Risk Hedging in Revenue Management Models with a Mean-Variance Approach

by

Gülce SARI

A Thesis Submitted to the
Graduate School of Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Master of Science

in

Industrial Engineering

Koç University

July, 2013

Koç University
Graduate School of Sciences and Engineering

This is to certify that I have examined this copy of a master's thesis by

Gülce SARI

and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
examining committee have been made.

Committee Members:

____
Prof. Fikri Karaesmen (Advisor)

____
Asst. Prof. Onur Kaya

____
Assoc. Prof. Semih Onur Sezer

Date: _______________________________

To my dearest parents Nevriye and İbrahim SARI,

who believed and supported me in every step I have taken in my life

ABSTRACT

Revenue management problems have become one of the mostly studied problems in the literature and has drawn considerable attention from any field of research and application. This is mainly due to the fact that the problem of allocating a fixed capacity can be encountered in many fields, some of which are hotel industry, healthcare operations and financial services. The randomness in demand creates uncertainty to the decision makers. Most of the related literature assumes that the decision maker is risk-neutral and aims to maximize the expected profit. However, in real life, most individuals may be risk-sensitive in a sense that they tend to avoid risk in exchange for a reduction in the return. Moreover, a common assumption in the related literature is independence of the demand variables; most studies do not account for the case where the demands are related. Nevertheless, it is known that in most cases, the demand variables tend to depend on each other. In this thesis, we present the results for both independent and dependent demand cases and follow a mean-variance approach to the revenue management problem to account for the risk-sensitivity of the decision maker. Furthermore, we incorporate hedging into the model by assuming that there is a correlation between the demand and the financial market, for which there is strong statistical evidence. The risk is hedged by investing in a portfolio of financial instruments. We determine the optimal portfolio and the optimal protection level simultaneously by solving the mean-variance objective function. While describing demand structure, we account for both the perfect and partial dependency cases with the financial market. Finally, we perform simulation studies to illustrate our findings. Numerical illustrations are presented to show the relationship between the risk aversion level and the optimal order quantity as well as to quantify the effect of the mean-variance approach and hedging on the variance of the revenue management cash flow.

Keywords : Revenue management model, mean-variance approach, risk management

ÖZETÇE

Gelir yönetimi problemi literatürde en çok işlenen konulardan biridir. Esas olarak bu problemde sabit bir kaynağın en iyi biçimde paylaştırılması ile gelirin eniyilenmesi hedeflenmektedir. Bu problem çok sıkça karşılaşılan bir problem olduğu için, çoğu araştırma ve uygulama alanları gelir yönetimine ihtiyaç duymaktadır; bu yüzden uygulama alanı geniştir. Bu problemde talepteki rassallık; belirsizlik ve ona bağlı riskli durumlar yaratmaktadır. Çoğu yöneticinin riskten kaçındığı bilinmesine rağmen ilgili literatürün çoğunda yöneticiler riske duyarsız olarak kabul edilmiştir. Literatürdeki çoğu çalışmadaki başka bir eksiklik ise talep değişkenlerinin birbirinden bağımsız kabul edilmesidir. Ancak bu varsayımın gerçekçi olmadığı birçok durumda gözlemlenmiştir. Bu iki eksikliği gidermek adına bu tezde biz gelir yönetimi modellerine ortalama-varyans yaklaşımını izlemekteyiz. Sonuçları bağımsız ve bağımlı talep değişkenleri için ayrı ayrı sunmaktayız. Sonrasında ise rassallığı oluşturan müşteri talebinin bazı finansal endeksler veya varlıklar ile korelasyonu olduğu durumlar incelenecektir. Yönetici, finansal marketlerdeki bu varlıklara veya onların türevlerine yatırım yaparak riskini azaltabilecektir. Ortalama-varyans amaç fonksiyonu çözülerek en iyi portföy ve en iyi koruma seviyesi aynı anda belirlenebilecektir. Riskten kaçınma seviyesi ile koruma seviyesi arasındaki ilişki numerik örneklerle gösterilecek, ardından ortalama-varyans yaklaşımının ve finansal işlemlerin modeldeki riske etkisi tartışılacaktır.

Anahtar kelimeler: Gelir yönetimi modeli, ortalama-varyans yaklaşımı, risk yönetimi

ACKNOWLEDGMENTS

First I would like to gratefully acknowledge the excellent supervision of my advisors Prof. Fikri Karaesmen and Prof. Süleyman Özekici who shared their knowledge and experience with me. Without them this work would not be possible. I consider myself very lucky to have advisors such as them and I am also very proud of being one of their students.

I would like to thank to Prof. Onur Kaya and Prof. Semih Onur Sezer for taking part in my thesis committee, for critical reading of this thesis and for their valuable suggestions and comments.

I also thank TUBITAK (The Scientific and Technological Research Council of Turkey) for their generous financial support.

I would like to thank to all my friends at Koç University, especially to Zehra, Can, Senem, Aysim, İrem, Efe, Abdullah, Caner, Gökhan and Müge for their valuable friendship which I will not forget and for all the fun and good times we shared together. I also want to thank to my best friends Gülru, Özlem, Çiğdem, Merve, İrem and Tolga for their support and encouragement throughout my study.

The best outcome from the past two years is finding my best friend and my love, Deniz. I truly thank him, whose love and encouragement allowed me to finish this work. I thank him, too, for his patience with the long hours when he should have been sleeping but instead stayed up with me to keep me awake while I was writing my thesis.

I also wish to express my gratitude to my parents, Nevriye and and İbrahim Sarı, for always believing in me and for their endless, supporting love. Finally, I would like to dedicate this work to my departed relatives including my grandmother Hamide Sarı and my grandfathers Emrullah Tuna and Niyazi Sarı, and lastly to my aunt Kıymet Tuna, who left us too soon. This thesis is also dedicated to my dearest aunt Semiha Sarı and my amazing grandmother Sıdıka Tuna. I hope that this work makes you proud.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

INTRODUCTION

Revenue management is the collection of strategies and tactics to scientifically coordinate the procedures of demand management to maximize the profit over the long run. The goal is to sell the right product to the right customer at the right time at the right price. Since its first emergence in the airline industry in the mid 1970s, the revenue management concept has become one of the biggest successes enabling higher profits to most firms. Firms have become dependent on this practice for their success in the long term. While growing in importance, the revenue management concept has spread to virtually any industry that aims to make a profit; such as automobile rental, broadcasting, cruise lines, Internet service provision, lodging and hospitality, non-profit sectors and passenger railways, as argued by McGill and Ryzin (1999). Statistical forecasting techniques and mathematical optimization methods developed after deregulation in the airline industry created significant returns to airline firms. Smith et al. (1992) state that these statistical forecasting techniques and optimization methods are found to generate 2%-8% more revenue compared to the case that no revenue management method is employed or manual methods are used. McGill and Ryzin (1999) divide the revenue management approach into four components; namely, forecasting, overbooking, seat inventory control, and pricing. The focus in this thesis is on the quantity decisions; more specifically, the allocation and management of the seat inventory.

Capacity allocation, also known as fare class management, is the process of deciding what portion of the capacity to reserve for the higher paying customers who may arrive to the system later. In static capacity allocation models, this decision must be made before the demands are realized, thus there is a potential loss due to this uncertainty. If many seats are reserved for the high-revenue customers, some of the low-revenue customers may be rejected and low-demand for high-revenue resources may result in unused capacity. If less seats are made available for the high-revenue customers, then some of the high-revenue customers may be rejected or they may decide to buy low-revenue seats, implying down-sell.

Revenue management and its applications have been widely studied in the literature due to the benefits and the easy application techniques that revenue management offers to any field of research. Most of the studies in this context make two important assumptions. The first is risk-neutrality. That is, the decision maker simply aims to maximize the returns ignoring the effects of the risk; he is completely indifferent to any form of risk. The second is the independence of the demand random variables for demand classes. However, these assumptions have been proved to be unrealistic in many cases. Most individuals may be risk-sensitive in a sense that they tend to avoid risk in exchange for a reduction in the return. Moreover, the independent demands assumption is challenged by some cases which indicate that there are significant reasons why the demands are mostly dependent. One of them is the case of scheduled events. In the presence of a scheduled event such as professional conferences, the demands for all fare classes tend to be higher. A positive correlation is expected among all fare classes. Another possibility is that some of the discount fare customers will upgrade to full fare if the discount fare booking limit is reached. There will be a positive correlation in this case, as well.

Risk models enable incorporating risk management into any decision subject to the uncertainty. There is a vast amount of literature on risk models, however in the revenue management context there are not many incorporating risk sensitivity. There are a number of approaches that can be used; for example, some of them are utility models, some are mean-variance (MV) models, and some are value-at-risk (VaR) models. Utility models represent the satisfaction of the decision maker. The aim is maximizing the expected utility of the cash flow rather than the cash flow itself. MV approach on the other hand, aims to capture the tradeoff between high return and low risk. The VaR model involves finding the maximum loss on a portfolio of financial assets for a given risk level. In this study we take the MV approach to the revenue management model due to its applicability and comprehensibility.

Studies show that there is correlation between the financial market and stochastic events of most business practices. The financial instruments can be bonds, futures, call options, forwards, along with many other available instruments. The decision maker may form a portfolio using these instruments to hedge the risk involved due to randomness. As a result, the variance of the cash flow is reduced, implying reduced risk. The reduction depends on the degree of correlation between randomness and the price of the financial instrument. Throughout this thesis, the market is assumed to be complete and arbitrage-free. Lastly, despite the number of studies on financial hedging, in the RM literature there are only few. This investigation makes our study novel and interesting.

This thesis explores the financial hedging strategies under an MV approach to mitigate the risks of demand uncertainty in RM model, where the demands are correlated with the price of a financial asset. This thesis contributes to the literature first by considering an MV approach to the revenue management model, and then by applying financial hedging strategies. The organization of this thesis is as follows. Chapter 2 presents the related literature on RM history, single-leg seat allocation models, risk management models, MV approach, and financial hedging. Chapter 3 describes an MV approach to the two-class RM problem without financial hedging. Chapter 4 explores an MV approach to the two-class RM model with financial hedging by presenting the hedging model and the main results. Chapter 5 analyzes the case of perfect hedging by characterizing its optimal protection levels and hedging strategies for both two-class and the generalized n-class cases. Finally, Chapter 6 presents numerical illustrations and Chapter 7 concludes the thesis.

Chapter 2

LITERATURE REVIEW

Seat allocation models are one of the mostly used basic models in revenue management. The classical seat allocation model does not incorporate risk sensitivity. It aims to maximize the expected profit, ignoring the risk created by the demand uncertainty. However, this is not the case in real life. Most individuals are known to be sensitive to risk, which makes the risk-neutrality assumption invalid in such cases. In this thesis, risk is taken into account in the model by using an MV approach. Then, to decrease the variations in the profit, financial hedging approach is employed. This thesis is mainly motivated by the work of Gaur and Seshadri (2005), who propose the idea that uncertainties in the profit function of the single-period inventory model, also known as the newsvendor model, can be hedged by investing in the instrument in the financial market which are correlated with the uncertainties. Section 2.1 discusses the literature on single-leg RM models. Section 2.2 presents the risk management models and its applications in RM. Section 2.3 provides an overview of the MV approach and related literature. Finally, Section 2.4 summarizes the studies on financial hedging, and Section 2.5 shows the important derivations of the basic RM models.

2.1 Single-leg RM Models

Single-leg seat allocation problems have been at the center of a growing interest despite that in airline industry, real-life problems encountered are mostly of a network-based nature. Birbil et al. (2009) explain this with two reasons. First, the network-based problems tend to be extremely challenging to solve, and they are mostly approximated by heuristic methods. Second, the emergence of small airline companies especially in Europe, having one-hub networks with single-legs, create a need for the development of single-leg literature tailored to their special characteristics.

The airline industry is where the RM concept emerged; the trigger was the Airline

Deregulation Act of 1978. Before this act, the Civil Aeronautics Board (CAB) was the primary body who controlled fares and schedules for all flights in the US. This federal law altered the way that the airline operations had been carried out by removing the controls on the pricing decisions of airline firms. This change stimulated the market; on one hand there were the existing firms now free to set up their own pricing policies, while on the other hand there were the newly established firms aiming to steal some of the share from the deregulated market. The history of RM (fare class management) began in this setting, as discussed in Kole and Lehn (1999).

Starting from the beginning of 1970s, some airline firms came up with a new pricing scheme. They started to offer different fares to the customers of the same flight depending on their booking time. One of them was British Overseas Airways Corporation (BOAC, now known as British Airways), which offered earlybird bookings of lower fares for customers who booked at least twenty-one days before the departure time, as stated by McGill and Ryzin (1999). The idea of protecting the seats for the latecomers first appeared following this event in an attempt to distribute the seats evenly between the passengers willing to pay lower and the ones willing to pay relatively higher fares to the same seat. Then came a success story from American Airlines (AA), providing an insight to grasp the importance of the RM applications, and how crucial and promising they are. In an attempt to compete with the companies offering lower fares, AA held a brainstorming session in 1976. The session concluded that the core problem is not excessive costs, but low profits due to the empty seats. Cross (1997) provides a detailed history of this success story. The main course of action should be pulling more customers by introducing a customized pricing scheme. The first attempt was introducing the 'Super Saver Fares' in 1977. The fares were capacity controlled, only 35 percent of sales were allowed to be Super Savers on a given flight; they were advance-purchase restricted, the bookings should be made at least 30 days before take-off as described in Bailey et al. (1985). In 1985, AA continued the price cuts with the Ultimate Super Saver fares to segment the market between leisure and business customers, and eventually pull customers to increase the profits. Jenkins and Ray (1995) point out that even the term 'Yield Management' was coined by the Senior Vice President of American Airlines, Bob Crandall, who was in charge of all this work. This program brought the 1991 Edelman Prize for the best application of management science to the developers of the program, as discussed in Li (2010).

The research before the deregulation covered mostly the overbooking concept in airline and hotel management due to the heavy supervision on pricing operations. Thus the work of Rothstein (1971) and Rothstein (1974), being some of the first works in the RM literature, mostly deal with characterizations of overbooking strategies. Being another one of the first studies in the RM literature, Littlewood (1972) of BOAC proposed a procedure to determine a booking limit on the lower revenue fare class on a single-leg flight with stochastic demand arrivals. The trade-off was whether obtaining a sure revenue of a lower fare class or the expected revenue of rejecting the request of lower fare class and reserving it for a higher fare class customer. Simply, the core idea is setting a booking limit that minimizes the expected losses in future revenue from using the capacity now rather than using it in the future. This method often referred to as Littlewood's rule. Littlewood's rule is discussed in detail with primary results at the end of Section 2.

Almost all the early work in seat inventory control literature requires a set of simplifying assumptions. McGill and Ryzin (1999) summarizes these assumptions as follows: 1) sequential booking classes, 2) low-before-high fare booking arrival pattern, 3) statistical independence of demands between booking classes, 4) no cancellations or no-shows, 5) single flight leg with no consideration of network effects, and, 6) no batch booking. There are a number of works related to the Littlewood's work, most studied under these assumptions. Bhatia and Parekh (1973) and Richter (1982) provide the derivations of the Littlewood's rule. Mayer (1976) and Titze and Griesshaber (1983) provide simulation studies evaluating the performance of the rule. The former suggests that successive uses of the rule for a flight yields results almost as good as the complex DP solution, while the latter work shows the robustness of the rule in settings where the low-before-high fare assumption is not valid.

Following the work of Littlewood, a series of work on airline seat inventory control is published by Belobaba. Belobaba (1987b) presents an extension of Littlewood's rule, introducing the seat importance control methods (EMSR-a and EMSR-b). EMSR-a and EMSR-b are heuristic methods created to set booking limits in static, single-leg yield management problems with multiple fare classes. The sole difference is the approximation method of the expected revenue. EMSR-a aggregates the protection levels while EMSR-b aggregates the demand. In fact, EMSR-b is found to yield more realistic approximations and has been used widely, according to Ryzin and McGill (2000). These methods do not provide optimal booking limits (except the two-fare case), as one might expect. Furthermore, Belobaba (1987a) and Belobaba (1989) provide a more detailed anaylsis involving the implementation of a computerized system to set the booking limits systemically.

Meanwhile, following the work of Belobaba, McGill (1989) and Wollmer (1992) provide numerical studies of the EMSR for typical airline demand distributions to prove the validity and soundness of the method. Wollmer (1992) discusses that EMSR provide quite close results to the optimal solution in most of the cases. On the other hand, the model may fail to provide reasonable results for more general distributions, as discussed by Robinson (1995). For these situations he suggests using a Monte Carlo simulation instead. He continues with arguing that it is critical to assess the method's validity for particular cases before employing it.

There are works characterizing the optimal booking limits for single leg flights, such as McGill (1989), Curry (1990), Wollmer (1992) and Brumelle and McGill (1993). Brumelle and McGill characterize the problem as a series of monotone optimal stopping problems (where all six assumptions discussed before are made). They further analyze the model to determine the optimal booking limits with a set of probability conditions, which were later generalized by Robinson (1995) easily in an attempt to relax the low-before-high fare assumption. For more information on the origins and history of revenue management may refer to Belobaba (1987a), Smith et al. (1992), Dunleavy (1995), Vinod (1995), Jenkins and Ray (1995), and finally the extensive book by Talluri and Van Ryzin (2005).

The literature discussed up to this point includes the milestones in revenue management. However, most of them suffer from a simplifying but unrealistic assumption that the consumer demands for each fare product is independent from each other. The first work to consider a setting that takes into account possible dependencies between demands is from Belobaba (1987b). He characterizes the optimality conditions in single-leg setting where the customers upgrade to the higher fares if the lower fare classes are not available (termed as buy-up later). Pfeifer (1989) provides a proof of the result. Stochastically dependent discount and full fare demands case is studied by McGill (1989) and Brumelle et al. (1990). They show that the optimality condition for the booking limit is a variant of the Littlewood's rule, with a monotonicity assumption on the dependency of demands. This assumption allows them to show that the problem is also a monotone optimal stopping problem. In this thesis we take an approach similar to that of Brumelle et al. (1990).

2.2 Risk Management Models in RM

The term risk management has been studied for decades; however, recent developments in methods and techniques of risk management has made this rise a more rapid one, discusses Merton (1995). Risk has been incorporated to the works in RM literature only in recent years. Bitran and Caldentey (2003, p. 224) state that "essentially all the (RM) models that we have discussed assume that the seller is risk-neutral." The literature available mostly assume a risk-neutral decision maker only aiming to maximize the expected total profit. Nonetheless, decision makers are concerned about the variations in profit. Actually, Schweitzer and Cachon (2000) present experimental evidence on this issue, and find that for some high-profit products, decision makers behave risk-aversely. A risk-neutral approach is justifiable for revenue management problems considering the long-term average effect for the case that the same problem occurs in numerous instances in a small time interval. However, there are cases in RM where the event occurs less frequently. Levin et al. (2008) discuss such cases. That is why, it is of critical importance to account for the risk-aversion to have a realistic approach for these cases, as well.

Being the first study on RM with risk sensitivity, Feng and Xiao (1999) present a risk-sensitive pricing model to maximize sales revenue of perishable commodities with two given prices and allowing for only one price change. They incorporate the risk factor by adding a weighted penalty to the objective function to account for the sales variance (the risk). Levin et al. (2008) look at a dynamic pricing model of perishable products with risk sensitivity. They incorporate a value-at risk approach in the form of a desired minimum level of revenue constraint with a minimum acceptable probability. To incorporate risk-sensitivity, they add a penalty term to the objective function to quantify the effect of the revenue falls below the minimum level defined. Lancaster (2003) discusses the significance of risk factors in airline revenue management using an analysis on the volatility of the historical data of revenue per available seat mile. Mitra and Wang (2005) present a risk-sensitive model for network revenue management, analyzing the objective function by employing mean-variance, mean-standard-deviation and mean-conditional-value-at-risk approaches. They developed the efficient frontier to characterize the effect of the risk-aversion level. Koenig and Meissner (2010) present a problem where multiple products consuming a single resource over a finite time period. Then, they compare the effectiveness of a dynamic pricing policy with a list-price capacity control policy. They discuss the riskiness of the expected revenue, standard deviation and conditional-value-at-risk strategies using numerical illustrations. They found that the list pricing strategy is useful when the dynamic pricing method is not feasible. Birbil et al. (2009) introduce the robust versions of the classical static and dynamic single-leg seat allocation models to account for the uncertain underlying probability distributions. The robust model is found to generate less variability (compared to the classical models) in exchange for a small reduction in average revenue.

Weatherford (2004) pioneered the implementation of the expected utility in RM setting. His main point was to optimize the expected utility of the decision maker to include his risk-aversion level. He also proposes a new heuristic called expected marginal seat utility (EMSU), a modified version of Belobaba (1989)'s expected marginal seat revenue (EMSR) to account for the risk-aversion. Barz and Waldmann (2007) also use the expected utility function in single-leg RM problem; specifically, they employ an exponential utility function to model the risk-aversion. They model both the static and the dynamic problem, and conclude that if the decision maker is risk-averse, he would be more inclined to accept lower prices earlier as time and remaining capacity decreases. Feng and Xiao (2008)'s work is quite similar to Barz and Waldmann (2007)'s; however, they provide the optimal solution in closed form. They also show that the risk-averse model produces more conservative pricing policies.

Huang and Chang (2009) propose the use of a risk-averse dynamic control policy involving a discount factor in the decision function instead of the exponential utility function. The discount factor acts as a risk premium for obtaining the certain revenue now, rather than an uncertain revenue in the future. Koenig and Meissner (2009) extend the work of Huang and Chang (2009) and show that no extra dynamic programming recursions are needed; the risk-averse decision rules can be directly applied using the results of the risk-neutral case. By this approach they offer reduced computational requirements and more convenience for the practitioners.

Robust optimization, maximizing the worst-case expected revenue over all parameter values in the uncertainty set, has also been studied in an RM setting. Demand function uncertainty using a robust optimization approach has been studied by Thiele (2006), Perakis and Roels (2010), Lai and Ng (2005), Lim and Shanthikumar (2007), and Lim et al. (2008).

Seshadri and Subrahmanyam (2005) present papers that contribute to the interdisciplinary area between operations and finance. Barz (2007) writes extensively on the RM problems with risk-sensitivity.

2.3 MV Approach in Operational Problems

The mean-variance approach first appeared in finance as a method for the portfolio management problem. Nobel prize winner Markowitz (1952) presented modern portfolio selection analysis for single period. The analytical expression for the mean-variance efficient frontier was constructed in Markowitz (1956) and Merton (1972). These works determine the portfolio weights for a given value of mean return, so as to minimize the variance of the return. The reason for choosing the MV model in our analysis is the applicability of the MV model. Van Mieghem (2003) and Buzacott et al. (2011) intepret the MV approach as a method which provides implementable solutions; only the mean and variance functions are needed. It also provides useful solutions when compared to expected utility theory. In the literature, the MV approach is frequently compared to the von Neumann-Morgenstern utility (VNMU) approach, which promises a more precise solution compared to MV. However, one drawback of using the utility functions is apparent: there is a vast number of different utility functions and it is hard to assess a proper function. And one advantage of the MV approach is that it is intuitive due to the use of the mean and variance functions in the formulation. Furthermore, the works Levy and Markowitz (1979), Kroll et al. (1984) and Van Mieghem (2003) show that the MV solution is actually in close proximity to the VNMU optimal solution. Another criticized aspect of the MV approach is that it also penalizes the upward deviation from the mean. This problem is addressed by Nawrocki (1999), presenting the downside risk approach. Yet, later, Grootveld and Hallerbach (1999) show that most of the time, the two methods hardly produce significant differences. Steinbach (2001) provides a complete list of works on single-period and multi-period mean-variance models in financial portfolio analysis.

The MV approach has drawn attention from various fields and has been studied extensively. One of the most widely studied areas is inventory management. Being the first work considering an approach similar to the MV, Lau (1980) maximizes an objective function consisting of the expected profit and the standard deviation of the profit for the newsvendor problem. He shows that a risk-averse newsvendor orders less than the risk-neutral counterpart, and the optimal order quantity can be found between the risk-neutral optimal order quantity and zero. Berman and Schnabel (1986) used the MV approach for both the risk-averse and risk-loving newsvendors. Chen and Federgruen (2000) discuss the trade-offs of the newsvendor models and some infinite-horizon models using an MV approach. Martínez-de-Albéniz and Simchi-Levi (2006) consider a manufacturer exposed to the mean-variance trade-offs who signs a portfolio of option contracts with its suppliers. Choi et al. (2008) perform a MV analysis of single supplier single retailer supply chains having the newsvendor setting under a returns policy. Then, Wu et al. (2009) incorporate the stockout cost into the mean-variance objective function in newsvendor model. They show that the risk-averse newsvendor does not necessarily order less than the risk-neutral counterpart in the presence of the stockout cost.

Despite its popularity and applicability, the applications of the MV approach is limited to a number of works in revenue management literature. The work by Feng and Xiao (1999) can be considered as a different form of mean-variance approach. They present a single-resource risk-sensitive pricing model that maximizes the sales revenue of perishable commodities. By adding a penalty to the objective function, they represent the effects of sales variance. In a different area, Mitra and Wang (2005) discuss risk modeling for traffic and revenue management in networks. They construct the objective function by using mean-variance, mean-standard-deviation, and mean-conditional-value-at-risk, and construct the efficient frontier for a truncated Gaussian demand distribution. Huang and Chang (2009) study the dynamic capacity control problem for risk-sensitive decision makers in terms of mean versus standard deviation. Koenig and Meissner (2010) discuss the trade-off between expected revenue and risk represented by standard deviation and conditional-value-at-risk.

2.4 Financial Hedging

Nonfinancial corporations have been known to hedge in financial markets for a long time. The idea of hedging operational risk using financial instruments has been extensively studied in single-period inventory models (known as newsvendor model), where the demand distribution is correlated with the return of the financial market. Anvari (1987) studies the newsvendor model with normal demand distribution and no set up costs, and employed the capital asset pricing model (CAPM). Chung (1990) finds the same result as Anvari's work, but with a different solution method. More recently, Caldentey and Haugh (2006) investigate the operations of a risk-averse nonfinancial company with an MV objective function. The method developed chooses the optimal operating policy and the optimal trading strategy at the same time, in financial markets. Caldentey and Haugh show that differences observed in solution methodologies depend on the use of different information assumptions. Gaur and Seshadri (2005) discuss hedging the inventory risk for the newsvendor problem when the demand is correlated with the price of a financial asset. They show strong statistical evidence that an inventory index (Redbook), which represents average sales is highly correlated with a financial index (S&P 500) representing the average asset prices. This fact allows constructing static hedging strategies in both MV and utility-maximization approaches. When the forecast demand and the price of the asset are linearly dependent, they construct the perfectly hedged cash flow for the arbitrage-free complete market. To make the derivation more realistic, they also provide the partially correlated solution. They show that the risk of inventory carrying can be replicated as a financial portfolio of a wide range of financial instruments like futures, bonds and call options. They derive the optimal ordering policy as well as the amount of investment in a portfolio of financial instruments correlated with the uncertainties in the model. Chu et al. (2009) present a single-product continuously reviewed inventory model. They employ financial hedging to mitigate the inventory risk with an MV approach. Ding et al. (2007) merge the operational and financial hedging approaches and use an MV utility function to model the company's risk-aversion. They find that the use of the operational hedge can increase the revenue while the use of the financial hedge can decrease the variance of the profit. More recently, Okyay et al. (2011, 2013) use the expected cash flow maximization and Tekin and Özekici (2012) takes the MV approach to the newsvendor model.

