Inventory Management, Pricing and Risk Hedging in the Presence of Price Fluctuations

by

Caner Canyakmaz

A Dissertation Submitted to the
Graduate School of Sciences and Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Doctor of Philosophy

in

Industrial Engineering and Operations Management

KOÇ UNIVERSITY

July, 2017

Inventory Management, Pricing and Risk Hedging in the Presence of Price Fluctuations

Koç University

Graduate School of Sciences and Engineering

This is to certify that I have examined this copy of a doctoral dissertation by

Caner Canyakmaz

and have found that it is complete and satisfactory in all respects,

and that any and all revisions required by the final

examining committee have been made.

Committee Members:

____

Prof. Süleyman Özekici

____

Asst. Prof. Uğur Çelikyurt

____

Asst. Prof. Pelin Gülşah Canbolat

____

Prof. Refik Güllü

____

Assoc. Prof. Semih Onur Sezer

Date: ____________________

Dedicated to my grandfather...

ABSTRACT

Price uncertainties are among the most critical challenges that retailers and manufacturers have to face. For instance, companies whose operations require procuring from commodity markets are exposed to commodity price fluctuations which experience sharp movements frequently. Besides random nature of customer demand, due to this input and/or selling price volatility, there might be considerable variability in firms' profits. It is vital for these firms to consider price fluctuations in adjusting inventory control and pricing policies, and take a variety of risk management measures. In this dissertation, we consider such a firm where continuous price changes during the planning horizon affect both unit payoff from sales as well as customer arrivals. In a multi-period setting, we first investigate optimal price-dependent inventory control policies and numerically illustrate how continuous price fluctuations affect optimal controls and resulting payoffs. Then, we analyze optimal pricing policies assuming that selling prices are determined both by market-driven random prices and firm's markup decision. We show that level of price variability has a negative effect on firms' final profits. Finally, in a minimum-variance framework, we explore financial hedging strategies of the risk-sensitive firm. We assume inherent price dynamics of the inventory item is correlated with prices of various products which are freely traded in financial markets. This presents an opportunity for the firm to invest in a financial portfolio of these products to manage its exposure to price and demand uncertainties by observing the current inventory, wealth and price levels. In this environment, we explicitly characterize dynamic variance-minimizing investment decisions of the firm using dynamic programming. We then explore the risk reduction effects of minimum-variance financial hedges through numerical examples and show that significant risk reductions may be possible by using the right hedge.

ÖZETÇE

Üretici ve perakendecilerin önlem alması gereken en kritik zorluklardan biri fiyat belirsizlikleridir. Örneğin operasyonları emtia piyasalarından alım yapılmasını gerektiren şirketler, sürekli değişen emtia fiyatlarına maruz kalmaktadırlar. Talepteki belirsizliklerin dışında alış/satış fiyatlarındaki belirsizlikler, şirketlerin nakit akışlarında önemli derecede değişkenliğe neden olmaktadır. Bu nedenle şirketler fiyatlardaki oynaklıkları göz önüne alan envanter ve fiyatlama politikaları geliştirmeli ve riskini kontrol edebileceği adımlar atmalıdırlar. Bu çalışmada, planlama dönemindeki sürekli fiyat oynaklıkları yüzünden hem birim satış getirisi hem de müşteri geliş zamanları etkilenen bir şirketin endüstriyel ve finansal operasyonları ele alınmaktadır. İlk kısımda, çeşitli durumlarda en iyi çoklu-dönem envanter kontrolü politikaları analiz edilip sayısal örneklerle fiyat değişimlerinin bu politikaları ve getirilerini nasıl etkiledikleri gösterilmektedir. İkinci kısımda, satış fiyatlarının piyasa-bazlı fiyatlardan ve şirketin kar payı kararlarından etkilendiği durumda şirketin en iyi fiyatlama stratejileri incelenmekte ve fiyat oynaklıklarının şirketin nihai getirilerine negatif etkisi teorik olarak gösterilmektedir. Son kısımda ise, riske-duyarlı bu şirketin finansal risk azaltımı politikaları analiz edilmektedir. Üretilen ve/veya satışı yapılan ürünün piyasa bazlı içsel fiyat hareketlerinin finansal piyasalarda alım-satımı yapılan çeşitli ürünlerin fiyatlarıyla karşılıklı ilişkilerinin olduğu kabul edilmektedir. Bu da riske duyarlı şirketin finansal ürünlerden bir yatırım portföyü oluşturup, fiyat ve talep risklerini yönetebilmesi için fırsat sağlamaktadır. Bu durumda şirketin envanter, varlık ve fiyat seviyelerini gözlemleyip dinamik bir şekilde oluşturduğu minimum-varyans yatırım kararları, dinamik programlama kullanılarak açık bir biçimde karakterize edilmektedir. Doğru finansal yatırım kararlarının şirketin nakit akışlarında dikkate değer risk azaltımları sağladıkları çeşitli sayısal örneklemelerle gösterilmektedir.

