Grothendieck's Dessin Theory

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Abstract

The bridge between algebraic geometry and complex geometry is built by Riemann on the following observation: compact Riemann surfaces and nonsingular complex projective curves can be considered to be same. After the celebrated theorem of Belyĭ, which is a bridge between curves defined over number fields and the existence of certain coverings of the projective line, Grothendieck launched in the 1980s, in his famous *Equisse d'un programme* that such coverings is completely determined by the pre-image of the real interval $[0, 1]$ which is named a *dessin d'enfant* (child's drawing) by him.

We give an introduction to the theory of dessins d'enfants. These combinatorial objects are simply graphs embedded into topological surfaces and provide an extraordinary link to a special topic of arithmetic geometry: curves defined over number fields can be described by such combinatorial objects. In addition to the initial equivalence built by Riemann given in section 3, in this thesis, we give several equivalences to built the general aspects of Grothendieck's dessin theory. The first equivalence is well-known *Grothendieck correspondence*: Any dessin d'enfant arises from a finite covering of the projective line $\mathbb{P}^1(\mathbb{C})$ by a Riemann surface $\mathcal{X}$ unramified except the points $0, 1, \infty$ and conversely, given a dessin one can construct such a covering of the projective line and vice versa. This is done in section 2.

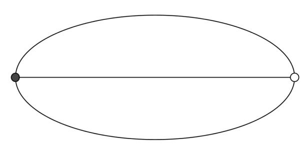

Figure 1: Dessin d'enfant corresponding to the covering $x \mapsto \dfrac{x^3}{x^3 - 1}$

The importance of these equivalences is due to Belyĭ theorem which is given in section 4: As essentially a consequence of Weil's descent theory, it was known that any dessin arises from a finite covering of the projective line $\mathbb{P}^1(\mathbb{C})$ that can be defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers. So the question is: Which algebraic curves arises in this way? Belyĭ showed that every algebraic curve defined over $\overline{\mathbb{Q}}$ can be represented as a covering of the projective line ramified at most three points. In other words, every algebraic curve defined over $\overline{\mathbb{Q}}$ contains an embedded

dessin d'enfant. One important consequence of this theorem is that $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ has a faithful action on the set of dessins.

A dessin can be regarded as an ordered pair of permutations generating a transitive subgroup of a symmetric group S_n. The group $\mathrm{PGL}_2(\mathbb{Z})$ has an action on these pairs of permutation, hence on dessins d'enfants. Our aim is to define and study an action of $\mathrm{PGL}_2(\mathbb{Z})$ on dessins which appears to have not been studied until now. The final section is dedicated to investigate combinatorial and arithmetic aspects of this action.

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Özet

Cebirsel geometri ile kompleks geometri arasında Riemann'ın kurduğu köprü şu gözleme dayanmaktadır: Kompakt Riemann yüzeyleri ve tekil olmayan kompleks projektif eğrileri aynıdır.

Belyi'nin, sayı cisimleri üzerine tanımlanan eğriler ile projektif doğrunun belli örtülerinin varlığı arasında köprü kuran meşhur teoreminden sonra, Grothendieck 1980'lerde, "Equisse d'un programme"'da bu tür örtülerin [0,1] reel aralığının öngörüntüsü ile belirlendiğini açıkladı ve bu öngörüntüleri *dessin d'enfant* (çocuk çizimleri ya da kısaca desen) olarak adlandırdı. Belyi göstermişti ki, rasyonel sayı cisminin cebirsel kapanışı üzerine tanımlı her cebirsel eğri, projektif doğrunun en fazla üç noktada dallanmış örtüleri ile temsil edilebilir. Başka bir deyişle, rasyonel sayı cisminin cebirsel kapanışı üzerine tanımlı her cebirsel eğri, içine gömülü bir desen barındırır. Bu tezde desen teorisini tanıtacağız.

Bir desen n harfli simetri grubundan sıralı, geçişken bir permütasyon çifti ile betimlenebilir. PGL(2,Z) grubunun bu çiftler zerinde bir etkisi vardır, böylelikle desenler üzerinde de bir etkisi vardır. Amacımız henüz incelenmemiş bu etkiyi tanımlamaktır ve incelemektir. Son bölüm bu etkinin kombinatoryel ve aritmetik doğasını anlamaya ayrıldı.

1 Riemann Surfaces and Meromorphic Functions

This section consists of basic ingredient concerning Riemann surfaces and morphisms on them. We recall the properties of these morphisms and refer to well-known sources. The examples at the end of this section will be useful for the rest of this thesis. In fact the figures for this examples are nothing but the corresponding dessins d'enfants.

1.1 Thrice Punctured Riemann Sphere

Let $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Introduce the following topology on $\widehat{\mathbb{C}}$. The open sets are of two type: the usual open sets $U \subseteq \mathbb{C}$ and the sets of the form $V \cup \{\infty\}$ where $V \subseteq \mathbb{C}$ is the complement of a compact set $K \subseteq \mathbb{C}$. With this topology, $\widehat{\mathbb{C}}$ is a compact Hausdorff, second countable topological space. Set

$$U_1 = \widehat{\mathbb{C}} \setminus \{\infty\} = \mathbb{C}$$
$$U_2 = \widehat{\mathbb{C}} \setminus \{0\} = \mathbb{C}^* \cup \{\infty\}$$

Define the maps $\varphi_1 \equiv Id_{\mathbb{C}}$ on U_1 and $\varphi_2(z) = \begin{cases} 1/z, & \text{if } z \in \mathbb{C}^* \\ 0, & \text{if } z = \infty \end{cases}$ on U_2. Then φ_1 and φ_2 are homeomorphisms so that $\widehat{\mathbb{C}}$ is a surface. Note that since U_1 and U_2 are connected and have a non-empty intersection, it follows that $\widehat{\mathbb{C}}$ is connected.

Now, the complex structure on $\widehat{\mathbb{C}}$ is defined by the atlas consisting of the charts (U_1, φ_1) and (U_2, φ_2) which are holomorphically compatible since $\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^*$ given by $z \mapsto 1/z$ is biholomorphic. The resulting compact Riemann surface is the **Riemann sphere**. Throughout this thesis we will deal with coverings of thrice punctured Riemann sphere.

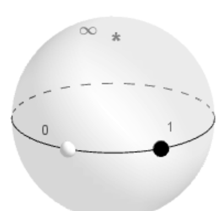


Figure 2: $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$

Let $\mathcal{X}$ be a two-dimensional manifold. A *complex chart* on $\mathcal{X}$ is a homeomorphism $\varphi : U \to V$ of an open subset $U \subseteq \mathcal{X}$ onto an open subset $V \subseteq \widehat{\mathbb{C}}$. Two complex charts $\varphi_i : U_i \to V_i$, $i = 1, 2$ are said to be *holomorphically compatible* if the map

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \longrightarrow \varphi_2(V_1 \cap V_2)$$

is biholomorphic. A *complex atlas* on $\mathcal{X}$ is a collection $\mathfrak{A} = \{\varphi_i : U_i \to V_i, \mathrm{i} \in I\}$ of complex charts which are holomorphically compatible and which cover $\mathcal{X}$ that is $\bigcup_{\mathrm{i} \in I} U_i = \mathcal{X}$. Two complex atlases $\mathfrak{A}$ and $\mathfrak{A}'$ on $\mathcal{X}$ are called *analytically equivalent* if every complex chart of $\mathfrak{A}$ is holomorphically compatible with every complex chart of $\mathfrak{A}'$. This is indeed an equivalence relation and an equivalence class of analytically equivalent atlases on $\mathcal{X}$ is a *complex structure* on $\mathcal{X}$. Hence a complex structure on $\mathcal{X}$ can be given by the choice of a complex atlas as done above for the Riemann sphere.

A **Riemann surface** is a pair $(\mathcal{X}, \Sigma)$ where $\mathcal{X}$ is a connected two-dimensional manifold and Σ is a complex structure on $\mathcal{X}$.

1.2 Meromorphic Functions on a Riemann Surface

Let $\mathcal{X}$ be a Riemann surface and $f : \mathcal{X} \longrightarrow \mathbb{C}$ be a function. We say that f is *holomorphic* on $\mathcal{X}$ if for every chart (U, φ), $f \circ \varphi^{-1} : \varphi(U) \longrightarrow \mathbb{C}$ is holomorphic in the usual sense. We denote the set of all holomorphic functions on $\mathcal{X}$ by $\mathcal{O}(\mathcal{X})$. It is clear that this definition extends the usual definition of holomorphic function in complex analysis by considering the atlas consisting only of $(\mathbb{C}, Id)$ on the complex line. We also note that since holomorphic functions are defined via coordinate charts, all the local properties of usual holomorphic functions on $\mathbb{C}$ are valid for the holomorphic functions on Riemann surfaces.

A function $f : \mathcal{X} \longrightarrow \mathbb{C}$ is *meromorphic* on $\mathcal{X}$ if there exists a subset $\mathcal{X}'$ of $\mathcal{X}$ such that

i. the restricted function $f : \mathcal{X}' \longrightarrow \mathbb{C}$ is holomorphic,

ii. $P = \mathcal{X} \backslash \mathcal{X}'$ is a discrete set whose elements are called *poles*.

iii. if $p \in P$ then $lim_{z \to p} |f(z)| = \infty$

Definition 1.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be Riemann surfaces. A function $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is called **holomorphic (resp. meromorphic)** if $\psi \circ f \circ \varphi^{-1}$ is holomorphic (resp.

meromorphic) for every $x \in \mathcal{X}$, φ a coordinate around x and ψ around $f(x)$. Finally, f is an **analytic isomorphism** if f is bijective and both f and f^{-1} are holomorphic.

In other words, a meromorphic function on a Riemann Surface $\mathcal{X}$ is a holomorphic function $\beta : \mathcal{X} \to \widehat{\mathbb{C}}$ which is not identically ∞. We denote the set of all meromorphic functions by $\mathcal{M}(\mathcal{X})$. It is easy to see that $\mathcal{M}(\mathcal{X})$ is a field called *the field of meromorphic functions* on $\mathcal{X}$. In section 2 we shall see that it is a field extension of $\mathbb{C}$ of transcendence degree 1.

Proposition 1.1. *We list the following properties concerning Riemann surfaces. We refer [5] and [19] for proofs.*

I. **(Identity Theorem)**
Let $\mathcal{X}$ and $\mathcal{Y}$ be Riemann surfaces and let $f, g : \mathcal{X} \to \mathcal{Y}$ be two holomorphic functions. Suppose that f and g coincide on a set $A \subseteq \mathcal{X}$ with non-empty interior. Then f and g identically equal.

II. **(Riemann removable singularity theorem)**
Let D be an open subset of a Riemann surface and let $a \in D$. Suppose that f is holomorphic on $D \setminus \{a\}$ and bounded on D. Then f extends uniquely to a holomorphic function on the whole D.

III. Let $f : \mathcal{X} \to \mathbb{C}$ be a meromorphic function and $p \in \mathcal{X}$ be a point. Let (U, z) be a complex chart centred at p. Then f can be expanded in a Laurent series:

$$f = \sum_{k=m}^{\infty} c_k z^k, \text{ where } c_k \in \mathbb{C}; m \in \mathbb{Z}$$

  i. The integer m is the **multiplicity** of f at p and is independent of the chosen chart (U, z) denoted by $m = \nu(f, p)$.
  ii. If m is negative then p is a pole of order $-m$.
  iii. If $m \geq 0$ then f is holomorphic and f has a zero of order m whenever m is non-zero.

IV. **(Local normal form for holomorphic functions)**
Let $\mathcal{X}$ and $\mathcal{Y}$ be Riemann surfaces and $f : \mathcal{X} \to \mathcal{Y}$ be a non-constant holomorphic function. Suppose that $p \in \mathcal{X}$ and $q := f(p)$. Then there exists an integer $m \geq 1$ and complex charts $\varphi : U \to V$ on $\mathcal{X}$ and $\psi : U^{'} \to V^{'}$ on $\mathcal{Y}$ satisfying

i. $p \in U$ and $q \in U^{'}$, $\psi(q) = 0$
ii. $f(U) \subseteq U^{'}$
iii. The function $F := \psi \circ f \circ \varphi^{-1} : V \to V^{'}$ is given by $F(z) = z^m$ for all $z \in V$

V. These last two properties characterise the local behaviour of a holomorphic function. Moreover, every non-constant meromorphic function $\beta : \mathcal{X} \to \widehat{\mathbb{C}}$ takes each value the same number of times counting multiplicity and so that β is called a meromorphic function of **degree n**, denoted by $deg\beta = n$. Therefore, for most points $z \in \widehat{\mathbb{C}}$, the set $\beta^{-1}(z)$ has n distinct points. More precisely, one has

$$n = \sum_{z \in f^{-1}(p)} \nu(f, z)$$

if $z_0 \in \widehat{\mathbb{C}}$ is a point for which the set $\beta^{-1}(z_0)$ has fewer than n points we say that z_0 is a **ramification point, *or critical point*** of β and any point in $\beta^{-1}(z_0)$ a **ramification value, *or critical value*** of β. We denote the set of ramification values of a meromorphic function f by **Ram(f)**.

