Categorical homotopy theory

by

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,

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ABSTRACT

In this thesis we present and discuss some basic aspects of homotopic algebra. We present how elements from category theory can be helpful in studying homotopy theory.

Homotopy theory arises in studying topological spaces. Here we present a meaningful way to extend the notion of homotopy to other categories such as chain complexes. Then we present the axioms for closed model categories, which capture common features of categories in which we can talk about homotopy.

After describing how we can do some homotopy theory starting with a few axioms, we construct the main object of study which is the homotopy category associated to a closed model category. After construction we will prove that the homotopy category is just a localization.

Finally we present some features of the homotopy category in order to emphasize the advantages of doing homotopy in closed model categories and give brief mention of homotopy limits and colimits in the final section. All of the above are done keeping in mind the main examples which are topological spaces and chain complexes.

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Chapter 1

INTRODUCTION

1.1 Why category theory?

The practice of algebraic topology is assigning algebraic objects to topological spaces in order to learn more about the space. Homotopy groups, chain complexes and homology groups associated with a space are fundamental instances of this practice. How does category theory help?

First, the class of topological spaces themselves can be seen as a category **Top**, where arrows between two spaces are continuous maps. Similarly we can talk about the category of pointed spaces, that as a matter of fact is the comma category $\star \uparrow$ **Top** where $\star$ is the one point space. In light of this, the associations of chain complexes $X \mapsto C_\star(X)$, homology groups $X \mapsto C_\star(X) \mapsto H_n(X)$ and homotopy groups $(X, x_0) \mapsto \pi_n(X, x_0)$ are functorial.

But of course there is a lot more. Imposing the homotopy relation between maps in **Top** yields a new category π**Top** where maps between objects are homotopy classes of maps rather than all maps. The isomorphisms in this category are called *homotopy equivalences*.

The notion of homotopy comes naturally in **Top**. But we can extend the notion to other categories as well in a "meaningful" way. The singular n-chain functors

$C_n : \mathbf{Top} \longrightarrow Mod_R$ that assigns to each space $X \in \mathbf{Top}$ the free R-module generated by the n-simiplicies $\Delta^n \to X$ assemble to a functor

$$C_\star : \mathbf{Top} \longrightarrow Ch_R$$

where Ch_R denotes the category of non-negatively graded chain complexes over a commutative ring R.

One observes that a homotopy $H : X \times I \longrightarrow Y$ between maps $f, g : X \to Y$ in **Top** gives rise to maps $s : C_{n+1}(X) \to C_n(Y)$ (called prism maps) having the property that

$$f_\sharp - g_\sharp = s\partial - \partial s$$

where $f_\sharp, g_\sharp : C_\star(X) \longrightarrow C_\star(Y)$ are the induced map and ∂ denotes the boundary maps of these chain complexes. Exactly as above we can define the homotopy relation between maps in Ch_R. And we are in a situation where the singular chain functor sends homotopic maps in **Top** to homotopic maps in Ch_R.

Moreover, homotopic maps in Ch_R have the property that they induce the same maps between homology groups. Therefore homotopic maps in **Top** induce the same maps between homology groups. There is a distinguished class of maps $f : A \longrightarrow B$ in Ch_R that induce isomorphisms between all homology groups $f_\star : H_n(A) \longrightarrow H_n(B)$ called quasi-isomorphisms. These maps will serve as weak equivalences for chain complexes.

Another distinguished class of maps is the collection of *weak homotopy equivalences* in **Top**. These are maps $f : X \longrightarrow Y$ that induce isomorphisms on all homotopy groups $f_n : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ for all $x \in X$ and a bijection $f_0 : \pi_0(X) \longrightarrow$

$\pi_0(Y)$ between path connected components. We would like to consider these maps as isomorphisms and we deal with this via localization of categories.

This being the situation, certain properties of maps are highlighted, for example *the homotopy lifting property*. Maps satisfying this property are called *Serre fibrations*. In Mod_R *projective* modules are central to homological algebra because of their lifting properties. On the level of chain complexes we can produce *projective resolutions* associated to each R-module.

Therefore it is useful to be able to talk about homotopy in an abstract setting, in many different categories. And of course if we want to talk about homotopy in some category $\mathcal{C}$ we want this category to share some features with **Top** and Ch_R. Quillen introduced the axioms for *closed model categories*, axioms that are satisfied not only by **Top** and Ch_R but by many other categories such as various diagram categories and simplicial sets. We will see closed model categories and the homotopy in these categories in some detail.

Chapter 2

SOME BACKGROUND AND TERMINOLOGY FROM CATEGORY THEORY

2.1 Colimits and limits

Definition 1. *Let $\mathcal{C}$ be a category. An initial object in $\mathcal{C}$ is an object $\emptyset$ such that for all objects $A \in \mathcal{C}$ there is a unique morphism $\emptyset \to A$. Dually, an object $\star$ is said to be terminal if for all objects $X \in \mathcal{C}$ there is a unique morphism $X \to \star$*

One can verify that initial and terminal objects are unique up to unique isomorphism, therefore in this sense we can say the terminal object $\star$ and the initial object $\emptyset$. Notation is inspired by the category **Set**, where the empty set is the initial object and the one point set is the terminal object. The same is true for the category of spaces **Top**.

We say that a category $\mathbf{D}$ is *small* if the classes of objects $Ob(\mathbf{D})$ and morphisms $Mor(\mathbf{D})$ are sets, and we say that $\mathbf{D}$ is *locally small* if for any two objects $X, Y \in \mathbf{D}$ the class of morphisms between these objects $hom_{\mathbf{D}}(X, Y)$ is a set.. The categories **Top** and **Set** are not small but just locally small. In general, categories of interest will not be small but rather locally small. However small diagrams in those categories may be of interest depending on the occasion.

Definition 2. *Let $\mathcal{C}$ be a category and $\boldsymbol{D}$ be a small category. A diagram of shape" $\boldsymbol{D}$ in $\mathcal{C}$ is a functor $X : \boldsymbol{D} \to \mathcal{C}$.*

For example, if $\mathbf{D}$ is the category $\{a \to b\}$ then a diagram of shape $\mathbf{D}$ in $\mathcal{C}$ is just a map $A \to B$ in $\mathcal{C}$. Given a small category $\mathbf{D}$ we can think of all diagrams $\mathbf{D} \to \mathcal{C}$ as objects in a category $\mathcal{C}^{\mathbf{D}}$. For two diagrams $X, Y : \mathbf{D} \to \mathcal{C}$ a map $f : X \to Y$ is the data of maps $f_i : X_i \to Y_i$ for all $i \in \mathbf{D}$ that respect the structure of the diagrams X and Y, i.e. f is a natural transformation between functors X and Y.

This being said, we can see the category $\mathcal{C}$ fully embedded in $\mathcal{C}^{\mathbf{D}}$, meaning that any object $A \in \mathcal{C}$ can be seen as a diagram $A : \mathbf{D} \to \mathcal{C}$ where $A_i = A$ for all $i \in \mathbf{D}$ and all maps in the diagram are id_A. This is called the constant diagram at A. It is easy to observe that a diagram map between constant diagrams A and B is the same as just a map of objects $A \to B$ in $\mathcal{C}$. Therefore we can talk about a well defined functor

$$\Delta : \mathcal{C} \longrightarrow \mathcal{C}^{\mathbf{d}}$$

that sends each object to the corresponding constant diagram. Δ is called the *constant diagram functor*.

Now we define two concepts that are ubiquitous in mathematics, colimits and limits.

Definition 3. *Let $\mathcal{C}$ be a category, $\boldsymbol{D}$ be a small category and $X : \boldsymbol{D} \longrightarrow \mathcal{C}$ a diagram of shape $\boldsymbol{D}$ in $\mathcal{C}$. A cone object over the diagram X consists of an object $A \in \mathcal{C}$ and structure maps $X_i \to A$ for $i \in \boldsymbol{D}$ that respect the maps in the diagram X.*

We can see a cone object as a constant diagram, and alternatively we can say that a cone object is a constant diagram $A : \mathbf{D} \longrightarrow \mathcal{C}$ equipped with a diagram map $X \to A$. The cone object which is initial with respect to this property is called the colimit of the functor X.

Definition 4. *Let $\mathcal{C}$ be a category, $\boldsymbol{D}$ be a small category and $X : \boldsymbol{D} \longrightarrow \mathcal{C}$ a diagram* *of shape* $\boldsymbol{D}$ *in* $\mathcal{C}$*. The colimit of the diagram* X *in* $\mathcal{C}$ *is a cone object* $colim_i X_i$ *such that for any other cone object* A *over* X *there is a unique morphism* $colim_i X_i \to A$ *between constant diagrams commuting with the structure maps of the colimit and* A

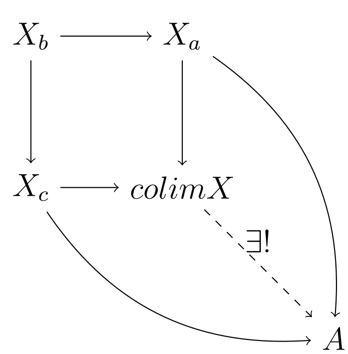
In other words, the colimit $colim_i X_i$ is an object in $\mathcal{C}$ such that for all objects $A \in \mathcal{C}$ there is a natural bijection

$$hom_{\mathcal{C}}(colim_i X_i, A) \cong hom_{\mathcal{C}^{\mathbf{D}}}(X, A)$$

We can see colimit as a functor $\mathcal{C}^{\mathbf{D}} \longrightarrow \mathcal{C}$, and instead of the above we can define colimit as just being the left adjoint to the constant diagram functor

$$colim : \mathcal{C}^{\mathbf{D}} \leftrightarrows \mathcal{C} : \Delta$$

Pushouts are a type of colimit we often encounter. If $\mathbf{D}$ is the three object category $\{a \leftarrow b \rightarrow c\}$ then the colimit of a diagram of this shape $X_a \leftarrow X_b \rightarrow X_c$ is called pushout. We can illustrate the universal property with the diagram:



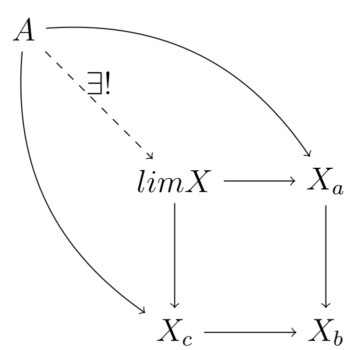
The top commutative square is referred to as a *pushout square*. In light of this the coproduct (or direct sum) $A \coprod B$ can be seen as a pushout over the diagram $A \leftarrow \emptyset \rightarrow B$, where $\emptyset$ is the initial object. Other examples of colimits are direct limits and arbitrary coproducts.

Dually we may define the limit functor.