ACKNOWLEDGMENTS

First and foremost, I am truly indebted to two great persons, my supervisors, Prof. Süleyman Özekici and Prof. Fikri Karaesmen for their inspiring mentorship and guidance. They have continuously supported me by sharing their wisdom, not only on this dissertation, but also on every part of my life here and always encouraged me to be better. They are my ultimate role models and I am very proud to be their student.

I also would like to thank Asst. Prof. Pelin Canbolat and Asst. Prof. Uğur Çelikyurt for taking part in my thesis committee, for patiently reading my reports and for their valuable suggestions and guidance which led this work to improve.

I also would like to thank to my office mates for bringing all the joy and laughter to our work environment.

Last, but not least, I am grateful to my family for their warmest love, patience and encouragement throughout my doctoral study. Finally, my special thanks are due to my late granfather Sabri, who insistedly supported me to follow this path. His memory will always be with me.

TABLE OF CONTENTS

LIST OF FIGURES

NOMENCLATURE

P	:	Market price process, $P = \{P_t; t \geq 0\}$
Λ	:	Intensity process, $\Lambda = \{\Lambda_t; t \geq 0\}$
N	:	Demand process, $N = \{N_t; t \geq 0\}$
$\lambda(.)$	:	Intensity function
$f(.)$	:	Selling price function
$\overline{T}$	:	Customer arrival times, $\overline{T} = \{T_n; n \geq 1\}$
T	:	Length of a single sales period
M	:	Number of sales periods
x	:	Inventory level at the beginning of a period
y	:	Inventory level after ordering
p	:	Observed price at the beginning of a period
r	:	Interest rate per unit time
γ	:	Discount factor for a period, $\gamma = e^{-rT}$
$h\,(p)$	:	Unit inventory holding cost
$b\,(p)$	:	Unit backorder (lost-sale) cost
R_t	:	Total discounted revenue until time t
$r_t(p)$	:	Expected total discounted revenue until time t
$c(y; p)$	:	Expected one-period backorder and holding cost function
$g(y; p)$	:	Expected one-period profit function
$V_k(x, p)$	:	Maximum expected total discounted profit from period k
$G_k(y; p)$	:	Expected total discounted profit from period k
$\Psi_k(y; p)$	:	Expected discounted future profits from period k

$S_k(p)$	:	Base-stock level in period k
$s_k(p)$	:	Reorder point in period k
K	:	Fixed ordering cost
$\triangle f$	:	Forward difference operator
$z_t(p)$	:	Expected discounted price at time t
W	:	Wiener process, $W = \{W_t; t \geq 0\}$
$r(y;p)$	:	Expected total discounted revenue function
α	:	Proportional sales markup
X	:	Individual customer demand process, $X = \{X_n; n \geq 1\}$
D_n	:	Cumulative demand until nth customer
$N(y)$	:	Order of the last customer who makes a purchase
σ_X	:	Standard deviation of investment return X
$\rho_{X,Y}$	:	Correlation coefficient between X and Y
$CF(y,N,P)$	:	Unhedged operational cash flow
θ	:	Financial investment strategy
S	:	Price process vector for financial securities
$G(\theta,S)$	:	Payoff from financial investment strategy
$HCF(y,\theta,N,P,S)$	:	Hedged cash flow
$\mathcal{T}$	:	Set of trading times, $\mathcal{T} = \{t_0, ..., t_n\}$
C	:	Covariance matrix of security prices, $C = \{C_{ij}\}$
$\mu(y)$	:	Covariance vector of cash flow and security prices
$\theta^*(y)$	:	Optimal hedge as a function of operational decision
$R_{[s,t]}$	:	Total revenue between times s and t
$N_{[s,t]}$	:	Total demand between times s and t
W_k	:	Wealth level at the beginning of period k
X_k	:	Inventory level at the beginning of period k

Chapter 1

INTRODUCTION

Today's global economy leaves the firms with many critical challenges to deal with in terms of uncertainties. Besides the random nature of customer demand, manufacturers ad retailers usually have to consider the fluctuations in commodity prices, exchange rates etc. as well for their supply chain operations. For instance, companies whose operations require procuring from commodity markets are exposed to commodity price fluctuations which experience sharp movements frequently. Unstable economies, supply disruptions due to uncontrolled factors such as earthquakes, strikes, fluctuating exchange rates are all contributing factors to volatile commodity prices or input materials. Considering the fact that a significant portion of manufacturers' expenses are due to raw material costs, it is vital for firms to take a variety of risk management measures against undesired price movements. Moreover, it is clear that in this sort of price-fluctuating environments, firms' operational policies such as ordering and pricing as well as financial strategies are greatly affected.

