Initial-Boundary Value Problem for Burgers' Original Equations

by

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
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To my family

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ABSTRACT

The thesis is devoted to the initial-boundary value problem for the Burgers' original model of turbulence. The problem of existence and uniqueness of initial-boundary value problem for Burgers' original equations modeling turbulence in fluid flow is studied. Uniform estimate of solutions and stability of a stationary state is established under some restrictions on parameters of the system.

ÖZETÇE

Bu tez Burgers'in orijinal türbülans modeli için başlangıç-sınır değer problemi ile ilgilidir. Türbülanslı akışı modelleyen Burgers'in orijinal denklemleri için başlangıç-sınır değer probleminin çözümünün varlığı ve tekliği incelenmiştir. Düzgün kestirimler elde edilmiş ve denge noktasının üstel kararlılığı ispat edilmiştir.

TABLE OF CONTENTS

Chapter 1

INTRODUCTION

In his paper [1] Burgers simplified the Navier-Stokes equation

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) + u(x,t)\cdot\nabla u(x,t) = -\nabla p(x,t) + \nu\Delta u(x,t), \\ \nabla\cdot u(x,t) = 0, \end{cases}$$

and got the following so called Burgers equation:

$$\frac{\partial}{\partial t}u(x,t) + u(x,t)\frac{\partial}{\partial x}u(x,t) = \nu\frac{\partial^2}{\partial x^2}u(x,t) + F(x,t) \tag{1.1}$$

Generally, this equation is considered without external force $F(x,t)$. This equation is non-linear, however it was shown by Hopf [7] and Cole [3] that the solutions of the equation does not depend on the initial conditions. Also the equations can be simplified by Cole–Hopf transformation:

$$u = -2\nu\frac{u_x}{u}$$

into linear heat equation. In the same paper [1], Burgers introduced and studied another model describing dynamics of fluid flow.This model consists a coupled system of nonlinear ordinary differential equations of the form

$$\begin{aligned} \frac{dU}{dt} &= P - \nu U - v^2, \\ \frac{dv}{dt} &= Uv - \nu v, \end{aligned}$$

where U represents the velocity of the mean motion, and v turbulent motion. P, v are constants representing the external force and a kinematic viscosity, respectively.

Later on in his famous paper [2] Burgers proposed a more sophisticated system that consists of an ordinary differential equation and a nonlinear second order parabolic equation:

$$\begin{cases} b\frac{dU}{dt} = P - \frac{\nu}{b}U - \frac{1}{b}\int_0^b v^2(t,y)dy, \\ \frac{\partial v}{\partial t} = \frac{1}{b}Uv + \nu\frac{\partial^2 v}{\partial y^2} - 2v\frac{\partial v}{\partial y}. \end{cases} \tag{1.2}$$

Here, $U(t), v(t,y)$ are the unknown functions:
$u(t)$ is the analogue of the primary or the mean motion in the case of a liquid flowing through a channel, $v(t,y)$ represents the secondary motion. The case $v \neq 0$ describes turbulence in the system. The variable y that occurs in v plays the part of the coordinate in the direction of the cross dimension of the channel. P, ν are given constants. P represents the exterior force acting upon the primary motion, and ν stands for frictional effects. Burgers considered the case when the domain of y is an interval $(0,b)$, and v vanishes at both ends of the interval.
There are many studies of the Cauchy problem and the initial boundary value problem for the viscous Burgers' equation

$$\frac{\partial}{\partial t}u(x,t) + u(x,t)\frac{\partial}{\partial x}u(x,t) - \nu\frac{\partial^2}{\partial x^2}u(x,t) = 0, \quad x \in \mathbb{R}, \ t > 0. \tag{1.3}$$

This equation is a special case of the Burgers' original equation. For the results on the local and global existence of solutions to the initial boundary value problems of this equation, we refer to the books [9], [8]. There are also some publications on generalized Burgers' equations:

$$\begin{cases} \frac{\partial}{\partial t}v(x,t) = \frac{\mu}{\rho(x,t)}\frac{\partial^2}{\partial x^2}v(x,t) - v(x,t)\frac{\partial}{\partial x}v(x,t), \\ \frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}(\rho(x,t)v(x,t)), \quad (\mu \text{ is a positive constant}), \end{cases} \tag{1.4}$$

see e.g [10].

Further study of initial boundary value problem for the Burger's original model of turbulence is done in the paper [5] and the book [6] of Eden. In the paper, the author found an estimate for the dimension of the attractor of the problem (1.2) which is of the same order as the square root of a Reynolds number. Also, in the book the author proved that the initial boundary value problem for the Burger's original model generates a continuous semi-group in a proper phase space $\mathbb{R} \times L^2(0,1)$. Moreover it is shown that the semi-group has a finite dimensional exponential attractor.

We are going to study the system of equations:

$$\frac{dU(t)}{dt} = P - \nu U(t) - \int_\Omega v^2(t,x)dx, \quad x \in (0,\pi), \quad t > 0, \tag{1.5}$$

$$\frac{\partial v(t,x)}{\partial t} = U(t)v(t,x) + \nu \frac{\partial^2 v(t,x)}{\partial x^2} - \frac{\partial}{\partial x}(v^2(t,x)), \quad x \in (0,\pi), \quad t > 0, \tag{1.6}$$

under the following initial and boundary conditions

$$U(0) = U_0,$$
$$v(0,x) = \phi(x) \text{ for } x \in (0,\pi),$$
$$v(t,0) = v(t,\pi) = 0 \text{ for } t \geq 0,$$

where $U = U(t) : [0,\infty) \to \mathbb{R}, v = v(t,x) : \overline{Q} \to \mathbb{R}$ are unknown functions. Here and in what follows we use the notations

$$Q := \Omega \times (0,\infty), \ \Omega := (0,\pi).$$

This system is called in the literature Burger's original model of turbulence. Following [4] we prove theorems on global unique solvability of initial boundary value problem for the system (1.5),(1.6), obtain uniform estimates for solutions of the problem. Finally we prove that under some restrictions on parameters the equilibrium solution of the problem is exponentially stable.

Chapter 2

EXISTENCE AND UNIQUENESS

In this chapter we consider the following system of equations

$$\frac{dU(t)}{dt} = P - \nu U(t) - \int_\Omega v^2(t,x)dx \quad x \in (0,\pi), t > 0, \tag{2.1.1}$$

$$\frac{\partial v(t,x)}{\partial t} = U(t)v(t,x) + \nu \frac{\partial^2 v(t,x)}{\partial x^2} - \frac{\partial}{\partial x}(v^2(t,x)), \quad x \in (0,\pi), t > 0, \tag{2.1.2}$$

under the following initial and boundary conditions

$$\begin{aligned} U(0) &= U_0, \\ v(0,x) &= \phi(x) \text{ for } x \in (0,\pi), \\ v(t,0) &= v(t,\pi) = 0 \text{ for } t \geq 0, \end{aligned} \tag{2.2}$$

where $U = U(t) : [0,\infty) \to \mathbb{R}, v = v(t,x) : \overline{Q} \to \mathbb{R}$ are unknown functions. Here and in what follows we use the notations

$$Q := \Omega \times (0,\infty), \ \Omega := (0,\pi).$$

Definition 2.1. A pair of functions (U,v) is said to be a weak solution of the problem (2.1.1), (2.1.2) if U is absolutely continuous function on each interval $[0,T], T > 0$, and

$$U(t) = U_0 + \int_0^t (P - \nu U(\tau) - \int_\Omega v^2(\tau,x)dx)d\tau,$$

$$v \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega)) \quad \forall T > 0, \text{ and,} \tag{2.3}$$

$$< v', w > + \nu a(v,w) + 2b(v,v,w) = c(U,v,w) \tag{2.4}$$

for any function $w \in H_0^1(\Omega)$, where $v' = \frac{\partial v}{\partial t}$, and

$$v(0,x) = \phi(x) \equiv v_0(x) \in L^2(\Omega), \tag{2.5}$$

where a, b, c defined as follows:

$$\begin{aligned} a(f,h) &:= \int_\Omega \frac{\partial f}{\partial x}\frac{\partial h}{\partial x} dx = \langle Af, h\rangle, \\ b(f,h,l) &:= \int_\Omega f\frac{\partial h}{\partial x} l dx = \langle g(f,h), l\rangle, \\ c(f,h,l) &:= \int_\Omega fhl dx = \langle C(f,h), l\rangle. \end{aligned}$$

Here and what follows $\langle\ ,\ \rangle$ stands for $\langle\ ,\ \rangle_{L^2(\Omega)}$.

