Long-Time Behaviour of Solutions to Phase Field Equations

by

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This is to certify that I have examined this copy of a master's thesis by

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and that any and all revisions required by the final

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ABSTRACT

Theorem on existence and uniqueness of global solutions to initial-boundary value problems for the phase field equations is proved.

Results on the stabilization of solutions and the existence of a global attractor of a continuous semigroup generated by the problem are also established.

ÖZETÇE

Faz alan denklemler için başlangıç-sınır değer problemlerin küresel çözümlerin varlığı ve tekliği teoremi kanıtlanmıştır.

Çözümlerin istikrarı ve problemin oluşturduğu sürekli yarı grubunun küresel çeker varlığı hakkında sonuçlar da elde edilmiştir.

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Chapter 1

PRELIMINARIES

1.1 Introduction

In [1], G.Caginalp has considered, as a model describing the phase transition with a separation surface of finite thickness, the following system of nonlinear parabolic differential equations known as the phase field equations :

$$\tau\varphi_t = \xi^2\Delta\varphi - f(\varphi) + 2u \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{1.1.1}$$

$$u_t + \frac{\ell}{2}\varphi_t = \kappa\Delta u \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{1.1.2}$$

where Ω is a bounded domain in $\mathbb{R}^d$ with a sufficiently smooth boundary $\partial\Omega$; φ is the phase function; u is the reduced temperature; $f(z) = \frac{1}{2}(z^3 - z)$ is the nonlinear term; τ, ξ, ℓ, and κ are positive constants which characterize the relaxation time, the length scale, the latent heat, and the thermal diffusivity, respectively. In [1], under the assumption $\xi^2/\tau < \kappa$, the global existence of classical solutions of the initial-boundary value problem for the system (1.1.1)-(1.1.2) has been proven with non-homogeneous Dirichlet boundary conditions of the form :

$$\varphi|_{\partial\Omega} = \varphi_\partial(x)\,, \quad u|_{\partial\Omega} = u_\partial(x)\,. \tag{1.1.3}$$

The investigation of the global behaviour of solutions of the initial-boundary value problems for the system (1.1.1)-(1.1.2) has been carried out by C. M. Elliott and Song-Mu Zheng in [3], where they have proved the global existence of smooth solutions within the class $C(\mathbb{R}^+; H^2(\Omega) \times H^2(\Omega))$, where $\Omega \subset \mathbb{R}^d$ and $d \leq 3$, without the assumption $\xi^2/\tau < \kappa$, for the boundary conditions of the form (1.1.3) as well as for the following boundary conditions :

$$\left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = 0\,, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0 \quad \text{or} \quad \varphi|_{\partial\Omega} = \varphi_\partial(x)\,, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0\,.$$

In [3], C. M. Elliott and Song-Mu Zheng have also studied the asymptotic behaviour of solutions of the system (1.1.1)-(1.1.2) as $t \to \infty$. They have investigated the corresponding stationary problems and have proved that as $t \to \infty$, each solution of the system (1.1.1)-(1.1.2) tends in the norm of $H^1(\Omega) \times H^1(\Omega)$ to the corresponding stationary solution.

The investigation of the existence of a global attractor for the system (1.1.1)-(1.1.2) has been carried by V. Kalantarov in [5], where he has proved the unique global solvability of the initial-boundary value problem for the system (1.1.1)-(1.1.2) within the class $C\big(\mathbb{R}^+; H^1(\Omega) \times H^1(\Omega)\big)$, where $\Omega \subset \mathbb{R}^d$ and $d \leq 3$, for the boundary conditions of the form (1.1.3) and showed that it generates a continuous semigroup for which there exists a global attractor which is connected and has finite fractal dimension.

In Chapter 1, we will provide a short background from functional analysis and the theory of partial differential equations so that one can follow the discussions in the subsequent chapters.

In Chapter 2, we will study the global existence and uniqueness of solutions for the following system of partial differential equations :

$$\tau\varphi_t = \xi^2\Delta\varphi - g(x,\varphi) + 2v \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{1.1.4}$$

$$v_t = \kappa\Delta v - \frac{\kappa\ell}{2}\Delta\varphi \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{1.1.5}$$

where g is a nonlinear function satisfying certain properties. We will consider the system of equations (1.1.4)-(1.1.5) together with the following homogeneous Dirichlet boundary and initial conditions :

$$\begin{aligned} \varphi|_{\partial\Omega} &= 0\,, \quad t \in \mathbb{R}^+, \\ v|_{\partial\Omega} &= 0\,, \quad t \in \mathbb{R}^+, \end{aligned} \tag{1.1.6}$$

$$\begin{aligned} \varphi(x,0) &= \varphi_0(x)\,, \quad x \in \Omega\,, \\ v(x,0) &= v_0(x)\,, \quad x \in \Omega\,. \end{aligned} \tag{1.1.7}$$

In Chapter 3, we will study the internal stabilization of the following system of phase field equations :

$$\tau\varphi_t = \xi^2\Delta\varphi - f(\varphi) + 2u - k\chi_{\varpi}\varphi \quad \text{in} \quad \Omega \times \mathbb{R}^+\,, \tag{1.1.8}$$

$$u_t + \frac{\ell}{2}\varphi_t = \kappa\Delta u \quad \text{in} \quad \Omega \times \mathbb{R}^+\,, \tag{1.1.9}$$

where f is a nonlinear function, k is a nonnegative real number and $\chi_{\bar{\omega}}$ is a characteristic function of a subdomain $\omega \subset \Omega$. Notice that the system of phase field equations (1.1.8)-(1.1.9) becomes equivalent to the system (1.1.4)-(1.1.5) with $v := u + \frac{\ell}{2}\varphi$ and $g(x,s) := f(s) - \ell s + k\chi_{\bar{\omega}}(x)s$. We will give a sufficient condition under which such a system can be exponentially stabilized by only one feedback controller acting on a subdomain in the first equation. In Chapter 4, we will study the existence of a global attractor for the system of equations (1.1.8)-(1.1.9) with $k = 0$.

1.2 Function Spaces

Here we will review the functions spaces which will be used in our discussions.

1. *Lebesgue spaces* $(1 \le p \le \infty)$

   $L^p(\Omega)$ is the Banach space (i.e. the complete linear normed space) consisting of all measurable (in the sense of Lebesgue) functions on Ω having the following finite norm:

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx\right)^{1/p} \quad (1 \le p < \infty),$$

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| \quad (p = \infty).$$

   The norm in $L^2(\Omega)$ will be abbreviated to $\|\cdot\|$ and the inner product to $(\, , \,)$.

2. *The space of test functions*

   The space $C_c^\infty(\Omega)$ is the following set of so called test functions:

$$C_c^\infty(\Omega) = \{\phi \in C^\infty(\Omega) \mid \text{supp}(\phi) \text{ is a compact set in } \Omega\}.$$

   Note that the space of test functions $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, $1 \le p < \infty$.

3. *Sobolev Spaces*

   For two functions $u, v \in L^2(\Omega)$, we say that v is the i^{th} weak partial derivative of u if the following identity holds :

$$\int_\Omega u\varphi_{x_i}\, dx = -\int_\Omega v\varphi\, dx, \quad \text{for all} \quad \varphi \in C_c^\infty(\Omega).$$

The space $H^1(\Omega)$ consists of functions from $L^2(\Omega)$ whose all weak partial derivatives also belong to $L^2(\Omega)$. The space $H^1(\Omega)$ is a Hilbert space when equipped with the following inner product :

$$(u,v)_{H^1(\Omega)} = \int_\Omega uv + \nabla u \cdot \nabla v \, dx \quad \text{for} \quad u,v \in H^1(\Omega).$$

The Hilbert space $H_0^1(\Omega)$ is defined to be the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$. The space $H_0^1(\Omega)$ has its own inner product :

$$(u,v)_{H_0^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v \, dx \quad \text{for} \quad u,v \in H_0^1(\Omega).$$

1.3 Some Useful Inequalities

Cauchy-Schwarz Inequality. *Let H be an inner product space. Then for any $u, v \in H$, the following inequality holds :*

$$|(u,v)| \leq \|u\|\|v\|.$$

Hölder's Inequality. *Suppose that $p \in [1,\infty]$ and $1/p + 1/q = 1$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and $\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)}$.*

Cauchy's Inequality with ϵ. *For any $\epsilon > 0$, the following inequality holds :*

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2, \quad \textit{for any} \quad a,b \in \mathbb{R}\,.$$

Young's Inequality with ϵ. *For any $\epsilon > 0$ and $p > 1$, the following inequality holds :*

$$ab \leq \epsilon a^p + C(\epsilon,p) b^q, \quad \textit{for any} \quad a,b \in \mathbb{R}^+\,,$$

where q is the conjugate of p and $C(\epsilon,p) = (\epsilon p)^{-q/p} q^{-1}$.

Inequality 1.3.1. *For any $\epsilon > 0$ and $p > 2$, the following inequality holds :*

$$ab \leq \epsilon a^2 + \epsilon b^p + C(\epsilon,p), \quad \textit{for any} \quad a,b \in \mathbb{R}^+\,.$$

Gronwall's Inequality. *Suppose that u is an absolutely continuous function which satisfies* *the following differential inequality :*

$$\frac{d}{dt}u(t) \leq g(t)u(t) + h(t)\,, \quad \textit{for almost all} \quad t \in [0,T].$$

where $g, h : [0,T] \to \mathbb{R}$ *are integrable functions. Then*

$$u(t) \leq u(0)\exp\left(\int_0^t g(\tau)\,d\tau\right) + \int_0^t \exp\left(\int_s^t g(\tau)\,d\tau\right) h(s)\,ds\,, \quad \textit{for all} \quad t \in [0,T].$$

Remark 1.3.1. *Absolute continuity is a strengthening of uniform continuity that provides a necessary and sufficient condition for the fundamental theorem of calculus to hold. A continuous function is absolutely continuous if and only if its weak (or distributional) derivative is integrable.*

Lemma 1.3.2. *Let* $h : \mathbb{R} \to \mathbb{R}$ *be a continuous function and* $H(z) := \int_0^z h(s)\,ds$*. Suppose also that* h *satisfies the inequality*

$$-\beta_0 + \beta_1 |z|^p \leq z h(z) \leq \beta_0 + \beta_2 |z|^p, \quad \textit{for all} \quad z \in \mathbb{R}\,, \tag{1.3.1}$$

where $p > 2$ *and* $\beta_0, \beta_1, \beta_2 > 0$*. Then there exist some positive constants* $\gamma_0, \gamma_1, \gamma_2$ *and* γ_3 *such that*

$$-\gamma_0 + \gamma_1 |z|^p \leq H(z) \leq \gamma_0 + \gamma_2 |z|^p, \quad \textit{for all} \quad z \in \mathbb{R} \tag{1.3.2}$$

and

$$|h(z)| \leq \gamma_3 (1 + |z|^{p-1}), \quad \textit{for all} \quad z \in \mathbb{R}. \tag{1.3.3}$$

Remark 1.3.3. *From Lemma 1.3.2 it is obvious that* H *is bounded from below :*

$$H(z) \geq -\gamma_0, \quad \textit{for all} \quad z \in \mathbb{R}.$$

Furthermore, if $\beta_2 < p\beta_1$*, then the function* $zh(z) - H(z)$ *is also bounded from below :*

$$zh(z) - H(z) \geq -\gamma_0, \quad \textit{for all} \quad z \in \mathbb{R}.$$

Poincaré Inequality. *If* $\Omega \subset \mathbb{R}^d$ *is a bounded domain, then there is a constant* C *depending* *only on* Ω *such that*

$$\int_{\Omega} |u(x)|^2 \, dx \leq C \int_{\Omega} |\nabla u(x)|^2 \, dx \,, \quad \text{for all } \ u \in H_0^1(\Omega) \,. \tag{1.3.4}$$

1.4 Auxiliary Theorems

Theorem 1.4.1. *Let* I *be a closed interval in* $\mathbb{R}$*. Suppose that* $g : \Omega \times I \longrightarrow \mathbb{R}$ *such that* $g(\cdot, z) : \Omega \longrightarrow \mathbb{R}$ *is integrable, for each* $z \in I$*. Let* $G(z) := \int_{\Omega} g(x, z) \, dx$*.*

(i) Suppose that there is $h \in L^1(\Omega)$ *such that* $|g(x, z)| \leq h(x)$*, for all* $(x, z) \in \Omega \times I$*. If* $\lim_{z \to z_0} g(x, z) = g(x, z_0)$*, for every* $x \in \Omega$*, then* $\lim_{z \to z_0} G(z) = G(z_0)$ *i.e. we can pass the limit inside integral. In particular, if* $g(x, \cdot)$ *is continuous, for each* $x \in \Omega$*, then* G *is continuous.*

(ii) Suppose that $\partial g / \partial z$ *exists and there is* $h \in L^1(\Omega)$ *such that* $|\frac{\partial g}{\partial z}(x, z)| \leq h(x)$*, for all* $(x, z) \in \Omega \times I$*. Then* G' *exists and*

$$G'(z) = \int_{\Omega} \frac{\partial g}{\partial z}(x, z) \, dx$$

i.e. we can differentiate under the integral sign.

Theorem 1.4.2. *Let* X *be a Banach space. Then the following are equivalent :*

1. *The space* X *is reflexive ;*
2. *The closed unit ball of* X *is compact in the weak topology* (**Banach - Alaoglu Theorem**) ;
3. *Every bounded sequence in* X *has a weakly convergent subsequence.*

Lebesgue's Dominated Convergence Theorem. *Let* $1 \leq p < \infty$ *and* g_m *be a sequence in* $L^p(\Omega)$ *such that* g_m *converges to some function* g *almost everywhere on* Ω*. Suppose that there is* $h \in L^p(\Omega)$ *such that* $|g_m| \leq h$ *almost everywhere on* Ω*, for all* $m \geq 1$*. Then* $g \in L^p(\Omega)$ *and* $\|g_m - g\|_{L^p(\Omega)} \to 0$ *as* $m \to \infty$*.*

Theorem 1.4.3. *Suppose that* $1 \leq p < \infty$ *and* $g_m \to g$ *in* $L^p(\Omega)$*. Then there exists a subsequence of* g_m *which converges pointwise to* g *almost everywhere on* Ω*.*

Weak Dominated Convergence Theorem. *Let Ω be a bounded domain in $\mathbb{R}^d$ and $p \in [1, \infty)$. Suppose that $g \in L^p(\Omega)$ and g_m is a bounded sequence in $L^p(\Omega)$ such that $g_m(x) \to g(x)$, for almost all $x \in \Omega$. Then $g_m \rightharpoonup g$ in $L^p(\Omega)$.*

Theorem 1.4.4. *Let Ω be a bounded domain in $\mathbb{R}^d$. If $1 \le p < q \le \infty$, then $L^q(\Omega)$ is continuously embedded in $L^p(\Omega)$.*

Sobolev Embedding Theorem. *Let Ω be a bounded domain in $\mathbb{R}^d$ of class C^1 and $k \in \mathbb{Z}^+$. Then $H^k(\Omega)$ is continuously embedded in $L^p(\Omega)$, where*

$$p \in \left[1, \frac{2d}{d-2k}\right] \quad \text{if} \quad k < \frac{d}{2} \qquad \text{and} \qquad p \in [1, \infty) \quad \text{if} \quad k \ge \frac{d}{2}\,.$$

Furthermore, $H^k(\Omega)$ is continuously embedded in $C(\overline{\Omega})$ if $k > d/2$.

Rellich's Compactness Theorem. *Let Ω be a bounded domain in $\mathbb{R}^d$. Then $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$.*

Cauchy-Picard Theorem. *Suppose that $G : \mathbb{R}^d \to \mathbb{R}^d$ satisfies*

$$\|G(y) - G(\bar{y})\|_{\mathbb{R}^d} \le L(B)\|y - \bar{y}\|_{\mathbb{R}^d}, \tag{1.4.1}$$

for all $y, \bar{y}$ in any bounded set $B \subset \mathbb{R}^d$. Then there exists $T = T(y_0)$ such that the initial value problem

$$\frac{dy}{dt} = G(y), \quad y(0) = y_0\,, \tag{1.4.2}$$

has a unique solution defined on the interval $[0, T]$.

Theorem 1.4.5. *A solution $y(t)$ of the initial value problem (1.4.2) has a finite maximal interval of existence $[0, S^*)$ if and only if $\|y(t)\|_{\mathbb{R}^d} \to \infty$ as $t \to S^*$.*

Proof. If $\|y(t)\|_{\mathbb{R}^d} \to \infty$ as $t \to S^*$, then there can be no continuous extension of $y(t)$ to an interval containing S^*. Conversely, suppose that $y(t)$ has a finite maximal interval of existence $[0, S^*)$. If $y(t)$ is bounded on $[0, S^*)$, then $G(y(t))$ is also bounded (being continuous) and the following limit exists :

$$\lim_{t \to S^*} y(t) = y_0 + \int_0^{S^*} G(y(s))\, ds\,.$$

Then by the Theorem (1.4.5), we can extend the interval of existence to $[0, S^*+\epsilon]$, for some $\epsilon > 0$, contradicting the maximality of $[0, S^*)$.

□

Ehrling's Lemma. *Let H, X and Y be Banach spaces such that X is compactly embedded in H and H is continuously embedded in Y. Then for each $\epsilon > 0$, there is a constant C_ϵ such that*

$$\|u\|_H \leq \epsilon\|u\|_X + C_\epsilon\|u\|_Y, \quad \textit{for all} \quad u \in X.$$

Proof. Suppose for a contradiction that there is $\epsilon_0 > 0$ such that for each $m \geq 1$, there is $u_m \in X$ with

$$\|u_m\|_H > \epsilon_0\|u_m\|_X + m\|u_m\|_Y,$$

Consider the normalized sequence $v_m := u_m/\|u_m\|_X$ which satisfies the following inequality:

$$\|v_m\|_H > \epsilon_0 + m\|v_m\|_Y, \quad \text{for all} \quad m \geq 1. \tag{1.4.3}$$

Since the sequence v_n is bounded in X, by compact embedding, there is a subsequence v_{n_k} such that $v_{n_k} \to v$ in H. By continuous embedding, we also have that $v_{n_k} \to v$ in Y. The only way the right hand side of (1.4.3) remains bounded is that $\lim_{m\to\infty}\|v_m\|_Y = 0$. Therefore, v must be zero. Then by taking limit in (1.4.3) as $n_k \to \infty$, we obtain a contradiction. □

Lemma 1.4.6. *Let X be a Banach space and $u \in L^1(0, T; X)$. Then u is weakly differentiable with integrable derivative $u_t = v \in L^1(0, T; X)$ if and only if*

$$u(t) = C_0 + \int_0^t v_s(s)ds,$$

for almost all $t \in (0, T)$. In that case, u is differentiable pointwise almost everywhere and its pointwise derivative coincides with its weak derivative.