Successful inventory management is an effective approach to mitigate risks due to input or selling price fluctuations. Besides its importance in managing the usual trade-off between holding, shortage and purchase costs, it can create additional value in fluctuating price environments by adjusting the order sizes in response to the price. Although it is common to design contracts with suppliers or use financial derivatives to hedge against possible price risks, a successful inventory policy should also take the evolution of prices into account in order to avoid downside risk or sometimes benefit from advantageous variations (Chod et al. (2010), Caldentey and Haugh (2006)). For instance a manufacturer, in anticipation of raw material price increases, can invest in inventory to avoid high purchase costs and thus benefit from high selling prices in future.

Classical inventory models usually take purchase and selling prices as constants. However, it is clear that for some products, input prices are volatile. Moreover, it is common that selling prices in some industries may be difficult to predict as well. For instance, a wholesaler that sells in a different currency will bear an exchange rate risk in selling prices. Some industries such as apparel and high technology face the problem of variable selling price due to rapid product substitution and short life cycles. Jewelry retailers which buy and sell products that are made of precious metals or stones such as gold, silver and diamond may reflect the fluctuation in input prices to customers and may charge different prices to customers arriving at different times. For inputs that are traded in commodity markets, there is a rich literature on modeling commodity price processes. Sophisticated models are proposed to take into account both long-term and short-term price volatilities. On the other hand, most inventory models ignore the full effects of such input price volatilities.

This dissertation addresses the inventory management and pricing policies of a firm where the inventory item faces price volatilities, especially within the ordering cycles. Due to the nature of the item, input and selling prices as well as customer demand are all affected from such price changes. After analyzing operational strategies, risk hedging policies are investigated. Next, we summarize the main topics studied in the subsequent chapters.

In Chapter 3 of this dissertation, we propose an inventory model that can potentially integrate sophisticated input price processes with price dependent demand. In particular, we propose and investigate a multi-period, single-item, periodic review inventory control model where we explicitly model a continuous-time stochastic market price process which determines both purchase and selling price and influences the customer demand. In this environment, the arrival times of customers and the value of the price process is important as they determine the sales revenue. The model, which includes both price and demand uncertainties in continuous time, covers many important simpler models as special cases. Our main contributions to the literature are as follows. First, we capture the effects of continuous stochastic input and selling price fluctuations and their effects on the continuous demand process in a tractable model. The resulting model has both continuous time and discrete time components and non-trivial within-period dynamics between fixed time points. Second, for this model, we characterize the optimal ordering policy using dynamic programming. Further, our characterization leads to numerically implementable solutions for practically relevant price processes. Using these solutions, we also generate insights on the effect of price volatilities on optimal expected profits and inventory decisions. To our knowledge, this has not been addressed before for such general price processes that are consistent with the finance literature.

In Chapter 4, we examine how such a firm coordinates its inventory and pricing decisions in a fluctuating-price environment. We assume that the selling price of the item consists of a market-driven random price which constantly changes during sales period and firm's operational markup. Rather than determining selling price on its own, the firm determines a proportional markup on the market price to reflect the effect of prevailing commodity prices on the retail price. For this setting, we explicitly characterize the optimal inventory-markup strategy and theoretically analyze the effect of price volatilities in firm's optimal controls and profitability.

In Chapter 5, we analyze the financial hedging problem of a firm which is exposed to commodity price fluctuations in its inventory operations. At prespecified trading times throughout the sales season, we assume that the firm has the opportunity to invest in available financial securities which are correlated with the commodity price process. The risk-averse firm then aims to minimize the variance of the cash flow at the end of the sales season for any inventory policy by exploiting these correlations. For this setting, we characterize optimal static and dynamic variance-minimizing trading policies which use the available information at each trading time. In a numerical setting, we also investigate the effect of the number of trading periods and price volatility on the effectiveness of financial hedging by using several derivative securities which are written on the underlying market price of the inventory item.

Apart from simple inventory models and special cases, this problem is known to be challenging and few general results exist. Our contributions to the literature in this part are two-fold. First, we apply a variance minimization approach to a rather general multi-period inventory model where there are both selling price and demand risks that are driven by a continuous stochastic price process. Second, we explicitly characterize the minimum-variance hedge at each period by solving a stochastic dynamic program. Despite the complexity of this dynamic program, its solution turns out to be relatively simple. This leads to useful characterizations in some important special cases and to implementable numerical solutions in general. Using this approach, we can then characterize the benefits of financial hedging for different plausible financial portfolios.

