\pi}{2}\sum_{k=1}^\infty\int_0^T \left(k^2F_k(\tau)\right)^2 d\tau \\ &= \frac{2}{\pi}\sum_{k=1}^\infty \int_0^T \left(k^2\int_0^\pi F(\xi,\tau)\sin k\xi d\xi\right)^2 d\tau.\end{aligned} \tag{3.0.27}$$

Therefore, we have

$$\begin{aligned}\sum_{k=1}^\infty \left[k^3 \max_{0\le\tau\le t}|u_k(\tau)|\right]^2 &\le \frac{8}{\pi}\|\varphi'''\|^2_{L_2(0,\pi)} + \frac{8}{\pi}\|\psi''\|^2_{L_2(0,\pi)} + \frac{8T}{\pi}\left\|\frac{\partial^2 F}{\partial x^2}\right\|^2_{L_2(Q)} \\ &+ \frac{T\pi^2}{3}\int_0^t\left(\left[\max_{0\le\eta\le\tau}|a(\eta)|\right]^2 + \sum_{k=1}^\infty\left[k^3\max_{0\le\eta\le\tau}|u_k(\eta)|\right]^2\right)^2 d\tau.\end{aligned} \tag{3.0.28}$$

From the equation (3.0.16) and by using the Hölder's Inequality, we obtain

$$\max_{0\le\tau\le t}|a(\tau)| \le \frac{1}{\min\limits_{0\le t\le T}|g(t)|}\Bigg\{\max_{0\le t\le T}|g''(t)| + \max_{0\le t\le T}|F_x(0,t)| + \sum_{k=1}^{\infty}|\varphi_k|k^3 + \sum_{k=1}^{\infty}|\psi_k|k^2 + \frac{2}{\pi}\sum_{k=1}^{\infty}\left[\int_0^t\left(k^2\int_0^\pi F(\xi,\tau)\sin k\xi d\xi\right)^2 d\tau\right]^{\frac{1}{2}}\left[\int_0^t \sin^2[k(t-\tau)]d\tau\right]^{\frac{1}{2}} + \sum_{k=1}^{\infty}\left[\int_0^t\left(k^2 a(\tau)u_k(\tau)\right)^2 d\tau\right]^{\frac{1}{2}}\left[\int_0^t \sin^2[k(t-\tau)]d\tau\right]^{\frac{1}{2}}\Bigg\}.$$

By using the inequality $\left(\sum\limits_{i=1}^{n} b_i\right)^2 \le n\sum\limits_{i=1}^{n} b_i^2$, and inequality (3.0.23), we obtain

$$\left(\max_{0\le\tau\le t}|a(\tau)|\right)^2 \le \frac{1}{\left(\min\limits_{0\le t\le T}|g(t)|\right)^2}\Bigg\{6\left\|g''\right\|^2_{C[0,T]} + 6\left\|F_x(0,\cdot)\right\|^2_{C[0,T]} + 6\left(\sum_{k=1}^{\infty}|\varphi_k|k^3\right)^2 + 6\left(\sum_{k=1}^{\infty}|\psi_k|k^2\right)^2 + \frac{24T}{\pi^2}\left(\sum_{k=1}^{\infty}\left[\int_0^T\left(k^2\int_0^\pi F(\xi,\tau)\sin k\xi d\xi\right)^2 d\tau\right]^{\frac{1}{2}}\right)^2 + 6T\left(\sum_{k=1}^{\infty}\left[\int_0^t\left(k^2 a(\tau)u_k(\tau)\right)^2 d\tau\right]^{\frac{1}{2}}\right)^2\Bigg\}. \tag{3.0.29}$$

By using the same methods in (3.0.26)-(3.0.27), we have

$$\left(\sum_{k=1}^{\infty}|\varphi_k|k^3\right)^2 = \left(\sum_{k=1}^{\infty}|\varphi_k|k^4\frac{1}{k}\right)^2 \le \sum_{k=1}^{\infty}\left(\varphi_k k^4\right)^2\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}\frac{2}{\pi}\left\|\varphi^{(4)}\right\|^2_{L_2(0,\pi)} = \frac{\pi}{3}\left\|\varphi^{(4)}\right\|^2_{L_2(0,\pi)}, \tag{3.0.30}$$