VI. A non-constant holomorphic function $f : \mathcal{X} \to \mathcal{Y}$ is open, i.e. it maps open sets of $\mathcal{X}$ to open sets of $\mathcal{Y}$ and discrete, i.e. the fibre $f^{-1}(y)$ of a point $y \in \mathcal{Y}$ is a discrete subset of $\mathcal{X}$.

VII. Another important consequence local property of morphisms between Riemann surfaces is the following: Suppose that $\mathcal{Y}^*$ is Riemann surface obtained from a compact Riemann surface $\mathcal{Y}$ by removing finitely many points and that $f^* : \mathcal{X}^* \to \mathcal{Y}^*$ is an holomorphic covering of finite degree n. Then Riemann removable singularity theorem guarantees that for each $y \in \mathcal{Y} \setminus \mathcal{Y}^*$ add n points to $\mathcal{X}^*$ that extend the holomorphic covering f^* to a morphism $f : \mathcal{X} \to \mathcal{Y}$ so that $\mathcal{X}$ being compact too. Moreover, if $\mathcal{X}_1$ and $\mathcal{X}_2$ obtained in this way then they are isomorphic. Hence the resulting compact Riemann surface is unique up to removing finitely many points. Details can be found in [3].

VIII. **(Riemann-Hurwitz Formula)**

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of degree n between compact Riemann surfaces of genera g_1 and g_2 respectively. Then

$$2g_1 - 2 = n(2g_2 - 2) + \sum_{x \in \mathcal{X}} \Big(\nu(f, x) - 1 \Big)$$

IX. The field of meromorphic functions on the Riemann sphere consists of only rational functions: $\mathcal{M}(\widehat{\mathbb{C}}) = \mathbb{C}(z)$,the field of fractions of the polynomial ring $\mathbb{C}[z]$.

Example 1.1. If f is a polynomial of degree n then $a \in \mathbb{C}$ has n distinct pre-image if and only if $f(z) = a$ and $f^{'}(z) = 0$ have no common solutions. Indeed, if

$$f(z) - a = c \prod_{k=1}^{n} (z - r_k)$$

Then by product rule,

$$f^{'}(z) = c \sum_{j=1}^{n} \Big(\prod_{k=1, k \neq j}^{n} (z - r_k) \Big)$$

so that

$$f^{'}(r_i) = c \Big(\prod_{k=1, k \neq i}^{n} (r_i - r_k) \Big)$$

Hence r_i's are distinct if and only if $f^{'}(r_i) \neq 0$. Recall that in order to investigate the local behaviour of $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ at the point ∞ one needs to consider the behaviour of the composition $f \circ J$ at 0, where $J(z) = 1/z$. Similarly, for the cases $f(p) = \infty$, $p \neq \infty$ and for $f(\infty) = \infty$ the behaviour around the point ∞ is given by the composition $J \circ f$ and $J \circ f \circ J$ respectively.

So in the case of this example, since p is a polynomial, it is clear that $p(\infty) = \infty$ so that $J \circ f \circ J$ examined at 0 gives the local behaviour of f at ∞.

Example 1.2. The polynomial $p_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, defined by $p_n(z) = z^n$ has two critical points namely 0 and ∞. We have $p_n(0) = 0$ with multiplicity n and for $z \neq 0$, $p_n(z) \neq 0$.Therefore 0 is a critical point. Let us see that ∞ is a critical point. Indeed, considering $(J \circ p_n \circ J)(z) = z^n$ which has a zero of order n, we get $p_n(\infty) = \infty$.

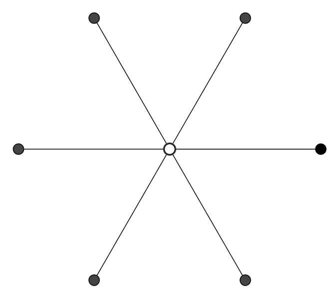


Figure 3: The [0,1]-pre-image of the mapping p_6

Example 1.3. Let m and n be integers with m, n, $m+n \neq 0$. Consider rational maps $\beta_{m,n} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by

$$\beta_{m,n}(z) = \frac{(m+n)^{m+n}}{m^m n^n} z^m (1-z)^n$$

These form an important class of Belyĭ functions. We shall investigate the nature of critical points and critical values of them. Indeed,

$$\beta'_{m,n}(z) = \frac{(m+n)^{m+n}}{m^m n^n} z^{m-1} (1-z)^{n-1} (m-(m+n)z)$$

so that the critical values are $0,1,\dfrac{m}{m+n},\infty$ which correspond to the critical points $0,0,1,\infty \in \widehat{\mathbb{C}}$ respectively.

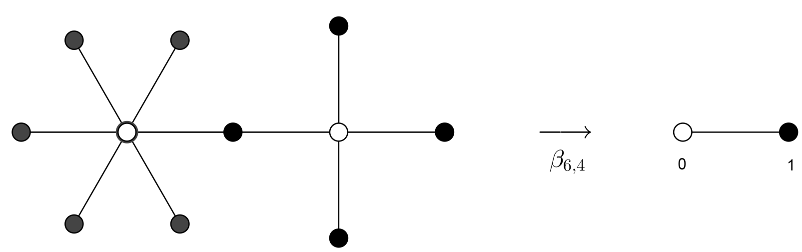


Figure 4: The [0,1]-pre-image of the mapping $\beta_{6,4}$

Example 1.4. (Elliptic Curves)
In this example at first we will determine the compact Riemann surface corresponding to a special case of elliptic curves and we will find the critical points and critical values of a certain morphism on this compact Riemann surface .

Consider the algebraic curve given by $y^2 = x(x-\lambda_1)(x-\lambda_2)$ where λ_1 and λ_2 are distinct complex numbers. Let

$$S_0 = \{(x, y) \in \mathbb{C} : y^2 = x(x-\lambda_1)(x-\lambda_2)\}.$$

We define the chart (U, φ) around each point $P_0 = (x_0, y_0)$ in the following cases:

Case 1: For $P_0 = (x_0, y_0)$, where $x_0 \neq 0, \lambda_1, \lambda_2$ we take

$$\varphi^{-1}(z) = (z + x_0, \sqrt{(z+x_0)(z+x_0-\lambda_1)(z+x_0-\lambda_2)})$$

defined in the open disc $B(0, \epsilon)$ with ϵ small enough to guarantee $z \neq 0, \lambda_1, \lambda_2$. The branch of the square root is chosen so that its value equals y_0 at x_0.

Case 2: For $P_0 = (a_i, 0)$, where $a_i = 0, \lambda_1$ or λ_2 the parametrization is given by

$$\varphi_i^{-1}(z) = (z^2 + a_i, \sqrt{\Pi_{j\neq i}(z^2 + a_i - a_j)})$$

defined in the open disc $B(0, \epsilon)$ with ϵ small enough so that $z^2 + a_i \neq a_j$ whenever $j \neq i$.

Now the transition function $(\varphi \circ \varphi_i^{-1})(z) = z^2 + a_i$ is holomorphic wherever it is defined. As in the compactification of $\widehat{\mathbb{C}}$, we add a point ∞ to S_0 and we obtain a compact Riemann surface $S = S_0 \cup \{\infty\}$ by the following parametrization around $P_0 = \infty$.

Case 3: For $P_0 = \infty$ we let

$$\phi^{-1}(z) = \begin{cases} (\frac{1}{z^2}, \frac{1}{z^3}\sqrt{(1-a_1 z_2)(1-a_2 z_2)(1-a_3 z_2)}) & \text{if } 0 < |z| < \epsilon \\ \infty & \text{if } z = 0 \end{cases}$$

Since the domain of ϕ and those of φ_i's can be chosen to be disjoint sets, these chart maps are compatible. In addition to this, we have the transition function, whose domain does not contain $z = 0$, $(\varphi \circ \phi^{-1})(z) = \dfrac{1}{z^2}$ is holomorphic.

Finally, we shall show that the Riemann surface $S = S_0 \cup \{\infty\}$ that we obtained in this way is compact. To do this, we decompose S as the union of two compact sets as follows:

$$S = \{(x, y) \in S_0 : |x| \leq \frac{1}{\epsilon}\} \cup (\{(x, y) \in S_0 : |x| \geq \frac{1}{\epsilon}\} \cup \{\infty\})$$

Note that the first set in the union is compact since it is closed and bounded in $\mathbb{C}^2$ and the second one equals $\phi^{-1}(\overline{B(0, \sqrt{\epsilon})})$ hence is compact.

Example 1.5. Now we exemplify a meromorphic function on the compact Riemann surface S. Consider the coordinate map $\pi_X : S \to \widehat{\mathbb{C}}$ defined as

$$\begin{aligned} (x, y) &\longmapsto x \\ \infty &\longmapsto \infty \end{aligned}$$

To determine the critical points we calculate

$$\begin{aligned} \varphi_1 \circ \pi_x \circ \varphi^{-1}(z) &= z \\ \varphi_1 \circ \pi_x \circ \varphi_i^{-1}(z) &= z^2 + \lambda_i \\ \varphi_2 \circ \pi_x \circ \psi^{-1}(z) &= z^2 \end{aligned}$$

so that we found critical points $(0, 0), (\lambda_1, 0), (\lambda_2, 0)$ and ∞ and corresponding critical values $0, \lambda_1, \lambda_2$ and ∞ respectively. Also we notice that these critical points are of multiplicity 2. Here, recall that φ_1 and φ_2 are the charts defined for the sphere at the beginning of this section.

2 Grothendieck's Correspondence

Now we aim to give a description of Grothendieck correspondence between dessin d'enfants and Belyĭ pairs. The notion of dessin d'enfant is a nice way to describe the coverings $\beta : \mathcal{X} \to \widehat{\mathbb{C}}$ from a compact Riemann surface $\mathcal{X}$ to the Riemann sphere $\widehat{\mathbb{C}}$ which are ramified at most over the set $\{0, 1, \infty\}$. We will show that a Belyĭ pair is up to equivalence determined by

1. a dessin d'enfant up to equivalence
2. a bipartite connected ribbon graph up to equivalence
3. a monodromy map $\Phi : \mathbb{F}_2 \to S_d$; i.e. a transitive action of $\mathbb{F}_2$, the free group on two letters, on the set $\{1, 2, ..., d\}$, up to conjugation in S_d,
4. a 3-constellation up to conjugation,
5. a finite index subgroup of $\mathbb{F}_2$ up to conjugation.

The first equivalence is often called the *Grothendieck correspondence*. In the following section we shall define these notions and figure out these equivalences.

2.1 Belyĭ Pairs as Coverings

Definition 2.1. A **Belyĭ pair** $(\mathcal{X}, \beta)$ is a pair of a compact Riemann surface $\mathcal{X}$ together with a meromorphic function $\beta : \mathcal{X} \to \widehat{\mathbb{C}}$ unramified outside three distinct points on $\mathcal{X}$. Indeed, these functions are ramified coverings called **Belyĭ morphisms** as we will see in this section.

We say that two Belyĭ pairs $(\mathcal{X}_1, \beta_1)$ and $(\mathcal{X}_2, \beta_2)$ are **equivalent** if there exists an isomorphism $f : \mathcal{X}_1 \to \mathcal{X}_2$ such that $\beta_2 \circ f = \beta_1$. We shall consider Belyĭ pairs up to equivalence.

Remark. The automorphism group $Aut(\widehat{\mathbb{C}}) = PSL(2, \mathbb{C})$ is 3-transitive on $\widehat{\mathbb{C}}$ that is given any set of three distinct point there is an automorphism of $\widehat{\mathbb{C}}$ that send this set to another set of three points. Thus, if β is a Belyĭ function, the set of ramification values can always be seen as the set $\{0, 1, \infty\}$ via the composition of this automorphism with β. Explicitly, if β is ramified at $\{w_1, w_2, w_3\}$ then $\tau \in PSL(2, \mathbb{C})$ defined by

$$\tau(w) = \frac{w_2 - w_3}{w_2 - w_1} \cdot \frac{w - w_1}{w - w_3}.$$

sends $w_2 \mapsto 0$, $w_2 \mapsto 1$ and $w_2 \mapsto \infty$ and hence $\beta \ = \tau \circ \beta$ is a new Belyĭ function branched at $\{0, 1, \infty\}$ which is obviously equivalent the previous one. So we may suppose that the set of ramification points of a Belyĭ function lies inside the set $\{0, 1, \infty\}$.

Example 2.1. The functions in the examples 1.3-1.5 are all Belyĭ morphisms since they are ramified at most three points.

Definition 2.2. Let $\mathcal{X}$ and $\mathcal{Y}$ be two path connected topological spaces. Let $f : \mathcal{X} \to \mathcal{Y}$ be a continuous mapping. A pair $(\mathcal{X}, f)$ is called a **covering** of $\mathcal{Y}$ by $\mathcal{X}$ if for any $y \in \mathcal{Y}$ there exists a neighbourhood V of y such that the pre-image $f^{-1}(V) \subseteq \mathcal{X}$ is homeomorphic to $V \times S$, where S is a discrete set. The connected components of $f^{-1}(V)$ are called **sheets**, the set $f^{-1}(y)$ of pre-images of a point $y \in \mathcal{Y}$ is called **the fibre over y** and finally the cardinality $|S|$ of the set is **the degree of the covering**.

Note that in the definition above the number of pre-images of a point in $\mathcal{Y}$ is locally constant. Since $\mathcal{X}$ is connected, it is constant.So the definition of *number of sheets* is well-defined and equal to $|S|$.