Proposition 2.2. For any functions $f \in H_0^1(\Omega), h, l \in L^2(\Omega)$, the following inequality holds true:

$$\Big|\int_\Omega fhldx\Big| \le \|f\|_{H_0^1}^{1/2}\|f\|_{L^2}^{1/2}\|h\|_{L^2}\|l\|_{L^2}.$$

Proof The following estimate is the consequence of Cauchy-Schwarz inequality and the fact that $H_0^1(\Omega) \subset L^\infty(\Omega)$:

$$\Big|\int_\Omega fhldx\Big| \le \|f\|_{L^\infty}\Big|\int_\Omega hldx\Big| \le \|f\|_{L^\infty}\|h\|_{L^2}\|l\|_{L^2}.$$

For any $f \in C_c^1(\Omega)$, we have:

$$\begin{aligned} |f(x)|^2 &= \int_0^x D_s f(s).f(s)ds + \int_\pi^x D_s f(s).f(s)ds \\ &\le \Big|\int_0^x D_s f(s).f(s)ds\Big| + \Big|\int_\pi^x D_s f(s).f(s)ds\Big| \\ &\le \int_\Omega |D_s.f(s)|ds \\ &\le \|D_x f\|_{L^2}\|f\|_{L^2} \\ &\le \|f\|_{H_0^1}\|f\|_{L^2}. \end{aligned}$$

Since it holds for all $f \in C_c^1(\Omega)$, it also holds for all $f \in H_0^1(\Omega)$. Moreover, for the functions $f, h \in H_0^1(\Omega)$, and $l \in L^2(\Omega)$ the following should be noted:

Remark 2.3. $|b(f,h,l)| = \Big|\int_\Omega fD_x hldx\Big| \le \|f\|_{H_0^1}^{1/2}\|f\|_{L^2}^{1/2}\|D_x h\|_{L^2}\|l\|_{L^2}.$

Remark 2.4. For any function v satisfying definition 2.1 we have

$$g(v,v) \in L^2(0,T;H^{-1}(\Omega)).$$

Indeed, if $w \in H_0^1(\Omega)$, we get:

$$\begin{aligned}|\langle g(v,v),w\rangle| = |b(v,v,w)| = \frac{1}{2}|b(v,w,v)| &\leq \frac{1}{2}\|v\|_{H_0^1}^{1/2}\|v\|_{L^2}^{1/2}\|w\|_{H_0^1}\|v\|_{L^2}\\ &\leq \frac{1}{2}\|v\|_{H_0^1}\|w\|_{H_0^1}\|v\|_{L^2}^{3/2}.\end{aligned}$$

Lemma 2.5. For any weak solution (U,v) of (2.1.1), (2.1.2), $v' \in L^2(0,T;H^{-1}(\Omega))$.

Proof Let $w \in H_0^1(\Omega)$, and consider $\langle v',w\rangle$. From condition (2.4), we have:

$$\begin{aligned}\Big|\int_\Omega v'(x,t)w(x)dx\Big| = \Big| &-\nu\int_\Omega \frac{\partial v(x,t)}{\partial x}\frac{\partial w(x)}{\partial x}dx\\ &-2\int_\Omega v(x,t)\frac{\partial v(x,t)}{\partial x}w(x)dx + \int_\Omega U(t)v(x,t)w(x)dx\Big|\\ &\leq \nu\|v\|_{H_0^1}\|w\|_{H_0^1} + \|v\|_{H_0^1}\|w\|_{H_0^1}\|v\|_{L^2}^{3/2} + \|u\|_{C^0}\|v\|_{L^2}\|w\|_{L^2}\\ &\leq \nu\|v\|_{H_0^1}\|w\|_{H_0^1} + \|v\|_{H_0^1}\|w\|_{H_0^1}\|v\|_{L^2}^{3/2} + \|u\|_{C^0}\|v\|_{H_0^1}\|w\|_{H_0^1}.\end{aligned}$$

Lemma 2.6. Any weak solution v is almost everywhere continuous from $[0,T]$ to $L^2(\Omega)$.

Proof follows from theorem 5.24.

Now, we start to prove the existence of the weak solution of the problem (2.1.1), (2.1.2) using the Galerkin method.

Theorem 2.7. *There exists a weak solution of the problem* (2.1.1), (2.1.2) *in the sense of definition* 2.1.

Proof We are looking for the function $v(x,t)$ as the limit of the approximate solutions of the form

$$v_m(t,x) = \sum_{k=1}^{m} c_{mk}(t)w_k(x), \tag{2.6}$$

where

$$w_1(x), w_2(x), ..., w_j(x), ...$$

are eigenfunctions of the Sturm-Liouville problem

$$-w''(x) = \lambda w(x) \quad x \in \Omega,$$
$$w(0) = w(\pi) = 0.$$

It is clear that they satisfy:

$$\langle w_j, \eta \rangle_{H_0^1} := a(w_j, \eta) = \lambda_j \langle w_j, \eta \rangle, \quad j = 1, 2, ...$$

for any $\eta \in H_0^1(\Omega)$. The functions $c_{mk}(t)$ and $U_m(t)$ satisfy the system of $(m+1)$ ordinary differential equations

$$\frac{d}{dt} U_m(t) = P - \nu U_m(t) - \|v_m(t)\|_{L^2}^2, \tag{2.7.1}$$

$$\langle v_m', w_l \rangle + \nu a(v_m, w_l) + 2b(v_m, v_m, w_l) = U_m \langle v_m, w_l \rangle, \; l = 1, ..., m \tag{2.7.2}$$

and the conditions

$$U_m(0) = U_{0m}, \; U_{0m} \to U_0, \; v_m(0) = v_{0m}, \; v_{0m} \to v_0 \text{ in } L^2(\Omega), \tag{2.7.3}$$

where the convergence of the sequence $\{v_{0m}\}$ follows from (2.6).
By classical existence theorem 5.25, we ensure that U_m and $\{c_{ml}\}_1^m$ exist for any m. c_{mk} exists in the intervals $[0, t_{mk})$ and U_m exists in $[0, \min_k t_{mk})$.
Now, we will show that U_m and $\{c_{ml}\}_1^m$ exist on the whole interval $[0, T]$, and

$$U_m \text{ is bounded in } L^\infty(0, T),$$

$$v_m \text{ is bounded in } L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)).$$

We multiply (2.7.1) by U_m, and integrate over $(0, t)$, where $t \in [0, T]$

$$\int_0^t U_m(\tau) \frac{d}{d\tau} U_m(\tau) d\tau = \int_0^t P U_m(\tau) d\tau - \nu \int_0^t U_m^2(\tau) d\tau - \int_0^t U_m(\tau) \|v_m(\tau)\|_{L^2}^2 d\tau$$

$$\frac{1}{2} \int_0^t \frac{d}{d\tau} U_m^2(\tau) d\tau + \nu \int_0^t U_m^2(\tau) d\tau = P \int_0^t U_m(\tau) d\tau - \int_0^t U_m(\tau) \|v_m(\tau)\|_{L^2}^2 d\tau \tag{2.8}$$

Multiply(2.7.2) by c_{ml}, integrate over $(0,t)$, and sum over l:

$$\begin{aligned}
\sum_{l=1}^{m}\int_0^t c_{ml}U_m\langle v_m,w_l\rangle d\tau &= \sum_{l=1}^{m}\int_0^t c_{ml}\langle v_m',w_l\rangle d\tau + \nu\sum_{l=1}^{m}\int_0^t c_{ml}a(v_m,w_l)d\tau \\
&+ 2\sum_{l=1}^{m}\int_0^t c_{ml}b(v_m,v_m,w_l)d\tau \\
\int_0^t U_m\langle v_m,\sum_{l=1}^{m}c_{ml}w_l\rangle d\tau &= \int_0^t\langle v_m',\sum_{l=1}^{m}c_{ml}w_l\rangle d\tau + \nu\int_0^t a(v_m,\sum_{l=1}^{m}c_{ml}w_l)d\tau \\
&+ 2\int_0^t\langle g(v_m,v_m),\sum_{l=1}^{m}c_{ml}w_l)\rangle d\tau \\
\int_0^t U_m\langle v_m,v_m\rangle d\tau &= \int_0^t\langle v_m',v_m\rangle d\tau + \nu\int_0^t a(v_m,v_m)d\tau \\
&+ 2\int_0^t\langle g(v_m,v_m),v_m\rangle d\tau \\
\int_0^t U_m(\tau)\|v_m(\tau)\|_{L^2}^2 d\tau &= \frac{1}{2}\int_0^t\frac{d}{d\tau}\|v_m(\tau)\|_{L^2}^2 d\tau + \nu\int_0^t\|v_m(\tau)\|_{H_0^1}^2 d\tau \qquad (2.9)
\end{aligned}$$