Theorem 1.4.7. *Let X be a Banach space and $p \in [1, \infty]$. If $u \in W^{1,p}(0, T; X)$, then $u \in C([0, T]; X)$ and $u(t) = u(s) + \int_s^t u_\tau(\tau)\,d\tau$, for every $0 \geq s \geq t \geq T$. Furthermore, we* *have the estimate*

$$\|u\|_{L^\infty(0,T;X)} \leq C\|u\|_{W^{1,p}(0,T;X)}, \quad \textit{for all} \quad u \in W^{1,p}(0,T;X).$$

Lemma 1.4.8. *Let X be a Banach space with dual X^*. If $u, v \in L^1(0,T;X)$, then u is weakly differentiable with $u_t = v$ if and only if for every $w \in X^*$, we have*

$$\frac{d}{dt}\langle w, u(t)\rangle = \langle w, v(t)\rangle \quad \textit{as a real-valued weak derivative in} \quad (0,T).$$

Lebesgue's Dominated Convergence Theorem (For Bochner Integrable Functions)**.** *Let X be a Banach space and $p \in [1,\infty)$. Let u_m be a sequence in $L^p(0,T;X)$ such that $u_m(t)$ converges to $u(t)$ for almost all $t \in (0,T)$. Suppose that there is $h \in L^p(0,T)$ such that $\|u_m(t)\|_X \leq h(t)$ for almost all $t \in (0,T)$ and for all $m \geq 1$. Then $u \in L^p(0,T;X)$ and $\|u_m - u\|_{L^p(0,T;X)} \to 0$ as $m \to \infty$.*

Compactness Theorem. *Let H, X and Y be reflexive Banach spaces such that X is compactly embedded in H and H is continuously embedded in Y. Suppose that u_m is a sequence that is uniformly bounded in $L^2(0,T;X)$ and u_{mt} is uniformly bounded in $L^p(0,T;Y)$, where $p \in (1,\infty)$. Then there is a subsequence of u_m that converges in $L^2(0,T;H)$.*

Proof. Since X and Y are reflexive, $L^2(0,T;X)$ and $L^p(0,T;Y)$ are also reflexive spaces. By the Banach-Alaoglu Theorem there is a subsequence u_m (with the same notation) such that

$$u_m \to u \quad \text{in} \quad L^2(0,T;X) \quad \text{and} \quad u_{mt} \to v \quad \text{in} \quad L^p(0,T;Y).$$

Let $w \in Y^*$ and $\phi \in C_c^\infty(0,T)$. Since $u_m, u_{mt} \in L^1(0,T;Y)$, by the Lemma (1.4.8), we can write

$$\int_0^T \langle w, u_m(t)\rangle \phi'(t)\, dt = -\int_0^T \langle w, u_{mt}(t)\rangle \phi(t)\, dt, \tag{1.4.4}$$

for all $m \geq 1$. Since $w\phi' \in L^2(0,T;X^*)$ and $L^q(0,T;Y^*)$, we can pass to the limit in (1.4.4) to obtain

$$\int_0^T \langle w, u(t)\rangle \phi'(t)\, dt = -\int_0^T \langle w, v(t)\rangle \phi(t)\, dt,$$

Since $w \in Y^*$ and $\phi \in C_c^\infty(0,T)$ are arbitrary, from the Lemma (1.4.8), it follows that u is weakly differentiable and $u_t = v$. Let $v_m := u_m - u$. Since $v_m, v_{mt} \in L^1(0,T;Y)$ i.e. $v_m \in$

$W^{1,1}(0,T;Y)$, from the Theorem (1.4.7) it follows that $v_m \in C([0,T];Y)$. Furthermore,

$$\|v_m\|_{L^\infty(0,T;Y)} \leq C\|v_m\|_{W^{1,1}(0,T;Y)} = C\left(\|v_m\|_{L^1(0,T;Y)} + \|v_{mt}\|_{L^1(0,T;Y)}\right)$$

for all $m \geq 1$. Since X is continuously embedded in Y, we have $\|v_m\|_{L^1(0,T;Y)} \leq C_1\|v_m\|_{L^1(0,T;X)}$, for all $m \geq 1$. Then

$$\|v_m\|_{L^\infty(0,T;Y)} \leq C\left(C_1\|v_m\|_{L^1(0,T;X)} + \|v_{mt}\|_{L^1(0,T;Y)}\right) \tag{1.4.5}$$

If we apply the Hölder's Inequality to the right hand side terms of (1.4.5), then we deduce that

$$\|v_m\|_{L^\infty(0,T;Y)} \leq C\left(C_1T^{1/2}\|v_m\|_{L^2(0,T;X)} + T^{1/q}\|v_{mt}\|_{L^p(0,T;Y)}\right) \leq M\,.$$

for all $m \geq 1$. Therefore, $\|v_m(t)\|_Y \leq M$ for all $t \in [0,T]$ and all $m \geq 1$. Fix $t \in (0,T]$. Then for any $0 \leq \sigma \leq t$, we can write

$$v_m(t) = v_m(\sigma) + \int_\sigma^t v_{mr}(r)\,dr\,. \tag{1.4.6}$$

If we integrate the equality (1.4.6) with respect to σ from $t-s$ to t, then we obtain

$$v_m(t) = \frac{1}{s}\int_{t-s}^t v_m(\sigma)\,d\sigma + \frac{1}{s}\int_{t-s}^t\int_\sigma^t v_{mr}(r)\,drd\sigma\,. \tag{1.4.7}$$

Let's denote the first and the second integrals on the right hand side of the equality (1.4.7) by a_m and b_m, respectively. Then

$$b_m = \frac{1}{s}\int_{t-s}^t\int_\sigma^t v_{mr}(r)\,dr = \frac{1}{s}\int_{t-s}^t v_m(t) - v_m(\sigma)\,d\sigma = \frac{1}{s}\left(sv_m(t) - \int_{t-s}^t v_m(\sigma)\,d\sigma\right) =$$

$$= \frac{1}{s}\int_{t-s}^t (\sigma - t + s)v_{m\sigma}(\sigma)\,d\sigma\,.$$

By the Hölder's Inequality, we deduce that

$$\|b_m\|_Y \leq \int_{t-s}^t \|v_{m\sigma}(\sigma)\|_Y\,d\sigma \leq s^{1/q}\left(\int_{t-s}^t \|v_{m\sigma}(\sigma)\|_Y^p\,d\sigma\right)^{1/p} \leq s^{1/q}\|v_{mt}\|_{L^p(0,T;Y)}\,.$$

Let $\epsilon > 0$ be given. Since v_{mt} is uniformly bounded in $L^p(0,T;Y)$, we can choose $s > 0$

small enough so that

$$\|b_m\|_Y \leq \frac{\epsilon}{2}, \quad \text{for all} \quad m \geq 1\,. \tag{1.4.8}$$

For this value of s, it follows that

$$\int_{t-s}^{t} v_m(\sigma)\,d\sigma \rightharpoonup 0 \quad \text{in} \quad X\,.$$

Indeed, if $w \in X^*$ and χ is the characteristic function of $[t-s, t]$, then $\chi w \in L^2(0,T;X^*)$ so that

$$\left\langle w, \int_{t-s}^{t} v_m(\sigma)\,d\sigma \right\rangle = \int_{t-s}^{t} \langle w, v_m(\sigma)\rangle\,d\sigma = \int_0^T \langle \chi w, v_m(\sigma)\rangle\,d\sigma \rightharpoonup 0$$

as $m \to \infty$ since $v_m \rightharpoonup 0$ in $L^2(0,T;X)$. Therefore, $a_m \rightharpoonup 0$ in X. Since X is compactly embedded in Y, there is a subsequence a_m (with same notation) such that $a_m \to 0$ in Y. Then for large enough m's, we have

$$\|a_m\|_Y < \frac{\epsilon}{2}\,. \tag{1.4.9}$$

Then from (1.4.7)-(1.4.9) we deduce that $\|v_m(t)\|_Y < \epsilon$, for large enough m's so that $v_m(t) \to 0$ in Y, for all $t \in (0,T]$. By Lebegue's Dominated Convergence Theorem, it follows that

$$v_m \to 0 \quad \text{in} \quad L^2(0,T;Y)\,.$$

By Ehrling's Lemma, for each $\epsilon > 0$, there is C_ϵ such that

$$\|v_m\|_H^2 \leq \epsilon\|v_m\|_X^2 + C_\epsilon\|v_m\|_Y^2, \quad \text{for each} \quad m\,. \tag{1.4.10}$$

Since v_m is uniformly bounded in $L^2(0,T;X)$, by integrating the inequality (1.4.10) from 0 to T, we deduce that

$$\|v_m\|_{L^2(0,T;H)}^2 \leq \epsilon\|v_m\|_{L^2(0,T;X)}^2 + C_\epsilon\|v_m\|_{L^2(0,T;Y)}^2 \leq \epsilon M + C_\epsilon\|v_m\|_{L^2(0,T;Y)}^2, \tag{1.4.11}$$

for each m. If we take lim sup from both sides of the last inequality (1.4.11), then we obtain

$$\limsup_m \|v_m\|_{L^2(0,T;H)}^2 \leq \epsilon M, \quad \text{for any} \quad \epsilon > 0\,.$$

If we let $\epsilon \to 0$ in the last inequality, then we deduce that $\lim_{m\to\infty} \|v_m\|^2_{L^2(0,T;H)} = 0$. Therefore,

$$u_m \to u \quad \text{in} \quad L^2(0,T;H)\,.$$

□

1.5 Spectral Theory of Unbounded Symmetric Operators

Definition 1.5.1. *Let X and Y be Banach spaces. A (nonlinear) operator $A : X \to Y$ is called compact if the image under A of any bounded set in X is precompact in Y.*

Lemma 1.5.2. *Any compact operator between two Banach spaces is bounded.*

Definition 1.5.3. *A bounded linear operator $A : H \to H$ is called symmetric if*

$$(u, Av)_H = (Au, v)_H\,, \quad \textit{for all} \quad u, v \in H\,.$$

Lemma 1.5.4. *Let $A : H \to H$ be a symmetric operator. Then*

$$\|A\| = \sup_{\|u\|=1} |(Au, u)|\,. \tag{1.5.1}$$

Definition 1.5.5. *We say that a complex number λ is an eigenvalue of a linear operator A if there is a nonzero vector u (the eigenvector) satisfying $Au = \lambda u$.*

Lemma 1.5.6. *If A is a compact symmetric operator, then at least one of $\pm\|A\|$ is an eigenvalue of A.*

Hilbert-Schmidt Theorem. *Let A be a compact symmetric operator acting on an infinite-dimensional Hilbert space H. Then all eigenvalues λ_k of A are real, $|\lambda_k|$ is monotonically decreasing and $\lim_{k\to\infty} \lambda_k = 0$. Furthermore, the eigenvectors w_k can be chosen so that they form an orthonormal basis for the range of A and the action of A on any $u \in H$ is given by*

$$Au = \sum_{k=1}^{\infty} \lambda_k (u, w_k) w_k\,.$$

Proof. By Lemma (1.5.6), there exits w_1 such that $\|w_1\| = 1$ and $Aw_1 = \lambda_1 w_1$, where

$\lambda_1 := \pm\|A\|$. Let H_1 be the subspace of H perpendicular to w_1. If $u \perp w_1$, then

$$(Au, w_1) = (u, Aw_1) = \lambda_1(u, w_1) = 0.$$

So, A maps H_1 into itself. If we consider $A_1 := A|_{H_1} : H_1 \to H_1$, then we have another compact symmetric operator such that $\|A_1\| \leq \|A\|$. Then by applying the same argument on A_1, we obtain an eigenvalue $\lambda_2 := \pm\|A_1\|$ and a corresponding unit eigenvector w_2 which is perpendicular to w_1. Let H_2 be the subspace of H_1 perpendicular to w_1 and w_2. Then A_1 maps H_2 into itself. Similarly, if we consider $A_2 := A_1|_{H_1} : H_2 \to H_2$, then again we have another compact symmetric operator such that $\|A_2\| \leq \|A_1\|$. Then by applying the same argument on A_2, we obtain an eigenvalue $\lambda_3 := \pm\|A_2\|$ and a corresponding unit eigenvector w_3 which is perpendicular to w_1 and w_2. If we continue in this way, then we obtain a sequence of orthonormal eigenvectors w_k such that

$$Aw_k = \lambda_k w_k \quad \text{and} \quad |\lambda_{k+1}| \leq |\lambda_k|\,, \quad \text{for} \quad k = 1, 2, 3, \ldots$$

Suppose that the monotone sequence λ_k does not converge to zero. Then λ_k must be bounded below by some positive constant γ. Since

$$\gamma w_k = A\left(\frac{\gamma}{\lambda_k} w_k\right) \quad \text{and} \quad \left\|\frac{\gamma}{\lambda_k} w_k\right\| < 1\,,$$

the orthogonal sequence γw_k is a subset of $A(B_1(0))$ and it has no convergent subsequences which contradicts to the compactness of A. So, $\lim_{k\to\infty} \lambda_k = 0$. If u is orthogonal to all eigenvectors w_k, then

$$\|Au\| \leq |\lambda_k|\|u\|, \quad \text{for all} \quad k \geq 1\,.$$

So, $Au = 0$ i.e. u must belong to the kernel of A. Therefore, there are no more nonzero eigenvalues of A. In particular, there are no complex eigenvalues of A. Finally, if we let W be the linear span of all eigenvectors w_k equipped with the inner product $(\cdot, \cdot)$, then W is a closed subspace of H with an orthonormal basis consisting of the eigenvectors w_k. Then H can be represented as a direct sum $H = W \oplus \ker A$ so that any element $u \in H$ can be written as

$$u = \sum_{k=1}^{\infty} (u, w_k) w_k + v \,,$$

where $v \in \ker A$. If we let $P_n u := \sum_{k=1}^{n} (u, w_k) w_k$, then

$$\|Au - \sum_{k=1}^{n} \lambda_k (u, w_k) w_k\| = \|A(u - v - P_n u)\| \leq \|A\| \|u - v - P_n u\| \to 0$$

as $n \to \infty$. Therefore,

$$Au = \sum_{k=1}^{\infty} \lambda_k (u, w_k) w_k$$

and the orthonormal sequence w_k form a basis for the range of A.

□

Corollary 1.5.7. *If A is invertible and satisfies the conditions of the Hilbert-Schmidt Theorem, then there is a basis of H consisting entirely of eigenvectors of A.*

Definition 1.5.8. *An unbounded operator $A : \mathcal{D}(A) \subset H \to H$ is called symmetric if*

$$(u, Av) = (Au, v) \,, \quad \textit{for all} \quad u, v \in \mathcal{D}(A) \,.$$

Lemma 1.5.9. *If A is an unbounded symmetric operator whose range is whole of H and whose inverse is well-defined, then A^{-1} is bounded and symmetric.*

Proof. If A^{-1} is not bounded, then there is a sequence of unit vectors $Av_k \in \mathcal{D}(A)$ such that $\|v_k\| \to \infty$ as $k \to \infty$. Let T_k be a bounded linear operator defined by

$$v \mapsto (v, v_k) \,.$$

If $v \in H = \text{ran} A$, then there is $u \in \mathcal{D}(A)$ with $Au = v$ so that

$$|(v_k, v)| = |(v_k, Au)| = |(Av_k, u)| \leq \|u\| \,, \quad \text{for all} \quad k \geq 1 \,.$$

Therefore, $\sup_{k \geq 1} \|T_k v\| \leq \|u\|$. By the Banach-Steinhaus Theorem, the sequence of operators T_k must be uniformly bounded on H i.e. $\sup_{k \geq 1} \|T_k\| \leq \infty$. Since $\|T_k\| \leq \|v_k\|$ and $\|T_k v_k\| = \|v_k\|^2$, we have $\|T_k\| = \|v_k\|$. Then the sequence v_k must be bounded which is a contradiction. So, A^{-1} is bounded. Let $v, \bar{v} \in H$ with $v = Au$ and $\bar{v} = A\bar{u}$, where $u, \bar{u} \in \mathcal{D}(\mathcal{A})$. Then

$$(A^{-1}v, \bar{v}) = (A^{-1}Au, A\bar{u}) = (u, A\bar{u}) = (Au, \bar{u}) = (v, A^{-1}\bar{v})\,.$$

So, A^{-1} is symmetric. Now, any eigenvector of A is an eigenvector of A^{-1} and vice versa :

$$Aw_k = \lambda_k w_k \quad \text{if and only if} \quad A^{-1}w_k = \lambda_k^{-1} w_k\,.$$

□

Corollary 1.5.10. *Let A be a symmetric operator acting on an infinite-dimensional Hilbert space H whose range is all of H. Suppose that H has compact inverse. Then A has infinite set of eigenvalues λ_k such that $|\lambda_k|$ is monotonically increasing and $\lim_{k\to\infty} |\lambda_k| = \infty$. Furthermore, the eigenvectors w_k can be chosen so that they form an orthonormal basis for H. In terms of this basis, the operator A can be represented as*

$$Au = \sum_{k=1}^{\infty} \lambda_k (u, w_k) w_k\,, \quad \textit{for all} \quad u \in \mathcal{D}(A)\,.$$

Definition 1.5.11. *An operator $A : \mathcal{D}(A) \subset H \to H$ is called positive if there $\mu > 0$ such that the following inequality is satisfied :*

$$(Au, u) \geq \mu \|u\|^2\,, \quad \textit{for all} \quad u \in \mathcal{D}(A)\,.$$

Remark 1.5.12. *For a positive operator A satisfying the conditions of the Corollary 1.5.10, we can define fractional powers of A as follows :*

$$A^s u := \sum_{k=1}^{\infty} \lambda_k^s (u, w_k) w_k\,, \quad u \in \mathcal{D}(A^s)\,,$$

where the domain of A^s is given by

$$\mathcal{D}(A^s) = \{u : \|A^s u\| < \infty\} = \left\{ u : u = \sum_{k=1}^{\infty} (u, w_k) w_k \ \text{ s.t. } \ \sum_{k=1}^{\infty} |(u, w_k)|^2 \lambda_k^{2s} < \infty \right\}.$$

Then $\mathcal{D}(A^s)$ becomes a Hilbert space when equipped with the following inner product and the *corresponding norm* :

$$(u,v)_{\mathcal{D}(A^s)} := (A^s u, A^s v) \quad \text{and} \quad \|u\|_{\mathcal{D}(A^s)} := \|A^s u\| \,.$$

1.6 Eigenfunctions of The Laplace Operator

Let $T : H_0^1(\Omega) \to H^{-1}(\Omega)$ be a linear operator defined by $\langle Tu, v\rangle := \int_\Omega \nabla u \nabla v \, dx$, $v \in H_0^1(\Omega)$, where $\langle \cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Then it follows that the operator T is a bijective isometry between the spaces $H_0^1(\Omega)$ and $H^{-1}(\Omega)$:

Theorem 1.6.1. *For any $h \in H^{-1}(\Omega)$, there is a unique $u \in H_0^1(\Omega)$ such that*

$$Tu = h \text{ in } H^{-1}(\Omega) \quad \text{and} \quad \|h\|_{H^{-1}(\Omega)} = \|u\|_{H_0^1(\Omega)} \,.$$

Remark 1.6.2. *If we restrict the domain of the operator T on the following set* :

$$\mathcal{D}(A) := \{u \in H_0^1(\Omega) : Tu \in L^2(\Omega)\} \,,$$

then we get an unbounded linear operator $A := T|_{\mathcal{D}(A)}$ on $L^2(\Omega)$ with full range :

$$A : \mathcal{D}(A) \subset L^2(\Omega) \to L^2(\Omega) \,.$$

Lemma 1.6.3. *The operator A is symmetric and has compact inverse.*

Proof. Every element h of $L^2(\Omega)$ give rise to a bounded linear operator on $H_0^1(\Omega)$ by the following definition :

$$\langle h, v\rangle := \int_\Omega hv \, dx \,, \quad \text{for all} \quad v \in H_0^1(\Omega) \,.$$

Therefore, for any $u, v \in H_0^1(\Omega)$, we have

$$(Au, v) = \int_\Omega Auv \, dx = \langle Au, v\rangle = \int_\Omega \nabla u \nabla v \, dx = \langle Av, u\rangle = \int_\Omega Avu \, dx = (u, Av) \,.$$

So, A is symmetric. Let $u \in \ker A$. By the Poincaré Inequality, we have

$$\frac{1}{C}\int_\Omega u^2 \, dx \le \int_\Omega |\nabla u|^2 \, dx = \langle Au, u\rangle = 0 \,,$$

where C is the Poincaré constant. Then $u = 0$. Therefore, the kernel of A is trivial and A must be invertible. Note that $L^2(\Omega)$ is continuously embedded in $H^{-1}(\Omega)$:

$$\|u\|_{H^{-1}(\Omega)} \leq C\|u\|\,, \quad \text{for all} \quad u \in L^2(\Omega)\,.$$

If $u \in L^2(\Omega)$, then there exists a unique $v \in H_0^1(\Omega)$ such that $Av = u$ (or $v = A^{-1}u$) and $\|u\|_{H^{-1}(\Omega)} = \|v\|_{H_0^1(\Omega)}$. Therefore,

$$\|A^{-1}u\|_{H_0^1(\Omega)} \leq C\|u\|\,, \quad \text{for all} \quad u \in L^2(\Omega)$$

i.e. A^{-1} is a bounded map from $L^2(\Omega)$ into $H_0^1(\Omega)$. Since bounded sets in $H_0^1(\Omega)$ are precompact in $L^2(\Omega)$, the inverse map $A^{-1} : L^2(\Omega) \to L^2(\Omega)$ is compact.

□

Remark 1.6.4. *If $u \in C^2(\overline{\Omega}) \cap H_0^1(\Omega)$, then from the Green's identities it follows that*

$$\int_\Omega Auv\,dx = \int_\Omega \nabla u \nabla v\,dx = \int_\Omega -\Delta uv\,dx\,, \quad \textit{for all} \quad v \in H_0^1(\Omega)\,.$$

Therefore, $Au = -\Delta u$ everywhere within Ω.

Theorem 1.6.5. *Suppose that $h \in H^k(\Omega)$ and u is the unique element of $H_0^1(\Omega)$ satisfying*

$$\langle h, v\rangle = \int_\Omega \nabla u \nabla v\,dx \quad \textit{for all} \quad v \in H_0^1(\Omega)\,.$$

Then $u \in H_{\text{loc}}^{k+2}(\Omega)$: for each $K \subset\subset \Omega$, we have the estimate

$$\|u\|_{H^{k+2}(K)} \leq C_K\|h\|_{H^k(\Omega)}\,.$$

Furthermore, if Ω is of class C^{k+2}, then $u \in H^{k+2}(\Omega)$ with the estimate

$$\|u\|_{H^{k+2}(\Omega)} \leq C\|h\|_{H^k(\Omega)}\,.$$

Definition 1.6.6. *We say that a real number λ is an eigenvalue of the Laplace operator $-\Delta$ with Dirichlet boundary condition if there is a nonzero function $u \in C^2(\Omega)$ (the eigenfunction) satisfying $-\Delta u = \lambda u$ and $u|_{\partial\Omega} = 0$.*

Theorem 1.6.7. *Let Ω be a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary. There is an orthonormal basis of $L^2(\Omega)$ consisting of the eigenfunctions of the Laplace operator $-\Delta$ with Dirichlet boundary condition. These eigenfunctions are elements of $C^\infty(\Omega) \cap H_0^1(\Omega)$.*

Proof. By the Corollary 1.5.10 and the Lemma 1.6.3, there is an orthonormal basis of $L^2(\Omega)$ which consists entirely of eigenfunctions $w_k \in H_0^1(\Omega)$ of the operator A. The smoothness of these eigenfunctions on the interior of Ω follows by applying the Theorem 1.6.5 over and over to the right hand side of the equality

$$Aw_k = \lambda_k w_k \quad \text{in} \quad H^{-1}(\Omega)\,,$$

where λ_k's are the eigenvalues of A. Then, $u \in C^\infty(\Omega) \cap H_0^1(\Omega)$ and $-\Delta w_k = \lambda_k w_k$, by the Remark 1.6.4.

