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Affirmative Action in School Choice: Reserve Seats and Vouchers

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M.A. Thesis

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May 2017

Koc University

Graduate School of Social Sciences and Humanities

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Date: 23/05/2017

Acknowledgements

I am indeed grateful to an excellent advisor, Professor Özgür Yılmaz for his support and guidance during these two years. The chance of working with him is invaluable and I have really learned a lot from him during my studies.

I am really indebted to all my professors at Koç University. Being their student have been surely a great experience not only for the great depth of knowledge they posses but also for their friendly attitude. Also, the friends I have gained in this program were very successful in making these years more fun and deserve special thanks.

My family of course deserves the deepest of the gratitudes. Without their support too few would be possible.

Last but not least, I would like to express my thanks to BIDEB 2210-A graduate scholarship program of The Scientific and Technological Research Council of Turkey (TÜBİTAK) for the financial support it provided during my graduate studies.

Affirmative action in school choice: reserve seats and vouchers

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May 2017

Abstract

Both school voucher and reserve programs have recently become popular affirmative action policy tools to provide low-income students with further alternatives. While the idea of voucher is simply to fund private school expenses of these students through public resources, affirmative action aims at making good public schools more accessible for them. The current practice is to treat these options separately, they are separate and independent programs. We argue that the two policy tools can be considered together because (1) both reserve seats and school vouchers aim at providing better alternatives for low-income students, and (2) there might be unintended consequences when they are considered separately, that is, when a disadvantaged student is endowed with the right of both reserve seats and vouchers. Our contribution is the design of a mechanism which simultaneously determines an assignment of school seats and voucher allocation. The outcome of this mechanism improves students' welfare compared to the outcome when these tools are considered separately.

Keywords : School choice, stability, voucher, affirmative action

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Özet

Devlet bursları ve devlet okullarındaki rezerv uygulamaları düşük gelirli öğrencilere daha iyi alternatifler sunma amacıyla günümüzde sıkça kullanılmaktadır. Devlet bursları, devlet kaynaklarıyla düşük gelirli öğrencilerin özel okul masraflarının karşılanması amacıyla kullanılırken, rezerv kontenjanlar yoluyla yapılan pozitif ayrımcılık bu öğrencilerin daha çok tercih edilen devlet okullarına girişini kolaylaştırmak amacıyla kullanılmaktadır. Mevcut uygulama bu iki farklı aracı ayrık ve bağımsız olarak kullanmaktır. Bizce bu iki araç birlikte dikkate alınarak uygulanmalıdır çünkü (1) her iki araç da finansal olarak dezavantajlı öğrencilere daha iyi alterna-tifler sunmayı amaçlamaktadır, (2) bu araçlar bağımsız kullanıldığında istenmeyen sonuçlar görülebilir: finansal anlamda dezavantajlı bir öğrenci hem rezerv hem de devlet bursu hakkına sahip olmuş olabilir. Bu çalışma devlet burslarını ve okul kontenjanlarını birlikte dağıtan bir mekanizma ortaya koymaktadır. Bu mekanizmanın verdiği sonuç öğrencilerin tercihlerini karşılama noktasında bu iki aracın bağımsız kullanımına göre daha iyi sonuç vermektedir.

Anahtar sözcükler: Okul seçimi, stabilite, burs, pozitif ayrımcılık

Contents

1 Introduction

School choice is an important policy tool to provide the parents with the opportunity to choose their child's school. Basically, parents reveal a list of schools in a preference ordering and school districts determine the assignment of school seats based on these preferences and schools' priorities over the students. Schools' priorities are obtained from a set of criteria, the most important of which is the walk-zone. Unfortunately, for the districts where there is strong spatial and economic segregation, this system with walk-zone priorities elevates segregation through schooling. Many school districts in the US (including Chicago, Boston, and New York City), concerned with a high level of segregation and the resulting lack of diversity, developed and embedded affirmative action systems in school choice. These systems typically introduce certain slots for disadvantaged groups.

A separate policy to extend choice for disadvantaged groups is a school voucher system which has recently become quite popular in some countries including the US, Sweden, Chile, and Hong Kong. The idea is to fund private school expenses of students from lower socioeconomic tiers through public resources. We argue that by considering affirmative action and school vouchers together (as opposed to the current practice, that is, using these two different systems as separate policy tools), it is possible to improve students' welfare. We propose a mechanism which simultaneously determines an assignment of school seats and voucher allocation. The outcome of this mechanism improves students' welfare compared to the outcome when these tools are considered separately.

School choice programs are widely using the deferred acceptance (DA) algorithm (Gale and Shapley, 1962) to determine the assignment of school seats. This mechanism has been popular in practice since it guarantees that no student ever envies a student with lower priority (*stability*). The mechanism is such that revealing true preferences is the dominant strategy, thus eliminating gains through strategic manipulation. It is not problem-free though: the outcome of the DA algorithm can be inefficient, although it is efficient when the assignments are restricted to be *stable*. An important part of the school choice literature is based on models of improving students' welfare by giving up on the full bite of *stability*.

Affirmative action policies are usually considered within the framework of the DA mechanism: how can school districts improve diversity and provide better choices for low-income and disadvantaged students, stability and students' welfare being the main concerns? The practice is often to reserve certain slots for these groups and weaken stability such that priorities can be violated by the students assigned to these seats. We adopt the same approach and model affirmative action as the reserved seats for the low-income students.

A different policy to provide low-income groups with better options is the school voucher system: students are given vouchers which they use to pay tuition for private schools. While there is a hot debate whether the voucher system is constitutional, and it has become a political issue in the US, it is currently an important component of schooling in the US. School voucher programs are adopted by 13 states: of those, eight states offer vouchers to special needs students, four states plus D.C. offer them to low-income students or students from failing schools, and two states offer them to certain rural students.[1] However, school voucher programs are not problem-free: in Louisiana, 9,100 scholarships were offered in 2014, but only a little less than 7,400 students chose to take advantage of the program and over 3.7 million USD of the related funding has not been used.[2]

Our model embeds the school voucher system into school choice with affirmative action. We argue that the two policy tools can be considered together because (1) both affirmative action and school vouchers aim at providing better alternatives for low-income students, and (2) there might be unintended consequences when they are considered separately, that is, when a disadvantaged student is endowed with the right to both reserve seats and vouchers. We show that there is a better way to utilize these programs compared to the current practice where vouchers are distributed by means of a lottery and separately from the affirmative action policy. We propose a mechanism to improve the efficiency aspect of the school assignment by a weakening of the notion of the stability with reserve seats further by differentiating a

[1] http://www.ncsl.org/research/education/school-choice-vouchers.aspx
[2] http://www.nola.com/politics/index.ssf/2014/11/louisiana-private-voucher-prog.html

disadvantaged student with a voucher from the one without a voucher. This weakening allows for possibilities to improve welfare; we analyze how these gains can be obtained by means of a mechanism.

1.1 Related Literature

The school choice problem is modeled with students with preferences on the one side and schools with priorities on the other side of the market (Abdulkadiroğlu and Sönmez, 2003). *Gale and Shapley Student Optimal Deferred Acceptance Algorithm* (Gale and Shapley, 1962) is the algorithm that gives the most efficient outcome for students among stable ones for these types of problems (Abdulkadiroğlu and Sönmez, 2003).

Affirmative action policies for student placements is prevalent in many parts of the world. A small body of literature examines affirmative action policies in school choice problem. Alternative affirmative action practices like implementing majority quotas, which imposes a maximum number of majority students that can be admitted, and preferential treatment to minority students, which increases the priority of minority students at some schools, can cause every minority student to be worse off compared to a no affirmative action case although it is contrary to the intention of affirmative action (Kojima, 2012). For diversity constraints in schools, when there is hard lower and upper bounds for constraints, assignments that satisfy standard fairness and non-wastefulness properties may not exist, therefore soft bounds which are flexible depending on the preferences of students are proposed to achieve fairness and non-wastefulness (Ehlers, Hafalir, Yenmez, and Yildirim, 2014). Affirmative action with minority reserves has been proposed as a policy superior to majority quotas in terms of efficiency and a new stability notion, *stability with minority reserves* is introduced for such a policy (Hafalir, Yenmez, and Yildirim, 2013). *Deferred Acceptance with Minority Reserves* mechanism gives the most efficient result among results that are stable with minority reserves (Hafalir, Yenmez, and Yildirim, 2013) and this rule can not be strictly Pareto dominated by *Deferred Acceptance* with no affirmative action. Since there are cases in which using minority reserves is Pareto inferior (not strictly) for minority students to not practice any affirmative action, a minimally responsive rule that makes efficiency improvement over *Deferred Acceptance with Minority Reserves* is developed (Doğan, 2015). The improvement of the mechanism is made through weakening the notion of *stability with minority reserves.* Using this rule ensures that an affirmative action policy with minority reserves cannot be Pareto dominated for minority students by a stable allocation with no affirmative action. In our work we use a fairness notion that is more suitable to the combined problem of affirmative action and allocation of vouchers, and this notion allows us to have efficiency improvement over a mechanism that gives outcomes that are *stable with minority reserves.* Diversity problems are also analyzed as a matching with contracts model where schools are seen as branches and slots are school seats with different priorities for students determining the affirmative action practice; a generalization of *Deferred Acceptance*, the *cumulative offer* mechanism gives an outcome in the school-student market that corresponds to student optimal outcome in one to one student-slot market (Kominers and Sonmez, 2016). In their analysis of high school placements in Chicago, Dur, Pathak, and Sonmez (2016) introduced affirmative action policies that are most preferable to disadvantaged students when affirmative action policy is explicit or implicit; the order of the processing of school seats plays a critical role in the analysis because of alternative types of seats in each school that alters the priority of disadvantaged students .