Two coverings $(\mathcal{X}_1, f_1)$ and $(\mathcal{X}_2, f_2)$ are **equivalent** if there exists a **morphism** ψ of coverings between them, i.e. there is a homeomorphism $\psi : \mathcal{X}_1 \to \mathcal{X}_2$ such that $\psi \circ f_2 = f_1$. If ψ from $\mathcal{X}$ to itself is a homeomorphism such that $\psi \circ f = f$,then it is called an **automorphism** of a covering $(\mathcal{X}, f)$. The group of all automorphisms of a covering $(\mathcal{X}, f)$ is denoted by $Aut(\mathcal{X}/\mathcal{Y})$.

A covering $(\mathcal{X}, f)$ of $\mathcal{Y}$ is called **Galois** if for every pair of points x_1 and x_2 in $\mathcal{Y}$ with $f(x_1) = f(x_2)$ there is a covering $g : \mathcal{X} \to \mathcal{Y}$ such that $g(x_1) = x_2$.

Example 2.2. The mapping $p_n : \mathbb{C}^* \to \mathbb{C}^*$ given as in the example 1.2 by $p_n(z) = z^n$ is a Galois covering. Indeed, for any $z_1, z_2 \in \mathbb{C}^*$ with $p_n(z_1) = p_n(z_2)$ we have $z_2 = \omega z_1$, where ω is a n^{th} root of unity and the function $z \mapsto \omega z$ is a morphism of coverings.

If $f : \mathcal{X} \to \mathcal{Y}$ is covering then it has the following **path-lifting property**: For any path $\gamma : \mathcal{I} \to \mathcal{Y}$ and any pre-image p of $\gamma(0)$ there is a path $\widehat{\gamma}$ on $\mathcal{X}$ such that $\widehat{\gamma}(0) = p$ and $f \circ \widehat{\gamma} = \gamma$. Roughly speaking, one can *lift* the path γ on $\mathcal{Y}$ to a a path on $\mathcal{X}$, starting at any pre-image of the starting point of γ.

Since Belyĭ functions are the coverings of the thrice punctured sphere which has already a complex structure, we may derive the properties inherited from covering theory (such as monodromy and classification of coverings) because the complex structure is already determined by the following:

Proposition 2.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a covering, where $\mathcal{Y}$ is a Riemann surface. Then $\mathcal{X}$ has a unique Riemann surface structure that makes f holomorphic.*

Proof. For a chart (V_j, ϕ_j) in $\mathcal{Y}$, we form a chart as $(U_i, \phi_j \circ f)$, where $f(U_i) = V_j$ for each pre-image of V_j. The transition functions $(\phi_k \circ f) \circ (\phi_j \circ f)^{-1} = \phi_k \circ \phi_j^{-1}$ are obviously holomorphic. Since the local expression of f in these charts is $(\phi_j \circ f) \circ (\phi_j \circ f)^{-1} = Id$, f is indeed holomorphic. Let us now show the uniqueness of this complex structure. Suppose that f is holomorphic to another chart (U, ϕ), that is, $(\phi_j \circ f) \circ \phi^{-1}$ is holomorphic. Then (U, ϕ) is compatible with the all the charts $(U_i, \phi_j \circ f)$ above. So we conclude that this complex structure on β is unique. □

Suppose that $\beta : \mathcal{X} \to \mathcal{Y}$ is a morphism between compact Riemann surfaces. By removing the ramification values from the space $\mathcal{Y}$ and their pre-images from $\mathcal{X}$ we get an unramified holomorphic covering $\beta^* : \mathcal{X}^* \to \mathcal{Y}^*$.

Let $x \in \mathcal{X}^*$ be an arbitrary point. By construction x is not a ramification point hence there is an open neighbourhood V of x such that the restriction map $\beta|_V$ is an injection. Being a holomorphic mapping β is open and continuous so that β maps V homeomorphically to $\beta(V)$. This shows that β^* is a local homeomorphism.

Indeed $\beta^* : \mathcal{X}^* \to \mathcal{Y}^*$ is a covering map in the topological sense. To see this let $y \in \mathcal{Y}^*$. Since β is discrete, $\beta^{-1}(y)$ being a discrete subset of the compact space $\mathcal{X}$ is finite so we may set $\beta^{-1}(y) = \{x_1, x_2, \ldots, x_n\}$. Now let V be a neighbourhood of y and $U_1, U_2, \ldots, U_n$ be the neighbourhoods of $x_1, x_2, \ldots, x_n$ respectively. Note that V can be chosen so that $f^{-1}(V) = \sqcup_i U_i$. Otherwise, there would be a sequence $(y_k)_k$ in V converging to y such that each fibre $f^{-1}(y_k)$ contains a point $x_k^{'} \notin \cup_i U_i$. Let x be a limit point of this sequence. Since f is continuous $f(x) = y$. Therefore $x = x_j$ for some $j \in \{1, 2, \ldots, n\}$. But then $x_k^{'} \in U_j$ for k large enough, a contradiction. So a Belyĭ function is a covering map in the following sense below. This construction describes pass to the topological covering theory from a given Belyĭ pair. Conversely if $\beta^* : \mathcal{X}^* \to \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ is a covering then one can add the missing points to extend it to a Belyĭ covering $\beta : \mathcal{X} \to \widehat{\mathbb{C}}$ by Proposition 1.1.

2.1.1 Monodromy Representation of Belyĭ coverings

Let $f : \mathcal{X} \to \mathcal{Y}$ be a covering. Let $y_0 \in \mathcal{Y}$ be a base point. We have the following monodromy action of $\pi_1(\mathcal{Y}, y_0)$ on the set $F = f^{-1}(y_0)$:

Let $\gamma \in \pi_1(\mathcal{Y}, y_0)$. We shall show that γ induces a bijection on the set E. Since γ is a loop (i.e. a closed oriented piecewise smooth curve), $f^{-1}(\gamma)$ consists of $|S|$-many oriented curves in $\mathcal{X}$ by the path-lifting property. Note that γ leads from y_0 to y_0, so each of the curves in the pre-image of γ permutes the points of F. The resulting map $g : F \to F$ is invertible since γ is invertible in $\pi_1(\mathcal{Y}, y_0)$. This gives an action of $\pi_1(\mathcal{Y})$. As $\mathcal{Y}$ is connected, this action is transitive. Indeed, if x_i and x_j are two points in the fibre of y_0, we can find a path $\widehat{\gamma}$ connecting x_i to x_j. Now, let $\gamma = f \circ \widehat{\gamma}$ be the image of $\widehat{\gamma}$ in $\mathcal{Y}$. Then γ is a loop based at y_0 since both x_i and x_j are sent to y_0 under the map f.

We see from this construction above that the correspondence $\gamma \mapsto g$ gives a group homomorphism from $\pi_1(\mathcal{Y}, y_0)$ to $Sym(F)$. The product of (equivalence classes of) paths corresponds to composition of bijections on E. We obtain in this way a **monodromy map** $\Phi : \pi_1(\mathcal{Y}, y_0) \to S_d$ which is independent of the chosen base point y_0 and of the way we choose the labelling its pre-images up to composition with a conjugation in the symmetric group. By definition, a **monodromy group** of the group generated by these g's.

In the context of Belyĭ pairs, we will deal with the case $\mathcal{Y} = \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. We start with a Belyĭ pair $(\mathcal{X}, \beta)$, and by removing the ramification points $0, 1$ and ∞ from $\widehat{\mathbb{C}}$ and all their pre-images from $\mathcal{X}$ we obtain an *unramified* covering: $\beta^* : \mathcal{X}^* \to \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. Here we denote the resulting punctured surface by $\mathcal{X}^*$ and the restricted map by β^*.

It is an application of Seifert-Van Kampen theorem that the fundamental group of $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ is $\mathbb{F}_2$. [8]. We fix an isomorphism between these groups. Then the unramified covering $\beta^* : \mathcal{X}^* \to \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ gives rise to the following monodromy map:

$$\mathbb{F}_2 \xrightarrow{\sim} \pi(\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}) \to Sym(F).$$

where y is a chosen base point of $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ and F the fibre of β'over y. Equivalently, this defines a finite index subgroup Γ of $\mathbb{F}_2$ up to conjugation, which is $\Gamma = Stab_{Sym(F)}(\{y_0\})$.

2.1.2 Classification of Belyĭ Coverings via Subgroups of $\mathbb{F}_2$

It is well-known that the subgroups of $\pi(\mathcal{Y}, y_0)$ characterise all coverings of the space $\mathcal{Y}$. Now, we shall construct this correspondence. Firstly, let $\mathcal{Y}$ be connected topological space. Let $\Gamma \leq \pi(\mathcal{Y}, y_0)$ be a subgroup. Consider the set $\mathcal{X}_0$ of all oriented paths in $\mathcal{Y}$ with the starting point y_0, i.e.

$$\mathcal{X}_0 = \{\gamma | \gamma \text{ is a path in } \mathcal{Y} \text{ starting at } y_0\}$$

. Define the following relation on $\mathcal{X}_0$: $\gamma_1 \sim \gamma_2$ if

(i) γ_1 and γ_2 have the same endpoint i.e. $\gamma_1(1) = \gamma_2(1)$ and

(ii) $[\gamma_1 * \gamma_2^{-1}] \in \Gamma$

It can be easily seen that this is indeed an equivalence relation. Denote the space of the set of equivalence classes of such paths by $\mathcal{X}_\Gamma$ and denote the equivalence class of a path γ by $\langle\gamma\rangle$. Define $f : \mathcal{X}_\Gamma \to \mathcal{Y}$ by $\langle\gamma\rangle_\Gamma \mapsto \gamma(1)$. We topologize $\mathcal{X}_\Gamma$ as follows: First, if $\gamma \in \mathcal{X}_0$ and if U is a neighbourhood of $\gamma(1)$ then a path $\widehat{\gamma} \in \mathcal{X}_0$ of the form $\widehat{\gamma} = \gamma * \lambda$ where $\lambda(0) = \gamma(1)$ and $\lambda([0,1]) \subset U$ is called *a continuation of* γ *in* U. Then for $\langle\gamma\rangle_\Gamma$ and a neighbourhood U of $\gamma(1)$ define

$$(U, \langle\gamma\rangle_\Gamma) := \left\{\langle\widehat{\gamma}\rangle_\Gamma \in \mathcal{X}_\Gamma | \widehat{\gamma} \text{ is a continuation of } \gamma \text{ in } U\right\}$$

It can be shown that these sets form a basis for a topology on $\mathcal{X}_\Gamma$ for which $f : \mathcal{X}_\Gamma \to \mathcal{Y}$ is a covering map.

In other words, $(\mathcal{X}_\Gamma, f)$ is a covering of $(\mathcal{Y}, y_0)$ where $f : \mathcal{X}_\Gamma \to \mathcal{Y}$ is the projection map, that is, the map sending each class of equivalent paths to their common endpoint. Details of this construction can be found in [9].

Conversely, let $(\mathcal{X}, f)$ be a covering of $(\mathcal{Y}, y_0)$. Consider the monodromy action of $\pi(\mathcal{Y}, y_0)$ on the fibre $f^{-1}(y_0)$. Fix $x_0 \in f^{-1}(y_0)$ and let Γ be the stabilizer of x_0 under this monodromy action. Note that another choice of $x_0^{'}$ in $f^{-1}(y_0)$ gives a conjugate of the group Γ in $\pi(\mathcal{Y}, y_0)$. Now the right cosets of Γ are in bijection with $f^{-1}(y_0)$. Indeed, for $[\gamma_1]$ and $[\gamma_2]$ in $\pi(\mathcal{Y}, y_0)$ the cosets $\Gamma[\gamma_1]$ and $\Gamma[\gamma_2]$ coincide if and only if $\gamma_1\gamma_2^{-1} \in \Gamma$ if and only if both γ_1 and γ_2 send the element x_0 to the same element $x \in f^{-1}(y_0)$. Therefore the index of Γ in $\pi(\mathcal{Y}, y_0)$ equals to the number of elements of $f^{-1}(y_0)$. Moreover, Γ is isomorphic to the fundamental group of $\mathcal{X}$ with base point x_0

since it consists of loops in $\pi(\mathcal{Y}, y_0)$ whose lifting to $\mathcal{X}$ starting at x_0 and return back to x_0.

We sum up this subsection with the following diagram

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{monodromy representation}\\ \text{up to conjugation}\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\text{Bely\u{i} pairs}\\ \text{up to equivalence}\end{array}\right\} \\
\updownarrow & & \updownarrow \\
\left\{\begin{array}{c}\text{finite index subgroups of } \mathbb{F}_2\\ \text{up to conjugation}\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\text{unramified coverings of } \widehat{\mathbb{C}} \setminus \{0,1,\infty\}\\ \text{up to equivalence}\end{array}\right\}
\end{array}$$

2.2 Ribbon structure associated to Dessins d'Enfants

Now we will define Grothendieck's dessins d'enfants and ribbon structure carried by them. This structure enables us to speak of the notion *"next-turn-edge"* and to draw dessins on a piece of paper whether it is embedded on sphere or a surface of genus greater than 0. Let us begin with the definitions of these two notions.