□

Theorem 1.6.8 (Orthogonal basis for $H_0^1(\Omega)$)**.** *For any $u \in H_0^1(\Omega)$, we have*

$$u = \sum_{k=1}^{\infty} (u, w_k) w_k \quad \textit{in} \quad H_0^1(\Omega).$$

Proof. Observe that the functions $\frac{w_k}{\sqrt{\lambda_k}}$ form an orthonormal set in $H_0^1(\Omega)$. Then for any $u \in H_0^1(\Omega)$, the identities

$$(u, w_k)_{H_0^1(\Omega)} = \int_\Omega \nabla u, \nabla w_k \, dx = \lambda_k (u, w_k) = 0 \quad k = 1, 2, 3, \ldots$$

force $u = 0$, since the eigenfunctions w_k form an orthonormal basis for $L^2(\Omega)$. Therefore, the functions $\frac{w_k}{\sqrt{\lambda_k}}$ form an orthonormal basis for $H_0^1(\Omega)$. Then any $u \in H_0^1(\Omega)$ can be written as

$$u = \sum_{k=1}^{\infty} \beta_n \frac{w_k}{\sqrt{\lambda_k}} \quad \text{in} \quad H_0^1(\Omega)\,,$$

where

$$\beta_k = (u, \frac{w_k}{\sqrt{\lambda_k}})_{H_0^1(\Omega)} = \frac{1}{\sqrt{\lambda_k}} \int_\Omega \nabla u \nabla w_k \, dx = \sqrt{\lambda_k}(u, w_k)\,, \quad \text{for each} \quad k \geq 1.$$

□

Remark 1.6.9. *The smallest eigenvalue* λ_1 *of the Laplace operator* $-\Delta$ *with Dirichlet boundary condition is positive and satisfies the following inequality :*

$$\int_\Omega u^2\,dx \le \lambda_1^{-1}\int_\Omega |\nabla u|^2\,dx\,, \quad \textit{for all} \quad u \in H_0^1(\Omega)$$

so that λ_1^{-1} *is the smallest Poincaré constant.*

Remark 1.6.10. *The operator* A *satisfies* $(Au,u) \ge \lambda_1 \|u\|^2$*, for all* $u \in \mathcal{D}(A)$*. Therefore,* A *is positive and it is possible to define its fractional powers.*

Lemma 1.6.11. *If* $k \in \mathbb{Z}^+$ *and* Ω *is of class* C^k*, then we have the following inequality :*

$$\|A^{k/2}u\| \le \|u\|_{H^k(\Omega)} \le C\|A^{k/2}u\|\,, \quad \textit{for all} \quad u \in \mathcal{D}(A^{k/2}).$$

Lemma 1.6.12. *Let* Ω *be a bounded domain in* $\mathbb{R}^d$ *with sufficiently smooth boundary and* $p \in [1,\infty)$*. Then the linear span of the eigenfunctions* w_k *of the Laplace operator* $-\Delta$ *with Dirichlet boundary condition is dense in* $L^p(\Omega)$*.*

Proof. By the Sobolev Embedding Theorem, we can choose $k \in \mathbb{Z}^+$ with $k \ge \frac{d(p-2)}{2p}$ such that $H^k(\Omega)$ is continuously embedded in $L^p(\Omega)$. If Ω is of class C^k, then from the Lemma 1.6.11 it follows that the Hilbert space $\mathcal{D}(A^{s/2})$ is continuously embedded in $H^s(\Omega)$. Therefore, $\mathcal{D}(A^{s/2})$ is continuously embedded in $L^p(\Omega)$. The space of test functions $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ and belong to $\mathcal{D}(A^{s/2})$. Therefore, $\mathcal{D}(A^{s/2})$ is dense in $L^p(\Omega)$. Since the eigenfunctions w_k form a basis for the Hilbert space $\mathcal{D}(A^{s/2})$, their linear span is dense in $\mathcal{D}(A^{s/2})$. Therefore, the linear span of the eigenfunctions w_k is dense in $L^p(\Omega)$.

□

Theorem 1.6.13. *Let* Ω *be a bounded domain in* $\mathbb{R}^d$ *with sufficiently smooth boundary. If* w_k *are the eigenfunctions of the Laplace operator* $-\Delta$ *with Dirichlet boundary condition, then for any* $u \in H_0^1(\Omega) \cap L^p(\Omega)$*, there is a sequence* $u_m = \sum_{k=1}^m c_{km} w_k$ *which converges to* u *in* $L^2(\Omega)$ *and has a subsequence that converges to* u *in* $H_0^1(\Omega) \cap L^p(\Omega)$*. Furthermore,*

$$\sup_{k\ge 1} |c_{km} - (u,w_k)| \to 0 \quad \textit{as} \quad m \to \infty\,.$$

1.7 *Gelfand Triples*

Let H be a Hilbert space and V be a linear subspace which is dense in H. Suppose that V has its own norm $\|\cdot\|_V$ and that V is a Banach space with respect to $\|\cdot\|_V$. Suppose also that the injection $V \hookrightarrow H$ is continuous. Then there is a canonical map $T : H^* \to V^*$ that is simply the restriction to V of continuous linear functionals u on H :

$$\langle Tu, v\rangle_{V^*,V} := \langle u, v\rangle_{H^*,H}, \quad \text{for all} \quad v \in V\,.$$

The canonical map T has the following properties : (i) T is injective; (ii) $\|Tu\|_{V^*} \leq C\|u\|_{H^*}$; (iii) if V is reflexive, then the range of T is dense in V^*. By identifying H^* with H and by using T as a canonical embedding from H^* into V^*, we can write

$$V \hookrightarrow H \simeq H^* \hookrightarrow V^*\,,$$

where all the embeddings are continuous and dense. We call such a triple a Hilbert triple.

Example Let $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ and $V^* = H^{-1}(\Omega)$. Then the canonical map $T : L^2(\Omega) \to H^{-1}(\Omega)$ is defined by the identification of a square integrable function u with its corresponding distribution :

$$\langle Tu, v\rangle_{V^*,V} := \int_\Omega u(x)v(x)\,dx \quad \text{for all} \quad v \in V\,.$$

Then the following embeddings are continuous and dense :

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \simeq (L^2(\Omega))^* \hookrightarrow H^{-1}(\Omega)\,.$$

Example Let $V = H_0^1(\Omega) \cap L^p(\Omega)$, $H = L^2(\Omega)$ and $V^* = H^{-1}(\Omega) + L^q(\Omega)$. The action of $u \in L^2(\Omega)$ on a test function $v \in H_0^1(\Omega) \cap L^p(\Omega)$ is given by

$$\langle Tu, v\rangle_{V^*,V} := \int_\Omega u(x)v(x)\,dx\,. \tag{1.7.1}$$

Since $H_0^1(\Omega) \cap L^p(\Omega)$ is reflexive, the embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega) + L^q(\Omega)$ which is given by (1.7.1) is dense. Therefore, the following embeddings are continuous and dense :

$$H_0^1(\Omega) \cap L^p(\Omega) \hookrightarrow L^2(\Omega) \simeq (L^2(\Omega))^* \hookrightarrow H^{-1}(\Omega) + L^q(\Omega)\,.$$

Example Let $V = H_0^2(\Omega)$, $H = H_0^1(\Omega)$ and $V^* = H^{-2}(\Omega)$. The action of $u \in H_0^1(\Omega)$ on a test function $v \in H_0^2(\Omega)$ is given by

$$\langle Tu, v\rangle_{V^*,V} := \int_\Omega \nabla u(x) \nabla v(x)\, dx\,.$$

Then the following embeddings are continuous and dense :

$$H_0^2(\Omega) \hookrightarrow H_0^1(\Omega) \simeq (H_0^1(\Omega))^* \hookrightarrow H^{-2}(\Omega)\,.$$

Theorem 1.7.1. *Let $V \hookrightarrow H \simeq H^* \hookrightarrow V^*$ be a Hilbert triple. If $u \in L^2(0,T;V)$ and $u_t \in L^2(0,T;V^*)$, then $u \in C([0,T];H)$. Furthermore,*

(i) for any $v \in V$, the real-valued function $t \mapsto (u(t),v)_H$ is weakly differentiable in $(0,T)$ and

$$\frac{d}{dt}(u(t),v)_H = \langle u_t(t), v\rangle_{V^*,V};$$

(ii) the real-valued function $t \mapsto \|u(t)\|_H^2$ is weakly differentiable in $(0,T)$ and

$$\frac{d}{dt}\|u(t)\|_H^2 = 2\langle u_t(t), u(t)\rangle_{V^*,V};$$

(iii) there is a constant $C = C(T)$ such that

$$\|u\|_{L^\infty(0,T;H)} \le C\left[\|u\|_{L^2(0,T;V)} + \|u_t\|_{L^2(0,T;V^*)}\right].$$

Theorem 1.7.2 (Integration by parts formula)**.** *Let $V \hookrightarrow H \simeq H^* \hookrightarrow V^*$ be a Hilbert triple. Suppose that $u, v \in L^2(0,T;V)$ and $u_t, v_t \in L^2(0,T;V^*)$. Then*

$$\int_0^T \langle u_t(t), v(t)\rangle_{V^*,V} dt = (u(T),v(T))_H - (u(0),v(0))_H - \int_0^T \langle v_t(t), u(t)\rangle_{V^*,V} dt\,.$$

1.8 Monotone Operators

Definition 1.8.1. *Let $V \hookrightarrow H \simeq H^* \hookrightarrow V^*$ be a Hilbert triple. A nonlinear operator f from V into V^* is called a monotone operator if it satisfies the following condition:*

$$\big(f(u) - f(v), u - v\big) \geq 0, \quad \textit{for all} \quad u, v \in V.$$

Theorem 1.8.2. *Let $1 < p < \infty$ and f be a monotone operator which satisfies the following conditions: (i) $\|f(u)\|_{V^*} \leq C\left[1 + \|u\|^{p-1}\right]$, for all $u \in V$; (ii) For fixed $u, v, w \in V$, the mapping $\lambda \mapsto \big(f(u + \lambda v, w)\big)$ is continuous on $\mathbb{R}$. Suppose that: (1) $u_m \rightharpoonup u$ in V; (2) $f(u_m) \rightharpoonup \psi$ in V^*; (3) $\limsup \big(f(u_m), u_m\big) \leq \big(\psi, u\big)$. Then $\psi = f(u)$.*

Proof. Since f is a monotone operator, it follows that

$$\big(f(u_m), u_m\big) - \big(f(u_m), v\big) - (f(v), u_m - v) \geq 0, \quad \text{for all} \quad v \in V. \tag{1.8.1}$$

If we take the limit superior of the inequality (1.8.1), then we obtain

$$(\psi, u) \geq \limsup \big(f(u_m), u_m\big) \geq \big(\psi, v\big) + (f(v), u - v)\,.$$

Therefore, $\big(\psi - f(v), u - v\big) \geq 0$. If we let $v := u + \lambda w$ with $w \in V$, then we get

$$\lambda\big(\psi - f(u + \lambda w), w\big) \geq 0, \quad \text{for all} \quad w \in V.$$

If we let $\lambda \to 0^+$, then from the last inequality we deduce that $\big(\psi - f(u), w\big) \geq 0$. Similarly, by letting $\lambda \to 0^-$, we obtain $\big(\psi - f(u), w\big) \leq 0$. Therefore, $\big(\psi - f(u), w\big) = 0$ for all $w \in V$ so that $\psi = f(u)$. □

Corollary 1.8.3. *Let $p, q \in (1, \infty)$ with $1/p + 1/q = 1$ and f be a monotone operator from a separable Banach space V into V^* which satisfies the following conditions: (i) $\|f(u)\|_{V^*} \leq C\left[1 + \|u\|^{p-1}\right]$, for all $u \in V$; (ii) For any fixed $u, v, w \in V$, the mapping $\lambda \mapsto \big(f(u + \lambda v, w)\big)$ is continuous on $\mathbb{R}$. Suppose that:*

$$u_m \rightharpoonup u \quad \textit{in} \quad L^p(0,T;V)\,; \quad f(u_m) \rightharpoonup \psi \quad \textit{in} \quad L^q(0,T;V^*)\,;$$

$$\limsup \int_0^T \big(f(u_m), u_m\big)\, dt \le \int_0^T \big(\psi, u\big)\, dt.$$

Then $\psi = f(u)$.

1.9 Semigroups and Attractors

Definition 1.9.1. *Let H be a Banach space. A family $\{S(t)\}_{t\in\mathbb{R}^+}$ of nonlinear operators, $S(t) : H \to H$, is called a continuous semigroup on H if the following properties are satisfied : (i) $S(0)$ is the identity map on H ; (ii) $S(t_1+t_2) = S(t_1)S(t_2)$, for all $t_1, t_2 \in \mathbb{R}^+$; (iii) The mapping $(t, u) \mapsto S(t)u$ is continuous on $[0, \infty) \times H$.*

Definition 1.9.2. *A continuous semigroup $\{S(t)\}_{t\in\mathbb{R}^+}$ on a Banach space H is called compact semigroup if for each $t \in \mathbb{R}^+$, the operator $S(t)$ is compact.*

Definition 1.9.3. *Let $\{S(t)\}_{t\in\mathbb{R}^+}$ be a continuous semigroup on a Banach space H.*
(1) A set B in H is called positively invariant if for all $t > 0$, $S(t)B \subset B$;
(2) A set B in H is called invariant if for all $t \in \mathbb{R}^+$, $S(t)B = B$;
(3) For $u \in H$, the set $\bigcup_{t\ge 0} S(t)u$ is called an orbit starting from u ;
(4) The ω-limit set $\omega(B)$ of a subset B of H is defined as follows :

$$\omega(B) := \bigcap_{s\ge 0} \overline{E(s)}\,, \quad \textit{where} \quad E(s) := \bigcup_{t\ge s} S(t)B.$$

Remark 1.9.4. *It is easy to see that $u \in \omega(B)$ if and only if there is a sequence $u_m \in B$ and a sequence $t_m \to \infty$ such that $S(t_m)u_m \to u$ in H as $m \to \infty$.*

Lemma 1.9.5. *Let $\{S(t)\}_{t\in\mathbb{R}^+}$ be a continuous semigroup on a Banach space H. If B is a nonempty subset of H, then the ω-limit set $\omega(B)$ is positively invariant.*

Proof. For any fixed $t > 0$, if $\psi \in S(t)\omega(B)$, then there is $\phi \in \omega(B)$ such that $\psi = S(t)\phi$. By the definition of ω-limit set, there exist sequences $\phi_m \in B$ and $t_m \to \infty$ such that $S(t_m)\phi_m \to \phi$ in H. Then by the continuity of $S(t)$, we have

$$S(t + t_m)\phi_m = S(t)S(t_m)\phi_m \to S(t)\phi \quad \text{as} \quad m \to \infty.$$

Therefore, $\psi = S(t)\phi \in \omega(B)$. Then $S(t)\omega(B) \subset \omega(B)$, for all $t \ge 0$.

□

Lemma 1.9.6. *Let $\{S(t)\}_{t\in\mathbb{R}^+}$ be a continuous semigroup on a Banach space H. If B be a nonempty subset of H and $s_0 > 0$ such that the set*

$$E(s_0) = \bigcup_{t\geq s_0} S(t)B$$

is relatively compact in H, then the ω-limit set $\omega(B)$ is nonempty, compact and invariant. Furthermore, if B is connected, then $\omega(B)$ is also connected.

Proof. Since B is nonempty, the set $E(s)$ is nonempty for each $s \geq 0$. Then we have a decreasing chain of nonempty compact sets :

$$\overline{E(s_0)} \supset \overline{E(s_1)} \supset \overline{E(s_2)}\,, \quad \text{for all} \quad s_0 \leq s_1 \leq s_2.$$

Hence, $\omega(B) = \bigcap_{s\geq 0} \overline{E(s)}$ is nonempty and compact. By the Lemma 1.9.5, $\omega(B)$ is positively invariant. Let $\phi \in \omega(B)$. Then there exist sequences $\phi_m \in B$ and $t_m \to \infty$ such that $S(t_m)\phi_m \to \phi$ in H. By the assumption, for any fixed $t > 0$, , ϕ_m is relatively compact in H. Hence, there is a subsequence $S(t_m - t)\phi_m$ (with the same notation) such that $S(t_m - t)\phi_m \to \psi$ in H. This implies that $\psi \in \omega(B)$. By the continuity of $S(t)$, we have

$$S(t_m)\phi_m = S(t)S(t_m - t)\phi_m \to S(t)\psi = \phi \quad \text{as} \quad m \to \infty.$$

Therefore, $\phi \in S(t)\omega(B)$, for any $t \geq 0$. Hence, $\omega(B)$ is invariant. Suppose that B is connected. Since the mapping $(t, u) \mapsto S(t)u$ is continuous on $[0, \infty) \times H$, it follows that $E(s)$ is also connected, for each $s \geq 0$. Suppose for a contradiction that $\omega(B)$ is not connected. Then there are two open sets U_1 and U_2 such that

$$\omega(B) \subset U_1 \cup U_2, \quad \omega(B) \cap U_1 \neq \emptyset \quad \omega(B) \cap U_2 \neq \emptyset, \quad U_1 \cap U_2 = \emptyset.$$

Let $V_{1\epsilon}$ be the ϵ-neighborhood of $\omega(B) \cap U_1$ and $V_{2\epsilon}$ be the ϵ-neighborhood of $\omega(B) \cap U_2$ such that

$$\omega(B) \cap V_{1\epsilon} \neq \emptyset, \quad \omega(B) \cap V_{2\epsilon} \neq \emptyset, \quad V_{2\epsilon} \cap V_{1\epsilon} = \emptyset.$$

Since $V_{2\epsilon} \cup V_{1\epsilon}$ contains some δ-neighborhood of $\omega(B)$, by the definition of $\omega(B)$, it follows that $E(s)$ eventually enters this δ-neighborhood so that

$$E(s) \subset V_{1\epsilon} \cup V_{2\epsilon}, \quad E(s) \cap V_{1\epsilon} \neq \emptyset \quad E(s) \cap V_{2\epsilon} \neq \emptyset, \quad V_{1\epsilon} \cap V_{2\epsilon} = \emptyset$$

which contradicts to the connectedness of $E(s)$. Therefore, $\omega(B)$ must be connected.

□

Definition 1.9.7 (Attractor)**.** *Let* $\{S(t)\}_{t\in\mathbb{R}^+}$ *be a continuous semigroup on a Banach space* H*. A set* $\mathcal{A}$ *is called attractor if it satisfies the following two properties :* (i) $\mathcal{A}$ *is invariant set ;* (ii) $\mathcal{A}$ *possesses an open set* $\mathcal{U}$ *such that for any element* $u \in \mathcal{U}$*,* $S(t)u$ *converges to* $\mathcal{A}$ *as* $t \to \infty$*, i.e.*

$$\text{dist}(S(t)u, \mathcal{A}) = \inf_{v\in\mathcal{A}} \|S(t)u - v\| \to 0 \quad \textit{as} \quad t \to \infty.$$

The maximal open set $\mathcal{U}$ *satisfying the property* (ii) *is called the basin of attraction of* $\mathcal{A}$*. If a subset* $B \subset \mathcal{U}$ *satisfies*

$$\text{dist}(S(t)B, \mathcal{A}) = \sup_{u\in S(t)B} \inf_{v\in\mathcal{A}} \|u - v\| \to 0 \quad \textit{as} \quad t \to \infty,$$

then we say that $\mathcal{A}$ *attracts* B*.*

Definition 1.9.8 (Global Attractor)**.** *If* $\mathcal{A}$ *is a compact attractor and it attracts bounded sets of* H*, then* $\mathcal{A}$ *is called a global attractor.*

Definition 1.9.9. *Let* $\{S(t)\}_{t\in\mathbb{R}^+}$ *be a continuous semigroup on a Banach space* H*. Suppose that* B_0 *is a subset of* H *and* $\mathcal{U}$ *is an open set containing* B_0*. If for any bounded set* $B \in \mathcal{U}$*, there exists* $t(B) \geq 0$ *such that*

$$S(t)B \subset B_0, \quad \textit{for all} \quad t \geq t(B),$$

then we say that B_0 *is an absorbing set in* $\mathcal{U}$*.*

Theorem 1.9.10. *Let* $\{S(t)\}_{t\in\mathbb{R}^+}$ *be a continuous semigroup on a Banach space* H *which satisfies the following two conditions :* (i) *there exists a bounded absorbing set* B_0 *in* H *and* (ii) *for any bounded set* B*, there exists* $t(B) > 0$ *such that the set* $\bigcup_{t\geq t(B)} S(t)B$ *is precompact in* H*. Then* $\mathcal{A} := \omega(B_0)$ *is a global attractor.*

Proof. From the conditions (i)-(ii) and the Lemma 1.9.6 it follows that $\mathcal{A} = \omega(B_0)$ is a nonempty compact invariant set. Then it is left to show that $\mathcal{A}$ attracts bounded sets of H. Suppose for a contradiction that there is a bounded set B such that when time goes to infinity, $\text{dist}(S(t)B, \mathcal{A})$ does not converge to zero. Then there exist $\delta > 0$ and a sequence $t_m \to \infty$ such that $\text{dist}(S(t_m)B, \mathcal{A}) \geq \delta > 0$, for all $m \geq 1$. Furthermore, for each $m \geq 1$, there is $u_m \in B$ such that

$$\text{dist}(S(t)B, \mathcal{A}) \geq \frac{\delta}{2} > 0. \tag{1.9.1}$$

Since B_0 is an absorbing set, there is $t(B) \geq 0$ such that $S(t_m + t(B))u_m \in B_0$, for all $m \geq 1$. By the condition (ii), the set $S(t_m + t(B))u_m$ is relatively compact. Hence, there is a subsequence $S(t_m + t(B))u_m$ (with the same notation) such that

$$S(t_m)S(t(B))u_m = S(t_m + t(B))u_m \to \phi\,, \quad \text{as} \quad m \to \infty.$$

Since $S(t(B))u_m \in B_0$, it follows that $\phi \in \mathcal{A} = \omega(B_0)$ which contradicts to (1.9.1). Therefore, $\mathcal{A} = \omega(B_0)$ must attract bounded sets of H.

□

Lemma 1.9.11. *If H is connected, then the global attractor $\mathcal{A} = \omega(B_0)$ is connected.*

Proof. If B_0 is a bounded absorbing set, then a ball B containing B_0 is also a bounded absorbing set. Since $\mathcal{A} = \omega(B_0)$ is maximal, we have $\mathcal{A} = \omega(B)$. By the Lemma 1.9.6, it follows that $\mathcal{A}$ is connected since the ball B is clearly connected.