2.5 Positioning of This Research

Up to now, we have summarized the works in the literature that are related to our study. In this section we will position our study among the literature we have just reviewed. This research surely contributes to the literature in many ways. The work in this thesis can be stated as a mean-variance approach to the revenue management model with hedging. We mainly study the single-leg two-class static models; however in Chapter 4 we also present an extension of the model accounting for the n-class case. In an attempt to incorporate the risk-sensitivity into the model, the MV approach is taken. Despite the number of papers on RM model accounting for risk-sensitivity, those taking the MV approach are rare in the literature. Thus, our work occupies voids in both the RM and the MV literature. Furthermore, in Chapter 5, we add the financial hedging option to the risk-sensitive RM model developed in the early chapters of this thesis. Financial hedging concept is studied extensively in inventory management literature. However, it is new in the RM literature. Indeed, to the extent of our knowledge, there is no published work that incorporates financial hedging to the seat allocation problem which is analyzed in this thesis. However, the financial hedging idea is worth investigating. Gaur and Seshadri (2005) provide examples showing that it is very likely for the demand or sales to be correlated with at least one financial instrument in the market. There are papers using similar approaches as our work which also make use of the idea of Gaur and Seshadri. For example, Okyay et al. (2011, 2013) analyze the newsvendor problem with risk-neutral and minimum-variance cases with hedging using a two-step procedure. This study differs from theirs in two ways; firstly, our aim is to maximize the hedged MV RM objective function and, secondly, the optimal protection level and hedging portfolio are determined together in one step.

2.6 A Detailed Overview of the Basic RM Models

In this section we will present static, single-leg seat allocation models. In static, single-leg models, the demand from each class is assumed to arrive in separate, non-overlapping intervals. Further, it is assumed that the low revenue classes arrive before high revenue classes. This assumption can be relaxed, though. Talluri and Van Ryzin (2005)'s book analyzes this topic extensively. We examine the two-class models in the next two subsections.

2.6.1 Two-class problem (Littlewood's two class model)

Being the pioneering work that all the literature on single-leg static seat allocation is built on, Littlewood's rule is a result which should be clear to the reader. For this purpose, the derivations of the basic results of Littlewood (1972) are presented in this section. Let Q be the number of seats on a scheduled flight. Assume there are only two product classes, class-1 and class-2, with revenues (price minus variable cost) r_1 and r_2, such that $r_2 > r_1$. The demand for class-j is denoted by D_j, which are assumed to be continuous with distribution F_j, and y represents the number of seats that are protected for class-2 customers. Q, the parameter for the capacity, is fixed. The aim in this problem is determine the optimal y^* value, also called *protection level*, so that the number of seats reserved for class-2 customers is determined. In other words, the decision maker is able to sell only $Q - y$ seats to the low-fare class-1 customers, which is defined as *booking limit*. Booking limit and the protection level, then, adds to Q. The revenue function or the cash flow is

$$CF(D_1, D_2, y) = r_1 \min\{D_1, Q - y\} + r_2 \min\{D_2, Q - \min\{D_1, Q - y\}\}. \tag{2.1}$$

Now assume further that demand for class-1 (low revenue class) arrives first. Via a simple marginal analysis, we observe that the optimal protection level, denoted by y^* satisfies

$$r_1 < r_2 P\{D_2 > y^*\}$$

and

$$r_1 \geq r_2 P\{D_2 \geq y^* + 1\}$$

if demand is discrete.

If demand has a continuous distribution, then the optimal protection level is given by

$$P\{D_2 > y^*\} = \frac{r_1}{r_2}, \tag{2.2}$$

and the booking limit is defined correspondingly as

$$b_1^* = Q - y^*.$$

Or we can simply find the expected value of the cash flow function

$$\mathrm{E}[CF(D_1, D_2, y)] = r_1\,\mathrm{E}[\min\{D_1, Q-y\}] + r_2\,\mathrm{E}[\min\{D_2, Q - \min\{D_1, Q-y\}\}]$$

and then take derivative with respect to y to obtain

$$\begin{aligned}\frac{d\,\mathrm{E}[CF(D_1, D_2, y)]}{dy} &= -r_1\,\mathrm{E}[1_{\{D_1>Q-y\}}] + r_2\,\mathrm{E}[1_{\{D_1>Q-y\}}]\,\mathrm{E}[1_{\{D_2>y\}}] \\ &= -r_1 P\{D_1 > Q-y\} + r_2 P\{D_2 > y\} P\{D_1 > Q-y\} \\ &= P\{D_1 > Q-y\}\left(-r_1 + r_2 P\{D_2 > y\}\right).\end{aligned}$$

We cannot comment on concavity in y since $P\{D_1 > Q-y\}$ increases while $P\{D_2 > y\}$ decreases in y. We implicitly assume that $P\{D_2 > y\} > 0$. However, the function is quasi-concave. It increases on the region $[0, y^*]$, since derivative is positive; it decreases on $[y^*, +\infty)$, since derivative is negative. The objective function being quasi-concave satisfies the sufficient condition for the existence of an optimal protection level maximizing the expected value of the cash flow. (2.2) then gives the optimal protection level.

It might also be the case that $y^* > Q$. In that case, the optimal protection level is Q.

2.6.2 Two-class problem with dependent demand

In the previous subsection, demands were assumed to be independent. However, in real life, demands tend to show dependent characteristics. In this subsection, we analyze the two-class problem where class-1 and class-2 demands are stochastically dependent, as studied in Brumelle et al. (1990).

In order to obtain the optimal y^* value, we need the first derivative of the expected cash flow expression (2.1). However this time, we do not impose anything on the demand random variables to analyze the most general case. The derivative of the cash flow is

$$\begin{aligned}\frac{d\,\mathrm{E}[CF(D_1, D_2, y)]}{dy} &= -r_1\,\mathrm{E}[1_{\{D_1>Q-y\}}] + r_2\,\mathrm{E}[1_{\{D_1>Q-y\}\cup\{D_2>y\}}] \\ &= -r_1 P\{D_1 > Q-y\} + r_2 P\{D_2 > y | D_1 > Q-y\} P\{D_1 > Q-y\} \\ &= P\{D_1 > Q-y\}\left(-r_1 + r_2 P\{D_2 > y | D_1 > Q-y\}\right).\end{aligned}$$

The expected RM cash flow function with dependent demand structures is quasi-concave if $P\{D_2 > y | D_1 > Q-y\}$ is decreasing in y. In that case, the above expression can be used to obtain the optimal solution as

$$P\{D_2 > y^*|D_1 > Q - y^*\} = \frac{r_1}{r_2}.$$

Chapter 3

MV APPROACH TO THE TWO-CLASS RM PROBLEM

Most of the RM literature is built on the assumption of risk-neutrality. However, it is unrealistic to assume that individuals are insensitive to the variability in the cash flow for a given return. In this section, we take the mean-variance approach which incorporates the risk-aversion of individuals. The mean-variance approach is a parametric optimization method where the mean measures the expected value of the cash flow while the variance captures the risk involved with the cash flow. In this method, we either maximize the mean subject to an upper bound on the variance, or minimize the variance of the cash flow subject to a lower bound on the mean of the cash flow.

In this two-class RM problem, our objective is accomplishing the optimal allocation of the fixed capacity between two fare classes. Let $CF(D_1, D_2, y)$ denote the random cash flow, where D_1 and D_2 denote the demand random variables for classes 1 and 2. Class-1 is the discount-fare class while class-2 is the full-fare. The decision variable y is the protection level of class-2, which is the number of seats reserved for class-2 requests. Initially, both classes are available for booking. The problem in this context is deciding how much of the fixed capacity to reserve for the full-fare requests.

The objective function of the mean-variance hedging is

$$\max_{y \geq 0} H(\theta, y) = \mathrm{E}[CF(D_1, D_2, y)] - \theta \,\mathrm{Var}[CF(D_1, D_2, y)] \tag{3.1}$$

where $\theta \geq 0$ is a parameter specifying the risk-sensitivity of the individual, called the risk-aversion rate. The larger the θ, the more conservative an individual's behavior will be. Section 3.1 presents an analysis on the static two-class seat allocation problem with the MV objective. Sections 3.2 and 3.3 discuss the cases of independent and dependent demand random variables, respectively. Finally, Section 3.4 investigates the sensitivity of the optimal solution to risk aversion.

3.1 MV Model Characterizations

In this section, we will discuss the classical revenue management (seat allocation) problem with two separate fare classes. Let Q be the number of seats on a scheduled flight, and let r_1 and r_2 denote the revenues (sales price minus variable cost) of the fare classes 1 and 2, respectively, with the assumption that $r_1 < r_2$. Requests for the lower fare class (class-1) are assumed to arrive earlier than class-2 requests. In this setting, we will characterize the optimal protection level, y^*, for the MV objective.

The random cash flow, denoted by $CF(D_1, D_2, y)$, can be written as

$$CF(D_1, D_2, y) = r_1 \min\{D_1, Q - y\} + r_2 \min\{D_2, Q - \min\{D_1, Q - y\}\},$$

where $\min\{D_1, Q-y\}$ gives the number of seats sold to class-1 customers, and $\min\{D_2, Q - \min\{D_1, Q - y\}\}$ gives the number of seats sold to class-2 customers. Note that the term $\min\{D_2, Q - \min\{D_1, Q - y\}\}$ is equivalent to $\min\{D_2, \max\{Q - D_1, y\}\}$ if we remove the inside parenthesis. Some of the following derivations will be done according to this form, however, in general, we will follow the former definition.

For simplicity, in some of the equations, the mean and the variance of the cash flow are denoted by $m(y)$ and $v(y)$, respectively. The mean and variance functions for the MV model are

$$m(y) = \mathrm{E}[CF(D_1, D_2, y)] = r_1\, \mathrm{E}[\min\{D_1, Q - y\}] + r_2\, \mathrm{E}[\min\{D_2, Q - \min\{D_1, Q - y\}\}],$$

and

$$\begin{aligned} v(y) &= \mathrm{Var}[CF(D_1, D_2, y)] \\ &= r_1^2\, \mathrm{Var}[\min\{D_1, Q - y\}] + r_2^2\, \mathrm{Var}[\min\{D_2, Q - \min\{D_1, Q - y\}\}] \\ &\quad + 2r_1 r_2\, \mathrm{Cov}[\min\{D_1, Q - y\}, \min\{D_2, Q - \min\{D_1, Q - y\}\}]. \end{aligned}$$

The objective function can be expressed as

$$\max_{y \geq 0} H(\theta, y) = \mathrm{E}[CF(D_1, D_2, y)] - \theta\, \mathrm{Var}[CF(D_1, D_2, y)] \tag{3.2}$$

where

$$\begin{aligned} H(\theta, y) = {} & r_1\, \mathrm{E}[\min\{D_1, Q - y\}] + r_2\, \mathrm{E}[\min\{D_2, Q - \min\{D_1, Q - y\}\}] \\ & -\theta\Big(r_1^2\, \mathrm{Var}[\min\{D_1, Q - y\}] + r_2^2\, \mathrm{Var}[\min\{D_2, Q - \min\{D_1, Q - y\}\}] \\ & + 2r_1 r_2\, \mathrm{Cov}[\min\{D_1, Q - y\}, \min\{D_2, Q - \min\{D_1, Q - y\}\}]\Big) \end{aligned} \tag{3.3}$$

for any fixed $\theta \geq 0$.

We denote the cumulative distribution function and probability density function of any random variable X by F_X and f_X respectively. Since the objective function contains minimum operators, the following derivations will be helpful in constructing the expected values and the variances. Below equations hold for any constant y and for positive valued random variables X and Z with continuous joint and marginal probability density functions. The probability density functions are given by f_X and f_Z. The conditional density function of Z given $\{X = x\}$ is denoted by $f_{Z|x}$. Suppose y and w are positive valued constants. The expected value of the minimum function is

$$\mathrm{E}[\min\{X, y\}] = \int_0^y x f_X(x)dx + y \int_y^{+\infty} f_X(x)dx,$$

and its derivative with respect to y being

$$\frac{d\,\mathrm{E}[\min\{X, y\}]}{dy} = \int_y^{+\infty} f_X(x)dx = \mathrm{P}\{X > y\} = \mathrm{E}[1_{\{X>y\}}] = 1 - F_X(y). \tag{3.4}$$

Furthermore one can show that

$$\begin{aligned}\mathrm{E}[\min\{Z, \max\{X, y\}\}] &= \int_0^\infty f_X(x)dx \left(\int_0^{\max\{x,y\}} z f_{Z|x}(z)dz + \max\{x, y\} \int_{\max\{x,y\}}^\infty f_{Z|x}(z)dz \right) \\ &= \int_0^y f_X(x)dx \left(\int_0^y z f_{Z|x}(z)dz + y \int_y^\infty f_{Z|x}(z)dz \right) \\ &\quad + \int_y^\infty f_X(x)dx \left(\int_0^x z f_{Z|x}(z)dz + x \int_x^\infty f_{Z|x}(z)dz \right),\end{aligned}$$

and the derivative is then

$$\begin{aligned}\frac{d\,\mathrm{E}[\min\{Z, \max\{X, y\}\}]}{dy} &= \int_0^y f_X(x)dx \int_y^\infty f_{Z|x}(z)dz \\ &= \mathrm{P}\{X \leq y, Z > y\} \\ &= \mathrm{E}[1_{\{X \leq y, Z > y\}}].\end{aligned} \tag{3.5}$$

For the variance terms the derivations are similar,

$$\begin{aligned}\mathrm{Var}[\min\{X, y\}] &= \mathrm{E}[(\min\{X, y\})^2] - (\mathrm{E}[\min\{X, y\}])^2 \\ &= \int_0^y x^2 f_X(x)dx + y^2 \int_y^{+\infty} f_X(x)dx - (\mathrm{E}[\min\{X, y\}])^2,\end{aligned} \tag{3.6}$$

$$
\begin{aligned}
\mathrm{Var}[\min\{Z,\max\{X,y\}\}] =& \mathrm{E}[(\min\{Z,\max\{X,y\}\})^2] - \mathrm{E}[\min\{Z,\max\{X,y\}\}]^2 \\
=& \int_0^\infty f_X(x)dx \left(\int_0^{\max\{x,y\}} z^2 f_{Z|x}(z)dz + \max\{x,y\}^2 \int_{\max\{x,y\}}^\infty f_{Z|x}(z)dz \right) \\
&- \mathrm{E}[\min\{Z,\max\{X,y\}\}]^2 \\
=& \int_0^y f_X(x)dx \left(\int_0^y z^2 f_{Z|x}(z)dz + y^2 \int_y^\infty f_{Z|x}(z)dz \right) \\
&+ \int_y^\infty f_X(x)dx \left(\int_0^x z^2 f_{Z|x}(z)dz + x^2 \int_x^\infty f_{Z|x}(z)dz \right) \\
&- \mathrm{E}[\min\{Z,\max\{X,y\}\}]^2.
\end{aligned}
$$

The covariance term can be derived as follows

$$
\begin{aligned}
\mathrm{Cov}[\min\{X,y\},\min\{Z,\max\{X,w\}\}] =& \mathrm{E}[\min\{X,y\}\min\{Z,\max\{X,w\}\}] \\
&- \mathrm{E}[\min\{X,y\}]\,\mathrm{E}[\min\{Z,\max\{X,w\}\}] \\
=& \int_0^\infty \min\{x,y\} f_X(x)dx \\
&\left(\int_0^{\max\{x,w\}} z f_{Z|x}(z)dz + \max\{x,w\} \int_{\max\{x,w\}}^\infty f_{Z|x}(z)dz \right) \\
&- \mathrm{E}[\min\{X,y\}]\,\mathrm{E}[\min\{Z,\max\{X,w\}\}] \\
=& \int_0^y x f_X(x)dx \left(\int_0^w z f_{Z|x}(z)dz + w \int_w^\infty f_{Z|x}(z)dz \right) \\
&+ \int_y^\infty y f_X(x)dx \left(\int_0^x z f_{Z|x}(z)dz + x \int_x^\infty f_{Z|x}(z)dz \right) \\
&- \mathrm{E}[\min\{X,y\}]\,\mathrm{E}[\min\{Z,\max\{X,w\}\}].
\end{aligned}
$$

The derivatives of the variance functions are found as

$$
\begin{aligned}
\frac{d\,\mathrm{Var}[\min\{X,y\}]}{dy} &= 2y \int_y^\infty f_X(x)dx - 2\,\mathrm{E}[\min\{X,y\}]\,\mathrm{E}[1_{\{X>y\}}] \\
&= 2\left(y\,\mathrm{E}[1_{\{X>y\}}] - \mathrm{E}[\min\{X,y\}]\,\mathrm{E}[1_{\{X>y\}}] \right), \\
&= 2\,\mathrm{Cov}[\min\{X,y\}, 1_{\{X>y\}}]
\end{aligned}
\tag{3.7}
$$

and

$$
\begin{aligned}
\frac{d\,\mathrm{Var}[\min\{Z,\max\{X,y\}\}]}{dy} =& 2y \int_0^y f_X(x)dx \int_y^\infty f_{Z|x}(z)dz \\
&- 2\,\mathrm{E}[\min\{Z,\max\{X,y\}\}]\,\mathrm{E}[1_{\{X\le y, Z>y\}}] \\
=& 2(y\,\mathrm{E}[1_{\{X\le y,Z>y\}}] - \mathrm{E}[\min\{Z,\max\{X,y\}\}]\,\mathrm{E}[1_{\{X\le y,Z>y\}}]) \\
=& 2\,\mathrm{Cov}[\min\{Z,\max\{X,y\}\}, 1_{\{X\le y,Z>y\}}].
\end{aligned}
\tag{3.8}
$$

Furthermore the derivative of the covariance function is

$$\begin{aligned}
\frac{d\,\mathrm{Cov}[\min\{X,y\},\min\{Z,\max\{X,w\}\}]}{dy} &= \int_y^\infty f_X(x)dx \left(\int_0^w z f_{Z|x}(z)dz + 2w\int_w^\infty f_{Z|x}(z)dz\right) \\
&\quad - \mathrm{E}[1_{\{X>y\}}]\,\mathrm{E}[\min\{Z,\max\{X,w\}\}] \\
&\quad - \mathrm{E}[\min\{X,y\}]\,\mathrm{E}[1_{\{X\le y, Z>w\}}] \\
&= \mathrm{E}[1_{\{X>y\}}\min\{X,Z,w\}] \\
&\quad - \mathrm{E}[1_{\{X>y\}}]\,\mathrm{E}[\min\{Z,\max\{X,w\}\}] \\
&\quad + \mathrm{E}[\min\{X,y\}1_{\{X\le y, Z>w\}}] \\
&\quad - \mathrm{E}[\min\{X,y\}]\,\mathrm{E}[1_{\{X\le y, Z>w\}}] \\
&= \mathrm{Cov}[\min\{Z,\max\{X,w\}\}, 1_{\{X>y\}}] \\
&\quad + \mathrm{Cov}[\min\{X,y\}, 1_{\{X\le y, Z>w\}}].
\end{aligned} \tag{3.9}$$

To solve the initial optimization problem (3.2), the derivative of (3.3) with respect to y is found, then is set equal to zero. By making use of (3.4), (3.5), (3.7), (3.8) and (3.9), the most general form of the first order condition becomes

$$\begin{aligned}
\frac{dH(\theta,y)}{dy} =& - r_1\,\mathrm{E}[1_{\{D_1>Q-y\}}] + r_2\,\mathrm{E}[1_{\{D_1\ge Q-y, D_2>y\}}] \\
& - \theta(-2r_1^2\,\mathrm{Cov}[\min\{D_1,Q-y\}, 1_{\{D_1>Q-y\}}] \\
& + 2r_2^2\,\mathrm{Cov}[\min\{D_2, Q-\min\{D_1,Q-y\}\}, 1_{\{D_1\ge Q-y, D_2>y\}}] \\
& - 2r_1r_2\,\mathrm{Cov}[\min\{D_2, Q-\min\{D_1,Q-y\}\}, 1_{\{D_1>Q-y\}}] \\
& + 2r_1r_2\,\mathrm{Cov}[\min\{D_1,Q-y\}, 1_{\{D_1\ge Q-y, D_2>y\}}]) \\
=& 0.
\end{aligned} \tag{3.10}$$

As can be seen above, the objective function and its first order condition do not yield neat equations. For this reason we take a simplifying approach. Depending on the values the risk-aversion rate θ, ranging from zero to $+\infty$; $H(\theta,y)$ takes different values. The two extreme cases $\theta=0$ and $\theta=+\infty$ will be helpful to understand the nature of the MV objective function. This analysis will be carried out separately for the two cases of independent and dependent demand structures.

3.2 MV Approach to the RM Problem with Independent Demands

In this section the RM problem will be analyzed by taking a MV approach, in which the demand random variables are independent. The demand random variables D_1 and D_2 are assumed further to be continuous with continuous densities on $(0, \infty)$.

If D_1 and D_2 are independent, the first order condition becomes

$$\begin{aligned}\frac{d\,\mathrm{E}[CF(D_1, D_2, y)]}{dy} &= \int_{Q-y}^{\infty} f_{D_1}(x)dx \left(-r_1 + r_2 \int_{y}^{\infty} f_{D_2}(z)dz\right) \\ &= \mathrm{P}\{D_1 > Q - y\}\left(-r_1 + r_2\,\mathrm{P}\{D_2 > y\}\right) \\ &= 0.\end{aligned}$$

We observe from the first derivative of the expected value function that it is quasi-concave. Analyzing the following further

$$\frac{d\,\mathrm{E}[CF(D_1, D_2, y)]}{dy} = \mathrm{P}\{D_1 > Q - y\}(-r_1 + r_2\,\mathrm{P}\{D_2 > y\}), \tag{3.11}$$

we observe that the first term $\mathrm{P}\{D_1 > Q-y\}$, being a probability, is always greater than or equal to zero. We denote the expression in parenthesis by $k(y) = -r_1 + r_2\,\mathrm{P}\{D_2 > y\}$. By taking into account the assumption $r_2 > r_1$, it follows that $k(y)$ changes sign from positive to negative at most once. This implies that the objective function is quasi-concave. The point of the maximum of $E[CF(D_1, D_2, y)]$ can be found by setting $k(y)$ equal to zero, and denoted by y^*_{RN}.

Analyzing $k(y)$ is equivalent to analyzing $m'(y)$ since $\mathrm{P}\{D_1 > Q - y\}$ is greater than or equal to zero. Having insight on $m'(y)$ we may determine the properties of $m(y)$.

When $k(y)$ is evaluated at $y = 0$ and $y = Q$ we get the two boundary values

$$k(0) = -r_1 + r_2\,\mathrm{P}\{D_2 > 0\} \tag{3.12}$$

and

$$k(Q) = -r_1 + r_2\,\mathrm{P}\{D_2 > Q\}. \tag{3.13}$$

It is clear that $k(0) \geq k(Q)$, and $k(y)$ is non-increasing in y since $P\{D_2 > y\}$ is non-increasing in y. Then there are two possibilities for the $k(0), k(Q)$ combination, as presented in Figure 3.1.

- If $k(0) > 0$ and $k(Q) < 0$, then the optimal protection level is found by setting the first order condition (3.11) of $m(y)$ equal to zero, as shown in Figure 3.1a.

- If $k(0) > 0$ and $k(Q) \geq 0$, then the optimal protection level y^*_{RN} is Q. It follows from that the function $k(y)$ is always positive between 0 and Q, as shown in Figure 3.1b, which means $m(y)$ is increasing in y.

We are mainly interested in the first case, in which the function is quasi-concave. The analysis below assumes that the first case holds.

The mean function, $m(y)$ being quasi-concave implies that the first order condition (3.11) can be solved to find the optimal protection level y^*_{RN} so that

$$\frac{d\,\mathrm{E}[CF(D_1, D_2, y^*_{RN})]}{dy} = 0$$

or

$$\mathrm{P}\{D_1 > Q - y^*_{RN}\}\,(-r_1 + r_2\,\mathrm{P}\{D_2 > y^*_{RN}\}) = 0. \tag{3.14}$$

Using (3.14), we find the optimal protection level maximizing the objective function as

$$P\{D_2 > y^*_{RN}\} = \frac{r_1}{r_2} \tag{3.15}$$

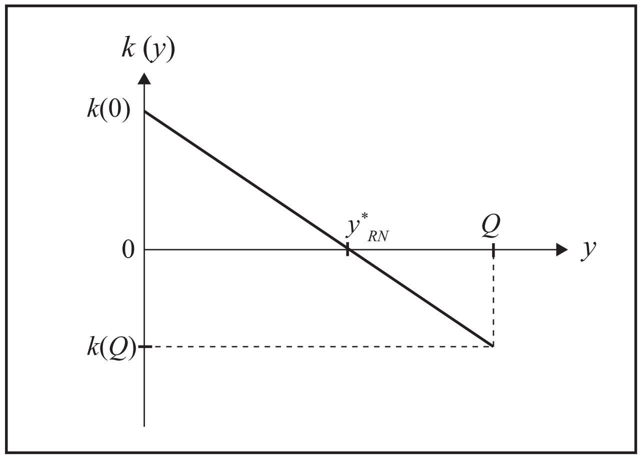
where y^*_{RN} is the optimal protection level for the risk-neutral decision maker, and r_1/r_2 is the critical ratio which satisfies $0 < r_1/r_2 < 1$.

If the probability $P\{D_2 > y\}$ is larger than r_1/r_2, equivalently, if y is less than $F^{-1}_{D_2}((r_2 - r_1)/r_2)$, $k(y)$ is positive. Thus $m(y)$ is increasing in y, on the region $[0, y^*_{RN}]$. If the probability $P\{D_2 > y\}$ is smaller than r_1/r_2, equivalently, if y is greater than $F^{-1}_{D_2}((r_2 - r_1)/r_2)$, $k(y)$ is negative. Thus $m(y)$ is decreasing in y, on the region $[y^*_{RN}, Q]$.

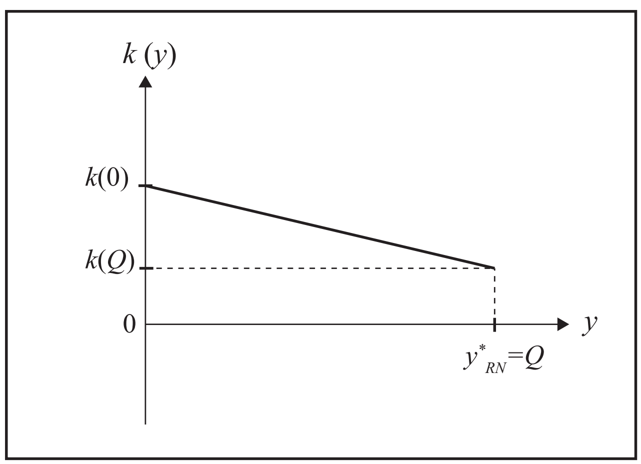
Furthermore, we observe that the mean function is concave on the region $[y^*_{RN}, Q]$. For a function to be concave, its first derivative must be decreasing. When we observe the first derivative of $m(y)$,

$$\mathrm{P}\{D_1 > Q - y\}(-r_1 + r_2\,\mathrm{P}\{D_2 > y\}) \tag{3.16}$$

it is straightforward to see that $\mathrm{P}\{D_1 > Q - y\}$ is increasing in y, and $\mathrm{P}\{D_2 > y\}$ is decreasing in y. Furthermore, the term $-r_1 + r_2\,\mathrm{P}\{D_2 > y\}$ is negative on the region $[y^*_{RN}, Q]$, thus it is negative decreasing in y. Multiplication of a positive increasing and a negative decreasing function yields a negative decreasing function. Then the function is concave on the region $[y^*_{RN}, Q]$.