Another practice in favor of disadvantaged students is allocating vouchers that enables disadvantaged students to afford private schools. The voucher allocation problem is first modeled as adding private schools with priorities and a voucher endowment structure, which indicates the students with vouchers, to the school choice problem (Afacan, 2016). A mechanism that allocates vouchers and school seats efficiently is found (Afacan, 2016). In our work we treat private schools as outside option rather than as schools in the system that has priorities. Also, we consider affirmative action problem besides the voucher allocation problem.

Literature that is closely related to ours investigates how efficiency gains over stable mechanisms can be achieved. The *Efficiency Adjusted Deferred Acceptance Algorithm* (EADAM), which allows students to consent to violation of their priorities which has no affect on their assignment, is proved to be an improvement over *Gale and Shapley Student Optimal Deferred Acceptance Algorithm* in terms of efficiency (Kesten, 2010). *Stable efficiency improvement cycles* is developed to be used after some tie-breaking to get a constrained efficient outcome in a case of indifferences in the priority structure of schools (Erdil and Ergin, 2008). A generalization of the idea of priority violation is made through a new notion, *partial stability*, and a class of mechanisms, *Student Exchange Under Partial Fairness* (SEPF), which gives constrained efficient results according to *partial stability* by using *efficiency improvement cycles*, is found (Dur, Gitmez, and Yilmaz, 2015). We also use the idea of priority violation as a part of our fairness definition along with efficiency improvement cycles and chains to improve the welfare of the students.

2 The Model

A **school choice problem with vouchers and reserves** consists of the following elements:

- a finite set of students $I = \{i_1, i_2, \ldots, i_n\}$
- a set of *disadvantaged* students $I^d \subset I$, who are from the lowest socioeconomic tier,
- a finite set of schools $S = \{s_1, s_2, ..., s_n\}$,
- a capacity vector $q = (q_s)_{s \in S}$, where q_s is the number of available seats at school s,
- a reserve vector $r = (r_s)_{s \in S}$, where $r_s \geq 0$ is the number of reserve seats for the *disadvantaged* students at school s,
- a strict priority structure of schools $\succ = (\succ_s)_{s \in S}$ where $\succ_s$ is the complete priority order of school s over I,
- a strict preference profile of students $P = (P_i)_{i \in I}$ such that P_i is student i's preferences over $S \cup \{\emptyset\}$, where $\emptyset$ stands for the student's outside option, and R is the associated

weak preference relation; that is, $s\ R_i\ s'$ if and only if $s\ P_i\ s'$ or $s = s'$,

- a voucher endowment structure $\pi_0 : I^d \to \{v, 0\}$, where student $i \in I^d$ is endowed with a voucher if and only if $\pi_0(i) = v$.

We fix all but R and π_0 throughout the paper, unless stated otherwise, and denote a *school choice problem with vouchers* or simply, a **problem**, by (R, π_0). A **matching** $\mu : I \to S \cup \emptyset$ is a function such that for each $s \in S$, $|\mu^{-1}(s)| \leq q_s$. A **voucher allocation** $\pi : I^d \to \{v, 0\}$ is a function such that a voucher is allocated to student $i \in I^d$ if and only if $\pi(i) = v$. An **allocation** (μ, π) is a pair of matching and voucher allocation functions. A **rule** is a systematic procedure of specifying an allocation for each problem.

An allocation (μ, π) is **feasible** if $|\pi^{-1}(v)| \leq \left|\pi_0^{-1}(v)\right|$ and for each $i \in I^d$, $\mu(i) = \emptyset$ implies $\pi(i) = v$. An allocation (μ, π) is **non-wasteful** if there are no $i \in I$ and $s \in S$ such that $s\ P_i\ \mu(i)$ and $|\mu^{-1}(s)| < q_s$. An allocation (μ, π) is **individually rational** if for each $i \in I$ such that $\emptyset\ P_i\ \mu(i)$, we have $i \in I^d \setminus \pi_0^{-1}(v)$ and $\pi(i) = 0$. An allocation (μ, π) **violates the priority** of student $i \in I$ at $s \in S$ via student $j \in \mu^{-1}(s)$ if $s\ P_i\ \mu(i)$ and $i \succ_s j$. An allocation (μ, π) is **fair** if there does not exist $i \in I$ and $s \in S$ such that priority of i is violated at s. An allocation (μ, π) is **stable** if μ is *non-wasteful*, *individually rational* and *fair*.

An allocation (μ, π) **respects the reserves** if there does not exist a pair of a student $i \in I^d$ and a school $s \in S$ such that $s\ P_i\ \mu(i)$ and $|\mu^{-1}(s) \cap I^d| < r_s$. An allocation (μ, π) is **fair under reserve seats** if the priority of i is violated at s via $j \in \mu^{-1}(s)$ only if $j \in I^d$, $i \in I \setminus I^d$ and $|\mu^{-1}(s) \cap I^d| \leq r_s$. An allocation (μ, π) is **stable under reserve seats** if it is *non-wasteful, individually rational, respects the reserves* and *fair under reserve seats.*

An allocation (μ, π) **Pareto dominates** another allocation (μ', π') if for each $i \in I$, $\mu(i)\ R_i\ \mu'(i)$ and for some $j \in I$, $\mu(j)\ P_j\ \mu'(j)$. An allocation (μ, π) is **Pareto efficient** if it is not *Pareto dominated* by another matching μ'. An allocation (μ, π) has **wasted vouchers** if there are students $i, j \in I^d$ such that $\pi(j) = v$, $\mu(j) = s$ for some $s \in S$ and $\emptyset\ P_i\ \mu(i)$. An allocation (μ, π) is **constrained efficient (under reserve seats)** if (μ, π) is *stable (under reserve seats)* and there does not exist another stable (under reserve seats) allocation (μ', π')

Pareto dominating (μ, π).

Reserve seats policies in the presence of vouchers present certain ambiguities in terms of how to treat students with vouchers in terms of their rights for reserve seats. This can be seen by the following example.

Example 1 *Let* $q_s = 1$ *for each* $s \in S$*. Let* $r_s = 1$ *for* s_1, s_2 *and* $r_s = 0$ *for* s_3, s_4, s_5, s_6*. Let* $I^d = \{i_1, i_2, i_3, i_5, i_6, i_7, i_8\}$ *and* $I \setminus I^d = \{i_4\}$*. Let* $\pi_0^{-1}(v) = \{i_3, i_5, i_8\}$*.*

s_1	s_2	s_3	s_4	s_5	s_6
i_4	i_4	i_4	i_6	i_5	i_7
i_1	i_2	.	.	.	i_8
i_3	i_3	.	.	.	.
i_2	i_1	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.

i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8
s_2	s_1	s_1	s_1	s_5	$\emptyset$	$\emptyset$	s_6
s_1	s_2	s_2	s_2	$\emptyset$	s_4	s_6	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	s_3	.	.	.	.
.	.	.	$\emptyset$	.	.	.	.
.	.	.	.	.	.	.	.

Suppose voucher transfers are not allowed. Then, there exists a unique allocation which is *stable under reserve seats* and induces the following matching:
$(s_1, i_1), (s_2, i_2), (s_3, i_4), (s_4, i_6), (s_5, i_5), (s_6, i_7), (i_3, \emptyset), (i_8, \emptyset)$.

By exchanging their seats, students i_1 and i_2 can be better off. But, since i_3 is a *disadvantaged* student and has a reserve seat priority over i_1 at s_2 and i_2 at s_1, this exchange leads to a violation of *stability under reserve seats*. Also, i_3 has a voucher and enrols in a private school. Thus, although student i_3 has an advantage over other *disadvantaged* students i_1 and i_2 in receiving a voucher, she blocks the exchange of these students, which is through the reserve seats rights (in the form of reserve seat priorities) given to her. Her priority in reserve seats harms other *disadvantaged* students. We argue that the reserve priority of a *disadvantaged* student enrolling in a private school (via the voucher given to her) can be violated if this leads to exchanges between other students like in this case since one might consider her voucher as a practised affirmative action and this should change her status of *disadvantaged* student at schools with reserves. Thus, to respect the reserve seat priority of such *disadvantaged* students with vouchers is too strong from fairness perspective.