□

Theorem 1.9.12. *Let $\{S(t)\}_{t\in\mathbb{R}^+}$ be a compact semigroup on a Banach space H which has a bounded absorbing set B_0 in H. Then $\mathcal{A} := \omega(B_0)$ is a global attractor.*

Proof. For any bounded set B, there is $t(B) > 0$ such that $\bigcup_{t\geq t(B)} S(t)B$ is a bounded set (being a subset of B_0). Since the operator $S(1)$ is compact, we deduce that the set

$$\bigcup_{t\geq t(B)+1} S(t)B = S(1)\left(\bigcup_{t\geq t(B)} S(t)B\right).$$

is precompact in H. Therefore, the result follows from Theorem 1.9.10. □

Chapter 2

GLOBAL EXISTENCE AND UNIQUENESS

In this chapter, we will consider the following initial-boundary value problem :

$$\tau\varphi_t = \xi^2\Delta\varphi - g(x,\varphi) + 2v \quad \text{in} \quad \Omega\times\mathbb{R}^+\,, \tag{2.0.1}$$

$$v_t = \kappa\Delta v - \frac{\kappa\ell}{2}\Delta\varphi \quad \text{in} \quad \Omega\times\mathbb{R}^+\,, \tag{2.0.2}$$

$$\begin{aligned} \varphi|_{\partial\Omega} &= 0\,, \quad t\in\mathbb{R}^+, \\ v|_{\partial\Omega} &= 0\,, \quad t\in\mathbb{R}^+, \end{aligned} \tag{2.0.3}$$

$$\begin{aligned} \varphi(x,0) &= \varphi_0(x)\,, \quad x\in\Omega\,, \\ v(x,0) &= v_0(x)\,, \quad x\in\Omega\,, \end{aligned} \tag{2.0.4}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$; φ_0, v_0 and g are given functions; under the assumption that the nonlinear term $g(x,s)$ is a function measurable in x and continuously differentiable in s satisfying the following conditions :

$$-\beta_0 + \beta_1|s|^p \le sg(x,s) \le \beta_0 + \beta_2|s|^p\,, \tag{2.0.5}$$

$$g(x,0) = 0\,, \tag{2.0.6}$$

$$g_s(x,s) \ge -\beta_3\,, \tag{2.0.7}$$

for all $x\in\Omega$ and all $s\in\mathbb{R}$, where $p>2$ and β_j's are positive constants. The sufficient smoothness of the boundary $\partial\Omega$ is the following : choose $k\in\mathbb{Z}^+$ and assume that Ω is of class C^k such that the Hilbert space $\mathcal{D}(A^{k/2})$ is continuously embedded in $L^p(\Omega)$ [See Appendix 1.6 for more details]. We will prove that the initial-boundary value problem (2.0.1)-(2.0.4) is uniquely globally solvable within the class

$$C\big(\mathbb{R}^+; H_0^1(\Omega)\times H_0^1(\Omega)\big)\,.$$

2.1 Weak Solutions

First, it will be shown that the initial-boundary value problem (2.0.1)-(2.0.4) has a unique solution from the class

$$C\big(\mathbb{R}^+; L^2(\Omega) \times L^2(\Omega)\big)\,.$$

Such a solution is defined as follows :

Definition 2.1.1 (Weak Solution)**.** *Assume* $[\varphi_0, v_0] \in L^2(\Omega) \times L^2(\Omega)$. *Then a pair of functions* $[\varphi, v] : [0, \infty) \to L^2(\Omega) \times L^2(\Omega)$ *is called a weak solution of the initial-boundary value problem (2.0.1)-(2.0.4) if*
(i) for each $T > 0$,

$$[\varphi, v] \in C\big([0,T]; L^2(\Omega) \times L^2(\Omega)\big) \cap L^2\big(0,T; H_0^1(\Omega) \times H_0^1(\Omega)\big)\,,$$
$$\varphi \in L^p(\Omega_T)\,,$$
$$[\varphi_t, v_t] \in L^2\big(0,T; H^{-1}(\Omega) + L^q(\Omega)\big) \times L^2\big(0,T; H^{-1}(\Omega)\big)\,;$$

(ii) for each $w \in H_0^1(\Omega) \cap L^p(\Omega)$*, the following equality holds for almost all* $t \in \mathbb{R}^+$:

$$\tau\langle\langle \varphi_t(t), w\rangle\rangle + \xi^2 \int_\Omega \nabla\varphi(t,x)\nabla w(x)\,dx = -\int_\Omega g(x,\varphi(t,x))w(x)\,dx + 2\int_\Omega v(t,x)w(x)\,dx\,,$$

where $\langle\langle\cdot,\cdot\rangle\rangle$ *denote the duality pairing between* $H_0^1(\Omega) \cap L^p(\Omega)$ *and* $H^{-1}(\Omega) + L^q(\Omega)$;
(iii) for each $w \in H_0^1(\Omega)$*, the following equality holds for almost all* $t \in \mathbb{R}^+$:

$$\langle v_t(t), w\rangle + \kappa \int_\Omega \nabla v(t,x)\nabla w(x)\,dx = \frac{k\ell}{2}\int_\Omega \nabla\varphi(t,x)\nabla w(x)\,dx\,,$$

where $\langle\cdot,\cdot\rangle$ *denote the duality pairing between* $H_0^1(\Omega)$ *and* $H^{-1}(\Omega)$;
(iv) $\varphi(0) = \varphi_0$ *and* $v(0) = v_0$.

Theorem 2.1.2. *If* $[\varphi_0, v_0] \in L^2(\Omega) \times L^2(\Omega)$*, then the initial-boundary value problem (2.0.1)-(2.0.4) has a unique weak solution* $[\varphi, v]$*. Furthermore, the mapping*

$$[\varphi_0, v_0] \mapsto [\varphi(t), v(t)]$$

is continuous on $L^2(\Omega) \times L^2(\Omega)$.

Proof. Consider the initial-value problem for the system of ordinary differential equations :

$$\tau \int_\Omega \varphi_{mt}(t,x) w_k(x)\, dx + \xi^2 \int_\Omega \nabla\varphi_m(t,x) \nabla w_k(x)\, dx = \\ = -\int_\Omega g(x, \varphi_m(t,x)) w_k(x)\, dx + 2\int_\Omega v_m(t,x) w_k(x)\, dx, \quad (2.1.1)$$

$$\int_\Omega v_{mt}(t,x) w_k(x) dx + \kappa \int_\Omega \nabla v_m(t,x) \nabla w_k(x) dx = \frac{\kappa\ell}{2} \int_\Omega \nabla\varphi_m(t,x) \nabla w_k(x) dx \quad (2.1.2)$$

$$\varphi_{mk}(0) = \int_\Omega \varphi_0(x) w_k(x)\, dx \quad \text{and} \quad v_{mk}(0) = \int_\Omega v_0(x) w_k(x)\, dx, \quad (2.1.3)$$

for $k = 1, 2, 3, \ldots, m$, where φ_m and v_m are the Galerkin approximations :

$$\varphi_m(t,x) = \sum_{k=1}^{m} \varphi_{mk}(t) w_k(x) \quad \text{and} \quad v_m(t,x) = \sum_{k=1}^{m} v_{mk}(t) w_k(x)\,.$$

We can rewrite the system (2.1.1)-(2.1.2) as follows :

$$\tau\varphi'_{mk}(t) = -\xi^2 \lambda_k \varphi_{mk}(t) - \int_\Omega g(x, \varphi_m(t,x) w_k(x)\, dx + 2v_{mk}(t)\,, \\ v'_{mk}(t) = -\kappa\lambda_k v_{mk}(t) + \frac{\kappa\ell}{2} \lambda_k \varphi_{mk}(t)\,,$$

where $\lambda_k = \int_\Omega |\nabla w_k|^2\, dx$ and $k = 1, 2, 3, \ldots, m$. According to the Cauchy-Picard Theorem, the initial value problem (2.1.1)-(2.1.3) has a unique solution on some finite time interval $[0, T^*]$ provided that the functions

$$G_k(s) := \int_\Omega g(x, s) w_k(x)\, dx$$

are locally Lipschitz, for each $k = 1, 2, 3, \ldots, m$. Since $g(x, s)$ is continuously differentiable in s variable, from Theorem 1.4.1, it follows that G_k's are continuously differentiable functions which justifies the existence of a unique solution on some interval $[0, T^*]$. By the Theorem 1.4.5, the time interval $[0, T^*]$ can be extended to infinity if φ_{mk} and v_{mk} remain bounded on every finite interval $[0, T]$.

Now, we will show that the approximate solutions $[\varphi_m, v_m]$ are bounded on every finite time interval $[0, T]$ and uniformly bounded in m. If we multiply the equation (2.1.1) by $\varphi_{mk}(t)$

and take sum over $k = 1, 2, 3, \ldots, m$, then we obtain the equality

$$\frac{\tau}{2}\frac{d}{dt}\int_\Omega |\varphi_m(t,x)|^2 dx + \xi^2 \int_\Omega |\nabla\varphi_m(t,x)|^2 dx = \\ = -\int_\Omega g(x,\varphi_m(t,x))\varphi_m(t,x)\,dx + 2\int_\Omega v_m(t,x)\varphi_m(t,x)\,dx \quad (2.1.4)$$

which holds for each $t \in (0,T]$. The equation (2.1.4) can also be written as follows :

$$\frac{\tau}{2}\frac{d}{dt}\|\varphi_m\|^2 + \xi^2\|\nabla\varphi_m\|^2 = -(g(x,\varphi_m),\varphi_m) + 2(v_m,\varphi_m) \quad (2.1.5)$$

By applying the inequality 1.3.1 and the Poincaré inequality, we deduce

$$2(v_m,\varphi_m) \le \epsilon\|v_m\|^2 + \epsilon\|\varphi_m\|^p_{L^p(\Omega)} + C_0 \le \frac{\epsilon}{\lambda_1}\|\nabla v_m\|^2 + \epsilon\|\varphi_m\|^p_{L^p(\Omega)} + C_0\,, \quad (2.1.6)$$

where $C_0 > 0$ depends only on ϵ, p and Ω. From (2.0.5) we obtain

$$-\,(g(x,\varphi_m),\varphi_m) \le \beta_0|\Omega| - \beta_1\|\varphi_m\|^p_{L^p(\Omega)}\,. \quad (2.1.7)$$

By taking into account (2.1.6) and (2.1.7), from (2.1.5) we deduce

$$\frac{\tau}{2}\frac{d}{dt}\|\varphi_m\|^2 + \xi^2\|\nabla\varphi_m\|^2 + [\beta_1 - \epsilon]\|\varphi_m\|^p_{L^p(\Omega)} \le \frac{\epsilon}{\lambda_1}\|\nabla v_m\|^2 + C_1\,, \quad (2.1.8)$$

where $C_1 := C_0 + \beta_0|\Omega|$.

If we multiply the equation (2.1.2) by $v_{mk}(t)$ and take sum over $k = 1, 2, 3, \ldots, m$, then we obtain the equality

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |v_m(t,x)|^2 dx + \kappa\int_\Omega |\nabla v_m(t,x)|^2 dx = \frac{\kappa\ell}{2}\int_\Omega \nabla\varphi(t,x)\nabla v_m(t,x)\,dx$$

which holds for all $t \in (0,T]$. It can also be written as follows :

$$\frac{1}{2}\frac{d}{dt}\|v_m\|^2 + \kappa\|\nabla v_m\|^2 = \frac{\kappa\ell}{2}(\nabla\varphi_m,\nabla v_m)\,. \quad (2.1.9)$$

By the Cauchy-Schwartz inequality and the Cauchy's inequality with $\epsilon = \ell/4$, we deduce that

$$\frac{\kappa\ell}{2}|(\nabla\varphi_m,\nabla v_m)| \le \frac{\kappa\ell}{2}\|\nabla\varphi_m\|\|\nabla v_m\| \le \frac{\kappa\ell^2}{8}\|\nabla\varphi_m\|^2 + \frac{\kappa}{2}\|\nabla v_m\|^2\,. \tag{2.1.10}$$

By adding (2.1.9) and (2.1.10) we obtain

$$\frac{1}{2}\frac{d}{dt}\|v_m\|^2 + \frac{\kappa}{2}\|\nabla v_m\|^2 \le \frac{\kappa\ell^2}{8}\|\nabla\varphi_m\|^2\,. \tag{2.1.11}$$

If we multiply (2.1.11) by $4\xi^2/\kappa\ell^2$ and add it to (2.1.8), then we obtain

$$\frac{d}{dt}\left[\frac{\tau}{2}\|\varphi_m\|^2 + \frac{2\xi^2}{\kappa\ell^2}\|v_m\|^2\right] + \frac{\xi^2}{2}\|\nabla\varphi_m\|^2 + \left[\frac{2\xi^2}{\ell^2} - \frac{\epsilon}{\lambda_1}\right]\|\nabla v_m\|^2 +$$

$$+[\beta_1-\epsilon]\|\varphi_m\|^p_{L^p(\Omega)} \le C_1\,.$$

If we integrate the last inequality from 0 to t, then we deduce

$$\frac{\tau}{2}\int_\Omega|\varphi_m(t,x)|^2\,dx + \frac{2\xi^2}{\kappa\ell^2}\int_\Omega|v_m(t,x)|^2\,dx + \frac{\xi^2}{2}\int_0^t\int_\Omega|\nabla\varphi_m(s,x)|^2\,dxds +$$

$$+\left[\frac{2\xi^2}{\ell^2} - \frac{\epsilon}{\lambda_1}\right]\int_0^t\int_\Omega|\nabla v_m(s,x)|^2\,dxds + [\beta_1-\epsilon]\int_0^t\int_\Omega\varphi_m(s,x)|^p\,dxds \le$$

$$\le \frac{\tau}{2}\|\varphi_0\|_{L^2(\Omega)} + \frac{2\xi^2}{\kappa\ell^2}\|v_0\|_{L^2(\Omega)} + C_1T\,.$$

If we choose $\epsilon > 0$ sufficiently small and take the supremum over (0,T], then we obtain

$$\|[\varphi_m,v_m]\|_{L^\infty(0,T;L^2(\Omega)\times L^2(\Omega))} + \|[\varphi_m,v_m]\|_{L^2\left(0,T;H^1_0(\Omega)\times H^1_0(\Omega)\right)} + \|\varphi_m\|_{L^p(\Omega_T)} \le C\,, \tag{2.1.12}$$

where C depends only on $\varphi_0, v_0, p, \beta_0, \ell, \kappa, \tau, \xi, \Omega$ and T. From the above estimate it follows that the interval of existence $[0,T^*]$ of functions φ_{mk} and v_{mk} can be extended to $[0,\infty)$. Now, it will be shown that the nonlinear term $f(\varphi_m)$ is also uniformly bounded in m. Due to the assumption (2.0.5), from the Lemma 1.3.2 it follows that g satisfies the following inequality :

$$|g(x,s)| \le \gamma_3(1+|s|^{p-1})\,, \tag{2.1.13}$$

for all $(x,s) \in \Omega\times\mathbb{R}$. Let q be the conjugate of p. Then, by taking into account the inequality (2.1.13), from the Jensen's inequality we obtain

$$\|g(x,\varphi_m)\|^q_{L^q(\Omega_T)} = \int_0^T \int_\Omega |g(x,\varphi_m(t,x))|^q \, dx\, dt \leq \gamma_3^q \int_0^T \int_\Omega (1+|\varphi_m(t,x)|^{p-1})^q \, dx\, dt \leq$$

$$\leq (2\gamma_3)^q \int_0^T \int_\Omega (\frac{1}{2} + \frac{1}{2}|\varphi_m(t,x)|^{p-1})^q \, dx\, dt \leq (2\gamma_3)^q \int_0^T \int_\Omega \frac{1}{2} + \frac{1}{2}|\varphi_m(t,x)|^{(p-1)q} \, dx\, dt\,.$$

From the last inequality, we deduce that

$$\|g(x,\varphi_m)\|^q_{L^q(\Omega_T)} \leq 2^{q-1}\gamma_3^q \left(T|\Omega| + \|\varphi_m\|^p_{L^p(\Omega_T)}\right). \tag{2.1.14}$$

Since φ_m is uniformly bounded in $L^p(\Omega_T)$, from the estimate (2.1.14) it follows that $g(x,\varphi_m)$ is uniformly bounded in $L^q(\Omega_T)$.
From (2.1.2) we deduce that

$$\|v_{mt}\|^2_{H^{-1}(\Omega)} \leq \big(\kappa\|\Delta v_m\|_{H^{-1}(\Omega)} + \frac{\kappa\ell}{2}\|\Delta\varphi_m\|_{H^{-1}(\Omega)}\big)^2 = \big(\kappa\|\nabla v_m\| + \frac{\kappa\ell}{2}\|\nabla\varphi_m\|\big)^2 \leq$$

$$\leq 2\kappa^2\|\nabla v_m\|^2 + \frac{\kappa^2\ell^2}{2}\|\nabla\varphi_m\|^2\,.$$

If we integrate the last inequality from 0 to T, then we obtain

$$\|v_{mt}\|^2_{L^2\left(0,T;H^{-1}(\Omega)\right)} \leq 2\kappa^2\|v_m\|^2_{L^2(0,T;H^1_0(\Omega))} + \frac{\kappa^2\ell^2}{2}\|\varphi_m\|^2_{L^2(0,T;H^1_0(\Omega))}\,.$$

Therefore, v_{mt} is uniformly bounded in $L^2\big(0,T;H^{-1}(\Omega)\big)$.

By taking into account the uniform estimates, thanks to the Banach-Alaoglu Theorem, we can extract a subsequence $[\varphi_n, v_m]$ such that

$$[\varphi_m, v_m] \rightharpoonup [\varphi, v] \quad \text{in} \quad L^2\big(0,T;H^1_0(\Omega)\big) \times L^2\big(0,T;H^1_0(\Omega)\big)\,;$$

$$\varphi_m \rightharpoonup \varphi \quad \text{in} \quad L^p(\Omega_T) \quad \text{and} \quad g(x,\varphi_m) \rightharpoonup \psi \quad \text{in} \quad L^q(\Omega_T)\,.$$

Remark 2.1.3. *Let $w \in H^1_0(\Omega)$ and $\phi \in C^\infty_c(0,T)$. By the definition of distributional derivative with respect to t variable, we have*

$$\int_0^T \langle\varphi_{mt}, w\rangle\phi(t)\, dt = -\int_0^T \int_\Omega \varphi_m(t,x) w(x)\phi'(t)\, dxdt \tag{2.1.15}$$

$$\int_0^T \langle \varphi_t, w\rangle \phi(t)\,dt = -\int_0^T \int_\Omega \varphi(t,x)w(x)\phi'(t)\,dxdt\,. \tag{2.1.16}$$

Since $\varphi_m \rightharpoonup \varphi$ in $L^2\big(0,T;H_0^1(\Omega)\big)$, the right hand side of (2.1.15) converges to the right hand side of (2.1.16). Therefore,

$$\varphi_{mt} \rightharpoonup \varphi_t \;\; in \;\; L^2\big(0,T;H^{-1}(\Omega)\big)\,.$$

Similarly, it follows that

$$u_{mt} \rightharpoonup u_t \;\; in \;\; L^2\big(0,T;H^{-1}(\Omega)\big) \quad and \quad \varphi_{mt} \rightharpoonup \varphi_t \;\; in \;\; L^q(\Omega_T)\,.$$

Remark 2.1.4. *We can replace w_k in the equations (2.1.1)-(2.1.2) with any*

$$w \in E_M := \operatorname{span}\{w_k\}_{k=1}^M\,, \quad where \quad M \le m\,.$$

Let $\phi \in C_c^\infty(0,T)$ be a test function and $w \in E_M$. If we multiply (2.1.1)-(2.1.2) by ϕ and integrate both equations from 0 to T, then we get

$$\tau \int_0^T \int_\Omega \varphi_{mt}(t,x)w(x)\phi(t)\,dxdt + \xi^2 \int_0^T \int_\Omega \nabla\varphi_m(t,x)\nabla w(x)\phi(t)\,dxdt =$$

$$-\int_0^T \int_\Omega g(x,\varphi_m(t,x))w(x)\phi(t)\,dxdt + 2\int_0^T \int_\Omega v_m(t,x)w(x)\phi(t)\,dxdt \tag{2.1.17}$$

and

$$\int_0^T \int_\Omega v_{mt}(t,x)w(x)\phi(t)\,dxdt + \kappa \int_0^T \int_\Omega v_m(t,x)w(x)\phi(t)\,dxdt =$$

$$= \frac{\kappa\ell}{2} \int_0^T \int_\Omega \varphi_m(t,x)w(x)\phi(t)\,dxdt \tag{2.1.18}$$

Since the function $t \mapsto w\phi(t)$ belongs to $C_c^\infty\big(0,T;C^\infty(\Omega)\cap H_0^1(\Omega)\big)$, we can pass to the limit in (2.1.17)-(2.1.18) to obtain

$$\tau \int_0^T \langle\langle \varphi_t, w\rangle\rangle \phi(t)\,dt + \xi^2 \int_0^T \int_\Omega \nabla\varphi(t,x)\nabla w(x)\phi(t)\,dxdt =$$

$$-\int_0^T \int_\Omega \psi(t,x)w(x)\phi(t)\,dxdt + 2\int_0^T \int_\Omega v(t,x)w(x)\phi(t)\,dxdt \tag{2.1.19}$$

and

$$\int_0^T \langle v_t, w\rangle \phi(t)\, dt + \kappa \int_0^T \int_\Omega v(t,x)w(x)\phi(t)\, dxdt = \frac{\kappa\ell}{2}\int_0^T \int_\Omega \varphi(t,x)w(x)\phi(t)\, dxdt \quad (2.1.20)$$

Since (2.1.19) and (2.1.20) hold for any $\phi \in C_c^\infty(0,T)$, we obtain the following equations :

$$\tau\langle\langle \varphi_t, w\rangle\rangle + \xi^2 \int_\Omega \nabla\varphi(t,x)\nabla w(x)\, dx = -\int_\Omega \psi(t,x)w(x)\, dx + 2\int_\Omega v(t,x)w(x)\, dx, \quad (2.1.21)$$

and

$$\langle v_t, w\rangle + \kappa \int_\Omega \nabla v(t,x)\nabla w(x)dx = \frac{\kappa\ell}{2}\int_\Omega \nabla\varphi(t,x)\nabla w(x)dx\,, \quad (2.1.22)$$

which hold for almost all $t \in (0,T)$ and every $w \in \bigcup_{M\geq 1} E_M$. Since the set of functions

$$E := \operatorname{span}\{w_k\}_{k=1}^\infty = \bigcup_{M\geq 1} E_M$$

is dense in $H_0^1(\Omega)$ as well as in $L^p(\Omega)$, the equality (2.1.21) holds for every $w \in H_0^1(\Omega)\cap L^p(\Omega)$ and the equality (2.1.22) holds for every $w \in H_0^1(\Omega)$. Since the equalities (2.1.21) and (2.1.22) hold almost everywhere on every finite time interval $(0,T)$, they are valid for almost all $t \in \mathbb{R}^+$.