$$\left(\sum_{k=1}^{\infty}|\psi_k|k^2\right)^2 = \left(\sum_{k=1}^{\infty}|\psi_k|k^3\frac{1}{k}\right)^2 \le \sum_{k=1}^{\infty}\left(\psi_k k^3\right)^2\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}\frac{2}{\pi}\left\|\psi^{(3)}\right\|^2_{L_2(0,\pi)} = \frac{\pi}{3}\left\|\psi^{(3)}\right\|^2_{L_2(0,\pi)}, \tag{3.0.31}$$

and

$$\left(\sum_{k=1}^{\infty}\left[\int_0^T\left(k^2\int_0^{\pi}F(\xi,\tau)\sin k\xi d\xi\right)^2 d\tau\right]^{\frac{1}{2}}\right)^2$$

$$\begin{aligned} &= \left(\sum_{k=1}^{\infty}\frac{1}{k}\left[\int_0^T\left(k^3\int_0^{\pi}F(\xi,\tau)\sin k\xi d\xi\right)^2 d\tau\right]^{\frac{1}{2}}\right)^2 \\ &\leq \sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{k=1}^{\infty}\int_0^T\left(k^3\int_0^{\pi}F(\xi,\tau)\sin k\xi d\xi\right)^2 d\tau \\ &= \frac{\pi^3}{12}\left\|\frac{\partial^3 F}{\partial x^3}\right\|_{L_2(Q)}^2 . \end{aligned} \tag{3.0.32}$$

By using the simple inequality $2ab \leq a^2 + 2ab + b^2 = (a+b)^2$, we have

$$\begin{aligned} \left(\sum_{k=1}^{\infty}\left[\int_0^t\left(k^2 a(\tau)u_k(\tau)\right)^2 d\tau\right]^{\frac{1}{2}}\right)^2 &= \left(\sum_{k=1}^{\infty}\frac{1}{k}\left[\int_0^t\left(k^3 a(\tau)u_k(\tau)\right)^2 d\tau\right]^{\frac{1}{2}}\right)^2 \\ &\leq \sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{k=1}^{\infty}\int_0^t\left(k^3 a(\tau)u_k(\tau)\right)^2 d\tau \\ &= \frac{\pi^2}{12}\int_0^t 2a^2(\tau)\sum_{k=1}^{\infty}\left(k^3u_k(\tau)\right)^2 d\tau \\ &\leq \frac{\pi^2}{12}\int_0^t\left(a^2(\tau)+\sum_{k=1}^{\infty}\left(k^3u_k(\tau)\right)^2\right)^2 d\tau \\ &\leq \frac{\pi^2}{12}\int_0^t\left(\left[\max_{0\leq\eta\leq\tau}|a(\eta)|\right]^2+\sum_{k=1}^{\infty}\left[k^3\max_{0\leq\eta\leq\tau}|u_k(\eta)|\right]^2\right)^2 d\tau. \end{aligned} \tag{3.0.33}$$

Therefore, by using the results (3.0.30)-(3.0.33), we obtain from the equation (3.0.29)

$$\begin{aligned} \left(\max_{0\leq\tau\leq t}|a(\tau)|\right)^2 \leq \frac{1}{H_*^2}\Bigg\{ & 6\left\|g''\right\|_{C[0,T]}^2 + 6\left\|F_x(0,\cdot)\right\|_{C[0,T]}^2 \\ & + 2\pi\left\|\varphi^{(4)}\right\|_{L_2(0,\pi)}^2 + 2\pi\left\|\psi^{(3)}\right\|_{L_2(0,\pi)}^2 + 2\pi T\left\|\frac{\partial^3 F}{\partial x^3}\right\|_{L_2(Q)}^2 \\ & + \frac{\pi^2 T}{2}\int_0^t\left(\left[\max_{0\leq\eta\leq\tau}|a(\eta)|\right]^2+\sum_{k=1}^{\infty}\left[k^3\max_{0\leq\eta\leq\tau}|u_k(\eta)|\right]^2\right)^2 d\tau\Bigg\}. \end{aligned} \tag{3.0.34}$$

where $H_* = \min_{0\leq t\leq T}|g(t)|$.