The rest of this dissertation is organized as follows. In Chapter 2, we review some of the related papers in inventory management and pricing literature that deal with price uncertainties. Furthermore, several risk management approaches including financial hedging are reviewed in the context of inventory operations. In Chapter 3, we introduce the problem formulation and analyze the optimal ordering policies of the inventory system that involves price fluctuations. In Chapters 4 and 5, we investigate pricing and dynamic financial hedging strategies for the models outlined before, respectively. Last, in Chapter 6, we give our concluding remarks and share our ideas for future studies. An appendix is provided at the end for some of the derivations and lengthy proofs.

Chapter 2

LITERATURE REVIEW

In this chapter, we review some of the available operations models in literature which mostly deal with volatile prices. Then, we outline several risk-sensitive models involving financial hedging framework.

2.1 Inventory Models with Random Prices

A number of papers explore the effect of volatile purchase prices on inventory control problems. Kalymon (1971) extends the classical inventory model of Scarf (1960) that involves fixed ordering costs by incorporating random purchase price which is governed by a Markov process. For such a system, he proves that a price dependent (s, S) policy is optimal. Golabi (1985) considers a single-item and deterministic-demand inventory system. He assumes that at the beginning of each cycle, the ordering price is a random variable with a known distribution function. He derives a policy where it is optimal to order for a number of next periods if the price falls into a certain interval. Gavirneni (2004) studies a periodic review inventory problem where unit purchase cost at each stage takes values from a discrete set according to a Markovian transition matrix. He shows the optimality of order-up-to policies and presents conditions that lead to monotone order-up-to levels. Chen et al. (2007) consider a multi-period pricing and inventory management problem in discrete time and show that price dependent base-stock policies are also optimal when additive exponential utility functions are used to represent the risk sensitivity of the decision maker. Their results extend to cases where demand and cost parameters are Markov-modulated. Berling and Martínez-de Albéniz (2011) investigate a Poisson demand system where the purchase price is a Markov process to study the effect of price evolution on the optimal policy and its parameters. Specifically, they consider both geometric Brownian motion and Ornstein-Uhlenbeck processes for the price. Utilizing the decomposition method of Muharremoglu and Tsitsiklis (2008) they characterize optimal base-stock levels as a series of thresholds and provide an algorithm to calculate them. Following their work, Berling and Xie (2014) present simple heuristics to calculate those threshold levels efficiently. Chen et al. (2014) study the impact of purchase price volatility in a multi-period stochastic inventory system where input prices at each period are random and independent of the demand distribution. They establish that higher price variability results in lower costs.

Another related stream of literature is on inventory models with Markov-modulated demand. In these papers typically demand distributions change over time, usually dependent on an external stochastic process (see Song and Zipkin (1993), Özekici and Parlar (1999), Cheng and Sethi (1999), Erdem and Özekici (2002), Gallego and Hu (2004) and Gayon et al. (2009)). In our model, an internal price process modulates the demand arrivals.

Our study differs from the above papers in following points. First, in addition to input price fluctuations, we also model selling price fluctuations and their impact on the demand process. In addition, in our case, the demand and the revenue within a period depend on the continuous price process which connects the optimal ordering policy to the properties of the price process.

A few authors study inventory systems with changing selling prices. Available models usually consider an inflation rate or a deterministic continuous price decrease, mostly in the context of the Economic Order Quantity (EOQ) model (Erel (1992), Hariga (1995) and Khouja and Park (2003), and Yang et al. (2011)). Banerjee and Meitei (2009) specifically dealt with the effect of changing selling price. Referring to the price history of Nokia's two mobile phone models, they consider a single period stochastic demand inventory model with random lead time and continuously decreasing selling price. They assume uniform demand over the selling season and investigate retailer profitability by proving the existence of optimal solutions. In contrast to the above papers, we model the case of random demand modulated by a randomly fluctuating price process and assume that the selling price is a general function of fluctuating input price.

There are also some papers that consider firms that can buy from or sell in spot markets where prices are constantly changing. Goel and Gutierrez (2006) consider a multi-period stochastic inventory model where a firm may purchase from both spot and future markets. Haksöz and Seshadri (2007) review existing models that incorporate spot market procurements with volatile prices in several supply chain operations. Katariya et al. (2014) consider a multi-period problem of a supplier that has access to a volatile spot market and more stable long-term contractual customers to sell items. In each period, the supplier decides on a production quantity and how much to liquidate in the spot market. They show that the optimal policy consists of two parameters and provide bounds on these. Guo et al. (2011), on the other hand, consider a firm that can purchase at any time from a spot market and faces a random demand at a later random time. The firm meets demand as much as possible and salvages the leftovers, if any. They prove that optimal policy is a price-dependent two-threshold policy. Secomandi (2010) studies the warehouse problem of a merchant that is involved in commodity-trading activities. He assumes that in each period, the spot price of the commodity evolves as a Markov process. In the presence of both space and buy-sell limits, he shows that operational and financial decisions can not be separated and the optimal policy is characterized by two-stage price-dependent base-stock targets.