Remark 2.1.5. *We have the following equality in $H^{-1}(\Omega) + L^q(\Omega)$ for almost all $t \in \mathbb{R}$:*

$$\tau\varphi_t = \xi^2 A\varphi - \psi + 2v\,. \quad (2.1.23)$$

Note that the Fubini's Theorem and the integration by parts in t variable yield

$$\tau \int_0^T \int_\Omega \varphi_{mt}(t,x)w(x)\phi(t)\, dxdt = \tau \int_\Omega \int_0^T \varphi_{mt}(t,x)w(x)\phi(t)\, dtdx =$$

$$= \tau \int_\Omega \varphi_m(T,x)w(x)\phi(T)\, dx - \tau \int_\Omega \varphi_m(0,x)w(x)\phi(0)\, dx - \tau \int_\Omega \int_0^T \varphi_m(t,x)w(x)\phi'(t)\, dtdx$$

$$= \tau \int_\Omega \varphi_m(T,x)w(x)\phi(T)\, dx - \tau \int_\Omega \varphi_m(0,x)w(x)\phi(0)\, dx - \tau \int_0^T \int_\Omega \varphi_m(t,x)w(x)\phi'(t)\, dxdt$$

Similarly,

$$\int_0^T \int_\Omega v_{mt}(t,x)w(x)\phi(t)\,dxdt = \int_\Omega \int_0^T v_{mt}(t,x)w(x)\phi(t)\,dtdx =$$

$$= \int_\Omega v_m(T,x)w(x)\phi(T)\,dx - \int_\Omega v_m(0,x)w(x)\phi(0)\,dx - \int_\Omega \int_0^T v_m(t,x)w(x)\phi'(t)\,dtdx$$

$$= \int_\Omega v_m(T,x)w(x)\phi(T)\,dx - \int_\Omega v_m(0,x)w(x)\phi(0)\,dx - \int_0^T \int_\Omega v_m(t,x)w(x)\phi'(t)\,dxdt\,.$$

If we perform the integration by parts for the first integrals on the left hand side of (2.1.17)-(2.1.18), where $\phi \in C^\infty([0,T])$ with $\phi(0) = 1$ and $\phi(T) = 0$, then we obtain

$$-\tau \int_0^T \int_\Omega \varphi_m(t,x)w(x)\phi'(t)\,dt + \xi^2 \int_0^T \int_\Omega \nabla\varphi_m(t,x)\nabla w(x)\phi(t)\,dxdt =$$

$$= \tau \int_\Omega \varphi_m(0,x)w(x)\,dx - \int_0^T \int_\Omega g(x,\varphi_m(t,x))w(x)\phi(t)\,dxdt + \tag{2.1.24}$$

$$+ 2\int_0^T \int_\Omega v_m(t,x)w(x)\phi(t)\,dxdt$$

and

$$- \int_0^T \int_\Omega v_m(t,x)w(x)\phi'(t)\,dxdt + \kappa \int_0^T \int_\Omega v_m(t,x)w(x)\phi(t)\,dxdt =$$

$$= \int_\Omega v_m(0,x)w(x)\,dx + \frac{\kappa\ell}{2} \int_0^T \int_\Omega \varphi_m(t,x)w(x)\phi(t)\,dxdt \tag{2.1.25}$$

If we pass to the limit in (2.1.24)-(2.1.25), then we obtain

$$-\tau \int_\Omega \int_0^T \varphi(t,x)w(x)\phi'(t)\,dtdx + \xi^2 \int_0^T \int_\Omega \nabla\varphi(t,x)\nabla w(x)\phi(t)\,dxdt =$$

$$= \tau \int_\Omega \varphi_0(x)w(x)\,dx - \int_0^T \int_\Omega \psi(t,x)w(x)\phi(t)\,dxdt + \tag{2.1.26}$$

$$+ 2\int_0^T \int_\Omega v(t,x)w(x)\phi(t)\,dxdt$$

and

$$- \int_\Omega \int_0^T v(t,x)w(x)\phi'(t)\,dtdx + \kappa \int_0^T \int_\Omega v(t,x)w(x)\phi(t)\,dxdt =$$

$$= \int_\Omega v_0(x)w(x)\,dx + \frac{\kappa\ell}{2} \int_0^T \int_\Omega \varphi(t,x)w(x)\phi(t)\,dxdt \tag{2.1.27}$$

In the similar way, if we multiply (2.1.21)-(2.1.22) by ϕ, integrate from 0 to T and apply the integration by parts formula to the first integrals on the left hand side, then we deduce

$$\begin{aligned} -\tau \int_\Omega \int_0^T \varphi(t,x) w(x) \phi'(t)\, dtdx + \xi^2 \int_0^T \int_\Omega \nabla\varphi(t,x) \nabla w(x) \phi(t)\, dxdt = \\ = \tau \int_\Omega \varphi(0,x) w(x)\, dx - \int_0^T \int_\Omega \psi(t,x) w(x) \phi(t)\, dxdt + \\ + 2 \int_0^T \int_\Omega v(t,x) w(x) \phi(t)\, dxdt \end{aligned} \tag{2.1.28}$$

and

$$\begin{aligned} - \int_\Omega \int_0^T v(t,x) w(x) \phi'(t)\, dtdx + \kappa \int_0^T \int_\Omega v(t,x) w(x) \phi(t)\, dxdt = \\ = \int_\Omega v(0,x) w(x)\, dx + \frac{\kappa \ell}{2} \int_0^T \int_\Omega \varphi(t,x) w(x) \phi(t)\, dxdt \end{aligned} \tag{2.1.29}$$

From (2.1.26)-(2.1.29) it follows that

$$\int_\Omega \varphi(0,x) w(x)\, dx = \int_\Omega \varphi_0(x) w(x)\, dx \quad \text{and} \quad \int_\Omega v(0,x) w(x)\, dx = \int_\Omega v_0(x) w(x)\, dx\,,$$

for any $w \in E = \text{span}\{w_k\}_{k=1}^\infty$. Therefore, the following functions are equal in $L^2(\Omega)$:

$$\varphi(0,x) = \varphi_0(x) \quad \text{and} \quad v(0,x) = v_0(x)\,.$$

We will use classical monotonicity argument to show that $\psi = g(x, \varphi)$. Without loss of generality we may assume that

$$g_s(x,s) \geq 0\,, \quad \text{for all} \quad s \in \mathbb{R}, \tag{2.1.30}$$

since from the beginning we could separate the linear part of g so that the rest would satisfy (2.1.30). If we consider the Gelfand triple $L^p(\Omega) \hookrightarrow L^2(\Omega) \simeq L^2(\Omega) \hookrightarrow L^q(\Omega)$, then g is monotone operator from $L^p(\Omega)$ into $L^q(\Omega)$. According to the Corollary 1.8.3, it is enough to demonstrate that

$$\limsup_{m\to\infty} \int_0^T \big(g(x,\varphi_m), \varphi_m\big)\, dt \leq \int_0^T \big(\psi, \varphi\big)\, dt. \tag{2.1.31}$$

If we integrate (2.1.4) from 0 to T, then we obtain the following equality :

$$\int_0^T \big(g(x,\varphi_m),\varphi_m\big)\,dt = \frac{\tau}{2}\int_\Omega |\varphi_m(0,x)|^2dx - \frac{\tau}{2}\int_\Omega |\varphi_m(T,x)|^2dx - \\ - \xi^2\int_0^T \|\nabla\varphi_m\|^2dt + 2\int_0^T \big(v_m,\varphi_m\big)\,dt \tag{2.1.32}$$

If we multiply (2.1.23) by $-\varphi$, integrate from 0 to T, use the integration by parts formula, and add the resulting relation to (2.1.32), then we obtain

$$\int_0^T \big(g(x,\varphi_m),\varphi_m\big)\,dt = \frac{\tau}{2}\int_\Omega |\varphi_m(0,x)|^2dx - \frac{\tau}{2}\|\varphi_0\|^2 - \frac{\tau}{2}\int_\Omega |\varphi_m(T,x)|^2dx + \frac{\tau}{2}\int_\Omega |\varphi(T,x)|^2dx - \\ - \xi^2\int_0^T \|\nabla\varphi_m\|^2dt + \xi^2\int_0^T \|\nabla\varphi\|^2dt + 2\int_0^T \big(v_m,\varphi_m\big)\,dt - 2\int_0^T \big(v,\varphi\big)\,dt. \tag{2.1.33}$$

By the Compactness Theorem with $X = H_0^1(\Omega)$, $H = L^2(\Omega)$ and $Y = H^{-1}(\Omega)$, there is a subsequence u_m such that $u_m \to u$ in $L^2(\Omega_T)$. Therefore,

$$\int_0^T \big(v_m,\varphi_m\big)\,dt \to \int_0^T \big(v,\varphi\big)\,dt. \tag{2.1.34}$$

Due to the lower semicontinuity of weak convergence, if we take the limit superior [over a subsequence so that (2.1.34) is valid] from both sides of the inequality (2.1.33), then we deduce (2.1.31). Therefore, $\psi = g(x,\varphi)$. Note that the following spaces are Hilbert triples :

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \simeq (L^2(\Omega))^* \hookrightarrow H^{-1}(\Omega)$$

as well as

$$H_0^1(\Omega)\cap L^p(\Omega) \hookrightarrow L^2(\Omega) \simeq (L^2(\Omega))^* \hookrightarrow H^{-1}(\Omega) + L^q(\Omega)\,,$$

Then, according to the Theorem 1.7.1, it follows that

$$[\varphi, v] \in C\big([0,T]; L^2(\Omega)\times L^2(\Omega)\big)\,.$$

It is left to show the uniqueness and the continuous dependence on the initial data of weak solutions. Let $[\varphi, v]$ and $[\overline{\varphi}, \overline{v}]$ be two weak solutions of the initial-boundary value problem (2.0.1)-(2.0.4) with initial data $[\varphi_0, v_0] \in L^2(\Omega)\times L^2(\Omega)$ and $[\overline{\varphi}_0, \overline{v}_0] \in L^2(\Omega)\times L^2(\Omega)$, respectively. If we subtract the equation for $\bar{\varphi}$ from the equation for φ and let $w = \varphi - \bar{\varphi}$, then we obtain

$$\frac{\tau}{2}\frac{d}{dt}\|\varphi - \bar{\varphi}\|^2 = -\xi^2\|\nabla\varphi - \nabla\bar{\varphi}\|^2 - \big(g(x,\varphi) - g(x,\bar{\varphi}), \varphi - \bar{\varphi}\big) + 2\big(v - \bar{v}, \varphi - \bar{\varphi}\big)\,. \quad (2.1.35)$$

Similarly, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v - \bar{v}\|^2 = -\kappa\|\nabla v - \nabla\bar{v}\|^2 + \frac{\kappa\ell}{2}(\nabla\varphi - \nabla\bar{\varphi}, \nabla v - \nabla\bar{v})\,. \quad (2.1.36)$$

If we apply the Cauchy-Schwartz inequality and the Cauchy's inequality with $\epsilon = \ell/4$, then we obtain the estimate

$$\frac{\kappa\ell}{2}(\nabla\varphi - \nabla\bar{\varphi}, \nabla v - \nabla\bar{v}) \le \frac{\kappa\ell}{2}\|\nabla\varphi - \nabla\bar{\varphi}\|\|\nabla v - \nabla\bar{v}\| \le \frac{\kappa\ell^2}{8}\|\nabla\varphi - \nabla\bar{\varphi}\|^2 + \frac{\kappa}{2}\|\nabla v - \nabla\bar{v}\|^2\,.$$

$$\le \frac{\kappa\ell^2}{8}\|\nabla\varphi - \nabla\bar{\varphi}\|^2 + \frac{\kappa}{2}\|\nabla v - \nabla\bar{v}\|^2\,. \quad (2.1.37)$$

Then from (2.1.36) and (2.1.37) we obtain

$$\frac{1}{2}\frac{d}{dt}\|v - \bar{v}\|^2 \le \frac{\kappa\ell^2}{8}\|\nabla\varphi - \bar{\varphi}\|^2\,. \quad (2.1.38)$$

If we multiply (2.1.37) by $\frac{4\xi^2}{\kappa\ell^2}$ and add the resulting inequality to (2.1.35), then we deduce

$$\frac{d}{dt}\left(\frac{\tau}{2}\|\varphi - \bar{\varphi}\|^2 + \frac{2\xi^2}{\kappa\ell^2}|v - \bar{v}\|^2\right) \le -\big(g(x,\varphi) - g(x,\bar{\varphi}), \varphi - \bar{\varphi}\big) + 2\big(v - \bar{v}, \varphi - \bar{\varphi}\big) \quad (2.1.39)$$

Since g_s is bounded below by $-\beta_3$, it follows that

$$(g(x,\varphi) - g(x,\bar{\varphi}), \varphi - \bar{\varphi}) = \int_\Omega \left(\int_{\bar{\varphi}}^{\varphi} g_s(x,s)\,ds\right)(\varphi - \bar{\varphi})\,dx \ge -\beta_3\|\varphi - \bar{\varphi}\|^2\,. \quad (2.1.40)$$

By the Cauchy-Schwartz inequality and the Cauchy's inequality with $\epsilon = \beta$, we obtain the estimate

$$2\big(v - \bar{v}, \varphi - \bar{\varphi}\big) \le 2\|v - \bar{v}\|\|\varphi - \bar{\varphi}\| \le \beta_3\|\varphi - \bar{\varphi}\|^2 + \frac{1}{4\beta_3}\|v - \bar{v}\|^2 \quad (2.1.41)$$

By taking into account (2.1.40) and (2.1.41), from (2.1.39) we deduce that

$$\frac{d}{dt}\left(\frac{\tau}{2}\|\varphi-\overline{\varphi}\|^2+\frac{2\xi^2}{\kappa\ell^2}\|v-\bar{v}\|^2\right)\leq 2\beta_3\|\varphi-\overline{\varphi}\|^2+\frac{1}{4\beta_3}\|v-\bar{v}\|^2\,.$$

Then for sufficiently large $D>0$, we have

$$\frac{d}{dt}\left(\frac{\tau}{2}\|\varphi-\overline{\varphi}\|^2+\frac{2\xi^2}{\kappa\ell^2}\|v-\bar{v}\|^2\right)\leq D\left(\frac{\tau}{2}\|\varphi-\overline{\varphi}\|^2+\frac{2\xi^2}{\kappa\ell^2}\|v-\bar{v}\|^2\right)\,. \tag{2.1.42}$$

If we apply the Gronwall's inequality, then from (2.1.42) we deduce that

$$\frac{\tau}{2}\int_\Omega|\varphi(t,x)-\overline{\varphi}(t,x)|^2\,dx+\frac{2\xi^2}{\kappa\ell^2}\int_\Omega|v(t,x)-\bar{v}(t,x)|^2\,dx\ \leq$$

$$\leq\ e^{-Dt}\left(\frac{\tau}{2}\|\varphi_0-\overline{\varphi}_0\|^2+\frac{2\xi^2}{\kappa\ell^2}\|v_0-\bar{v}_0\|^2\right), \tag{2.1.43}$$

for all $t\in[0,T]$. If $[\varphi_0,v_0]=[\overline{\varphi}_0,\bar{v}_0]$, then from (2.1.43) we obtain the uniqueness of weak solutions. For otherwise, (2.1.43) gives us the continuous dependence of weak solutions on the initial data.

□

2.2 *Strong Solutions*

Definition 2.2.1 (Strong Solutions)**.** *Assume* $[\varphi_0,v_0]\in[H_0^1(\Omega)\cap L^p(\Omega)]\times[H_0^1(\Omega)\cap L^p(\Omega)]$. *We say that a pair of functions* $[\varphi,v]:[0,\infty)\to H_0^1(\Omega)\times H_0^1(\Omega)$ *is a strong solution of the initial-boundary value problem (2.0.1)-(2.0.4) if*
(i) for each $T>0$,

$$[\varphi,v]\in C([0,T];H_0^1(\Omega)\times H_0^1(\Omega))\cap L^2(0,T;H_0^2(\Omega)\times H_0^2(\Omega))\,,$$

$$\varphi\in L^\infty(0,T;L^p(\Omega))\quad\text{and}\quad[\varphi_t,v_t]\in L^2(\Omega_T)\times L^2(\Omega_T)\,;$$

(ii) the following equations hold for almost all $t\in\mathbb{R}^+$ *and every* $w\in L^p(\Omega)$:

$$\tau\int_\Omega\varphi_t(t,x)w(x)dx-\xi^2\int_\Omega\Delta\varphi(t,x)w(x)dx=-\int_\Omega g(x,\varphi(t,x))w(x)dx+2\int_\Omega v(t,x)w(x)dx;$$

(iii) the following equations hold for almost all $t \in \mathbb{R}^+$ and every $w \in L^2(\Omega)$:

$$\int_\Omega v_t(t,x)w(x)dx - \kappa \int_\Omega \Delta v(t,x)w(x)dx = -\frac{\kappa\ell}{2}\int_\Omega \Delta\varphi(t,x)w(x)dx;$$

(iv) $\varphi(0) = \varphi_0$ and $v(0) = v_0$.

Remark 2.2.2. *For $\varphi_0 \in H_0^1(\Omega) \cap L^p(\Omega)$, by the Theorem 1.6.13, there is a sequence $\sum_{k=1}^m c_{mk} w_k$ that converges to φ_0 in $L^2(\Omega)$ and has a subsequence that converges to φ_0 in $H_0^1(\Omega) \cap L^p(\Omega)$.*

Theorem 2.2.3. *If $[\varphi_0, v_0] \in [H_0^1(\Omega) \cap L^p(\Omega)] \times [H_0^1(\Omega) \cap L^p(\Omega)]$, then the initial-boundary value problem (2.0.1)-(2.0.4) has a unique strong solution $[\varphi, v]$. If $d \le 3$, then we also have the following equalities in $L^2(\Omega)$ for almost all $t \in \mathbb{R}^+$:*

$$\tau\varphi_t = \xi^2\Delta\varphi - g(x,\varphi) + 2v, \tag{2.2.1}$$

$$v_t = \kappa\Delta v - \frac{\kappa\ell}{2}\Delta\varphi. \tag{2.2.2}$$

Proof. We consider the initial-value problem (2.1.1)-(2.1.3) with a slight modification on the initial conditions :

$$\varphi_{mk}(0) := c_{mk}\,, \quad \text{for} \quad k = 1, 2, 3, \ldots, m,$$

where c_{mk} are as in the Remark 2.2.2. Under this modification we still have that

$$\varphi_m(0,x) = \sum_{k=1}^m \varphi_{mk}(0)w_k(x) = \sum_{k=1}^m c_{mk}w_k(x) \to \varphi_0 \quad \text{in} \quad L^2(\Omega)\,.$$

Therefore, by the Theorem 2.1.2, there is a unique weak solution $[\varphi, v]$ of the initial-boundary value problem (2.0.1)-(2.0.4). In addition, according to the Remark 2.2.2, we know that there is a subsequence of the Galerkin approximations :

$$\varphi_m(t,x) = \sum_{k=1}^m \varphi_{mk}(t)w_k(x) \quad \text{and} \quad v_m(t,x) = \sum_{k=1}^m v_{mk}(t)w_k(x)$$

such that

$$\varphi_m(0,x) = \sum_{k=1}^{m} c_{mk} w_k(x) \to \varphi_0 \quad \text{in} \quad H_0^1(\Omega) \cap L^p(\Omega)\,. \tag{2.2.3}$$

Therefore, before applying the Banach-Alaoglu Theorem, we take a subsequence of the Galerkin approximations so that (2.2.3) is valid. Then it is enough to establish better estimates for the Galerkin approximations $[\varphi_m, v_m]$ to prove that $[\varphi, v]$ is actually a strong solution. From (2.1.1)-(2.1.2) it follows that the equations

$$\tau \int_\Omega \varphi_{mt}(t,x) w(x)\, dx - \xi^2 \int_\Omega \Delta\varphi_m(t,x) w(x)\, dx =$$

$$= - \int_\Omega g(x, \varphi_m(t,x)) w(x)\, dx + 2 \int_\Omega v_m(t,x) w(x)\, dx, \tag{2.2.4}$$

$$\int_\Omega v_{mt}(t,x) w(x) dx - \kappa \int_\Omega \Delta v_m(t,x) w(x) dx = -\frac{\kappa\ell}{2} \int_\Omega \Delta\varphi_m(t,x) w(x) dx\,, \tag{2.2.5}$$

hold for any $w \in E_m = \text{span}\{w_k\}_{k=1}^m$. If we let $w = -\Delta\varphi_m$ in (2.2.4), then we deduce the equality

$$\frac{\tau}{2}\frac{d}{dt} \int_\Omega |\nabla\varphi_m(t,x)|^2\, dx + \xi^2 \int_\Omega |\Delta\varphi_m(t,x)|^2\, dx =$$

$$= \int_\Omega g(x, \varphi_m(t,x)) \Delta\varphi_m(t,x)\, dx - 2 \int_\Omega v_m(t,x) \Delta\varphi_m(t,x)\, dx \tag{2.2.6}$$

which holds for all $t \in (0, T]$. By the Green's identity, we obtain the estimate

$$\int_\Omega g(x, \varphi_m(t,x)) \Delta\varphi_m(t,x)\, dx = - \int_\Omega g_s(x, \varphi_m(t,x)) |\nabla\varphi_m(t,x)|^2\, dx +$$

$$+ \int_{\partial\Omega} g(x, \varphi_m(t,x)) \frac{\partial\varphi_m}{\partial\nu}(t,x) dS(x) \le \beta_3 \int_\Omega |\nabla\varphi_m(t,x)|^2 dx = -\beta_3 \int_\Omega \varphi_m(t,x) \Delta\varphi_m(t,x) dx\,,$$

where we have used the assumptions (2.0.6)-(2.0.7). If we apply the Cauchy-Schwartz inequality and the Cauchy's inequality with ϵ, then the terms on the right hand side of (2.2.6) can be estimated as follows :

$$\int_\Omega g(x, \varphi_m(t,x)) \Delta\varphi_m(t,x)\, dx \le -\beta_3 \int_\Omega \varphi_m(t,x) \Delta\varphi_m(t,x)\, dx \le \beta_3 \epsilon_1 \|\varphi_m\|^2 + \frac{\beta_3}{4\epsilon_1} \|\Delta\varphi_m\|^2$$

and

$$-2 \int_\Omega v_m(t,x) \Delta\varphi_m(t,x)\, dx \le 2\epsilon_2 \|v_m\|^2 + \frac{1}{2\epsilon_2} \|\Delta\varphi_m\|^2$$