Definition 5. *Let $\mathcal{C}$ be a category, $\boldsymbol{D}$ be a small category. The limit functor is defined to be the right adjoint of the constant diagram functor*

$$\Delta : \mathcal{C} \rightleftarrows \mathcal{C}^{\boldsymbol{D}} : lim$$

If we let $\mathbf{D}$ be the category $\{a \to b \leftarrow c\}$ the limit gives the dual notion of the pushout, called the pullback. For a diagram $X_a \to X_b \leftarrow X_c$ of shape $\mathbf{D}$ in $\mathcal{C}$ we illustrate the limit with the diagram:



The bottom commutative square is called a pullback square. The product of two objects A and B, denoted $A \times B$, can be seen as a pullback of the diagram $A \to \star \leftarrow B$, where $\star$ is the terminal object. Arbitrary products and inverse limits are examples of limits as well.

Of course, limits and colimits might not always exist. They are objects defined in an abstract context with a universal property.

Definition 6. *A category $\mathcal{C}$ is said to be complete (resp. cocomplete) if colimits (resp. limits) exist over all diagrams $\boldsymbol{D} \longrightarrow \mathcal{C}$ for all small categories $\boldsymbol{D}$.*

Many categories of interest are complete and cocomplete. For example **Top**, **Set**, Mod_R and Ch_R are complete and cocomplete. Another important example of a category admitting all small limits and colimits is the category **Cat**, whose objects are locally small categories and arrows are functors between these categories.

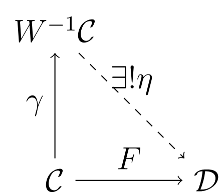
2.2 Localization of categories

Let $\mathcal{C}$ be a category and $W \subset \mathcal{C}$ be a class of morphisms called *weak equivalences.* Assume we are in a situation where we want to consider weak equivalences as isomorphisms. What we do is construct a new category $W^{-1}\mathcal{C}$ with the same objects as $\mathcal{C}$ such that maps in W are isomorphisms in $W^{-1}\mathcal{C}$.

Definition 7. *The locatization of $\mathcal{C}$ with respect to the class of weak equivalences W is a category $W^{-1}\mathcal{C}$ equipped with a functor*

$$\gamma : \mathcal{C} \longrightarrow W^{-1}\mathcal{C}$$

called the localization functor, satisfying the property "$\gamma(f)$ is an isomorphism for all $f \in W$", and universal with respect to this property, i.e. if $\mathcal{D}$ is another category equipped with a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ such that $F(f)$ is an isomorphism for all $f \in W$, then there is a unique functor $\eta : W^{-1} \longrightarrow \mathcal{D}$ such that $\eta\gamma = F$



Just like limits and colimits localization of a category is defined in an abstract context satisfying some universal property. But unlike limits and colimits the existence of localization is always guaranteed because limits and colimits exist in the category of locally small categories **Cat**.

Lemma 2.2.1. *In the sense of the above definition $W^{-1}\mathcal{C}$ exists.*

Proof. Let $I = \{u : a \to b\}$ be a category with two objects and a single map between them and $\tilde{I} = \{\tilde{u} : a \to b\}$ be a category with two objects and a single isomorphism between them. There is a functor $I \to \tilde{I}$ sending $u \rightarrowtail \tilde{u}$. Let the category $\coprod_{s\in W} I$ be the coproduct of copies of I indexed over maps $s \in W$.

Then observe that by definition $W^{-1}\mathcal{C}$ is given by the pushout diagram:

$$\begin{array}{ccc} \coprod_{s\in W} I & \to & \coprod_{s\in W} \tilde{I} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{\gamma} & W^{-1}\mathcal{C} \end{array}$$

where the functor $\coprod_{s\in W} I \longrightarrow \mathcal{C}$ is the one that sends the map $u_s : a \to b$ to the map $s \in \mathcal{C}$. □

The above proof tells us that the objects in $W^{-1}\mathcal{C}$ are the same as the objects in $\mathcal{C}$ and that the localization functor γ is the identity map on the objects. Arrows between objects however are not easy to describe in $W^{-1}\mathcal{C}$. All we can say in general is that they are zig-zags of arrows in $\mathcal{C}$ subject to some equivalence relation [2].

The most trivial, and not interesting at the same time, case of localization is when W coincides with the class of all isomorphisms. In that case the localization $W^{-1}\mathcal{C}$ is just $\mathcal{C}$ and γ the identity functor $id_{\mathcal{C}}$ in **Cat**.

A more interesting case is when W is the class of weak equivalences in a closed model category $\mathcal{C}$. Its localization is non-trivial and the maps in the localization have nice description. This will be our case of interest.

Chapter 3

CLOSED MODEL CATEGORIES

3.1 Definitions and terminology

The notion of a closed model category was developed by Quillen [8].

Definition 8. *A closed model category is a category $\mathcal{C}$ containing three typer of morphisms called fibrations, cofibrations and weak equivalences subject to the following axioms:*

MC1 *$\mathcal{C}$ has all finite limits and colimits*

MC2 *For all commutative triangles*

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle f} & \\ & Z & \end{array}$$

if any two of f, g, h are weak equivalences then so is the third

MC3 *If f is a retract of g, then f is a fibration/ cofibration weak equivalence whenever g is.*

MC4 *For solid arrow commutative squares*

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ V & \longrightarrow & Y \end{array}$$

where p is a fibration and i is a cofibration, the dotted arrow exists in case p or i is a weak equivalence.

MC5 *Any morphism $f : X \longrightarrow Y$ can be factored as:*

1. *$f = pi$ where p is a fibration and i is a trivial fibration.*
2. *$f = qj$ where p is a trivial fibration and j is a cofibration.*

The above axioms will allow us to develop a notion of homotopy and a homotopy category associated to $\mathcal{C}$. As we will see, weak equivalences will play a central role, because we will consider weakly equivalent spaces as being of the same "type".

Definition 9. *We say a fibration (resp. cofibration) is acyclic if it is a weak equivalence as well.*

Observe that MC1 guarantees the existence of an initial and terminal objects, which wee will denote $\emptyset$ and $*$ respectively.

Definition 10. *We call an object A cofibrant in case the map $\emptyset \to A$ is a cofibration, and an object X fibrant in case $X \to *$ is a fibration.*

Let $f : X \to Y$ and $g : A \to B$ be two maps in $\mathcal{C}$. We say that f has *the right lifting property* with respect to g (resp. g has the *left lifting property* with respect to f) if for all solid arrow commutative squares

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle g}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

the dotted arrow exists. Shortly we will write that f has RLP with respect to g and g has LLP with respect to f. So in this terminology **MC4** says that acyclic fibrations have RLP with respect to all cofibrations and acyclic cofibrations have LLP with respect to all fibrations.

Before defining homotopy in closed model categories, let us first establish model category structures on topological spaces and chain complexes. This way we will see how notions we develop and terminology makes sense.

3.2 Our main examples

3.2.1 Topological spaces

It is not straightforward to show that some category has a model structure. To establish a model category structure on **Top** first we have to establish what the fibrations, cofibrations and weak equivalences will be. First we fix the class of weak equivalences. As mentioned in the introduction, a map $f : X \to Y$ in **Top** is a weak equivalence if the induced maps $f_n : \pi_n(X, x) \to \pi_n(Y, f(x))$ are isomorphisms for $n > 0$ and $f_0 : \pi_0(X, x) \to \pi_0(Y, f(x))$ is a bijection.

Definition 11. *A map of spaces $p : X \to Y$ is said to be a Serre fibration if it has RLP with respect to all inclusions $A \times 0 \to A \times I$, for all CW-complexes A. This is also called the homotopy lifting property.*

In particular, a Serre fibration has RLP with respect to all inclusions $D^n \times 0 \to D^n \times I$. But each CW-complex is a colimit of its building blocks, so RLP will be inherited from the inclusions $D^n \times 0 \to D^n \times I$ to the inclusions $A \times 0 \to A \times I$.

Lemma 3.2.1. *A map of spaces $p : X \to Y$ is a Serre fibration if and only if it has RLP with respect to all inclusions $D^n \times 0 \to D^n \times I$.*

Proof. Assume p has RLP with respect to all inclusions $D^n \times 0 \to D^n \times I$. Let A be a CW-complex and let A_n be the n-skeleton of A. $A = colim_n A_n$, is a filtered colimit of all its skeletons. And since filtered colimits commute with finite limits,the inclusion $A \times 0 \to A \times I$ is the colimit of all the inclusions $A_n \times 0 \to A_n \times I$. So it is enough to show that p has RLP with respect to inclusions $A_n \times 0 \to A_n \times I$.

Moreover, we know that A_n is obtained by attaching copies of D^n to A_{n-1} along the boundaries. What we mean is that for cells D^n we attach, that are indexed by some set J_n, there is a pushout square

$$\begin{array}{ccc} \coprod_{J_n} \partial D^n & \longrightarrow & A_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{J_n} D^n & \longrightarrow & A_n \end{array}$$

Therefore to show that p has RLP with respect to $A_n \times 0 \to A_n \times I$ it is enough to show p has RLP with respect to $A_{n-1} \times 0 \to A_{n-1} \times I$ and $\coprod_{J_n} D^n \times 0 \to \coprod_{J_n} D^n \times I$. Since the latter is given by assumption, by induction we reduce the proof only for $A_0 \times 0 \to A_0 \times I$, which is easy to see. $\square$

We fix the class of fibrations in **Top** to be the class of Serre fibrations and the class of cofibrations to be the class of maps that have LLP with respect to all acyclic fibrations.

As we did with fibrations, we characterize acyclic fibrations with lifting properties.

Lemma 3.2.2. *A map of spaces $p : X \to Y$ is a Serre fibration and a weak homotopy equivalence if and only if one of the following equivalent conditions holds:*

i) p has RLP with respect to all inclusions of CW-complexes $A \to B$

ii) p has RLP with respect to all boundary inclusions $S^{n-1} \to D^n$

Theorem 3.2.3. *With the above classes of weak equivalences, fibrations and cofibrations* ***Top*** *is a closed model category.*

Proof. (of **MC1 - MC3**)

MC1 and **MC2** are not a problem. Regarding **MC3**, assume $f : X \to Y$ is a retract of $f' : X' \to Y'$, meaning that we have a commutative diagram

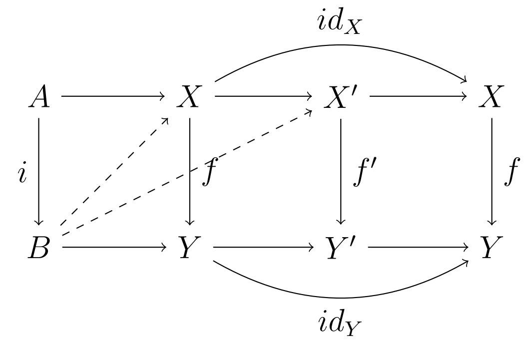
$$\begin{array}{ccccc} & & \overset{id_X}{\frown} & & \\ X & \longrightarrow & X' & \longrightarrow & X \\ \downarrow f & & \downarrow f' & & \downarrow f \\ Y & \longrightarrow & Y' & \longrightarrow & Y \\ & & \underset{id_Y}{\smile} & & \end{array}$$

First, it is clear that if f' is a weak equivalence then so is f.This is so because the retract diagram will be preserved to the level of homotopy groups and isomorphisms are closed under retract.