Definition 2.3. A **dessin d'enfant (or a dessin)** is a pair $(\mathcal{X}, \mathcal{G})$ of bipartite connected graph $\mathcal{G}$ embedded into an orientable closed topological surface $\mathcal{X}$ such that $\mathcal{X} \setminus \mathcal{G}$ is a disjoint union of open cells. We say that two dessins $(\mathcal{X}_1, \mathcal{G}_1)$ and $(\mathcal{X}_2, \mathcal{G}_2)$ are **equivalent** if there is an orientation-preserving homeomorphism $f : \mathcal{X}_1 \to \mathcal{X}_2$ such that $f(\mathcal{G}_1) = \mathcal{G}_2$ is a graph isomorphism. The genus of a dessin $(\mathcal{X}, \mathcal{G})$ is the *genus* of the topological surface $\mathcal{X}$ by definition.

Example 2.3. Consider graph below embedded into the sphere at the left and to torus on the right. The first one is a dessin whereas the graph drawn on torus is not a dessin since cutting along edges of the graph does not produce disjoint union of open cells.

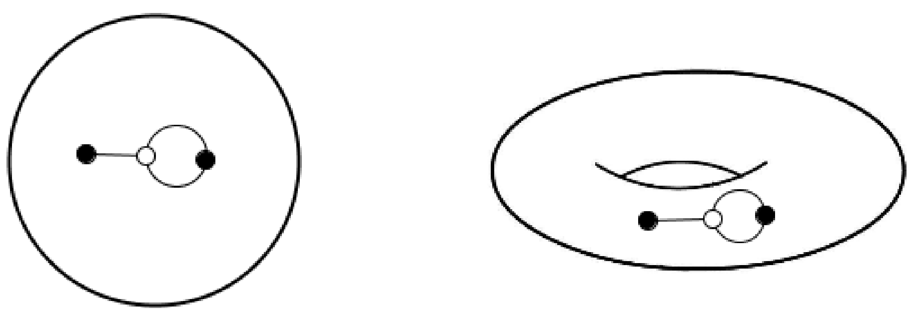


Figure 5: a dessin d'enfant on sphere

Definition 2.4. A **ribbon graph** (G, O) is a connected graph G together with a **ribbon structure** $O = \{\sigma_v | v \text{ is a vertex of } G\}$ which assigns to each vertex v of G a cyclic permutation σ_v of the half edges incident to v. We say that two ribbon graphs (G_1, O_1) and (G_2, O_2) are equivalent if there exists a graph isomorphism $f : G_1 \to G_2$ such that the pull-back of O_2 is equal to O_1.

Example 2.4. The following figure shows a graph G with different ribbon structures $O_1 = \{(132), (456), (14), (25), (36)\}$ and $O_2 = \{(132), (465), (14), (25), (36)\}$. **(Figure 6)**

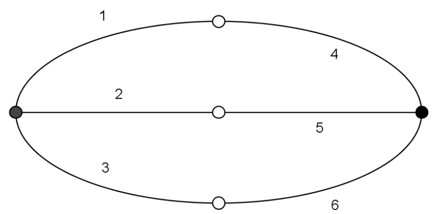
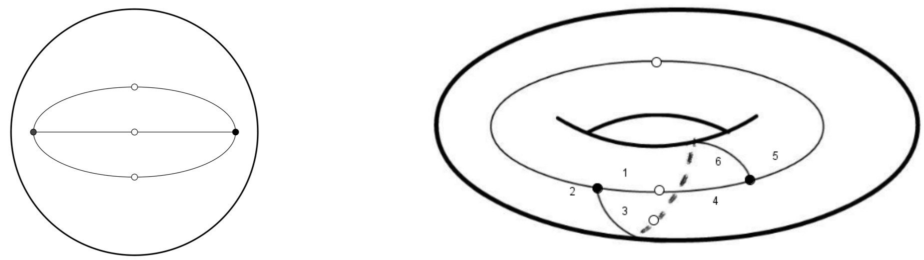


Figure 6: Two different ribbon graphs

From a dessin $\mathcal{G}(\beta)$ to a bipartite ribbon graph (G, O)

We first note that the abstract graph $\mathcal{G}$ itself does not uniquely determine the dessin, in other words, two different dessins may have two isomorphic abstract graphs such as the graphs Example 2.5. So we must focus on the way we embed the graph $\mathcal{G}$ into the surface. To do this, it suffices to assign to each vertex a cyclic permutation of edges which are incident to this vertex. Indeed, suppose that $\mathcal{G}$ has n edges. first we enumerate the n edges of the graph. For each vertex v, we take sufficiently small neighbourhood U of v so that we get a chart (U, ψ) such that $\psi(U \cap \mathcal{G}) \subseteq \mathbb{R}^2$ is a star centred at $\psi(v)$. In this small vicinity of $\psi(v)$ we encircle counter-clockwise order the vertex and hence we get a permutation σ_v which is in S_n. The set of all these permutations is the ribbon structure O of the resulting bipartite ribbon graph (G, O). We note that this description of dessins will enable us to think of a Belyĭ pair in a purely combinatorial way.

Example 2.5. The construction above applied to the ribbon graphs given in Example 2.5 produces two dessins. The first ribbon structure determines the dessin on the sphere and that of second ribbon graph is a dessin on torus. Evidently, these are different dessins since these are dessins of different genera, hence they are not equivalent.



(a) dessin corresponding to (G, O_1) on sphere (b) dessin corresponding to (G, O_2) on torus

Figure 7: Two dessins corresponding to ribbon graphs given in Example 2.5

From a bipartite ribbon graph (G, O) to dessin $\mathcal{G}(\beta)$

For the converse construction we first observe that the ribbon structure of a *bipartite* ribbon graph consists exactly of the permutations around white point and the permutations around black points. This leads us to the following definition:

Definition 2.5. A sequence $[g_1, g_2, g_3]$ where $g_i \in S_d$ is called a **3-constellation** if

i. The group $G = \langle g_1, g_2, g_3 \rangle$ acts transitively on the set of d letters.

ii. The permutations g_1, g_2 and g_3 satisfy the relation: $g_1 g_2 g_3 = id$.

Now, let (G, O) be a bipartite ribbon graph. Let $\sigma_\circ$ be the product of all $\sigma_v^{'}s$ in O where v is white vertex and similarly let $\sigma_\bullet$ be the product of all $\sigma_v^{'}s$ in O where v is black vertex and we let $\sigma_\infty = \sigma_\bullet^{-1}\sigma_\circ^{-1}$. Then $[\sigma_\circ, \sigma_\bullet, \sigma_\infty]$ is a 3-constellation. Conversely, if we are given a 3-constellation $[\alpha, \sigma, \varphi]$, it is possible to construct a corresponding topological surface as follows:

1. First, for each cycle of φ^{-1} of length m we take a polygon with m sides and assign the letters of this cycle to each side in the counter-clockwise direction.

2. Considering the permutation α, glue the sides of these polygons in such a way that the orientation of the sides glued together is always opposite so that the resulting surface will be oriented. The cycles of σ at each vertex will be automatically glued.

So if we are given a bipartite ribbon graph then we have indeed a 3-constellation and we may construct a topological surface as above. The dessin G is already embedded into the surface $\mathcal{X}$ as $\mathcal{X} \setminus G$ consists of disjoint open cells. If C is one of these cells, considering the ribbon structure of the dessin, one observes that the cell C is encircled by the edges $i, \sigma_\circ\sigma_\bullet(i), (\sigma_\circ\sigma_\bullet)^2(i), \ldots, (\sigma_\circ\sigma_\bullet)^m(i) = i$. This shows that the number of edges bounding C is produced by an m-cycle of the permutation $\sigma_\circ\sigma_\bullet$. Therefore, these two procedures are inverse to each other.

We conclude these equivalences as:

$$\left\{\begin{array}{c}\text{dessins d'enfants with n edges}\\ \text{up to equivalence}\end{array}\right\} \rightarrow \left\{\begin{array}{c}\text{bipartite ribbon graphs}\\ \text{up to equivalence}\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{bipartite ribbon graphs}\\ \text{up to equivalence}\end{array}\right\} \downarrow \left\{\begin{array}{c}\text{ordered pairs of permutations}\\ \text{generating transitive subgroups of } S_n\\ \text{up to conjugation}\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{ordered pairs of permutations}\\ \text{generating transitive subgroups of } S_n\\ \text{up to conjugation}\end{array}\right\} \rightarrow \left\{\begin{array}{c}\text{dessins d'enfants with n edges}\\ \text{up to equivalence}\end{array}\right\}$$

From these constructions we can calculate the genus of the dessin $(\mathcal{X}, \mathcal{G})$ only considering the corresponding permutation pair $(\sigma_0, \sigma_\bullet)$ on N letters by the formula: If g is the genus of $\mathcal{X}$ then

$$\begin{aligned} 2 - 2g &= \big(\text{number of cycles of } \sigma_\circ + \text{number of cycles of } \sigma_\bullet\big) \\ &- N \\ &+ \text{number of cycles of } \sigma_\circ\sigma_\bullet \end{aligned}$$

We conclude this section as follows. A dessin d'enfant has immediately a corresponding ribbon structure and vice versa. As this section suggests, we will regard dessins of n edges as a pair of two permutations $(\sigma_0, \sigma_\bullet)$ transitive on n letters which is actually a 3-constellation attached to the corresponding ribbon structure. Moreover, as the next section suggest dessins being the lift of arc $\mathcal{I} = [0, 1] \subseteq \widehat{\mathbb{C}}$ can be represented as a pair of permutation (σ_0, σ_1) called the *permutational representation* of the dessin given.

2.3 Belyĭ Pairs and Dessins d'Enfants

2.3.1 From the Belyĭ pair $(\mathcal{X}, \beta)$ to a dessin $\mathcal{G}(\beta)$:

Let $(\mathcal{X}, \beta)$ be a Belyĭ pair. We will observe that $(\mathcal{X}, \beta)$ naturally determines a dessin $\mathcal{G}$ on $\mathcal{X}$. Let $\mathfrak{T}_0$ be a triangulation of $\widehat{\mathbb{C}}$ performed by three vertices 0, 1 and ∞; three edges along the line segments $[0, 1]$, $[1, \infty]$ and $[0, \infty]$. Then $\beta^{-1}(\mathfrak{T}_0)$ is a triangulation $\mathfrak{T}$ of $\mathcal{X}$. Next, we delete the vertex ∞ and its two edges in $\mathfrak{T}_0$ and consider the closed real interval $\mathcal{I} = [0, 1] \subseteq \widehat{\mathbb{C}}$. Then β lifts $\mathcal{I}$ to a graph $\mathcal{G} = \beta^{-1}(\mathcal{I})$ on $\mathcal{X}$ whose vertices are the pre-images of 0 and 1. By colouring the pre-images of 0 with one color, say white; and the pre-images of 1 with another, say black, we naturally get a bipartite graph structure. Moreover, $\mathcal{X} \setminus \mathcal{G}$ is a disjoint union of connected components each of them containing precisely one pre-image of ∞. In other words, each of these components is an open cell and hence is holomorphic to the open unit disc. We also observe that equivalent Belyĭ pairs give rise to equivalent dessins.

2.3.2 From a dessin $\mathcal{G}$ to a Belyĭ pair $(\mathcal{X}, \beta)$

We have seen that a dessin $\mathcal{G}$, say with n letters, have a natural bipartite ribbon graph structure $O = \{\sigma_1, \sigma_2, ..., \sigma_s\}$. We have already determine a 3-constellation from this ribbon structure. We shall determine the monodromy representation from this ribbon structure as follows: Recall that $\sigma_\circ$ is the product of all cycles around each white vertex and $\sigma_\bullet$ is the product of all cycles around each white vertex. Let y be a base point of the $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. As the generators of the group $\pi(\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}) \simeq \mathbb{F}_2$, we take two loops γ_0, the loop circling around 0 and γ_1, the loop circling around 1 and both starts at y and both are oriented counter-clockwise, see the figure below. We let $\Phi(\gamma_0) := \sigma_\circ$ and $\Phi(\gamma_1) := \sigma_\bullet$. Now,

$$\Phi : \mathbb{F}_2 \longrightarrow S_n$$

defined in this way is a monodromy map hence defines a Belyĭ pair as constructed in Section 2.2.

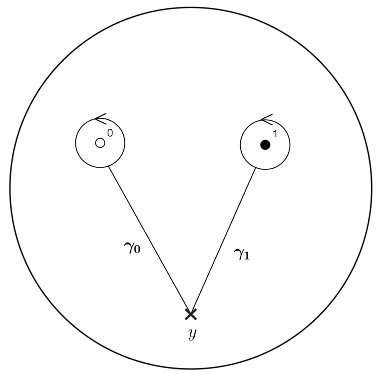


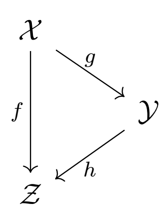
Figure 8: generators of thrice punctured sphere

2.3.3 Ritt's Theorem

An advantage of dealing with the monodromy group of a covering follows from the Ritt's theorem. Before stating this theorem we recall some definitions from basic group theory.

Definition 2.6. Let G be a permutation group on n letters. If the underlying set may split into disjoint subsets of equal size different from 1 and n which are called *blocks* , such that for any $g \in G$ the image of a block is always a block, then the group G is called **imprimitive**. Otherwise, it is **primitive**.