If we take $\epsilon_1 = \beta_3/\xi^2$ and $\epsilon_2 = 2/\xi^2$, then from the equation (2.2.6) and the above estimates we obtain

$$\frac{\tau}{2}\frac{d}{dt}\|\nabla\varphi_m\|^2 + \frac{\xi^2}{2}\|\Delta\varphi_m\|^2 \leq \frac{\beta_3^2}{\xi^2}\|\varphi_m\|^2 + \frac{4}{\xi^2}\|v_m\|^2 . \tag{2.2.7}$$

If we let $w = -\Delta v_m$ in (2.2.5), then we obtain the equality

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v_m(t,x)|^2\,dx + \kappa\int_\Omega |\Delta v_m(t,x)|^2\,dx = \frac{\kappa\ell}{2}\int_\Omega \Delta\varphi_m(t,x)\Delta v_m(t,x)\,dx\,, \tag{2.2.8}$$

which holds for each $t \in (0,T]$. By the Cauchy-Schwartz inequality and the Cauchy's inequality with $\epsilon = \ell/4$, we obtain

$$\frac{\kappa\ell}{2}\int_\Omega \Delta\varphi_m(t,x)\Delta v_m(t,x)\,dx \leq \frac{\kappa\ell^2}{8}\|\Delta\varphi_m\|^2 + \frac{\kappa}{2}\|\Delta v_m\|^2 . \tag{2.2.9}$$

Then from (2.2.8) and (2.2.9) we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla v_m\|^2 + \frac{\kappa}{2}\|\Delta v_m\|^2 \leq \frac{\kappa\ell^2}{8}\|\Delta\varphi_m\|^2 . \tag{2.2.10}$$

If we multiply (2.2.10) by $2\xi^2/\kappa\ell^2$ and add the resulting inequality to (2.2.10), then we obtain

$$\frac{d}{dt}\left[\frac{\tau}{2}\|\nabla\varphi_m\|^2 + \frac{\xi^2}{\kappa\ell^2}\|\nabla v_m\|^2\right] + \frac{\xi^2}{4}\|\Delta\varphi_m\|^2 + \frac{\xi^2}{\ell^2}\|\Delta v_m\|^2 \leq \frac{\beta_3^2}{\xi^2}\|\varphi_m\|^2 + \frac{4}{\xi^2}\|v_m\|^2 .$$

If we integrate the last inequality from 0 to t, then we obtain the inequalities :

$$\frac{\tau}{2}\int_\Omega |\nabla\varphi_m(t,x)|^2\,dx + \frac{\xi^2}{\kappa\ell^2}\int_\Omega |\nabla v_m(t,x)|^2\,dx +$$

$$+\ \frac{\xi^2}{4}\int_0^t\int_\Omega |\Delta\varphi_m(s,x)|^2\,dxds + \frac{\xi^2}{\ell^2}\int_0^t\int_\Omega |\Delta v_m(s,x)|^2\,dxds\ \leq$$

$$\leq\ \frac{\tau}{2}\int_\Omega |\nabla\varphi_m(0,x)|^2\,dx + \frac{\xi^2}{\kappa\ell^2}\int_\Omega |\nabla v_m(0,x)|^2\,dx + \frac{\beta_3^2}{\xi^2}\|\varphi_m\|^2_{L^2(\Omega_T)} + \frac{4}{\xi^2}\|v_m\|^2_{L^2(\Omega_T)}\ \leq$$

$$\leq\ \frac{\tau}{2}\int_\Omega |\nabla\varphi_0(x)|^2\,dx + \frac{\xi^2}{\kappa\ell^2}\int_\Omega |\nabla v_0(x)|^2\,dx + \frac{\beta_3^2}{\xi^2}\|\varphi_m\|^2_{L^2(\Omega_T)} + \frac{4}{\xi^2}\|v_m\|^2_{L^2(\Omega_T)} .$$

If we take the supremum over $(0,T]$ and use the estimate (2.1.12), then it is easy to see that

$$\|[\varphi_m, v_m]\|_{L^\infty(0,T;H_0^1(\Omega)\times H_0^1(\Omega))} + \|[\varphi_m, v_m]\|_{L^2(0,T;H_0^2(\Omega)\times H_0^2(\Omega))} \le C\,, \tag{2.2.11}$$

where C depends only on $\varphi_0, v_0, p, \beta_0, \ell, \kappa, \tau, \xi, \Omega$ and T. If we let $w = \varphi_{mt}$ in (2.2.11), then we obtain the equality

$$\tau \int_\Omega |\varphi_{mt}(t,x)|^2 dx + \frac{\xi^2}{2}\frac{d}{dt}\int_\Omega |\nabla \varphi_m(t,x)|^2 dx =$$

$$= -\int_\Omega g(x,\varphi_m(t,x))\varphi_{mt}(t,x)\,dx + 2\int_\Omega v_m(t,x)\varphi_{mt}(t,x)\,dx \tag{2.2.12}$$

which holds for each $t \in (0,T]$. The equation (2.2.12) can also be written as follows :

$$\tau\|\varphi_{mt}\|^2 + \frac{\xi^2}{2}\frac{d}{dt}\|\nabla\varphi_m\|^2 + (g(x,\varphi_m),\varphi_{mt}) = 2(v_m,\varphi_{mt})\,. \tag{2.2.13}$$

By the Cauchy-Schwartz inequality and the Cauchy's inequality with $\epsilon = 1/\tau$, we obtain

$$2(v_m,\varphi_{mt}) \le 2\|v_m\|\|\varphi_{mt}\| \le \frac{2}{\tau}\|v_m\|^2 + \frac{\tau}{2}\|\varphi_{mt}\|^2\,. \tag{2.2.14}$$

From (2.2.13) and (2.2.14) we deduce that

$$\frac{\tau}{2}\|\varphi_{mt}\|^2 + \frac{\xi^2}{2}\frac{d}{dt}\|\nabla\varphi_m\|^2 + (g(x,\varphi_m),\varphi_{mt}) \le \frac{2}{\tau}\|v_m\|^2\,. \tag{2.2.15}$$

If we integrate the inequality (2.2.15) from 0 to t, then we obtain

$$\frac{\tau}{2}\int_0^t\int_\Omega |\varphi_{ms}(s,x)|^2\,dxds + \int_0^t\int_\Omega g(x,\varphi_m(s,x))\varphi_{ms}(t,x)\,dxds \le$$

$$\le \frac{\xi^2}{2}\int_\Omega |\nabla\varphi_m(0,x)|^2 dx + \frac{2}{\tau}\|v_m\|^2_{L^2(\Omega_T)}\,. \tag{2.2.16}$$

For each fixed $x \in \Omega$, let $G(x,s) := \int_0^s g(x,\rho)\,d\rho$ be the primitive of $g(x,\cdot)$. By the Lemma (1.3.2), $G(x,s)$ satisfies the inequality

$$-\gamma_0 + \gamma_1|s|^p \le G(x,s) \le \gamma_0 + \gamma_2|s|^p\,, \tag{2.2.17}$$

for all $(x,s)\in\Omega\times\mathbb{R}$, where γ_0,γ_1 and γ_2 are positive constants. Note that

$$\frac{\partial}{\partial s}\left(G(x,\varphi_m(s,x))\right)=G_s(x,\varphi_m(s,x))\varphi_{ms}(s,x)\,.$$

Then, by the Fubini's Theorem, it follows that

$$\int_0^t\int_\Omega G(x,\varphi_m(s,x))\varphi_{ms}(s,x)\,dxds=\int_\Omega\int_0^t G(x,\varphi_m(s,x))\varphi_{ms}(s,x)\,dsdx=$$

$$=\int_\Omega\int_0^t\frac{\partial}{\partial s}\left(G(x,\varphi_m(s,x))\right)\,dsdx=\int_\Omega G(x,\varphi_m(t,x))\,dx-\int_\Omega G(x,\varphi_m(0,x))\,dx\,. \quad (2.2.18)$$

By the inequality (2.2.17), we deduce that

$$-\gamma_0|\Omega|+\gamma_1\int_\Omega|\varphi_m(t,x)|^p dx\le\int_\Omega G(x,\varphi_m(t,x))\,dx\le\gamma_0|\Omega|+\gamma_1\int_\Omega|\varphi_m(t,x)|^p dx\,, \quad (2.2.19)$$

for all $(t,x)\in(0,T]\times\Omega$. Then from (2.2.16)-(2.2.19) we obtain the inequalities

$$\frac{\tau}{2}\int_0^t\int_\Omega|\varphi_{ms}(s,x)|^2\,dxds+\int_\Omega|\varphi_m(t,x)|^p\,dx\le 2\gamma_0|\Omega|+\int_\Omega|\varphi_m(0,x)|^p\,dx+$$

$$+\frac{\xi^2}{2}\int_\Omega|\nabla\varphi_m(0,x)|^2dx+\frac{2}{\tau}\|v_m\|^2_{L^2(\Omega_T)}\le \quad (2.2.20)$$

$$\le 2\gamma_0|\Omega|+\int_\Omega|\varphi_0(x)|^p\,dx+\frac{\xi^2}{2}\int_\Omega|\nabla\varphi_0(x)|^2dx+\frac{2}{\tau}\|v_m\|^2_{L^2(\Omega_T)}$$

If we take the supremum of the inequality (2.2.20) over $(0,T]$, then we deduce that φ_{mt} is uniformly bounded in $L^2(\Omega_T)$ and φ_m is uniformly bounded in $L^\infty(0,T;L^p(\Omega))$. If we take $w=v_{mt}$ in (2.2.18), then we obtain the equality

$$\int_\Omega|v_{mt}(t,x)|^2\,dx=\kappa\int_\Omega\Delta v_m(t,x)v_{mt}(t,x)\,dx-\frac{\kappa\ell}{2}\int_\Omega\Delta\varphi_m(t,x)v_{mt}(t,x)\,dx \quad (2.2.21)$$

which holds for each $t\in(0,T]$. If we apply the Cauchy-Schwartz inequality and the Cauchy's inequality with ϵ, then the terms on the left hand side of (2.2.20) can be estimated as follows:

$$\kappa\int_\Omega\Delta v_m(t,x)v_{mt}(t,x)\,dx\le\kappa^2\|\Delta\varphi_m\|^2+\frac{1}{4}\|v_{mt}\|^2,\quad\text{with}\quad\epsilon=\kappa\,, \quad (2.2.22)$$

$$-\frac{\kappa\ell}{2}\int_{\Omega}\Delta\varphi_m(t,x)v_{mt}(t,x)\,dx \leq \frac{\kappa^2\ell^2}{4}\|\Delta v_m\|^2 + \frac{1}{4}\|v_{mt}\|^2, \quad \text{with} \quad \epsilon = \frac{2}{\kappa\ell}. \tag{2.2.23}$$

If we integrate (2.2.21) from 0 to T and use the estimates (2.2.22)-(2.2.23), then we obtain

$$\frac{1}{2}\|v_{mt}\|^2_{L^2(\Omega_T)} \leq \kappa^2\|\Delta\varphi_m\|^2_{L^2(\Omega_T)} + \frac{\kappa^2\ell^2}{4}\|\Delta v_m\|^2_{L^2(\Omega_T)}. \tag{2.2.24}$$

Due to the the estimate (2.2.23), from (2.2.24) it follows that v_{mt} is uniformly bounded in $L^2(\Omega_T)$. Note that the following spaces form a Hilbert triple :

$$H_0^2(\Omega) \hookrightarrow H_0^1(\Omega) \simeq (H_0^1(\Omega))^* \hookrightarrow H^{-2}(\Omega).$$

Since $[\varphi, v] \in L^2\big(0,T; H_0^2(\Omega)\times H_0^2(\Omega)\big)$ and $[\varphi_t, v_t] \in L^2(\Omega_T)\times L^2(\Omega_T)$, from the Theorem 1.7.1 it follows that $[\varphi, v] \in C\big([0,T]; H_0^1(\Omega)\times H_0^1(\Omega)\big)$. Therefore, the pair of functions $[\varphi, v]$ is a unique strong solution of the initial-boundary value problem (2.0.1)-(2.0.4). If $d \leq 3$, then from the Sobolev Embedding Theorem it follows that $H^2(\Omega)$ is continuously embedded in $C(\overline{\Omega})$. Hence, $\varphi \in L^2(0,T; C(\overline{\Omega}))$ so that the integral $\int_\Omega g\big(x,\varphi(t,x)\big)w(x)\,dx$ is finite for any $w \in L^2(\Omega)$. This is the reason why the equalities (2.2.1)-(2.2.2) are in $L^2(\Omega)$. $\square$

Theorem 2.2.4. *Suppose that $d \leq 3$ and the derivative of f satisfies the following condition:*

$$|f'(s)| \leq \alpha_4\big(1+|s|^{p-2}\big), \quad \textit{for all} \quad s\in\mathbb{R}, \tag{2.2.25}$$

where $p \in (2,4]$ if $d=3$ and $p\in(2,\infty)$ if $d=1$ or $d=2$. Then the mapping

$$[\varphi_0, v_0] \mapsto [\varphi(t), v(t)]$$

is continuous on $H_0^1(\Omega)\times H_0^1(\Omega)$.

Proof. For $d \leq 3$ and $p\in(2,4]$, by the Sobolev Embedding Theorem, $H_0^1(\Omega)$ is continuously embedded in $L^p(\Omega)$. In this case, we take the initial conditions for the initial-boundary value problem (2.0.1)-(2.0.4) from the space $H_0^1(\Omega)\times H_0^1(\Omega)$. Let $[\varphi, v]$ and $[\overline{\varphi}, \overline{v}]$ be two strong solutions of the initial-boundary value problem (2.0.1)-(2.0.4) with initial data $[\varphi_0, v_0] \in H_0^1(\Omega)\times H_0^1(\Omega)$ and $[\overline{\varphi}_0, \overline{v}_0] \in H_0^1(\Omega)\times H_0^1(\Omega)$, respectively. If we subtract the equation for

$\overline{\varphi}$ from the equation for φ and then take the inner product in $L^2(\Omega)$ of the resulting relation by $-(\Delta\varphi - \Delta\overline{\varphi})$, then we obtain

$$\frac{\tau}{2}\frac{d}{dt}\|\nabla\varphi - \nabla\overline{\varphi}\|^2 = -\xi^2\|\Delta\varphi - \Delta\overline{\varphi}\|^2 + \int_\Omega \big(g(x,\varphi) - g(x,\overline{\varphi})\big)\big(\Delta\varphi - \Delta\overline{\varphi}\big)\,dx + $$
$$+\ 2\int_\Omega \big(\nabla\varphi - \nabla\overline{\varphi}\big)\big(\nabla v - \nabla\bar{v}\big)\,dx\,. \tag{2.2.26}$$

By the Cauchy-Schwartz inequality and the Cauchy's inequality with $\epsilon = \frac{1}{2\xi^2}$, we obtain the following estimate :

$$\int_\Omega \big(g(x,\varphi) - g(x,\overline{\varphi})\big)\big(\Delta\varphi - \Delta\overline{\varphi}\big)\,dx \le \|g(x,\varphi) - g(x,\overline{\varphi})\|\|\Delta\varphi - \Delta\overline{\varphi}\| \ \le$$
$$\le\ \frac{1}{2\xi^2}\|g(x,\varphi) - g(x,\overline{\varphi})\|^2 + \frac{\xi^2}{2}\|\Delta\varphi - \Delta\overline{\varphi}\|^2, \tag{2.2.27}$$

Similarly, we obtain

$$2\int_\Omega \big(\nabla\varphi - \nabla\overline{\varphi}\big)\big(\nabla v - \nabla\bar{v}\big)\,dx \le 2\|\nabla\varphi - \nabla\overline{\varphi}\|\|\nabla v - \nabla\bar{v}\| \ \le$$
$$\le\ \|\nabla\varphi - \nabla\overline{\varphi}\|^2 + \|\nabla v - \nabla\bar{v}\|^2\,. \tag{2.2.28}$$

By taking into account the estimates (2.2.27)-(2.2.28), from (2.2.26) we deduce that

$$\frac{\tau}{2}\frac{d}{dt}\|\nabla\varphi - \nabla\overline{\varphi}\|^2 \le -\frac{\xi^2}{2}\|\Delta\varphi - \Delta\overline{\varphi}\|^2 + \frac{1}{2\xi^2}\|g(x,\varphi) - g(x,\overline{\varphi})\|^2 +$$
$$+\ \|\nabla\varphi - \nabla\overline{\varphi}\|^2 + \|\nabla v - \nabla\bar{v}\|^2\,. \tag{2.2.29}$$

If we subtract the equation for $\bar{v}$ from the equation for v and then take the inner product in $L^2(\Omega)$ of the resulting relation by $-(\Delta v - \Delta\bar{v})$, then we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla v - \nabla\bar{v}\|^2 = -\kappa\|\Delta v - \Delta\bar{v}\|^2 + \frac{\kappa\ell}{2}\int_\Omega \big(\Delta\varphi - \Delta\overline{\varphi}\big)\big(\Delta v - \Delta\bar{v}\big)\,dx \tag{2.2.30}$$

By the Cauchy-Schwartz inequality and the Cauchy's inequality with $\epsilon = \frac{\ell}{4}$, we obtain the following estimate :

$$\frac{\kappa\ell}{2}\int_\Omega \big(\Delta\varphi - \Delta\overline{\varphi}\big)\big(\Delta v - \Delta\bar{v}\big)\,dx \le \frac{\kappa\ell}{2}\|\Delta\varphi - \Delta\overline{\varphi}\|\|\Delta v - \Delta\bar{v}\| \ \le$$

$$\leq \frac{\kappa\ell^2}{8}\|\Delta\varphi - \Delta\overline{\varphi}\|^2 + \frac{\kappa}{2}\|\Delta v - \Delta\overline{v}\|^2 . \tag{2.2.31}$$

Then from (2.2.30) and (2.2.31) we obtain the following inequality :

$$\frac{1}{2}\frac{d}{dt}\|\nabla v - \nabla\overline{v}\|^2 \leq \frac{\kappa\ell^2}{8}\|\Delta\varphi - \Delta\overline{\varphi}\|^2. \tag{2.2.32}$$

If we multiply the inequality (2.2.32) by $\frac{2\xi^2}{\kappa\ell^2}$ and add the resulting relation to (2.2.29), then we obtain

$$\frac{d}{dt}\left(\frac{\tau}{2}\|\nabla\varphi - \nabla\overline{\varphi}\|^2 + \frac{\xi^2}{\kappa\ell^2}\|\nabla v - \nabla\overline{v}\|^2\right) \leq \frac{1}{2\xi^2}\|g(x,\varphi) - g(x,\overline{\varphi})\|^2 +$$

$$+ \|\nabla\varphi - \nabla\overline{\varphi}\|^2 + \|\nabla v - \nabla\overline{v}\|^2 . \tag{2.2.33}$$

From the assumption (2.2.25) it follows that

$$\|g(x,\varphi) - g(x,\overline{\varphi})\|^2 = \int_\Omega |g(x,\varphi(t,x)) - g(x,\overline{\varphi}(t,x))|^2 dx \leq$$

$$\leq \int_\Omega \left[\int_{\overline{\varphi}(t,x)}^{\varphi(t,x)} |g_s(x,s)|\, ds\right]^2 dx \leq \alpha_4^2 \int_\Omega \left[\int_{\overline{\varphi}(t,x)}^{\varphi(t,x)} \left(1 + |s|^{p-2}\right) ds\right]^2 dx \leq$$

$$\leq \alpha_4^2 \int_\Omega \left[1 + |\overline{\varphi}(t,x)|^{p-2} + |\varphi(t,x)|^{p-2}\right]^2 |\varphi(t,x) - \overline{\varphi}(t,x)|^2 dx \leq$$

$$\leq 3\alpha_4^2 \int_\Omega \left[1 + |\overline{\varphi}(t,x)|^{2(p-2)} + |\varphi(t,x)|^{2(p-2)}\right] |\varphi(t,x) - \overline{\varphi}(t,x)|^2 dx .$$

If we apply the Hölder's inequality, then from the last inequality we obtain

$$\|g(x,\varphi) - g(x,\overline{\varphi})\|^2 \leq 3\alpha_4^2 \left[|\Omega|^{(p-1)/(p-2)} + \|\overline{\varphi}\|^{2(p-2)}_{L^{2(p-1)}(\Omega)} + \|\varphi\|^{2(p-2)}_{L^{2(p-1)}(\Omega)}\right] \|\varphi - \overline{\varphi}\|^2_{L^{2(p-1)}(\Omega)} .$$

By the Sobolev Embedding Theorem, if $d = 3$, then $H_0^1(\Omega)$ is continuously embedded in $L^{2(p-1)}(\Omega)$, for any $p \in (2,4]$. If $d = 1$ or $d = 2$, then $H_0^1(\Omega)$ is continuously embedded in $L^r(\Omega)$, for any $r \in [1,\infty)$. In any case, from the last inequality we deduce that

$$\|g(x,\varphi) - g(x,\overline{\varphi})\|^2 \leq C_1 \left[1 + \|\nabla\overline{\varphi}\|^{2(p-2)} + \|\nabla\varphi\|^{2(p-2)}\right] \|\nabla\varphi - \nabla\overline{\varphi}\|^2 . \tag{2.2.34}$$

Then for sufficiently large $C > 0$, from (2.2.33) and (2.2.34) we obtain the inequality

$$\frac{d}{dt}\left(\frac{\tau}{2}\|\nabla\varphi - \nabla\overline{\varphi}\|^2 + \frac{\xi^2}{\kappa\ell^2}\|\nabla v - \nabla\bar{v}\|^2\right) \leq CD(t)\left(\frac{\tau}{2}\|\nabla\varphi - \nabla\overline{\varphi}\|^2 + \frac{\xi^2}{\kappa\ell^2}\|\nabla v - \nabla\bar{v}\|^2\right),$$

where $D(t) := 1 + \|\nabla\varphi\|^{2(p-2)} + \|\nabla\overline{\varphi}\|^{2(p-2)}$ is a real-valued continuous function on $[0, T]$ since $\varphi, \overline{\varphi} \in C([0,T]; H_0^1(\Omega))$. By the Gronwall's inequality, from the last inequality we deduce that

$$\frac{\tau}{2}\int_\Omega |\nabla\varphi(t,x) - \nabla\overline{\varphi}(t,x)|^2\,dx + \frac{\xi^2}{\kappa\ell^2}\int_\Omega |\nabla v(t,x) - \nabla\bar{v}(t,x)|^2\,dx \;\leq$$

$$\leq\; \exp\left(C\int_0^t D(s)ds\right)\left(\frac{\tau}{2}\|\nabla\varphi_0 - \nabla\overline{\varphi}_0\|^2 + \frac{\xi^2}{\kappa\ell^2}\|\nabla v_0 - \nabla\bar{v}_0\|^2\right),$$

for all $t \in [0, T]$. If we choose initial data close to each other within the space $H_0^1(\Omega)\times H_0^1(\Omega)$, then from the last inequality it follows that the corresponding strong solutions are close to each other within the space $C\big([0,T]; H_0^1(\Omega) \times H_0^1(\Omega)\big)$. This is what we aimed to prove.