Therefore, it follows from (3.0.28) and (3.0.34)

$$\left(\max_{0\le\tau\le t}|a(\tau)|\right)^2+\sum_{k=1}^{\infty}\left[k^3\max_{0\le\tau\le t}|u_k(\tau)|\right]^2$$
$$\le C_1+\left(\frac{\pi^2}{3}+\frac{\pi^2}{2H_*^2}\right)T\int_0^t\left(\left[\max_{0\le\eta\le\tau}|a(\eta)|\right]^2+\sum_{k=1}^{\infty}\left[k^3\max_{0\le\eta\le\tau}|u_k(\eta)|\right]^2\right)^2d\tau \tag{3.0.35}$$

where

$$C_1=\frac{8}{\pi}\|\varphi'''\|^2_{L_2(0,\pi)}+\frac{8}{\pi}\|\psi''\|^2_{L_2(0,\pi)}+\frac{8T}{\pi}\left\|\frac{\partial^2F}{\partial x^2}\right\|^2_{L_2(Q)}+\frac{6}{H_*^2}\|g''\|^2_{\mathcal{C}[0,T]}$$
$$+\frac{6}{H_*^2}\|F_x(0,\cdot)\|^2_{\mathcal{C}[0,T]}+\frac{2\pi}{H_*^2}\left\|\varphi^{(4)}\right\|^2_{L_2(0,\pi)}+\frac{2\pi}{H_*^2}\left\|\psi^{(3)}\right\|^2_{L_2(0,\pi)}$$
$$+\frac{2\pi T}{H_*^2}\left\|\frac{\partial^3F}{\partial x^3}\right\|^2_{L_2(Q)}.$$

Since $T_1\ge T$, inequality (3.0.35) holds if we replace T by T_1. Then, by using the Lemma 1.3.3 and under the assumption of the Theorem 3.0.9, we get

$$\underbrace{\left(\max_{0\le\tau\le T}|a(\tau)|\right)^2+\sum_{k=1}^{\infty}\left[k^3\max_{0\le\tau\le T}|u_k(\tau)|\right]^2}_{\|v\|_E^2}\le\frac{C_1}{1-C_1\left(\frac{\pi^2}{3}+\frac{\pi^2}{2H_*^2}\right)T_1^2}$$

Then,

$$\|v\|_E\le M$$

□

We now restrict our attention to the class of functions defined as

$$\mathcal{K}=\left\{v\in E=\mathcal{B}_2^3(Q)\times\mathcal{C}[0,T]\ :\ \|v\|_E\le M\right\}$$

Note that K is complete since K is a closed subset of the complete metric space E.

Let us show that ϕ maps $\mathcal{K}$ into $\mathcal{K}$.

Lemma 3.0.12. *Under the assumption of Theorem 3.0.9, ϕ maps $\mathcal{K}$ into $\mathcal{K}$.*

Proof. Let v be in $\mathcal{K}$. Then, by the definition of ϕ in (3.0.19), we have

$$\|\phi_1v\|^2_{\mathcal{B}_2^3(Q)}=\|u\|^2_{\mathcal{B}_2^3(Q)}\le\frac{8}{\pi}\|\varphi'''\|^2_{L_2(0,\pi)}+\frac{8}{\pi}\|\psi''\|^2_{L_2(0,\pi)}+\frac{8T}{\pi}\left\|\frac{\partial^2F}{\partial x^2}\right\|^2_{L_2(Q)}$$
$$+\frac{T\pi^2}{3}\int_0^t\left(\left[\max_{0\le\eta\le\tau}|a(\eta)|\right]^2+\sum_{k=1}^{\infty}\left[k^3\max_{0\le\eta\le\tau}|u_k(\eta)|\right]^2\right)^2d\tau, \tag{3.0.36}$$

and

$$\|\phi_2 v\|^2_{C[0,T]} = \|a\|^2_{C[0,T]} \le \frac{1}{H_*^2}\Bigg\{ 6\left\|g''\right\|^2_{C[0,T]} + 6\left\|F_x(0,\cdot)\right\|^2_{C[0,T]} + 2\pi\left\|\varphi^{(4)}\right\|^2_{L_2(0,\pi)} + 2\pi\left\|\psi^{(3)}\right\|^2_{L_2(0,\pi)} + 2\pi T\left\|\frac{\partial^3 F}{\partial x^3}\right\|^2_{L_2(Q)} + \frac{\pi^2 T}{2}\int_0^t\left(\left[\max_{0\le\eta\le\tau}|a(\eta)|\right]^2 + \sum_{k=1}^{\infty}\left[k^3\max_{0\le\eta\le\tau}|u_k(\eta)|\right]^2\right)^2 d\tau\Bigg\}. \tag{3.0.37}$$