Another issue with this allocation is that the lack of voucher transfers between students leads to an allocation with *wasted vouchers.* Student i_5 is endowed with a voucher but she does not use it since the public school system assigns her to a state school which she prefers over the private school that she can enrol with her voucher. If the voucher would be transferred from student i_5 to student i_6, student i_6 would be better off since $\emptyset \; P_{i_6} \; s_4$. Since the welfare of other students would not be affected by this transfer, this is a Pareto improvement. Similarly if the voucher of i_8 is transferred to i_7, then i_8 is assigned to s_6 and i_7 to her outside option ($\emptyset$), which is a Pareto improvement. In the first case, a voucher is wasted if a transfer from i_5 to i_6 is not allowed. In the second case, voucher is not wasted but there is a potential efficiency gain via voucher transfer from i_8 to i_7.

Respecting the reserve priorities of a *disadvantaged* student i with a voucher can be considered as a very strong form of affirmative action rights, since this might hurt other *disadvantaged* students without improving student i. *To improve students' welfare, we argue for a two-fold weakening of the current reserve seats and voucher systems:* (i) *to allow for violation of reserve-seat priorities of voucher users,* (ii) *to allow voucher transfers.* The first weakening is a weaker stability notion.

The following example shows that if we require stability with reserves then we may have either a wasted voucher or a disadvantaged student who is both using voucher and blocking the exchange of other students through her reserve seat rights.

Example 2 *Let* $S = \{s_1, s_2, s_3\}$ *with* $q_{s_1} = q_{s_3} = q_{s_2} = r_{s_2} = 1$ *and* $r_{s_1} = r_{s_3} = 0$. *Let*

$I^d = \{i_1, i_2\}$ *and* $I \setminus I^d = \{i_3\}$ *with* $\pi_0^{-1}(v) = \{i_2\}$. *Preference and priority profile is as follows:*

s_1	s_2	s_3
i_3	i_1	i_2
i_1	i_3	i_1
i_2	i_2	i_3

i_1	i_2	i_3
s_1	s_1	s_2
s_2	s_2	s_1
s_3	$\emptyset$	s_3
$\emptyset$	s_3	$\emptyset$

Fairness with reserves implies $\mu(i_2) \neq s_1$ *and* $\mu(i_2) \neq s_2$*: Suppose* $\mu(i_2) = s_1$ *then this contradicts fairness with reserves since* $s_1\ P_{i_1}\ \mu(i_1)$ *and* $i_1 \succ_{s_1} i_2$. *Suppose* $\mu(i_2) = s_2$ *then fairness with reserves implies* $\mu(i_1) = s_1$ *which implies* $\mu(i_3) = s_2$ *by fairness with reserve again and this is a contradiction. By stability we need* $\mu(i_3) = s_1$*: if* $\mu(i_3) = s_2$ *this will violate fairness with reserve because of* i_2*; and if* $\mu(i_3) = s_3$ *or* $\mu(i_3) = \emptyset$ *this will violate non-wastefulness (because we know that either* s_2 *or* s_1 *will be empty since* $\mu(i_2) \neq s_1$ *and* $\mu(i_2) \neq s_2$*). Also, by stability we need* $\mu(i_1) = s_2$ *because if* $\mu(i_1) = s_3$ *or* $\mu(i_1) = \emptyset$ *this will again violate non-wastefulness. Notice that the reserve seat right of* i_2 *is effective since we have* (i) $s_2\ P_{i_3}\ \mu(i_3) = s_1$, $s_2\ P_{i_2}\ \mu(i_2)$ *and* (ii) *if we had* $\mu(i_3) = s_2$ *and* $\mu(i_1) = s_1$ *this would violate fairness with reserve unless we change the status of* i_2 *to non-disadvantaged. Therefore, to satisfy not giving disproportional benefits to students we need* $\mu(i_2) \neq \emptyset$ *which implies* $\mu(i_2) = s_3$. *So we have* $\emptyset\ P_{i_2}\ \mu(i_2) = s_3$. *Thus, vouchers are wasted whoever is assigned to voucher among disadvantaged students: if* $\pi(i_2) = v$ *we will have wasted voucher since* $\mu(i_2) = s_3$ *and if* $\pi(i_1) = v$ *we will have wasted vouchers since* $\mu(i_1) = s_2$.

2.1 Fairness with reserves and vouchers

An allocation (μ, π) is **fair under reserve seats and vouchers** if the priority violation of i at s via j implies that j is disadvantaged, $|\mu^{-1}(s) \cap I^d| \leq r_s$, and i is either not disadvantaged or a voucher user. An allocation (μ, π) is **stable under reserve seats and vouchers** if it is *non-wasteful, individually rational, respects the reserves* and *fair under reserve seats and* *vouchers.*

An allocation (μ, π) is **constrained efficient** under reserve seats and vouchers if it is stable under reserve seats and vouchers and there is no other allocation (μ', π') that is stable under reserve seats and vouchers and *Pareto dominates* (μ, π).

3 An Allocation Mechanism

3.1 Modified school choice problem

Each school $s \in S$ is replaced by two schools representing reserve seats, s^1, and regular seats, s^2, with $q'_{s^1} = r_s$ and $q'_{s^2} = q_s - r_s$. Let $S^1 = \{s^1 : s \in S\}$ be the set of all reserve seat schools and $S^2 = \{s^2 : s \in S\}$ be the set of all regular seat schools. For each $s \in S^1$, each student's priority within her group is the same with $\succ_s$, and for each $i \in I^d$ and $j \in I \setminus I^d$, i has higher priority than j. Thus, for each $s^1 \in S^1$, let $\succ'_{s^1}$ be defined as follows:

- $i, j \in I^d$ or $i, j \in I \setminus I^d$, and $i \succ_s j \Rightarrow i \succ'_{s^1} j$,
- $i \in I^d$ and $j \in I \setminus I^d \Rightarrow i \succ'_{s^1} j$.

For each $s^2 \in S^2$, priorities are given by $\succ_s$, thus $\succ'_{s^2} = \succ_s$. The modified priority structure is given by $\succ' = (\succ'_{s^1}, \succ'_{s^2})_{s \in S}$. The modified preferences of students $P' = (P'_i)_{i \in I}$ are as follows: for each $i \in I, s_1\ P_i\ s_2 \Rightarrow s_1^1\ P'_i\ s_1^2\ P'_i\ s_2^1\ P'_i\ s_2^2$. We denote the associated weak preference relation by R': $s\ R'_i\ s'$ if and only if $s\ P'_i\ s'$ or $s = s'$. We denote the modified school choice problem by $(I, I^d, R', S^1 \cup S^2, \succ', q', \pi_0)$.

Our fairness definition is equivalent to the following definition in the modified school choice problem: An allocation (μ, π) is **fair under reserve seats and vouchers** if the following condition is satisfied:

If there exist $i, j \in I$ and $s \in S^1 \cup S^2$ such that $s\ P'_i\ \mu(i)$, $\mu(j) = s$ and $i \succ'_s j$, then we have $s \in S^1$, $i \in I^d$, $\mu(i) = \emptyset$ and $j \in I^d$.

For a given voucher endowment structure π_0, the Deferred Acceptance mechanism can be modified in a straightforward way. The only (trivial) difference is that the preferences of disadvantaged students change with the voucher endowment structure: while each student $i \in I^d$ such that $\pi_0(i) = v$ and each student $j \in I \setminus I^d$ apply (to state schools) only if she prefers them to $\emptyset$, the students in I^d with $\pi_0(i) = 0$ can not get their outside options and must apply to (state) schools.

The Deferred Acceptance (DA) algorithm:

Step 1: Each disadvantaged student without a voucher applies to her most preferred school and each other student applies to her most preferred school if she prefers it to her outside option $\emptyset$; otherwise, she gets $\emptyset$. Each school $s \in S^1 \cup S^2$ tentatively accepts its most preferred students among applicants until its capacity is filled or applicants are exhausted. The rest of the applicants, if any remain are rejected.

Step k: Among the rejected students of Step k-1, each disadvantaged student student without a voucher applies to her most preferred school by which she has not been rejected yet, and each other student applies to her most preferred school by which she has not been rejected yet, if she prefers it to her outside option $\emptyset$; otherwise, she gets $\emptyset$. Each school $s \in S^1 \cup S^2$ tentatively accepts its most preferred students among new applicants and the students it has tentatively accepted at Step k-1 until its capacity is filled or students are exhausted and it rejects others if any student remains. The assignment is final if no student is rejected.

The algorithm stops when no rejection occurs and tentative matching in the last step becomes the outcome of the mechanism. Let μ_0 be this matching. There are finite schools and students can not apply to any school more than once. At each step except the last, some student is rejected. Thus, the mechanism terminates at a finite step. Note that voucher endowment structure π_0 has not changed at the end of the algorithm.