It is also easy to see that left and right lifting properties with respect to a class of maps are preserved under retract. For example assume f' has RLP with respect to a map $i : A \to B$ and assume we have the commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & & \downarrow f \\ B & \longrightarrow & Y \end{array}$$

Then we have a commutative diagram



The right dotted arrow exists because f' has RLP with respect to i. Composing with the map $X' \to X$ we get the other dotted arrow commuting with everything. In particular, the left square commutes, proving that f has RLP with respect to i.

With a similar argument we can show that left lifting property is preserved under retract as well. But cofibrations are defined to be maps having LLP with respect to acyclic fibrations are defined as maps having RLP with respect to some inclusions. Therefore these classes are preserved by retracts. □

Part of **MC4** is immediate by definition of cofibrations. The profs of the rest of **MC4**, that fibrations have RLP with respect to acyclic cofibrations, and **MC5** depend upon an argument introduced by Quillen called the small object argument.

The small abject argument:

For the setting we take $\mathcal{C}$ to be a category containing all small limits and colimits. Assume we are given a set of morphisms $\mathcal{F} = \{f_i : A_i \to B_i\}_{i \in I}$ in $\mathcal{C}$ and a map $p : X \to Y$. We wish to decompose p as $X \to X' \to Y$ such that $X' \to Y$ has RLP with respect to all maps in $\mathcal{F}$ and X' is similar" enough to X (otherwise we may just choose $X = X'$).

Let $B : \mathbb{Z}^+ \to \mathcal{C}$ be a functor and $A \in \mathcal{C}$ be an object. The maps $B(n) \to colim_n B(n)$ induce maps $hom_{\mathcal{C}}(A, B(n)) \to hom_{\mathcal{C}}(A, colim_n B(n))$ compatible enough and therefore we have a map

$$colim_n hom_{\mathcal{C}}(A, B(n) \longrightarrow hom_{\mathcal{C}}(A, colim_n B(n))$$

Definition 12. *An object A is said to be sequentially small if for all functors $B : \mathbb{Z}^+ \to \mathcal{C}$ the above map is a bijection.*

Now consider all commutative squares D of the form:

$$\begin{array}{ccc} A_i & \xrightarrow{g_D} & X \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle p} \\ B_i & \xrightarrow{h_D} & Y \end{array}$$

with $f_i \in \mathcal{F}$. Define the object $G^1(\mathcal{F}, p)$ to be given by attaching B_i's to X with the pushout diagram

$$\begin{array}{ccc} \coprod_D A_i & \xrightarrow{\coprod_D g_D} & X \\ {\scriptstyle \coprod_D f_i}\downarrow & & \downarrow{\scriptstyle i_1} \\ \coprod_D B_i & \to & G^1(\mathcal{F}, p) \end{array}$$

We have a map $p_1 = (\coprod_D h_D, p) : G^1(\mathcal{F}, p) \to Y$. We repeat the same process as above for p_1 to get the object $G^2(\mathcal{F}, p) := G^1(\mathcal{F}, p_1)$ and maps $i_2 : G^1(\mathcal{F}, p) \to G^2(\mathcal{F}, p)$ and $p_2 : G^2(\mathcal{F}, p) \to Y$. We can go on inductively repeating the process and define the object $G^\infty(\mathcal{F}, p) := colim_n G^n(\mathcal{F}, p)$ together with the induced maps $i^\infty : X \to G^\infty(\mathcal{F}, p)$ and $p^\infty : G^\infty(\mathcal{F}, p) \to Y$ fitting in the commutative diagram:

$$\begin{array}{ccccccccc} X & \xrightarrow{i_1} & G^1(\mathcal{F}, p) & \xrightarrow{i_2} & G^2(\mathcal{F}, p) & \longrightarrow & \cdots\cdots & \longrightarrow & G^\infty(\mathcal{F}, p) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & & & \downarrow{\scriptstyle p^\infty} \\ Y & \xrightarrow{id_Y} & Y & \xrightarrow{id_Y} & Y & \longrightarrow & \cdots\cdots & \longrightarrow & Y \end{array}$$

Claim: In the above situation, if all A_i's are sequentially small then p^∞ has the RLP with respect to all maps in $\mathcal{F}$.

Proof. (of the claim)

Indeed, if we have a commutative square

$$\begin{array}{ccc} A_i & \xrightarrow{g} & G^\infty(\mathcal{F}, p) \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle p^\infty} \\ B_i & \xrightarrow{h} & Y \end{array}$$

since A_i is sequentially small, there is $n > 0$ and a map $g' : A_i \to G^n(\mathcal{F}, p)$ such that g factors as $A \to G^n(\mathcal{F}, p) \to G^\infty(\mathcal{F}, p)$. Finally we have a diagram

$$\begin{array}{ccccccccc} A_i & \xrightarrow{g'} & G^n(\mathcal{F}, p) & \xrightarrow{i_{n+1}} & G^{n+1}(\mathcal{F}, p) & \to & \cdots\cdots & \to & G^\infty(\mathcal{F}, p) \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle p_{n+1}} & & & & \downarrow{\scriptstyle p^\infty} \\ B_i & \xrightarrow{h} & Y & \xrightarrow{id_Y} & Y & \longrightarrow & \cdots\cdots & \longrightarrow & Y \end{array}$$

But (g', h), by definition of $G^{n+1}(\mathcal{F}, p)$, will contribute in attaching B_i to $G^n(\mathcal{F}, p)$, and therefore we get a map $B_i \to G^{n+1}(\mathcal{F}, p)$ lifting the diagram. Hence we find the desired map $B_i \to G^\infty(\mathcal{F}, p)$ by composition, ending the proof of the claim.

□

In our case we consider the family of inclusions $\mathcal{F} = \{j_n : D^n \times 0 \to D^n \times I\}$. And by the above construction, the map $i_1 : X \to G^1(\mathcal{F}, p)$ is a relative CW inclusion because $G^1(\mathcal{F}$ is obtained by attaching cylinders to X along the bottom So this map will have RLP with respect to all Serre fibrations and it will be a weak equivalence and hence i^∞ inherits the same properties as a colimit map. Moreover, since the objects D^n are sequentially small, p^∞ is a Serre fibration. Therefore we have proven the following lemma:

Lemma 3.2.4. *Each map f in **Top** can be factored as a $f = pi$, where p is an acyclic fibration and i has LLP with respect to all Serre fibrations.*

Using this lemma we can finish the proving that **Top** is a model category.

Proof. (of **MC4** and **MC 5**) Regarding **MC4** we have to show that acyclic cofibration have LLP with respect to Serre fibrations. So let $j : A \to B$ be an acyclic cofibration. We can write $j = pi$ as in Lemma 2.4. We have the commutative square

$$\begin{array}{ccc} A & \xrightarrow{i} & A' \\ {\scriptstyle j}\downarrow & \overset{\exists g}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow{id_B} & B \end{array}$$

and the map g exists because p is an acyclic fibration and j a cofibration. By **MC2**, i is acyclic since j and p are. Moreover j is a retract of i according to the diagram

$$\begin{array}{ccccc} A & \xrightarrow{id_A} & A & \xrightarrow{id_A} & A \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ B & \xrightarrow{g} & A' & \xrightarrow{p} & B \end{array}$$

And since left lifting properties are preserved by retracts, j has LLP with respect to Serre fibrations because i does.

MC5 i) is an immediate consequence of Lemma 2.4. To prove **MC5** ii) we can consider the set of inclusions $\mathcal{F} = \{i_n : S^{n-1} \to D^n\}$ and use the small object argument similarly. □

3.2.2 Chain complexes

Let Ch_R be the category of bounded below chain complexes R-modules for some ring R. We will put a closed model structure on this category as follows:

Definition 13. *Let $f : M \to N$ be a morphism in Ch_R. We say that:*

(i) f is a weak equivalence if f is a quasi-isomorphism

(ii) f is a fibration if f is an epimorphism

(iii) f is a cofibration if f is a monomorphism with cokernel projective in each degree

Recall that quasi-isomorphisms are maps that induce isomorphisms on homology groups, epimorphisms (resp. monomorphisms) in Ch_R are maps that are surjective (resp. injective) in each degree. In the sense of the above definition, we can say that cofibrations are inclusions of the form $A \hookrightarrow A \oplus P$ where P is a complex which is projective in each degree, because an exact sequence of R-modules with projective cokernel always splits.

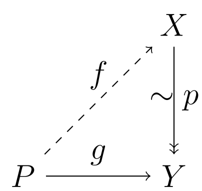
Theorem 3.2.5. *With the above notions of weak equivalence, fibration and cofibration, Ch_R is a closed model category.*

Proof. (of **MC1-MC4**) MC1 holds because Mod_R, the category of R-modules, contains all small limits and colimits, and therefore Ch_R does. MC2 is clear as well. MC3 holds in virtue of the fact that inclusions, surjections, and isomorphisms are preserved by retracts in Mod_R, and therefore are preserved under retract in Ch_R.

MC4 (i): Assume we have a commutative square:

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ {\scriptstyle i}\big\downarrow & \overset{f}{\nearrow} & \big\downarrow{\scriptstyle \sim\, p} \\ A \oplus P & \xrightarrow{g} & Y \end{array}$$

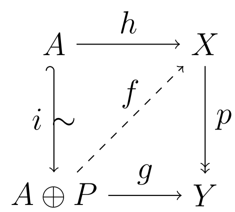
where P is a chain complex which is projective in every degree, and p is a surjective quasi-isomorphism i.e. $ker(p)$ is acyclic (by the homology long exact sequence). We wish to show the dashed arrow exists making the whole diagram commute. We need to find the structure maps of f and we already know that the structure map corresponding to A is just g. Therefore without loss of generality, we may assume that $A = 0$, and just show that the lifting problem:



has a solution.