(a) $k(0) > 0, k(Q) < 0$



(b) $k(0) > 0, k(Q) > 0$

Figure 3.1: y vs. $k(y)$

Lemma 3.2.1 *(a)* $\mathrm{E}[CF(D_1, D_2, y)]$ *is quasi-concave in y; it is increasing on* $[0, y^*_{RN}]$ *and decreasing on* $[y^*_{RN}, \infty]$*. (b)* $\mathrm{E}[CF(D_1, D_2, y)]$ *is concave on the region* $[y^*_{RN}, Q]$*.*

Proof. The analysis in (3.11) and the conditions (3.12) and (3.13) show that $\mathrm{E}[CF(D_1, D_2, y)]$ is quasi-concave and have a unique maximizer y^*_{RN}. Moreover, it is seen from (3.16) that $\mathrm{E}[CF(D_1, D_2, y)]$ is concave on the region $[y^*_{RN}, Q]$. ■

Now consider the case where $\theta \nearrow +\infty$; that is, the individual becomes extremely risk-averse. As θ increases to $+\infty$, the mean part of the objective function becomes negligible, and the problem becomes minimizing the variance of the cash flow, which is

$$\min_{y \geq 0} v(y) = \mathrm{Var}[CF(D_1, D_2, y)].$$

The first order condition is obtained to further analyze the function

$$\begin{aligned} v'(y) = &- 2r_1^2 \,\mathrm{Cov}[\min\{D_1, Q - y\}, 1_{\{D_1 > Q - y\}}] \\ &+ 2r_2^2 \,\mathrm{Cov}[\min\{D_2, Q - \min\{D_1, Q - y\}\}, 1_{\{D_1 \geq Q - y\}} 1_{\{D_2 > y\}}] \\ &- 2r_1 r_2 \,\mathrm{Cov}[\min\{D_2, Q - \min\{D_1, Q - y\}\}, 1_{\{D_1 > Q - y\}}] \\ &+ 2r_1 r_2 \,\mathrm{Cov}[\min\{D_1, Q - y\}, 1_{\{D_1 \geq Q - y\}} 1_{\{D_2 > y\}}] = 0. \end{aligned} \tag{3.17}$$

However, in the RM model, variance of the cash flow function cannot be characterized easily. The variance function does not necessarily possess convexity or concavity. Consider increasing y. Some of the terms may increase in y while some of them may decrease. However, to carry out further analysis, we need a minimum-variance protection level. Actually, numerical examples in Chapter 6 show that for most cases, the variance function is convex. Here the variance is assumed to be quasi-convex to have a unique y^*_{MV} value minimizing the variance.

Assumption 3.2.1 *The function* $v(y) = \mathrm{Var}[CF(D_1, D_2, y)]$ *is quasi-convex on* $[0, Q]$*.*

Then $v(y)$ has a minimizer, denoted by y^*_{MV}, satisfying the fist order condition (3.17).

It is clear from part (a) of Lemma 3.2.1 that there is a finite, positive point maximizing $\mathrm{E}[CF(D_1, D_2, y)]$. We also assumed that the variance function is quasi-convex; a finite, positive point y^*_{MV} minimizing $\mathrm{Var}[CF(D_1, D_2, y)]$ exists. Nonetheless, the objective here is finding the protection level maximizing the MV objective function, which requires a common solution incorporating both the mean and the variance of the cash flow. Due to the nature of the MV problem, there are two conflicting objectives, which makes obtaining an optimal solution harder. For instance, the optimal solution maximizing the mean of the cash flow may be the worst solution for minimizing the variance and vice versa.

Knowing that $m(y)$ is quasi-concave and assuming that $v(y)$ is quasi-convex, we can comment on the efficient set of solutions. For the multi-objective problems, a feasible solution is called efficient (Pareto optimal or non-dominated) if there is no other feasible solution where all the objectives get a better value. We state that y is dominated if and only if there exists y' that satisfies $\mathrm{E}[CF(D_1, D_2, y')] \geq \mathrm{E}[CF(D_1, D_2, y)]$ and $\mathrm{Var}[CF(D_1, D_2, y')] \leq \mathrm{Var}[CF(D_1, D_2, y)]$. Let y^* be an optimal solution to (3.1). Then the following must be true

$$\mathrm{E}[CF(D_1, D_2, y^*)] - \theta\,\mathrm{Var}[CF(D_1, D_2, y^*)] \geq \mathrm{E}[CF(D_1, D_2, y')] - \theta\,\mathrm{Var}[CF(D_1, D_2, y')] \tag{3.18}$$

for all y'. Now suppose there exists a y' such that

$$\mathrm{E}[CF(D_1, D_2, y')] \geq \mathrm{E}[CF(D_1, D_2, y^*)] \tag{3.19}$$

and

$$\mathrm{Var}[CF(D_1, D_2, y')] \leq \mathrm{Var}[CF(D_1, D_2, y^*)]. \tag{3.20}$$

If we multiply (3.20) by $\theta \geq 0$ and subtract it from (3.19), the following inequality is obtained

$$\mathrm{E}[CF(D_1, D_2, y')] - \theta\,\mathrm{Var}[CF(D_1, D_2, y')] \geq \mathrm{E}[CF(D_1, D_2, y^*)] - \theta\,\mathrm{Var}[CF(D_1, D_2, y^*)]$$

where at least one of the inequalities is strict. The above condition contradicts (3.18). Thus, it can be concluded that the solution to the MV problem is from the non-dominated region.

Proposition 3.2.1 *For the MV problem, the non-dominated region is $[y^*_{RN}, y^*_{MV}]$ if $y^*_{RN} \leq y^*_{MV}$, while it is $[y^*_{MV}, y^*_{RN}]$ if $y^*_{MV} < y^*_{RN}$. The optimal solution to the MV problem lies on the region between the risk neutral optimal protection level y^*_{RN} and the minimum variance optimal protection level y^*_{MV}.*

Proof. Assume $y^*_{RN} < y^*_{MV}$. Now suppose that there exists a protection level y such that $y < y^*_{RN}$. Then by quasi-concavity of the cash flow and the optimality of y^*_{RN} for $m(y)$ below condition is true

$$\mathrm{E}[CF(D_1, D_2, y)] \leq \mathrm{E}[CF(D_1, D_2, y^*_{RN})].$$

Furthermore, from the quasi-convexity of the variance and optimality of y^*_{MV} for $v(y)$ the following is true

$$\mathrm{Var}[CF(D_1, D_2, y)] \geq \mathrm{Var}[CF(D_1, D_2, y^*_{RN})].$$

The return obtained at the point y is smaller than that of y^*_{RN}, and the risk at y is greater than that of y^*_{RN}. Thus we can infer that the region $[0, y^*_{RN}]$ is dominated by the point y^*_{RN}. Similarly suppose that there exists a protection level such that $y > y^*_{MV}$. Then by quasi-concavity of the cash flow and the optimality of y^*_{RN} for $m(y)$ below condition is true

$$\mathrm{E}[CF(D_1, D_2, y)] \leq \mathrm{E}[CF(D_1, D_2, y^*_{MV})].$$

Furthermore, from the quasi-convexity of the variance and optimality of y^*_{MV} for $v(y)$ the following is true

$$\mathrm{Var}[CF(D_1, D_2, y)] \geq \mathrm{Var}[CF(D_1, D_2, y^*_{MV})].$$

The return obtained at the point y is smaller than that of y^*_{MV}, and the risk at y is greater than that of y^*_{MV}. Thus we can infer that the region $[y^*_{MV}, Q]$ is dominated. Combining the results we find the non-dominated region as $[y^*_{RN}, y^*_{MV}]$. The same analysis can be carried out for the case $y^*_{MV} < y^*_{RN}$, which will yield the same results. ■

Figure 3.2: Possible orderings for y_{MV} and y_{RN}.

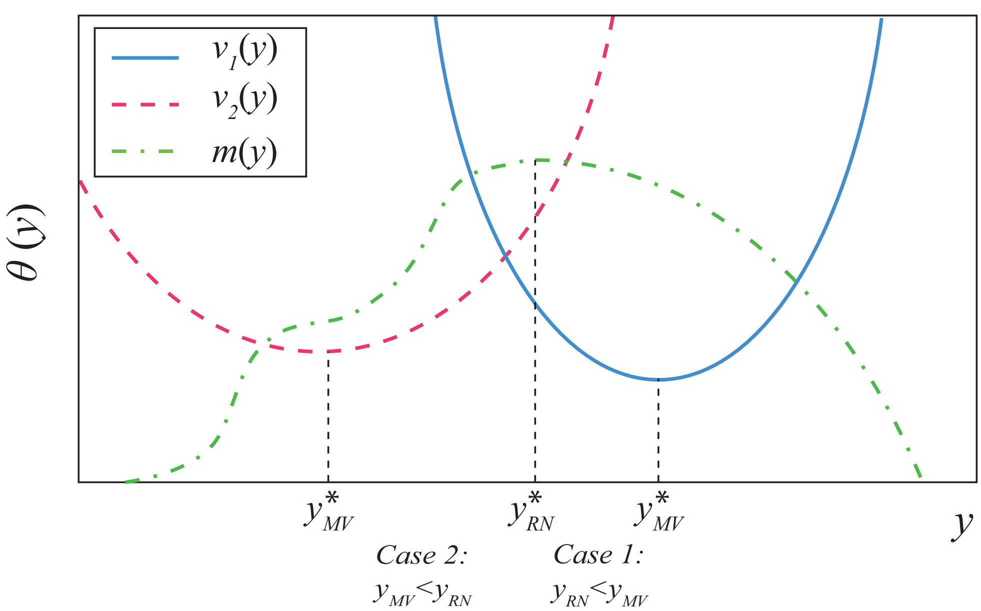


Figure 3.2 illustrates the findings both for $y^*_{MV} \leq y^*_{RN}$ and $y^*_{RN} < y^*_{MV}$. To be able to characterize the relationship between protection level y and risk-aversion rate θ, we need to further assume that $m(y)$ is concave and $v(y)$ is convex between y^*_{RN} and y^*_{MV}.

Assumption 3.2.2 *Between y^*_{MV} and y^*_{RN}, the functions $m(y) = \mathrm{E}[CF(D_1, D_2, y)]$ and $v(y) = \mathrm{Var}[CF(D_1, D_2, y)]$ are concave and convex, respectively.*

The possible arrangements for y^*_{RN} and y^*_{MV} pairs can be observed from Figure 3.2. If $m(y)$ is concave and $v(y)$ is convex between y^*_{RN} and y^*_{MV}, then there is a unique $y(\theta)$ for any risk-aversion rate $\theta \geq 0$.

Define a risk-aversion function $\theta(y)$ which satisfies the optimality condition of (3.10) as follows

$$m'(y) - \theta(y)v'(y) = 0.$$

Then $\theta(y)$ is

$$\theta(y) = \frac{m'(y)}{v'(y)}. \tag{3.21}$$

The first order condition of (3.21) with respect to y is obtained as

$$\theta'(y) = \frac{m''(y)v'(y) - m'(y)v''(y)}{(v'(y))^2} = 0.$$

The above condition allows us to characterize $\theta(y)$. However, it is not straightforward because we do not have much information on $m(y)$ and $v(y)$ except the one that the mean function is quasi-concave, and it is concave on $[y^*_{RN}, Q]$. Assumption 3.2.1 and 3.2.2 are of great value in the analysis of $\theta(y)$.

The characterization of $\theta(y)$ function is the same for both dependent and independent demand cases, and will be presented in Section 3.4.

3.3 MV Approach to the RM Problem with Dependent Demands

In this section, we will analyze the RM problem with the MV approach, where the demand random variables are not necessarily independent.

Consider the case that $\theta = 0$. The objective function takes the form

$$\max_{y \geq 0} m(y) = \mathrm{E}[CF(D_1, D_2, y)],$$

same as in Section 3.2.

The first order condition is

$$\begin{aligned}\frac{d\,\mathrm{E}[CF(D_1,D_2,y)]}{dy} &= -r_1\int_{Q-y}^{\infty} f_{D_1}(x)dx + r_2\int_{Q-y}^{\infty} f_{D_1}(x)dx\int_{y}^{\infty} f_{D_2|D_1}(z|x)dz\\ &= r_1\left(-E[1_{\{D_1>Q-y\}}]\right) + r_2\left(\mathrm{E}[1_{\{D_1>Q-y,D_2>y\}}]\right)\\ &= \mathrm{P}\{D_1>Q-y\}(-r_1+r_2\,\mathrm{P}\{D_2>y|D_1>Q-y\}.\end{aligned}$$

We observe that the expected value function of the cash flow is quasi-concave, dependent on one assumption. Analyzing the first order condition

$$\frac{d\,\mathrm{E}[CF(D_1,D_2,y)]}{dy} = \mathrm{P}\{D_1>Q-y\}\left(-r_1+r_2\,\mathrm{P}\{D_2>y|D_1>Q-y\}\right) \tag{3.22}$$

the first term, $\mathrm{P}\{D_1>Q-y\}$, being a probability, is greater than or equal to zero. We first denote the expression in parenthesis by $k(y)$, such that $k(y)=-r_1+r_2\,\mathrm{P}\{D_2>y|D_1>Q-y\}$.

Assumption 3.3.1 $k(y)=-r_1+r_2\,\mathrm{P}\{D_2>y|D_1>Q-y\}$ *is decreasing in* y.

Analyzing $k(y)$ by taking into account the assumption that $r_2>r_1$ and Assumption 3.3.1, it changes sign from positive to negative at one point. Thus this expression has a threshold point, which is the maximizer of the function $m(y)$, denoted by y^*_{RN}. This clearly shows that the mean function is quasi-concave. The threshold point can be found by setting $k(y)$ equal to zero. Analyzing $k(y)$ will surely prove insight on $m(y)$.

If the probability $P\{D_2>y|D_1>Q-y\}$ is larger than r_1/r_2, equivalently, if y is less than y^*_{RN}, $k(y)$ is positive. Thus $m(y)$ is increasing in y, on the region $[0,y^*_{RN}]$. If the probability $P\{D_2>y\}$ is smaller than r_1/r_2, equivalently, if y is greater than y^*_{RN}, $k(y)$ is negative. Thus $m(y)$ is decreasing in y, on the region $[y^*_{RN},Q]$.

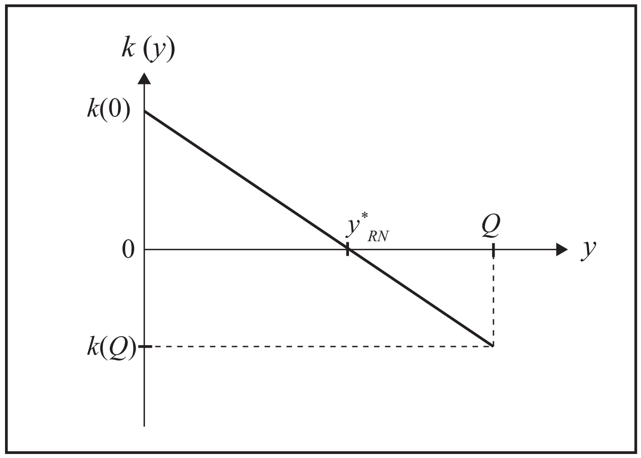
Furthermore, when $k(y)$ is evaluated at $y=0$ and $y=Q$, we get the two boundary values

$$k(0)=-r_1+r_2\,\mathrm{P}\{D_2>0|D_1>Q\} \tag{3.23}$$

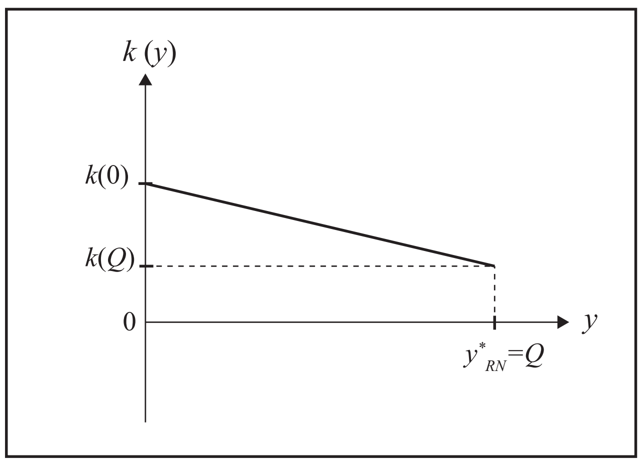
and

$$k(Q)=-r_1+r_2\,\mathrm{P}\{D_2>Q|D_1>0\}. \tag{3.24}$$

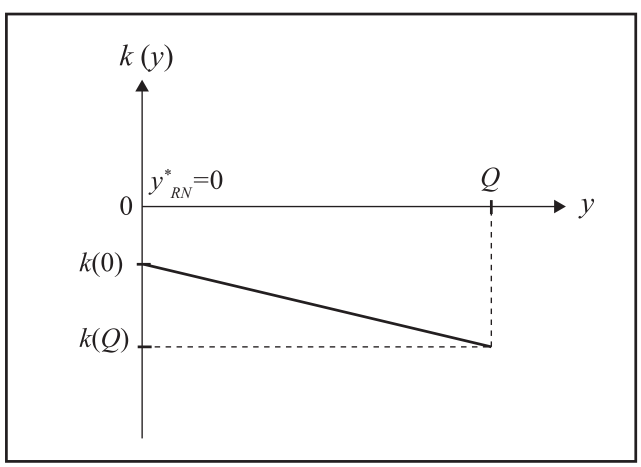
It is clear from Assumption 3.3.1 that $k(0)\geq k(Q)$, and $k(y)$ is decreasing in y. Then there are three possibilities for the $k(0),k(Q)$ combination.



(a) $k(0) > 0, k(Q) < 0$



(b) $k(0) > 0, k(Q) > 0$



(c) $k(0) > 0, k(Q) > 0$

Figure 3.3: y vs. $k(y)$

- If $k(0) > 0$ and $k(Q) < 0$, then the optimal protection level is found by setting the first order condition (3.22) of the $m(y)$ equal to zero. This case is shown in Figure 3.3a.

- If $k(0) > 0$ and $k(Q) \geq 0$, then the optimal protection level y^*_{RN} is Q. It follows from that the function $k(y)$ is always positive between 0 and Q, as shown in Figure 3.3b, which means $m(y)$ is increasing in y.

- If $k(0) \leq 0$ and $k(Q) < 0$, then the optimal protection level y^*_{RN} is 0. It follows from that the function $k(y)$ is always negative between 0 and Q, as shown in Figure 3.3c, which means $m(y)$ is decreasing in y.

We are mainly interested in the first case, in which the function is quasi-concave.

The mean function, $m(y)$ being quasi-concave implies that it has a maximizer y^*_{RN}. The first derivative (3.22) is set equal to zero to find the optimal protection level y^*_{RN} so that

$$\frac{d\,\mathrm{E}[CF(D_1, D_2, y^*_{RN})]}{dy} = 0,$$

or

$$\mathrm{P}\{D_1 > Q - y^*_{RN}\}\,(-r_1 + r_2\,\mathrm{P}\{D_2 > y^*_{RN}|D_1 > Q - y^*_{RN}\}) = 0. \tag{3.25}$$

Using (3.25), we find the optimal protection level maximizing the objective function as

$$\mathrm{P}\{D_2 > y^*_{RN}|D_1 > Q - y^*_{RN}\} = \frac{r_1}{r_2}, \tag{3.26}$$

where y^*_{RN} is the optimal protection level for the risk-neutral decision maker, and r_1/r_2 is the critical ratio which satisfies $0 < r_1/r_2 < 1$. Equation (3.26) gives the optimality condition provided that $k(0) > 0$ (equivalently $m'(0) > 0$) and $k(Q) < 0$ (equivalently $m'(Q) < 0$).

Apart from being quasi-concave, we see that the function is concave on the region $[y^*_{RN}, Q]$. For a function to be concave, its first derivative must be decreasing. When we observe the first derivative of $m(y)$,

$$\mathrm{P}\{D_1 > Q - y\}(-r_1 + r_2\,\mathrm{P}\{D_2 > y|D_1 > Q - y\}) \tag{3.27}$$

it is straightforward to see that $\mathrm{P}\{D_1 > Q - y\}$ is increasing in y, and by Assumption 3.3.1, $P\{D_2 > y|D_1 > Q - y\}$ is decreasing in y. Furthermore, the term $-r_1 + r_2 P\{D_2 > y|D_1 >$

$Q - y\}$ is negative on $[y^*_{RN}, Q]$, thus it is negative decreasing in y. Multiplication of a positive increasing and a negative decreasing function yields a negative decreasing function. Then the function is concave on the region $[y^*_{RN}, Q]$.

Lemma 3.3.1 *(a)* $\mathrm{E}[CF(D_1, D_2, y)]$ *is quasi-concave in* y*; it is increasing on* $[0, y^*_{RN}]$ *and decreasing on* $[y^*_{RN}, \infty]$*. (b)* $\mathrm{E}[CF(D_1, D_2, y)]$ *is concave on* $[y^*_{RN}, Q]$*.*

Proof. The analysis in (3.22) and the conditions (3.23) and (3.24) show that $\mathrm{E}[CF(D_1, D_2, y)]$ is quasi-concave and thus have a unique maximizer y^*_{RN}. Moreover, it is seen from (3.27) that $\mathrm{E}[CF(D_1, D_2, y)]$ is concave on the region $[y^*_{RN}, Q]$. ■

Now consider the case where $\theta \nearrow +\infty$; that is, the individual becomes extremely risk-averse. As θ increases to $+\infty$, the mean part of the objective function becomes negligible and the problem becomes minimizing the variance of the cash flow, which is

$$\min_{y \geq 0} v(y) = \mathrm{Var}[CF(D_1, D_2, y)].$$

The first order condition is obtained to further analyze the function

$$\begin{aligned} v'(y) = & -2r_1^2 \,\mathrm{Cov}[\min\{D_1, Q-y\}, 1_{\{D_1 > Q-y\}}] \\ & + 2r_2^2 \,\mathrm{Cov}[\min\{D_2, Q - \min\{D_1, Q-y\}\}, 1_{\{D_1 \geq Q-y, D_2 > y\}}] \\ & - 2r_1 r_2 \,\mathrm{Cov}[\min\{D_2, Q - \min\{D_1, Q-y\}\}, 1_{\{D_1 > Q-y\}}] \\ & + 2r_1 r_2 \,\mathrm{Cov}[\min\{D_1, Q-y\}, 1_{\{D_1 \geq Q-y, D_2 > y\}}]. \end{aligned} \tag{3.28}$$

When we look at the above expression, the variance function does not promise much to give nice results. It does not necessarily possess convexity or concavity. With an increase in y, the function both gains and loses in some amount. Since we cannot determine if (3.28) increases or decreases in y, we cannot prove that there is a y value minimizing the variance function that can be obtained from the first order condition, which is denoted by y^*_{MV}. However, to carry out further analysis, we need a minimum-variance protection level. Actually, numerical examples in Chapter 6 show that for most cases, the variance function is convex. Here the variance is assumed to be quasi-convex to have a unique y^*_{MV} value minimizing the variance.

Assumption 3.3.2 *The function* $v(y) = \mathrm{Var}[CF(D_1, D_2, y)]$ *is quasi-convex on* $[0, Q]$*.*

Then $v(y)$ has a minimizer, denoted by y^*_{MV}, satisfying the condition that (3.28) is set equal to zero.

It is clear from part (a) of Lemma 3.3.1 and (3.26) that there is a finite, positive point maximizing $\mathrm{E}[CF(D_1, D_2, y)]$. Moreover, by Assumption 3.3.2 a finite, positive point minimizing $\mathrm{Var}[CF(D_1, D_2, y)]$, y^*_{MV} exists. Nonetheless, the objective here is finding the protection level maximizing the MV objective function, which requires a common solution incorporating both the mean and the variance of the cash flow. Due to the nature of the MV problem, there are two conflicting objectives, which makes obtaining an optimal solution harder. For instance, the optimal solution maximizing the mean of the cash flow may be the worst solution for minimizing the variance.

Knowing $m(y)$ is quasi-concave and assuming $v(y)$ is quasi-convex, we can comment on the efficient set of solutions. The derivation of the efficient set of solutions is presented in Section 3.2. For the multi-objective problems, a feasible solution is called efficient (Pareto optimal or non-dominated) if there is no other feasible solution where all the objectives get a better value. We state that y is dominated if and only if there exists y' that satisfies $\mathrm{E}[CF(D_1, D_2, y')] \geq \mathrm{E}[CF(D_1, D_2, y)]$ and $\mathrm{Var}[CF(D_1, D_2, y')] \leq \mathrm{Var}[CF(D_1, D_2, y)]$ where at least one of the inequalities is strict. As it is proven in Section 3.2, the solution to the MV problem is from the non-dominated region, which is proven to be $[y^*_{RN}, y^*_{MV}]$, if $y^*_{RN} < y^*_{MV}$, while it is $[y^*_{MV}, y^*_{RN}]$, if $y^*_{MV} < y^*_{RN}$, in Proposition 3.2.1. Figure 3.2 illustrates the findings both for $y^*_{MV} < y^*_{RN}$ and $y^*_{RN} < y^*_{MV}$.

To be able to characterize the relationship between protection level y and risk-aversion rate $\theta(y)$, we need to make another assumption.

Assumption 3.3.3 *Between y^*_{MV} and y^*_{RN}, the functions $m(y) = \mathrm{E}[CF(D_1, D_2, y)]$ and $v(y) = \mathrm{Var}[CF(D_1, D_2, y)]$ are concave and convex, respectively.*

The possible arrangements for y^*_{RN} and y^*_{MV} pairs can be observed from Figure 3.2. If $m(y)$ is concave and $v(y)$ is convex between y^*_{RN} and y^*_{MV}, then there is a unique y for any risk-aversion rate θ the individuals may have.

Let $\theta(y)$ satisfy the optimality condition of (3.10) as follows

$$m'(y) - \theta(y)v'(y) = 0.$$

Then $\theta(y)$ is

$$\theta(y) = \frac{m'(y)}{v'(y)}. \tag{3.29}$$

Moreover, the first order condition of (3.29) with respect to y is

$$\theta(y)' = \frac{m''(y)v'(y) - m'(y)v''(y)}{(v'(y))^2} = 0.$$

We should be able to use the above condition to characterize $\theta(y)$. However, since the mean function is quasi-concave and the variance function is neither quasi-convex or convex, we are not able to accomplish it using the above condition. Assumption 3.3.2 and 3.3.3 will be helpful in analyzing $\theta(y)$. The characterization of $\theta(y)$ is presented in the next section as the continuation for both Section 3.2 and 3.3.

3.4 Characterization of the Risk-aversion Function $\theta(y)$

The two cases of dependent and independent demands are analyzed up to here. To be able to further analyze $\theta(y)$, we have made assumptions on $m(y)$ and $v(y)$ in the last two sections. Between y^*_{RN} and y^*_{MV}, $m(y)$ and $v(y)$ are assumed to be concave and convex, respectively.