Adding (2.8) to (2.9) we get:

$$\frac{1}{2}\int_0^t\frac{d}{d\tau}U_m^2(\tau)d\tau+\nu\int_0^t U_m^2(\tau)d\tau+\frac{1}{2}\int_0^t\frac{d}{d\tau}\|v_m(\tau)\|_{L^2}^2 d\tau+\nu\int_0^t\|v_m(\tau)\|_{H_0^1}^2 d\tau = P\int_0^t U_m(\tau)d\tau$$

$$\frac{1}{2}U_m^2(t)+\frac{1}{2}\|v_m(t)\|_{L^2}^2+\nu\int_0^t[U_m^2(\tau)+\|v_m(\tau)\|_{H_0^1}^2] = \frac{1}{2}U_m^2(0)+\frac{1}{2}\|v_{0m}\|_{L^2}^2+P\int_0^t U_m(\tau)d\tau$$

Using Young's inequality:

$$\frac{1}{2}U_m^2(t)+\frac{1}{2}\|v_m(t)\|_{L^2}^2+\nu\int_0^t[U_m^2(\tau)+\|v_m(\tau)\|_{H_0^1}^2]d\tau \le \frac{1}{2}U_m^2(0)+\frac{1}{2}\|v_{0m}\|_{L^2}^2+\frac{\epsilon T}{2}P^2+\frac{1}{2\epsilon}\int_0^t U_m^2(\tau)d\tau$$

$$\frac{1}{2}U_m^2(t)+\frac{1}{2}\|v_m(t)\|_{L^2}^2+\nu\int_0^t\|v_m(\tau)\|_{H_0^1}^2 d\tau+(\nu-\frac{1}{2\epsilon})\int_0^t U_m^2(\tau)d\tau \le \frac{1}{2}U_m^2(0)+\frac{1}{2}\|v_{0m}\|_{L^2}^2+\frac{\epsilon T}{2}P^2$$

If we choose ϵ such that $(\nu-\frac{1}{2\epsilon})>0$, we get:

$$\begin{aligned}
U_m^2(t) &\le U_m^2(0)+\|v_{0m}\|_{L^2}^2+\epsilon TP^2, \\
\nu\int_0^t\|v_m(\tau)\|_{H_0^1}^2 d\tau &\le U_m^2(0)+\|v_{0m}\|_{L^2}^2+\epsilon TP^2, \\
\|v_m(t)\|_{L^2}^2 &\le U_m^2(0)+\|v_{0m}\|_{L^2}^2+\epsilon TP^2.
\end{aligned}$$

Hence $\{U_m\}$ is bounded in $L^\infty(0,T)$, and $\{v_m\}$ is bounded in $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. By theorem 5.26, these estimates guarantee the existence of $\{U_m\}$ and $\{v_m\}$ on the interval $[0,T]$ for any $T > 0$. Since the sequence $\{v_m\}$ is bounded in $L^2(0,T;H_0^1(\Omega))$, we can extract a subsequence by remark 5.7, still denoted $\{v_m\}$, that weakly converges to v in $L^2(0,T;H_0^1(\Omega))$. Now, let $\psi \in C_c^\infty(0,T;H_0^1(\Omega))$ which is a dense subset of $L^2(0,T;H_0^1(\Omega))$.Then,

$$\begin{aligned}
\int_0^T \int_\Omega \frac{dv_m}{dt}(x,t)\psi(x,t)dxdt &= \int_\Omega \int_0^T \frac{dv_m}{dt}(x,t)\psi(x,t)dtdx \\
= -\int_\Omega \int_0^T v_m(x,t)\frac{d\psi}{dt}(x,t)dtdx &\to -\int_\Omega \int_0^T v(x,t)\frac{d\psi}{dt}(x,t)\ dtdx \text{ and,} \\
-\int_\Omega \int_0^T v(x,t)\frac{d\psi}{dt}(x,t)\ dtdx &= \int_\Omega \int_0^T \frac{dv}{dt}(x,t)\psi(x,t)dtdx \text{ so,} \\
\int_0^T \int_\Omega \frac{dv_m}{dt}(x,t)\psi(x,t)dxdt &\to \int_0^T \int_\Omega v(x,t)\frac{d\psi}{dt}(x,t)\ dtdx,
\end{aligned}$$

i.e

v'_m converges weakly to v' in $L^2(0,T;H^{-1}(\Omega))$, hence $\{v'_m\}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. Now, choosing appropriate subsequences at each step, by remark 5.7 and theorem 5.22 we get:

$$v_m \to v_2 \text{ in } L^2(0,T;L^2(\Omega)),\ \text{and, } v_m \to v_3 \text{ weak* in } L^\infty(0,T;L^2(\Omega)).$$

Notice that $L^\infty(0,T;L^2(\Omega)) \subset L^2(0,T;L^2(\Omega))$, so one can easilly show that $v_2 = v_3$. Same argument works for v, and v_3. So, we have:

$$v_m \to v \text{ weak in } L^2(0,T;H_0^1(\Omega)), \tag{2.10}$$

$$v_m \to v \text{ weak* in } L^\infty(0,T;L^2(\Omega)), \tag{2.11}$$

$$v_m \to v \text{ in } L^2(0,T;L^2(\Omega)), \tag{2.12}$$

$$v'_m \to v' \text{ weak in } L^2(0,T;H^{-1}(\Omega)). \tag{2.13}$$

After extracting an appropriate subsequence, we will show that v satisfies

$$\langle v', w\rangle + \nu a(v, w) + 2b(v, v, w) = c(U, v, w)$$

for any function $w \in H_0^1(\Omega)$. We know that

$$\langle v_m', w_j\rangle + \nu a(v_m, w_j) + 2b(v_m, v_m, w_j) = U_m\langle v_m, w_j\rangle \text{ for } 1 \le j \le m,$$

and it holds for any $w \in span\{w_j\}_{j=1}^M$ where $M \le m$. So, we have

$$\langle v_m', w\rangle + \nu a(v_m, w) + 2b(v_m, v_m, w) = U_m\langle v_m, w\rangle \text{ for all } w \in span\{w_j\}_{j=1}^M.$$

We multiply the last equation by $\varphi \in C_c^\infty(0, T)$ and integrate over the interval $(0, T)$ with respect to t and get:

$$\int_0^T \langle v_m', w_j\rangle \varphi(t)dt + \nu \int_0^T a(v_m, w_j)\varphi(t)dt + 2\int_0^T b(v_m, v_m, w_j)\varphi(t)dt$$
$$= \int_0^T U_m(t)\langle v_m, w_j\rangle \varphi(t)dt \tag{3.14}$$

Our aim is to pass to the limit as $m \to \infty$. In order to do that we will deal with each term separately. First, notice that

$$w(x)\varphi(t) \in C_c^\infty(0, T; C^\infty(\Omega) \cap H_0^1(\Omega)) \subset L^2(0, T; H_0^1(\Omega)).$$

(i)

$$\lim_{m\to\infty} \int_0^T \int_\Omega v_m'(x, t)w(x)\varphi(t)dxdt = \int_0^T \int_\Omega v'(x, t)w(x)\varphi(t)dxdt$$

(ii)

$$\lim_{m\to\infty} \int_0^T a(v_m, w)\varphi(t)dt = \lim_{m\to\infty} \int_0^T \int_\Omega \frac{\partial}{\partial x} v_m(x, t)w'(x)\varphi(t)dxdt$$
$$= -\lim_{m\to\infty} \int_0^T \int_\Omega v_m(x, t)w''(x)\varphi(t)dxdt = -\int_0^T \int_\Omega v(x, t)w''(x)\varphi(t)dxdt$$
$$= \int_0^T \int_\Omega \frac{\partial}{\partial x} v(x, t)w'(x)\varphi(t)dxdt = \int_0^T a(v, w)\varphi(t)dt$$