One key difference of our models is that we explicitly model the customer demand process in detail and the firm do not have access to an ample spot market. Our model in that sense is more appropriate for firms who may use commodities to manufacture specialized products which are not easily traded in spot markets.

Overall, the models presented in this dissertation differ from existing models in the sense that they incorporate the effect of random selling prices by explicitly modeling a continuous-state stochastic price process. They also relate random purchase and selling prices through a multiplicative retail markup unlike classical models that take selling price as constants or as a random variable unrelated to the purchase costs. This case is applicable to situations where a firm is selling in a foreign currency or selling a commodity-based item such that any price fluctuations in the material cost also pass to customers as in the jewelry industry. In addition, instead of having random demands that are realized at the end of sales periods, we model the individual customer demand also as a stochastic process that depends on the prevailing stochastic prices. Finally, to our knowledge, there are few results for the challenging lost sales problem when demand is price dependent.

2.2 Joint Inventory Management and Pricing

A significant portion of operations management literature focuses on coordinated inventory and pricing decisions. In all of these models, whether stochastic or deterministic, customer demand is a function of selling price. Whitin (1955) was the first to allow selling price to be set simultaneously with order quantity in the newsvendor model. His method is to first determine the optimal ordering quantity as a function of price and then find the corresponding optimal price. Reviews by Petruzzi and Dada (1999), Chan et al. (2004) and Yano and Gilbert (2005) provide the current models in joint inventory-pricing literature. Focusing on the newsvendor model, Petruzzi and Dada (1999) investigate both additive and multiplicative demand models and provide conditions that are sufficient to ensure unimodality of the profit function. With the introduction of a base price, they are able show that the optimal price can be interpreted as the sum of base price and a premium. Federgruen and Heching (1999), on the other hand, consider a multi-period inventory model with backorders where the distributions of independent demand functions of each period depend on the price charged at the current period. They show that optimal inventory pricing strategy is a base-stock list-price policy which suggests to order up to the optimal base-stock level and charge the optimal price of a given period if the state of inventory level is less than that optimal base-stock level. Otherwise, it is optimal to order nothing and charge the unique optimal corresponding price for the current inventory level. Some of available models incorporate risk-aversion in the problem as well. Agrawal and Seshadri (2000) decide both order quantity and selling price to maximize the expected utility in a newsvendor setting. They show that a risk-averse retailer will set a higher selling price and order less compared to a risk-neutral retailer. Chen et al. (2007) consider joint pricing and ordering in a multi-period model with an exponential utility objective.

With our model, we contribute to the literature by incorporating the effect of fluctuating commodity prices into inventory and pricing decisions. Unlike traditional models that use a demand function that has an error term and a deterministic part which changes with respect to pricing decision, we decide on a proportional markup and model a customer arrival process that is modulated by a stochastic price process as well as the markup decision.

2.3 Risk-Sensitive Inventory Management

In this dissertation, we mainly consider inventory systems where a continuous and Markovian commodity price process determines both the selling prices and instantaneous arrival rates of the customers. These both contribute to the total risk of the final cash flow. Although in the next two chapters we analyze risk-neutral ordering and pricing policies, in Chapter 5 we take a risk-sensitive approach and examine minimum-variance hedging policies. In this part it is worth reviewing some of the available risk-sensitive inventory management models.

Although there are several papers that incorporate price related risks into inventory models, the majority of the risk-sensitive inventory management literature deals with alleviating demand related risks. Numerous approaches have been proposed in which the objective function is adjusted to reflect risk preferences of the decision maker. The most notable approaches include expected utility maximization, Mean-Variance (MV) criterion, satisficing probability maximization, Value-at-Risk (VaR) and conditional-Value-at-Risk (CVaR).

Due to its analytical tractability, the expected utility criterion is relatively more frequently used among the approaches outlined. Lau (1980) was the first to analyze the newsvendor problem under the expected utility criterion. He uses nth-degree polynomial approximation to a general utility function and provides a numerical solution mechanism. Similarly, Eeckhoudt et al. (1995) use utility theory and consider a risk-averse newsvendor to investigate the effects of changes in riskiness of background wealth and demand. They conclude that as risk-aversion increases, the resultant optimal order quantity decreases. Under decreasing absolute risk aversion, they show that increases in initial wealth also increase the number of orders. Bouakiz and Sobel (1992) consider a multi-period inventory control problem where the objective is to minimize the expected utility of discounted costs in finite and infinite planning horizons. They characterize the optimal replenishment strategy as a base-stock policy provided that ordering costs are linear. Chen et al. (2007) extend the finite horizon problem studied in Bouakiz and Sobel (1992). They take a different economics perspective and aim to maximize expected utilities from consumption flows rather than typical cash flows to avoid the so called temporal risk problem. As thoroughly explained in Smith (1998), this problem occurs when decision makers are sensitive to the time at which uncertainties are resolved. Chen et al. (2007) prove that when the additive utility is exponential, the optimal replenishment policies have the same structure as those for risk-neutral cases, i.e., base-stock policies are optimal. When general utility functions are used, on the other hand, optimal policies become wealth-dependent.