Remark 1 *The allocation (μ_0, π_0) is fair under reserve seats and non-wasteful.*

Let (μ_0, π_0) be not fair. So $\exists i, j \in I$ such that $s \; P'_i \; \mu_0(i)$, $i \succ'_s j$ and $\mu_0(j) = s$. This means that i is rejected from s at some step of the mechanism in favour of another student. Note that at every step, accepted students have higher priority in s than the rejected students. Therefore by transitivity of the priority structure, we have $j \succ'_s i$ which is a contradiction.

Suppose we get a wasteful outcome (μ_0, π_0) at the end of the D.A.. Hence, $\exists i \in I$ such that $s \; P'_i \; \mu_0(i)$ for some $s \in S^1 \cup S^2$ and $|\mu_0^{-1}(s)| < q'(s)$. That means i is rejected at some step from s and at that step s accepted $q'(s)$ many students tentatively. In each of the remaining steps s considers its tentatively accepted students and new students applied and tentatively accepts $q'(s)$ of them. Therefore, we can not have$|\mu^{-1}(s)| < q'(s)$.

3.2 A Constrained Efficient Mechanism Class

Given an allocation (μ, π) be an allocation, we define the following sets:

- $C_{\mu,\pi}(s^1) = \{i \in I^d : \mu(i) = \emptyset\}$ (the set of students whose priorities can be violated by disadvantaged students at $s^1 \in S^1$)
- For $k = 1, 2$: $D_{\mu,\pi}(s^k) = \{i \in I : s^k \; P'_i \; \mu(i)\}$ (the set of students who prefer $s^k \in S^k$ to their assignment at μ)
- $X_{\mu,\pi}(s^1) = \{i \in D_{\mu,\pi}(s^1) : i \in I^d$ and $\forall j \in D_{\mu,\pi}(s^1) \setminus (C_{\mu,\pi}(s^1) \cup \{i\}), i \succ_{s^1} j,$ or $\forall j \in D_{\mu,\pi}(s^1) \setminus \{i\}, i \succ_{s^1} j\}$ (the set of students who are eligible to take a seat at $s^1 \in S^1$)
- $X_{\mu,\pi}(s^2) = \{i \in D_{\mu,\pi}(s^2) : \forall j \in D_{\mu,\pi}(s^2) \setminus \{i\}), i \succ_{s^2} j\}$ (the set of students who are eligible to take a seat at $s^2 \in S^2$)

A student $i \in I^d$ is a **voucher user** if $\mu(i) = \emptyset$. The set of voucher users is denoted by $I^d_\emptyset(\mu, \pi)$. For a student $i \in I^d$ with $\mu(i) \in S^1 \cup S^2$ and $\pi(i) = v$, we denote the **unused voucher** of i by $v_i(\pi)$.

Let $G = (V, E)$ be a directed graph with the set of vertices V, and the set of directed edges E, which is a set of ordered pairs of V. For each allocation (μ, π), we define $G(\mu, \pi) = (I \cup \{v_i(\pi) : i \in \pi^{-1}(v) \setminus I^d_\emptyset(\mu, \pi)\}, E(\mu, \pi))$ be the **(directed) application graph** associated with (μ, π) where the set of directed edges $E(\mu, \pi)$ is as follows: $ix \in E(\mu, \pi)$ (that is, *i points to x*) if and only if

- $x \in I$ and, for $k = 1, 2$: $i \in X_{\mu,\pi}(s^k)$ and $s^k = \mu(x)$, or
- $x \in I^d_\emptyset(\mu, \pi) \cup \{v_i(\pi) : i \in \pi^{-1}(v) \setminus I^d_\emptyset(\mu, \pi)\}$ and $i \in I^d$ with $\emptyset \; P'_i \; \mu(i)$

A set of edges in E, $\{i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}\}$, is a **cycle** if the vertices $i_1, i_2 \ldots, i_n$ are distinct and $i_1 = i_{n+1}$. We say that a **cycle** $\phi = \{i_1 i_2, i_2 i_3, \ldots, i_k i_1\} \subseteq E(\mu, \pi)$ **is solved** when for each $ij \in \phi$, student i is assigned to $\mu(j)$ if $\mu(j) \in S^1 \cup S^2$, and to her outside option (by a voucher transfer from j to i) if i is disadvantaged and j is a voucher user, towards a new matching. Formally, we denote the *solution of a cycle* $\phi = \{i_1 i_2, i_2 i_3, \ldots . i_k i_1\}$ by the operation $\circ$; that is, $(\mu', \pi') = \phi \circ (\mu, \pi)$ if and only if

- for each $ij \in \phi$, $\mu(j) \in S^1 \cup S^2$ implies $\mu'(i) = \mu(j)$,
- $i \in I^d$ and $j \in I^d_\emptyset(\mu, \pi)$ imply $\mu'(i) = \emptyset$ and $\pi'(i) = v$, $\pi'(j) = 0$,
- $i' \notin \{i_1, i_2, \ldots, i_k\}$ implies $\mu'(i') = \mu(i')$ and $\pi'(i') = \pi(i')$.

Note that the fact that there can not be an unused voucher in a cycle is due to the way the application graph is constructed: unused vouchers do not point.

A set of ordered pairs $(i_1 i_2, i_2 i_3, \ldots . i_n v_i(\pi))$ is a **chain** if no student points to i_1 and, $\{i_1, \ldots, i_n\} \subseteq I \setminus I^d_\emptyset(\mu, \pi)$ and $i_n \in I^d$. We call i_1 as the **tail** of the chain. For a chain ϕ, the solution of the chain $(\mu', \pi') = \phi \circ (\mu, \pi)$ is the same as the solution of a cycle where $\mu'(i_n) = \emptyset$ and $\pi'(i_n) = v$.

A cycle is **implementable** if there is not a voucher user in the cycle whose priority will be violated if the cycle is solved. A cycle would not be implementable at (μ, π) in two ways. First, if a cycle at (μ, π) is constructed by allowing priority violation of a disadvantaged student who uses a voucher and is present in the cycle to give away her voucher. Second, if the cycle contains a voucher user whose priority is already violated at (μ, π) unless solving the cycle prevents violating the priority of the voucher user.

Let $(I, I^d, R', S^1 \cup S^2, \succ', q', \pi_0)$ be a modified school choice problem and (μ_0, π_0) be the initial allocation given by the modified Deferred Acceptance algorithm. The following algorithm is built on *solving* cycles and chains iteratively in the appropriately defined graph:

The VSERS (Voucher and Seat Exchange under Reserve Seats) Algorithm:[3]

Step 0 Let (μ_0, π_0) be the initial allocation.

Step k Given an allocation (μ_{k-1}, π_{k-1}),

- **(k.1)** if there is no chain or implementable cycle in $G(\mu_{k-1}, \pi_{k-1})$, then the algorithm terminates and (μ_{k-1}, π_{k-1}) is the allocation obtained;
- **(k.2)** otherwise, choose one of the chains or implementable cycles in $G(\mu_{k-1}, \pi_{k-1})$, say ϕ_k, and let $(\mu_k, \pi_k) = \phi_k \circ (\mu_{k-1}, \pi_{k-1})$.

Note that this algorithm is well defined since at each step we have chains or implementable cycles and we implement them or we stop. This algorithm gives a class of mechanisms since we did not specify which cycles or chains among the others will be chosen at each step. This class of mechanisms ends in finite step since at every step except the last, at least one student gets better off and we have finitely many schools and students.

Theorem 1 *An allocation (μ, π) obtained by the VSERS is constrained efficient under reserve seats and vouchers.*

4 Improvement over efficient voucher allocation

It is desirable to make improvement over a mechanism that gives an allocation in which vouchers are efficiently distributed (by taking into account initial voucher endowments) without any priority violation. Achieving this gives further incentives for the usage of such mechanism.

An allocation (μ, π) satisfies **improvement over efficient voucher allocation** if for each student $i \in I^d$ whose priority at some school $s \in S^1$ is violated there does not exist another

[3] The example in Appendix A demonstrates how the algorithm works.

allocation (μ', π') which is *constrained efficient* with respect to usual fairness definition (so that for the original problem the allocation is *constrained efficient* with respect to *fairness with reserve*) such that $\mu'(i)\ P'_i\ \mu(i)$.

In the modified school choice problem, satisfying this property means that if there is a student whose priority is violated at an allocation then this student should not be better off at an allocation in which there is no priority violation. The following example shows the importance of this property.

Example 3 *Let $I = I^d = \{i_1, i_2, i_3, i_4, i_5\}$ and $S = \{s_1, s_2, s_3\}$ with $q_s = r_s = 1$ for each $s \in S$. We have $\pi_0 = ((i_1, 0), (i_2, 0), (i_3, 0), (i_4, v), (i_5, v))$. Since each student is a disadvantaged student and each school has reserve seats only, the modified problem is exactly equivalent to the original problem.*

s_1	s_2	s_3
i_1	i_3	i_2
i_4	i_4	i_4
i_5	i_2	i_3
.	.	.
.	.	.
.	.	.

i_1	i_2	i_3	i_4	i_5
$\emptyset$	s_2	s_3	s_1	s_1
s_1	s_3	s_2	s_2	$\emptyset$
.	$\emptyset$	s_1	s_3	.
.	.	.	$\emptyset$	.