Definition 2.7. Let $f : \mathcal{X} \to \mathcal{Z}$ be a covering. A **subcovering** of $(\mathcal{X}, f)$ is covering $(\mathcal{Y}, g)$ of $\mathcal{Z}$ such that $f = g \circ h$. i.e. if the following diagram is commutative:



We say that $f : \mathcal{X} \to \mathcal{Z}$ is **decomposable** if it has a subcovering. Otherwise, it is called **indecomposable**.

Example 2.6. The following covering is decomposable where $\beta_{m,n}$ defined as in example 1.3

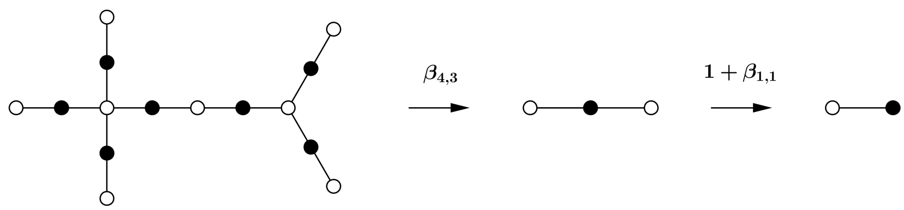


Figure 9: a decomposable covering

Theorem 2.2. ***(Ritt's Theorem)*** *A covering is decomposable if and only if its monodromy group is imprimitive.*

Proof. See [38] □

2.3.4 Belyĭ functions in genus 0 case

Suppose that $\mathcal{G}$ is a dessin on the sphere $\widehat{\mathbb{C}}$. We want to find a Belyĭ morphism that realizes the covering associated to that dessin. Recall that the morphisms on the sphere are exactly rational functions so we want to find a rational function $\frac{p(x)}{q(x)}$ which the given dessin $\mathcal{G}$ is associated to. Considering the white vertices on the dessin, one can figure out the ramification indices for 0 which are equal to the number of edges at each white point. This shows that p is of the form $p(X) = a\prod_i (X-a_i)^{r_i}$ where $a \neq 0$. By a similar observation for ∞, one can see that $q(X) = b\prod_j (X-b_j)^{s_j}$, $b \neq 0$. By considering the black vertices one finds the ramification indices and sees that $p - q(x) = c\prod_k (X-c_k)^{t_k}$. Solving the equation

$$a\prod_i (X-a_i)^{r_i} - b\prod_j (X-b_j)^{s_j} = c\prod_k (X-c_k)^{t_k}$$

in terms of a, b, c, a_i, b_j, c_k one can find out the resulting Belyĭ morphism.

3 Compact Riemann Surfaces and Algebraic Curves

In this section we shall show the categorical equivalences of the following class of objects:

1. Compact Riemann surfaces,
2. Algebraic function fields of transcendence degree 1,
3. Complex algebraic nonsingular curves

3.1 Complex Algebraic Curves and Riemann Surfaces

In this section we will show that every algebraic curve determines a compact Riemann surface . The crucial point in doing this is the use of the following Implicit Function Theorem. Before giving the details of this construction, we shall explain the idea from a local viewpoint. Suppose first that g is a holomorphic function defined on an open connected subset V of $\mathbb{C}$. The graph of $g : V \to \mathbb{C}$ is the set

$$Gr(g) = \{(x, g(x)) \in \mathbb{C}^2 \mid x \in V\} \subseteq \mathbb{C}^2$$

We give $Gr(g)$ the subspace topology so that the projection map $\pi : Gr(g) \to V$ is a homeomorphism, whose inverse sends the point x to $(x, g(x))$. In fact, by Proposition 2.1. π is a *complex* coordinate chart on $Gr(g)$, whose domain covers all of $Gr(g)$. Consisting only of this single chart, we see that $Gr(g)$ is equipped with a Riemann surface structure in this way. The converse procedure of this construction is stated as:

3.1. ***(The Implicit Function Theorem)*** *Let f be holomorphic function of two variables on the rectangle $R = \{(x, y) \in \mathbb{C}^2 \mid |x - a| < r_1$ and $|y - b| < r_2\}$ and assume that*

$$f(a, b) = 0 \quad \textit{and} \quad \frac{\partial f}{\partial y}(a, b) \neq 0.$$

Then there exist $\epsilon > 0$ and $\delta > 0$ such that for all $x \in D_\epsilon(a) = \{z \in \mathbb{C} \mid |z - a| < \epsilon\}$ there is a unique solution $y(x)$ of the equation $f(x, y) = 0$ with $|y - b| < \delta$ and $y(a) = b$. The function defined by $x \mapsto y(x)$ is holomorphic on $D_\epsilon(a)$

Proof. We refer [20] for a proof. □

In order to define complex charts on the vanishing set of a polynomial, this theorem enables us to determine local complex structure on this set where the complex charts are locally the graphs of holomorphic functions. Throughout this section let

$$\begin{aligned} F(X,Y) &= a_0(X)Y^n + a_1(X)Y^{n-1} + \ldots + a_n(X) \\ &= b_0(Y)X^m + b_1(Y)X^{m-1} + \ldots + b_m(Y) \end{aligned}$$

be an irreducible polynomial. Consider $F(X,Y)$ as a function in two variables and let $a \in \mathbb{C}$ be such that $a_0(a) \neq 0$ and there is no $b \in \mathbb{C}$ with $F(a,b) = 0 = \dfrac{\partial F}{\partial y}(a,b)$. Then the polynomial $F(a,Y)$ has exactly n roots $b_1, b_2, \ldots, b_n$ because $\dfrac{\partial F}{\partial y}(a,b) \neq 0$. Now, there is a holomorphic function $y_i(x)$ with $y_i(a) = b_i$ and $F(x, y_i(x)) \equiv 0$ defined on a sufficiently small neighbourhood of a. In other words, $y_1(x), y_2(x), \ldots, y_n(x)$ are roots of $F(x,Y)$ when considered as a polynomial in one variable Y thus a has a neighbourhood which is covered by n pre-image sets. We recollect this as:

3.2. *Let $F(X,Y) = a_0(X)Y^n + a_1(X)Y^{n-1} + \ldots + a_n(X)$ be an irreducible polynomial in two variables with complex coefficients. Let $a \in \mathbb{C}$ be such that $a_0(a) \neq 0$ and such that there is no $b \in \mathbb{C}$ with $F(a,b) = 0 = \dfrac{\partial F}{\partial y}(a,b)$.*

Then there is $\epsilon > 0$ and n holomorphic functions $y_1(x), y_2(x), \ldots y_n(x)$ defined on the disc $D_\epsilon(a)$ satisfying

i. $F(x, y_i(x)) \equiv 0$ *for* $x \in D_\epsilon(a)$, $i = 1, 2, \ldots, n$.

ii. $y_i(x) \neq y_j(x')$ *if* $i \neq j$ *and* $x, x' \in D_\epsilon(a)$

iii. *If* $\eta \in \mathbb{C}$ *and* $F(x,\eta) = 0$ *for some* $x \in D_\epsilon(a)$ *then* $\eta = y_i(x)$ *for a unique* i.

Let $S_F^X = \{(x,y) \in \mathbb{C}^2 \mid F(x,y) = 0, \dfrac{\partial F}{\partial y}(x,y) \neq 0, a_0(x) \neq 0\}$, that is, the vanishing set of F after removing possible singular points with respect to the y-coordinate. Let $P = (x_0, y_0)$ be a point in S_F^X so that $\dfrac{\partial F}{\partial y}(x_0, y_0)) \neq 0$. Now, we can solve y in terms of x. In other words, there is a holomorphic function $y(x)$ defined on a neighbourhood $V \subseteq \mathbb{C}$ of x_0 such that S_F^X is locally equal to the set $\{(x, y(x)) \in \mathbb{C}^2 \mid x \in V\}$. Hence, we have determined the coordinate charts on S_F^X, which are locally a graph of a holomorphic function. It is straightforward to check that these charts are compatible.

Similarly, one defines $S_F^Y = \{(x,y) \in \mathbb{C}^2 \mid F(x,y) = 0, \dfrac{\partial F}{\partial x}(x,y) \neq 0, b_0(y) \neq 0\}$ and the complex structure on it. Note that both spaces S_F^X and S_F^Y are Hausdorff and second countable being as a subset of $\mathbb{C}^2$. As a conclusion, we have

3.3. *S_F^X and S_F^Y are Riemann surfaces on which the coordinate functions π_X and π_Y where $\pi_X : S_F^X \to \widehat{\mathbb{C}}$ is defined by $(x,y) \mapsto x$ are holomorphic, where π_Y is defined similarly.*

3.4. *The function $\pi_X : S_F^X \to \pi_X(S_F^X) \subseteq \widehat{\mathbb{C}}$ is a covering map with degree equal to $deg_Y F = n$ and $\pi_Y : S_F^Y \to \pi_Y(S_F^Y) \subseteq \widehat{\mathbb{C}}$ is a covering map with degree equal to $deg_X F = m$.*

Let $F(X,Y)$ and $G(X,Y)$ be two polynomials in $\mathbb{C}[X,Y]$. a weak version of Bezout's Theorem says that: If F and G are relatively prime then the curves $F(X,Y) = 0$ and $G(X,Y) = 0$ intersect only at finitely many points, whose coordinates are in $\mathbb{C}$. Another basic fact concerning curves is the following weak version of Nullstellensatz: If F is irreducible and G vanishes at all points of the curve $F(X,Y) = 0$ then F divides G. These are standard facts and the proofs can be found in [12].

3.5. *Both S_F^X and S_F^Y are connected.*

Let us see that S_F^X is connected. Note that the polynomials F and F_Y have only finitely many common zeros by Bezout's theorem. So $\widehat{\mathbb{C}} \setminus \pi_X(S_F^X)$ is a finite set. Set $\widehat{\mathbb{C}} \setminus \pi_X(S_F^X) = \{a_1, a_2, \ldots, a_r, \infty\}$. Let W be a connected component of S_F^X. We will show that $W = S_F^X$ and hence, that S_F^X is connected. Clearly, the restriction $\pi_X : W \to \widehat{\mathbb{C}} \setminus \{a_1, a_2, \ldots, a_r, \infty\}$ is a covering map with degree $d \leq n$. Moreover, by Proposition 1.1 (VII) given in the first section there is a unique morphism of compact Riemann surface $\pi_X : \widehat{W} \to \widehat{\mathbb{C}}$. For $x \in \widehat{\mathbb{C}} \setminus \{a_1, a_2, \ldots, a_r, \infty\}$ there are d holomorphic functions $y_1(x), y_2(x), \ldots, y_d(x)$ such that $(x, y_1(x)), (x, y_2(x)), \ldots, (x, y_d(x))$ are the pre-images of x under the first coordinate function π_X. Now consider the symmetric functions

$$\begin{aligned} s_1(x) &= \textstyle\sum_i y_i(x) \\ s_2(x) &= \textstyle\sum_{i,j} y_i(x) y_j(x) \\ &\vdots \\ s_d(x) &= \textstyle\prod_i y_i(x) \end{aligned}$$

Note that $s_1(x), s_2(x), \ldots, s_d(x)$ are holomorphic functions defined on the whole $\widehat{\mathbb{C}} \setminus \{a_1, a_2, \ldots, a_r, \infty\}$ We will see that each function $s_i(x)$ extends to a meromorphic function defined in the whole $\widehat{\mathbb{C}}$. Indeed, in a small neighbourhood of a_k the roots $y_k(x)$ are bounded in terms of coefficients of the polynomial $F(x, Y)$ by the lemma below. Then $1/y_k(x)$ is bounded near ∞. Therefore each of the holomorphic functions $s_i(x)$ extends to meromorphic functions defined on the whole $\widehat{\mathbb{C}}$. Since $\mathcal{M}(\widehat{\mathbb{C}}) = \mathbb{C}(x)$, it follows that each $s_i(x)$ can be identified to a rational function. Let $s(X)$ be the least common multiple of the denominators of these rational functions and define

$$G(X,Y) = s(X)[Y^d - s_1(X)Y^{d-1} + s_2(X)Y^{d-2} - \ldots + (-1)^{d-1}s_d(X)]$$

Let $P = (x, y_j(x)) \in W$ be a point. Then

$$\begin{aligned} G(P) &= s(x)[y_j^d(x) - s_1 y_j^{d-1}(x) + \ldots + (-1)^{d-1}s_d(x)] \\ &= s(x)\prod_{i=1}^{d}(y_j(x) - y_i(x)) \\ &= 0 \end{aligned}$$

This shows that $G(X, Y)$ vanishes at all points of W. Clearly, the irreducible polynomial $F(X, Y)$ also vanishes at all points of W then by Nullstellensatz, F divides G. In particular, $d = deg_Y G \geq deg_Y F = n$. It follows that $d = n$. We conclude that $F = G$ hence $W = S_F^X$, which shows S_F^X is connected. Similarly, S_F^Y is connected as well.