(iii) For $\int_0^T b(v_m, v_m, w)\varphi(t)dt$, first we need to show that v_m^2 converges weakly to v^2 in $L^2(0,T;L^2(\Omega))$. As in the proof of proposition 2.2, for $f \in C_c^1[0,\pi]$, we have the inequality: $\max_{x\in[0,\pi]} |f(x)|^2 \le \|f\|_{H_0^1(\Omega)}\|f\|_{L^2(\Omega)}$ that holds for all $f \in H_0^1(\Omega)$.

$$\left| \int_0^T \|v_m^2\|_{L^2}^2 dt \right| = \left| \int_0^T \int_\Omega v_m^4 dxdt \right| \le \int_0^T \int_\Omega \|v_m\|_{H_0^1(\Omega)}^2 \|v_m\|_{L^2(\Omega)}^2 dxdt < \infty,$$

and $v_m \to v$ almost everywhere in $L^2(0,T;L^2(\Omega))$, so $v_m^2 \to v^2$ almost everywhere in $L^2(0,T;L^2(\Omega))$. Therefore we have:

$$\begin{aligned} \lim_{m\to\infty} \int_0^T b(v_m, v_m, w)\varphi(t)dt &= \lim_{m\to\infty} \int_0^T \int_\Omega v_m(x,t)\frac{\partial}{\partial x} v_m(x,t)w(x)\varphi(t)dxdt \\ = -\lim_{m\to\infty} \int_0^T \int_\Omega \frac{1}{2} v_m^2(x,t)w'(x)\varphi(t)dxdt &= -\frac{1}{2}\int_0^T \int_\Omega v^2(x,t)w'(x)\varphi(t)dxdt \\ = \int_0^T \int_\Omega v(x,t)\frac{\partial v}{\partial x}(x,t)w(x)\varphi(t)dxdt &= \int_0^T b(v,v,w)\varphi(t)dt. \end{aligned}$$

(iv) Notice that

$$\int_0^T U_m(t)\langle v_m, w\rangle \varphi(t)dt = \int_0^T \int_\Omega u_m(t)v_m(t)w(x)\varphi(t)dxdt = \langle w\varphi U_m, v_m\rangle_{H^{-1},H_0^1}.$$

We know that v_m converges weakly to v in $L^2(0,T;H_0^1(\Omega))$ and if we can show that $w(x)\varphi(t)U_m(t)$ converges to $w(x)\varphi(t)U(t)$ in $L^2(0,T;H^{-1}(\Omega))$, we are done. First, we need to show that $U_m \to U$ uniformly in $[0,T]$. Recall that U_m satisfies

$$U_m(t) = U_m(0) + \int_0^t [p - \nu U_m(\tau) - \|v_m(\tau)\|_{L^2}^2]d\tau.$$

So,

$$\begin{aligned} \big|U_m(t) - U_n(t)\big| &= \Big|U_m(0) - U_n(0) - \int_0^t \nu(U_m(\tau) - U_n(\tau))d\tau - \int_0^t (\|v_m(\tau)\|_{L^2}^2 - \|v_n(\tau)\|_{L^2}^2)d\tau\Big| \\ &\le \big(|U_m(0) - U_n(0)| + \epsilon_{m,n}\big) + \int_0^t \nu\big|U_m(\tau) - U_n(\tau)\big|d\tau, \\ &\le \big(|U_m(0) - U_n(0)| + \epsilon_{m,n}\big)e^{\nu T} \to 0 \text{ as } m,n \to \infty, \end{aligned}$$

since $\epsilon_{mn} = |\|v_m\|_{L^2(Q)}^2 - \|v_n\|_{L^2(Q)}^2| \to 0$, and $U_m(0)$ is convergent. So, U_m is uniformly convergent to U in $[0,T]$. Now, we will show that $w(x)\varphi(t)U_m(t)$ converges to $w(x)\varphi(t)U(t)$

in $L^2(0,T;H^{-1}(\Omega))$.

$$\begin{aligned}
&\|w(x)\varphi(t)U_m(t) - w(x)\varphi(t)U(t)\|^2_{L^2(0,T;H^{-1}(\Omega))} \\
&= \int_0^T \|\|w(x)\varphi(t)U_m(t) - w(x)\varphi(t)U(t)\|^2_{H^{-1}} dt \\
&\leq \int_0^T \left|U_m(t) - U(t)\right|^2 \|w(x)\varphi(t)\|^2_{H^{-1}} dt \\
&\leq \|U_m - U\|^2_{L^\infty} \int_0^T \|w\varphi\|^2_{H^{-1}} dt.
\end{aligned}$$

The term on the right hand side of the last inequality tends to zero as $m \to \infty$ since $\|w\varphi\|_{H^{-1}}$ is bounded and U_m uniformly convergent to U. So,

$$\begin{aligned}
\lim_{m\to\infty} \int_0^T U_m(t)\langle v_m, w\rangle \varphi(t)dt &= \lim_{m\to\infty} \langle w\varphi U_m, v_m\rangle_{H^{-1},H_0^1} \\
&= \langle w\varphi U, w\rangle_{H^{-1},H_0^1} \\
&= \int_0^T \int_\Omega w(x)\varphi(t)U(t)v(x,t)dxdt \\
&= \int_0^T U(t)\langle v, w\rangle \varphi(t)dt.
\end{aligned}$$

Putting $i - iv$ together and passing to the limit in 3.14, we get:

$$\begin{aligned}
\int_0^T \int_\Omega v'(x,t)w(x)\varphi(t)dxdt &= \nu \int_0^T a(v,w)\varphi(t)dt + \int_0^T 2b(v,v,w)\varphi(t)dt \\
&= \int_0^T U(t)\langle v, w\rangle \varphi(t)dt
\end{aligned}$$

for all $\varphi(t) \in C_0^\infty(0,T)$ and for all $w \in \bigcup_{M\geq 1}\text{span } \{w_k\}_{k=1}^M$. Since this holds for all $\varphi \in C_0^\infty(0,T)$, we get:

$$\langle v'(t), w\rangle + \nu a(v(t), w) + 2b(v(t), v(t), w) = U(t)\langle v(t), w\rangle.$$

for almost every $t \in [0,T]$ and for all $w \in H_0^1(\Omega)$.

Next, we need to show that $v(0,x) = v_0(x)$. Let $\varphi(t) \in C^1([0,T])$ where $\varphi(0) = 1$, and $\varphi(T) = 0$.

$$\begin{aligned}\int_0^T \langle v'_m, w\rangle \varphi(t)dt &= \int_0^T \int_\Omega v'_m(x,t)w(x)\varphi(t)dxdt\\ &= \int_\Omega w(x)\int_0^T v'_m(x,t)\varphi(t)dtdx\\ &= \int_\Omega w(x)\Big(- v_m(x,0) - \int_0^T v_m(x,t)\varphi'(t)dt\Big)dx\\ &= - \int_\Omega w(x)v_m(x,0)dx - \int_\Omega\int_0^T w(x)v_m(x,t)\varphi'(t)dtdx.\end{aligned}$$

Passing to the limit,

$$\begin{aligned}\int_0^T \int_\Omega v'(x,t)w(x)\varphi(t)dxdt &= -\lim_{m\to\infty}\int_\Omega w(x)v_m(x,0)dx\\ &\quad - \int_\Omega\int_0^T w(x)v(x,t)\varphi'(t)dtdx,\\ -\int_\Omega w(x)v(x,0)\varphi dx - \int_\Omega\int_0^T w(x)v(x,t)\varphi'(t)dtdx &= -\lim_{m\to\infty}\int_\Omega w(x)v_m(x,0)dx\\ &\quad - \int_\Omega\int_0^T w(x)v(x,t)\varphi'(t)dtdx,\\ -\int_\Omega w(x)v(x,0)\varphi(t)dx &= -\lim_{m\to\infty}\int_\Omega w(x)v_m(x,0)dx\\ &= -\int_\Omega w(x)v_0(x)dx.\end{aligned}$$

Hence we get: $v(0,x) = v_0(x)$.