Although a significant portion of available models use the expected utility framework for capturing risk sensitivity of the decision maker, it only helps to reduce the risks by adjusting the order quantity appropriately. However, these deviations in these control variables may cause service levels to fall considerably and it is usually better to have other risk reduction plans which also work for any ordering decision. Finally, expected utility approach is often criticized being impractical since it is usually hard to estimate the utility function of an individual and usually they are not mathematically tractable for arbitrary utility functions.

Another approach that has received considerable attention in risk management is the use of MV approach that Markowitz (1959) introduced on the portfolio selection problem. In his study, Markowitz (1959) considers an investor who wants to allocate his initial wealth among a number of risky assets to minimize the variance of the return while expecting a predetermined level of return. By utilizing the mean-variance framework, he determines a set of efficient portfolios offering appropriate risk and return levels. Although it originated in finance, the mean-variance method is becoming useful for any problem involving conflicting objectives. In the context of inventory management, this approach is very similar to portfolio selection model in which the returns are now the expected profits from inventory operations and the risk is the variation of profits.

The use of MV approach in inventory management began with Lau (1980) who uses an MV objective function for the newsvendor problem to incorporate risk-aversion. He shows that the expected profit maximizer yields an upper bound on the optimal order quantity. Choi et al. (2008) consider both mean-downside risk and mean-variance on two cases where the selling price is a decision variable or not. They conclude that for both cases optimal order quantities are the same regardless of the two objective functions. Berman and Schnabel (1986), Choi et al. (2008) and Wu et al. (2009) also utilize MV framework on single-period inventory problems. Few works employ MV approach in multi-period inventory models. Chen and Federgruen (2000) consider a base-stock policy for a single item inventory system. Using the MV method, they construct the efficient frontier for two performance measures: long-run holding costs and a measure that is a function of both the expectation and the variance of customer waiting time.

A different line of research is the use of satisficing probability maximization method, which is basically maximizing the probability of achieving a target profit level. This method is particularly useful for a decision maker if satisfying a certain profit level is more important than the level of extra profit. Lau (1980) and Sankarasubramanian and Kumaraswamy (1983) consider satisficing probability maximization objective in newsvendor models. Lau and Lau (1988) and Li et al. (1991) extend their works to two-product cases of the newsvendor model. Parlar and Weng (2003) use probability of achieving the expected profit instead of a fixed target profit. This approach is not very popular compared to the expected utility and the MV method since it does not specifically deal with the variations in operational profits.

A more recent approach used in the context of risk-averse inventory management is value-at-risk (VaR), a downside risk measure that is commonly used in financial risk management. By definition, VaR is the lowest amount that will not be exceeded with a given probability level (see Jorion (2007)). In the context of inventory management, lowest amount refers to the lowest profits associated with a specific inventory replenishment policy. Luciano et al. (2003) apply VaR on an infinite-horizon problem with no lead time and constant order quantity at the beginning of each replenishment cycle. Özler et al. (2009) investigate a multi-product newsvendor problem using VaR as a risk measure. They derive exact distribution functions for the two-product case and develop an approximation for the general N-product case. An alternative to VaR is the conditional value-at-risk (CVaR) measure which is defined as the conditional expectation of losses above the VaR value. Gotoh and Takano (2007) consider the minimization of CVaR in the context of the newsvendor problem. They show that, due to its convexity, usage of CVaR leads to tractable problems. Ahmed et al. (2007) analyze coherent risk measures such as CVaR and mean-absolute deviation in a multi-period single-item inventory problem. Coherent risk measures are a class of functions satisfying certain axioms introduced in Artzner et al. (1999). According to these axioms, although CVaR is coherent, VaR is not, since it does not satisfy convexity and subadditivity.

2.4 Financial Hedging

There is a growing interest in integrating supply chain operations and risk hedging with financial instruments. It is understood that the existence of financial instruments whose movements are correlated with the random components in firms' operations, usually allow decision makers to hedge against possible risks (Gaur and Seshadri (2005), Caldentey and Haugh (2006)). It is well known that firms often use financial products to hedge apparent risks involving price or exchange-rate uncertainty. For instance, a firm procuring from a commodity market may hedge their risk using commodity futures. Another example would be a producer selling to a foreign market in foreign currency units. This also presents an opportunity to invest in derivatives of the particular foreign currency as it will be correlated with producer's profits in the domestic currency.