Then, from (3.0.36) and (3.0.37), we obtain

$$\begin{aligned}
\|\phi v\|^2_E &= \|\phi_1 v\|^2_{\mathcal{B}_2^3(Q)} + \|\phi_2 v\|^2_{C[0,T]} \\
&\le C_1 + \left(\frac{\pi^2}{3} + \frac{\pi^2}{2H_*^2}\right)T\int_0^t\left(\left[\max_{0\le\eta\le\tau}|a(\eta)|\right]^2 + \sum_{k=1}^{\infty}\left[k^3\max_{0\le\eta\le\tau}|u_k(\eta)|\right]^2\right)^2 d\tau \\
&\le C_1 + \left(\frac{\pi^2}{3} + \frac{\pi^2}{2H_*^2}\right)T\int_0^t M^4 d\tau \\
&\le C_1 + \left(\frac{\pi^2}{3} + \frac{\pi^2}{2H_*^2}\right)M^4T^2 \\
&\le C_1 + \left(\frac{\pi^2}{3} + \frac{\pi^2}{2H_*^2}\right)M^4T_2^2.
\end{aligned}$$

So, under the conditions of Theorem, we have

$$\|\phi v\|_E \le M.$$

□

Let us show that some iteration of ϕ is a contraction.

Lemma 3.0.13. *For some $n \in \mathbb{N}$, ϕ^n is a contraction, i.e., there exits a nonnegative real number $k < 1$ such that for all $v, w \in E$,*

$$\|\phi^n v - \phi^n w\| \le k\|u - w\|.$$

Proof. Let $u, w \in \mathcal{K}$ be arbitrary elements. Consider the sequences

$$v^{(0)} = v, \quad v^{(1)} = \phi(v^{(0)}), \quad v^{(2)} = \phi(v^{(1)}), \quad \cdots \quad v^{(n)} = \phi(v^{(n-1)}), \quad \cdots,$$

and

$$w^{(0)} = w, \quad w^{(1)} = \phi(w^{(0)}), \quad w^{(2)} = \phi(w^{(1)}), \quad \cdots \quad w^{(n)} = \phi(w^{(n-1)}), \quad \cdots.$$

Now consider the n-th iteration

$$\left\|v^{(n)} - w^{(n)}\right\|_E^2 = \left\|u^{(n)} - \tilde{u}^{(n)}\right\|_{\mathcal{B}_2^3(Q)}^2 + \left\|a^{(n)} - \tilde{a}^{(n)}\right\|_{\mathcal{C}[0,T]}^2$$

where

$$\begin{aligned} v^{(n)} &= \{u^{(n)}, a^{(n)}\},\ a^{(0)} = a,\ u^{(0)} = u \\ w^{(n)} &= \{\tilde{u}^{(n)}, \tilde{a}^{(n)}\},\ \tilde{a}^{(0)} = \tilde{a},\ \tilde{u}^{(0)} = \tilde{u} \end{aligned}$$

By using the definition of ϕ_1 and the equation (3.0.10), we have

$$\begin{aligned} u^{(1)} - \tilde{u}^{(1)} &= \phi_1(u, a) - \phi_1(\tilde{u}, \tilde{a}) \\ &= \sum_{k=1}^{\infty} \frac{2}{k\pi} \int_0^t \int_0^{\pi} [a(\tau)u(\xi,\tau) - \tilde{a}(\tau)\tilde{u}(\xi,\tau)] \sin k\xi \sin[k(t-\tau)] d\xi d\tau \sin kx \\ &= \sum_{k=1}^{\infty} \frac{2}{k\pi} \int_0^t \int_0^{\pi} \left[\big(a(\tau) - \tilde{a}(\tau)\big)u(\xi,\tau)\right] \sin k\xi \sin[k(t-\tau)] d\xi d\tau \sin kx \\ &+ \sum_{k=1}^{\infty} \frac{2}{k\pi} \int_0^t \int_0^{\pi} \left[\tilde{a}(\tau)\big(u(\xi,\tau) - \tilde{u}(\xi,\tau)\big)\right] \sin k\xi \sin[k(t-\tau)] d\xi d\tau \sin kx. \end{aligned} \tag{3.0.38}$$