At Step 0 we get the result of Deferred Acceptance *as* $\mu_0 = ((i_1, s_1), (i_2, s_3), (i_3, s_2), (i_4, \emptyset), (i_5, \emptyset))$ *and* π_0 *as allocation.*

$\Rightarrow D_{\mu_0,\pi_0}(s_1) = \{i_4, i_5\}, D_{\mu_0,\pi_0}(s_2) = \{i_2, i_4\}, D_{\mu_0,\pi_0}(s_3) = \{i_3, i_4\}$ *and* $C_{\mu_0,\pi_0}(s) = \{i_4, i_5\}$ *for each* $s \in S$.

$\Rightarrow X_{\mu_0,\pi_0}(s_1) = \{i_4, i_5\}, X_{\mu_0,\pi_0}(s_2) = \{i_2, i_4\}, X_{\mu_0,\pi_0}(s_3) = \{i_3, i_4\}$.

$\Rightarrow i_1$ *points to* i_4 *and* i_5 *for their vouchers.*

We have 3 implementable cycles in $G(\mu_0, \pi_0)$ *to apply:*

1) $(i_2 i_3, i_3 i_2)$,

2) $(i_4 i_1, i_1 i_4)$,

3) $(i_1 i_5, i_5 i_1)$

Suppose we applied the 3rd cycle first. Then we get $\mu_1 = ((i_2, s_3), (s_2, i_3), (i_4, \emptyset), (i_1, \emptyset), (i_5, s_1))$ *and*

$\pi_1 = ((i_1, v), (i_2, 0), (i_3, 0), (i_4, v), (i_5, 0))$.

$\Rightarrow D_{\mu_1,\pi_1}(s_1) = \{i_4\}, D_{\mu_1,\pi_1}(s_2) = \{i_2, i_4\}, D_{\mu_1,\pi_1}(s_3) = \{i_3, i_4\}$ *and* $C_{\mu_1,\pi_1}(s) = \{i_1, i_4\}$ *for each* $s \in S$.

$\Rightarrow X_{\mu_1,\pi_1}(s_1) = \{i_4\}, X_{\mu_1,\pi_1}(s_2) = \{i_2, i_4\}, X_{\mu_1,\pi_1}(s_3) = \{i_3, i_4\}$.

$\Rightarrow$ *No one points to voucher users.*

We have only one cycle in $G(\mu_1, \pi_1)$ *to apply:* $(i_2 i_3, i_3 i_2)$.

We apply this cycle and get $\mu_2 = ((i_1, \emptyset), (i_2, s_2), (i_3, s_3), (i_4, \emptyset), (i_5, s_1))$ *with* $\pi_2 = \pi_1$.

$\Rightarrow$ *This is the final allocation we get since there are no cycles or chains.*

Now suppose we do not let priority violation, we apply the Deferred Acceptance *and get* $\mu'_0 = ((i_1, s_1), (i_2, s_3), (i_3, s_2), (i_4, \emptyset), (i_5, \emptyset))$ *and* π_0 *as allocation at Step 0 same as previously.*

$\Rightarrow D_{\mu_0,\pi_0}(s_1) = \{i_4, i_5\}, D_{\mu_0,\pi_0}(s_2) = \{i_2, i_4\}, D_{\mu_0,\pi_0}(s_3) = \{i_3, i_4\}$.

$\Rightarrow X_{\mu_0,\pi_0}(s_1) = \{i_4\}, X_{\mu_0,\pi_0}(s_2) = \{i_4\}, X_{\mu_0,\pi_0}(s_3) = \{i_4\}$.

$\Rightarrow$ i_1 *points to* i_4 *and* i_5 *for their vouchers.*

We have only 1 cycle to apply in $G(\mu'_0, \pi_0)$*:* $(i_4 i_1, i_1 i_4)$.

We apply this cycle and get $\mu'_1 = ((i_2, s_3), (s_2, i_3), (i_4, s_1), (i_1, \emptyset), (i_5, \emptyset))$ *and*

$\pi'_1 = ((i_1, v), (i_2, 0), (i_3, 0), (i_4, 0), (i_5, v))$.

$\Rightarrow D_{\mu'_1,\pi'_1}(s_1) = \{i_5\}, D_{\mu_1,\pi_1}(s_2) = \{i_2\}, D_{\mu_1,\pi_1}(s_3) = \{i_3\}$.

$\Rightarrow X_{\mu'_1,\pi'_1}(s_1) = \{i_5\}, X_{\mu'_1,\pi'_1}(s_2) = \{i_2\}, X_{\mu'_1,\pi'_1}(s_3) = \{i_3\}$.

$\Rightarrow$ *No one points to voucher users.*

We have only 1 cycle in $G(\mu'_1, \pi'_1)$ *to apply:* $(i_2 i_3, i_3 i_2)$.

We get $\mu'_2 = ((i_1, \emptyset), (i_2, s_2), (i_3, s_3), (i_4, s_1), (i_5, \emptyset))$ *and* $\pi'_2 = \pi'_1$.

$\Rightarrow$ *This is the final allocation we get since there are no cycles or chains.*

When we did not let priorities of students to be violated, i_4 whose priority was violated in the first final allocation gets a better match in the second final allocation. Thus, if we use a cycle selection rule resulting as in the first case then this would be worse for some students whose priorities are violated compared to a matching got by just efficiently allocating vouchers. So a rule that results in the first final allocation may not be giving enough incentive people to let their priorities be violated. Furthermore, in the second case inefficiency caused by priority of i_4 is solved by just efficiently allocating vouchers without using priority violation. So if we use a cycle selection rule as in the first case i_4 may have a right to say that the inefficiency was not due to her priority but to inefficient allocation of vouchers which will hurt our justification to violate priorities.

We will try to find a subclass in VSERS that will satisfy such a property. A candidate is a mechanism in which at every step if we can find cycles or chains in which no priority violation occurs we apply them, if not we can apply implementable cycles or chains at every step until no implementable cycle or chain remains. Such a mechanism may have close relationship with the Top Priority Rule.

5 Incentives to apply vouchers

We say a disadvantaged student *has incentive to apply to voucher* if by not applying to voucher he does not get a better match.

In this example we will show that by not applying to vouchers a disadvantaged student can get a better match if she is not given a voucher during the mechanism at any step. If instead we are allowed to give the student voucher at some step then this example would not work.

Example 4 *Let all the students be in the set* $I^d = \{i_1, i_2, i_3, i_4\}$ *and* $S = \{s_1, s_2\}$ *be the set of state schools such that* $q_s = r_s = 1$ *for* $s = s_1, s_2$ *which means all seats are reserve seats in both* *schools. And suppose that* $\pi_0(i) = v$ *for* $i = i_1, i_3, i_4$ *and* $\pi_0(i_2) = 0$.

s_1	s_2
i_2	i_1
i_4	i_3
i_1	i_2
i_3	i_4

i_1	i_2	i_3	i_4
s_1	s_2	s_2	s_1
$\emptyset$	s_1	$\emptyset$	$\emptyset$
s_2	$\emptyset$	s_1	s_2

Since s_1 *and* s_2 *has only reserve seats and we have only disadvantaged students in this example the modified problem will be exactly the same as the original problem.*

When we apply D.A. step of our mechanism we get $\mu_0 = ((i_1, \emptyset), (i_2, s_1), (i_3, s_2), (i_4, \emptyset))$

$\Rightarrow D_{\mu_0,\pi_0}(s_1) = \{i_1, i_4\}$, $D_{\mu_0,\pi_0}(s_2) = \{i_2\}$ *and* $C_{\mu_0,\pi_0}(s) = \{i_1, i_4\}$ *for all* $s \in S$.

$\Rightarrow X_{\mu_0,\pi_0}(s_1) = \{i_1, i_4\}$, $X_{\mu_0,\pi_0}(s_2) = \{i_2\}$.

The mechanism terminates since there is no cycle or chain in $G(\mu_0, \pi_0)$. *We get* (μ_0, π_0).

Now, instead suppose that i_1 *did not apply to voucher and did not have one and the other initial voucher assignments are the same, then when we apply the D.A. step we will get:*

$\mu_0 = ((i_3, \emptyset), (i_4, \emptyset), (i_1, s_2), (i_2, s_1))$ *and* $\pi_0 = ((i_1, 0), (i_2, 0), (i_3, v), (i_4, v))$

$\Rightarrow D_{\mu_0,\pi_0}(s_1) = \{i_1, i_4\}$, $D_{\mu_0,\pi_0}(s_2) = \{i_2\}$ *and* $C_{\mu_0,\pi_0}(s) = \{i_3, i_4\}$ *for all* $s \in S$.

$\Rightarrow X_{\mu_0,\pi_0}(s_1) = \{i_1, i_4\}$, $X_{\mu_0,\pi_0}(s_2) = \{i_2\}$.