Unlike the expected utility and the MV methods which only adjust the order quantities to manage the inventory risks, financial hedging approach takes a more proactive role and promises to reduce the risks for any inventory policy in the presence of relevant financial products. Considering the variety of financial products offered in developed economies, financial hedging may provide a good opportunity for risk-sensitive firms to control their risks.

The first study that investigates the effect of financial markets on inventory policies is Anvari (1987). He uses the well-known Capital Asset Pricing Model (CAPM) in a newsvendor setting with no setup cost. In case of normally distributed demand, he shows that optimal order quantity varies as the covariance between demand and market returns change. Chung (1990) enhances Anvari's model by sharpening the optimality conditions and showing that the optimal strategy can be simplified to a single equation regardless of the sign of covariance. Gaur and Seshadri (2005) consider hedging demand risk in inventory models motivated by the statistical finding that demand for discretionary purchase items and the S&P 500 index are highly correlated. In a newsvendor framework, they analyze both perfect and imperfect hedging cases and characterize the optimal hedge. Caldentey and Haugh (2006) examine the problem of dynamically hedging a risk-averse firm's operational profits in continuous time. Their central modeling insight is to view operational profits as an asset in the firm's portfolio and address the hedging problem as one of the most extensively studied problems in the finance literature: financial hedging in incomplete markets. They give a fairly general modeling insight where the decision maker's objective is to maximize the expected utility from profits by simultaneously considering both operational and trading strategies. Chen et al. (2007) consider the opportunity of financial hedging in their multi-period inventory control models where inventory and trading decisions are made at discrete time points. They further assume that security prices and cost parameters are world-driven, i.e., they are modulated by an external environmental process. They show that under a partially complete financial market, full hedging is possible if additive exponential utility functions are used. Ding et al. (2007) attack the problem of integrating the operational and financial hedging decisions of a firm selling to both foreign and domestic markets. Such a firm will obviously face demand and exchange rate uncertainties. As an operational hedge, the firm may postpone their capacity commitment until uncertainties are resolved. This can be done using capacity allocation option. As a financial hedge, the firm may use currency options for protection against exchange-rate risk. Using MV approach in their models, the authors conclude that expected profit increase and risk reduction is possible via operational and financial hedging. Chod et al. (2010) investigate when operational and financial hedging are substitutes and complements. Ni et al. (2016) introduce a certainty equivalent operator to find optimal hedging-consistent decisions in presence of non-financial random factors that can not be hedged through financial markets. In an illustrative example on commodity procurement and storage, they show the optimality of base-stock policies and characterize the financial hedging portfolio.

Recent works by Okyay et al. (2014), Sayın et al. (2014) and Tekin and Özekici (2015) extend Gaur and Seshadri (2005) by incorporating supply uncertainty and discuss the implication of using expected utility and mean-variance objective functions in financial hedging context. In particular, Okyay et al. (2014) find the variance minimizing financial portfolio for a given order quantity. We also take a similar approach but our models are more general since they capture continuous-time price fluctuations and their influence on demand as well as multi-period cases with dynamic decision making.

Few papers consider both random demand and fluctuating prices in the context of financial hedging. Kouvelis et al. (2013) analyze the inventory operations of a risk-averse firm that procures from both a volatile spot market and a long-term supplier. Exposed to both demand and price uncertainty, the firm is assumed to have access to financial securities written on the commodity price. In a multi-period setting, the objective of the decision maker is to dynamically maximize the interperiod mean-variance utility of the firm's cash flow. In a similar work, Kouvelis et al. (2015) study the same setting in Kouvelis et al. (2013) without a long-term supplier. Assuming that the objective is to maximize the mean-variance utility of the terminal wealth, they characterize optimal time-consistent inventory and financial hedging policies. Our work differs from Kouvelis et al. (2013) and Kouvelis et al. (2015) in two major ways. First, we analyze a rather different operational model where the main randomness is due to within-period price fluctuations. More specifically, in each decision interval, a continuous stochastic price process drives both selling prices and demand arrivals in our model. Moreover, in the multi-period case, leftover inventory is not liquidated at each period and is carried over to satisfy customer demand which makes the dynamic program of joint-optimization rather intractable Second, in the context of financial hedging, we focus on the specific objective of variance minimization where the decision maker seeks a minimum-variance hedge for any given operational policy. This objective has some nice features. First, by definition, the operational policies are independent of the financial portfolio and operational profits can be accounted for independently. Therefore, the financial hedge supports the operation towards variance reduction without enforcing operational policy changes. Second, it turns out that the minimum-variance hedge leads to tractable solutions. This allows experimenting with different operational policies to computationally explore non-dominated mean-variance policies. Moreover, the formulation also leads to a tractable solution in a multi-period setting with a dynamic ordering policy where inventory carrying is allowed. This provides a hedging approach for general multi-period inventory problems.