By using the equation (3.0.38) and the inequality (3.0.24) and (3.0.25), we have

$$\begin{aligned} &\left\|u^{(1)} - \tilde{u}^{(1)}\right\|_{\mathcal{B}_2^3(Q)} \\ &\leq \left\|\sum_{k=1}^{\infty} \frac{2}{k\pi} \int_0^t \int_0^{\pi} \left[\big(a(\tau) - \tilde{a}(\tau)\big)u(\xi,\tau)\right] \sin k\xi \sin[k(t-\tau)] d\xi d\tau \sin kx\right\|_{\mathcal{B}_2^3(Q)} \\ &+ \left\|\sum_{k=1}^{\infty} \frac{2}{k\pi} \int_0^t \int_0^{\pi} \left[\tilde{a}(\tau)\big(u(\xi,\tau) - \tilde{u}(\xi,\tau)\big)\right] \sin k\xi \sin[k(t-\tau)] d\xi d\tau \sin kx\right\|_{\mathcal{B}_2^3(Q)} \\ &\leq \left\{\frac{4T\pi^2}{6} \int_0^t \big(a(\tau) - \tilde{a}(\tau)\big)^2 \sum_{k=1}^{\infty} \left[k^3 \max_{0 \leq \eta \leq \tau} |u_k(\eta)|\right]^2 d\tau\right\}^{\frac{1}{2}} \\ &+ \left\{\frac{4T\pi^2}{6} \int_0^t \big(\tilde{a}(\tau)\big)^2 \sum_{k=1}^{\infty} \left[k^3 \max_{0 \leq \eta \leq \tau} |u_k(\eta) - \tilde{u}_k(\eta)|\right]^2 d\tau\right\}^{\frac{1}{2}} \\ &\leq \left\{\frac{4T\pi^2 M^2}{6} \int_0^t \left(\max_{0 \leq \eta \leq \tau} |a(\eta) - \tilde{a}(\eta)|\right)^2 d\tau\right\}^{\frac{1}{2}} \\ &+ \left\{\frac{4T\pi^2 M^2}{6} \int_0^t \sum_{k=1}^{\infty} \left[k^3 \max_{0 \leq \eta \leq \tau} |u_k(\eta) - \tilde{u}_k(\eta)|\right]^2 d\tau\right\}^{\frac{1}{2}}. \end{aligned}$$

Therefore, by using the inequality $(a+b)^2 \leq 2(a^2+b^2)$, we have

$$\left\|u^{(1)}-\tilde{u}^{(1)}\right\|^2_{\mathcal{B}^3_2(Q)} \leq \frac{8T\pi^2M^2}{6}\left\{\int_0^t\left(\left\|a^{(0)}-\tilde{a}^{(0)}\right\|^2_{\mathcal{C}[0,T]} + \left\|u^{(0)}-\tilde{u}^{(0)}\right\|^2_{\mathcal{B}^3_2(Q)}\right)d\tau\right\}. \tag{3.0.39}$$

By using the definition of ϕ_2 and the equation (3.0.16), we have

$$\begin{aligned}
a^{(1)}-\tilde{a}^{(1)} &= \phi_2(u,a)-\phi_2(\tilde{u},\tilde{a}) \\
&= \frac{1}{g(t)}\sum_{k=1}^{\infty}\frac{2k^2}{\pi}\int_0^t\int_0^\pi [a(\tau)u(\xi,\tau)-\tilde{a}(\tau)\tilde{u}(\xi,\tau)]\sin k\xi \sin[k(t-\tau)]d\xi d\tau \\
&= \frac{1}{g(t)}\sum_{k=1}^{\infty}\frac{2k^2}{\pi}\int_0^t\int_0^\pi \left[\big(a(\tau)-\tilde{a}(\tau)\big)u(\xi,\tau)\right]\sin k\xi \sin[k(t-\tau)]d\xi d\tau \\
&+ \frac{1}{g(t)}\sum_{k=1}^{\infty}\frac{2k^2}{\pi}\int_0^t\int_0^\pi \left[\tilde{a}(\tau)\big(u(\xi,\tau)-\tilde{u}(\xi,\tau)\big)\right]\sin k\xi \sin[k(t-\tau)]d\xi d\tau.
\end{aligned} \tag{3.0.40}$$