Remember that we have defined $\theta(y)$ symbolically in Section 3.2 and 3.3, such as

$$m'(y) - \theta(y)v'(y) = 0$$

so that

$$\theta(y) = \frac{m'(y)}{v'(y)}. \tag{3.30}$$

Then we obtained the derivative as

$$\theta'(y) = \frac{m''(y)v'(y) - m'(y)v''(y)}{(v'(y))^2}. \tag{3.31}$$

We will now describe the characteristics of $\theta(y)$ and $y(\theta)$.

Consider the case $y^*_{RN} < y^*_{MV}$. Then as shown in Figure 3.2, on the region $[y^*_{RN}, y^*_{MV}]$, $m(y)$ is decreasing, and we assumed earlier that $m(y)$ is concave between y^*_{RN} and y^*_{MV}. Then both $m'(y)$ and $m''(y)$ are negative. For the variance function, on the region $[y^*_{RN}, y^*_{MV}]$, $v(y)$ is also decreasing, and we assumed earlier that $v(y)$ is convex between y^*_{RN} and y^*_{MV}. Then $v'(y)$ is negative and $v''(y)$ is positive.

Now we assess the characteristics of $\theta(y)$ by observing the signs of the function and its first order derivative as follows, for $y^*_{RN} \leq y \leq y^*_{MV}$

$$\begin{aligned} \text{sgn}(\theta(y)) &= \frac{\text{sgn}(m'(y))}{\text{sgn}(v'(y))} \\ &= \frac{(-1)}{(-1)} \\ &= (+1), \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} \text{sgn}(\theta'(y)) &= \frac{\text{sgn}(m''(y)v'(y) - m'(y)v''(y))}{\text{sgn}((v'(y))^2)} \\ &= \frac{(-1)(-1) - (-1)(+1)}{(-1)^2} \\ &= (+1). \end{aligned} \tag{3.33}$$

Then for $y^*_{RN} \leq y^*_{MV}$, $\theta(y)$ is positive and increasing as it can be seen from Figure 3.4a.

Now consider the opposite case, $y^*_{MV} < y^*_{RN}$. Then as shown in Figure 3.2, on the region $[y^*_{MV}, y^*_{RN}]$, $m(y)$ is increasing, and we assumed earlier that $m(y)$ is concave between y^*_{RN} and y^*_{MV}. Then $m'(y)$ is positive while $m''(y)$ is negative. For the variance function, on the region $[y^*_{MV}, y^*_{RN}]$, $v(y)$ is also increasing, and we assumed earlier that $v(y)$ is convex between y^*_{RN} and y^*_{MV}. Then both $v'(y)$ and $v''(y)$ are positive. The sign of the $\theta(y)$ function can be found, for $y^*_{MV} \leq y \leq y^*_{RN}$, such that

$$\begin{aligned} \text{sgn}(\theta(y)) &= \frac{\text{sgn}(m'(y))}{\text{sgn}(v'(y))} \\ &= \frac{(+1)}{(+1)} \\ &= (+1), \end{aligned} \tag{3.34}$$

while the sign of the first derivative is

$$\begin{aligned} \text{sgn}(\theta'(y)) &= \frac{\text{sgn}(m''(y)v'(y) - m'(y)v''(y))}{\text{sgn}((v'(y))^2)} \\ &= \frac{(-1)(+1) - (+1)(+1)}{(+1)^2} \\ &= (-1). \end{aligned} \tag{3.35}$$

Then for $y^*_{MV} < y^*_{RN}$, $\theta(y)$ is positive and decreasing as it can be seen from Figure 3.5a.

Moreover, using (3.30) we determine the range of $\theta(y)$ function for the boundary values $y = y^*_{RN}$ and $y = y^*_{MV}$ as follows

$$\theta(y^*_{RN}) = 0$$

Table 3.1: The analysis of the function $\theta(y)$ for $y^*_{RN} < y^*_{MV}$

	Interval		
Signs of	$[\mathbf{0}, \boldsymbol{y^*_{RN}}]$	$[\boldsymbol{y^*_{RN}}, \boldsymbol{y^*_{MV}}]$	$[\boldsymbol{y^*_{MV}}, \boldsymbol{Q}]$
$\boldsymbol{m'(y)}$	(+)	(-)	(-)
$\boldsymbol{v'(y)}$	(-)	(-)	(+)
$\boldsymbol{\theta(y)}$	(-)	(+)	(-)
$\boldsymbol{\theta'(y)}$		(+)	

Table 3.2: The analysis of the function $\theta(y)$ for $y^*_{MV} < y^*_{RN}$

	Interval		
Signs of	$[\mathbf{0}, \boldsymbol{y^*_{MV}}]$	$[\boldsymbol{y^*_{MV}}, \boldsymbol{y^*_{RN}}]$	$[\boldsymbol{y^*_{RN}}, \boldsymbol{Q}]$
$\boldsymbol{m'(y)}$	(+)	(+)	(-)
$\boldsymbol{v'(y)}$	(-)	(+)	(+)
$\boldsymbol{\theta(y)}$	(-)	(+)	(-)
$\boldsymbol{\theta'(y)}$		(-)	

since $m'(y^*_{RN}) = 0$ and

$$\theta(y^*_{MV}) = \infty$$

since $v'(y^*_{MV}) = 0$.

$\theta(y)$ being an increasing or decreasing function of y means that we have proved the existence of an optimal protection level for any risk-aversion parameter. Now we can find the optimal protection level for the risk-aversion rate θ by taking the inverse Θ^{-1} of $\theta(y)$, for any θ greater than or equal to zero so that

$$y(\theta) = \Theta^{-1}(\theta). \tag{3.36}$$

Theorem 3.4.1 *Under Assumptions 3.2.2 and 3.3.3, the following results are true:*

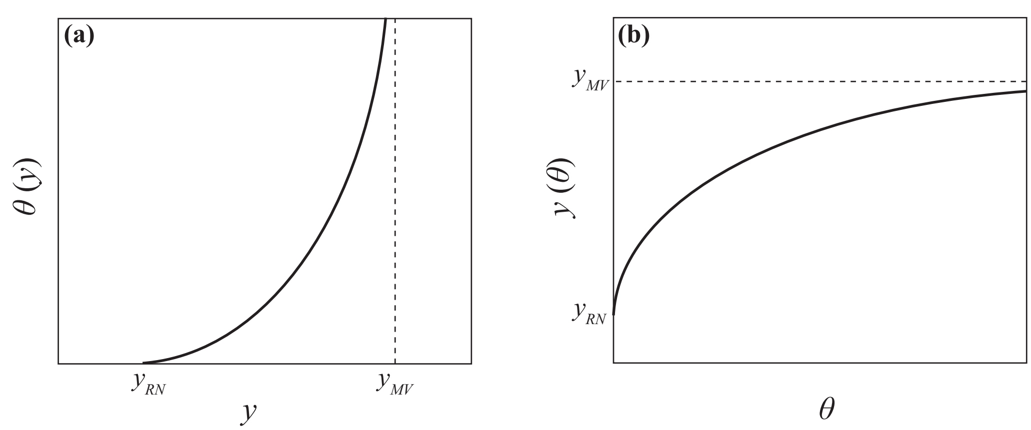
1. *If $y^*_{RN} \leq y^*_{MV}$, then $y(\theta)$ is increasing in θ on $[y^*_{RN}, y^*_{MV}]$*

2. If $y^*_{RN} > y^*_{MV}$, then $y(\theta)$ is decreasing in θ on $[y^*_{MV}, y^*_{RN}]$.

Proof. The analyses shown in (3.32), (3.33), (3.34) and (3.35), show that $\theta(y)$ is increasing in y if $y^*_{RN} \leq y^*_{MV}$, and $\theta(y)$ is decreasing in y if $y^*_{RN} > y^*_{MV}$. Tables 3.1 and 3.2 summarize the signs of the mean, variance and $\theta(y)$ functions and their derivatives in certain regions, confirming the claim. Moreover, by making use of (3.36), $y(\theta)$ can be determined to be increasing in θ if $y^*_{RN} \leq y^*_{MV}$, and decreasing in θ if $y^*_{RN} > y^*_{MV}$. ■

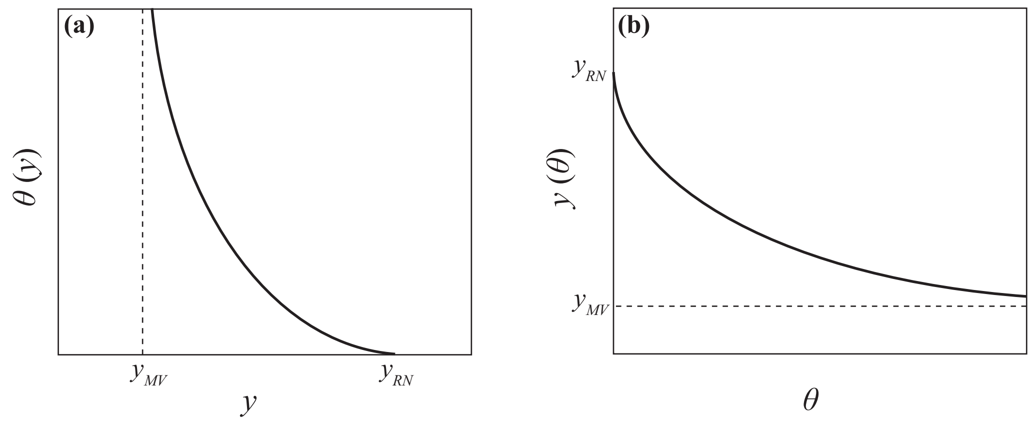
If we are to comment of the meaning of the relationship between $y(\theta)$ and θ, for $y^*_{RN} \leq y^*_{MV}$ (Case I), $\theta(y)$ increases in y. This also implies that $y(\theta)$ increases in θ. In other words, as the risk-aversion level increases, the optimal protection level increases from y^*_{RN} to y^*_{MV} (Figure 3.4b). While for $y^*_{MV} < y^*_{RN}$ (Case II), $\theta(y)$ decreases in y, meaning that $y(\theta)$ decreases in θ. As the risk-aversion level increases, the optimal protection level decreases from y^*_{RN} to y^*_{MV} (Figure 3.5b).

Figure 3.4: $\theta(y)$ vs y and $y(\theta)$ vs θ figures for $y_{RN} < y_{MV}$



To sum up, individuals move from the risk-neutral protection level towards to the minimum-variance protection level as their risk-aversion increases. That means the ones with higher risk aversion are likely to reserve more seats for class-2 customers; they are willing to bear the uncertainty in second period of sales because for Case I, the minimum variance is obtained by setting a higher protection level. For Case II, individuals with higher risk-aversion level tends to reserve smaller number of seats for class-2, since the minimum variance is obtained by setting a lower protection level.

Figure 3.5: $\theta(y)$ vs y and $y(\theta)$ vs θ figures for $y_{MV} < y_{RN}$



In this chapter, we presented a set of analyses on the optimal protection level and θ function, as well as the expected profit and variance functions of the cash flow. We aimed to better understand the nature of the MV function in a seat management (revenue management, in general) environment. Inclusion of the variance function is important because it enables us to incorporate risk into the model, in order to account for the uncertainty created by the stochastic demands. The risk-aversion rate brings a different angle to the problem by allowing us to add the relative risk-sensitivity of the individuals. We observed how the optimal protection level changes with an increase or decrease in the risk-aversion level of individuals. We concluded that for Case I, the optimal protection level of the MV objective function increases as the risk-aversion level increases. On the other hand for Case II, the optimal protection level of the MV objective function decreases as the risk-aversion level increases. The assumptions 3.2.1, 3.2.2, 3.3.2 and 3.3.3 made in this chapter will be verified numerically in Chapter 6.

Chapter 4

PERFECT HEDGING IN SINGLE-LEG STATIC RM PROBLEM

In Chapter 3, we studied the RM cash flow which has uncertainty due to the random demands. The aim was maximizing the expected return of the cash flow while penalizing the deviations (positive or negative) in the cash flow. There we did not account for the hedging opportunities that may be applied to the MV problem. Gaur and Seshadri (2005) claim that the demand for discretionary purchase items are correlated with economic and financial indicators. They provide support for their claim and show that the correlation between a financial index (S&P 500) and a sales index (Redbook) is significant. Furthermore, they apply this idea to the single-period, single-item inventory problem. In this chapter, we benefit from this insight.

In real life, individuals may tend to avoid risk in most cases. Although a high variance in the cash flow implies a chance of a higher return than the expected return, it may also result in a huge loss. This downside risk is what the investors fear of. For this reason, many firms perform hedging activities to systematically manage the risk. Hedging offers means for reducing the risk without sacrificing from the expected return. There are two types of hedging in the literature; namely, operational hedging and financial hedging. Van Mieghem (2003) defines operational hedging to be "mitigating risk by counterbalancing actions in a processing network that do not involve financial instruments." However, in this thesis we use financial hedging, which involves the use of financial instruments such as futures, options and derivatives to mitigate the risk. The type of financial hedging and the extent of the benefits it offers depends on the conditions of the financial market and the demand data. The two types of financial hedging are namely; perfect hedging, employed when the financial instruments and demands are perfectly correlated, and imperfect hedging, used when there is partial correlation between financial instruments and demands.

In this chapter, we discuss financial hedging with perfect correlation, and obtain results for its application on the RM model. Section 4.1 discusses the general idea behind the hedging operations. After that Section 4.2 and Section 4.3 analyze the RM problem for two-class and generalized n-class cases respectively. Lastly, Subsection 4.3.1 presents an algorithm for systematically analyzing any cash flow and offers hedging strategies.

4.1 General Idea

In real life, the operations that the firms carry out include many sources of randomness. For this reason, the benefits of hedging may be of great value practically in any industry, in any operation. In this section, we will discuss problems involving functions of random variables, and how they are interpreted when the random variables are correlated with financial variables.

For the sake of simplicity, we start by defining a function $h_T(X, y)$ which denotes the random cash flow which will be received at time T, where X denotes the random factor and y denotes the decision variable. Then our problem can be defined as

$$\max_y \mathrm{E}[h_T(X, y)].$$

Now suppose that there is a financial variable S, which denotes the price of a financial instrument at time T. Let S be correlated perfectly with the random variable X, so that $X = H(S)$, for some known deterministic function H. Furthermore, suppose there is a payoff function $R_i(S, y)$ for derivative i of the financial variable S. Then if this function satisfies

$$h_T(X, y) = h_T(H(S), y) = \sum_{i=1}^{n} \alpha_i R_i(S, y), \tag{4.1}$$

then perfect hedging can be performed, where α_i is the amount of derivative i used in the portfolio.

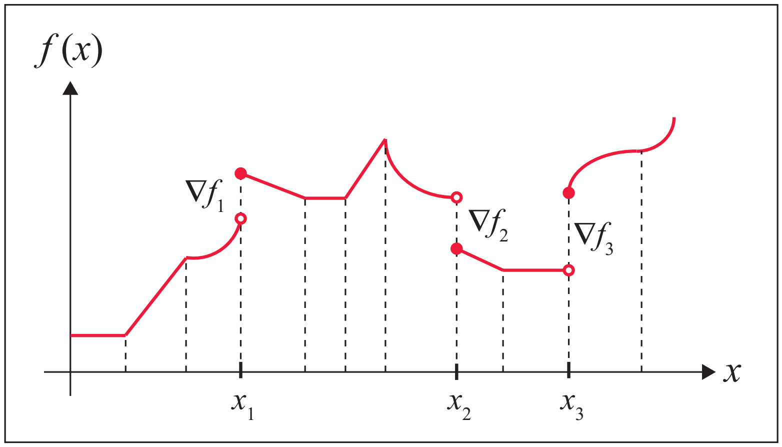
Moreover, the value of the portfolio at time 0 is

$$\sum_{i=1}^{n} \alpha_i r_i(S_0, y) \tag{4.2}$$

where $r_i(S_0, y)$ is the price of derivative i at time zero.

The main idea in this section is that we are able to represent any cash flow as a replicating portfolio of financial instruments. Let $f(x) = h_T(H(x), y)$ for any fixed y. To clarify the idea, consider the function in Figure 4.1 with m jumps at $x_1, x_2, ..., x_m$ where the difference between the right and left ends of the function being $\nabla f_1, \nabla f_2, ..., \nabla f_m$, respectively. We

Figure 4.1: The cash flow representation



assume that the function f is twice differentiable over each interval (x_i, x_{i+1}), and the first derivative $f'(x)$ has bounded variation. Then, we can replicate the random cash flow $f(S) = h_T(H(S), y)$ by investing in financial derivatives, such as futures, call and put options, and digital claims.

The function $f(x)$ can be expressed as a function of the financial instruments using

$$f(x) = f(0) + \sum_{k=1}^{m} \nabla f_k 1_{\{x \le x_k\}} + f'_+(0)x + \int_0^\infty (x-z)^+ f'(\mathrm{d}z). \tag{4.3}$$

Actually, (4.2) and (4.3) are presenting the same idea, where (4.3) gives the replicating portfolio for the cash flow function in (4.2). Note that in (4.3), payoffs are summed up, which are obtained using the bonds, digital claims, futures, and European call options used in the portfolio, respectively, as discussed in the book by Protter (2005).

Throughout the next two sections, we will build up the replicated cash flow functions that can be easily hedged in the light of this idea presented in this section.

4.2 *Analysis of the Two-class Problem and Perfect Hedging*

In Chapter 3, we presented the RM problem with two classes of demands. In this section, we will discuss the perfect hedging strategies for this model, assuming that the demand variables are perfectly correlated with some financial instruments.

As defined in Chapter 3, D_1 and D_2 be the demand random variables for class-1 and class-2, respectively. Let both of the demand random variables are correlated with the price

S of a primary asset in the market. y is the protection level for class-2. Sales period starts at time 0, and ends at time T, when the plane takes off. Suppose further that all cash flows occur at the end of the sales period, which is T. Remember the assumption we made in the previous chapter that $r_1 < r_2$. The RM cash flow was

$$CF(D_1, D_2, y) = r_1 \min\{D_1, Q - y\} + r_2 \min\{D_2, Q - \min\{D_1, Q - y\}\}.$$

The idea of describing functions as a replicating portfolio can be applied to many functions with different forms. However, for the sake of simplicity, throughout this section we will assume a linear dependency, such as $D_i = a_i + b_i S$, where $a_i, b_i \geq 0$, for $i = 1, 2$.

Substituting the demand expressions in terms of the price of the financial instrument the total cash flow becomes

$$CF(S, y) = r_1 \min\{a_1 + b_1 S, Q - y\} + r_2 \min\{a_2 + b_2 S, Q - \min\{a_1 + b_1 S, Q - y\}\}.$$

Define

$$\begin{aligned} f_1(S) &= \min\{a_1 + b_1 S, Q - y\} \\ f_2(S) &= \min\{a_2 + b_2 S, Q - \min\{a_1 + b_1 S, Q - y\}\} \end{aligned} \tag{4.4}$$

where $f_1(S)$ is the number of class-1 seats sold, and $f_2(S)$ is the number of class-2 seats sold, so that the cash flow becomes

$$CF(S, y) = r_1 f_1(S) + r_2 f_2(S).$$

Note that class-1 has $Q - y$ seats available for sale, where Q is the whole capacity, and y is the number of seats protected for class-2. On the other hand, class-2 has y seats protected for itself, plus it has the number of seats unsold to class-1, if any. Thus, the number of seats sold to each class depend on each other. When the functions for number of seats sold to each class, $f_1(S)$ and $f_2(S)$, are closely observed it is seen that $f_2(S)$ contains $f_1(S)$ inside itself. This fact suggests us the idea that $f_2(S)$ in a way depends on $f_1(S)$, so that

$$f_2(S) = \min\{a_2 + b_2 S, Q - f_1(S)\}.$$

In order to find the replicating portfolio of the cash flow we should first find the replicating portfolio of the functions $f_1(S)$ and $f_2(S)$. Thus, a thorough analysis for both of them is required, which will be presented in the next two subsections.

4.2.1 The analysis of $f_1(S)$

Note that $f_1(S)$, as in (4.4), is a continuous and piecewise linear function of S. The breakpoint of $f_1(S)$ is $B_1 = (Q - y - a_1)/b_1$.

Since $S \geq 0$, we will consider only the region $[0, +\infty)$ in all derivations throughout the remainder of this chapter.

Then, $f_1(S)$ can be described as

$$f_1(S) = \begin{cases} a_1 + b_1 S, & \{S \leq B_1\} \\ \\ Q - y, & \{S > B_1\}\,. \end{cases}$$

The values of $f_1(S)$ are only valid in their defined regions. For example, $a_1 + b_1 S$ is valid on $\{0 \leq S \leq B_1\}$. If $B_1 < 0$, then $f_1(S)$ will be equal to $Q - y$.

Thus, we have two possible cases for $f_1(S)$.

Case 1. If $B_1 > 0$, then

Then, $f_1^1(S)$ can be shown to have the structure below

$$f_1^1(S) = \begin{cases} a_1 + b_1 S, & \{0 \leq S \leq B_1\} \\ \\ Q - y, & \{S > B_1\}\,, \end{cases}$$

as shown in Figure 4.2.

Case 2. If $B_1 \leq 0$, then

$$f_1^2(S) = Q - y$$

represented by Figure 4.3.

Figure 4.2: $f_1^1(S)$ vs S

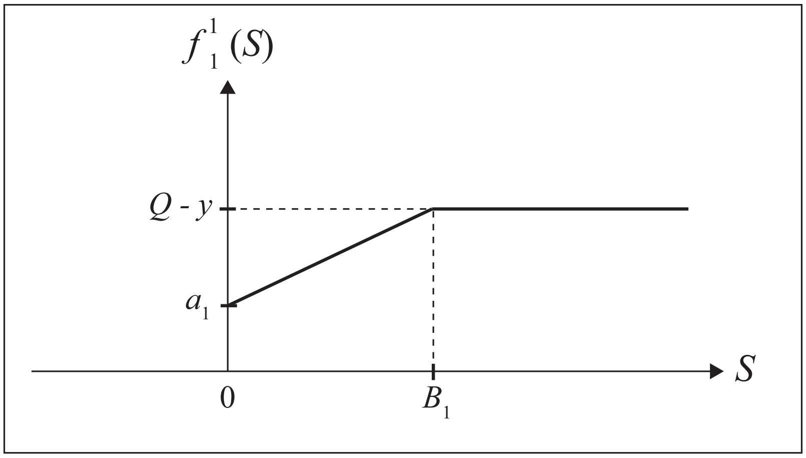
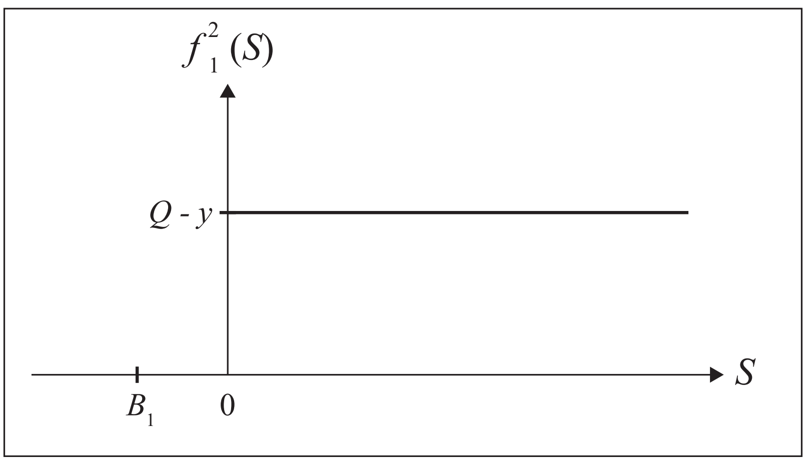


Figure 4.3: $f_1^2(S)$ vs S



4.2.2 The analysis of $f_2(S)$

In this section, we will discuss the second function, $f_2(S)$. It includes the first function, $f_1(S)$, thus during the analysis of $f_2(S)$, we begin by evaluating $f_1(S)$ as we did in the previous subsection.

As seen in (4.4), $f_2(S)$ has two possible breakpoints, B_2 and B_3, depending on the value of $f_1(S)$.

1. For $f_1(S) = a_1 + b_1 S$, the function becomes

$$f_2(S) = \min\{a_2 + b_2 S, Q - a_1 - b_1 S\}.$$

   The breakpoint is then

$$B_2 = \frac{Q - a_1 - a_2}{b_1 + b_2} \tag{4.5}$$

   on the region $[0, B_1]$.

2. For $f_1(S) = Q - y$, the function becomes

$$f_2(S) = \min\{a_2 + b_2 S, y\}.$$

   Then, another breakpoint is given by

$$B_3 = \frac{y - a_2}{b_2} \tag{4.6}$$

   on the region $[B_1, +\infty)$.

$f_2(S)$ can be defined as a piecewise linear function which includes all cases, given by

$$f_2(S) = \begin{cases} a_2 + b_2 S, & \{S \leq \min\{B_1, B_2\}\} \\ Q - a_1 - b_1 S, & \{\min\{B_1, B_2\} < S \leq B_1\} \\ a_2 + b_2 S, & \{B_1 < S \leq \max\{B_1, B_3\}\} \\ y, & \{S > \max\{B_1, B_3\}\}. \end{cases}$$

In the above definition, we accounted for all possible breakpoints. It provides a general representation of $f_2(S)$. Indeed, at most one of the breakpoints $B_2 = (Q - a_1 - a_2)/(b_1 + b_2)$ and $B_3 = (y - a_2)/b_2$ may exist in their defined regions, depending on the values of the constants, which is proven below.

If $\{B_2 \leq B_1\}$, then $\{B_3 \leq B_1\}$. However, in (4.6) it is shown that B_3 can only lie on the region $[B_1, +\infty)$. Then, if B_2 exits, B_3 cannot exist. If $\{B_3 \geq B_1\}$, then $\{B_2 \geq B_1\}$. However, in (4.5) it is shown that B_2 can only lie on the region $[0, B_1]$. Then, if B_3 exits, B_2 cannot exist.

These statement follow from the analysis which will be presented here; suppose $\{B_2 \leq B_1\}$ is true, which means B_2 lies in its allowed region $[0, B_1]$. Then the following is true

$$\frac{Q - a_1 - a_2}{b_1 + b_2} \leq \frac{Q - y - a_1}{b_1}.$$

After simple mathematical operations and simplifications, the following is obtained

$$\frac{y - a_2}{b_2} < \frac{Q - y - a_1}{b_1},$$

meaning that the breakpoint $B_3 < B_1$, which cannot be true since B_3 can only exist in $[B_1, +\infty)$.

Similarly, suppose $\{B_1 \leq B_3\}$ is true, which means B_3 lies in its allowed region $[B_1, +\infty)$. Then the following is true

$$\frac{Q - y - a_1}{b_1} \leq \frac{y - a_2}{b_2}.$$

After multiplying the numerators by the denominators and adding the term $Qb_1 - a_1b_1$ to the both sides we obtain

$$\frac{Q - a_1 - a_2}{b_1 + b_2} > \frac{Q - y - a_1}{b_1},$$

meaning that the breakpoint $B_2 > B_1$, which cannot be true since B_2 can only exist in $[0, B_1]$.

We conclude that if the function breaks at B_2, then it does not break on B_3. The converse is also true.