We can construct f component wise by induction. First, we have a lift $f_0 : X_0 \to Y_0$ such that $p_0 f_0 = g_0$ since P_0 is projective and p_0 surjective. Then we also have a map $\tilde{f}_1 : P_1 \to X_1$ such that $p_1 \tilde{f}_1 = g_1$. Consider the error map $\epsilon = \partial \tilde{f}_1 - f_0 \partial : P_1 \to X_0$. We have $p_0 \epsilon = 0$ and therefore ϵ lifts to a map $\epsilon' : P_1 \to K_0$, where $K = kernel(p)$. Since K is acyclic, $K_1 \to K_0$ is surjective and therefore we can lift $\epsilon^{p}rime$ to a map $\epsilon'' : P_1 \to K_1$ and we let $f_1 = \tilde{f}_1 - \epsilon''$ (we think of ϵ'' as a map $P_1 \to X_1$). Now it is easy to check that $\partial f_1 = f_0 \partial$. In a similar manner we can construct all components of f inductively.

MC4 (ii): Suppose we have the commutative square



where P is a chain complex projective at every degree and p is surjective and i is a quasi-isomorphism i.e. P is acyclic. Again without loss of generality we will assume $A = 0$ and we get the lifting problem:

$$\begin{array}{ccc} & & X \\ & \overset{f}{\nearrow} & \downarrow p \\ P & \xrightarrow{g} & Y \end{array}$$

with P acyclic and projective at each degree and p surjective. Again, we construct f component wise inductively. First, there is some $f_0 : P_0 \to Y_0$ such that $p_0 f_0 = g_0$. Since P is acyclic, $P_1 \to P_0$ is surjective and therefore there is a splitting map $s : P_0 \to P_1$ such that $\partial s = id_{P_0}$. Now lift the map $g_1 s$ with respect to the surjection $p_1 : X_1 \to Y_1$ to get a map $P_0 \to X_1$ commuting with everything, and let $f_1 : P_1 \to P_0 \to X_1$ be given by composition with ∂. It is easy to verify that $\partial f_1 = f_0 \partial$. Similarly, we construct the other components of f inductively. □

To prove **MC5** we will use the small object argument one more time. In order to implement the small object argument we need a couple or results.

Important observation: Let M be an R-module. Choosing an element $m \in M$ is exactly the same thing as giving a map $R \to M$ that sends 1 to m. In other words, R represents the forgetful functor $Mod_R \to Set$.

In light of our observation, we can talk about surjectivity simply in terms of lifting properties. A map of R-modules $f : M \to N$ will be surjective if and only if it has RLP with respect to the map $0 \to R$. But what about chain complexes?

Definition 14. *i) The n-disc in Ch_R, denoted by $D_n(R)$ or simply D^n, is defined with $D^n_n = D^n_{n-1} = R$ with the identity map as differential and* 0 *elsewhere.*

ii) The n-sphere in Ch_R, denoted by $S^n(R)$ or simply S^n, is defined with $S^n_n = R$ and 0 *elsewhere.*

Observe that in the case $R = \mathbb{Z}$ and $Mod_R = Ab$, the singular chain complex associated to the topological n-disc is $D^n(\mathbb{Z})$ (D^n has one n-cell and its boundary as an $(n-1)$-cell) and the one associated to the topological n-sphere is exactly $S^n(\mathbb{Z})$ (S^n is composed of one n-cell).

Let M be a chain complex of R-modules. Observe that choosing an element $m \in M_n$ is the same thing as giving a map $D^n(R) \to M$. Therefore we have the following lemma:

Lemma 3.2.6. *A map in Ch_R is a fibration if and only if it has RLP with respect to inclusions $0 \to D^n$.*

So we characterize fibrations in terms of lifting properties. Moreover, a simple diagram chase reveals a characterization of acyclic fibrations:

Lemma 3.2.7. *A map in Ch_R is an acyclic fibration if and only if it has RLP with respect to all inclusions $S^{n-1} \to D^n$.*

We also need the following lemma.

Lemma 3.2.8. *An R-module is sequentially small in case it is finitely presented.*

Proof. Recall that an R-module M is finitely presented if there is an exact sequence

$$R^m \to R^n \to M \to 0$$

So assume this is the case. We want to show that for all functors $B : \mathbb{Z}^+ \to Mod_R$ we have a bijection

$$colim_n hom_{Mod_R}(M, B_n) \cong hom_{Mod_R}(M, colim_n B)$$

Colimits are right exact, so without loss of generality we may assume M is finitely generated and free (R^n and R^m being sequentially small implies that M is sequentially small by an exact sequence argument). But since colimits commute with each other we may assume that $M = R$ without loss of generality. And since R represents the forgetful functor $Mod_R \to Set$ and this functor preserves colimits, the bijection is obvious. □

Using these lemmas it is not difficult to prove **MC5** using the small object argument. We will not present a proof here because what is important for us is the fact that CH_R has the proposed model category structure. In next sections we will see how this structure enables us to do some homotopy theory.

3.3 New model categories from old

Let $\mathcal{C}$ be a closed model category. There are other categories that admit a closed model structure induced by the one in $\mathcal{C}$.

The dual category $\mathcal{C}^{op}$ has a natural model structure. Let an arrow in $\mathcal{C}^{op}$ be a weak equivalence if the dual arrow in $\mathcal{C}$ is a weak equivalence, a fibration if its dual is a cofibration and a cofibration if its dual is a fibration. A mere observation and **MC1** - **MC5** are satisfied by these three classes of morphisms in $\mathcal{C}^{op}$. We call this the dual model category on $\mathcal{C}$.

Let $A \in \mathcal{C}$ be an object. The comma category $\mathcal{C} \downarrow A$ admits a natural model structure. For two object $f : B \to A$ and $g : C \to A$, say a morphism

$$\begin{array}{ccc} B & \xrightarrow{h} & C \\ {\scriptstyle f}\searrow & & \swarrow{\scriptstyle g} \\ & A & \end{array}$$

is a weak equivalence (resp. fibration of cofibration) in case h is a weak equivalence (resp. fibration or cofibration). Clearly these classes satisfy **MC1 - MC5**. Nothing changes much except that in the comma category the terminal element becomes the map $id_A : A \to A$, and hence an object $f : B \to A$ is fibrant if and only if f is a fibration in $\mathcal{C}$. Similarly we may impose a closed model structure on $A \uparrow \mathcal{C}$. As a matter of fact, this category has the dual model structure with respect to $\mathcal{C} \downarrow A$.

Let $\mathbf{D}$ be a small category. Does the functor category $\mathcal{C}^{\mathbf{D}}$ admit a natural model structure as well? By natural we mean that for two objects $X, Y : \mathbf{D} \to \mathcal{C}$ a map $f : X \to Y$ is a weak equivalence (resp. fibration or cofibration) if the maps $f_i : X_i \to Y_i$ are weak equivalences (resp. fibrations or cofibrations) for all $i \in \mathbf{D}$. Again, verification of **MC1- MC4** is a mere observation. But **MC5** is not guaranteed to be satisfied. If we have a natural transformation $f : X \to Y$, since $\mathcal{C}$ is a model category we can factor each map $f_i : X_i \to Y_i$ as $f_i = p_i j_i$ where p_i is a fibration and j_i is a cofibration and one of them is acyclic. However This data is not enough to factor f because **MC5** just postulates the existence of factorizations but it does not postulate naturality of those factorizations. Functorial factorizations in $\mathcal{C}$ would guarantee **MC5** for $\mathcal{C}^{\mathbf{D}}$.

When are the factorizations provided by **MC5** natural? It turns out we are OK when $\mathcal{C}$ is *cofibrantly generated*. Let us define the notion precisely.

Definition 15. *Let $\mathcal{C}$ be a category and $\mathcal{F} \subset \mathcal{C}$ be a class of morphisms. An object $A \in \mathcal{C}$ is said to be small for $\mathcal{F}$ if for all sequences of arrows in $\mathcal{F}$*

$$X_1 \longrightarrow X_2 \longrightarrow X_3 ... \longrightarrow X_n \longrightarrow ...$$

the map

$$colim_n hom_{\mathcal{C}}(A, X_n) \longrightarrow hom_{\mathcal{C}}(A, colim_n X)$$

is a bijection

Note that an object is sequentially small as in Definition 5 if it is small for $\mathcal{C}$.

Definition 16. *A closed model category $\mathcal{C}$ is said to be cofibrantly generated in case there are two classes of morphisms I and J in $\mathcal{C}$ such that:*
(i) Sources of the maps in I are small for the class of cofibrations and a map in $\mathcal{C}$ is an acyclic fibration if and only if it has RLP with respect to all maps in I
(ii) Sources of the maps in J are small for the class of acyclic cofibrations and a map in $\mathcal{C}$ is a fibration if and only if it has RLP with respect to all maps in J

In case a category is cofibrantly generated, the class I "generates" cofibrations and the class J "generates" acyclic cofibrations, in the sense that the saturated class generated by I coincides with cofibrations and the saturated class generated by J coincides with acyclic cofibrations.

As mentioned previously cofibrations are stable under retract or coproduct. We can easily see that cofibrations are preserved by pushout as well. Can we produce all cofibrations like this? The answer is yes for cofibrantly generated categories. Starting with some cofibrations we can saturate" the class of cofibrations. Saturation is made precise as follows.

Definition 17. *Let $\mathcal{C}$ be a category and $M \subset \mathcal{C}$ be a class of monomorphisms. We say that M is saturated if the following axioms are satisfied:*
A: *All isomorphisms are in M.*
B: *M is closed under pushout, i.e if we have a pushout square*

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_\star} \\ B & \longrightarrow & B\coprod_A C \end{array}$$

then $i_\star \in M$ *in case* $i \in M$.

C: *Each retract of an element of* M *is in* M.

D: M *is closed under countable composition and coproducts.*

It is not difficult to see that the class of cofibrations satisfies these axioms. Also, the intersection of two saturated classes will still be saturated. Therefore we can talk about a saturated class generated by a bunch of monomorphisms, which we can define to be the smallest class containing those monomorphisms. In the case of **Top**, the class of cofibrations coincides with the saturated class generated by inclusions $D^n \times 0 \to D^n \times I$. For a detailed proof see [6].

The details of why cofibrantly generated model categories admit functorial factorizations can be found in [6] and [4]. The proofs for the model structures on **Top** and Ch_R indicate that both these model categories are cofibrantly generated. Therefore in such model categories one can talk about the model structure on small diagram categories.

Chapter 4

HOMOTOPY IN CLOSED MODEL CATEGORIES

4.1 The meaning of "closed"

Theorem 4.1.1. *In a closed model category $\mathcal{C}$*

a) A map is a fibration (resp. acyclic fibration) if and only if it has the right lifting property with respect to all acyclic cofibrations (resp. cofibrations).

b) A map is a cofibration (acyclic cofibration) if it has the left lifting property with respect to all acyclic fibrations (fibrations).