3.6. *There exists a unique connected compact Riemann surface $S = S_F$ that contains S_F^X and S_F^Y and the coordinate functions π_X and π_Y extend to meromorphic functions on S.*

Consider the holomorphic unramified covering $\pi_X : S_F^X \to \widehat{\mathbb{C}} \setminus \{a_1, a_2, \ldots, a_r, \infty\}$ of degree n. Then there exists a unique compact Riemann surface S_F and a unique morphism $\pi_X : S_F \to \widehat{\mathbb{C}}$. Similarly, one can extend S_F^Y to a compact Riemann surface uniquely. Since S_F^X and S_F^Y differ for only finitely many points, the compact Riemann surface obtained in these two constructions are isomorphic by Proposition 1.1 (VII). This shows **3.6.** By the construction above we also have:

3.7. *The set of ramification points π_X is equal to $\{(x,y) \in S | F_Y(x,y) = 0\}$ and that of π_Y $\{(x,y) \in S | F_X(x,y) = 0\}$.*

Lemma 3.8. *If* α *is a root of the polynomial* $p(X) = X^n + c_1 X^{n-1} + \ldots + c_n \in \mathbb{C}[X]$ *then*

$$|\alpha| < 2.\boldsymbol{max}\{|c_i|^{1/i},\ i = 1, \ldots, n\}$$

Proof. Let $c = 2.\boldsymbol{max}\{|c_i|^{1/i},\ i = 1, \ldots, n\}$. If $y = \alpha/c$ then $y^n + \dfrac{c_1}{c} y + \ldots + \dfrac{c_n}{c} = 0$. Since $|c_i| < c^i$ it follows that $|y|^n \leq |y|^{n-1} + \ldots + 1$.

Now if $|y| \geq 2$ then

$$1 \leq \frac{1}{|y|} + \ldots \frac{1}{|y|^n} \leq \frac{1}{2} + \ldots + \frac{1}{2^n} < 1$$

a contradiction. So $|y| < 2$ thus, $|\alpha| < 2c$. □

In conclusion, we have shown that any complex algebraic curve give rise to a compact Riemann surface. In the following section we will see that any compact Riemann surface arises in this way.

3.2 Compact Riemann Surfaces and Algebraic Function Fields

An **algebraic function field** of one variable over $\mathbb{C}$ is a field extension K of transcendence degree one over $\mathbb{C}$. In other words, K is a finite extension of $\mathbb{C}(x)$ where x is an element of K such that x is transcendental over $\mathbb{C}$. Let S be a compact Riemann surface. We will prove that $\mathcal{M}(S)$, the field of meromorphic functions on S is an algebraic function field of one variable. Firstly, we notice that each non-constant meromorphic function f on $\mathcal{X}$ is transcendental over $\mathbb{C}$. To see this, suppose on the contrary that $f \in \mathcal{M}(S) \setminus \mathbb{C}$ satisfies a polynomial $G(T) = T^n + a_1 T^{n-1} + \ldots + a_{n-1} T + a_0$ with coefficients in $\mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, $G(T)$ factors as $G(T) = (T - b_1)(T - b_2) \ldots (T - b_n) = 0$, and it follows that $f \equiv b_i$ for some i, which is not possible. Therefore, $\mathbb{C}(f)$ is a transcendental extension over $\mathbb{C}$. For a general account of algebraic function fields see [15].

3.9. *Suppose that* $f \in \mathcal{M}(S)$ *has degree* n*. Let* $h \in \mathcal{M}(S)$*. Then* (f, h) *satisfies a polynomial of degree* $\leq n$*. In particular,* $[\mathbb{C}(f, h) : \mathbb{C}(f)] \leq n$*.*

Proof. Define S_0 to be the set of points of S at which f is nonsingular. Note that $S \setminus S_0$ is finite as S is compact. Let $p \in S_0$, let U be a small neighbourhood of p and let $\zeta_0 = f(p)$. Then there are n neighbourhoods $U_1 = U, U_2, \ldots, U_n$ of $p_1 = p, p_2, \ldots, p_n$

respectively, such that $f_{|U_i}$ is holomorphic and locally, there are n inverse functions $y_i : f(U_i) \to U_i$ such that $q = y_i(\zeta)$ if and only if $\zeta = f(q)$. Now, consider the symmetric functions

$$\begin{aligned} g_1(\zeta) &= \sum h(y_i(\zeta)) \\ g_2(\zeta) &= \sum h(y_i(\zeta))h(y_j(\zeta)) \\ &\vdots \\ g_n(\zeta) &= \prod h(y_i(\zeta)) \end{aligned}$$

Since each y_i is locally holomorphic and h is meromorphic, it follows that each g_i is locally meromorphic. As in the proof of **3.5.** each $g_i(\zeta)$ is a meromorphic function on $\widehat{\mathbb{C}}$ hence, is a rational function of ζ. Therefore for each i,

$$g_i(\zeta) = r_i(\zeta)/s_i(\zeta)$$

where r_i and s_i are polynomial with complex coefficients. Composing with f we have $(g_i \circ f) = (r_i \circ f)/(s_i \circ f) : S \to \widehat{\mathbb{C}}$

Now consider

$$p(f) = (h - h(y_1(f)))(h - h(y_2(f)))\ldots(h - h(y_n(f)))$$

At each point of S_0, $p(f)$ equals zero. On the other hand, it is easy to see that

$$p(f) = \sum (-1)^k g_k(f)h$$

Since this vanishes on $\mathcal{X}_0$ and the functions h and each g_i are meromorphic on S, $p(f) \equiv 0$.

Hence h satisfies the polynomial,

$$G(H) = H^n + H^{n-1}r_1(f)/s_1(f) + \ldots + Hr_{n-1}(f)/s_{n-1}(f) + r_n(f)/s_n(f)$$

By multiplying $G(H)$ by the least common multiple of $s_1, s_2, \ldots, s_n$ we obtain a new polynomial $F(X,Y)$ of degree n in Y for which $F(f,h) = 0$ □

3.10. *Let $\mathcal{X}$ and f be as in the theorem. Then $[\mathcal{M}(S) : \mathbb{C}(f)] \leq n$.*

Proof. Suppose that $\mathcal{M}(S)$ is a finite field extension of $\mathbb{C}(f)$ with $[\mathcal{M}(S) : \mathbb{C}(f)] > n$. Since the characteristic of the field $\mathbb{C}(f)$ is 0, by primitive element theorem $\mathcal{M}(S)$ is a primitive extension of $\mathbb{C}(f)$, that is $\mathcal{M}(S) = \mathbb{C}(f)(g) = \mathbb{C}(f, g)$ for some $g \in \mathcal{M}(S)$. But by the previous theorem g satisfies a polynomial of degree n and this contradicts that $[\mathcal{M}(S) : \mathbb{C}(f)] > n$.

Now let us assume that $\mathcal{M}(S)$ is an infinite field extension of $\mathbb{C}(f)$. Since $\mathcal{M}(S)$ is algebraic over $C(f)$, there are infinitely many field extensions K such that $\mathbb{C}(f) < K < \mathcal{M}(S)$ and $[K : \mathbb{C}(f)] > n$, which is not possible as in the previous argument. □

Now, we state a corollary of the well-known Riemann-Roch theorem (we refer [15]) which is known as the *separation property of the field of meromorphic functions.*

Theorem 3.11. *Let S be a compact Riemann surface and P_1, P_2 be distinct points of S. Then there exists a meromorphic function $\varphi \in \mathcal{M}(S)$ such that $\varphi(P_1) = 0$ and $\varphi(P_2) = \infty$.*

3.12. *Let S be a compact Riemann surface for which $\mathcal{M}(S) = \mathbb{C}(f, h)$ and let S_F be the compact Riemann surface corresponding to $F(X, Y)$, where $F(X, Y)$ is an irreducible polynomial such that $F(f, h) = 0$. Then the map $\Psi : S \to S_F$ defined by $\Psi(P) = (f(P), h(P))$ is an analytic isomorphism.*

Proof. First let us see that Ψ is well defined. As before, we consider $\pi_X : S_F^X \to \pi_X(S_F^X) \subseteq \widehat{\mathbb{C}} \setminus B$ where $B = \{a_1, a_2, \ldots, a_r, \infty\}$. Set $S^* = S \setminus f^{-1}(B)$. The following diagram is commutative:

$$\begin{array}{ccc} S^* & \xrightarrow{\Psi} & S_F^X \\ & {}_{f}\searrow & \downarrow{\scriptstyle \pi_X} \\ & & \widehat{\mathbb{C}} \setminus B \end{array}$$

Note that $f(P) = a \in \widehat{\mathbb{C}} \setminus B$ then $h(P)$ is equal to one of the n distinct roots of $F(a, Y)$. Therefore $\Psi(P)$ is well-defined for $P \in S^*$.

Since both π_X and f are covering maps, $\Psi : S^* \to S_F^X$ is also a covering. Thus, we can extend Ψ to whole S. Notice that up to now we have not used the fact that f and h generate $\mathcal{M}(S)$, therefore Ψ always defines a morphism for every pair of functions f and h such that $F(f, h) = 0$.

We shall prove that Ψ is indeed an isomorphism by showing that its degree equals 1 under the assumption that $\mathcal{M}(S) = \mathbb{C}(f, h)$. Suppose not. Then the fibres $\Psi^{-1}(a, b)$ of all but finitely many points $(a, b) \in S_F^X$ would contain at least two distinct points, say Q_1 and Q_2. Let φ be an arbitrary meromorphic function on S. Since $\mathcal{M}(S) = \mathbb{C}(f, h)$, it follows that φ can be expressed as

$$\varphi = \frac{\sum a_{ij} f^i h^j}{\sum b_{ij} f^i h^j}$$

so that

$$\varphi(Q_1) = \frac{\sum a_{ij} a^i b^j}{\sum b_{ij} a^i b^j} = \varphi(Q_2)$$

Since φ is arbitrarily chosen, this contradicts the separation property of $\mathcal{M}(S)$. □

We conclude

3.13. *Let (F) denote the ideal of $\mathbb{C}[X, Y]$ generated by F. Then*

(i) the map

$$\begin{array}{rcl} \mathbb{C}[X,Y]/(F) & \longrightarrow & \mathcal{M}(S_F) \\ X & \longrightarrow & f \\ Y & \longrightarrow & h \end{array}$$

is a $\mathbb{C}$-algebra isomorphism.

(ii) the map

$$\begin{array}{rcl} \mathbb{C}[X,Y]/(F) & \longrightarrow & \mathcal{M}(S) \\ X & \longrightarrow & \pi_X \\ Y & \longrightarrow & \pi_Y \end{array}$$

is also $\mathbb{C}$-algebra isomorphism. In particular, $\mathcal{M}(S_F) = \mathbb{C}(\pi_X, \pi_Y)$

(iii) $F(\pi_X, Y)$ is the minimal polynomial of π_Y over $\mathbb{C}(\pi_X)$

(iv) deg$(f) = [\mathcal{M}(S) : \mathbb{C}(f)]$

Proof. Since $F(f,h) \equiv 0$ as in the proof of the first theorem this correspondence defines a homomorphism of $\mathbb{C}$-algebras. Let $G(X,Y)$ be in the kernel so that $G(f,h) \equiv 0$. But this means as before that $G(X,Y)$ vanishes on the curve $F(x,y) = 0$. By weak Nullstellensatz ,it follows that F divides G, that is, $G \in (F)$. Therefore this correspondence is indeed an isomorphism of $\mathbb{C}$-algebras. This proves (i) and, (ii) follows directly from the previous theorem. Now (iii) is obvious. To see (iv) note that $[\mathcal{M}(S) : \mathbb{C}(f)]$ equals the degree of the minimal polynomial h over $\mathbb{C}(f)$. Since the latter equal to $deg_Y(F)$ which is the degree of the function π_X, again by the previous theorem, $deg(f) = \pi_X$ we see that $deg(f) = [\mathcal{M}(S) : \mathbb{C}(f)]$.

□

3.3 Summary

We will summarize these equivalences of the these categories in this section. Given an irreducible polynomial $F(X,Y)$ one defines complex charts by solving y in terms of x thanks to the Implicit function theorem. This gives a compact Riemann surface. Then given a compact Riemann surface $\mathcal{X}$, passing to its function field namely $\mathcal{M}(\mathcal{X})$, we get an algebraic field extension of transcendence degree 1. It is an algebraic field extension of $\mathbb{C}(f)$ where f is a meromorphic function on $\mathcal{X}$. By choosing a pair of generators f and h we obtained an irreducible polynomial $F(X,y)$ satisfying $F(f,h) \equiv 0$ we get $\mathcal{M}(\mathcal{X}) = \mathbb{C}(f,h)$. So we have $S \simeq S_F$ if and only if $\mathcal{M}(S)$ has generators f and h such that $F(f,h) \equiv 0$.

4 Belyĭ's Theorem

4.1 Belyĭ Surfaces

In section 3, we have seen that any compact Riemann surface arises from an irreducible polynomial in $\mathbb{C}[X, Y]$, in other words, any compact Riemann surface is already defined on the field of complex numbers $\mathbb{C}$. Recall that $S \simeq S_F$ if and only if $\mathcal{M}(S)$ has generators f and h such that $F(f, h) \equiv 0$.