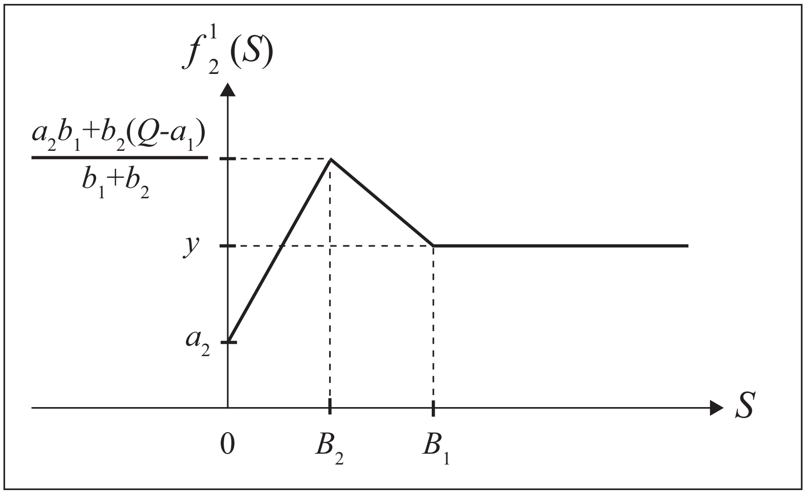
In the light of the analyses up to now, we may define all possible structures that $f_2(S)$ can have.

Case 1: If $0 \leq B_2 \leq B_1$, then

$$f_2^1(S) = \begin{cases} a_2 + b_2 S, & [0, B_2] \\ Q - a_1 - b_1 S, & [B_2, B_1] \\ y, & [B_1, +\infty), \end{cases}$$

where the representation of $f_2^1(S)$ is given in Figure 4.4.

Figure 4.4: $f_2^1(S)$ vs S

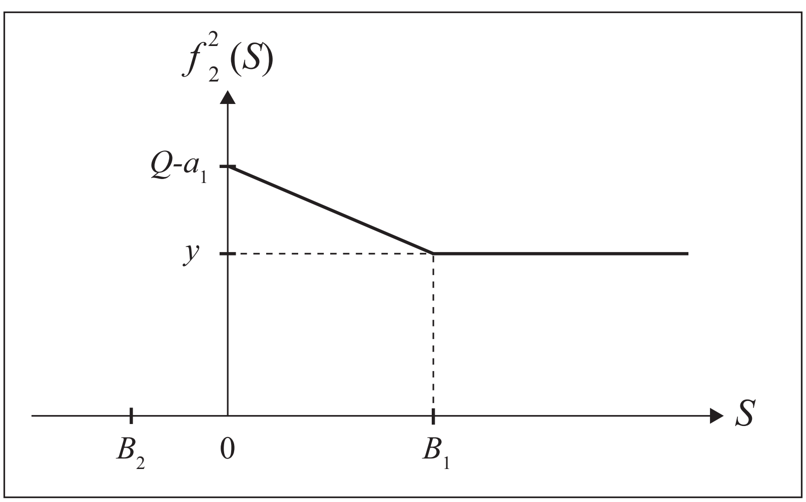


Case 2: If $B_2 \leq 0 \leq B_1$, then

$$f_2^2(S) = \begin{cases} Q - a_1 - b_1 S, & [0, B_1] \\ \\ y, & [B_1, +\infty), \end{cases}$$

which is also given by Figure 4.5.

Figure 4.5: $f_2^2(S)$ vs S

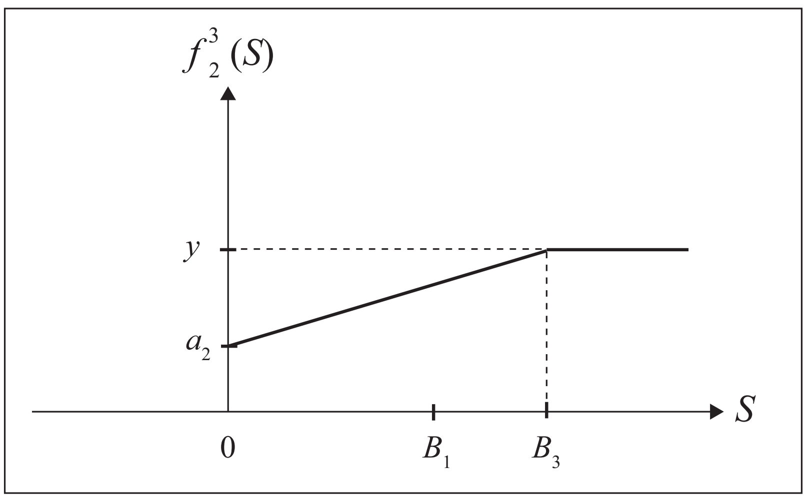


Case 3: If $0 \leq B_1 \leq B_3$, then

$$f_2^3(S) = \begin{cases} a_2 + b_2 S, & [0, B_1] \\ a_2 + b_2 S, & [B_1 . B_3] \\ y, & [B_3, +\infty), \end{cases}$$

where the graphical representation is given by Figure 4.6.

Figure 4.6: $f_2^3(S)$ vs S



Case 4: If $B_1 \leq 0 \leq B_3$, then

$$f_2^4(S) = \begin{cases} a_2 + b_2 S, & [0, B_3] \\ y, & [B_3, +\infty), \end{cases}$$

whose representation is given by Figure 4.7.

Case 5: If $B_1 \leq B_3 \leq 0$, then

$$f_2^5(S) = y,$$

which is simply given by Figure 4.8.

Figure 4.7: $f_2^4(S)$ vs S

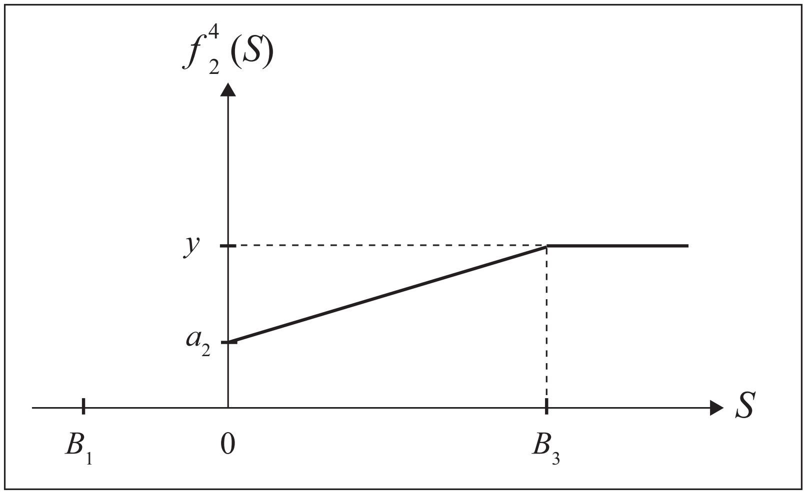
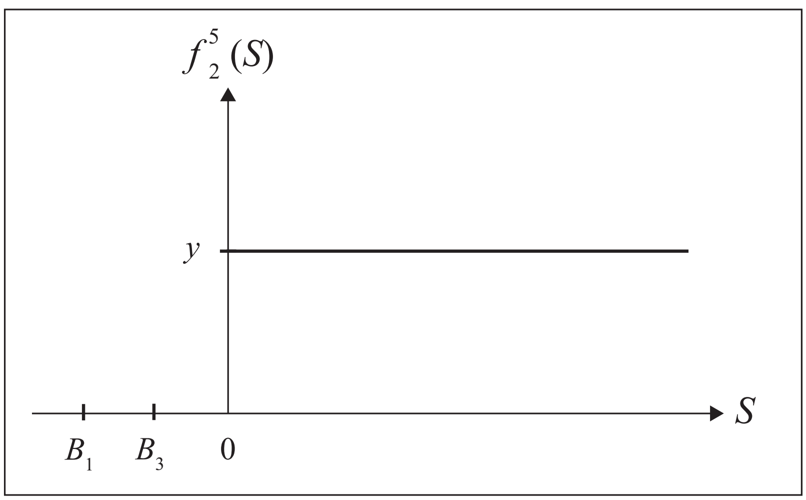


Figure 4.8: $f_2^5(S)$ vs S



Only one of the piecewise functions found for $f_2(S)$ is valid depending on the values of a_1, a_2, b_1, b_2, y, and Q. As we have shown, the functions $f_1(S)$ and $f_2(S)$ can be written as piecewise linear functions.

We have analyzed the two-period static single leg revenue management problem and found an explicit representation of the cash flow function. Then it can be replicated by a portfolio of financial instruments.

Using the idea presented in (4.3) to rearrange the functions $f_1^1(S)$, $f_1^2(S)$, $f_2^1(S)$, $f_2^2(S)$,

$f_2^3(S)$, $f_2^4(S)$ and $f_2^5(S)$, we obtain

$$f_1^1(S) = a_1 + b_1 S - b_1 (S - B_1)^+$$
$$f_1^2(S) = Q - y$$

while

$$f_2^1(S) = a_2 + b_2 S - (b_1 + b_2)(S - B_1)^+ + b_1 (S - B_3)^+$$
$$f_2^2(S) = Q - a_1 - b_1 S + b_1 (S - B_1)^+$$
$$f_2^3(S) = f_2^4(S) = a_2 + b_2 S - b_2 (S - B_1)^+$$
$$f_2^5(S) = y.$$

Then the cash flow function is simply the summation of $f_1(S)$ and $f_2(S)$ functions, such that

$$CF(S, y) = r_1 f_1(S) + r_2 f_2(S),$$

which is also a piecewise function. Then we can obtain the optimal perfect hedging portfolio using this piecewise cash flow.

There are ten possible expressions for $CF(S, y)$ function. Suppose $f_1^1(S)$ and $f_2^1(S)$ are true for our problem. Then our cash flow becomes

$$\begin{aligned} CF(S, y) =& r_1 f_1^1(S) + r_2 f_2^1(S) \\ =& r_1 \left(a_1 + b_1 S - b_1 (S - B_1)^+\right) \\ &+ r_2 \left(a_2 + b_2 S - (b_1 + b_2)(S - B_2)^+ + b_1 (S - B_3)^+\right) \\ =& (r_1 a_1 + r_2 a_2) + (r_1 b_1 + r_2 b_2) S - r_1 b_1 (S - B_1)^+ \\ &- (r_2 b_1 + r_2 b_2)(S - B_2)^+ + r_2 b_1 (S - B_3)^+. \end{aligned}$$

Now suppose the sales period is between 0 and T. Throughout this chapter an important assumption made is that the market is complete and arbitrage-free with some risk-neutral probability measure Q. Let S_0 and S_T denote the current price of the financial asset and its price at time T, respectively. Furthermore, let r be the risk-free rate of return per year. Then the hedging transactions at time 0 is

1. Borrow and sell $(r_1 b_1 + r_2 b_2)$ units of the underlying asset at the current price S_0. Then replace the borrowed asset at time T, by purchasing $(r_1 b_1 + r_2 b_2)$ units of the asset at the price S_T.

2. Buy (r_1b_1) call options on this asset with strike price B_1 and settlement date T.

   Buy $(r_2b_1 + r_2b_2)$ call options on this asset with strike price B_2 and settlement date T.

   Sell (r_2b_1) call options on this asset with strike price B_3 and settlement date T.

3. Borrow a sum of money equal to $(r_1a_1 + r_2a_2)e^{-rT}$ at the risk-free rate and repay it at time T.

By performing these hedging operations, we obtain the hedged profit at time zero, $\mathrm{E}[CF_H(S,y)]$, as

$$\begin{aligned}\mathrm{E}[CF_H(S,y)] =& (r_1b_1 + r_2b_2)S_0 \\ &+ (r_1a_1 + r_2a_2)e^{-rT} \\ &- r_1b_1\,\mathrm{E}[(S_T - B_1)^+] - (r_2b_1 + r_2b_2)\,\mathrm{E}[(S_T - B_2)^+] + (r_2b_1)\,\mathrm{E}[(S_T - B_3)^+].\end{aligned}$$

Note that the hedging cash flow has an expected value equal to the expected value of the unhedged cash flow under the risk-neutral probability measure Q

$$\mathrm{E}_Q[CF(S,y)] = \mathrm{E}[CF_H(S,y)]. \tag{4.7}$$

Furthermore, the variance of the hedging cash flow is zero so that

$$\mathrm{Var}[CF_H(S,y)] = 0. \tag{4.8}$$

Perfect hedging does not affect the expected value of the cash flow, furthermore, it enables us to obtain a zero variance. We further conclude that, if there is perfect correlation between the demands and the financial market, the RM problem can be perfectly hedged using the futures and call options.

4.3 Analysis of the n-class Problem and Perfect Hedging

In this section, we are going to analyze the n-class seat allocation problem and possible hedging strategies that can be employed to perfectly hedge the risks involved.

Now suppose that demand for each class is correlated with a financial instrument S which can be traded in the market. In this setting, further assume that the demands are linear in S such that $D_i = b_i S$ (for simplicity, the constant term a_i is removed, for $i = 1, 2, ..., n$). The results will apply for any linear function. The protection level for class i is denoted by y_i, and represents the total number of seats reserved for class i and higher. It has a nested structure given by

$$y_n^* \leq y_{n-1}^* \leq ... \leq y_1^* = Q.$$

The number of seats sold to each class can be expressed as

$$\begin{aligned} f_1(S) &= \min\{a_1 + b_1 S, Q - y_2\} \\ f_2(S) &= \min\{a_2 + b_2 S, Q - y_3 - f_1(S)\} \\ &\ ... \\ f_{n-1}(S) &= \min\{a_{n-1} + b_{n-1} S, Q - y_n - \sum_{j=1}^{n-2} f_j(S)\} \\ f_n(S) &= \min\{a_n + b_n S, Q - \sum_{j=1}^{n-1} f_j(S)\}. \end{aligned} \tag{4.9}$$

In Section 4.2, we have examined the functions $f_1(S)$ and $f_2(S)$. Similar derivations can be performed for the rest of the functions that we defined in (4.9). Then the total revenue function could be written as

$$CF(S, y) = r_1 f_1(S) + r_2 f_2(S) + ... + r_{n-1} f_{n-1}(S) + r_n f_n(S).$$

Now we can state an algorithm to explicitly define all functions by finding the breakpoints and sub-functions they have. Later, these functions will build up the total revenue function.

4.3.1 An Algorithm

In this subsection we propose an algorithm to enumerate all the possibilities for all the functions in the general case.

An algorithm for defining the $f_j(S)$

1. Initialize: $j = 1$

   Set up the initial range of the functions as $R_0 = [0, \infty)$. Define the set of breakpoints as $\mathrm{P}_0 = \varnothing$, empty initially.

   Find the breakpoint B_1 of the first function $f_1(S)$.

- Check if $B_1 \in R_0$
  - If true, update $\text{P}_1 = \{B_1\}$,
  - Else, $\text{P}_1 = \text{P}_0$.

Define $f_1(S)$ accordingly.

2. $j = 2$

If P is updated in the previous step, it means that $f_1(S)$ was a two-piece function. For each piece, find the corresponding breakpoints B_{21} and B_{22} and define their regions to be $R_1 = [0, B_1]$ and $R_2 = [B_1, \infty)$, respectively.

- Check if $B_{21} \in R_1$
  - If true, update $B_2 = B_{21}$ and $\text{P}_2 = \{\text{P}_1, B_2\}$
- Check if $B_{22} \in R_2$
  - If true, update $B_2 = B_{22}$ and $\text{P}_2 = \{\text{P}_1, B_2\}$
- Else, $\text{P}_2 = \text{P}_1$.

If P is not updated in the previous step, it means that $f_1(S)$ was a one-piece function. Find the corresponding breakpoint B_2 defined in the region $R_1 = R_0$.

- Check if $B_2 \in R_1$
  - If true, update $\text{P}_2 = \{B_2\}$,
  - Else, $\text{P}_2 = \text{P}_1 = \text{P}_0$.

Define $f_2(S)$ accordingly.

3. $j = j + 1$

If P is updated in the previous step, it means that $f_{j-1}(S)$ was a two-piece function. For each piece, find the corresponding breakpoints B_{j1} and B_{j2} and define their regions to be $R_1 = [0, B_{j-1}]$ and $R_2 = [B_{j-1}, \infty)$, respectively.

- Check if $B_{j1} \in R_1$

  If true, update $B_j = B_{j1}$ and $\text{P}_j = \{\text{P}_{j-1}, B_j\}$

- Check if $B_{j2} \in R_2$

  If true, update $B_j = B_{j2}$ and $\mathrm{P}_j = \{\mathrm{P}_{j-1}, B_j\}$

- Else, $\mathrm{P}_j = \mathrm{P}_{j-1}$.

If P is not updated in the previous step, it means that $f_{j-1}(S)$ was a one-piece function. Find the corresponding breakpoint B_j defined in the region $R_j = R_{j-1}$.

- Check if $B_j \in R_j$
  - If true, update $\mathrm{P}_j = \{B_j\}$,
  - Else, $\mathrm{P}_j = \mathrm{P}_{j-1}$.

Define $f_j(S)$ accordingly.

Repeat this step until $j = n + 1$.

4. Now that all functions $f_1(S)$ through $f_n(S)$ are found, calculate $CF(S, y) = r_1 f_1(S) + r_2 f_2(S) + ... + r_{n-1} f_{n-1}(S) + r_n f_n(S)$.

Since all functions $f_j(S)$ are piecewise linear in S, their linear combination $CF(S, y)$ is also piecewise linear in S. Then it can be represented as a replicating portfolio composed of bonds, futures and European call options, as illustrated in Section 4.2.

Chapter 5

MV APPROACH TO THE TWO-CLASS RM PROBLEM WITH HEDGING

In Chapter 3, we presented revenue management models analyzed under the MV approach to account for the uncertainty created by stochastic demands. We did not consider any correlation between demand random variables and financial markets. However, Gaur and Seshadri (2005) provide substantial evidence that in a retail environment the sales amount in inventory models may be correlated with the financial markets in reality. In this chapter, inspired by this evidence we investigate a two-period RM problem, where the demand random variables are correlated with the prices of financial instruments in a financial market. In our model, the optimal protection level and the hedging strategies are determined simultaneously.

Section 5.1 examines the general MV approach with hedging in general form. Section 5.2 further analyzes the MV approach with hedging for the RM problem specifically, with independent demand random variables. Section 5.3 presents the counterpart of the results of Section 5.2, for dependent demands. Finally Section 5.4 discusses more on this topic regarding the structural properties of the problem.

5.1 MV Model Characterizations with Hedging

In this section, we present the RM model with the MV approach including financial hedging opportunities. In mean-variance hedging, the aim is to choose a portfolio of financial securities to maximize a weighted sum of the expected cash flow and the variance of the cash flow. In our analysis, the time convention is such that at time 0 the sales period starts, and at time T, the plane takes off. We assume that all cash flows occur at the end of the sales period T. The risk-free interest rate is a. Furthermore, as in Chapter 3, the net revenues earned from the sales of seats from class-1 and class-2 are denoted by r_1 and r_2, respectively. We assume $r_1 < r_2$ as before. Let D_1 and D_2 denote the random demands of classes 1 and

2, and $\mathbf{X} = (D_1, D_2)$ denote the vector of demand random variables. Furthermore, let y be the protection level, S be the price of a primary asset in the market, $f_i(S)$ be the net payoff of the ith derivative security, α_i denote the amount of security i in the hedging portfolio, and $CF(\mathbf{X}, y)$ denote the unhedged cash flow. Here S denotes the price of a single asset at time T, however our analysis is also valid when S is a vector representing the prices of a number of primary assets in the market. We assume that there is at least one derivative security in the market. We assume that the random vector $\mathbf{X}$ is correlated with the price of the financial variable S. Then the hedged cash flow is given by

$$CF_\alpha(\mathbf{X}, S, y) = CF(\mathbf{X}, y) + \sum_{i=1}^{n} \alpha_i f_i(S).$$

The net payoff $f_i(S)$ is defined as the difference between the payoff received at time T ($\hat{f}_i(S)$) and the investment cost at time zero (f_i^0) continuously compounded to time T, such that, $f_i(S) = \hat{f}_i(S) - e^{aT} f_i^0$. Suppose a call option with strike price K will be used in hedging, whose value at time T is $\hat{f}_i(S) = (S-K)^+ = \max\{S-K, 0\}$. Then the net payoff is $f_i(S) = \max\{S-K, 0\} - e^{aT} f_i^0$. Throughout this chapter an important assumption made is that the market is complete and arbitrage-free with some risk-neutral probability measure Q. Under this risk-neutral probability measure, the price of the ith derivative security is $f_i^0 = e^{-aT}\,\mathrm{E}_Q[\hat{f}_i(S)]$, which implies $\mathrm{E}_Q[f_i(S)] = \mathrm{E}_Q[\hat{f}_i(S) - f_i^T] = 0$, where $f_i^T = e^{aT} f_i^0$.

The decision maker aims to find the optimal protection level and the optimal hedging portfolio $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$. The objective could be either maximizing the expected return subject to a maximum-variance constraint or minimizing the variance subject to a minimum-return constraint. By taking the MV approach, the trade-off between the mean and the variance can be captured, and a compromise solution to these conflicting objectives can be found. The MV optimization problem is then

$$\begin{aligned}
\max_{y\geq 0, \boldsymbol{\alpha}} H(\theta, y, \boldsymbol{\alpha}) &= \mathrm{E}[CF_{\boldsymbol{\alpha}}(\mathbf{X}, S, y)] - \theta\,\mathrm{Var}[CF_{\boldsymbol{\alpha}}(\mathbf{X}, S, y)] \\
&= \mathrm{E}\left[CF(\mathbf{X}, y) + \sum_{i=1}^{n} \alpha_i f_i(S)\right] - \theta\,\mathrm{Var}\left[CF(\mathbf{X}, y) + \sum_{i=1}^{n} \alpha_i f_i(S)\right] \qquad (5.1) \\
&= \mathrm{E}\left[CF(\mathbf{X}, y)\right] - \theta\,\mathrm{Var}\left[CF(\mathbf{X}, y) + \sum_{i=1}^{n} \alpha_i f_i(S)\right]
\end{aligned}$$

where the last equation follows from the arbitrage-free market assumption implying $\mathrm{E}_Q[f_i(S)] =$

0. The variance term of the MV objective function can be presented in detail as follows

$$\begin{aligned} H(\theta, y, \boldsymbol{\alpha}) = \ & \mathrm{E}\left[CF(\mathbf{X}, y)\right] \\ & - \theta \Bigg\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \operatorname{Cov}(f_i(S), f_j(S)) \\ & + 2 \sum_{i=1}^{n} \alpha_i \operatorname{Cov}(f_i(S), CF(\mathbf{X}, y)) + \operatorname{Var}[CF(\mathbf{X}, y)] \Bigg\} \end{aligned} \tag{5.2}$$

for any fixed $\theta \geq 0$. In matrix notation, (5.2) can be expressed as

$$\begin{aligned} H(\theta, y, \boldsymbol{\alpha}) &= \mathrm{E}\left[CF(\mathbf{X}, y)\right] - \theta \left[\operatorname{Var}[CF(\mathbf{X}, y) + \boldsymbol{\alpha}^{\mathbf{T}} \mathbf{f}(S)]\right] \\ &= \mathrm{E}\left[CF(\mathbf{X}, y)\right] - \theta \left[\boldsymbol{\alpha}^{\mathbf{T}} \mathbf{C} \boldsymbol{\alpha} + 2\boldsymbol{\alpha}^{\mathbf{T}} \boldsymbol{\mu}(y) + \operatorname{Var}[CF(\mathbf{X}, y)]\right] \end{aligned}$$

where $\boldsymbol{\alpha}^{\mathbf{T}}$ denotes the transpose of $\boldsymbol{\alpha}$, $\mathbf{f}(S)$ is a column vector such that

$$\mathbf{f}(S) = (f_1(S), f_2(S), ..., f_n(S))$$

$\mathbf{C}$ denote the positive definite covariance matrix of the financial securities with elements

$$C_{ij} = \operatorname{Cov}(f_i(S), f_j(S))$$

and finally $\boldsymbol{\mu}(y)$ is a vector defined as

$$\mu_i(y) = \operatorname{Cov}(f_i(S), CF(\mathbf{X}, y)).$$

Theorem 5.1.1 *The optimal financial portfolio for the variance minimizing hedging is*

$$\boldsymbol{\alpha}^*(y) = -\mathbf{C}^{-1} \boldsymbol{\mu}(y) \tag{5.3}$$

for any protection level y.

Proof. The first order condition of the objective function is obtained by the gradient

$$\frac{dH(\theta, y, \boldsymbol{\alpha})}{d\boldsymbol{\alpha}} = -2\theta(\mathbf{C}\boldsymbol{\alpha} + \boldsymbol{\mu}(y)) = 0 \tag{5.4}$$

while the second order condition is

$$\frac{d^2 H(\theta, y, \boldsymbol{\alpha})}{d\boldsymbol{\alpha}^2} = -2\theta \mathbf{C} \leq 0$$

where the non-positivity is ensured because the covariance matrix is positive definite. Then we obtain (5.3) by (5.4). ■

Theorem 5.1.1 provides the optimal portfolio vector, which gives the optimal amounts of the financial instruments to invest in order to obtain the minimum variance. Note that the optimal portfolio depends both on the relationship between the derivatives and the relationship between the derivatives and the cash flow function.

The second step is finding the optimal protection level maximizing the MV objective function.

Theorem 5.1.2 *The optimal protection level $y^*(\theta)$ satisfies the equation*

$$\frac{d\,\mathrm{E}[CF(\boldsymbol{X},y)]}{dy} - \theta\left(-2\boldsymbol{\mu}(y)^{\boldsymbol{T}}\mathbf{C}^{-1}\frac{d\boldsymbol{\mu}(y)}{dy} + \frac{d\,\mathrm{Var}[CF(\boldsymbol{X},y)]}{dy}\right) = 0 \tag{5.5}$$

for all $\theta \geq 0$.

Proof. To obtain the optimal protection level, the derivative of the objective function with respect to y is obtained and is set equal to zero so that

$$\frac{dH(\theta,y,\boldsymbol{\alpha})}{dy} = \frac{d\,\mathrm{E}[CF(\mathbf{X},y)]}{dy} - \theta\left(2\boldsymbol{\alpha}^{\mathbf{T}}\frac{d\boldsymbol{\mu}(y)}{dy} + \frac{d\,\mathrm{Var}[CF(\mathbf{X},y)]}{dy}\right) = 0. \tag{5.6}$$

We substitute the optimal portfolio obtained in (5.3) to the above condition, then (5.5) is obtained. ■

We see from Theorem 5.1.2 that the optimality condition for the MV function with hedging is similar to the optimality condition for the MV function without hedging. It follows from that their mean functions are the same under the risk-neutral probability measure Q, and the only difference is in the variance function. Rewriting 5.5 we see that only the term $2\theta\boldsymbol{\mu}(y)^{\mathbf{T}}\mathbf{C}^{-1}\frac{d\boldsymbol{\mu}(y)}{dy}$ is added to the first order condition, given by

$$\frac{d\,\mathrm{E}[CF(\mathbf{X},y)]}{dy} - \theta\left(\frac{d\,\mathrm{Var}[CF(\mathbf{X},y)]}{dy}\right) + 2\theta\boldsymbol{\mu}(y)^{\mathbf{T}}\mathbf{C}^{-1}\frac{d\boldsymbol{\mu}(y)}{dy} = 0. \tag{5.7}$$

We may conclude that $y^*(\theta)$ of the hedged MV function will be different than the $y^*(\theta)$ of the unhedged MV function, due to the added term. Whether it is less than or greater than the unhedged $y^*(\theta)$ depends on the correlation between the stock and the cash flow, obviously (the difference is linked to the $\boldsymbol{\mu}(y)$ term).

In the remainder of this section, the results above are adapted for hedging with only one financial security ($n = 1$) case.