Proof. Suppose a map $f : X \to Y$ has RLP with respect to all cofibrations. By **MC5** we can write $f = pi$ where i is a cofibration and p is a trivial fibration. For the commutative square

$$\begin{array}{ccc} X & \xrightarrow{id_X} & X \\ {\scriptstyle i}\downarrow & \overset{\exists r}{\nearrow} & \downarrow{\scriptstyle f} \\ V & \xrightarrow{p} & Y \end{array}$$

the dotted arrow exists by **MC4** and therefore f is a retract of p

$$\begin{array}{ccccc} X & \xrightarrow{i} & V & \xrightarrow{r} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{id_Y} & Y & \xrightarrow{id_Y} & Y \end{array}$$

By **MC3** we get that f is an acyclic fibration.

Exactly the same argument works when f has RLP with respect to all acyclic cofibrations. In that case we write $f = qj$ where q is a fibration and j an acyclic cofibration using **MC5**, then using **MC4** we get that f is a retract of q, and by **MC3** we get that f is a fibration. For the rest of the lemma a similar argument works.

□

Our definitions and proofs for the model structures of **Top** and chain complexes already indicated that such a fact might be true. In light of this, in **Top** the inclusions $D^n \times 0 \to D^n \times I$ are cofibrations.

4.2 Cylinder objects and left homotopy

Let $\mathcal{C}$ be a fixed closed model category.

Definition 18. *A cylinder object for $A \in Ob(\mathcal{C})$ is a commutative diagram*

$$\begin{array}{ccc} & & A \wedge I \\ & \overset{i}{\nearrow} & \downarrow{\scriptstyle \sim\, q} \\ A \coprod A & \xrightarrow{id_A + id_A} & A \end{array}$$

where q is a weak equivalence.

A cylinder object is called *good* in case i is a cofibration and *very good* in case q is a fibration. Write i_0 and i_1 the structure maps of i. By **MC5** there is at least one very good cylinder object for each object of $\mathcal{C}$.

Definition 19. *Two maps $f, g \in hom_{\mathcal{C}}(A, X)$ are said to be left homotopic is there is a commutative diagram*

$$\begin{array}{ccc} A\coprod A & & \\ \downarrow i & \searrow f+g & \\ A\wedge I & \xrightarrow{H} & X \end{array}$$

H is called a good/ very good left homotopy if $A \wedge I$ is a good/ very good cylinder object. We denote the relation $f \sim_l g$. We wish to show that when A is cofibrant left homotopy is an equivalence relation. We do this with the help of some lemmas.

Lemma 4.2.1. *i_0 and i_1 are acyclic cofibrations in case A is cofibrant.*

Proof. Recall that A cofibrant means that the map from the initial object $\emptyset \to A$ is a cofibration. $A \coprod A$ is given by the pushout

$$\begin{array}{ccc} \emptyset & \longrightarrow & A \\ \downarrow & & \downarrow in_0 \\ A & \xrightarrow{in_1} & A\coprod A \end{array}$$

and since cofibrations are preserved bu pushouts in_0 and in_1 are cofibrations. We have $i_0 = iin_0$ and $i_1 = iin_1$ and therefore both maps are cofibrations as composition of cofibrations. □

Lemma 4.2.2. *If we have a left homotopy $f \sim_l g$ then:*

(i) we can find a good left homotopy f to g

(ii) we can find a very good left homotopy f to g in case X is fibrant.

Proof. (i) Assume H is a left homotopy f to g in $hom_{\mathcal{C}}(A, X)$ with cylinder object $A \wedge I$. Using MC5 we decompose $i = pj$ where $j : A \wedge I' \longrightarrow A \wedge I$ is a cofibration. Immediately we can verify that $A \wedge I'$ is a cylinder object, which is good by construction, and that Hp gives a good homotopy as desired.

(ii) Choose a good homotopy $H : f \sim_l g$ with cylinder object $A \wedge I$ and choose a very good cylinder object $A \wedge I'$. Since X is fibrant (it has the lifting property with respect to all acyclic cofibrations) we can solve the following lifting problem:

$$\begin{array}{ccc} A \wedge I' & \overset{\exists H'}{\dashrightarrow} & X \\ \Big\uparrow \sim & & \Big\uparrow H \\ A \coprod A & \xrightarrow{\sim} & A \wedge I \end{array}$$

and H' gives the desired very good left homotopy. $\square$

Proposition 4.2.3. *If A is cofibrant, the relation $\sim_l$ is an equivalence relation on $hom_{\mathcal{C}}(A, X)$.*

Proof. It is easy to see reflexivity and symmetry To show transitivity, assume $f, g, h \in hom_{\mathcal{C}}(A, X)$ are three maps and we have left homotopies $H : f \sim_l g$ and $H' : g \sim_l h$ with corresponding cylinder objects $A \wedge I$ and $A \wedge I'$ (we may assume both cylinder objects are good without loss of generality). We have diagrams

$$\begin{array}{ccc} A & & \\ i_0 \downarrow & \searrow f & \\ A \wedge I & \xrightarrow{H} & X \\ i_1 \uparrow & \nearrow g & \\ A & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & & \\ i'_0 \downarrow & \searrow g & \\ A \wedge I' & \xrightarrow{H'} & X \\ i'_1 \uparrow & \nearrow h & \\ A & & \end{array}$$

Let $A \wedge I''$ be the pushout of the maps i_1 and i'_0 (it will be a cylinder object for A) and $H'' : A \wedge I'' \to X$ the map induced by H and H'

$$\begin{array}{ccccc} & & A \wedge I & & \\ & i_1 \nearrow & & \sim \searrow & \overset{H}{\searrow} \\ A & & & & A \wedge I'' \overset{H''}{\dashrightarrow} X \\ & i'_0 \searrow & & \sim \nearrow & \overset{H'}{\nearrow} \\ & & A \wedge I' & & \end{array}$$

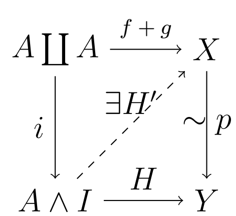
and H'' gives the desired homotopy $f \sim_l h$ proving the transitivity of left homotopy. □

We will write $\pi^l(A, X)$ for the set of equivalence classes with respect to left homotopy on $hom_{\mathcal{C}}(A, X)$ in case A is cofibrant, and for the set of classes of the equivalence relation generated by left homotopy in case A is not cofibrant.

Recall that if $p : X \to Y$ is a morphism, we have an induced map $p^* : hom_{\mathcal{C}}(A, X) \to hom_{\mathcal{C}}(A, Y)$ given by composition. But composition preserves homotopy, therefore we have an induced map on the level of homotopy classes $p^* : \pi^l(A, X) \to \pi^l(A, Y)$.

Lemma 4.2.4. *Let A be cofibrant. Then p^* is a bijection in case p is an acyclic fibration.*

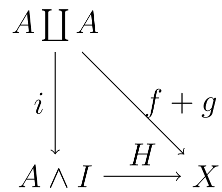
Proof. Let $[f], [g] \in \pi^l(A, X)$ be two classes and assume $p^*[f] = p^*[g]$. Choose a good homotopy $H : pf \sim_l pg$. Then we can lift the homotopy in $\pi^l(A, X)$ since p is an acyclic fibration according to the diagram:



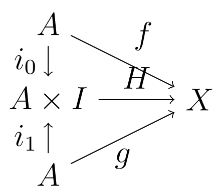
Therefore p^* is injective.

Since p is an acyclic fibration and A cofibrant, if $h : A \to Y$ is a map, there is a lift $\tilde{h} : A \to X$, and hence p^* is surjective. □

It is easy to guess what a cylinder object for a space $X \in$ **Top** is. As terminology suggests, $X \times I$, the cylinder with bases X, together with the projection map to X and the structure maps i_0 and i_1 being the top and bottom inclusions, is a very good cylinder object for X. Also, observe that the homotopy diagram



is equivalent to the classical homotopy diagram



Every topological space is fibrant, and every CW-complex is cofibrant.

Regarding chain complexes, the cofibrant objects are those complexes P such that P_k is acyclic for all $k > 0$ and $H_n(P) = 0$ for all $n > 0$. And all chain complexes are cofibrant objects.

Let M be a chain complex. Then define the complex $Cyl(M)$ with

$$Cyl(M)_n = M_n \oplus M_{n-1} \oplus M_n$$

and differential

$$\partial(a, b, c) = (\partial a + b, -\partial b, -b + \partial c)$$

Let the map $i : M \oplus M \to Cyl(M)$ be given by the structure maps $i_0(a) = (a, 0, 0)$ and $i_1(c) = (0, 0, c)$. And let the map $p : Cyl(M) \to M$ be given by $p(a, b, c) = a + c$. It is easy to verify that $Cyl(M)$ is a cylinder object for M with the maps i and p defined as above. Moreover, it is also easy to see that the left homotopy relation induced by $Cyl(M)$ is the same as the usual one on maps of chain complexes.

But it is not the case that this is always a good cylinder object. We can verify that

p is a quasi-isomorphisms all the time. However, by construction $coker(i) = M[-1]$ and hence i is a cofibration if and only if M is cofibrant.

As weird as the definition of the cylinder object looks, it is actually quite natural. For a space X we constructed the cylinder $X \times I$. For an n-simplex of X, there are three simplicies of $X \times I$: two n-simplicies on the top and bottom of the cylinder, and one $n-1$-simplex in $X \times (0,1)$. If we scrutinize this carefully we see that $C_\star(X \times I) = Cyl(C_\star(X))$. For more details see [9].

4.3 Path objects and right homotopy

Definition 20. *A path object for $X \in Ob(\mathcal{C})$ is a commutative triangle*

$$\begin{array}{ccc} & & X^I \\ & {}^{s}\nearrow^{\sim} & \downarrow p \\ X & \xrightarrow{(id_X, id_X)} & X \times X \end{array}$$

where s is a weak equivalence.

A path object is called *good* is p is a fibration and *very good* if in addition s is a cofibration. Write p_0 and p_1 for the structure maps of p.

Definition 21. *Two maps $f, g : A \longrightarrow X$ are said to be right homotopic if there is a map $H : A \longrightarrow X^I$ for some path object X^I, such that $p_0 H = f$ and $p_1 H = g$.*

$$\begin{array}{ccc} & {}^{f}\nearrow & X \\ & & \uparrow p_0 \\ A & \xrightarrow{H} & X^I \\ & & \downarrow p_1 \\ & {}_{g}\searrow & X \end{array}$$

H is called a good/ very good homotopy if X^I is good / very good. We denote the right homotopy relation $\sim_r$

Of course, right homotopy is dual to left homotopy. Instead of giving the above definition we can just say that a path object is just a cylinder object in $\mathcal{C}^{op}$ with the closed model structure induced by the one in $\mathcal{C}$. Therefore without doing any work, we can just state the results dual to the ones proved for cylinder objects:

Lemma 4.3.1. *If X is fibrant then the structure maps of p, p_0 and p_1, are both acyclic fibrations.*

Lemma 4.3.2. *If we have a right homotopy $f \sim_r g$ then:*

(i) we can find a good right homotopy f to g

(ii) we can find a very good left homotopy f to g in case A is cofibrant.