Theorem 2.8. *The weak solution of the problem* (2.1.1), (2.1.2) *is unique.*

Proof Let (U_1, v_1), and (U_2, v_2) be two different solutions of (2.1.1), (2.1.2), and let

$$U = U_1 - U_2,\ U(0) = 0,$$
$$v = v_1 - v_2,\ v(0,x) = 0 \text{ in } L^2(\Omega).$$

Then, U and v satisfies

$$U' = -\nu U - \|v_1\|_{L^2}^2 + \|v_2\|_{L^2}^2 \quad \text{(almost everywhere, when } U' \text{ exists)}, \tag{3.15}$$

$$\langle v', w\rangle = U_1\langle v, w\rangle + U\langle v_2, w\rangle - \nu a(v, w) - 2b(v1, v, w) - 2b(v, v_2, w) \quad \text{for all } w \in H_0^1(\Omega). \tag{3.16}$$

We multiply (3.15) by U, and get:

$$U(t)U'(t) = -\nu U^2(t) - U(t)\int_\Omega (v_1^2(x,t) - v_2^2(x,t))dx$$

$$(\frac{1}{2}U^2(t))' = -\nu U^2(t) - U(t)\int_\Omega v(x,t)(v_1(x,t) + v_2(x,t))dx$$

$$(\frac{1}{2}U^2(t))' + \nu U^2(t) \leq \frac{\epsilon_1}{2}U^2(t)\|v_1(t) + v_2(t)\|_{L^2}^2 + \frac{1}{2\epsilon_1}\|v(t)\|_{L^2}^2. \tag{3.17}$$

Now, since v is an element of $H_0^1(\Omega)$ for almost all $t \in [0, T]$, we can put $w = v$ in (3.16). Then, we get:

$$\langle v', v\rangle = U_1\langle v, v\rangle + U\langle v_2, v\rangle - \nu a(v, v) - 2b(v_1, v, v) - 2b(v, v_2, v)$$

$$\langle v', v\rangle = U_1(t)\|v(t)\|_{L^2}^2 + U(t)\langle v_2, v\rangle - \nu\|v(t)\|_{H_0^1}^2 - 2b(v_1, v, v) + 4b(v_2, v, v)$$

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 + \nu\|v(t)\|_{H_0^1}^2 \leq \epsilon_2\|v(t)\|_{L^2}^2 + \frac{\epsilon_3}{2}U^2(t)\|v_2(t)\|_{L^2}^2 + \frac{1}{2\epsilon_3}\|v(t)\|_{L^2}^2$$
$$+ |2b(v_1, v, v)| + |4b(v_2, v, v)|. \tag{3.18}$$

Note that ϵ_1, and ϵ_3 follows from Young's inequality, and ϵ_2 comes from the fact that $U(t)$ is bounded.Also, by propositon 2.2, and Young's inequality, for $i = 1, 2$:

$$\begin{aligned}|b(v_i, v, v)| &= \Big|\int_\Omega v_i(x,t)v'(x,t)v(x,t)dx\Big| \\ &\leq \|v(t)\|_{H_0^1}^{1/2}\|v(t)\|_{H_0^1}\|v_i(t)\|_{L^2}\|v_i(t)\|_{L^2} \\ &\leq \frac{\delta_i}{2}\|v(t)\|_{H_0^1}^2\|v_i(t)\|_{L^2}^2 + \frac{1}{2\delta_i}\|v(t)\|_{H_0^1}\|v_i(t)\|_{L^2} \\ &\leq \frac{\delta_i}{2}\|v(t)\|_{H_0^1}^2\|v_i(t)\|_{L^2}^2 + \frac{\delta_{ii}}{4\epsilon_i}\|v(t)\|_{L^2}^2 + \frac{1}{4\delta_i\delta_{ii}}\|v_i(t)\|_{H_0^1}^2.\end{aligned}$$

We add (3.17) to (3.18), and get:

$$\Big(\frac{1}{2}U^2(t)\Big)' + \Big(\frac{1}{2}\|v(t)\|^2_{L^2}\Big)' \leq U^2(t)\Big(-\nu + \frac{\epsilon_1}{2}\|v_1(t)+v_2(t)\|^2_{L^2} + \frac{\epsilon_3}{2}\|v_2(t)\|^2_{L^2}\Big)$$
$$+ \|v(t)\|^2_{H^1_0}\Big(-\nu + \delta_1\|v_1(t)\|^2_{L^2} + 2\delta_2\|v_2(t)\|^2_{L^2} + \frac{1}{2\delta_1\delta_{11}} + \frac{1}{\delta_2\delta 22}\Big)$$
$$+ \|v(t)\|^2_{L^2}\Big(\frac{1}{2\epsilon_1} + \epsilon_2 + \frac{1}{2\epsilon_3} + \frac{\delta_{11}}{2\delta_1} + \frac{\delta_{22}}{\delta_1}\Big).$$

We can choose constants to satisfy

$$\alpha_1 = -\nu + \text{ess supp}(\frac{\epsilon_1}{2}\|v_1(t)+v_2(t)\|^2_{L^2} + \frac{\epsilon_3}{2}\|v_2(t)\|^2_{L^2}) > 0,$$

$$\alpha_2 = -\nu + \text{ess supp}(\delta_1\|v_1(t)\|^2_{L^2} + 2\delta_2\|v_2(t)\|^2_{L^2} + \frac{1}{2\delta_1\delta_{11}} + \frac{1}{\delta_2\delta_{22}}) > 0.$$

Let $\alpha_3 = (\frac{1}{2\epsilon_1} + \epsilon_2 + \frac{1}{2\epsilon_3} + \frac{\delta_{11}}{2\delta_1} + \frac{\delta_{22}}{\delta_1})$. Then, we get:

$$(\frac{1}{2}U^2(t))' + (\frac{1}{2}\|v(t)\|^2_{L^2})' \leq \alpha_1 U^2(t) + \alpha_2\|v(t)\|^2_{H^1_0} + \alpha_3\|v(t)\|^2_{L^2}.$$

Now, we integrate the last inequality over $[0,t]$, and using the fact that $U(0)=0, v(0,x)=0$ in $L^2(\Omega)$, we obtain:

$$\frac{1}{2}U^2(t) + \frac{1}{2}\|v(t)\|^2_{L^2} \leq \alpha_1\int_0^t U^2(\tau)d\tau + \alpha_2\int_0^t \|v(\tau)\|^2_{H^1_0}d\tau + \alpha_3\int_0^t \|v(\tau)\|^2_{L^2}d\tau$$
$$\leq constant. \int_0^t \big(U^2(\tau) + \|v(\tau)\|^2_{L^2}\big)d\tau.$$

Finally, by Gronwall's lemma, we conclude that $U^2(t) + \|v(t,x)\|^2_{L^2} = 0$ on $[0,T]$.

Chapter 3

STABILITY

In this chapter, we will study the stability of the solution $\left(\frac{P}{\nu}, 0\right)$ of (2.1.1), (2.1.2). As the solution we mean the functions U, v in the sense of definition 2.1. $U(t)$ satisfies (2.1.1) in the classical sense and the function $z(t) := \|v(t,x)\|_{L^2}^2$ is continuous. We let $w = v$ in 2.4 and integrate from T_1 to T_2:

$$\int_{T_1}^{T_2}\int_\Omega U(t)v^2(x,t)dxdt = \int_{T_1}^{T_2}\int_\Omega v'(x,t)v(x,t)dxdt + \nu\int_{T_1}^{T_2}\int_\Omega \big(\frac{\partial}{\partial x}v(x,t)\big)^2 dxdt+$$
$$2\int_{T_1}^{T_2}\int_\Omega v^2(x,t)\frac{\partial}{\partial x}v(x,t)dxdt,$$
$$\frac{1}{2}z(T_2) - \frac{1}{2}z(T_1) = \int_{T_1}^{T_2} U(t)z(t)dt - \nu\int_{T_1}^{T_2}\|v(x,t)\|_{H_0^1}^2 dt,$$

and we use (2.1.1) to obtain the following system:

$$\begin{cases} \frac{d}{dt}U(t) = P - \nu U(t) - z(t), \\ \frac{1}{2}z(T_2) - \frac{1}{2}z(T_1) = \int_{T_1}^{T_2} U(t)z(t)dt - \nu\int_{T_1}^{T_2}\|v(x,t)\|_{H_0^1}^2 dt, \end{cases} \tag{3.1}$$

with the conditions

$$U(0) = U_0, \quad z(0) = \|v_0\|_{L^2}^2 = z_0,$$

where $0 \le T_1 \le T_2$ arbitrary.
We want to show the global exponential stability of the solution $\left(\frac{P}{\nu}\right)$ of (3.1) when $\frac{P}{\nu} < \nu$. Using the transformation $W(t) = U(t) - \frac{P}{\nu}$, we have the problem of the stability of the zero solution $(0,0)$ of:

$$\frac{d}{dt}W(t) = -\nu W(t) - z(t), \tag{3.2}$$

$$\frac{1}{2}z(T_2) - \frac{1}{2}z(T_1) = \int_{T_1}^{T_2}\Big(W(t) + \frac{P}{\nu}\Big)z(t)dt - \nu\int_{T_1}^{T_2}\|v(x,t)\|_{H_0^1}^2 dt, \tag{3.3}$$

with the conditions

$$W(0) = U(0) - \frac{P}{\nu}, \quad z(0) = z_0.$$

Theorem 3.1. *When $\frac{P}{\nu} \leq \nu$, the solution of* (3.2), (3.3) *is uniformly bounded.*