In 1978, Belyĭ proved that a compact Riemann surface S is defined over a number field, i.e. a finite extension of $\mathbb{Q}$ if and only if S admits a morphism with at most three ramification values lying in the set $\{0, 1, \infty\}$. This result lead people deal with the Belyĭ pairs and the corresponding dessins. The surprising part of the Belyĭ theorem is the *only if* direction. In [2], Belyĭ gives a simple algorithm to calculate the morphism β. For the *if* part of the proof, Belyĭ refers to a general a result of A.Weil which is known as *Weil's criterion.* This part of the theorem was already known by Grothendieck, see [31]. Later on, B. Köck clarified his proof in the language of algebraic geometry in [33] and J. Wolfart using uniformisation theory in [36].

Definition 4.1. A compact Riemann surface S is **defined over a field** K where $K \subseteq \mathbb{C}$ if $S \simeq S_F$ for some irreducible polynomial $F(X, Y) = \sum a_{ij} X^i Y^j \in K[X, Y]$. The smallest field that S is defined over is called **field of definition** of S.

Theorem 4.1. *(Belyĭ) Let $\mathcal{X}$ be a complex algebraic nonsingular curve. Then the following statements are equivalent:*

(i) $\mathcal{X}$ is defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers.

(ii) there exists a finite morphism $\beta : \mathcal{X} \to \widehat{\mathbb{C}}$ from $\mathcal{X}$ to the projective line $\widehat{\mathbb{C}}$ which is ramified at most over $0, 1$ and ∞.

Before mentioning the Belyĭ contribution to the proof of this theorem, recall that $\beta_{m,n} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by

$$\beta_{m,n}(z) = \frac{(m+n)^{m+n}}{m^m n^n} z^m (1-z)^n$$

where m and n are integers with m, n, $m + n \neq 0$. We have found the set ramification points as $0, 1, m/(m+n), \infty$. Note also that $\beta_{m,n}(0) = 0$, $\beta_{m,n}(1) = 0$, $\beta_{m,n}(\infty) = \infty$ and $\beta_{m,n}(\frac{m}{m+n}) = 1$. Concerning ramification values, it is easy to see the following:

$$Ram(f \circ g) = f(Ram(g)) \cup Ram(f)$$

So if a function f has ramification points $0, 1, m/(m+n), \infty$ then the composition $\beta_{m,n} \circ f$ has ramification values $0, 1, \infty$ and it sends possible remaining rational ramification values of f to rational values.

Proof. ***(only if* part)**
We aim to show that if S is defined over $\overline{\mathbb{Q}}$ then there exists a Belyĭ morphism on S with at most three ramification values $0, 1, \infty$. As a first step, we shall show that if $f : S \to \widehat{\mathbb{C}}$ is ramified over $\{0, 1, \infty, \lambda_1, \lambda_2, \ldots, \lambda_n\} \subseteq \mathbb{Q} \cup \{\infty\}$ then we can reduce this set of ramification points to the set $0, 1, \infty$ by composing f with suitable functions.

Without loss of generality, we can assume that $0 < \lambda_1 < 1$ by composing with the Mobius functions

$$T(x) = 1 - x$$
$$M(x) = \frac{1}{x}$$

So we can write $\lambda_1 = \dfrac{m}{m+n}$ where $m, n \in \mathbb{N}$. Now, composing f with $\beta_{m,n}$, we get a morphism

$$\begin{array}{rcl} S \longrightarrow \widehat{\mathbb{C}} & \longrightarrow & \widehat{\mathbb{C}} \\ 0 & \longmapsto & 0 \\ 1 & \longmapsto & 0 \\ \infty & \longmapsto & 0 \\ \lambda_1 & \longmapsto & 1 = \beta_{m,n}(\lambda_1) \\ \lambda_2 & \longmapsto & \beta_{m,n}(\lambda_2) \\ \vdots & & \vdots \\ \lambda_n & \longmapsto & \beta_{m,n}(\lambda_n) \end{array}$$

Therefore, $\beta_{m,n} \circ f : S \to \widehat{\mathbb{C}}$ has strictly less ramification points than f. Hence, in such a case we are done by induction. So it suffices to show that such an f exists and this is the next step.

Suppose that $S \simeq S_F$ where $F(X,Y) = p_0(X)Y^n + p_1(X)Y^{n-1} + \ldots + p_n(X) \in \overline{\mathbb{Q}}[X,Y]$ and consider the morphism

$$\begin{array}{rcl} \pi_X : S_F & \longrightarrow & \widehat{\mathbb{C}} \\ (X,Y) & \longmapsto & X \end{array}$$

Consider the set $Ram(\pi_X) = \{\mu_1, \mu_2, \ldots \mu_s\}$ of ramification points of π_X. Let us see that $Ram(\pi_X) \subseteq \overline{\mathbb{Q}} \cup \{\infty\}$. By the last part of the theorem 3.4, each μ_i either a zero of $p_0(x)$ or equals ∞ or equals the first coordinate of a common zero of $F, \dfrac{\partial F}{\partial Y} \in \overline{\mathbb{Q}}[X,Y]$. In the first case, being a root of $p_0(x)$, we have $\mu_i \in \overline{\mathbb{Q}}$ hence we see that $Ram(\pi_X) \subseteq \overline{\mathbb{Q}} \cup \{\infty\}$

If $Ram(\pi_X) \subseteq \mathbb{Q} \cup \{\infty\}$ then the first step guarantees the existence of a Belyĭ morphism. If not, we proceed the following inductive argument.

Let $m_1 \in \mathbb{Q}[T]$ be the minimal polynomial of $Ram(\pi_X) \setminus \{\infty\} = \{\mu_1, \mu_2, \ldots \mu_s\} \setminus \{\infty\}$. By definition, m_1 is the monic polynomial of smallest degree vanishing at $\mu_1, \mu_2, \ldots \mu_s$ or equivalently it is the product of the minimal polynomials of all algebraic numbers μ_i omitting the repeating factors.

Let $\alpha_1, \alpha_2, \ldots \alpha_d$ be roots of the derivative $m_1^{'}$ of m_1. Let $p(T)$ be their minimal polynomial. Clearly, $deg(p(T)) \leq deg(m_1^{'})$.

We have

$$\begin{array}{rcl} S_F \longrightarrow \widehat{\mathbb{C}} & \longrightarrow & \widehat{\mathbb{C}} \\ (X,Y) \longmapsto X & \longmapsto & m_1^{'}(X) \\ \mu_1, \ldots, \mu_s & \longmapsto & 0 \\ \infty & \longmapsto & \infty \\ & \longmapsto & \{m_1^{'}(\alpha_1), \ldots m_1^{'}(\alpha_d)\}) \end{array}$$

Therefore $Ram(m_1 \circ \pi_X) = \{0, \infty \cup m_1^{'}(\{\alpha_1, \ldots \alpha_d\})$ So the composition $m_1 \circ \pi_X$ sends each μ_i to zero but we have new ramification values $\{m_1^{'}(\alpha_1), \ldots m_1^{'}(\alpha_d)\})$.

If $R_1 = Ram(m_1 \circ \pi_x) \subseteq \mathbb{Q} \cup \{\infty\}$ then we are done. If not, we let the minimal polynomial of $m_1^{'}(\alpha_1), \ldots m_1^{'}(\alpha_d)$ be $m_2(T)$.

Note that $[\mathbb{Q}(m_1(\alpha_i)) : \mathbb{Q}] \leq [\mathbb{Q}(\alpha_i) : \mathbb{Q}]$. In other words, the degree of the minimal polynomial of $m_1(\alpha)$ is less than or equal to the degree of the minimal polynomial of α_i. Moreover, suppose that α_i and α_j have the same minimal polynomial. Then $\sigma(\alpha_i) = \alpha_j$ for some field embedding $\sigma : \mathbb{Q}(\alpha_i) \to \overline{\mathbb{Q}}$. Since σ is a monomorphism, it extends to a field embedding $\overline{\sigma} : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ so that $\sigma(m(\alpha_i)) = m(\alpha_j)$ and hence $m(\alpha_i)$ and $m(\alpha_j)$ have the same minimal polynomials.

Now we have

$$\text{deg} m_2(T) \leq \text{deg} P(T) \leq \text{deg} m_1^{'}(T) < \text{deg} m_1(T)$$

Consider the new morphism $m_2 \circ m_1 \circ \pi_X$. As before the set of ramification values is $R_2 = m_2(\{\text{roots of } m_2^{'}\}) \cup m_2(R_1)$. Again, if this set lies inside $\mathbb{Q} \cup \{\infty\}$ then we are done. Otherwise after finitely many steps by the inequality above, this process ends.

□

Example 4.1. Consider the compact Riemann surface S obtained from $y^2 = x(x-1)(x-\lambda)$ where $\lambda = 1+\sqrt{2}$ and consider the morphism $\pi_x : S \to \widehat{\mathbb{C}}$. By the example 1.5 the ramification points of π_x are $0, 1, \lambda$ and ∞. The minimal polynomial of $\lambda = 1+\sqrt{2}$ is $m_1(x) = x^2 - 2x - 1$ which can be easily verified. Proceeding as in the algorithm above we compute the ramification points of m_1. Indeed, we have $m_1^{'}(x) = 2x - 2$ so that $m_1(1) = -2$ is a ramification value. We have $M \circ T(-2) = 1/2$ so that we must take $m = 1$ and $n = 2$. Therefore the required Belyǐ morphism is

$$\beta = \beta_{1,1} \circ M \circ T \circ m_1 \circ \pi_x$$

Which is computed as $\beta(x, y) = \dfrac{9(-x^2 + 2x + 1)}{4(x^2 - 2x - 2)^2}$.

A Belyǐ morphism is called a **pre-clean Belyǐ morphism** if all the ramification orders over 0 are less than or equal to 2 and a **clean Belyǐ morphism** if they are all exactly equal to 2. A Belyǐ pair $(\mathcal{X}, \beta)$ is called a **clean Belyǐ pair** if β is clean. Any Belyǐ morphism gives rise to a clean one. Indeed, if $\alpha : \mathcal{X} \to \widehat{\mathbb{C}}$ is a Belyǐ morphism, then $\beta = 4\alpha(1-\alpha)$ is a clean Belyǐ morphism. All the white vertices of the dessin corresponding to a clean Belyǐ pair are of degree 2. With this observation, for instance all platonic solids can be seen as dessins by adding a white point into the middle of each edge.

4.2 The absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

Now we shall describe a very important action of the profinite group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on dessins. Before doing this we recall some related definitions from algebra.

The algebraic numbers are the elements $a \in \mathbb{C}$ which give rise to a finite extension $\mathbb{Q}(a)$ over the field of rational numbers $\mathbb{Q}$ and they form a field $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. For each element $a \in \overline{\mathbb{Q}}$, a has a minimal polynomial over $\mathbb{Q}$ and a splitting field $K_a \subset \overline{\mathbb{Q}}$. Note that the extension K_a over $\mathbb{Q}$ is finite and normal and hence Galois. Conversely, by primitive element theorem every Galois extension K over $\mathbb{Q}$ arises in this way. Therefore

$$\overline{\mathbb{Q}} = \bigcup_{K \in \mathcal{K}} K$$

where $\mathcal{K}$ is the set of all Galois extension K over $\mathbb{Q}$ in $\mathbb{C}$. For each $K \in \mathcal{K}$ the Galois group $Gal(K/\mathbb{Q})$ is a finite group of degree equal to $|K : \mathbb{Q}|$. For $K, L \in \mathcal{K}$ with $L \subseteq K$, the restriction map $Res_{K,L} : Gal(K/\mathbb{Q}) \to Gal(L/\mathbb{Q})$ is a homomorphism which is indeed a monomorphism since every automorphism of L can be extended to an automorphism of K.

The finite groups $Gal(K/\mathbb{Q})$, $K \in \mathcal{K}$ and the monomorphisms $Res_{K,L}$ form a inverse system and we have

$$Gal(\overline{\mathbb{Q}}/\mathbb{Q}) = \varprojlim Gal(K/\mathbb{Q})$$

as $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ being identified with its inverse limit. In other words, it is the subgroup of the Cartesian product

$$\prod_{K \in \mathcal{K}} Gal(K/\mathbb{Q})$$

consisting of all elements (g_K) such that $Res_{K,L}(g_K) = g_L$ whenever $L \subseteq K$, and each element $g \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ identified with the element (g_K) where g_K is the restriction of g to K. The finite groups are topologized with the discrete topology and this impose a topology on $\prod_{K \in \mathcal{K}} Gal(K/\mathbb{Q})$ which is compact by *Tychonoff's theorem*. The subgroup $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ inherits a topology called *Krull topology*, which is also compact since $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ is closed. Moreover infinite Galois theory insists that the subfields of $\overline{\mathbb{Q}}$ corresponds to the closed subgroups of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ and vice versa. For further details see [7].

4.3 Galois Action on Dessins

In Section 3 we have seen that compact Riemann surfaces correspond to complex algebraic curves. It is also well-known that meromorphic functions on a compact Riemann surface S corresponds to rational functions on the corresponding curve S_F. Moreover, in the case of Belyĭ surfaces, the polynomial $F(X,Y)$ can be chosen so as to have coefficients in a number field. We shall briefly describe the well-known action of *absgal* on Belyĭ pairs or equivalently dessins d'enfants.