Here, i_1 *does not point to* i_3 *for her voucher although she likes her outside option more than* s_2 *because at first she did not apply to vouchers so we think that we can not give her voucher at any step.*

Then we apply the only cycle $(i_1 i_2, i_2 i_1)$ $G(\mu_0, \pi_0)$ *and get* $\mu_1 = ((i_3, \emptyset), (i_4, \emptyset), (i_1, s_1), (i_2, s_2))$ *with* $\pi_1 = \pi_0$.

$\Rightarrow i_1$ *is better off when he does not apply to voucher.*

If we assumed that we can give voucher to a disadvantaged student, who did not apply to vouchers, in a step other than 0 then we would have another cycle to carry at the end of the step 0 which is $(i_1 i_3, i_3 i_1)$. *This example would not work for a cycle selection rule that selects* $(i_1 i_3, i_3 i_1)$.

Note: We assumed that every disadvantaged student will apply to vouchers but this example shows that there are cases in which a disadvantaged student may benefit from not applying to vouchers. Therefore applying to vouchers can be treated as an endogenous decision which we leave for future work.

In general we show that a disadvantaged student may gain from strategically not applying to vouchers if the mechanism at hand is restricted to find constrained efficient results.

Theorem 2 *Constrained efficiency and having an incentive to apply to vouchers are incompatible.*

Proof.

Let all the students be in the set $I^d = \{i_1, i_2, i_3\}$ and $S = \{s_1, s_2\}$ be the set of state schools such that $q_s = r_s = 1$ for $s = s_1, s_2$ which means all seats are reserve seats in both schools. And suppose that $\pi_0(i) = v$ for $i = i_1, i_3$ and $\pi_0(i_2) = 0$.

s_1	s_2
i_2	i_1
i_1	i_3
i_3	i_2

i_1	i_2	i_3
s_1	s_2	s_2
$\emptyset$	s_1	$\emptyset$
s_2	$\emptyset$	s_1

Notice that there are 3 constrained efficient allocations in this example:

1)$\mu = ((i_1, \emptyset), (i_2, s_1), (i_3, s_2))$ and $\pi = ((i_1, v), (i_2, 0), (i_3, v))$

2)$\mu = ((i_1, s_1), (i_2, s_2), (i_3, \emptyset))$ and $\pi = ((i_1, v), (i_2, 0), (i_3, v))$

3)$\mu = ((i_1, s_1), (i_2, \emptyset), (i_3, s_2))$ and $\pi = ((i_1, 0), (i_2, v), (i_3, v))$ or $\pi = ((i_1, v), (i_2, v), (i_3, 0))$.

If the 1st one is chosen by a mechanism then i_1 will be better off by not applying to vouchers. We will have 2 possible constrained efficient allocations and the mechanism will have to choose one of these in which i_1 is better off:

1)$\mu = ((i_1, s_1), (i_2, s_2), (i_3, \emptyset))$ and $\pi = ((i_1, 0), (i_2, 0), (i_3, v))$

2)$\mu = ((i_1, s_1), (i_2, \emptyset), (i_3, s_2))$ and $\pi = ((i_1, 0), (i_2, v), (i_3, 0))$

If the 2nd one is chosen by a mechanism then i_3 will be better off by not applying to vouchers. We will have 2 possible constrained efficient allocations and the mechanism will have to choose one of these in which i_3 is better off:

1)$\mu = ((i_1, \emptyset), (i_2, s_1), (i_3, s_2))$ and $\pi = ((i_1, v), (i_2, 0), (i_3, 0))$

2)$\mu = ((i_1, s_1), (i_2, \emptyset), (i_3, s_2))$ and $\pi = ((i_1, 0), (i_2, v), (i_3, 0))$

If the 3rd one is chosen by a mechanism then i_2 will be better off by not applying to vouchers. We will have 2 possible constrained efficient allocations and the mechanism will have to choose one of these in which i_3 is better off:

1)$\mu = ((i_1, \emptyset), (i_2, s_1), (i_3, s_2))$ and $\pi = ((i_1, v), (i_2, 0), (i_3, v))$

2)$\mu = ((i_1, s_1), (i_2, s_2), (i_3, \emptyset))$ and $\pi = ((i_1, v), (i_2, 0), (i_3, v))$

■

Notice that the first allocation among the three constrained efficient allocations is the unique outcome of the VSERS mechanism and when i_1 does not apply to vouchers VSERS yields uniquely the second allocation. Here i_1 gains from manipulation not because of preventing his priority to be violated but by first getting into a less desirable school than his outside option which will enable him to get into his most desired school thorough a mutual trade which becomes possible by violation of the priority of another student in the course of the mechanism.

6 An Alternative Fairness Definition

In this section we present an alternative fairness definition and compare it to the one we presented in the text (we combine the definition in the text with respecting reserves property).

The alternative definition is as follows:

An allocation (μ, π) is **fair with reserves and vouchers** if the following are satisfied:

(i) for each $i, j \in I$ and $s \in S$ such that $s \; P_i \; \mu(i)$, $i \succ_s j$ and $\mu(j) = s$ we have either

1)$j \in I^d$, $i \in I \setminus I^d$ and $|\mu^{-1}(s) \cap I^d| \leq r_s$ or

2)$i, j \in I^d$, $\mu(i) = \emptyset$ and $|\mu^{-1}(s) \cap I^d| \leq r_s$

(ii) there does not exist $i, j \in I$ and $s \in S$ such that $i \in I^d$, $j \in I \setminus I^d$, $s \; P_i \; \mu(i)$, $\mu(j) = s$ and $|\mu^{-1}(s) \cap I^d| < r_s$ unless we have $\mu(i) = \emptyset$ and $j \succ_s i$.

If we use this definition it is possible to see a disadvantaged student whose priority at a reserve seat is violated by a non-disadvantaged student although reserve seats are not exhausted at that school. Even though in a subclass of our mechanism class we will try to give precedence to students whose priorities are violated there will be situations in which the disadvantaged student with voucher will not be able to get into a school she desires in which she has the highest priority to get into if any space occurs (whichever fairness definition we use), in this case the definition in the text will keep the reserve seats exhausted at the school by allowing a priority violation by another disadvantaged or by not allowing a priority violation (priority from reserves only) if there is only a non-disadvantaged student who wants to trade mutually with a disadvantaged student at the school. The definition we have just written allows the priority of the voucher user to be violated in both cases and resulting in exhausted reserves in the first case and non-exhausted reserves in the second case. In both cases we know that the voucher user will not be able to get into the school she desires (we can say this for the subclass of our mechanism class we try to find but not in general) but we differ in treating disadvantaged and non-disadvantaged in terms of allowing priority violation if we use the definition in the text. However, the definition in the text prevents having a situation in which the priority of a voucher user is violated at some school because of his voucher and there are non-exhausted reserves at that school. Although as mechanism designer we know that even if he did not get a voucher the student would not be able to get the seat he desires it might be hard to understand from the point of view of the parents and students.

If we can not find the desired subclass then uniformity is still an issue. There may be situations in which we violate the priority of a voucher user and not being able to guarantee that he would get at best his outside option if he did not allow priority violation. In such a case if we allow the priority of voucher user to be violated by a non-disadvantaged student the situation will seem worse because the voucher user will see that reserve seats are not exhausted and at the same time we will not be able to say that he would not be getting into this school in any case.

7 Conclusion

In this paper we improve the efficiency aspect of school assignments thorough two channels: We allow for voucher transfers and we allow for the priority violation at reserve seats for the case of a voucher user. Our mechanism VSERS makes every student weakly better off compared to the case in which we do not allow such priority violations and voucher transfers. It distributes the seats and vouchers a way that the resulting outcome is constrained efficient in the class of allocations that are stable under reserve seats and vouchers.

Another result we found is that if we require constrained efficiency then a disadvantaged student may gain from not applying to vouchers. Hence, if we use VSERS a disadvantaged student can also gain from not applying to vouchers. This seems counter-intuitive since vouchers enlarge the choice sets of disadvantaged students but the reason behind the manipulability of VSERS is the following:During the course of the mechanism a disadvantaged student may have a better trade opportunity at a state school that he prefers less than his outside option.

Lastly, in future work we aim to find a subclass of VSERS which guarantees a voucher user, whose priority is violated, that she will not be able get a better school if she did not allow her priority to be violated. Apparently, students will have further incentives to use such a mechanism.

Appendix A The VSERS algorithm: An illustrative example

Example 5 *Let $S = \{s_1, s_2, s_3, s_4\}$, $I = I^d = \{i_1, i_2, i_3, i_4, i_5, i_6\}$, $\pi_0^{-1}(v) = \{i_1, i_3\}$ and $q_s = r_s = 1$ for all $s \in S$. Since all students are disadvantaged and all seats of schools are reserve seats the modified school choice problem is exactly equivalent to the original problem.*

s_1	s_2	s_3	s_4
i_5	i_2	i_6	i_4
i_1	i_5	i_3	i_6
i_3	.	i_1	.
i_2	.	i_4	.
.	.	.	.
.	.	.	.

i_1	i_2	i_3	i_4	i_5	i_6
s_3	$\emptyset$	s_1	$\emptyset$	s_2	s_4
s_1	s_1	s_3	s_3	s_1	s_3
$\emptyset$	s_2	$\emptyset$	s_4	.	.
.	.	.	.	.	.
.	.	.	.	.	.