Proposition 4.3.3. *If X is fibrant, the relation $\sim_r$ is an equivalence relation on $hom_{\mathcal{C}}(A, X)$.*

We will write $\pi^r(A, X)$ for the set of equivalence classes with respect to left homotopy on $hom_{\mathcal{C}}(A, X)$ in case X is fibrant, and for the set of classes of the equivalence relation generated by left homotopy in case X is not fibrant. If $i : C \to A$ is a morphism, we have an induced map $i_* : \pi^l(A, X) \to \pi^l(C, X)$ given by composition.

Proposition 4.3.4. *Let X be fibrant. Then i_* is a bijection in case i is an acyclic cofibration.*

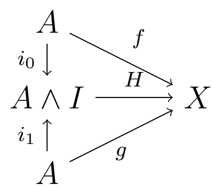
The relation between left and right homotopy is given by the following proposition, which is an "exponential law" very good path and cylinder objects satisfy:

Proposition 4.3.5. *Let $f, g : A \to X$ be two maps.*

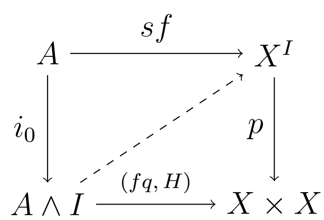
(i) In case A is cofibrant, $f \sim_l g$ implies $f \sim_r g$.

(ii) In case X is fibrant, $f \sim_r g$ implies $f \sim_l g$.

Proof. The two statements are dual so it is enough to prove one of them, say the first one. Choose a good cylinder object $A \wedge I$ for A and assume we have a homotopy $H : A \wedge I \longrightarrow X$ from f to g. Then choose a good path object $ps : X \to X^I \to X \times X$.



We can lift the diagram



to get a map $K : A^I \longrightarrow X^I$ and the map $Ki_1 : A \longrightarrow X^I$ gives the desired right homotopy. □

The following theorem describes a fundamental aspect of weak equivalence, that is being homotopy invertible. This is a version of Whitehead's theorem for closed model categories.

Theorem 4.3.6. *Let $A, X \in \mathcal{C}$ be two objects which are both fibrant and cofibrant. A map $f : A \to X$ is a weak equivalence if and only if f is homotopy invertible, i.e. if there is a map $g : X \to A$ such that $fg \sim id_X$ and $gf \sim id_A$.*

Proof. Assume first that f is a weak equivalence. We may factor f as $pi : A \to B \to X$ where p is an acyclic fibration and i an acyclic cofibration. The object B will be both fibrant and cofibrant: the map $\emptyset \to B$ is a composition of two cofibrations, i and $\emptyset \to A$, and hence it is a cofibration and the map $B \to \star$ is the composition of two fibrations, p and $X \to \star$, and hence it is a fibration. If i and p are homotopy invertible then so is f. So without loss of generality we can assume that f is a cofibration (the result for fibrations is dual).

In case f is an acyclic cofibration, we can lift the diagram

$$\begin{array}{ccc} A & \xrightarrow{id_A} & A \\ {\scriptstyle f}\big\downarrow{\scriptstyle \sim} & \nearrow{\scriptstyle \exists g} & \big\downarrow \\ X & \longrightarrow & \star \end{array}$$

to get a map $g : X \to A$ such that $gf = id_A$. But Proposition 2.17 tells us that we have a bijection $f_\star : \pi(X, X) \to \pi(A, X)$. And we know that $f_\star[id_X] = [f]$ and $f_\star[fg] = [fgf] = [f]$ and therefore $fg \sim id_X$, so that g is a two-sided homotopy inverse of f.

Conversely assume that f is homotopy invertible and let g be its homotopy inverse. Again we may factor $f = pi$ where p is a fibration and i an acyclic cofibration. So without loss of generality, we may assume that f is a fibration because by **MC2** if p is a weak equivalence then so is f.

In case f is a fibration, choose a good homotopy $H : X \wedge I \to X$ between fg and id_X. Then lift the homotopy according to the diagram using $MC4$:

$$\begin{array}{ccc} X & \xrightarrow{g} & A \\ {\scriptstyle i_0}\big\downarrow{\scriptstyle \sim} & \nearrow{\scriptstyle \exists H'} & \big\downarrow{\scriptstyle f} \\ X \wedge I & \xrightarrow{H} & X \end{array}$$

H' gives a homotopy from g to a map $s = H'i_1$. Moreover $fs = fH'i_1 = Hi_1 = id_X$, so that we get a retract diagram

$$\begin{array}{ccccc} A & \xrightarrow{id_A} & A & \xrightarrow{id_A} & A \\ \Big\downarrow f & & \Big\downarrow sf & & \Big\downarrow f \\ X & \xrightarrow{s} & A & \xrightarrow{f} & X \end{array}$$

But sf is a weak equivalence since it is homotopic to the weak equivalence id_A. Therefore f is a weak equivalence by **MC3**.

□

For a space X a path object, as we can imagine, is the space $C(I, X)$ of continuous maps $I \to X$ equipped with the compact open topology. The map s sends all points in X to the constant path at that point. The maps p_0 and p_1 send each path to its endpoints respectively.

For a chain complex M, we define a path object M^I as

$$M_n^I = M_n \oplus M_n \oplus M_{n+1}$$

with boundary map

$$\partial(a, b, c) = (\partial x, \partial y, (-1)^n(a - b) + \partial z)$$

Again, careful scrutiny will reveal the naturality of this definition. If we consider the singular chain complex of the path object for a space X, an n-simplex of $C(I, X)$ is the same as two n-simplicies (top and bottom) and another $(n + 1)$-simplex.

Chapter 5

THE HOMOTOPY CATEGORY

Again we fix a closed model category $\mathcal{C}$. We wish to consider objects up to weak equivalence and maps up to homotopy. We will construct the homotopy category $Ho(\mathcal{C})$ using the results we proved above. First lets define the following useful subcategories of $\mathcal{C}$:

$\mathcal{C}_f$: the full subcategory of fibrant objects

$\mathcal{C}_c$: the full subcategory of cofibrant objects

$\mathcal{C}_cf$: the full subcategory of objects which are both fibrant and cofibrant

$\pi\mathcal{C}_f$: the full subcategory of fibrant objects with morphisms right homotopy classes

$\pi\mathcal{C}_c$: the full subcategory of cofibrant objects with morphisms left homotopy classes

$\pi\mathcal{C}_cf$: the full subcategory of objects which are both fibrant and cofibrant with morphisms homotopy classes

5.1 *Cofibrant and fibrant replacement*

For each object $X \in Ob(\mathcal{C})$, we can decompose the maps $\emptyset \to X$ and $X \to *$ as

$$\begin{array}{ccc} & & QX \\ & \nearrow & \downarrow \sim p_X \\ \emptyset & \longrightarrow & X \end{array} \quad \text{and} \quad \begin{array}{ccc} RX & & \\ i_X \uparrow \sim & \searrow & \\ X & \longrightarrow & * \end{array}$$

using MC5(ii) and MC5(i) respectively. By construction QX is cofibrant and RX is fibrant. Basically, we are saying that $\mathcal{C}$ has "enough" fibrant and cofibrant objects in virtue of MC5.

Lemma 5.1.1. *If $f : X \to Y$ is a map, then there is a map $\bar{f} : QX \to QY$ making the following diagram commute:*

$$\begin{array}{ccc} QX & \overset{\bar{f}}{\dashrightarrow} & QY \\ {\scriptstyle p_X}\big\downarrow \wr & & {\scriptstyle p_Y}\big\downarrow \wr \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

Moreover, $\bar{f}$ is unique up to right or left homotopy and is a weak equivalence if and only if f is. If Y is fibrant then $\bar{f}$ depends only on the homotopy class of f.

Proof. The existence of $\bar{f}$ is guaranteed by the lifting property of the cofibrant object QX with respect to the acyclic fibration p_Y. The fact that the left homotopy class of $\bar{f}$ depends solely on f is a consequence of the fact that $p_Y^* : \pi_l(QX, QY) \to \pi_l(QX, Y)$ is a bijection, and that since QX is cofibrant left homotopy induces right homotopy. If Y is cofibrant, then we have a bijection $p_X^{*-1} p_Y^* : \pi_l(QX, QY) \to \pi_l(QX, Y) \to \pi_l(X, Y)$, which tells us that the homotopy class of $\bar{f}$ is determined uniquely by the homotopy class of f.

□

The lemma is just a recapitulation of the work done in previous sections but now we can speak of a well-defined functor

$$Q : \mathcal{C} \longrightarrow \pi\mathcal{C}_c$$

which is called the cofibrant replacement functor. The lemma also tells us that we have a well defined functor

$$\pi Q_f : \pi\mathcal{C}_f \longrightarrow \pi\mathcal{C}_{cf}$$

given by first restricting Q to fibrant objects, and then using the last part of the lemma we can restrict morphisms to homotopy classes of morphisms.

Dually, we can talk about functors:

$$R : \mathcal{C} \longrightarrow \pi\mathcal{C}_f$$

called the fibrant replacement functor and

$$\pi R_c : \pi\mathcal{C}_c \longrightarrow \pi\mathcal{C}_{cf}$$

In Top the cofibrant replacement is a manifestation of the well known fact that every space has the homotopy type of a CW-complex [5].

In Ch_R cofibrant replacement has much more meaning. Let M be an R-module. We can see M as a chain complex concentrated at degree 0. The cofibrant replacement QM has to be an acyclic complex projective in each degree and $p_M : QM \to M$ has to be a quasi-isomorphism. Therefore giving a cofibrant replacement for M is exactly the same thing as giving a projective resolution in the classical sense. Moreover the functoriality of the cofibrant replacement Q corresponds to the fact that lifts of maps of modules to the corresponding projective resolutions are unique up to chain map homotopy [9].

The model structure we established on Ch_R is called projective model structure for this reason. The dual model structure is called the injective model structure.

In the injective model structure, weak equivalences do not change, all inclusions are cofibrations and fibrations are surjective maps with injective kernel. With this dual structure, the fibrant replacement gives injective resolutions for modules. But no matter which structure we use we will eventually end up with the same homotopy category, which is the real object of interest.