Corollary 5.1.1 *Assuming only one security is used for hedging, the optimal portfolio becomes*

$$\alpha^*(y) = -\frac{\operatorname{Cov}(f(S), CF(\mathbf{X}, y))}{\operatorname{Var}[f(S)]}. \tag{5.8}$$

Furthermore, the optimal protection level satisfies

$$\begin{aligned} &\frac{d\,\mathrm{E}[CF(\boldsymbol{X}, y)]}{dy} \\ &- \theta\left(-2\left(\frac{\operatorname{Cov}(f(S), CF(\mathbf{X}, y))}{\operatorname{Var}[f(S)]}\right)\frac{d\operatorname{Cov}(f(S), CF(\mathbf{X}, y))}{dy} + \frac{d\operatorname{Var}[CF(\boldsymbol{X}, y)]}{dy}\right) = 0 \end{aligned} \tag{5.9}$$

for all $\theta \geq 0$.

Proof. The results above are the simplified versions of Theorem 5.1.1 and 5.1.2. When there is one security, the covariance term becomes $C = \operatorname{Cov}(f(S), f(S)) = \operatorname{Var}[f(S)]$ and $\mu(y)$ becomes $\operatorname{Cov}(f(S), CF(\mathbf{X}, y))$. It also follows from the initial hedged objective function (5.1). To obtain the optimal hedging portfolio and the optimal protection level, first the gradient of (5.1) is obtained as

$$\frac{dH(\theta, y, \alpha)}{d\alpha} = -2\theta[\alpha \operatorname{Var}[f(S)] + \operatorname{Cov}(f(S), CF(\mathbf{X}, y)] \tag{5.10}$$

and the Hessian is

$$\frac{d^2H(\theta, y, \alpha)}{d\alpha^2} = -2\theta \operatorname{Var}[f(S)] \leq 0.$$

The first order condition obtained by setting (5.10) equal to zero gives the optimal portfolio. Then the gradient of the objective function is found with respect to y, and is set equal to zero

$$\begin{aligned} \frac{dH(\theta, y, \alpha)}{dy} &= \frac{d\,\mathrm{E}[CF(\mathbf{X}, y)]}{dy} - \theta\left(2\alpha\frac{d\operatorname{Cov}(f(S), CF(\mathbf{X}, y))}{dy} + \frac{d\operatorname{Var}[CF(\mathbf{X}, y)]}{dy}\right) \\ &= 0. \end{aligned} \tag{5.11}$$

Lastly, $\alpha^*(y)$ is substituted into (5.11), obtaining (5.9). ■

Corollary 5.1.1 makes our point in the discussion of Theorem 5.1.2 more clear. Let us examine (5.9) by rewriting it as

$$\begin{aligned} &\frac{d\,\mathrm{E}[CF(\mathbf{X}, y)]}{dy} - \theta\left(\frac{d\operatorname{Var}[CF(\mathbf{X}, y)]}{dy}\right) \\ &+ 2\theta\left(\frac{\operatorname{Cov}(f(S), CF(\mathbf{X}, y))}{\operatorname{Var}[f(S)]}\right)\frac{d\operatorname{Cov}(f(S), CF(\mathbf{X}, y))}{dy} = 0 \end{aligned}$$

where $y(\theta)$ in the hedged MV function is dependent on the relationships between the financial instruments and the cash flow. Furthermore, we observe that the optimal portfolio is a function of the covariance between the stock price and the cash flow. The higher the covariance, more amount of investment is made.

5.2 MV Model with Independent Random Demands for the Two-class RM Model

In this section we will analyze the two-class RM problem with independent random demands in each class. The demand random variables D_1 and D_2 are assumed to be correlated with the financial variable S. The hedged cash flow for the RM MV model is given by

$$\begin{aligned} CF(\mathbf{X}, S, y) &= CF(D_1, D_2, y) + \boldsymbol{\alpha}^{\mathbf{T}}\mathbf{f}(S) \\ &= r_1 \min\{D_1, Q-y\} + r_2 \min\{D_2, Q - \min\{D_1, Q-y\}\} + \boldsymbol{\alpha}^{\mathbf{T}}\mathbf{f}(S), \end{aligned}$$

where $\mathbf{X} = (D_1, D_2)$. The hedged MV objective function that we are interested in is

$$\max_{y\geq 0, \boldsymbol{\alpha}} H(\theta, y, \boldsymbol{\alpha}) = \mathrm{E}[CF(D_1, D_2, y)] - \theta[\mathrm{Var}(CF(D_1, D_2, y) + \boldsymbol{\alpha}^{\mathbf{T}}\mathbf{f}(S)]).$$

The optimal portfolio found in (5.3) is the same for this case, while $\boldsymbol{\mu}(y)$ becomes

$$\mu_i(y) = \mathrm{Cov}(f_i(S), CF(D_1, D_2, y)). \tag{5.12}$$

Then the first order condition of the hedged objective function can be obtained using (5.6) as

$$\begin{aligned} \frac{dH(\theta, y, \boldsymbol{\alpha})}{dy} =& \mathrm{P}\{D_1 > Q-y\}(-r_1 + r_2 \mathrm{P}\{D_2 > y\}) \\ & - 2\theta \begin{pmatrix} -\boldsymbol{\alpha}^{\mathbf{T}}\boldsymbol{\mu}'(y) \\ -r_1^2 \mathrm{Cov}[\min\{D_1, Q-y\}, 1_{\{D_1 > Q-y\}}] \\ +r_2^2 \mathrm{Cov}[\min\{D_2, Q-\min\{D_1, Q-y\}\}, 1_{\{D_1 \geq Q-y\}}1_{\{D_2 > y\}}] \\ -r_1 r_2 \mathrm{Cov}[\min\{D_2, Q-\min\{D_1, Q-y\}\}, 1_{\{D_1 > Q-y\}}] \\ +r_1 r_2 \mathrm{Cov}[\min\{D_1, Q-y\}, 1_{\{D_1 \geq Q-y\}}1_{\{D_2 > y\}}] \end{pmatrix} \\ =& 0 \end{aligned} \tag{5.13}$$

where $\mu_i'(y)$ is the derivative of $\mu_i(y)$, given by

$$\mu_i'(y) = \frac{d\mu_i(y)}{dy} = -r_1 \mathrm{Cov}\left(f_i(S), 1_{\{D_1 > Q-y\}}\right) + r_2 \mathrm{Cov}\left(f_i(S), 1_{\{D_2 > y\}}1_{\{D_1 > Q-y\}}\right)$$

using (5.12). In the following equations, for simplicity, we define the mean and the variance of the hedged cash flow as $m_\alpha(y)$ and $v_\alpha(y)$, respectively.

Now we can obtain the first order condition by substituting the optimal portfolio into (5.13) such that

$$\frac{dH(\theta,y)}{dy} = m'_{\alpha^*}(y) - \theta v'_{\alpha^*}(y) = 0. \tag{5.14}$$

Mean function of the hedged cash flow can be found as

$$\begin{aligned} m_{\alpha^*}(y) &= \mathrm{E}_Q[CF_{\alpha^*}(D_1,D_2,S,y)] = \mathrm{E}_Q[CF(D_1,D_2,y) + \alpha^* f(S)] \\ &= m(y) + \alpha^* \mathrm{E}_Q[f(S)] \\ &= m(y) \end{aligned} \tag{5.15}$$

since

$$\mathrm{E}_Q[f(S)] = 0$$

under the risk-neutral probability measure Q. The variance of the hedged cash flow is given by

$$v_{\alpha^*}(y) = \mathrm{Var}[CF(D_1,D_2,y) + \alpha^* f(S)]. \tag{5.16}$$

Furthermore, the derivatives are

$$m'(y) = \mathrm{P}\{D_1 > Q-y\}(-r_1 + r_2\,\mathrm{P}\{D_2 > y\}),$$

and

$$\begin{aligned} v'_{\alpha^*}(y) = -2\boldsymbol{\mu}(y)^{\mathbf{T}}\mathbf{C}^{-1}\boldsymbol{\mu}'(y) &- 2r_1^2\,\mathrm{Cov}[\min\{D_1, Q-y\}, 1_{\{D_1>Q-y\}}] \\ &+ 2r_2^2\,\mathrm{Cov}[\min\{D_2, Q-\min\{D_1,Q-y\}\}, 1_{\{D_1\geq Q-y\}}1_{\{D_2>y\}}] \\ &- 2r_1r_2\,\mathrm{Cov}[\min\{D_2, Q-\min\{D_1,Q-y\}\}, 1_{\{D_1>Q-y\}}] \\ &+ 2r_1r_2\,\mathrm{Cov}[\min\{D_1, Q-y\}, 1_{\{D_1\geq Q-y\}}1_{\{D_2>y\}}]. \end{aligned}$$

It is shown in Lemma 3.2.1 and 3.3.1 that $m(y)$ is quasi-concave, and it is concave on the region $[y^*_{RN}, Q]$. To conduct the analyses in the rest of this chapter, $m_{\alpha^*}(y)$, equivalently $m(y)$, and $v_{\alpha^*}(y)$ should have certain properties. Throughout this chapter, the following assumption is made to ensure them.

Assumption 5.2.1 *The function $v_{\alpha^*}(y) = \mathrm{Var}[CF_{\alpha^*}(D_1,D_2,y)]$ is quasi-convex. Furthermore, $m(y)$ and $v_{\alpha^*}(y)$ are concave and convex, respectively, between y^*_{RN} (optimal risk-neutral threshold) and y^*_{MV} (optimal minimum-variance threshold).*

Proposition 5.2.1 *An optimal protection level $y(\theta)$ maximizing the hedged MV objective function always lies on the region between y^*_{RN} and y^*_{MV}.*

Proof. $m(y)$ is quasi-concave as proven earlier, and $v_{\alpha^*}(y)$ is quasi-convex by Assumption 5.2.1. Consider the case $y^*_{RN} \leq y^*_{MV}$. Then in $[0, y^*_{RN}]$, any y satisfies

$$m(y) = \mathrm{E}[CF(D_1, D_2, y)] \leq \mathrm{E}[CF(D_1, D_2, y^*_{RN})]$$

and

$$v_{\alpha^*}(y) = \mathrm{Var}[CF_{\alpha^*}(D_1, D_2, y)] \geq \mathrm{Var}[CF_{\alpha^*}(D_1, D_2, y^*_{RN})].$$

Then the region $[0, y^*_{RN}]$ is dominated given at least one of the above inequalities is strict. Similarly, on the region $[y^*_{MV}, Q]$, y satisfies

$$m(y) = \mathrm{E}[CF(D_1, D_2, y)] \leq \mathrm{E}[CF(D_1, D_2, y^*_{MV})]$$

and

$$v_{\alpha^*}(y) = \mathrm{Var}[CF_{\alpha^*}(D_1, D_2, y)] \geq \mathrm{Var}[CF_{\alpha^*}(D_1, D_2, y^*_{MV})].$$

Then the region $[y^*_{MV}, Q]$ is dominated where at least one of the inequalities above is strict.

For $y^*_{MV} < y^*_{RN}$, the same arguments can be made, that the regions $[0, y^*_{MV}]$ and $[y^*_{RN}, Q]$ are dominated. ■

Theorem 5.2.1 *The optimal protection level $y(\theta)$ maximizing the MV objective function is found using* (5.14) *such that*

$$m'(y(\theta)) - \theta v'_{\alpha^*}(y(\theta)) = 0.$$

Furthermore, $y(\theta)$ is increasing in θ if $y^*_{RN} < y^*_{MV}$, and it is decreasing in θ if $y^*_{MV} < y^*_{RN}$.

Proof. Define $\theta(y)$ as

$$\theta(y) = \frac{m'(y)}{v'_{\alpha^*}(y)}.$$

To have more information on $\theta(y)$, the derivative is obtained as

$$\theta'(y) = \frac{m''(y(\theta))v'_{\alpha^*}(y(\theta)) - m'(y(\theta))v''_{\alpha^*}(y(\theta))}{(v'_{\alpha^*}(y(\theta)))^2}. \tag{5.17}$$

From Assumption 5.2.1, the hedged mean and variance functions are concave and convex, respectively, between y^*_{RN} and y^*_{MV}. Then Tables 5.1 and 5.2 are valid for our problem. It is seen from Table 5.1 that for $y^*_{RN} < y^*_{MV}$, $\theta'(y)$ is positive, thus $\theta(y)$ is increasing on $[y^*_{RN}, y^*_{MV}]$. And it is seen from Table 5.2 that for $y^*_{MV} < y^*_{RN}$, $\theta'(y)$ is negative, thus $\theta(y)$ is decreasing on $[y^*_{MV}, y^*_{RN}]$.

Table 5.1: The analysis of the function $\theta(y)$ for $y^*_{RN} \leq y^*_{MV}$

	Interval		
Signs of	$[\boldsymbol{0, y^*_{RN}}]$	$[\boldsymbol{y^*_{RN}, y^*_{MV}}]$	$[\boldsymbol{y^*_{MV}, Q}]$
$\boldsymbol{m'(y)}$	(+)	(-)	(-)
$\boldsymbol{v'_{\alpha^*}(y)}$	(-)	(-)	(+)
$\boldsymbol{\theta(y)}$	(-)	(+)	(-)
$\boldsymbol{\theta'(y)}$		(+)	

Table 5.2: The analysis of the function $\theta(y)$ for $y^*_{MV} < y^*_{RN}$

	Interval		
Signs of	$[\boldsymbol{0, y^*_{MV}}]$	$[\boldsymbol{y^*_{MV}, y^*_{RN}}]$	$[\boldsymbol{y^*_{RN}, Q}]$
$\boldsymbol{m'(y)}$	(+)	(+)	(-)
$\boldsymbol{v'_{\alpha^*}(y)}$	(-)	(+)	(+)
$\boldsymbol{\theta(y)}$	(-)	(+)	(-)
$\boldsymbol{\theta'(y)}$		(-)	

Moreover we may infer the following results for $y = y^*_{RN}$ and $y = y^*_{MV}$

$$\theta(y^*_{RN}) = 0$$

since $m'(y^*_{RN}) = 0$, and

$$\theta(y^*_{MV}) = +\infty$$

since $v'_{\alpha^*}(y^*_{MV}) = 0$, which is in the denominator of $\theta(y)$.

The function $\theta(y)$ being increasing or decreasing in the non-dominated region suggests that there is an optimal protection level for any risk-aversion level $\theta \geq 0$. In the dominated regions $\theta(y)$ is less than or equal to zero.

Continuing to the analysis of the hedged objective function, the second order condition is given as

$$\frac{d^2 H(\theta, y)}{dy^2} = m''(y) - \theta v''_{\alpha^*}(y).$$

$m(y)$ is concave, thus $m''(y) \leq 0$, and $v_{\alpha^*}(y)$ is convex, thus $v''_{\alpha^*}(y) \geq 0$. Then the second order condition is satisfied, meaning that the MV objective function is concave between y^*_{MV} and y^*_{RN}.

Then the inverse Θ^{-1} of the $\theta(y)$ function becomes

$$y(\theta) = \Theta^{-1}(\theta),$$

which gives the optimal protection level for that θ value.

Thus, if $y^*_{RN} \leq y^*_{MV}$, since $\theta(y)$ increases on $[y^*_{RN}, y^*_{MV}]$, the inverse, $y(\theta)$, also does. And if $y^*_{MV} < y^*_{RN}$, since $\theta(y)$ decreases on $[y^*_{MV}, y^*_{RN}]$, its inverse, $y(\theta)$, also does. Figures 5.1 and 5.2 clearly demonstrates our findings. ■

Figure 5.1: $\theta(y)$ vs y and $y(\theta)$ vs θ for $y_{RN} \leq y_{MV}$

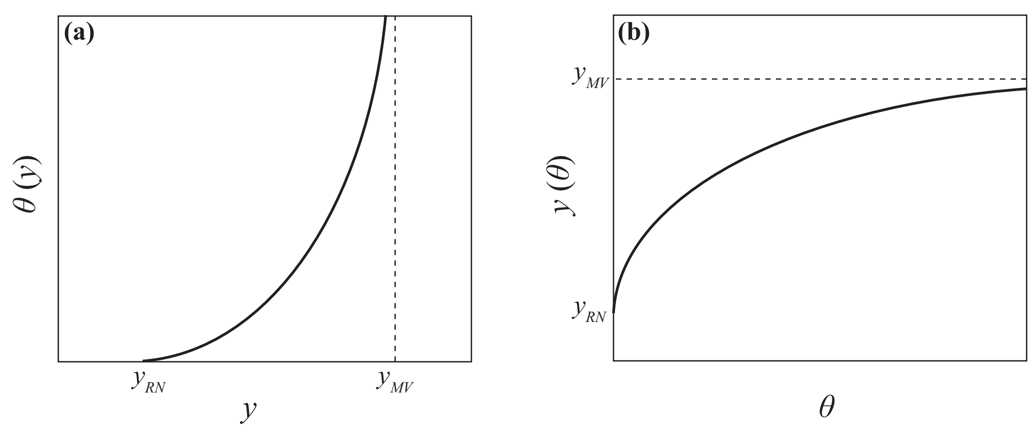
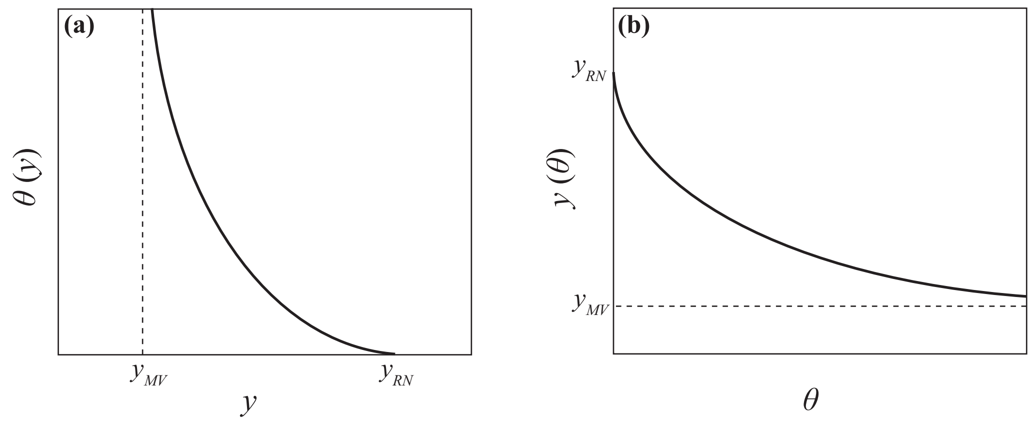


Figure 5.2: $\theta(y)$ vs y and $y(\theta)$ vs θ for $y_{MV} < y_{RN}$



5.2.1 *Hedging with a single financial security for the two-class RM problem with independent demands*

The analyses up to now were about the case in which multiple financial securities are used for hedging. However, the investor may choose to invest in a single financial security since it is practically useful and easier. Another reason could be that there may be a single asset that reduces the variance in a great amount, leaving no need for the multiple securities. Suppose now the hedging is performed with only one financial asset ($n = 1$), then the objective function becomes

$$\max_{y\geq 0,\alpha} H(\theta, y, \alpha) = \mathrm{E}[CF(D_1, D_2, y) + \alpha f(S)] - \theta \operatorname{Var}[CF(D_1, D_2, y) + \alpha f(S)], \tag{5.18}$$

where variance function can be written as

$$\begin{aligned}\operatorname{Var}[CF(D_1, D_2, y) + \alpha f(S)] =& \alpha^2 \operatorname{Var}[f(S)] + 2\alpha r_1 \operatorname{Cov}(f(S), \min\{D_1, Q - y\}) \\ &+ 2\alpha r_2 \operatorname{Cov}(f(S), \min\{D_2, Q - \min\{D_1, Q - y\}\}) \\ &+ \operatorname{Var}[CF(D_1, D_2, y)].\end{aligned} \tag{5.19}$$

Differentiating the objective function with respect to α, we obtain

$$\begin{aligned}\frac{d}{d\alpha} H(\theta, y, \alpha) =& -2\theta(\alpha \operatorname{Var}[f(S)] + r_1 \operatorname{Cov}(f(S), \min\{D_1, Q - y\}) \\ &+ r_2 \operatorname{Cov}(f(S), \min\{D_2, Q - \min\{D_1, Q - y\}\})).\end{aligned} \tag{5.20}$$

Furthermore, we observe that the objective function is concave in α from the second order condition

$$\frac{d^2}{d\alpha^2} H(\theta, y, \alpha) = -2\theta \operatorname{Var}[f(S)] \leq 0$$

which implies that the optimal α^* value obtained from setting (5.20) equal to zero is the unique maximizer of the hedged cash flow and satisfies the first order condition

$$\alpha \operatorname{Var}[f(S)] + r_1 \operatorname{Cov}(f(S), \min\{D_1, Q-y\}) + r_2 \operatorname{Cov}(f(S), \min\{D_2, Q-\min\{D_1, Q-y\}\}) = 0.$$

The optimal α^* is given by

$$\alpha^*(y) = \frac{-r_1 \operatorname{Cov}(f(S), \min\{D_1, Q-y\}) - r_2 \operatorname{Cov}(f(S), \min\{D_2, Q-\min\{D_1, Q-y\}\})}{\operatorname{Var}[f(S)]}. \tag{5.21}$$

Simplifying the expression above, we obtain

$$\alpha^*(y) = -r_1\beta_{D_1}(y) - r_2\beta_{D_1,D_2}(y)$$

where

$$\beta_{D_1}(y) = \frac{\operatorname{Cov}(f(S), \min\{D_1, Q-y\})}{\operatorname{Var}[f(S)]} \tag{5.22}$$

and

$$\beta_{D_1,D_2}(y) = \frac{\operatorname{Cov}(f(S), \min\{D_2, Q-\min\{D_1, Q-y\}\})}{\operatorname{Var}[f(S)]}. \tag{5.23}$$

Theorem 5.2.2 *The optimal protection level satisfies*

$$\begin{aligned} \frac{dH(\theta, y)}{dy} =& m'(y) \\ & - \theta\Big(- 2(r_1\beta_{D_1}(y) + r_2\beta_{D_1,D_2}(y))(r_1\beta'_{D_1}(y) + r_2\beta'_{D_1,D_2}(y)) \operatorname{Var}[f(S)]) + v'(y)\Big) \\ =& 0 \end{aligned} \tag{5.24}$$

where

$$\beta'_{D_1}(y) = \frac{\operatorname{Cov}(f(S), -1_{\{D_1 > Q-y\}})}{\operatorname{Var}(f(S))}$$

and

$$\beta'_{D_1,D_2}(y) = \frac{\operatorname{Cov}(f(S), 1_{\{D_1 \geq Q-y\}} 1_{\{D_2 > y\}})}{\operatorname{Var}[f(S)]}.$$

Proof. By Assumption 5.2.1, the objective function is concave between y^*_{RN} and y^*_{MV} and there is a unique y^* value maximizing the function. We can obtain y^* by taking the derivative of the objective function, (5.18), with respect to y and then setting it equal to zero. The next step is substituting the optimal α^* into this first order condition.

Taking the derivative of the objective function with respect to y, and setting equal to zero we obtain

$$\frac{dH(\theta,y)}{dy} = m'(y) - \theta\Big(2\alpha\,\mathrm{Var}[f(S)](r_1\beta'_{D_1}(y) + r_2\beta'_{D_1,D_2}(y)) + v'(y)\Big) = 0.$$

Substituting the optimal α^* found in (5.21), we obtain (5.24), which provides the optimal protection level. ■

Theorem 5.2.3 *The optimal hedging portfolio α^* reduces the variance of the cash flow by the amount $(-r_1\beta_{D_1}(y) - r_2\beta_{D_1,D_2}(y))^2 Var[f(S)]$.*

Proof. We substitute the optimal α^* into the variance function (5.19) and obtain the variance function of the hedged cash flow as

$$v_\alpha(y) = v(y) - (-r_1\beta_{D_1}(y) - r_2\beta_{D_1,D_2}(y))^2 Var[f(S)]. \tag{5.25}$$

$Var[f(S)]$ is positive, and the term containing the β functions is squared, meaning that it is also positive. Then the variance of hedged cash flow is less than the variance of the unhedged cash flow, where the amount of decrease being $(-r_1\beta_{D_1}(y) - r_2\beta_{D_1,D_2}(y))^2 Var[f(S)]$. ■

The reduction amount, $(-r_1\beta_{D_1}(y) - r_2\beta_{D_1,D_2}(y))^2 Var[f(S)]$, contains positive β functions. From (5.22) and (5.23), we observe that β functions show the correlation between the stock price and the cash flow. Then we can say that reduction amount increases as the correlation between the stock price and the cash flow increases.

5.3 MV Model with Dependent Random Demands for the Two-class RM Model

In this section we will discuss the dependent demands case of the same problem discussed in the previous section. The demand random variables D_1 and D_2 are again assumed to be correlated with the financial variable S. The hedged objective function for the RM MV model is given by the general formula

$$\max_{y\geq 0,\boldsymbol{\alpha}} H(\theta,y,\boldsymbol{\alpha}) = \mathrm{E}[CF(D_1,D_2,y)] - \theta[\mathrm{Var}(CF(D_1,D_2,y) + \boldsymbol{\alpha}^{\mathbf{T}}\mathbf{f}(S)])$$

The optimal portfolio found in (5.3) and the $\boldsymbol{\mu}(y)$ found in (5.12) are the same.

The derivative of the hedged objective function can be obtained using (5.6) as

$$
\begin{aligned}
\frac{dH(\theta, y, \boldsymbol{\alpha})}{dy} =& - r_1 \,\mathrm{P}\{D_1 > Q - y\} + r_2 \,\mathrm{P}\{D_1 \geq Q - y, D_2 > y\} \\
& - 2\theta \begin{pmatrix} -\boldsymbol{\alpha}^{\mathbf{T}} \boldsymbol{\mu}'(y) \\ -r_1^2 \,\mathrm{Cov}[\min\{D_1, Q-y\}, 1_{\{D_1 > Q-y\}}] \\ +r_2^2 \,\mathrm{Cov}[\min\{D_2, Q - \min\{D_1, Q-y\}\}, 1_{\{D_1 \geq Q-y, D_2 > y\}}] \\ -r_1 r_2 \,\mathrm{Cov}[\min\{D_2, Q - \min\{D_1, Q-y\}\}, 1_{\{D_1 > Q-y\}}] \\ +r_1 r_2 \,\mathrm{Cov}[\min\{D_1, Q-y\}, 1_{\{D_1 \geq Q-y, D_2 > y\}}] \end{pmatrix} \\
=& 0
\end{aligned}
\tag{5.26}
$$

where $\mu_i'(y)$ is the derivative of $\mu_i(y)$, such that

$$\mu_i'(y) = \frac{d\mu_i(y)}{y} = -r_1 \,\mathrm{Cov}\left(f_i(S), 1_{\{D_1 > Q-y\}}\right) + r_2 \,\mathrm{Cov}\left(f_i(S), 1_{\{D_2 > y, D_1 > Q-y\}}\right).$$

Now we can obtain the first order condition by substituting the optimal portfolio into (5.26) to obtain

$$\frac{dH(\theta, y)}{dy} = m_{\alpha^*}'(y) - \theta v_{\alpha^*}'(y) = 0 \tag{5.27}$$

where the mean and the variance functions of the hedged cash flow can be found as (5.15) and (5.16) in the previous section.