At Step 0 we get this allocation as the result of Modified Deferred Acceptance*:*

$\mu_0 = ((i_1, \emptyset), (i_2, s_2), (i_3, \emptyset), (i_4, s_4), (i_5, s_1), (i_6, s_3))$ *and* π_0

$\Rightarrow D_{\mu_0,\pi_0}(s_1) = \{i_1, i_2, i_3\}, D_{\mu_0,\pi_0}(s_2) = \{i_5\}, D_{\mu_0,\pi_0}(s_3) = \{i_1, i_3, i_4\}$ *and* $D_{\mu_0,\pi_0}(s_4) = \{i_6\}$ $C_{\mu_0,\pi_0}(s) = \{i_1, i_3\}$ *for each* $s \in S$.

$\Rightarrow X_{\mu_0,\pi_0}(s_1) = \{i_1, i_2, i_3\}, X_{\mu_0,\pi_0}(s_2) = \{i_5\}, X_{\mu_0,\pi_0}(s_3) = \{i_1, i_3, i_4\}$ *and* $X_{\mu_0,\pi_0}(s_4) = \{i_6\}$.

$\Rightarrow$ i_2 *and* i_4 *point to* i_1 *and* i_3 *for their vouchers.*

We have 8 cycles in $G(\mu_0, \pi_0)$ and all of them are implementable:

1) (i_5i_2, i_2i_5), *2)* (i_4i_6, i_6i_4), *3)* $(i_4i_1, i_1i_6, i_6, i_4)$, *4)* (i_3i_6, i_6i_4, i_4i_3), *5)* (i_2i_1, i_1i_5, i_5i_2), *6)* (i_2i_3, i_3i_5, i_5i_2) *7)* $(i_6i_4, i_4i_3, i_3i_5, i_5i_2, i_2i_1, i_1i_6)$, *8)* $(i_6i_4, i_4i_1, i_1i_5, i_5i_2, i_2i_3, i_3i_6)$

Suppose we applied the 1st one:

$\Rightarrow \mu_1 = ((i_1, \emptyset), (i_2, s_1), (i_3, \emptyset), (i_4, s_4), (i_5, s_2), (i_6, s_3))$ *and* $\pi_1 = \pi_0$

$\Rightarrow D_{\mu_1,\pi_1}(s_1) = \{i_1, i_3\}, D_{\mu_1,\pi_1}(s_2) = \{\emptyset\}, D_{\mu_1,\pi_1}(s_3) = \{i_1, i_3, i_4\}$ *and* $D_{\mu_1,\pi_1}(s_4) = \{i_6\}$ $C_{\mu_1,\pi_1}(s) = \{i_1, i_3\}$ *for each* $s \in S$.

$\Rightarrow X_{\mu_1,\pi_1}(s_1) = \{i_1, i_3\}, X_{\mu_1,\pi_1}(s_2) = \{\emptyset\}, X_{\mu_1,\pi_1}(s_3) = \{i_1, i_3, i_4\}$ *and* $X_{\mu_1,\pi_1}(s_4) = \{i_6\}$.

$\Rightarrow$ *i_2 and i_4 point to i_1 and i_3 for their vouchers.*

In $G(\mu_1, \pi_1)$ we have 7 cycles and all of them are implementable except the 4th one:

1) (i_1i_2, i_2i_1), *2)* (i_2i_3, i_3i_2), *3)* (i_4i_6, i_6i_4), *4)* (i_4i_3, i_3i_6, i_6i_4), *5)* (i_4i_1, i_1i_6, i_6i_4), *6)* $(i_4i_3, i_3i_2, i_2i_1, i_1i_6, i_6i_4)$, *7)* $(i_4i_1, i_1i_2, i_2i_3, i_3, i_6, i_6i_4)$

Suppose we applied the 3rd cycle:

$\Rightarrow \mu_2 = ((i_1, \emptyset), (i_2, s_1), (i_3, \emptyset), (i_4, s_3), (i_5, s_2), (i_6, s_4))$ *and* $\pi_2 = \pi_0$

$\Rightarrow D_{\mu_2,\pi_2}(s_1) = \{i_1, i_3\}, D_{\mu_2,\pi_2}(s_2) = \{\emptyset\}, D_{\mu_2,\pi_2}(s_3) = \{i_1, i_3\}$ *and* $D_{\mu_2,\pi_2}(s_4) = \{\emptyset\}$ $C_{\mu_2,\pi_2}(s) = \{i_1, i_3\}$ *for each* $s \in S$.

$\Rightarrow X_{\mu_2,\pi_2}(s_1) = \{i_1, i_3\}, X_{\mu_2,\pi_2}(s_2) = \{\emptyset\}, X_{\mu_2,\pi_2}(s_3) = \{i_1, i_3\}$ *and* $X_{\mu_2,\pi_2}(s_4) = \{\emptyset\}$.

$\Rightarrow$ *i_2 and i_4 point to i_1 and i_3 for their vouchers.*

In $G(\mu_2, \pi_2)$ we have 5 cycles and all of them are implementable except the first two:1) (i_1i_2, i_2i_1), *2)* (i_4i_3, i_3i_4), *3)* (i_2i_3, i_3i_2), *4)* (i_4i_1, i_1i_4), *5)* $(i_4i_3, i_3i_2, i_2i_1, i_1i_4)$

Applying 5 or 3 and 4 consecutively in two steps we will get the final allocation as:

$\mu = ((i_1, s_3), (i_2, \emptyset), (i_3, s_1), (i_4, \emptyset), (i_5, s_2), (i_6, s_4))$ *and* $\pi = ((i_1, 0), (i_2, v), (i_3, 0), (i_4, v), (i_5, 0), (i_6, 0))$

Notice that being unable to implement a cycle does not bring an efficiency loss in this example.

Appendix B The proof of Theorem 1

Remark 2 *At every step of the mechanism every student gets weakly better off.*

Remark 3 *For all $s \in S^1 \cup S^2$ we have $D_{\mu_t,\pi_t}(s) \subseteq D_{\mu_{t-1},\pi_{t-1}}(s)$*

Let (μ, π) be an allocation obtained by the VSERS.

Lemma 1 *(μ, π) is individually rational.*

Proof. We show that by D.A. at step 0 we get results that are individually rational. Each student $i \in I \setminus I^d$ or $i \in I^d$ with $\pi_0(i) = v$ can apply to only schools that they prefer to $\emptyset$ so they can not get a school that they do not weakly prefer to $\emptyset$. So for each $i \in I$ such that

$\emptyset \ P'_i \ \mu_0(i)$ we have $i \in I^d$ and $\pi_0(i) = 0$. Hence we have individual rationality at the end of step 0. Since at the following steps every agent gets weakly better off the students who can afford their outside option will get schools that they weakly prefer to their outside option. Also, at any step if a student $i \in I^d \setminus \pi_0^{-1}(v)$, gets a voucher then i will get into $\emptyset$ at that step and will get a weakly better outcome in the following steps. So at any step k of the mechanism if there is a student i with $\emptyset \ P'_i \ \mu_0(i)$ then we have $i \in I^d \setminus \pi_0^{-1}(v)$ and $\pi_k(i) = 0$. Thus our mechanism gives results that are individually rational. ■

Lemma 2 (μ, π) *is non-wasteful.*

Proof. Let $(\mu_0, \pi_0), (\mu_1, \pi_1), (\mu_2, \pi_2), ..., (\mu_k, \pi_k), ...(\mu, \pi)$ be the results we get in the application of the mechanism. At the end of step 0 we get a non-wasteful outcome by Deferred Acceptance. Suppose that (μ_{k-1}, π_{k-1}) is non-wasteful as an inductive hypothesis. At every step we apply only a chain or only a cycle.

Case 1: Suppose we applied a cycle at step k and get (μ_k, π_k). When we apply the cycle number of students in any school do not change. If we have $|\mu_{k-1}^{-1}(s)| = q_s$ for all s then we have non-wastefulness. Suppose $|\mu_{k-1}^{-1}(s)| < q_s$ for some s. Then since (μ_{k-1}, π_{k-1}) is non-wasteful we have $D_{\mu_{k-1}, \pi_{k-1}}(s) = \emptyset$. That means $D_{\mu_k, \pi_k}(s) = \emptyset$ by Remark 2. So (μ_k, π_k) is non-wasteful.