Proof We multiply (3.2) by W and integrate over $[T_1, T_2]$:

$$\int_{T_1}^{T_2} W)(t)W'(t)dt = -\nu \int_{T_1}^{T_2} W^2(t)dt - \int_{T_1}^{T_2} W(t)z(t)dt,$$
$$\frac{1}{2}W^2(T_2) - \frac{1}{2}W^2(T_1) = -\nu \int_{T_1}^{T_2} W^2(t)dt - \int_{T_1}^{T_2} W(t)z(t)dt,$$

adding the last equality to (3.3), we get:

$$\frac{1}{2}z(T_2)+\frac{1}{2}W^2(T_2)+\nu \int_{T_1}^{T_2} \left[\|v(t)\|^2_{H^1_0}+W^2(t)\right]dt = \frac{1}{2}z(T_1)+\frac{1}{2}W^2(T_1)+\frac{P}{\nu}\int_{T_1}^{T_2} z(t)dt. \quad (3.4)$$

Since $\frac{p}{\nu} < \nu$, and $z \leq \|v(t)\|^2_{H^1_0}$, it follows that:

$$\frac{1}{2}z(T_2) + \frac{1}{2}W^2(T_2) \leq \frac{1}{2}z(T_1) + \frac{1}{2}W^2(T_1)$$

for every $0 \leq T_1 \leq T_2$, so if we let $T_1 = 0$, and $M := z(0) + W^2(0)$, we get:

$$z(t) \leq M, \text{ and } W^2(t) \leq M \text{ for any } t \geq 0.$$

Also, notice that $h(t) := \frac{1}{2}z(t) + \frac{1}{2}W^2(t)$ is a decreasing function.

Lemma 3.2. For $\nu - \frac{P}{\nu} := \gamma > 0$ the solution $(0,0)$ of (3.2), (3.3) exponentially decays to zero.

Proof From (3.4) we have

$$\frac{1}{2}z(T_2) + \frac{1}{2}W^2(T_2) + \nu \int_{T_1}^{T_2} \left[\|v(t)\|^2_{H^1_0} + W^2(t)\right]dt = \frac{1}{2}z(T_1) + \frac{1}{2}W^2(T_1) + \frac{P}{\nu}\int_{T_1}^{T_2} z(t)dt.$$

So,

$$\begin{aligned}
h(T_2) = h(T_1) - \nu \int_{T_1}^{T_2} \left[\|v(t)\|_{H_0^1}^2 + W^2(t)\right] dt + \frac{P}{\nu} \int_{T_1}^{T_2} z(t)dt \\
\leq h(T_1) - \nu \int_{T_1}^{T_2} \left[z(t) + W^2(t)\right] dt + \frac{P}{\nu} \int_{T_1}^{T_2} z(t)dt \\
\leq h(T_1) - \nu \int_{T_1}^{T_2} W^2(t)dt + \left(\frac{P}{\nu} - \nu\right) \int_{T_1}^{T_2} z(t)dt \\
= h(T_1) - \nu \int_{T_1}^{T_2} W^2(t)dt - \gamma \int_{T_1}^{T_2} z(t)dt \\
\leq h(T_1) - c_1 \int_{T_1}^{T_2} \left[W^2(t) + z(t)\right] dt \\
= h(T_1) - c_1 \int_{T_1}^{T_2} h(t)dt,
\end{aligned}$$

where $c_1 = min\{\nu, \gamma\}$. Let $f(t) = h(0)e^{-c_1 t}$. We want to show that $h(t) \leq f(t)$. We can write $f(t) = h(0) - c_1 \int_0^t f(\tau)d\tau$, and we let $g(t) = h(t) - f(t)$ to be the difference. Recall that

$$h(t) \leq h(0) - c_1 \int_0^t h(\tau)d\tau.$$

So,

$$\begin{aligned}
h(t) - f(t) \leq h(0) - c_1 \int_0^t h(\tau)d\tau - f(t) \\
g(t) \leq -c_1 \int_0^t h(\tau)d\tau + h(0) - f(t) \\
g(t) \leq -c_1 \int_0^t h(\tau)d\tau + c_1 \int_0^t f(\tau)d\tau \\
g(t) \leq -c_1 \int_0^t g(\tau)d\tau.
\end{aligned}$$

Suppose that $g > 0$ in $(0, \alpha)$, then $\int_0^t g(\tau)d\tau < 0$ for any $t < \alpha$ which leads to a contradiction. Changing 0 to t_1, and α to t_2 gives the same contradiction. Hence we get $g(t) \leq 0$ for any $t \geq 0$. So, we have:

$$g(t) = h(t) - f(t) \leq 0.$$

i.e.

$$h(t) \leq f(t) = h(0)e^{-c_1 t}.$$

So, we conclude that $h(t)$ tends to zero with an exponential rate as $t \to \infty$.

Definition 3.3. A solution U_0 of a problem is called globally asymptotically stable if it is stable and all solutions tend to U_0 as $t \to \infty$.

Theorem 3.4. *When* $\frac{P}{\nu} = \nu$, *the solution* $(0,0)$ *of* $(3.2), (3.3)$ *is globally asymptotically stable.*

Proof We will consider three cases:

$$(i)\ W(t) \leq 0 \text{ for all } t \geq 0,$$
$$(ii)\ W(t_0) = 0 \text{ for some } t_0 > 0,$$
$$(iii\ )W(t) > 0 \text{ for all } t \geq 0.$$

For the second case, we know that $W(t) \leq 0$ for all $t > t_0$ if $W(t_0) = 0$. So we will consider first and second case together. Let T_1 denote $max\{0, t_0\}$. By (3.3)) we have:

$$\begin{aligned}
\frac{1}{2}z(T_2) &= \frac{1}{2}z(T_1) + \int_{T_1}^{T_2} \big(W(\tau) + \frac{P}{\nu}\big)z(\tau)d\tau - \nu \int_{T_1}^{T_2} \|v(\tau)\|^2_{H_0^1} d\tau \\
&= \frac{1}{2}z(T_1) + \int_{T_1}^{T_2} \big(W(\tau) + \nu\big)z(\tau)d\tau - \nu \int_{T_1}^{T_2} \|v(\tau)\|^2_{H_0^1} d\tau \\
&= \frac{1}{2}z(T_1) + \int_{T_1}^{T_2} W(\tau)z(\tau)d\tau - \nu \int_{T_1}^{T_2} \big(\|v(\tau)\|^2_{H_0^1} - z(\tau)\big)d\tau \\
&\leq \frac{1}{2}z(T_1) + \int_{T_1}^{T_2} W(\tau)z(\tau)d\tau \qquad (3.5)\\
&\leq \frac{1}{2}z(T_1).
\end{aligned}$$

We showed that $z(t)$ is decreasing when $t > T_1$. So, it has a limit $\alpha \geq 0$.
If $\alpha > 0$, by (3.2) we get:

$$W'(t) \leq -\nu W(t) - \frac{\alpha}{2} \text{ for sufficiently large } t > 0.$$

It follows from the last inequality that

$$W(t) \leq -\frac{\alpha}{4} \text{ for sufficiently large } t.$$

Really:

$$\begin{aligned}\left(e^{\nu t}W(t)\right)' &\leq -\frac{\alpha}{2}e^{\nu t}\\ e^{\nu t}W(t)-W(0) &\leq -\frac{\alpha}{2\nu}e^{\nu t}+\frac{\alpha}{2\nu}\\ W(t) &\leq W(0)e^{-\nu t}-\frac{\alpha}{2\nu}+\frac{\alpha}{2\nu}e^{-\nu t}\\ W(t) &\leq -\frac{\alpha}{4\nu}\quad \text{for sufficiently large t.}\end{aligned}$$

But then (3.5) implies that $Z(T_2) \to -\infty$ as $T_2 \to \infty$ which is a contradiction since $z \geq 0$. Hence, $\alpha = 0$.