□

Chapter 3

STABILIZATION WITH ONE FEEDBACK CONTROLLER

In this Chapter, we will study the internal stabilization of the following initial-boundary value problem for the system of phase field equations :

$$\tau\varphi_t = \xi^2\Delta\varphi - f(\varphi) + 2u - k\chi_{\bar{\omega}}\varphi \quad \text{in} \quad \Omega\times\mathbb{R}^+, \tag{3.0.1}$$

$$u_t + \frac{\ell}{2}\varphi_t = \kappa\Delta u \quad \text{in} \quad \Omega\times\mathbb{R}^+, \tag{3.0.2}$$

$$\begin{aligned} \varphi|_{\partial\Omega} &= 0\,, \quad t\in\mathbb{R}^+,\\ u|_{\partial\Omega} &= 0, \quad t\in\mathbb{R}^+, \end{aligned} \tag{3.0.3}$$

$$\begin{aligned} \varphi(x,0) &= \varphi_0(x)\,, \quad x\in\Omega\,,\\ u(x,0) &= u_0(x)\,, \quad x\in\Omega, \end{aligned} \tag{3.0.4}$$

where $\Omega\subset\mathbb{R}^d$ $(d\le 3)$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$; $\omega\subset\Omega$ is a nonempty subdomain of Ω with smooth boundary $\partial\omega$ such that $\bar{\omega}\subset\Omega$; k is a non-negative number; $[\varphi_0,u_0]\in H_0^1(\Omega)\times H_0^1(\Omega)$ is a given pair of initial functions; the nonlinear term f in (3.0.1) is continuously differentiable and satisfies the following conditions :

$$-\alpha_0+\alpha_1|z|^p \le zf(z)\le \alpha_0+\alpha_2|z|^p,$$

$$f(0)=0,$$

$$f'(z)\ge -\alpha_3,$$

for some positive constants α_j's and $p>2$. As mentioned in the introduction, the initial-boundary value problem (3.0.1)-(3.0.4) is equivalent to the problem (2.0.1)-(2.0.4) of the previous chapter with $v := u+\frac{\ell}{2}\varphi$, $v_0 := u_0+\frac{\ell}{2}\varphi_0$ and $g(x,s) := f(s)-\ell s+k\chi_{\bar{\omega}}(x)s$. From Chapter 2 we know that there exists a unique strong solution $[\varphi,u]\in C(\mathbb{R}^+;H_0^1(\Omega)\times H_0^1(\Omega))$ of the initial-boundary value problem (3.0.1)-(3.0.4) such that (3.0.1) and (3.0.2) hold as equalities in $L^2(\Omega)$, for almost all $t \in \mathbb{R}^+$. We also assume that f is of the following form :

$$f(z) = h(z) - \lambda z\,, \qquad \lambda > 0\,, \tag{3.0.5}$$

where h is a continuously differentiable function satisfying the following condition :

$$h(z)z \geq H(z) := \int_0^z h(\sigma)\,d\sigma \geq 0\,, \quad \text{for all} \quad z \in \mathbb{R}. \tag{3.0.6}$$

For example, f can be either of the following functions :

$$f(z) = \frac{1}{2}(z^3 - z),$$
$$f(z) = |z|^{p-2}z - \lambda z\,, \qquad p > 2\,, \quad \lambda > 0.$$

Our aim is to prove that there exist a feedback controller $-k\varphi$ which acts on a subdomain of Ω such that the corresponding solution $[\varphi, u]$ of the initial-boundary value problem (3.0.1)-(3.0.4) exponentially decays in the $H_0^1(\Omega) \times H_0^1(\Omega)$ norm.

3.1 Preliminaries

Let A_ω be the Laplace operator with the Dirichlet boundary condition defined on $\Omega_\omega := \Omega \backslash \bar{\omega}$, i.e.,

$$A_\omega \varphi := -\Delta\varphi, \quad \varphi \in D(A_\omega)\,,$$

where $D(A_\omega) := H^2(\Omega_\omega) \cap H_0^1(\Omega_\omega)$. We will denote by $\lambda_1(A_\omega)$ the smallest eigenvalue of the operator A_ω. By the Rayleigh's principle, we have

$$\lambda_1(A_\omega) = \inf_{\substack{\varphi \in H_0^1(\Omega_\omega) \\ \varphi \neq 0}} \frac{\int_{\Omega_\omega} |\nabla\varphi|^2\,dx}{\int_{\Omega_\omega} |\varphi|^2\,dx} =$$

$$= \inf \left\{ \int_{\Omega_\omega} |\nabla\varphi|^2\,dx : \varphi \in H_0^1(\Omega_\omega),\ \|\varphi\|_{L^2(\Omega_\omega)} = 1 \right\}. \tag{3.1.1}$$

Remark 3.1.1. *It follows that $\lambda_1(A_\omega) \to \infty$ as $d_H(\partial\Omega, \partial\omega) \to 0$, where d_H is the Hausdorff distance.*

Lemma 3.1.2. *For any $\epsilon > 0$, there exists $K > 0$ such that for $k > K$, the following* *inequality holds*

$$(\lambda_1(A_\omega) - \epsilon) \int_\Omega |\varphi|^2 \, dx \le \int_\Omega (|\nabla\varphi|^2 + k\chi_{\bar\omega}|\varphi|^2) dx \,, \quad \textit{for all} \quad \varphi \in H_0^1(\Omega) \,. \tag{3.1.2}$$

Proof. Define the operator A_k with the Dirichlet boundary condition on Ω by

$$A_k\varphi = -\Delta\varphi + k\chi_{\bar\omega}\varphi, \quad \varphi \in D(A_k)\,,$$

where $D(A_k) = H^2(\Omega) \cap H_0^1(\Omega)$. Let $\lambda_1(A_k)$ be the smallest eigenvalue of the operator A_k, i.e.,

$$\lambda_1(A_k) = \inf_{\substack{\varphi \in H_0^1(\Omega) \\ \varphi \ne 0}} \frac{\int_\Omega (|\nabla\varphi|^2 + k\chi_{\bar\omega}|\varphi|^2) dx}{\int_\Omega |\varphi|^2 \, dx} =$$

$$= \inf \left\{ \int_\Omega (|\nabla\varphi|^2 + k\chi_{\bar\omega}|\varphi|^2) dx : \varphi \in H_0^1(\Omega), \; \|\varphi\|_{L^2(\Omega)} = 1 \right\}. \tag{3.1.3}$$

Let $\phi_k \in H_0^1(\Omega)$ be the eigenfunction corresponding to the eigenvalue $\lambda_1(A_k)$, i.e.,

$$\lambda_1(A_k) = \|\nabla\phi_k\|^2_{L^2(\Omega)} + k\|\phi_k\|^2_{L^2(\bar\omega)}, \quad \|\phi_k\|_{L^2(\Omega)} = 1\,. \tag{3.1.4}$$

From (3.1.1),(3.1.3) and (3.1.4) it follows that

$$\lambda_1(A_k) = \|\nabla\phi_k\|^2_{L^2(\Omega)} + k\|\phi_k\|^2_{L^2(\bar\omega)} \le \lambda_1(A_\omega)\,, \quad \text{for any} \quad k > 0\,, \tag{3.1.5}$$

As a consequence of (3.1.5), there exists a subsequence ϕ_k (with the same notation) such that

$$\begin{aligned} \phi_k &\rightharpoonup \phi \quad \text{in} \quad H_0^1(\Omega) \\ \phi_k &\to \phi \quad \text{in} \quad L^2(\Omega) \end{aligned} \tag{3.1.6}$$

as $k \to \infty$, where we have applied the Banach-Alaoglu Theorem and the compact embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$. By the Poincaré inequality, from (3.1.5) we obtain

$$\lambda_1\|\phi_k\|^2_{L^2(\Omega)} + k\|\phi_k\|^2_{L^2(\bar\omega)} \le \lambda_1(A_\omega)\,, \quad \text{for any} \quad k > 0\,, \tag{3.1.7}$$

Then from (3.1.6) and (3.1.7) it follow that

$$\phi_k \to 0 \quad \text{in} \quad L^2(\bar\omega)\,. \tag{3.1.8}$$

From (3.1.6) and (3.1.8) we obtain

$$\|\phi\|_{L^2(\bar{\omega})} \leq \|\phi - \phi_k\|_{L^2(\bar{\omega})} + \|\phi_k\|_{L^2(\bar{\omega})} \leq \|\phi - \phi_k\|_{L^2(\Omega)} + \|\phi_k\|_{L^2(\bar{\omega})} \to 0$$

as $k \to \infty$. Therefore, $\phi = 0$ almost everywhere in $\bar{\omega}$ so that

$$\phi \in H_0^1(\Omega_\omega) \quad \text{and} \quad \|\phi\|_{L^2(\Omega_\omega)} = \|\phi\|_{L^2(\Omega)} = 1\,. \tag{3.1.9}$$

From (3.1.1),(3.1.4),(3.1.6) and (3.1.9) we deduce that

$$\liminf \lambda_1(A_k) \geq \liminf \|\nabla\phi_k\|^2_{L^2(\Omega)} \geq \|\nabla\phi\|^2_{L^2(\Omega)} = \|\nabla\phi\|^2_{L^2(\Omega_\omega)} \geq \lambda_1(A_\omega)\,. \tag{3.1.10}$$

From (3.1.5) and (3.1.10) we obtain

$$\lim_{k\to\infty} \lambda_1(A_k) = \lambda_1(A_\omega)\,. \tag{3.1.11}$$

Hence, the desired inequality (3.1.2) follows from (3.1.5) and (3.1.11).

□

3.2 The Stabilization Result

Here we prove that the system (3.0.1)-(3.0.4) can be exponentially stabilized by only one feedback controller acting on a subdomain in the first equation. To simplify notations we let $\psi := -k\chi_{\bar{\omega}}\varphi$.

Theorem 3.2.1. *There exists a feedback controller $-k\varphi$ (for some large $k > 0$) such that the corresponding strong solution $[\varphi, u]$ of (3.0.1)-(3.0.4) satisfies the inequality*

$$\int_\Omega |\nabla\varphi(t,x)|^2 dx + \int_\Omega |\nabla u(t,x)|^2 dx \leq Me^{-\delta t},$$

for any $t > 0$, where M and δ are positive constants.

Proof. If we take the inner product in $L^2(\Omega)$ of (3.0.1) with φ_t and of (3.0.2) with $(4/\ell)u$

and we add these relations, then we obtain

$$\tau\|\varphi_t\|^2 + \frac{d}{dt}\left[\frac{\xi^2}{2}\|\nabla\varphi\|^2 + (H(\varphi),1) - \frac{\lambda}{2}\|\varphi\|^2 + \frac{2}{\ell}\|u\|^2 - \frac{1}{2}(\psi,\varphi)\right] + \frac{4\kappa}{\ell}\|\nabla u\|^2 = 0. \quad (3.2.1)$$

If we take the inner product in $L^2(\Omega)$ of (3.0.2) with $\epsilon_2 u_t$ ($\epsilon_2 > 0$), then we obtain

$$\epsilon_2\|u_t\|^2 + \frac{\epsilon_2\ell}{2}(\varphi_t, u_t) + \frac{d}{dt}\left[\frac{\epsilon_2\kappa}{2}\|\nabla u\|^2\right] = 0\,. \quad (3.2.2)$$

By the Cauchy-Schwartz inequality and the Cauchy's inequality with $\epsilon = \ell/8$, we deduce the estimate :

$$\frac{\ell}{2}|(\varphi_t, u_t)| \leq \frac{\ell}{2}\|\varphi_t\|\|u_t\| \leq \frac{\ell^2}{16}\|\varphi_t\|^2 + \|u_t\|^2\,. \quad (3.2.3)$$

Then from (3.2.2) and (3.2.3) we obtain

$$\frac{d}{dt}\left[\frac{\epsilon_2\kappa}{2}\|\nabla u\|^2\right] \leq \frac{\epsilon_2\ell^2}{16}\|\varphi_t\|^2\,. \quad (3.2.4)$$

If we add inequalities (3.2.1) and (3.2.4) with $\epsilon_2 = 16\tau/\ell^2$, then we obtain

$$\frac{d}{dt}\left[\frac{\xi^2}{2}\|\nabla\varphi\|^2 + \frac{8\tau\kappa}{\ell^2}\|\nabla u\|^2 + (H(\varphi),1) - \frac{\lambda}{2}\|\varphi\|^2 + \frac{2}{\ell}\|u\|^2 - \frac{1}{2}(\psi,\varphi)\right] + \frac{4\kappa}{\ell}\|\nabla u\|^2 \leq 0. \quad (3.2.5)$$

If we take the inner product in $L^2(\Omega)$ of (3.0.1) with $\epsilon_3\varphi$ ($\epsilon_3 > 0$) , then we deduce

$$\frac{d}{dt}\left[\frac{\epsilon_3\tau}{2}\|\varphi\|^2\right] + \epsilon_3\xi^2\|\nabla\varphi\|^2 + \epsilon_3(f(\varphi),\varphi) - \epsilon_3\lambda\|\varphi\|^2 - \epsilon_3(\psi,\varphi) = 2\epsilon_3(u,\varphi)\,. \quad (3.2.6)$$

By the Cauchy-Schwartz inequality and the Cauchy's inequality with $\epsilon = \frac{1}{\lambda_1\xi^2}$, we obtain an estimate for the right hand side of the equation (3.2.6) as follows :

$$2|(u,\varphi)| \;\leq\; 2\|u\|\|\varphi\| \leq \frac{2}{\lambda_1}\|\nabla u\|\|\nabla\varphi\| \leq \frac{2}{(\lambda_1\xi)^2}\|\nabla u\|^2 + \frac{\xi^2}{2}\|\nabla\varphi\|^2\,. \quad (3.2.7)$$

By taking into account the assumption (3.0.6) and the estimate (3.2.7), from (3.2.6) we deduce that

$$\frac{d}{dt}\left[\frac{\epsilon_3\tau}{2}\|\varphi\|^2\right] + \frac{\epsilon_3\xi^2}{2}\|\nabla\varphi\|^2 + \epsilon_3(H(\varphi),1) - \epsilon_3\lambda\|\varphi\|^2 - \epsilon_3(\psi,\varphi) \leq \frac{2\epsilon_3}{(\lambda_1\xi)^2}\|\nabla u\|^2\,. \quad (3.2.8)$$

If we add the inequalities (3.2.5) and (3.2.8) with $\epsilon_3 = \kappa\xi^2\lambda_1^2/2\ell$, then we obtain

$$\frac{d}{dt}\left[\frac{\xi^2}{2}\|\nabla\varphi\|^2 + \frac{8\tau\kappa}{\ell^2}\|\nabla u\|^2 + (H(\varphi),1) + \left[\frac{\epsilon_3\tau}{2} - \lambda\right]\|\varphi\|^2 + \frac{2}{\ell}\|u\|^2 - \frac{1}{2}(\psi,\varphi)\right] +$$

$$+ \frac{\epsilon_3\xi^2}{2}\|\nabla\varphi\|^2 + \frac{3\kappa}{\ell}\|\nabla u\|^2 + \epsilon_3(H(\varphi),1) - \epsilon_3\lambda\|\varphi\|^2 - \epsilon_3(\psi,\varphi) \le 0\,. \tag{3.2.9}$$

By the Remark 3.1.1, we can choose Ω_ω "sufficiently thin" so that

$$\lambda_1(A_\omega) - \frac{4}{\xi^2}\lambda - 1 > 0\,.$$

By the Lemma 3.1.2, for sufficiently large $k > 0$, we have

$$[\lambda_1(A_\omega) - 1]\|\varphi\|^2 \le \|\nabla\varphi\|^2 - \frac{2}{\xi^2}(\psi,\varphi)\,. \tag{3.2.10}$$

The last estimate we need follows from Poincaré inequality :

$$\frac{3\lambda_1\kappa}{2\ell}\|u\|^2 \le \frac{3\kappa}{2\ell}\|\nabla u\|^2\,. \tag{3.2.11}$$

Then by taking into account (3.2.10) and (3.2.11), from (3.2.9) we deduce that

$$\frac{d}{dt}\left[\frac{\xi^2}{2}\|\nabla\varphi\|^2 + \frac{8\tau\kappa}{\ell^2}\|\nabla u\|^2 + (H(\varphi),1) + \left[\frac{\epsilon_3\tau}{2} - \lambda\right]\|\varphi\|^2 + \frac{2}{\ell}\|u\|^2 - \frac{1}{2}(\psi,\varphi)\right] +$$

$$+ \frac{\epsilon_3\xi^2}{4}\|\nabla\varphi\|^2 + \frac{3\kappa}{2\ell}\|\nabla u\|^2 + \epsilon_3(H(\varphi),1) + \tag{3.2.12}$$

$$\frac{\epsilon_3\xi^2}{4}\left[\lambda_1(A_\omega) - \frac{4}{\xi^2}\lambda - 1\right]\|\varphi\|^2 + \frac{3\lambda_1\kappa}{2\ell}\|u\|^2 - \frac{\epsilon_3}{2}(\psi,\varphi) \le 0\,.$$

Let

$$Y(t) := \frac{\xi^2}{2}\|\nabla\varphi\|^2 + \frac{8\tau\kappa}{\ell^2}\|\nabla u\|^2 + (H(\varphi),1) + \left[\frac{\epsilon_3\tau}{2} - \lambda\right]\|\varphi\|^2 + \frac{2}{\ell}\|u\|^2 - \frac{1}{2}(\psi,\varphi)\,.$$

Then, for sufficiently small $\delta > 0$, from (3.2.12) we obtain

$$\frac{d}{dt}Y(t) + \delta Y(t) \le 0\,.$$

By the Gronwall's inequality, we deduce that

$$Y(t) \leq Y(0)e^{-\delta t}\,, \quad \text{for all} \quad t > 0\,. \tag{3.2.13}$$

In particular, by taking into account the estimate (3.2.10) once again, from (3.2.13) we deduce that

$$\int_\Omega |\nabla\varphi(t,x)|^2 dx + \int_\Omega |\nabla u(t,x)|^2 dx \leq \frac{Y(0)}{C} e^{-\delta t}\,,$$

for all $t > 0$, where $C := \min\left\{\frac{\xi^2}{4}, \frac{8\tau\kappa}{\ell^2}\right\}$.

□

Chapter 4

GLOBAL ATTRACTOR

In this Chapter, we will study the problem of existence of a global attractor of the semigroup generated by the following initial-boundary value problem for the system of phase field equations :

$$\tau\varphi_t = \xi^2\Delta\varphi - f(\varphi) + 2u \quad \text{in} \quad \Omega \times \mathbb{R}^+\,, \tag{4.0.1}$$

$$u_t + \frac{\ell}{2}\varphi_t = \kappa\Delta u \quad \text{in} \quad \Omega \times \mathbb{R}^+\,, \tag{4.0.2}$$

$$\begin{aligned} \varphi|_{\partial\Omega} &= 0\,, \quad t \in \mathbb{R}^+, \\ u|_{\partial\Omega} &= 0\,, \quad t \in \mathbb{R}^+, \end{aligned} \tag{4.0.3}$$

$$\begin{aligned} \varphi(x,0) &= \varphi_0(x)\,, \quad x \in \Omega\,, \\ u(x,0) &= u_0(x)\,, \quad x \in \Omega\,, \end{aligned} \tag{4.0.4}$$

where Ω is a bounded domain in $\mathbb{R}^d$ $(d \leq 3)$ with sufficiently smooth boundary $\partial\Omega$, under the assumption that the nonlinear term f in (4.0.1) is continuously differentiable and satisfies the following conditions :

$$-\alpha_0 + \alpha_1|z|^p \leq zf(z) \leq \alpha_0 + \alpha_2|z|^p, \tag{4.0.5}$$

$$f(0) = 0,$$

$$f'(z) \geq -\alpha_2,$$

$$|f'(z)| \leq \alpha_4\big(1 + |z|^{p-2}\big), \tag{4.0.6}$$

for all $z \in \mathbb{R}$, where $\alpha_j > 0$, $p \in (2,4]$ if $d = 3$ and $p \in (2,\infty)$ if $d = 1$ or $d = 2$. From Chapter 2, we know that if $[\varphi_0, u_0] \in H_0^1(\Omega) \times H_0^1(\Omega)$, then there exists a unique strong solution $[\varphi, u] \in C(\mathbb{R}^+; H_0^1(\Omega) \times H_0^1(\Omega))$ of the above initial-boundary value problem such that (4.0.1) and (4.0.2) hold as equalities in $L^2(\Omega)$, for almost all $t \in \mathbb{R}^+$. From the global existence of unique solutions which depend continuously on initial functions it follows that the problem (4.0.1)-(4.0.4) generates a continuous semigroup $\{S(t)\}_{t\geq 0}$ which consists of the operators $S(t) : H_0^1(\Omega)\times H_0^1(\Omega) \to H_0^1(\Omega)\times H_0^1(\Omega)$ defined by $S(t)[\varphi_0, u_0] := [\varphi(t,\cdot), u(t,\cdot)]$.