Case 2: Suppose we applied a chain at step k and get (μ_k, π_k). By definition of a chain either a student $i \in I$ with $\mu_{k-1}(i) = s$ for some $s \in S^1 \cup S^2$ can be the tail of the chain or a student $i \in I \setminus I^d$ with $\mu_{k-1}(i) = \emptyset$ can be the tail of the chain. If the second is true then the number of students in any school do not change and proof becomes same with Case 1. If the first is true then number of students in s decreases by one and it will have unfilled capacity at (μ_k, π_k), but by definition of the chain $D_{\mu_{k-1}, \pi_{k-1}}(s) = \emptyset$. That means $D_{\mu_k, \pi_k}(s) = \emptyset$ by Remark 2. For other schools number of students assigned to them do not change so by the argument in Case 1 we can show that (μ_k, π_k) is non-wasteful. So by induction we proved that the outcome we get at every step, hence at the end of the mechanism is non-wasteful. ■

Lemma 3 (μ, π) *is fair under reserve seats and vouchers.*

Proof. We prove by induction. By Remark 1, (μ_0, π_0) is fair under reserve seats. Since fairness under reserve seats and vouchers is a weaker property, the DA satisfies it as well. For inductive hypothesis we assume that (μ_{k-1}, π_{k-1}) is fair. Suppose that there exist $i, j \in I$ and $s \in S^1 \cup S^2$ such that $s\ P'_i\ \mu_k(i)$, $\mu_k(j) = s$ and $i \succ'_s j$. Since $s\ P'_i\ \mu_k(i)$ we have $i \in D_{\mu_k,\pi_k}(s)$. Thus we have $i \in D_{\mu_{k-1},\pi_{k-1}}(s)$ since everyone gets weakly better off at every step. Suppose that:

Case 1: $\mu_{k-1}(j) = s$. Then, since (μ_{k-1}, π_{k-1}) is fair we must have *(i)*$i \in I^d$, $s \in S^1$ *(ii)* $\mu_{k-1}(i) = \emptyset$ Now we must have $\mu_k(i) = \emptyset$ otherwise that means a cycle in which i gives away his voucher is implemented although j still violates the priority of i after that cycle is implemented. But such a cycle is unimplementable. It contradicts to the fact that we get (μ_k, π_k) by our mechanism. Hence we have $\mu_k(i) = \emptyset$. Therefore (μ_k, π_k) is fair.

Case 2: $\mu_{k-1}(j) \neq s$. Since $\mu_k(j) = s$ we have $j \in X_{\mu_{k-1},\pi_{k-1}}(s)$. Therefore we have *(i)* $j \succ'_s i'$ for all $i' \in D_{\mu_{k-1},\pi_{k-1}}(s) \setminus j$, if $s \in S^2$ or $s \in S^1$ and $j \in I \setminus I^d$; *(ii)* $j \succ'_s i'$ for all $i' \in D_{\mu_{k-1},\pi_{k-1}}(s) \setminus (C_{\mu_{k-1},\pi_{k-1}}(s) \cup j)$, if $s \in S^1$ and $j \in I^d$; We can not have $s \in S^2$ or $s \in S^1$ and $j \in I \setminus I^d$ since otherwise we will have $j \succ'_s i$ which is a contradiction. So we have $s \in S^1$ and $j \in I^d$. Since we have $i \succ'_s j$ we must have $i \in C_{\mu_{k-1},\pi_{k-1}}(s)$. Thus $i \in I^d$ and $\mu_{k-1}(i) = \emptyset$. Now we must have $\mu_k(i) = \emptyset$ otherwise that means a cycle in which i gives away his voucher is implemented although j violates the priority of i after that cycle is implemented. But such a cycle is unimplementable. It contradicts to the fact that we get (μ_k, π_k) by our mechanism. Thus, we have $\mu_k(i) = \emptyset$. Thus, (μ_k, π_k) is fair. So by induction we proved that our mechanism gives fair results. ■

Suppose there exists an allocation $(\widetilde{\mu}, \widetilde{\pi})$ which is *stable under reserve seats and vouchers* and *weakly Pareto dominates* (μ, π). Let I' be the set of students who are better off at $\widetilde{\mu}$ than at μ.

Lemma 4 *If there exists an unused voucher at (μ, π), then each such voucher remains unused at $(\widetilde{\mu}, \widetilde{\pi})$.*

Proof. Suppose there exists an unused voucher $v_j(\pi)$ (for some $j \in I^d$) at (μ, π) and a student $i \in I^d$ such that $\mu(i) \neq \emptyset$ and $\widetilde{\mu}(i) = \emptyset$. Since $(\widetilde{\mu}, \widetilde{\pi})$ *weakly Pareto dominates* (μ, π), we have

$\emptyset\ P_i'\ \mu(i)$. But then, in the application graph $G(\mu,\pi)$, student i points to that unused voucher. Note that if a student is better off by using a voucher, the constraints due to *stability under reserve seats and vouchers* weakly shrink. Thus, since the allocation (μ,π) satisfies *stability under reserve seats and vouchers*, the allocation, where the only difference from (μ,π) is that student i is a voucher user instead of being assigned to a public school $\mu(i)$, satisfies *stability under reserve seats and vouchers* as well. Thus, $(iv_j(\pi))$ is a chain, which contradicts that the allocation (μ,π) is an outcome obtained by the VSERS. ■

Lemma 5 *If a voucher is used at (μ,π), then it is used at $(\widetilde{\mu},\widetilde{\pi})$ as well.*

Proof. Suppose a voucher is used at (μ,π), but not at $(\widetilde{\mu},\widetilde{\pi})$. By Lemma 4, this implies that there exists a school s such that $|\widetilde{\mu}^{-1}(s)| > |\mu^{-1}(s)|$. Thus, school s has an empty seat at μ. Moreover, each student is weakly better off at $\widetilde{\mu}$ than at μ. Thus, a student prefers school s to her assigned school at μ and μ has an empty seat, which contradicts non-wastefulness of (μ,π).

■

Lemma 6 *If $I' \neq \emptyset$, then there exists an implementable cycle in the graph $G(\mu,\pi)$.*

Proof. Let $i \in I'$ be a student assigned to a public school under $\widetilde{\mu}$. There exists at least one student who prefers school $\widetilde{\mu}(i)$ to her current assignment (e.g. i is such a student, since by definition of the set I', $\widetilde{\mu}(i)\ P_i\ \mu(i)$). We claim that $\mu^{-1}(\widetilde{\mu}(i)) \cap I' \neq \emptyset$. Suppose not. Then, each student in $\mu^{-1}(\widetilde{\mu}(i))$ remains at school $\widetilde{\mu}(i)$ under $\widetilde{\mu}$. Moreover, student i is placed at school $\widetilde{\mu}(i)$ under $\widetilde{\mu}$. This implies that there is an empty seat at school $\widetilde{\mu}(i)$ under μ and student i prefers school $\widetilde{\mu}(i)$ to $\mu(i)$, which contradicts with non-wastefulness. Let $j \in \mu^{-1}(\widetilde{\mu}(i)) \cap I'$. We claim that in the graph $G(\mu,\pi)$, a student in I' points to j. First, since there is at least one student who prefers school $\widetilde{\mu}(i)$ to her current school at μ, the set $X_{\mu,\pi}(\widetilde{\mu}(i))$ is non-empty. Thus, there exists at least one student who points to j in the graph $G(\mu,\pi)$. Since the allocation $(\widetilde{\mu},\widetilde{\pi})$ is stable under reserve seats and vouchers, by definition, student i being assigned to school $\widetilde{\mu}(i)$ does not violate any student's priority at that school. Since each student in $I \setminus I'$ is assigned to the same school both under μ and $\widetilde{\mu}$, no student in this set can have a higher priority than student i at school $\widetilde{\mu}(i)$. Thus, the set $X_{\mu,\pi}(\widetilde{\mu}(i))$ cannot include any student in the set $I \setminus I'$. Thus, student j is pointed by a student in I'. Let $i \in I'$ be a student assigned to a private school under $\widetilde{\mu}$. Since i is better off at $\widetilde{\mu}$, she must be a disadvantaged student, since otherwise, individual rationality is violated at (μ, π). Thus, i points to a voucher at μ. By Lemma 4 and 5, this voucher cannot be an unused voucher. Thus, each student in I' is pointed by another student in the same set. Thus, in the graph $G(\mu, \pi)$, there exists a cycle containing students from the set I'. If this cycle contains no student with a voucher at μ, then by definition, this is an implementable cycle and this contradicts that (μ, π) is the outcome of the VERS algorithm, since the graph $G(\mu, \pi)$ contains an implementable cycle and the algorithm cannot terminate. We claim that if the cycle contains a student with a voucher and is not implementable, then there is another cycle containing the same student with a voucher and is implementable. This follows simply by choosing the top priority student in I' in constructing the cycles in I' and treating a disadvantaged student at a private school such that her reserve seat priorities at a public school cannot be violated by another disadvantaged student. By the same argument above, a student at a public school cannot be pointed by a student in $I \setminus I'$. Thus, an implementable cycle forms. $\blacksquare$

By Lemma 6, the algorithm does not terminate at (μ, π), which is a contradiction. Thus, (μ, π) is constrained efficient.

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