Let $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Given a polynomial $f(X,Y) = \sum_{i,j} a_{i,j} X^i Y^j \in \overline{\mathbb{Q}}[X,Y]$. Evaluating $\sigma(a_{i,j})$ for each coefficient we get a new polynomial $f^\sigma(X,Y) = \sum_{i,j} \sigma(a_{i,j}) X^i Y^j$. The similar process can be applied to the corresponding rational function. This action on the coefficients of polynomials and the *coefficients* of the rational functions induces an action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on Belyĭ pairs equivalently on dessins d'enfants:

$$\begin{array}{ccc} (\mathcal{X},\mathcal{G}(\beta)) & \dashrightarrow & (\mathcal{X}_\sigma,\mathcal{G}(\beta_\sigma)) \\ \updownarrow & & \updownarrow \\ (\mathcal{X},\beta) & \longrightarrow & (\mathcal{X}_\sigma,\beta_\sigma) \end{array}$$

Under this action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, the number of edges, the number of white vertices and of black vertices, the genus and the monodromy group remain invariant.

For *if part* of the theorem, the following criterion can be found in [3]. An irreducible variety $X \subset \mathbb{P}^n(\mathbb{C})$ (resp. a morphism $f : X \to Y$) can be defined over a number field if and only if $Gal(\mathbb{C})$-orbit of X (resp. $Gal(\mathbb{C})$-orbit of f) contains only finitely many isomorphism classes of complex projective varieties (resp. of morphisms).

5 Action of $\mathrm{PGL}_2(\mathbb{Z})$ on Dessins d'Enfants

We have seen in section 2 that the monodromy representation of a Belyĭ pair determines the whole information about the corresponding dessin. In this section we shall investigate the following action of the automorphism group of the free group $\mathbb{F}_2 = \langle X, Y \rangle$ on dessins and its invariants.

6.1. Let $\tau \in Aut(\mathbb{F}_2)$ and $\mathfrak{D}$ be dessin with n edges corresponding to its monodromy representation $\Phi : \mathbb{F}_2 \to S_n$. Note the map $\Phi \circ \tau^{-1} : \mathbb{F}_2 \to S_n$ is also a monodromy map so that we define $\tau \cdot \Phi := \Phi \circ \tau^{-1}$. In this way we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{F}_2 & \xrightarrow{\Phi} & S_n \\ {\scriptstyle\tau}\downarrow & \nearrow_{\Phi\circ\tau^{-1}} & \\ \mathbb{F}_2 & & \end{array}$$

Indeed this gives an action of $Aut(\mathbb{F}_2)$ on monodromy maps hence on dessins. We have

(i) $Id \cdot \Phi = \Phi$

(ii) If τ_1 and τ_2 are in $Aut(\mathbb{F}_2)$, then $(\tau_1\tau_2) \cdot \Phi = \Phi \circ (\tau_1 \circ \tau_2)^{-1} = \Phi \circ \tau_2^{-1} \circ \tau_1^{-1}$ and $\tau_1 \cdot (\tau_2 \cdot \Phi) = (\Phi \circ \tau_2^{-1}) \circ \tau_1^{-1}$ so that $(\tau_1\tau_2) \cdot \Phi = \tau_1 \cdot (\tau_2 \cdot \Phi)$

Notation. As we mention above this gives equivalently an action of $Aut(\mathbb{F}_2)$ on dessins. We denote the dessin corresponding to the monodromy map $\Phi \circ \tau^{-1}$ by ${}^{\tau}\mathfrak{D}$. [1]

Remark. If α is an inner automorphism of the free group $\mathbb{F}_2$ then α acting on dessin $\mathfrak{D}$ gives rise to an equivalent dessin. In other words, an inner automorphism fixes equivalence classes. Therefore we can omit inner automorphisms and consider only outer automorphisms of the free group $\mathbb{F}_2$ up to an inner automorphism.

5.1 Calculation of $\mathrm{PGL}_2(\mathbb{Z})$-Orbits

6.2. It is well-known that

$$Out(\mathbb{F}_2) \simeq \mathrm{PGL}_2(\mathbb{Z}) \simeq \langle T, U \,|\, U^2 = (UTUT^{-2})^2 = (UTUT^{-1})^3 = 1 \rangle$$

[1]In this way we have ${}^{\tau_1}({}^{\tau_2}\mathfrak{D}) = {}^{\tau_1\tau_2}\mathfrak{D}$

where the actions of the generators are defined by

$$T(X,Y) = (XY,Y), \quad U(X,Y) = (Y,X)$$

By τ-orbit of a dessin, we mean the orbit of the subgroup generated by τ. Since any element in $\mathrm{PGL}_2(\mathbb{Z})$ is a product of T's and U's, we shall describe the action of generators in the context of dessins and focus only on orbits of these generators. If $\mathfrak{D}$ is dessin given with the permutational representation (σ_0, σ_1), which is an ordered pair of permutations transitively acting on edges , then the generator U produce a dessin with permutational representation (σ_1, σ_0). So U simply interchanges white points with black points. On the other hsnd, T produce a dessin with permutational representation $(\sigma_0\sigma_1^{-1}, \sigma_1)$. However, it is better to depict the action of T^{-1} instead of T. Since T^{-1} produce a dessin with permutational representation $(\sigma_0\sigma_1, \sigma_1)$, it fixes the darts with black points but turn each face into a white point of new dessin. Because the given dessin has permutation σ_0 around white points and permutation $(\sigma_0\sigma_1)^{-1}$ around the faces, the new dessin produced by the action of T^{-1} has permutation $\sigma_0\sigma_1$ around white points.

6.3. It is possible to compute genus of a given dessin (i.e. genus of the surface that the dessin is embedded) in terms of permutations σ_0, σ_1 and $(\sigma_0\sigma_1)^{-1}$ determined by white vertices , black vertices and faces respectively. Indeed, if g is the genus the underlying surface then

$$2 - 2g = \big(\#\text{ of cycles of } \sigma_\circ + \#\text{ of cycles of } \sigma_1\big) - N + (\#\text{ of cycles of } (\sigma_\circ\sigma_1)^{-1})$$

U preserves genus. This is clear since number of vertices and edges remains unchanged. But T does not.

action of T	***white vertices***	***black vertices***	***faces***
$\mathfrak{D}$	σ_0	σ_1	$(\sigma_0\sigma_1)^{-1}$
${}^T\mathfrak{D}$	$\sigma_0\sigma_1^{-1}$	σ_1	σ_0^{-1}

Note that σ_0 is conjugate to σ_0^{-1}. However $\sigma_0\sigma_1^{-1}$ and $\sigma_1^{-1}\sigma_0^{-1}$ need not to be conjugate. In the affirmative case, we know that both has the same cycle structure which implies two dessins have equal genera. Nevertheless in some cases it should. Examining that will enable us to find out exactly that how $\mathrm{PGL}_2(\mathbb{Z})$ changes genus.

6.4. Recall that for a monodromy representation (σ_0, σ_1), the monodromy group is defined to be the permutational group generated by σ_0 and σ_1. Note that the group generated by σ_0 and σ_1 equals to the group $\langle \sigma_0 \sigma_1^{-1}, \sigma_1 \rangle$, hence both T and U respects the monodromy group. So proceeding the action of a product consisting of T and U, the monodromy group of the resulting pair remains fixed. We conclude. We conclude

Corollary 5.1. *The monodromy group is invariant under this action. In particular, imprimitivity is preserved under this action.*

Corollary 5.2. *Galois dessins are sent to Galois dessins under this action. In other words, it sends normal subgroups to normal subgroups, preserving the quotient groups.*

$\mathrm{PGL}_2(\mathbb{Z})$ preserves monodromy group and the number of edges because of transitivity of monodromy pairs. So there cannot exist two dessins with the same monodromy group in different orbits. Moreover, this implies that the action of on dessins is not transitive. And as the examples shows this action is not transitive even fixing the number of edges.

6.5. ***Chebychev Dessins.*** Now we shall see how $\mathrm{PGL}_2(\mathbb{Z})$ acts on certain classes of dessins such as stars and Chebychev dessins (i.e. linear trees). With some calculation of orbits, one can see that the genus of $\mathrm{PGL}_2(\mathbb{Z})$-orbit of Chebychev dessins is fixed and equal to 0. We begin with an example.

Example 5.1. Consider the Chebychev dessin corresponding to the Chebychev polynomial T_4 and T_8.

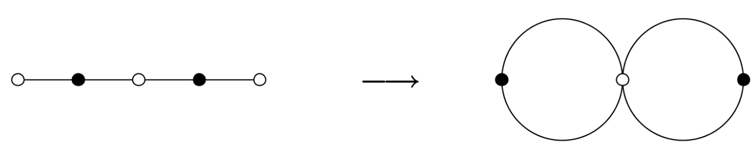


Figure 10: T acting on Chebychev dessin corresponding to T_4

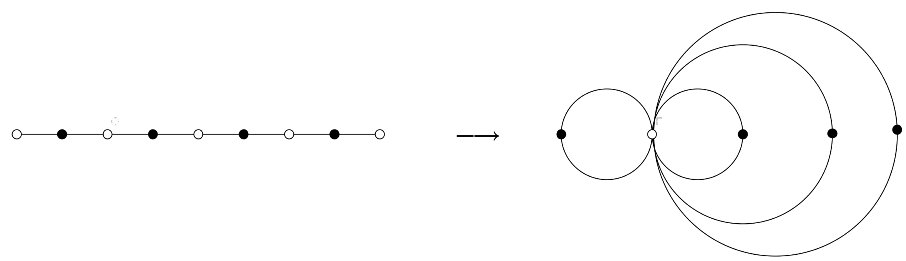


Figure 11: T acting on Chebychev dessin corresponding to T_8

6.5.*n-Stars.* We shall describe $\mathrm{PGL}_2(\mathbb{Z})$-action on another class of dessin d'enfants namely n-stars and we will see that the group $\mathrm{PGL}_2(\mathbb{Z})$ does not preserve genus of n-stars.

Example 5.2. Consider the 3-star given by $(Id, (123))$. T acts on this constellation as:

$$(Id, (123)) \to ((132), (123)) \to ((123), (123)) \to (Id, (123))$$

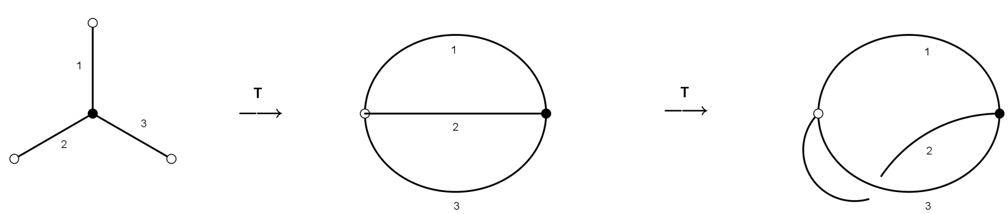


Figure 12: T acting on a 3-star

Here we observe that ${}^{T^3}\mathfrak{D} = \mathfrak{D}$. Since the dessin ${}^{T^2}\mathfrak{D}$ is on torus, we also see that this action does not preserve genera of dessins.

Proposition 5.3. *After some calculation, we get the following propositions.*

1. If $\mathfrak{D}$ is n-star then the T-orbit of $\mathfrak{D}$ contains a dessin of genus $(n-1)/2$, the dessin being ${}^{T^{n-1}}\mathfrak{D}$.

2. If p is a prime then the T-orbit of p-star contains only dessins of genus 0, 1 and $(p-1)/2$.

5.2 Arithmetic of $\mathrm{PGL}_2(\mathbb{Z})$-Action

6.5 Having described $\mathrm{PGL}_2(\mathbb{Z})$-action on dessins combinatorially, we now investigate the arithmetic point of view of this action. In other words, the next problem is given a dessin d'enfant $(\mathcal{X}, \mathfrak{D})$ how can we find the Belyĭ pair $({}^{\tau}\mathcal{X}, {}^{\tau}\beta)$ if $(\mathcal{X}, \beta)$ is a Belyĭ pair corresponding the given dessin.

As in the combinatorial case, it easy to describe the action of the generator U on Belyĭ pairs. If $(\mathcal{X}, \beta)$ corresponds to the dessin $\mathfrak{D}$ then $(\mathcal{X}, 1-\beta)$ corresponds to the dessin ${}^{U}\mathfrak{D}$

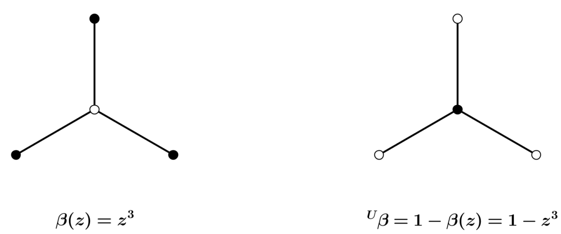


Figure 13: U acting on a 3-star

However there seems to be no straight-forward calculation of action of the generator T on Belyĭ pairs.

Another way of defining this action is as follows. A dessin is equivalently described by a conjugacy class of a subgroup of $\mathbb{F}_2$ (where the elements of the conjugacy class is parametrized by the edges of the dessin). An outer automorphism is an automorphism modulo an inner automorphism, so it acts on the set of conjugacy classes of subgroups of $\mathbb{F}_2$, i.e. on dessins.

Many questions arises from this action of $\mathrm{PGL}_2(\mathbb{Z})$ on dessins d'enfants: Given two dessins how can one decide whether they are in the same orbit? Is it possible to describe the action on the Belyĭ maps, explicitly?

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