Now, we continue with the third case: $W(t) > 0$.

From (3.2)) we deduce that $W'(t) < 0 \ \ \forall t > 0$, i.e. W is strictly decreasing. Thus, it has a limit $\alpha \geq 0$. If $\alpha > 0$, using again (3.2), we get:

$$\frac{dW}{dt} < -\nu\alpha.$$

So, $W(t) \to -\infty$ as $t \to \infty$, which is a contradiction since $W(t) > 0$. Hence, $\alpha = 0$, i.e. $W(t) \to 0$ as $t \to \infty$. We also need to show that $z(t) \to 0$ as $t \to \infty$. For a contradiction, assume not.

Then, there exists $\epsilon_0 > 0$ such that for all $T_0 > 0$ there exists $T_1 > T_0$ with $z(T_1) > 2\epsilon_0$.

For some ϵ_0, let the last condition to be satisfied. We know that $W(t)$ decreases to 0 as $t \to \infty$, so there exists a $\tau > 0$ such that

$$W(\tau) = \delta, \quad W(t) < \delta \text{ for all } t > \tau \tag{3.6}$$

, where $\delta > 0$, $\delta^2 < \frac{\epsilon_0^2}{8M}$, $M = z(0) + W^2(0)$, and

$$z(\tau) < \epsilon_0. \tag{3.7}$$

If (3.6) and (3.7) are not satisfied together for any $\tau > 0$, then $z(t) > \epsilon_0$ for sufficiently large t, and by (3.2) we have:

$$\begin{aligned}\frac{dW}{dt} &< -\nu W - \epsilon_0 \\ \left(W(t)e^{\nu t}\right)' &< -\epsilon_0 e^{\nu t} \\ W(t)e^{\nu t} - W(0) &< -\frac{\epsilon_0}{\nu}e^{\nu t} + \frac{\epsilon_0}{\nu} \\ W(t) &< W(0)e^{-\nu t} - \frac{\epsilon_0}{\nu} + \frac{\epsilon_0}{\nu}e^{-\nu t} \\ W(t) &< -\frac{\epsilon_0}{2\nu} \text{ for sufficiently large } t,\end{aligned}$$

which is a contradiction. Thus, we ensure the existence of τ which satisfies (3.6) and (3.7) together. Now, by $(*)$, there exists $T_\tau > \tau$ such that $z(T_\tau) > 2\epsilon_0$. We try to find a sufficiently large interval (t, T_τ), with $z(t) > \epsilon_0$, which will make $W < 0$. From (3.5)

$$\frac{1}{2}z(T_\tau) \le \int_t^{T_\tau} W(s)z(s)ds + \frac{1}{2}z(t) \le (T_\tau - t)M\delta + \frac{1}{2}z(t),$$

the estimate $W(t) < \delta$ follows from (3.6), and we showed earlier that $z(t) < M$. If we can show that

$$\frac{1}{2}z(T_\tau) - \frac{1}{2}z(t) \le (T_\tau - t)M\delta \le \frac{\epsilon_0}{2}, \tag{3.8}$$

we get $z(t) > \epsilon_0$ in some interval (t, T_τ) since $z(T_\tau) > 2\epsilon_0$. In order to satisfy (3.8), we must have $T_\tau - t \le \frac{\epsilon_0}{2M\delta}$. Let $D := \left[T_\tau - \frac{\epsilon_0}{4M\delta}\right]$. Clearly, for $t \in D$, (3.8) holds. Now, if we can show that $W < 0$ in D, we get the desired contradiction, and we are done. We have:

$$\begin{aligned}W(T_\tau) &= W(T_\tau - \frac{\epsilon_0}{4M\delta}) + \int_{T_\tau - \frac{\epsilon_0}{4M\delta}}^{T_\tau} W'(s)ds \\ W(T_\tau) &\le W(T_\tau - \frac{\epsilon_0}{4M\delta}) + \max_{t\in D} W'(t).\frac{\epsilon_0}{4M\delta},\end{aligned} \tag{3.9}$$

and from (3.2), we know $W' = -\nu W - z$. Since $W > 0$, we get:

$$\max_{t\in D} W'(t) \le \max_{t\in D}(-z(t)) = -\min_{t\in D} z(t) \le \epsilon_0$$

,i.e. $W' \leq -\epsilon_0$ in D. So, by (3.9)

$$W(T_\tau) \leq W(T_\tau - \frac{\epsilon_0}{4M\delta}) - \epsilon_0 . \frac{\epsilon_0}{4M\delta}$$
$$W(T_\tau) \leq \delta - \frac{\epsilon_0^2}{4M\delta} < 0,$$

which is a contradiction. Hence $z(t) \to 0$ as $t \to \infty$.

Chapter 4

CONCLUSION

A lot has been done on the Burgers' equation since its first appearance in 1939 [1]. In 1948 [2], Burgers introduced a more sophisticated model describing dynamics of fluid flow. This model consists of an ordinary differential equation and a nonlinear second order parabolic equation:

$$\begin{cases} b\frac{dU}{dt} = P - \frac{\nu}{b}U - \frac{1}{b}\int_0^b v^2(t,y)dy, \\ \frac{\partial v}{\partial t} = \frac{1}{b}Uv + \nu\frac{\partial^2 v}{\partial y^2} - 2v\frac{\partial v}{\partial y}. \end{cases}$$

In this paper, we considered the problem:

$$\frac{dU(t)}{dt} = P - \nu U(t) - \int_\Omega v^2(t,x)dx, \quad x \in (0,\pi), \quad t > 0,$$

$$\frac{\partial v(t,x)}{\partial t} = U(t)v(t,x) + \nu\frac{\partial^2 v(t,x)}{\partial x^2} - \frac{\partial}{\partial x}(v^2(t,x)), \quad x \in (0,\pi), \quad t > 0,$$

under the following initial and boundary conditions

$$U(0) = U_0,$$

$$v(0,x) = \phi(x) \text{ for } x \in (0,\pi),$$

$$v(t,0) = v(t,\pi) = 0 \text{ for } t \geq 0,$$

where $U = U(t) : [0,\infty) \to \mathbb{R}, v = v(t,x) : \overline{Q} \to \mathbb{R}$ are unknown functions, and

$$Q := \Omega \times (0,\infty), \ \Omega := (0,\pi).$$

We proved theorems on global unique solvability of initial boundary value problem for the last system, obtained uniform estimates for solutions of the problem. Finally we proved that under some restrictions on parameters the equilibrium solution of the problem is exponentially stable.

Chapter 5

APPENDIX

Definition 5.1. Let f be a real valued function on a compact interval $[a, b]$. We say that f is absolutely continuous if there exists a Lebesgue integrable function g on $[a, b]$ such that

$$f(x) = f(a) + \int_a^x g(t)dt$$

for all x on $[a, b]$.

Definition 5.2. Let X be a real Banach space. The space $L(X, \mathbb{R})$ of all linear functionals on X is denoted as X^* and called the dual space of X.

Definition 5.3. Let X be a Banach space. A sequence $x_n \in X$ converges weakly to x , written

$$x_n \rightharpoonup x \text{ in } X,$$

if $f(x_n) \to f(x)$ for every $f \in X^*$.

Definition 5.4. Let X be a real Banach space, and X^* be its dual. A sequence $f_n \in X^*$ converges weakly-* to f, written

$$f_n \rightharpoonup^* f,$$

if $f_n(x) \to f(x)$ for every $x \in X$.

Theorem 5.5. ***(Alaoglu weak-* compactness)*** *Let X be a seperable Banach space and let f_n be a bounded sequence in X^*. Then f_n has a weakly-* convergent subsequence.*

Theorem 5.6. *Let X be a reflexive Banach space and x_n a bounded sequence in X. Then x_n has a subsequence that converges weakly in X.*

Remark 5.7. Since any Hilbert space H is reflexive, a bounded sequence in H has a weakly convergent subsequence.

Sobolev Spaces

Definition 5.8. Let Ω be an open set in $\mathbb{R}^n, u \in L^1_{loc}(\Omega)$, and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ be a multi-index. The α-th distributional derivative or weak derivative of u is a linear functional $T : C_c^\infty(\Omega) \to \mathbb{R}$ defined by

$$T(\varphi) = (-1)^{|\alpha|} \int_\Omega u(x) D^\alpha \varphi(x) dx$$

for all $\varphi \in C_c^\infty(\Omega)$. We say $v \in L^1_{loc}(\Omega)$ is the α-th weak derivative of u if

$$T(\varphi) = \int_\Omega v(x) \varphi(x) dx,$$

that is,

$$\int_\Omega v(x)\varphi(x)dx = (-1)^{|\alpha|} \int_\Omega u(x) D^\alpha \varphi(x) dx$$

for all $\varphi \in C_c^\infty(\Omega)$.