Their derivatives are given by

$$m'(y) = \mathrm{P}\{D_1 > Q - y\}(-r_1 + r_2 \,\mathrm{P}\{D_2 > y | D_1 > Q - y\})$$

and

$$
\begin{aligned}
v_{\alpha^*}'(y) = -2\boldsymbol{\mu}(y)^{\mathbf{T}} \mathbf{C}^{-1} \boldsymbol{\mu}'(y) &- 2r_1^2 \,\mathrm{Cov}[\min\{D_1, Q-y\}, 1_{\{D_1 > Q-y\}}] \\
&+ 2r_2^2 \,\mathrm{Cov}[\min\{D_2, Q - \min\{D_1, Q-y\}\}, 1_{\{D_1 \geq Q-y, D_2 > y\}}] \\
&- 2r_1 r_2 \,\mathrm{Cov}[\min\{D_2, Q - \min\{D_1, Q-y\}\}, 1_{\{D_1 > Q-y\}}] \\
&+ 2r_1 r_2 \,\mathrm{Cov}[\min\{D_1, Q-y\}, 1_{\{D_1 \geq Q-y, D_2 > y\}}].
\end{aligned}
$$

By Assumption 5.2.1, Proposition 5.2.1 is also valid here. Then the statement that the optimal protection level $y(\theta)$ maximizing the hedged MV objective function always lies on the region between y_{RN}^* and y_{MV}^* is also true for this case. The non-dominated region is the region between y_{RN}^* and y_{MV}^*, regardless of their order.

From Theorem 5.2.1, the optimal protection level $y(\theta)$ maximizing the MV objective function is found using (5.27) such that

$$m'(y(\theta)) - \theta v'_{\alpha^*}(y(\theta)) = 0.$$

Moreover, $y(\theta)$ is increasing in θ if $y^*_{RN} \leq y^*_{MV}$, and it is decreasing in θ if $y^*_{MV} < y^*_{RN}$, by the use of the inverse Θ^{-1} of the $\theta(y)$ function

$$y(\theta) = \Theta^{-1}(\theta),$$

which gives the optimal protection level for that θ value. We observe that, as in Chapter 3, as the risk aversion level increases, the $y(\theta)$ value always move towards the y^*_{MV}, where the minimum-variance is obtained.

5.3.1 Hedging with a single financial security for the two-class RM problem with dependent demands

In this subsection, the dependent counterpart of Subsection 5.2.1 is presented. The optimal α^* is the same as in (5.21), which can be expressed simply as

$$\alpha^*(y) = -r_1\beta_{D_1}(y) - r_2\beta_{D_1,D_2}(y) \tag{5.28}$$

where

$$\beta_{D_1}(y) = \frac{\mathrm{Cov}(f(S), \min\{D_1, Q-y\})}{\mathrm{Var}[f(S)]}$$

and

$$\beta_{D_1,D_2}(y) = \frac{\mathrm{Cov}(f(S), \min\{D_2, Q - \min\{D_1, Q-y\}\})}{\mathrm{Var}[f(S)]}.$$

Theorem 5.3.1 *The optimal protection level of the MV objective satisfies*

$$\begin{aligned}\frac{dH(\theta,y)}{dy} =& m'(y) \\ & - \theta\Big(- 2(r_1\beta_{D_1}(y) + r_2\beta_{D_1,D_2}(y))(r_1\beta'_{D_1}(y) + r_2\beta'_{D_1,D_2}(y))\,\mathrm{Var}[f(S)]) + v'(y)\Big) \\ =& 0,\end{aligned} \tag{5.29}$$

where

$$\beta'_{D_1}(y) = \frac{\mathrm{Cov}(f(S), -1_{\{D_1 > Q-y\}})}{\mathrm{Var}(f(S))}$$

and

$$\beta'_{D_1,D_2}(y) = \frac{\mathrm{Cov}(f(S), 1_{\{D_1 \geq Q-y, D_2 > y\}})}{\mathrm{Var}[f(S)]}.$$

Proof. By Assumption 5.2.1, the objective function is concave between y^*_{RN} and y^*_{MV} and there is a unique y^* value maximizing the MV objective function. Now we obtain y^* from the first order condition of the objective function with respect to y, and substitute (5.28) into this condition.

Taking the derivative of the objective function with respect to y, setting it equal to zero, substituting the optimal α^* we obtain (5.29) above, which provides the optimal protection level. ■

The $y(\theta)$ for the hedged MV objective will be different from the unhedged $y(\theta)$ value without hedging. The difference depends on the β values which represent the correlation between the stock price and the cash flow.

Also note that the variance of the unhedged cash flow in this case is reduced by the same amount as shown in Theorem 5.2.3 in Subsection 5.2.1, due to the hedging operations.

5.4 Final Remarks on the Problem

In this section, we will shed more light on the structural properties of the hedged MV objective function of the RM model. As we have shown already, the objective function may simply be defined as

$$H(\theta, y, \alpha) = m(y) - \theta v_\alpha(y). \tag{5.30}$$

The mean function of the hedged cash flow is the same as the mean function of the unhedged cash flow under the risk-neutral probability measure Q. However, the variance function is

$$\begin{aligned} v_{\alpha^*}(y) &= \mathrm{Var}[CF(D_1, D_2, y) + \alpha f(S)] \\ &= v(y) + \alpha^2 \mathrm{Var}[f(S)] + 2\alpha \mathrm{Cov}(f(S), CF(D_1, D_2, y)). \end{aligned}$$

We have proved that the hedged objective function is concave in α. From the first order condition of (5.30) we obtain the optimal portfolio as

$$\alpha^*(y) = -\frac{\mathrm{Cov}(f(S), CF(D_1, D_2, y))}{\mathrm{Var}[f(S)]}$$

for all $\theta \geq 0$.

Theorem 5.4.1 *The amount of reduction in the variance function due to the hedging operations is proportional to the correlation coefficient between the payoff of the financial asset used, and the cash flow, such that*

$$v_{\alpha^*}(y) = (1 - \rho^2_{f(S),CF(D_1,D_2,y)})v(y)$$

Proof. Differentiating the objective function with respect to the protection level y, and substituting $\alpha^*(y)$ into the objective function, we obtain

$$\frac{d}{dy}H(\theta, y) = \frac{dm(y)}{dy} - \theta\frac{dv_{\alpha^*}(y)}{dy}.$$

The hedged variance function is then

$$\begin{aligned} v_{\alpha^*}(y) &= v(y) + \frac{\text{Cov}^2(f(S), CF(D_1, D_2, y))}{\text{Var}[f(S)]} - \frac{2\,\text{Cov}^2(f(S), CF(D_1, D_2, y))}{\text{Var}[f(S)]} \\ &= v(y) - \frac{\text{Cov}^2(f(S), CF(D_1, D_2, y))}{\text{Var}[f(S)]}. \end{aligned} \tag{5.31}$$

Since the covariance term is squared, it is always positive. $\text{Var}[f(S)]$ is also positive. The above expression for v_{α^*} shows that hedging reduces the variance of the cash flow for both negative and positive association. The amount of reduction is $\text{Cov}^2(f(S), CF(D_1, D_2, y))$ / $\text{Var}[f(S)]$. Furthermore, we can infer that the greater the correlation (either positive or negative), the greater the reduction in the variance.

Analyzing (5.31) further, we can rearrange it as follows

$$\begin{aligned} v_{\alpha^*}(y) &= v(y) - \rho^2_{f(S),CF(D_1,D_2,y)}v(y) \\ &= (1 - \rho^2_{f(S),CF(D_1,D_2,y)})v(y) \end{aligned}$$

where $\rho_{f(S),CF(D_1,D_2,y)})$ is the correlation coefficient between the payoff of the derivative security $f(S)$ and the unhedged cash flow $CF(D_1, D_2, y)$. The above result is obtained using the definition of correlation coefficient as

$$\begin{aligned} \rho_{f(S),CF(D_1,D_2,y)} &= \text{Cov}(f(S), CF(D_1, D_2, y))/\sqrt{\text{Var}[f(S)]\,\text{Var}[CF(D_1, D_2, y)]} \\ \rho^2_{f(S),CF(D_1,D_2,y)} &= \overset{2}{\text{Cov}}(f(S), CF(D_1, D_2, y))/\,\text{Var}[f(S)]\,\text{Var}[CF(D_1, D_2, y)], \end{aligned}$$

then

$$\overset{2}{\text{Cov}}(f(S), CF(D_1, D_2, y))/\operatorname{Var}[f(S)] = \rho^2_{f(S),CF(D_1,D_2,y)} \operatorname{Var}[CF(D_1, D_2, y)]. \blacksquare$$

To observe the effect of the correlation coefficient on hedging, let us investigate the hedged variance function on the extreme values that the correlation coefficient may take, namely $\rho = 0$ and $\rho = \{-1, 1\}$. If $\rho_{f(S),CF(D_1,D_2,y)}$ is zero, meaning that there is no correlation between the payoff of the derivative security and the cash flow, we obtain

$$\begin{aligned} v_{\alpha^*}(y) &= (1 - \rho^2_{f(S),CF(D_1,D_2,y)})v(y) \\ &= v(y) \end{aligned}$$

implying that the hedging operation does not reduce the variance at all. On the other hand, if $\rho_{f(S),CF(D_1,D_2,y)}$ is either -1 or 1, meaning that there is perfect correlation between the payoff of the derivative security and the cash flow, we obtain

$$\begin{aligned} v_{\alpha^*}(y) &= (1 - \rho^2_{f(S),CF(D_1,D_2,y)})v(y) \\ &= 0 \end{aligned}$$

which means the variance is reduced to zero. The operation has zero risk.

Another result regarding the optimal protection level of the hedged MV objective function is presented in the following theorem.

Theorem 5.4.2 *The optimal protection level after hedging is not the same as in the unhedged case.*

Proof. This primarily follows from Theorem 5.4.1, where it is shown that the hedged variance function $v_\alpha(y)$ is the scaled form of $v(y)$, the scale being $(1 - \rho^2_{f(S),CF(D_1,D_2,y)})$. Since the scale depends on the decision variable y, the optimal protection level will be different form that in the unhedged case. It could either be less than or greater than the old optimal protection level, depending on the structure of the correlation. $\blacksquare$

In this chapter, we analyzed the RM problem with an MV approach incorporating the hedging operations. The optimal portfolio and the optimal protection levels are determined simultaneously for the general MV approach, and for the RM model with independent and dependent demand random variables. Remarks on hedging with single and multiple financial instruments are presented. The special structure of the variance function obtained after the hedging operations is investigated. Moreover the hedged variance function is proved to be decreasing after performing hedging, and the amount of decrease is determined explicitly. Finally, it is shown that the amount of decrease is proportional to the correlation coefficient between the payoff of the derivative security used, and the unhedged cash flow, where perfect correlation implies a zero variance.

Chapter 6

NUMERICAL ILLUSTRATIONS

Theoretical results are of great importance in obtaining general features of any problem. However, to be able to have more intuition in the problem or describe the effects of the parameters on the model, numerical illustrations are proven to be very helpful. They are even more beneficial if the results on hand are complicated, or if the analytical results are hard to obtain. In this chapter, we aim to illustrate the results found in Chapter 3, Chapter 4 and Chapter 5. In Section 6.1, we provide simulation results for the MV model with independent and dependent demands, then in Section 6.2 we illustrate the results for the perfect and imperfect hedging on the MV model.

6.1 MV Model

The main results obtained in Chapter 3 will be illustrated in this section. Demand random variables D_1 and D_2 are assumed to have Poisson distributions with means λ and $\mu(x)$, having probability mass functions

$$\mathrm{P}\{D_1 = x\} = \frac{e^{-\lambda}\lambda^x}{x!} \tag{6.1}$$

and

$$\mathrm{P}\{D_2 = z | D_1 = x\} = \frac{e^{-\mu(x)}\mu(x)^z}{z!} \tag{6.2}$$

in the dependent demands case.

If the demands are independent, D_2 simplifies to a basic Poisson variable, having a distribution given by

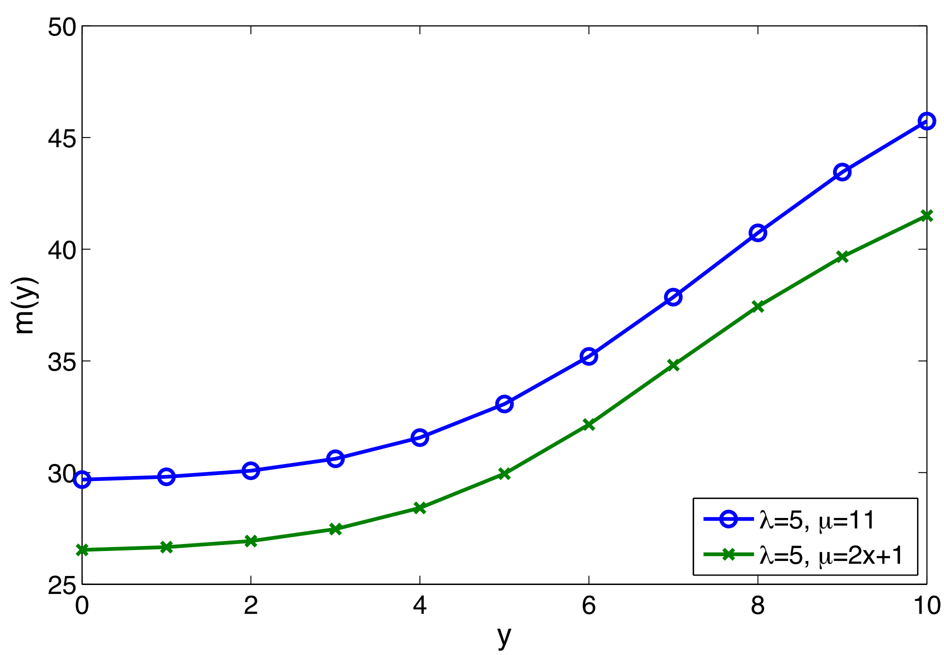
$$\mathrm{P}\{D_2 = z\} = \frac{e^{-\mu}\mu^z}{z!}. \tag{6.3}$$

By generating 10,000 instances, the demand values are obtained via MATLAB. The mean and the variance functions are then found by following the expressions in Section 3.1. They were obtained for the continuous demand random variables to better understand the general structure of the revenue management problem. However, for the simulation purposes, a discrete random variable will be more appropriate, since the demands for seats arrive in discrete amounts. Thus, for the numerical illustrations here, the discrete counterparts of the expressions is obtained and used.

The dependent and independent cases will be presented together, in the same figures. Suppose the seat prices for class-1 and class-2 are $r_1 = 1$, $r_2 = 5$, the fixed seat capacity in a plane is $Q = 10$. The demand parameter for class-1 is $\lambda = 5$, while the demand parameter for class-2 is $\mu = 11$, for the independent case. If the demand parameters are dependent, then μ is defined as a function of the demand realizations of class-1, such that $\mu(x) = 2x+1$. The decision variable is the protection level of class-2 (number of seats reserved for class-2), denoted by y.

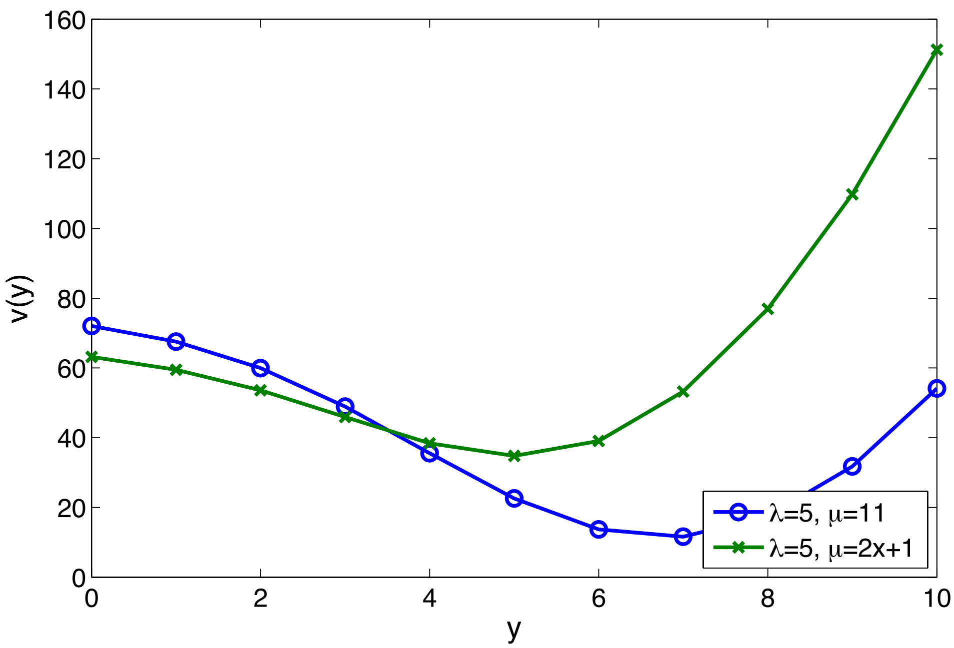
It is obvious from Figure 6.1 that the mean function, $m(y)$, is concave both in the independent and dependent cases. The protection level maximizing the expected return is $y^*_{RN} = 10$, for the independent and the dependent case. This implies, a risk-neutral decision maker protects all the seats for class-2 customers, in this case.

Figure 6.1: $m(y)$ vs y for the independent and dependent demands



The variance function is also obtained, given by Figure 6.2. The point minimizing the

Figure 6.2: $v(y)$ vs y for the independent and dependent demands



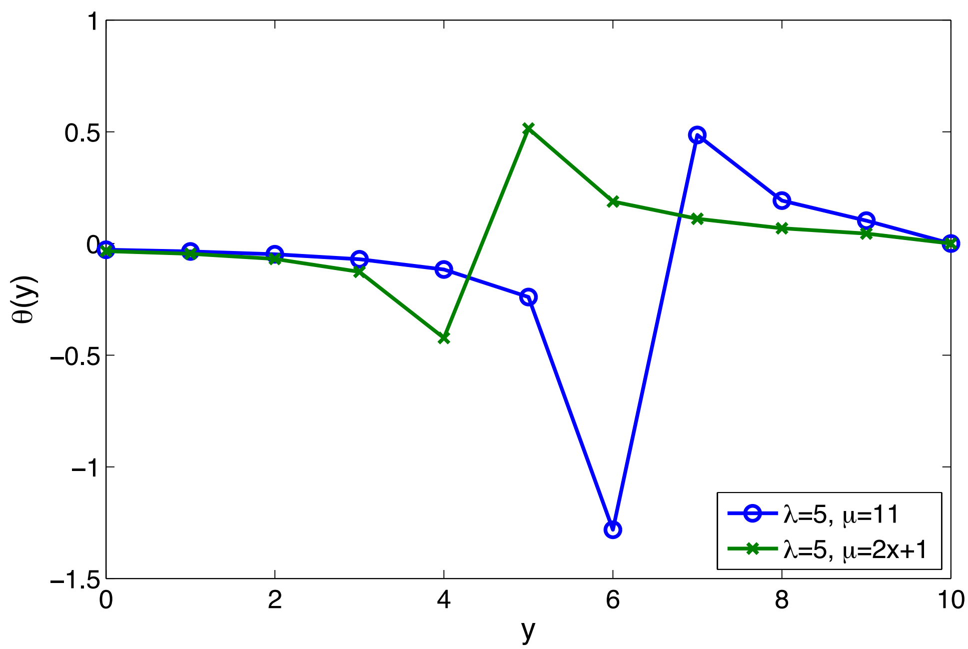
variance function is $y^*_{MV} = 7$ for the independent case, and $y^*_{MV} = 5$ for the dependent case. Then we have the case that $y^*_{MV} \leq y^*_{RN}$. From Proposition 3.2.1 of Chapter 3 the non-dominated region is $[7, 10]$ for the independent case, and it is $[5, 10]$ for the dependent case. Note that, in this regions, the mean functions are concave and the variance functions are convex. Then, from Theorem 3.4.1, it should be true that $y(\theta)$ is decreasing in θ, on the non-dominated regions, which is illustrated in Figure 3.5 of Section 3.4.

$\theta(y)$ is defined as

$$\theta(y) = \frac{m'(y)}{v'(y)}. \tag{6.4}$$

Figure 6.3 depicts $\theta(y)$ function. As anticipated, for the independent case, $\theta(y)$ is decreasing on $[7, 10]$; for the dependent case, it is again decreasing on $[5, 10]$. As the Tables 3.1 and 3.2 of Section 3.4 suggest, on the dominated regions $\theta(y)$ function takes minus sign, which implies $\theta(y)$ is only defined on the non-dominated regions since we have made the assumption $\theta \geq 0$ earlier. We can conclude that if $y^*_{MV} \leq y^*_{RN}$ is true, then the optimal protection level decreases as the risk- aversion level increases. It makes sense, because by decreasing, the optimal protection level actually gets closer to y^*_{MV}, where we obtain the minimum variance.

Figure 6.3: $\theta(y)$ vs y for the independent and dependent demands

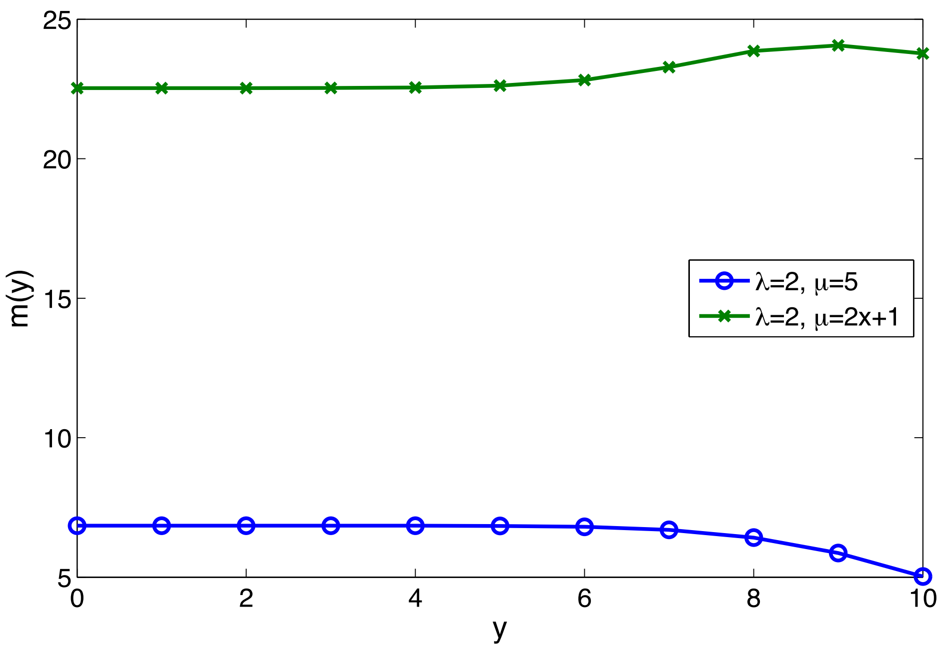


To illustrate the opposite case $y^*_{MV} > y^*_{RN}$, the following example will be presented. Suppose $r_1 = 1$, $r_2 = 1.01$, and $Q = 10$. The demand parameter for class-1 is $\lambda = 2$, while the demand parameter for class-2 is $\mu = 5$, for the independent case. If the demand parameters are dependent, then $\mu(x) = 2x+1$. The decision variable is again the protection level of class-2 (number of seats reserved for class-2), denoted by y.

It is obvious from Figure 6.4 that the mean function, $m(y)$, is concave for the independent case and it is quasi-concave for the dependent case. The protection level maximizing the expected return is $y^*_{RN} = 1$, for the independent case and $y^*_{RN} = 9$ the dependent case.

The variance function is also obtained, given by Figure 6.5. The point minimizing the variance function is $y^*_{MV} = 8$ for the independent case, and $y^*_{MV} = 3$ for the dependent case. Then we have the case that $y^*_{MV} > y^*_{RN}$ for independent demands. From Proposition 3.2.1 of Chapter 3 the non-dominated region is $[1, 8]$ for the independent case. Note that, in this regions, the mean function is concave and the variance function is convex. Then, from Theorem 3.4.1, it should be true that $y(\theta)$ is increasing in θ, on the non-dominated region,

Figure 6.4: $m(y)$ vs y for the independent and dependent demands



which is illustrated in Figure 3.5 of Section 3.4.

Figure 6.6 depicts $\theta(y)$ function. As anticipated, for the independent case, $\theta(y)$ is increasing on $[1, 8]$. As the Tables 3.1 and 3.2 of Section 3.4 suggest, on the dominated regions $\theta(y)$ function takes minus sign, which implies $\theta(y)$ is only defined on the non-dominated regions since we have made the assumption $\theta \geq 0$ earlier. We can conclude that if $y^*_{MV} > y^*_{RN}$ is true, then the optimal protection level increases as the risk-aversion level increases. It makes sense, because by increasing, the optimal protection level actually gets closer to y^*_{MV}, where we obtain the minimum variance.

Figure 6.5: $v(y)$ vs y for the independent and dependent demands

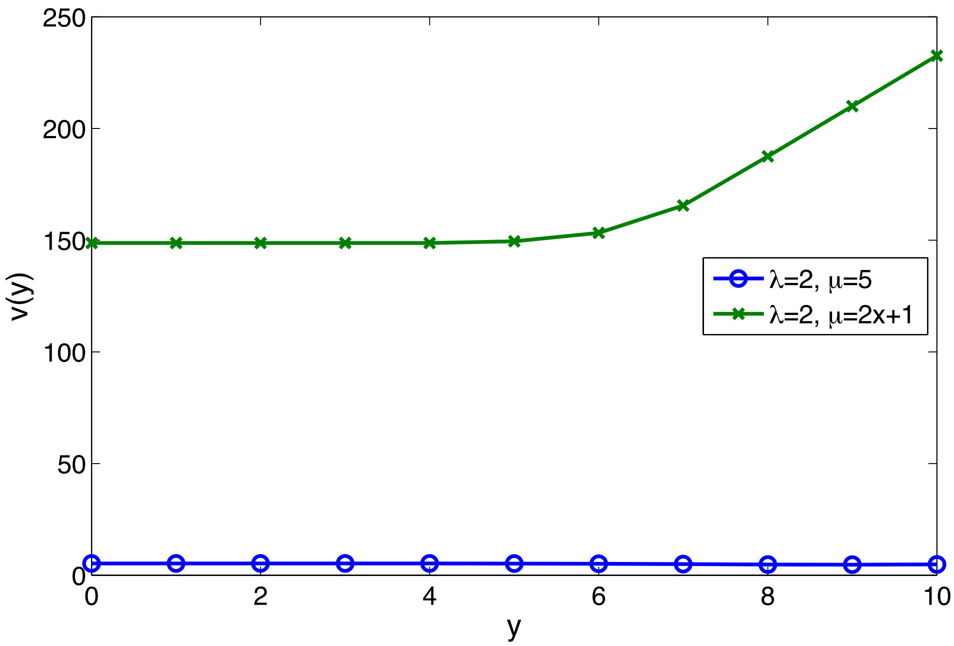
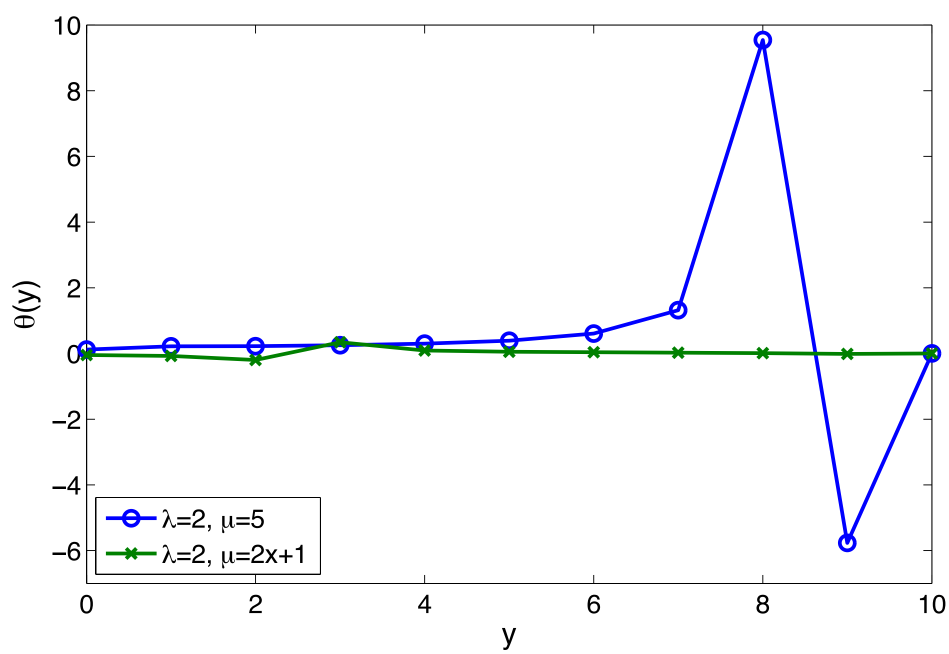


Figure 6.6: $\theta(y)$ vs y for the independent and dependent demands



6.2 *Perfect and Imperfect Hedging on MV Model*

In this section, we present numerical illustrations of the results in Chapter 4 and Chapter 5. Some of the parameters and structural properties of the model are obtained using an example similar to the one in Gaur and Seshadri (2005), in which they hedge the demand risk of a single-period single-item inventory problem using a stock. The interest rate is $r = 10\%$ per year and the initial stock price is $S_0 = 3$. Suppose that the return of the stock S_T/S_0 is lognormally distributed under the risk-neutral measure having mean $(r - \frac{\sigma^2}{2})T$ and standard deviation $\sigma\sqrt{T}$, such that

$$\ln\left(\frac{S_T}{S_0}\right) \sim\ N\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right) = N(0.0275, 0.2121)$$

where $\sigma = 30\%$ per year.