Definition 22. *The homotopy category associated to a closed model category $\mathcal{C}$ is a category $Ho(\mathcal{C})$ whose objects are the same as the objects of $\mathcal{C}$ and the set of morphisms is defined to be*

$$hom_{Ho(\mathcal{C})}(X,Y) = \pi(RQX, RQY)$$

5.2 Inverting weak equivalences

We have seen that for a map $f : X \to Y$ in $\mathcal{C}$, we have a corresponding map $\bar{f} : QX \to QY$ whose homotopy class depends only on f. Moreover we have an associated map $\tilde{\bar{f}} : RQX \to RQY$ whose homotopy class depends only on the homotopy class of $\bar{f}$. We illustrate this with the diagram:

$$\begin{array}{ccc}
RQX & \overset{\exists \tilde{\bar{f}}}{\dashrightarrow} & RQY \\
{\scriptstyle i_{QX}} \uparrow \wr & & {\scriptstyle i_{QY}} \uparrow \wr \\
QX & \overset{\exists \bar{f}}{\dashrightarrow} & QY \\
{\scriptstyle p_X} \downarrow \wr & & {\scriptstyle p_Y} \downarrow \wr \\
X & \xrightarrow{f} & Y
\end{array}$$

Therefore we can talk about a well defined functor:

$$\gamma : \mathcal{C} \longrightarrow HO(\mathcal{C})$$

which is identity on the objects and sends maps f to classes $[\bar{\bar{f}}] \in \pi(RQX, RQY)$. We know that f is a weak equivalence if and only if $\bar{f}$ is if and only if $\bar{\bar{f}}$ is. And we know that a map in $\mathcal{C}_{]\{}$ is a weak equivalence if and only if it has a homotopy inverse. Therefore we conclude the following simple but important conclusion:

Lemma 5.2.1. *$\gamma(f)$ is an isomorphism if and only if f is a weak equivalence*

We wish to prove that γ is universal with respect to this property. We will do that with the help of a couple of lemmas.

Lemma 5.2.2. *Let $\mathcal{C}$ be a closed model category. Assume a functor $F : \mathcal{C} \to \mathcal{D}$ takes weak equivalences to isomorphisms in $\mathcal{D}$. If $f \sim_l g$ or $f \sim_r g$ in $hom_{\mathcal{C}}(A, X)$, then $F(f) = F(g)$.*

Proof. Choose a left homotopy $H : A \wedge I \to X$ between f and g, with a good cylinder object $A \coprod A \hookrightarrow A \wedge I \to A$, $q : A \wedge I \to A$ being a weak equivalence. We have $fq = H = gq$. Apply F to get $Ff \cdot Fq = Fg \cdot Fq$. Since Fq is an isomorphism in $\mathcal{D}$, we get $Ff = Fg$. The right homotopy case is dual.

□

Lemma 5.2.3. *The maps in $HO(\mathcal{C})$ are generated via composition by images of maps in $\mathcal{C}$.*

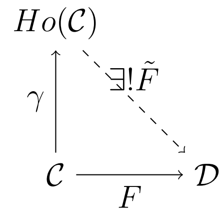
Proof. Every morphism $[g] \in hom_{Ho(\mathcal{C})}(X, Y) = \pi(RQX, RQY)$ can be written as

$$[g] = \gamma(p_y)\gamma(i_{QY})^{-1}\gamma(g)\gamma(i_{QX})\gamma(p_X)^{-1}$$

following the diagram above.

□

Proposition 5.2.4. *Let $\mathcal{C}$ be a closed model category and $F : \mathcal{C} \to \mathcal{D}$ a functor that takes weak equivalences to isomorphisms in $\mathcal{D}$. Then there is a unique functor $\tilde{F} : Ho(\mathcal{C}) \to \mathcal{D}$ lifting F with respect to γ.*



Proof. We construct $\tilde{F}$. It is obvious what $\tilde{F}$ is on the objects. Each morphism $[g] \in hom_{Ho(\mathcal{C})}(X, Y)$ can be written as

$$[g] = \gamma(p_y)\gamma(i_{QY})^{-1}\gamma(g)\gamma(i_{QX})\gamma(p_X)^{-1}$$

by the above lemma. So we know where to send $[g]$ since the choice of representative g is unique up to homotopy, and we know that F sends homotopic maps to the same map in $\mathcal{D}$. It is easy to see that $\tilde{F}$ is a functor and uniqueness follows by construction. □

In other words we have proved that the homotopy category $Ho(\mathcal{C})$ is just the localization of $\mathcal{C}$ with respect to the class of weak equivalences. We could have defined the homotopy category as a localization from the beginning but then the maps in a localized category are not easy to handle. But now we have a nice description of the maps in $Ho(\mathcal{C})$ as homotopy classes. This description is more meaningful and practical.

Proposition 5.2.5. *Let A be a cofibrant object and X a fibrant object in $\mathcal{C}$. Then the map $\gamma : hom_{\mathcal{C}}(A, X) \to hom_{Ho(\mathcal{C})}(A, X)$ is surjective and induces a bijection of homotopy classes $\gamma : \pi(A, X) \to hom_{Ho(\mathcal{C})}(A, X)$*

Proof. We have a commutative square

$$\begin{array}{ccc} \pi(RA, QX) & \longrightarrow & \pi(A, X) \\ \Big\downarrow \gamma & & \Big\downarrow \gamma \\ hom_{Ho(\mathcal{C})}(RA, QX) & \longrightarrow & hom_{Ho(\mathcal{C})}(A, X) \end{array}$$

By definition the horizontal arrows are bijections. The left arrow is a bijection as well by definition. Hence the right arrow is a bijection. □

In light of the these results we can characterize elements in the extension groups $Ext^n(A, B)$ for R-modules A and B as follows:

Proposition 5.2.6. *Let A and B be two R-modules and let $S^n(B)$ be the chain complex concentrated at degree n. Then there is a natural bijection*

$$hom_{Ho(Ch_R)}(A, S^n(B)) \cong Ext^n(A, B)$$

Proof. Recall that a cofibrant replacement for an R-module A is just a projective resolution. So choose a cofibrant replacement $P_\star \to A \to 0$. Observe that giving a chain map $f : P_\star \to S^n(B)$ is the same thing as giving a map $f : P_n \to B$ such that $f\partial = 0$.

$$\begin{array}{ccccc} P_{n+1} & \xrightarrow{\partial} & P_n & \xrightarrow{\partial} & P_{n-1} \\ \downarrow & & f\downarrow & \overset{s}{\swarrow} & \downarrow \\ 0 & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

Moreover, homotopy relations by definition will be given by maps $s : P_{n-1} \to B$. In other words, if we look at the complex

$$\cdots \to hom(P_{n-1}, B) \xrightarrow{\partial_\star} hom(P_n, B) \xrightarrow{\partial_\star} hom(P_{n+1}, B) \to \cdots$$

we see that $hom_{Ch_R}(P_\star, S^n(B)) \cong kernel\partial_\star$, and the image of $\partial_\star : hom(P_{n-1}, B) \to hom(P_n, B)$ gives the homotopy relations. Therefore choosing a homotopy class in $\pi(P_\star, S^n(B))$ is the same thing as choosing a homology class in $H_n(hom(P_\star, B))$. By definition, homotopy classes are just maps in the homotopy category and homology classes are just elements of the extension groups. Hence we have proven the bijection in the proposition. $\square$

In the category of topological spaces Proposition 3.6 tells us that for a CW-complex A (a cofibrant object in **Top**) and any space X, $hom_{Ho(\mathbf{Top})}(A, X)$ is just the set of conventional homotopy classes of maps $A \to X$.

In Ch_R we can conclude that in the derived category maps between two cofibrant objects, which are complexes of projective modules, are just homotopy classes between these complexes.

5.3 Derived functors and Quillen pairs

We have seen that the homotopy category $Ho(\mathcal{C})$ associated to a closed model category $\mathcal{C}$ is the localization with respect to the class of weak equivalences. Therefore we can talk about left, right and total derived functors of a functor $F : \mathcal{C} \to \mathcal{D}$.

Definition 23. *Let $\mathcal{C}$ be a closed model category and $F : \mathcal{C} \to \mathcal{D}$ be a functor. A left derived functor of F is a pair (LF, t) where $LF : Ho(\mathcal{C}) \to D$ is a functor, and $t : LF\gamma \to F$ is a natural transformation, and this pair is universal with respect to this property i.e. if we have another pair (G, s) of a functor $G : Ho(\mathcal{C}) \to D$ and a natural transformation $s : G\gamma \to F$, then there is a unique natural transformation $s' : G\gamma \to LF\gamma$ such that $ts' = s$.*

The right derived functor (Rf, t) is defined dually. The ideal case is when F itself factors through γ (we know this happens if and only if F sends weak equivalences to isomorphisms). In that case F itself induces the left derived functor LF and we actually have the identity $LF\gamma = F$. We will see that derived functors exist even if this is not the case.

Lemma 5.3.1. *Let $\mathcal{C}$ be a model category and $F : \mathcal{C}_c \to \mathcal{D}$ be a functor such that $F(f)$ is an isomorphism whenever f is an acyclic cofibration. Then if $f, g : A \to B$ are right homotopic maps in $\mathcal{C}_c$, we have $F(f) = F(g)$.*

Proof. Since A is cofibrant, we can choose a very good right homotopy $H : A \to B^I$. The acyclic cofibration $s : B \to B^I$ is such that $sf = sg = H$. And since B^I has to be cofibrant, we get $F(s)F(f) = F(s)F(g)$ with $F(s)$ an isomorphism. Hence, $F(f) = F(g)$.

□

In other words, the lemma asserts that functors F as above factor through the homotopy category as

$$\begin{array}{ccc} \mathcal{C}_c & \xrightarrow{F} & \mathcal{D} \\ & \searrow \quad \nearrow \pi F & \\ & \pi\mathcal{C}_c & \end{array}$$

Proposition 5.3.2. *Let $\mathcal{C}$ be a closed model category and $F : \mathcal{C} \to \mathcal{D}$ a functor such that $F(f)$ is an isomorphism whenever f is a weak equivalence between cofibrant objects. Then the left derived functor (LF, t) exists and moreover, for each cofibrant object X,*

$$t_X : LF(X) \to F(X)$$

is an isomorphism.

Proof. By the previous lemma, the restriction $F|_{\mathcal{C}_c}$ can be factors through the homotopy category. Let us denote $F' = \pi F|_{\mathcal{C}_c}$. We also have the cofibrant replacement functor $Q : \mathcal{C} \to \pi\mathcal{C}_c$. The composite $F'Q : \mathcal{C} \to \mathcal{D}$ sends weak equivalences to isomorphisms, therefore it factors through the localization functor $\gamma : \mathcal{C} \to Ho(\mathcal{C})$ to give us a left derived functor $LF : Ho(\mathcal{C}t \to \mathcal{D}$. The structure natural transformation t is given by $t_X = F(p_X) : LF(X) \to F(X)$. By construction, we have of course $Lf(X) = QX$.