Definition 5.9. The Sobolev space $W^{k,p}(\Omega)$ is defined as

$$W^{k,p}(\Omega) := u : D^\alpha u \in L^p(\Omega) \text{ for all } 0 \leq |\alpha| \leq k,$$

with norm

$$\|u\|_{W^{k,p}} = \left(\sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|^p_{L^p} \right)^{1/p}.$$

When $p = 2$, we have $W^{k,2} = H^k$. H^k is a Hilbert space when equipped with the inner product

$$\langle u, v \rangle_{H^k} = \sum_{0 \leq |\alpha| \leq k} \langle D^\alpha u, D^\alpha v \rangle_{L^2}.$$

The H^k norm corresponding to this inner product is

$$\|u\|_{H^k} = \left(\sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|^2_{L^2} \right)^{1/2}.$$

Definition 5.10. The Sobolev space $H^k(\Omega)$ is defined by

$$H^k(\Omega) := \{u : D^\alpha u \in L^2(\Omega), \text{ for all } 0 \leq |\alpha| \leq k\}.$$

Definition 5.11. The space of test functions $C_c^\infty(\Omega)$ is defined as

$$C_c^\infty(\Omega) := \{\varphi \in C^\infty(\Omega) : \ \mathrm{supp}(\varphi) \text{ is a compact set in } \Omega\}.$$

Proposition 5.12. $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for any $1 \leq p < \infty$.

Definition 5.13. The space $H_0^k(\Omega)$ is the completion of the space $C_c^\infty(\Omega)$ in $H^k(\Omega)$.

Definition 5.14. The space $H^{-k}(\Omega)$ is the dual space of $H_0^k(\Omega)$.

Proposition 5.15. (Poincare's inequality) Let Ω be a bounded domain. Then, there is a constant C such that

$$\|u\|_{L^2} \leq C\|Du\|_{L^2} \text{ for all } u \in H_0^1(\Omega)$$

.

Remark 5.16. Now, we have the above inequality, we can use

$$\|u\|_{H_0^1}^2 = \sum_{|\alpha|=1} |D^\alpha u|^2 = \|Du\|_{L^2}^2$$

as an alternative norm on $H_0^1(\Omega)$, equivalent to the standart H^1 norm. This follows since

$$\|u\|_{H_0^1}^2 \leq \|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|u\|_{H_0^1}^2 \leq (1+C)\|u\|_{H_0^1}^2.$$

Theorem 5.17. ***(Rellich's compactness theorem)*** *Let Ω be a bounded domain in $\mathbb{R}^d$. Then $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$.*

Remark 5.18. As a consequence of the Sobolev embedding theorems, we have:

$$H_0^1(\Omega) \subset C^0(\overline{\Omega}) \subset L^\infty(\Omega) \text{ for bounded } \Omega \in \mathbb{R}$$

$$H_0^1(\Omega) \subset L^p(\Omega) \text{ for } p \geq 1.$$

Vector-valued functions

Suppose that X is a real Banach space with norm $\|\|_X$ and dual space X^*. Let $0 < T < \infty$, and consider functions $f : (0,T) \to X$.

Definition 5.19. A simple function $f : (0,T) \to X$ is a function of the form,

$$f = \sum_{j=1}^{N} c_j \chi_{E_j},$$

where $E_1, E_2, ..., E_N$ are Lebesgue measurable subsets of $(0,T)$ and $c_1, c_2, ..., c_N \in X$.

Definition 5.20. A function $f : (0,T) \to X$ is strongly measurable, if there is a sequence $\{f_n : n \in \mathbb{N}\}$ of simple functions such that $f_n(t) \to f(t)$ strongly in X for $a.e\ t \in (0,T)$.

Definition 5.21. For $1 \leq p < \infty$ the space $L^p(0,T;X)$ consists of all strongly measurable functions $f : (0,T) \to X$ such that

$$\int_0^T \|f\|_X^p dt < \infty,$$

equipped with the norm

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f\|_X^p dt \right)^{1/p}.$$

The space $L^\infty(0,T;X)$ consists of all strongly measurable functions $f : (0,T) \to X$ such that

$$\|f\|_{L^\infty(0,T;X)} = \sup_{t \in (0,T)} \|f(t)\|_X < \infty,$$

where sup denotes the essential supremum.

Theorem 5.22. *1 Let $X \subset\subset H \subset Y$ be Banach spaces where X, and Y are reflexive. Suppose that u_n is a sequence that is uniformly bounded in $L^2(0,T;X)$, and du_n/dt is uniformly bounded in $L^p(0,T;Y)$, for some $p > 1$. Then there is a subsequence that converges strongly in $L^2(0,T;H)$.*

Proposition 5.23. Suppose that $u \in W^{1,p}(0,T;X)$, $1 \leq p \leq \infty$. Then

$$u(t) = u(s) + \int_s^t \frac{du}{dt}(\tau) d\tau \text{ for every } 0 \leq s \leq t \leq T,$$

and $u \in C^0([0,T];X)$(for almost every $t \in [0,T]$). Furthermore we have the estimate

$$\sup_{0\le t\le T} \|u(t)\|_X \le C\|u\|_{W^{1,p}(0,T;X)}.$$

Theorem 5.24. *Suppose that*

$$u \in L^2(0,T;H^1(\Omega)) \text{ and } du/dt \in L^2(0,T;H^{-1}(\Omega))$$

Then

(i) u is almost everywhere continuous from $[0,T]$ into $L^2(\Omega)$, with

$$\sup_{t\in[0,T]} |u(t)| \le C\big(\|u\|_{L^2(0,T;H^1)} + \|du/dt\|_{L^2(0,T;H-1)}\big),$$

and

(ii) $\frac{d}{dt}|u|^2 = 2\langle du/dt, u\rangle$ for almost every $t \in [0,T]$, that is,

$$|u(t)|^2 = |u(0)|^2 + 2\int_0^t \langle du/dt(s), u(s)\rangle ds.$$

Existence and uniqueness of solution of ODE's

Theorem 5.25. ***(Cauchy-Picard)*** *Suppose that $G : \mathbb{R}^d \to \mathbb{R}^d$ satisfies*

$$\|G(y) - G(y')\|_{\mathbb{R}^d} \le L(B)\|y - y'\|_{\mathbb{R}^d},$$

for all y, y' in any bounded set $B \subset \mathbb{R}^d$. Then there exists $T = T(y_0)$ such that the initial value problem

$$\frac{dy}{dt} = G(y),\ \ y(0) = y_0$$

has a unique solution defined on the interval $[0,T]$.

Theorem 5.26. *A solution $y(t)$ of the initial value problem*

$$\frac{dy}{dt} = G(y),\ \ y(0) = y_0$$

has a finite maximal interval of existence $[0,S^*)$ if and only if $\|y(t)\|_{\mathbb{R}^d} \to \infty$ as $t \to S^*$.

Some inequalities

Cauchy-Schwarz Inequality Let H be an inner product space. Then for any $u, v \in H$, the following inequality holds:

$$|\langle u, v\rangle| \leq \|u\|\|v\|.$$

Holder's Inequality Suppose that $p \in [1, \infty)$ and $1/p + 1/q = 1$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)}$$

.

Young's Inequality If a, b are nonnegative real numbers and p, q are positive real numbers such that $1/p + 1/q = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Young's inequality with ϵ If a, b, and ϵ are nonnegative real numbers, then

$$ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}.$$

Gronwall's Inequality Let $f(t) \in \mathbb{R}$ satisfy the differential inequality

$$\frac{d}{dt_+} f(t) \leq g(t)f(t) + h(t).$$

Then

$$f(t) \leq f(0)\exp[G(t)] + \int_0^t \exp[G(t) - G(s)]h(s)ds,$$

where

$$G(t) = \int_0^t g(r)dr.$$

In particular, if a and b are constants and

$$\frac{d}{dt_+} f(t) \leq a f(t) + b,$$

then

$$f(t) \leq \Big(f(0) + \frac{b}{a} \Big) e^{at} - \frac{b}{a}.$$

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