By using the equation (3.0.40) and the inequality (3.0.29) and (3.0.33), we have

$$\begin{aligned}
&\left\|a^{(1)}-\tilde{a}^{(1)}\right\|_{\mathcal{C}[0,T]} \\
&\leq \left\|\frac{1}{g(t)}\sum_{k=1}^{\infty}\frac{2k^2}{\pi}\int_0^t\int_0^\pi \left[\big(a(\tau)-\tilde{a}(\tau)\big)u(\xi,\tau)\right]\sin k\xi \sin[k(t-\tau)]d\xi d\tau\right\|_{\mathcal{C}[0,T]} \\
&+ \left\|\frac{1}{g(t)}\sum_{k=1}^{\infty}\frac{2k^2}{\pi}\int_0^t\int_0^\pi \left[\tilde{a}(\tau)\big(u(\xi,\tau)-\tilde{u}(\xi,\tau)\big)\right]\sin k\xi \sin[k(t-\tau)]d\xi d\tau\right\|_{\mathcal{C}[0,T]} \\
&\leq \left\{T\pi^2\int_0^t \big(a(\tau)-\tilde{a}(\tau)\big)^2\sum_{k=1}^{\infty}\left[k^3\max_{0\leq\eta\leq\tau}|u_k(\eta)|\right]^2 d\tau\right\}^{\frac{1}{2}} \\
&+ \left\{T\pi^2\int_0^t \big(\tilde{a}(\tau)\big)^2\sum_{k=1}^{\infty}\left[k^3\max_{0\leq\eta\leq\tau}|u_k(\eta)-\tilde{u}_k(\eta)|\right]^2 d\tau\right\}^{\frac{1}{2}} \\
&\leq \left\{T\pi^2M^2\int_0^t \left(\max_{0\leq\eta\leq\tau}|a(\eta)-\tilde{a}(\eta)|\right)^2 d\tau\right\}^{\frac{1}{2}} \\
&+ \left\{T\pi^2M^2\int_0^t \sum_{k=1}^{\infty}\left[k^3\max_{0\leq\eta\leq\tau}|u_k(\eta)-\tilde{u}_k(\eta)|\right]^2 d\tau\right\}^{\frac{1}{2}}.
\end{aligned}$$

Therefore, by using the inequality $(a+b)^2 \leq 2(a^2+b^2)$, we have

$$\left\|a^{(1)}-\tilde{a}^{(1)}\right\|^2_{\mathcal{C}[0,T]} \leq 2T\pi^2M^2\left\{\int_0^t\left(\left\|a^{(0)}-\tilde{a}^{(0)}\right\|^2_{\mathcal{C}[0,T]}+\left\|u^{(0)}-\tilde{u}^{(0)}\right\|^2_{\mathcal{B}^3_2(Q)}\right)d\tau\right\}. \tag{3.0.41}$$

Then, from (3.0.39) and (3.0.41), we have

$$\begin{aligned}\left\|v^{(1)}-w^{(1)}\right\|^2_E &= \left\|u^{(1)}-\tilde{u}^{(1)}\right\|^2_{\mathcal{B}^3_2(Q)}+\left\|a^{(1)}-\tilde{a}^{(1)}\right\|^2_{\mathcal{C}[0,T]}\\ &\leq \frac{10T\pi^2M^2}{3}\left\{\int_0^t\left(\left\|a^{(0)}-\tilde{a}^{(0)}\right\|^2_{\mathcal{C}[0,T]}+\left\|u^{(0)}-\tilde{u}^{(0)}\right\|^2_{\mathcal{B}^3_2(Q)}\right)d\tau\right\}\\ &= \frac{10T\pi^2M^2}{3}\int_0^t\left\|v^{(0)}-w^{(0)}\right\|^2_E d\tau.\end{aligned}$$

By induction, we get

$$\left\|v^{(n)}-w^{(n)}\right\|^2_E \leq \left\{\frac{10T\pi^2M^2}{3}\right\}^n\frac{T^n}{n!}\left\|v^{(0)}-w^{(0)}\right\|^2_E,$$

or

$$\left\|\phi^n v-\phi^n w\right\|_E \leq \left\{\left(\frac{10\pi^2M^2}{3}\right)^n\frac{T^{2n}}{n!}\right\}^{\frac{1}{2}}\left\|v-w\right\|_E.$$

Clearly, for large enough n

$$\left\{\left(\frac{10\pi^2M^2}{3}\right)^n\frac{T^{2n}}{n!}\right\}^{\frac{1}{2}} < 1.$$

Hence, ϕ^n is a contraction on $\mathcal{K}$ in the norm E. □

Thus ϕ has a unique fixed point in $\mathcal{K}$ by Theorem 1.2.7

Conclusion

In this work, our main goal is to study the problem of existence and uniqueness of various inverse problems for second order parabolic and hyperbolic equations. We consider the identification of the initial temperature distribution of heat equations from the final data. We study the problem of the identification of the source function of inhomogeneous heat equation. Then we study inverse problems of the determination of the time-dependent coefficients in various heat equations and a wave equation.

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