We assume that demand random variables are linearly correlated with the financial market, such that, $D_1 = a_1 + b_1 S + \epsilon_1$ and $D_2 = a_2 + b_2 S + \epsilon_2$, where $a_1 = 3$, $a_2 = 1$, $b_1 = 4$, and $b_2 = 1$. The error ϵ_i is normally distributed with mean zero and standard deviation σ_{ϵ_i}. The capacity is fixed, which is $Q = 10$, and the price parameters for the class-1 and class-2 are $r_1 = 50$ and $r_2 = 100$, respectively.

In the examples throughout this chapter, we use two instruments; futures and call options. In Chapter 4, we have proven that if the demands are linearly correlated with the financial market, the cash flow is continuous and piecewise linear, and it can be perfectly hedged using futures and call options.

We define three types of portfolios in this chapter. In the first portfolio, only the futures are used, which has a payoff of $f_1(S)$. Second portfolio consists of only the call options. As we have shown in Chapter 4, the cash flow function may have some of the three breakpoints $B_1 = (Q - y - a_1)/b_1$, $B_2 = (Q - a_1 - a_2)/(b_1 + b_2)$ and $B_3 = (y - a_2)/b_2$; but not the all of them. The corresponding payoff functions are $f_{21}(S)$, $f_{22}(S)$, and $f_{23}(S)$, with strike prices $\kappa_1 = B_1$, $\kappa_2 = B_2$ and $\kappa_3 = B_3$. Lastly, in the third portfolio, both the futures and the call options are used, with the payoffs $f_1(S)$, $f_{21}(S)$, $f_{22}(S)$, and $f_{23}(S)$, as given by

$$f_1(S) = S - e^{rT} S_0$$

and

$$f_{21}(S) = \max\{S - \kappa_1, 0\} - e^{rT} C_1$$
$$f_{22}(S) = \max\{S - \kappa_2, 0\} - e^{rT} C_2$$
$$f_{23}(S) = \max\{S - \kappa_3, 0\} - e^{rT} C_3$$

where C_1, C_2, and C_3 are the prices of the call options at time 0. We assume that in the market, there is no arbitrage opportunity, which implies that the expected values of the payoff functions of the instruments are zero. Under this assumption, the following are true

$$C_1 = E\left[e^{-rT} \max\{S - \kappa_1\}\right]$$
$$C_2 = E\left[e^{-rT} \max\{S - \kappa_2\}\right]$$
$$C_3 = E\left[e^{-rT} \max\{S - \kappa_3\}\right]$$

and

$$E[f_1(S)] = E[f_{21}(S)] = E[f_{22}(S)] = E[f_{23}(S)] = 0.$$

The numerical illustrations in the rest of this section are obtained by Monte Carlo method, using MATLAB. The demand values and stock prices are generated to obtain the unhedged and hedged cash flows. We have defined eight different scenarios to compare to each other. They are defined as follows:

Scenario 1: No portfolio is used, and the aim is maximizing the expected cash flow ($\theta = 0$),

Scenario 2: The first portfolio is used (future), and the aim is maximizing the expected cash flow ($\theta = 0$),

Scenario 3: The second portfolio is used (call options), and the aim is maximizing the expected cash flow ($\theta = 0$),

Scenario 4: The third portfolio is used (future and call options), and the aim is maximizing the expected cash flow ($\theta = 0$),

Scenario 5: No portfolio is used, and the aim is maximizing the MV cash flow ($\theta = 0.1$),

Scenario 6: The first portfolio is used (future), and the aim is maximizing the MV cash flow ($\theta = 0.1$),

Scenario 7: The second portfolio is used (call options), and the aim is maximizing the MV cash flow ($\theta = 0.1$),

Scenario 8: The third portfolio is used (future and call options), and the aim is maximizing the MV cash flow ($\theta = 0.1$).

6.2.1 The MV model

The MV model discussed in the earlier chapters will be used here as well. The unhedged cash flow at time T is given by

$$CF(D_1, D_2, y) = r_1 \min\{D_1, Q - y\} + r_2 \min\{D_2, Q - \min\{D_1, Q - y\}\}$$

and the hedged cash flow is

$$CF_{\boldsymbol{\alpha}^*(y)}(D_1, D_2, S, y) = CF(D_1, D_2, y) + \boldsymbol{\alpha}^*(y)\mathbf{f}(S).$$

As suggested in Chapter 4, the variance of the cash flow becomes zero when the demands are perfectly correlated with the financial market. We first assume that the standard deviation of the demand error is $\sigma_{\epsilon_1} = \sigma_{\epsilon_2} = 0$, so that the demands D_1 and D_2 are perfectly correlated with the financial variable S. The risk aversion level is chosen to be $\theta = 0.1$. By generating 50,000 instances, the stock prices, demand values, payoffs of the financial instruments, and profits are calculated for a stream of optimal protection levels. The optimal portfolios are calculated using (5.3) and (5.21) of Chapter 5. The mean, variance and MV values of each scenario are calculated for the corresponding optimal protection levels.

For scenarios 1-4, the decision maker is risk-neutral, thus the optimal protection level is determined by the means of the cash flows. On the other hand, in scenarios 5-8, the decision maker is risk averse, and uses an MV approach with $\theta = 0.1$. Optimal protection levels ($y(\theta)$) are the values maximizing the MV function.

Table 6.1 summarizes the results for $\sigma_{\epsilon_i} = 0$. The most important result is that the third portfolio of future and call options (scenarios 4 and 8) completely eliminates the variance of the cash flow for both the risk-neutral and the risk-averse (MV) decision maker. We have shown the validity and impact of our statement in Chapter 4.

Let us move on to comparing the risk-neutral scenarios 1-4. Their mean values can be considered to be the same, thus, we can compare their variances. We clearly observe the risk reduction by the use of financial hedging; the variance is reduced to zero (reduced by 100%) for scenario 4, and for the others it is reduced by up to 58%. For scenarios 5-8, we

Table 6.1: The mean and variance values of the cash flows, MV values, and the optimal portfolios for the optimal protection levels for $\sigma_{\epsilon_i} = 0$

$\sigma_{\epsilon_i} = 0$	$y(0.1)$	Mean	Variance	MV	Opt. Portfolio (α)
S1	10	949.31	6005.50	-	-
S2	10	949.19	2509.43	-	-65.52
S3	10	949.08	4741.84	-	0, -54.15, 0
S4	10	949.79	0	-	-200, 0, 200, 0
S5	8	896.40	300.28	866.37	-
S6	8	896.39	257.37	870.65	-7.25
S7	8	896.38	274.78	868.90	0, -5.78, 0
S8	10	949.79	0	949.79	-200, 0, 200, 0

compare the MV values to see the effect of financial hedging. It is obvious that hedging leads significant increases in the MV objective function values. Finally, to see the effect of the risk aversion (MV approach) on the variances, we compare the variances of scenarios 1 and 5. The variance decreases drastically by 95%.

The same analysis is conducted for different standard deviation values of the demand error $\sigma_{\epsilon_i} = 0.20$ and $\sigma_{\epsilon_i} = 0.40$. As the σ_{ϵ_i} value increases, the degree of correlation between the demands and the stock price decreases. The results are presented in Table 6.2 and 6.3.

Suppose $\sigma_{\epsilon_i} = 0.20$. The correlation is no more perfect, thus, the variance of scenarios 4 and 8 will not be zero, however, we expect all the variance values to decrease as a result of the hedging operations. Indeed, by comparing the variance values in scenarios 2-4 to scenario 1, it is seen that the variance is reduced by up to 97%. Comparing the variance values of scenarios 1 and 5 to quantify the effect of the risk aversion, we see that the variance is reduced by 94%.

Now suppose $\sigma_{\epsilon_i} = 0.40$. By comparing the variance values in scenarios 2-4 to scenario 1, it is seen that the variance is reduced by up to 89%. Comparing the variance values of scenarios 1 and 5 to quantify the effect of the risk aversion, we see that the variance is reduced by 93%.

Table 6.2: The mean and variance values of the cash flows, MV values, optimal protection levels and the optimal portfolios for $\sigma_{\epsilon_i} = 0.20$

$\sigma_{\epsilon_i} = 0.20$	$y(0.1)$	Mean	Variance	MV	Opt. Portfolio (α)
S1	10	948.90	6143.71	-	-
S2	10	948.77	2635.23	-	-65.63
S3	10	948.67	4860.62	-	0, -54.56, 0
S4	10	949.38	173.37	-	-199.82, 0, 200.34, 0
S5	8	896.19	328.65	863.32	-
S6	8	896.17	281.86	867.98	-7.58
S7	8	896.16	300.11	866.15	0, -6.11, 0
S8	10	949.38	173.37	932.04	-199.82, 0, 200.34, 0

Table 6.3: The mean and variance values of the cash flows, MV values, optimal protection levels and the optimal portfolios for $\sigma_{\epsilon_i} = 0.40$

$\sigma_{\epsilon_i} = 0.40$	$y(0.1)$	Mean	Variance	MV	Opt. Portfolio (α)
S1	10	947.57	6541.43	-	-
S2	10	947.45	2995.26	-	-65.98
S3	10	947.34	5199.75	-	0, -55.79, 0
S4	10	948.06	673.45	-	-199.37, 0, 201.41, 0
S5	8	895.55	413.45	854.21	-
S6	8	895.53	354.72	860.06	-8.49
S7	8	895.52	375.38	857.98	0, -7.06, 0
S8	9	931.54	347.79	896.77	-205.62, 0, 209.93, 0

The examples for the positive σ_{ϵ_i} values suggest that the variance reductions decrease when the standard deviation of the demand error increases.

If we are to comment on the optimal protection levels, we see that as the σ_{ϵ_i} value increases, the optimal protection level tends to decrease. This follows from Theorem (3.4.1) of Section 3.4 that as the risk aversion level increases, the optimal protection level move from the y^*_{RN} (risk-neutral solution), to the y^*_{MV} (minimum variance solution). For our example, the y^*_{RN} and y^*_{MV} values are given by Table 6.4. Clearly, $y^*_{MV} \leq y^*_{RN}$ for all four scenarios. Then as θ increases, we expect $y(\theta)$ to move towards the y^*_{MV} points, in other words, we expect them to decrease. From scenario 8, we observe that the MV optimal solution decreases as the standard deviation of the demand error increases.

Table 6.4: y^*_{RN} and y^*_{MV} values for scenarios 1-4

	y^*_{RN}	y^*_{MV}
S1	10	0
S2	10	0
S3	10	2
S4	10	3

As a last comment, the optimal portfolios found suggest that for the first portfolio (future), the optimal action is always selling the future. The reason behind is the positive correlation between the demands and the stock price. For the second portfolio (call options) we always sell the option where in the third portfolio (future and call options) we prefer to buy the call options. Furthermore, for all three examples here, the portfolio consisting of only futures provide a better representation of the cash flow, thus, it is able to decrease the variance more, compared to the portfolio consisting of call options only.

Chapter 7

CONCLUSIONS

In this thesis, we use the MV framework for the two-class revenue management problem when the demands are random. This work can be divided into three parts. In the first part, we present the results for the RM model with an MV approach. The second part discusses the structure of the problem when there is perfect correlation between the demands and the financial market. Finally, the third part deals with the RM problem using the MV approach along with the hedging opportunities, when the demands are partially correlated with the demand random variables.

In the first part, the RM model using an MV approach is analyzed. Optimal protection levels are characterized for each risk aversion level and analyzed to study the effects of risk-aversion parameter. The mean function turns out to be quasi-concave for the independent demands case, while the dependent demands case requires an assumption. We find explicit characterizations for the optimal protection level. Characterizations for the variance function, however, require certain assumptions. By assuming the quasi-concavity of the variance function, we find the non-dominated region as the region between the risk-neutral and minimum-variance optimal order quantities. To be able to characterize the relationship between the risk aversion level and the optimal order quantity, we further assume that the mean function is concave and the variance function is convex on the non-dominated region. It is observed that the optimal protection level increases in the risk-aversion parameter if the risk-neutral protection level is less than or equal to the minimum variance protection level. Similarly, the optimal protection level decreases in the risk-aversion parameter if the risk-neutral protection level is greater than the minimum variance protection level.

In the second part, we investigate the case where the demand variables and financial market are perfectly correlated. We describe the RM cash flow as continuous and piecewise linear functions of a financial variable in the market. Furthermore, we show that we can replicate these functions by using the financial instruments in the market. We find the optimal portfolio for one cash flow function. We see that the RM cash flow can be replicated using only the future and call options. By investing in them, we can hedge the risk perfectly. Afterwards, we extend our results by providing expressions for the n-class RM problem. Lastly, we present an algorithm for the n-class case.

In the third part, we consider the case that that the randomness in demand is partially correlated with the financial markets. The decision maker again invests in a portfolio of various financial instruments as in the second part. In this case, the perfect hedge is not possible. But variance reduction in the cash flow can be achieved. The investor decides on the optimal protection level and the financial portfolio to maximize the MV objective function. The optimal protection level and the optimal hedging portfolio are determined simultaneously. We obtain results both for independent and dependent demands cases, and both for portfolios with a single asset and the portfolios with multiple assets. The mean function appears to be same as in the first part, only the variance function changes. Under the same assumptions made in the first part, we obtain similar results for the optimal order quantity and the risk aversion level. Lastly, we quantify the amount of reduction in variance.

In the numerical illustrations part, the mean and variance functions are investigated for an illustrative example. Using Monte Carlo simulation, the assumptions made in the previous chapters are justified and the theorems are illustrated. The effects of the risk aversion parameter on the optimal protection level is observed. Moreover, the effects of the MV model and the hedging, on the variance of the cash flow are illustrated. We conclude that hedging operations reduce the variance in great amounts in addition to increasing the value of the MV objective function.

Future research on this topic may aim to clear the work off the assumptions to see the case if they do not hold. Moreover it would be interesting to study the multi-period, continuous time models or n-class models with the MV approach.

BIBLIOGRAPHY

Anvari, M. (1987), "Optimality criteria and risk in inventory models: The case of the newsboy problem." *Journal of the Operational Research Society*, 38, 625–632.

Bailey, E.E., D.R. Graham, and D.P. Kaplan (1985), *Deregulating the airlines*, volume 10. The MIT Press.

Barz, C. (2007), *Risk-averse capacity control in revenue management*, volume 597. Springer.

Barz, C. and K-H. Waldmann (2007), "Risk-sensitive capacity control in revenue management." *Mathematical Methods of Operations Research*, 65, 565–579.

Belobaba, P. (1987a), "Air travel demand and airline seat inventory management." Technical report, Cambridge, MA: Flight Transportation Laboratory, Massachusetts Institute of Technology.

Belobaba, P. (1987b), "Airline yield management-an overview of seat inventory control." *Transportation Science*, 21, 63–73.

Belobaba, P. (1989), "Application of a probabilistic decision model to airline seat inventory control." *Operations Research*, 37, 183–197.

Berman, O. and J.A. Schnabel (1986), "Mean-variance analysis and the single-period inventory problem." *International Journal of Systems Science*, 17, 1145–1151.

Bhatia, A.V. and S.C. Parekh (1973), "Optimal allocation of seats by fare." *Presentation by Trans World Airlines to AGIFORS Reservations Study Group.*

Birbil, Ş.İ., J.B.G. Frenk, J.A.S. Gromicho, and S. Zhang (2009), "The role of robust optimization in single-leg airline revenue management." *Management Science*, 55, 148–163.

Bitran, G. and R. Caldentey (2003), "An overview of pricing models for revenue management." *Manufacturing & Service Operations Management*, 5, 203–229.

Brumelle, S.L. and J.I. McGill (1993), "Airline seat allocation with multiple nested fare classes." *Operations Research*, 41, 127–137.

Brumelle, S.L., J.I. McGill, T.H. Oum, K. Sawaki, and M.V. Tretheway (1990), "Allocation of airline seats between stochastically dependent demands." *Transportation Science*, 24, 183–192.

Buzacott, J., H. Yan, and H. Zhang (2011), "Risk analysis of commitment–option contracts with forecast updates." *IIE Transactions*, 43, 415–431.

Caldentey, R. and M.B. Haugh (2006), "Optimal control and hedging of operations in the presence of financial markets." *Mathematics of Operations Research*, 31, 285–304.

Chen, F. and A. Federgruen (2000), "Mean-variance analysis of basic inventory models." Technical report, Columbia University.

Choi, T.M., D. Li, and H. Yan (2008), "Mean-variance analysis for the newsvendor problem." *IEEE Transactions on Systems, Man, and Cybernetics Part A: Systems and Humans*, 38, 1169–1180.

Chu, L-K., J. Ni, Y. Shi, and Y. Xu (2009), "Inventory risk mitigation by financial hedging." *Proceedings of the International Conference on Systems Engineering and Engineering Management (ICSEEM'09)*, 1260–1263.

Chung, K. (1990), "Risk in inventory models: The case of the newsboy problem, optimality conditions." *The Journal of the Operational Research Society*, 41, 173–176.

Cross, R.G. (1997), "Revenue management hard-core tactics for market domination."

Curry, R.E. (1990), "Optimal airline seat allocation with fare classes nested by origins and destinations." *Transportation Science*, 24, 193–204.

Ding, Q., L. Dong, and P. Kouvelis (2007), "On the integration of production and financial hedging decisions in global markets." *Operations Research*, 55, 470–489.

Dunleavy, H.N. (1995), "Airline passenger overbooking." *The Handbook of Airline Economics*, 469–482.

Feng, Y. and B. Xiao (1999), "Maximizing revenues of perishable assets with a risk factor." *Operations Research*, 47, 337–341.

Feng, Y. and B. Xiao (2008), "a risk-sensitive model for managing perishable products." *Operations Research*, 56, 1305–1311.

Gaur, V. and S. Seshadri (2005), "Hedging inventory risk through market instruments." *Manufacturing & Service Operations Management*, 7, 103–120.

Grootveld, H. and W. Hallerbach (1999), "Variance vs downside risk: is there really that much difference?" *European Journal of Operational Research*, 114, 304–319.

Huang, K. and K. Chang (2009), "A simulation analysis for the airline seat control considering expected revenue and risk." *Journal of Revenue and Pricing Management*.

Jenkins, D. and C.P. Ray (1995), *Handbook of airline economics*. Aviation Week Group.

Koenig, M. and J. Meissner (2009), "Risk management policies for dynamic capacity control."

Koenig, M. and J. Meissner (2010), "List pricing versus dynamic pricing: Impact on the revenue risk." *European Journal of Operational Research*, 204, 505–512.

Kole, S.R. and K.M. Lehn (1999), "Deregulation and the adaptation of governance structure: the case of the us airline industry." *Journal of Financial Economics*, 52, 79–117.

Kroll, Y., H. Levy, and H.M. Markowitz (1984), "Mean-variance versus direct utility maximization." *The Journal of Finance*, 39, 47–61.

Lai, K. and W. Ng (2005), "A stochastic approach to hotel revenue optimization." *Computers & Operations Research*, 32, 1059–1072.

Lancaster, J. (2003), "The financial risk of airline revenue management." *Journal of Revenue and Pricing Management*, 2, 158–165.

Lau, H.S. (1980), "The newsboy problem under alternative optimization objectives." *Journal of the Operational Research Society*, 525–535.

Levin, Y., J. McGill, and M. Nediak (2008), “Risk in revenue management and dynamic pricing.” *Operations Research*, 56, 326–343.

Levy, H. and H. M. Markowitz (1979), “Approximating expected utility by a function of mean and variance.” *American Economic Review*, 69, 308–317.

Li, L. (2010), *On the Use of Some Misspecified Models of Customer Choice in Revenue Management*. Ph.D. thesis, University of Minnesota.

Lim, A.E.B. and J.G. Shanthikumar (2007), “Relative entropy, exponential utility, and robust dynamic pricing.” *Operations Research*, 55, 198–214.

Lim, A.E.B., J.G. Shanthikumar, and T. Watewai (2008), “Robust multi-product pricing.” *Working paper*.

Littlewood, K. (1972), “Forecasting and control of passengers.” *In Proceedings of the 12th Annual AGIFORS Symposium*, 95–128.

Markowitz, H. (1952), “Portfolio selection.” *Journal of Finance*, 7, 77–91.

Markowitz, H. (1956), “The optimization of a quadratic function subject to linear constraints.” *Naval Research Logistics Quarterly*, 3, 111–133.

Martínez-de-Albéniz, V. and D. Simchi-Levi (2006), “Mean-variance trade-offs in supply contracts.” *Naval Research Logistics*, 53, 603–616.

Mayer, M. (1976), “Seat allocation, or a simple model of seat allocation via sophisticated ones.” In *AGIFORS Symposium Proceedings*, volume 16.

McGill, J. and G. Van Ryzin (1999), “Revenue management: Research overview and prospects.” *Transportation science*, 33, 233–256.

McGill, J.I. (1989), “Optimization and estimation problems in airline yield management.”

Merton, R.C. (1972), “An analytic derivation of the efficient portfolio frontier.” *Journal of Financial and Quantitative Analysis*, 7, 1851–1872.

Merton, R.C. (1995), “A functional perspective of financial intermediation.” *Financial Management*, 23–41.

Mitra, D. and Q. Wang (2005), "Stochastic traffic engineering for demand uncertainty and risk-aware network revenue management." *IEEE/ACM Transactions on Networking (TON)*, 13, 221–233.

Nawrocki, D.N. (1999), "A brief history of downside risk measures." *The Journal of Investing*, 8, 9–25.

Okyay, H.K., F. Karaesmen, and S. Özekici (2011), "Hedging demand and supply risks in the newsvendor model." Technical report, Koç University, Department of Industrial Engineering, Istanbul, Turkey.

Okyay, H.K., F. Karaesmen, and S. Özekici (2013), "Newsvendor models with dependent random supply and demand." *To appear in Optimization Letters.*

Perakis, G. and G. Roels (2010), "Robust controls for network revenue management." *Manufacturing & Service Operations Management*, 12, 56–76.

Pfeifer, P.E. (1989), "The airline discount fare allocation problem." *Decision Sciences*, 20, 149–157.

Protter, P.E. (2005), *Stochastic Integration and Differential Equations.* Springer-Verlag, Heidelberg.

Richter, H. (1982), "The differential revenue method to determine optimal seat allotments by fare type." In *AGIFORS Proceedings.*

Robinson, L.W. (1995), "Optimal and approximate control policies for airline booking with sequential nonmonotonic fare classes." *Operations Research*, 43, 252–263.

Rothstein, M. (1971), "An airline overbooking model." *Transportation Science*, 5, 180–192.

Rothstein, M. (1974), "Hotel overbooking as a markovian sequential decision process." *Decision Sciences*, 5, 389–404.

Ryzin, G. Van and J. McGill (2000), "Revenue management without forecasting or optimization: An adaptive algorithm for determining airline seat protection levels." *Management Science*, 46, 760–775.

Schweitzer, M.E. and G.P. Cachon (2000), "Decision bias in the newsvendor problem with a known demand distribution: Experimental evidence." *Management Science*, 46, 404–420.

Seshadri, S. and M. Subrahmanyam (2005), "Introduction to the special issue on risk management in operations." *Production and Operations Management*, 14, 1–4.

Smith, B.C., J.F. Leimkuhler, and R.M. Darrow (1992), "Yield management at american airlines." *Interfaces*, 22, 8–31.

Steinbach, M.C. (2001), "Markowitz revisited: Mean-variance models in financial portfolio analysis." *Society for Industrial and Applied Mathematics Review*, 43, 31–85.

Talluri, K.T. and G.J. Van Ryzin (2005), *The theory and practice of revenue management*, volume 68. Springerverlag Us.

Tekin, M. and S. Özekici (2012), *Mean-Variance Approach to the Newsvendor Problem with Random Supply and Financial Hedging*. Master's thesis.

Thiele, A. (2006), "Single-product pricing via robust optimization." *submitted to Operations Research*.

Titze, B. and R. Griesshaber (1983), "Realistic passenger booking behaviour and the simple low-fare high-fare seat allotment model." In *AGIFORS Proceedings*.

Van Mieghem, J.A. (2003), "Capacity management, investment, and hedging: Review and recent developments." *Manufacturing and Service Operations Management*, 5, 269–302.

Vinod, B. (1995), "Origin-and-destination yield management." *The Handbook of Airline Economics*, 459–468.

Weatherford, L.R. (2004), "Additional paper: Emsr versus emsu: Revenue or utility?" *Journal of Revenue and Pricing Management*, 3, 277–284.

Wollmer, R.D. (1992), "An airline seat management model for a single leg route when lower fare classes book first." *Operations Research*, 40, 26–37.

Wu, J., J. Li, S. Wang, and T.C.E. Cheng (2009), "Mean-variance analysis of the newsvendor model with stockout cost." *Omega*, 724–730.

VITA

Gülce SARI was born in Eskisehir, Turkey on July 25, 1988. She received her B.Sc. degree in Industrial Engineering from Middle East Technical University, Ankara, in 2011. During her undergraduate study, she studied Business and Management as an exchange student in University of Groningen, Netherlands, in 2010. From August 2011 to August 2013, she worked as a teaching and research assistant in Koc University, Turkey, receiving scholarship from The Scientific and Technological Research Council of Turkey (TUBITAK). She received her M.Sc. degree in Industrial Engineering from Koc University, in July 2013.