4.1 Existence of a Global Attractor

To prove the existence of a global attractor $\mathcal{A}$ for the semigroup $\{S(t)\}_{t\geq 0}$ we will use the Theorem 1.9.12. Therefore, we will proceed as follows : (i) Show that the semigroup $\{S(t)\}_{t\geq 0}$ has an absorbing ball in $H_0^1(\Omega)\times H_0^1(\Omega)$; (ii) Show that the semigroup $\{S(t)\}_{t\geq 0}$ is a compact semigroup.

Lemma 4.1.1. *The semigroup $\{S(t)\}_{t\geq 0}$ has an absorbing ball in $H_0^1(\Omega) \times H_0^1(\Omega)$.*

Proof. If we take the inner product in $L^2(\Omega)$ of (4.0.1) with φ_t and of (4.0.2) with $(4/\ell)u$ and we add these relations, then we obtain

$$\tau\|\varphi_t\|^2 + \frac{d}{dt}\left[\frac{\xi^2}{2}\|\nabla\varphi\|^2 + \big(F(\varphi), \mathbf{1}_\Omega\big) + \frac{2}{\ell}\|u\|^2\right] + \frac{4\kappa}{\ell}\|\nabla u\|^2 = 0. \tag{4.1.1}$$

If we take the inner product in $L^2(\Omega)$ of (4.0.2) with $\epsilon_2 u_t$ ($\epsilon_2 > 0$), then we obtain

$$\epsilon_2\|u_t\|^2 + \frac{\epsilon_2\ell}{2}(\varphi_t, u_t) + \frac{d}{dt}\left[\frac{\epsilon_2\kappa}{2}\|\nabla u\|^2\right] = 0\,. \tag{4.1.2}$$

By the Cauchy-Schwartz inequality and the Cauchy's inequality with $\epsilon = \ell/4$, we deduce the estimate :

$$-\frac{\ell}{2}(\varphi_t, u_t) \leq \frac{\ell}{2}\|\varphi_t\|\|u_t\| \leq \frac{\ell^2}{8}\|\varphi_t\|^2 + \frac{1}{2}\|u_t\|^2\,. \tag{4.1.3}$$

Then from (4.1.2) and (4.1.3) we obtain

$$\frac{\epsilon_2}{2}\|u_t\|^2 + \frac{d}{dt}\left[\frac{\epsilon_2\kappa}{2}\|\nabla u\|^2\right] \leq \frac{\epsilon_2\ell^2}{8}\|\varphi_t\|^2\,. \tag{4.1.4}$$

If we add inequalities (4.1.1) and (4.1.4) with $\epsilon_2 = 4\tau/\ell^2$, then we obtain

$$\frac{\tau}{2}\|\varphi_t\|^2 + \frac{2\tau}{\ell^2}\|u_t\|^2 +$$

$$+\frac{d}{dt}\left[\frac{\xi^2}{2}\|\nabla\varphi\|^2 + \frac{2\tau\kappa}{\ell^2}\|\nabla u\|^2 + \big(F(\varphi), \mathbf{1}_\Omega\big) + \frac{2}{\ell}\|u\|^2\right] + \frac{4\kappa}{\ell}\|\nabla u\|^2 \leq 0. \tag{4.1.5}$$

If we take the inner product in $L^2(\Omega)$ of (4.0.1) with $\epsilon_3\varphi$ ($\epsilon_3 > 0$) , then we deduce

$$\frac{d}{dt}\left[\frac{\epsilon_3\tau}{2}\|\varphi\|^2\right] + \epsilon_3\xi^2\|\nabla\varphi\|^2 + \epsilon_3\big(f(\varphi),\varphi\big) = 2\epsilon_3(u,\varphi)\,. \tag{4.1.6}$$

By the Cauchy-Schwartz inequality and the Cauchy's inequality with $\epsilon = \frac{1}{\lambda_1\xi^2}$, we obtain an estimate for the right hand side of the equation (4.1.6) as follows :

$$2(u,\varphi) \quad\leq\quad 2\|u\|\|\varphi\| \leq \frac{2}{\lambda_1}\|\nabla u\|\|\nabla\varphi\| \leq \frac{2}{(\lambda_1\xi)^2}\|\nabla u\|^2 + \frac{\xi^2}{2}\|\nabla\varphi\|^2\,. \tag{4.1.7}$$

By taking into account the estimate (4.1.7), from (4.1.6) we deduce that

$$\frac{d}{dt}\left[\frac{\epsilon_3\tau}{2}\|\varphi\|^2\right] + \frac{\epsilon_3\xi^2}{2}\|\nabla\varphi\|^2 + \epsilon_3\big(f(\varphi),\varphi\big) \leq \frac{2\epsilon_3}{(\lambda_1\xi)^2}\|\nabla u\|^2\,. \tag{4.1.8}$$

If we add the inequalities (4.1.5) and (4.1.8) with $\epsilon_3 = \frac{\kappa(\xi\lambda_1)^2}{2\ell}$, then we obtain

$$\frac{\tau}{2}\|\varphi_t\|^2 + \frac{2\tau}{\ell^2}\|u_t\|^2 + \frac{d}{dt}\left[\frac{\xi^2}{2}\|\nabla\varphi\|^2 + \frac{2\tau\kappa}{\ell^2}\|\nabla u\|^2 + \big(F(\varphi),\mathbf{1}_\Omega\big) + \frac{\epsilon_3\tau}{2}\|\varphi\|^2 + \frac{2}{\ell}\|u\|^2\right] +$$

$$+\ \frac{\epsilon_3\xi^2}{2}\|\nabla\varphi\|^2 + \frac{3\kappa}{\ell}\|\nabla u\|^2 + \epsilon_3\big(f(\varphi),\varphi\big) \leq 0\,. \tag{4.1.9}$$

The last estimates we need follows from Poincaré inequality :

$$\frac{\epsilon_3\lambda_1\xi^2}{4}\|\varphi\|^2 \leq \frac{\epsilon_3\xi^2}{4}\|\nabla\varphi\|^2 \quad \text{and} \quad \frac{3\lambda_1\kappa}{2\ell}\|u\|^2 \leq \frac{3\kappa}{2\ell}\|\nabla u\|^2\,. \tag{4.1.10}$$

Then by taking into account the estimates (4.1.10), from (4.1.9) we obtain

$$\frac{\tau}{2}\|\varphi_t\|^2 + \frac{2\tau}{\ell^2}\|u_t\|^2 + \frac{d}{dt}\left[\frac{\xi^2}{2}\|\nabla\varphi\|^2 + \frac{2\tau\kappa}{\ell^2}\|\nabla u\|^2 + \big(F(\varphi),\mathbf{1}_\Omega\big) + \frac{\epsilon_3\tau}{2}\|\varphi\|^2 + \frac{2}{\ell}\|u\|^2\right] +$$

$$+\ \frac{\epsilon_3\xi^2}{4}\|\nabla\varphi\|^2 + \frac{3\kappa}{2\ell}\|\nabla u\|^2 + \epsilon_3\big(f(\varphi),\varphi\big) + \frac{\epsilon_3\lambda_1\xi^2}{4}\|\varphi\|^2 + \frac{3\lambda_1\kappa}{2\ell}\|u\|^2 \leq 0\,. \tag{4.1.11}$$

Let

$$Y(t) := \frac{\xi^2}{2}\|\nabla\varphi\|^2 + \frac{2\tau\kappa}{\ell^2}\|\nabla u\|^2 + \big(F(\varphi),\mathbf{1}_\Omega\big) + \frac{\epsilon_3\tau}{2}\|\varphi\|^2 + \frac{2}{\ell}\|u\|^2\,.$$

Due to the assumption (4.0.5), from the Lemma 1.3.2 it follows that

$$\big(F(\varphi), \mathbf{1}_\Omega)\big) \geq -C_1 \quad \text{and} \quad \big(f(\varphi), \varphi\big) - \big(F(\varphi), \mathbf{1}_\Omega\big) \geq -C_2\,,$$

where C_1 and C_2 depend only on Ω. For sufficiently small $\delta > 0$, from (4.1.11) we obtain

$$\frac{d}{dt}Y(t) + \delta Y(t) \leq \delta[\big(F(\varphi), \mathbf{1}_\Omega\big) - \big(f(\varphi), \varphi\big)] + [\delta - \epsilon_3]\big(f(\varphi), \varphi\big) \leq \delta C_2 + [\epsilon_3 - \delta]\alpha_0 := C_3\,.$$

By the Gronwall's inequality, from the last inequality we deduce that

$$Y(t) \leq Y(0)e^{-\delta t} + C_4\,, \tag{4.1.12}$$

where $C_4 := \frac{C_3}{\delta}$. Then we obtain the following inequality :

$$\frac{\xi^2}{2}\|\nabla\varphi\|^2 + \frac{8\tau\kappa}{\ell^2}\|\nabla u\|^2 \leq Y(t) - \big(F(\varphi), \mathbf{1}_\Omega\big) \leq Y(0)e^{-\delta t} + C_1 + C_4\,. \tag{4.1.13}$$

From the assumption (4.0.5) it follows that

$$|F(s)| \leq \gamma_0 + \gamma_1|s|^p, \quad \text{for all} \quad s \in \mathbb{R}. \tag{4.1.14}$$

Since $H_0^1(\Omega)$ is continuously embedded in $L^p(\Omega)$, from (4.1.14) we deduce that

$$|(F(\varphi_0), \mathbf{1}_\Omega)| \leq \gamma_0|\Omega| + \gamma_2\|\nabla\varphi_0\|^p. \tag{4.1.15}$$

Then, by the Poincaré inequality and the estimate (4.1.15), we obtain

$$Y(0) \leq C_5 + C_6\|[\varphi_0, u_0]\|^2_{H_0^1(\Omega)\times H_0^1(\Omega)} \tag{4.1.16}$$

Finally, from (4.1.13) and (4.1.16) we deduce that

$$\int_\Omega |\nabla\varphi(t,x)|^2 dx + \int_\Omega |\nabla u(t,x)|^2 dx \leq C\left(\int_\Omega |\nabla\varphi_0(x)|^2 dx + \int_\Omega |\nabla u_0(x)|^2 dx\right) e^{-\delta t} + R$$

□

Theorem 4.1.2. *There exists a global attractor $\mathcal{A}$ for the continuous semigroup $\{S(t)\}_{t\geq 0}$. The global attractor $\mathcal{A}$ is a bounded, closed and connected subset of the space $H_0^2(\Omega)\times H_0^2(\Omega)$.*

Proof. From now on we will denote by K_i the positive constants which depend on $H_0^1(\Omega)\times H_0^1(\Omega)$ norm of initial functions $[\varphi_0, u_0]$. If we set $D(t) := \frac{\tau}{2}\|\varphi_t\|^2 + \frac{2\tau}{\ell^2}\|u_t\|^2$ and $E(t) := Y(t) - \frac{\epsilon_3\tau}{2}\|\varphi\|^2$, then from the inequality (4.1.5) it follows that

$$D(t) + \frac{d}{dt}E(t) \leq 0\,. \tag{4.1.17}$$

We integrate the inequality (4.1.17) from 0 to t and obtain

$$\int_0^t D(s)ds \leq E(0) - E(t) \leq Y(0) - \big(F(\varphi), \mathbf{1}_\Omega\big) \leq K_1. \tag{4.1.18}$$

If we multiply the inequality (4.1.17) by t and use the estimate (4.1.12), then we obtain

$$tD(t) + \frac{d}{dt}[tE(t)] \leq Y(0)e^{-\delta t} + C_4\,. \tag{4.1.19}$$

If we integrate the inequality (4.1.19) from 0 to t, then we obtain

$$\int_0^t sD(s)ds \leq K_2 + C_7 t\,, \tag{4.1.20}$$

where $C_7 := C_1 + C_4$. The rest of the derivations are so called "a priory estimates" validity of which can verified by using the Galerkin approximations. If we differentiate (4.0.1) with respect to t and take the inner product in $L^2(\Omega)$ of the resulting relation by $t\varphi_t$, then we obtain

$$\frac{\tau t}{2}\frac{d}{dt}\|\varphi_t\|^2 + \xi^2 t\|\nabla\varphi_t\|^2 + t\big(f'(\varphi), \varphi_t^2\big) = 2t\big(u_t, \varphi_t\big). \tag{4.1.21}$$

If we take the inner product of (4.0.2) with $(4/\ell)u_t$, then we obtain

$$\frac{4t}{\ell}\|u_t\|^2 + 2t\big(u_t, \varphi_t\big) + \frac{2\kappa t}{\ell}\frac{d}{dt}\|\nabla u\|^2 = 0. \tag{4.1.22}$$

From the last two relations we deduce the following inequality :

$$\frac{d}{dt}\left[\frac{\tau t}{2}\|\varphi_t\|^2 + \frac{2\kappa t}{\ell}\|\nabla u\|^2\right] - \frac{\tau}{2}\|\varphi_t\|^2 - \frac{2\kappa}{\ell}\|\nabla u\|^2 - \alpha_2 t\|\varphi_t\|^2 \leq 0\,, \tag{4.1.23}$$

where we have used the lower bound $-\alpha_2$ for the derivative of f. If we integrate the inequality (4.1.23) from 0 to t and use the estimates (4.1.13), (4.1.18), (4.1.20), then we deduce the following inequality :

$$\frac{\tau t}{2}\|\varphi_t\|^2 + \frac{2\kappa t}{\ell}\|\nabla u\|^2 \leq K_3 + C_8 t\,.$$

Therefore, for any $t > 0$ we have

$$\frac{\tau}{2}\|\varphi_t\|^2 \leq K_3 t^{-1} + C_8\,. \tag{4.1.24}$$

Note that f satisfies the following inequality :

$$\|f(\varphi)\|^2 \leq C_9\left[1 + \|\varphi\|^{2(p-1)}_{L^{2(p-1)}(\Omega)}\right], \tag{4.1.25}$$

where the constant C_{11} depends only on $|\Omega|$. If $d = 1$ or $d = 2$, then $2 < 2(p-1) < \infty$. If $d = 3$, then $2 < 2(p-1) \leq 6$. In any case, $H^1_0(\Omega)$ is continuously embedded in $L^{2(p-1)}(\Omega)$. Therefore, from (4.1.13) and (4.1.25) we obtain

$$\|f(\varphi)\|^2 \leq C_{10}\left[1 + \|\nabla\varphi\|^{2(p-1)}\right] \leq K_4 e^{-\gamma t} + C_{11}, \tag{4.1.26}$$

for $\gamma := 2\delta(p-1)$. From (4.1.13), (4.1.24) and (4.1.26) it follows that

$$\|\Delta\varphi\|^2 \leq K_6 t^{-1} + K_7 e^{-\delta t} + C_{12}, \quad \text{for all} \quad t > 0. \tag{4.1.27}$$

If we set $P(t) := \frac{\tau}{2}\|\varphi_t\|^2 + \frac{2\kappa}{\ell}\|\nabla u\|^2$, then from (4.1.21)-(4.1.22) we deduce the following inequality:

$$\xi^2\|\nabla\varphi_t\|^2 + \frac{d}{dt}P(t) - C_{13}P(t) \leq 0. \tag{4.1.28}$$

If we multiply (4.1.28) by $e^{-C_{13}t}$, then we obtain

$$\frac{d}{dt}\left[P(t)e^{-C_{13}t}\right] \leq 0.$$

Let $\epsilon \in (0,1)$. If we integrate the last inequality from ϵ to t, then we get

$$P(t) \leq C_{14} P(\epsilon) e^{C_{13}t} \leq K_8 e^{C_{13}t}, \quad \text{for all} \quad t \geq \epsilon. \tag{4.1.29}$$

From (4.1.28) and (4.1.29) it follows that

$$\xi^2 \|\nabla \varphi_t\|^2 + \frac{d}{dt} P(t) \leq c_{13} K_8 e^{C_{13}t}, \quad \text{for all} \quad t \geq \epsilon. \tag{4.1.30}$$

By integrating (4.1.30) from ϵ to t, we get

$$\xi^2 \int_\epsilon^t \|\nabla \varphi_s\|^2 ds \leq 2K_8 e^{C_{13}t}, \quad \text{for all} \quad t \geq \epsilon. \tag{4.1.31}$$

It we multiply (4.1.30) by t, then we obtain

$$\xi^2 t \|\nabla \varphi_t\|^2 + \frac{d}{dt}[tP(t)] \leq c_{14} K_8 t e^{C_{13}t}, \quad \text{for all} \quad t \geq \epsilon.$$

By integrating the last inequality from ϵ to t, we obtain

$$\xi^2 \int_\epsilon^t s \|\nabla \varphi_s\|^2 ds \leq K_9 t e^{C_{13}t}, \quad \text{for all} \quad t \geq \epsilon. \tag{4.1.32}$$

Now we differentiate the equations (4.0.1)-(4.0.4) with respect to t, take the inner product in $L^2(\Omega)$ with $t\varphi_{tt}$ and $(4/\ell)tu_t$, respectively, and then we add the obtained relations :

$$t\frac{d}{dt}\left[\frac{\xi^2}{2}\|\nabla \varphi_t\|^2 + \frac{2}{\ell}\|u_t\|^2\right] + \tau t \|\varphi_{tt}\|^2 + \frac{4\kappa t}{\ell}\|\nabla u_t\|^2 + t\big(f'(\varphi)\varphi_t, \varphi_{tt}\big) = 0. \tag{4.1.33}$$

By the Cauchy's inequality with $\epsilon = \frac{1}{2\tau}$, we obtain

$$\left|\big(f'(\varphi)\varphi_t, \varphi_{tt}\big)\right| \leq \frac{1}{2\tau}\|f'(\varphi)\varphi_t\|^2 + \frac{\tau}{2}\|\varphi_{tt}\|^2.$$

By the assumption (4.0.6) and the Hölders's inequality, we obtain

$$\|f'(\varphi)\varphi_t\|^2 \leq C_{15}\left[\|\varphi_t\|^2 + \|\varphi^{p-2}\varphi_t\|^2\right] \leq C_{15}\left[\|\varphi_t\|^2 + \|\varphi\|^{2(p-2)}_{L^{2(p-1)}(\Omega)}\|\varphi_t\|^2_{L^{2(p-1)}(\Omega)}\right].$$

Since $H_0^1(\Omega)$ is continuously embedded in $L^{2(p-1)}(\Omega)$, from the last two inequalities we deduce the following estimate :

$$|(f'(\varphi)\varphi_t, \varphi_{tt})| \leq C_{16}\left[\|\varphi_t\|^2 + \|\nabla\varphi\|^{2(p-2)}\|\nabla\varphi_t\|^2\right] + \frac{\tau}{2}\|\varphi_{tt}\|^2 \leq K_{10} + K_{11}\|\nabla\varphi_t\|^2 + \frac{\tau}{2}\|\varphi_{tt}\|^2.$$

From (4.1.33) and the last estimate we deduce the following inequality :

$$\frac{d}{dt}\left[\frac{\xi^2 t}{2}\|\nabla\varphi_t\|^2 + \frac{2t}{\ell}\|u_t\|^2\right] - \frac{\xi^2}{2}\|\nabla\varphi_t\|^2 - \frac{2}{\ell}\|u_t\|^2 \leq K_{10}t + K_{11}t\|\nabla\varphi_t\|^2. \tag{4.1.34}$$

If we integrate (4.1.34) from ϵ to t, then from the estimates (4.1.18),(4.1.31) and (4.1.32) we obtain

$$\frac{\xi^2 t}{2}\|\nabla\varphi_t\|^2 + \frac{2t}{\ell}\|u_t\|^2 \leq K_{12} + K_{13}t^2 + K_{14}te^{c_{13}t}.$$

From the last inequality it follows that

$$\frac{2}{\ell}\|u_t\|^2 \leq K_{12}t^{-1} + K_{13}t + K_{14}e^{c_{13}t}, \quad \text{for all} \quad t \geq \epsilon.$$

Therefore,

$$\|\Delta u\|^2 \leq K_{15}t^{-1} + K_{16}t + K_{17}e^{c_{13}t}, \quad \text{for all} \quad t \geq \epsilon. \tag{4.1.35}$$

From the estimates (4.1.27) and (4.1.35) it follows that the operator $S(t)$ is compact, for each $t \geq \epsilon > 0$, where ϵ is arbitrarily small. Therefore, the existence of a global attractor $\mathcal{A}$ follows from the Corollary 1.9.12. The attractor $\mathcal{A}$ is connected being a subset of a connected phase space $H_0^1(\Omega) \times H_0^1(\Omega)$. Furthermore, from the estimates we obtained it is clear that $\mathcal{A}$ is a closed and bounded subset of the space $H_0^2(\Omega) \times H_0^2(\Omega)$.

□

Remark 4.1.3. *The idea we have used here to estimate the terms $\Delta\varphi$ and Δu is due to Prof. O.A. Ladyzhenskaya (See [4]).*

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