All we are left to do is verify the universal property. So assume $G : \mathcal{C} \to \mathcal{D}$ is a functor that sends weak equivalences to isomorphisms and $s : G \to F$ is a natural transformation. For each object $X \in \mathcal{C}$ we have a commutative square

$$\begin{array}{ccc} F(QX) & \xrightarrow{F(p_X)} & F(X) \\ {\scriptstyle s_{QX}}\uparrow & & \uparrow{\scriptstyle s_X} \\ G(QX) & \xrightarrow{G(p_X)} & G(X) \end{array}$$

and we know that $G(p_X)$ is an isomorphism. Therefore we define for each $X \in \mathcal{C}$ the map $s'_X = s_{QX} G(p_X)^{-1}$, which defines a natural transformation $s' : G \to LF\gamma$. Uniqueness is immediate. Moreover, by construction, if X is cofibrant then p_X is a weak equivalence between cofibrant objects and therefore $t_X = F(p_X)$ is an isomorphism.

□

Now we define morphisms between model categories, which are called *Quillen functors.*

Definition 24. *Let $\mathcal{C}$ and $\mathcal{D}$ be closed model categories. A Quillen functor (a mor-*

phism between model categories) is an adjoint pair of functors

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

such that

(i) F preserves cofibrations and weak equivalences between cofibrant objects

(ii) G preserves fibrations and weak equivalences between fibrant objects.

Definition 25. *A Quillen functor (F, G) is said to be a Quillen equivalence if in addition we have*

(iii) For all $A \in \mathcal{C}_c$ and $Y \in \mathcal{D}_f$ we have "a map $A \to GY$ is a weak equivalence if and only if the adjoint map $FA \to Y$ is a weak equivalence".

Definition 26. *Let $\mathcal{C}$ and $\mathcal{D}$ be closed model categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. A total left derived functor of F is a functor*

$$LF : Ho(\mathcal{C}) \to Ho(\mathcal{D})$$

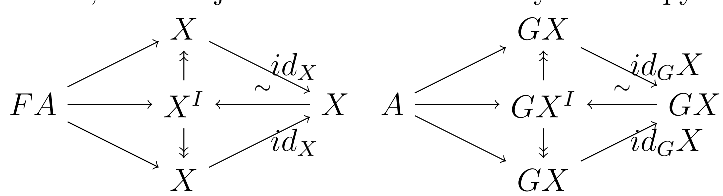
which is the left derived functor of $\gamma_{\mathcal{D}} F$.

By the above proposition (and its dual), for a Quillen functor (F, G) the total left derived functor LF and the total right derived functor RG exist. However, we can deduce much more than just existence.

Theorem 5.3.3. *Let (F, G) be a Quillen functor between model categories $\mathcal{C}$ and $\mathcal{D}$. Then the total derived functors LF and RG are adjoint functors. Moreover, if (F, G) is a Quillen equivalence, then LF and RG are equivalences of categories.*

Proof. First, we show that adjointness of F and G respects the homotopy relation, meaning that for $A \in \mathcal{C}_c$ and $X \in \mathcal{D}_f$ we have $\pi_{\mathcal{D}}(FA, X) \cong \pi_{\mathcal{C}}(A, GX)$.

Indeed, from adjointness we can identify homotopy diagrams



since X^I will be fibrant, and G preserves fibrations and weak equivalences between fibrant objects.

Keeping in mind the functors $Q : \mathcal{C} \to \mathcal{C}_c$ and $R : \mathcal{D} \to \mathcal{D}_f$, we may write the following:

$hom_{Ho(\mathcal{D})}(LF(A), X) \cong hom_{Ho(\mathcal{D})}(F(QA), X) \cong hom_{Ho(\mathcal{C})}(QA, G(X)) \cong$

$hom_{Ho(\mathcal{C})}(QA, RGX) \cong hom_{Ho(\rfloor)}(A, RG(X))$

proving the adjointness.

Now assume (F, G) is a Quillen equivalence. Let $A \in \mathcal{C}_c$. Consider the fibrant replacement $F(A) \to RF(A)$ in $\mathcal{D}$. By assumption, the adjoint map $A \to G(RF(A))$ is a weak equivalence in $\mathcal{C}$ and since $RG(LF(A)) = G(RF(A))$ by construction, we have an isomorphism $A \cong RG(LF(A))$. We get the natural equivalence we wanted. □

The most famous example of a Quillen equivalence is the pair consisting of the geometric realization functor and singular simplicial set functor from the category of simplicial set to the category of topological spaces. Unfortunately this is too big of a topic to be treated here but in [3] we can find all the constructions and proofs.

5.4 Homotopy limits and colimits

Let $j_n : S^{n-1} \hookrightarrow D^n$ be the boundary inclusion in **Top**. Consider the following diagram:

$$\begin{array}{ccccc} D^n & \xleftarrow{j_n} & S^{n-1} & \xrightarrow{j_n} & D^n \\ \downarrow & & \downarrow{\scriptstyle id_{S^{n-1}}} & & \downarrow \\ \star & \longleftarrow & S^n & \longrightarrow & \star \end{array}$$

The shown diagram is just a morphism between pushout diagrams in **Top** where all vertical arrows are weak equivalences. However, the pushout of the top row is just S^n and the pushout of the bottom row is $\star$ and the induced map on colimits $S^n \to \star$ is not a weak equivalence in this case.

In an abstract setting, we have a model category $\mathcal{C}$ and the category $\mathbf{D} = \{a \leftarrow b \to c\}$ and we look at the functor $colim : \mathcal{C}^{\mathbf{D}} \longrightarrow \mathcal{C}$. Assume for the moment that $\mathcal{C}^{\mathbf{D}}$ admits the natural model category structure, with a map $f : X \to Y$ being a weak equivalence (resp. fibration, cofibration) if and only if $f_\Box : X_\Box \to Y_\Box$ is a weak equivalence (resp. fibration, cofibration) for $\Box = a, b, c$. Then the above example shows that the colimit functor between model categories does not preserve weak equivalences and therefore it cannot be lifted to the level of homotopy categories. However, we can construct its left derived functor.

By definition, the colimit functor is left adjoint to the constant diagram functor.

$$colim : \mathcal{C}^{\mathbf{D}} \rightleftarrows \mathcal{C} : \Delta$$

Besides being an adjoint pair, $(colim, \Delta)$ is a Quillen pair between model categories. Indeed, $colim$ will preserve left lifting properties and therefore will preserve acyclic cofibrations between cofibrant objects and cofibrations themselves. And by definition of the model structure on $\mathcal{C}^{\mathbf{D}}$ the functor Δ preserves everything. Therefore the derived functors exist and they form an adjoint pair:

$$L(colim) : Ho(\mathcal{C}^{\mathbf{D}}) \leftrightarrows Ho(\mathcal{C}) : R(\Delta)$$

This is called the homotopy pushout. Dually, we can talk about homotopy pullback. If we let $\mathbf{D} = \{a \to b \leftarrow c\}$ we have a Quillen pair

$$\Delta : \mathcal{C} \rightleftarrows \mathcal{C}^{\mathbf{D}} : lim$$

which gives us the homotopy pushout as a right derived functor

$$L(\Delta) : Ho(\mathcal{C}) \leftrightarrows Ho(\mathcal{C}^{\mathbf{D}}) : R(lim)$$

We are left to prove that $\mathcal{C}^{\mathbf{D}}$ is a model category for the above cases. We do just the case of pushout diagrams because the other case is similar.

Proposition 5.4.1. *Let $\mathcal{C}$ be a model category and $\boldsymbol{D}$ be the category $\{a \leftarrow b \to c\}$. Then $\mathcal{C}^{\boldsymbol{D}}$ has a model category structure with a map $f : X \to Y$ being a weak equivalence (resp. fibration, cofibration) in case $f_\square : X_\square \to Y_\square$ is a weak equivalence (resp. fibration, cofibration) for $\square = a, b, c$.*

Proof. **MC1 - MC4** obviously hold by construction. The problem is factoring a given $f : X \to Y$ as required by **MC5** without having functorial factorizations n $\mathcal{C}$.

We construct a factorization of $f : X \to Y$

$$\begin{array}{ccccc}
X_a & \longleftarrow & X_b & \longrightarrow & X_c \\
{\scriptstyle f_a}\downarrow & & {\scriptstyle f_b}\downarrow & & {\scriptstyle f_a}\downarrow \\
Y_a & \longleftarrow & Y_b & \longrightarrow & Y_c
\end{array}$$

as follows: first factor the map $f_b : X_b \to Y_b$ as $f_b = p_b i_b : X_b \hookrightarrow Z_b \twoheadrightarrow Y_b$ with i_b acyclic cofibration and p_b fibration. Then construct pushouts squares

$$\begin{array}{ccc}
X_a & \longleftarrow & X_b \\
\sim\Big\downarrow{\scriptstyle i_a} & & \sim\Big\downarrow{\scriptstyle i_b} \\
Z_a & \longleftarrow & Z_b
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
X_b & \longrightarrow & X_c \\
\sim\Big\downarrow{\scriptstyle i_a} & & \sim\Big\downarrow{\scriptstyle i_c} \\
Z_b & \longrightarrow & Z_c
\end{array}$$

to get maps $i_a : X_a \hookrightarrow Z_a$ and $i_c : X_c \to Z_c$ which will be acyclic cofibrations since i_a is an acyclic cofibration and pushouts preserve acyclic cofibration. We have induced maps $Z_a \to Y_a$ and $Z_c \to Y_c$ and a commutative diagram

$$\begin{array}{ccccc}
X_a & \longleftarrow & X_b & \longrightarrow & X_c \\
\sim\Big\downarrow{\scriptstyle i_a} & & \sim\Big\downarrow{\scriptstyle i_b} & & \sim\Big\downarrow{\scriptstyle i_c} \\
Z_a & \longleftarrow & Z_b & \longrightarrow & Z_c \\
\Big\downarrow & & \Big\downarrow{\scriptstyle p_b} & & \Big\downarrow \\
Y_a & \longleftarrow & Y_b & \longrightarrow & Y_c
\end{array}$$

Finally use **MC5** to factor the maps $Z_a \to Y_a$ and $Z_c \to Y_c$ as $p_a i'_a : Z_a \hookrightarrow Z'_a \twoheadrightarrow Y_a$ and $p_c i'_c : Z_c \hookrightarrow Z'_c \twoheadrightarrow Y_c$ where i'_a, i'_c are acyclic cofibrations and p_a, p_c are fibrations, to get the desired factorization

$$
\begin{array}{ccccc}
X_a & \longleftarrow & X_b & \longrightarrow & X_c \\
\sim \big\downarrow i'_a i_a & & \sim \big\downarrow i_b & & \sim \big\downarrow i'_c i_c \\
Z'_a & \longleftarrow & Z_b & \longrightarrow & Z'_c \\
\big\downarrow p_a & & \big\downarrow p_b & & \big\downarrow p_c \\
Y_a & \longleftarrow & Y_b & \longrightarrow & Y_c
\end{array}
$$

□

Note that we do not need the above to talk about homotopy limits and colimits in cofibrantly generated categories because of the existence of functorial factorizations. In particular in **Top** and Ch_R all homotopy limits and colimits exists.

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