Chapter 3

INVENTORY MODELS WITH RANDOMLY FLUCTUATING PRICES

In this chapter, we investigate the ordering policies and their implications for a series of inventory systems where the main randomness is due to randomly fluctuating prices during sales cycles. More specifically, we consider systems such that a stochastic price process affect both input and selling prices as well as customer arrivals. These models may be relevant for firms which manufacture or procure items which consists of a market-driven component whose price is constantly changing according to an external process, and then the firm sells those items considering its own operational costs and random market price of the component. When price fluctuations are reflected in the selling price of the item, then consequently the customer arrivals are affected.

The models presented in this chapter belong to the class of periodic-review inventory models where an ordering decision is made at predefined time points. The main distinction from the current inventory management literature is that these models also incorporate the effect of random selling prices by modeling customer demand as a process rather than a single random variable. This type of modeling which explicitly considers within-period price fluctuations proves to be very important as these fluctuations affect ordering, pricing as well as risk-hedging policies of the firm which will be examined in the upcoming chapters. In the next section, we present the specifics of these models.

3.1 Model Setting

In all of the models considered in upcoming sections, we assume that there exists a nonnegative Markov process $P = \{P_t; t \geq 0\}$ with state space $\mathbb{R}_+ = [0, \infty)$ which models the input price of the inventory item. We call this the market price of the item to denote its relation to the market-driven component of the inventory item. We also assume that items are sold at arriving customers according to a nonnegative and deterministic selling price function $f > 0$, that is if a customer arrives at time t, then the corresponding selling price at that time is $f(P_t)$. The general selling price function f may include any potential proportional and/or incremental markups that the firm sets. Note that unless f is a constant function, the firm passes any fluctuations in market price of the item to customers. A constant selling price function is the standard assumption in most basic inventory management models. For now, we do not assume any particular form for the selling price function f. However in some special cases in this chapter and throughout Chapter 4, we will specifically assume that the firm uses a multiplicative selling price function $f(p) = \alpha p$ where $\alpha \geq 1$ is the proportional retail markup.

For the inventory system, we assume that there are M periods whose lengths are equal to T units of time where at the beginning of each sales period the firm places an order. Based on our definition, the values of the random prices at review times (i.e., $P_T, P_{2T}, ..$) are the random purchase prices for the firm. Unlike most of the inventory management papers that model customer demand as a random variable to be realized at the end of each review period, we assume that there is a customer arrival process and it is modulated by the random price process that we consider. This also explicitly incorporates the effect of fluctuating selling prices into the operational model. More specifically, we assume that the unit customer demand process is a modulated Poisson process where stochastic arrival rate at time t is $\Lambda_t = \lambda(P_t)$ and $\lambda(.)$ is a deterministic, nonnegative function of random price realizations. Customers arrive according to this process and at each arrival they demand one unit of the item. This is relaxed at Section 3.5 as we investigate the compound Poisson case where each customer demands a random amount of the item.

We do not necessarily assume that $\lambda(.)$ is a decreasing function of price. If, for example, the firm deals with commodity-based items whose prices are constantly changing and can be freely traded in markets, the demand for these types of products does not necessarily decrease as price increases. Customers may also be willing to buy in anticipation of future price increases even if prices have already increased.

In our setting, since arrival rate is a function of stochastic price, we have a stochastic arrival rate process $\Lambda = \{\Lambda_t; t \geq 0\}$ which modulates the customer arrival process. These types of models are referred as doubly stochastic Poisson processes introduced by Cox (1955), or shortly, Cox processes. If Λ is a deterministic function rather than a stochastic process, we have a non-stationary Poisson process. Since we assume that P is a Markov process, Λ is also Markovian.

We denote the customer purchase process by $N = \{N_t; t \geq 0\}$ where N_t denotes the number of sales by time t and $N_0 = 0$. The arrival times of the customers form a random sequence $\overline{T} = \{T_n; n \geq 1\}$ where $\overline{T}$ and N are related as $\{T_n \leq t\} = \{N_t \geq n\}$. Here we remark that equal period length assumption is not a necessity and the models presented in this chapter can easily be extended to cover variable period lengths. The structure of the optimal policies remains unchanged.

At the beginning of each period, the decision maker observes the current price and inventory level to make an ordering decision. We assume that there is no lead time and the entire order is received immediately at the beginning of each period. We investigate two distinct cases where, at first, we allow backordering for unsatisfied demands. Secondly, we consider the lost-sale case. We specify the basics of these models in Section 3.2 and Section 3.3, respectively.

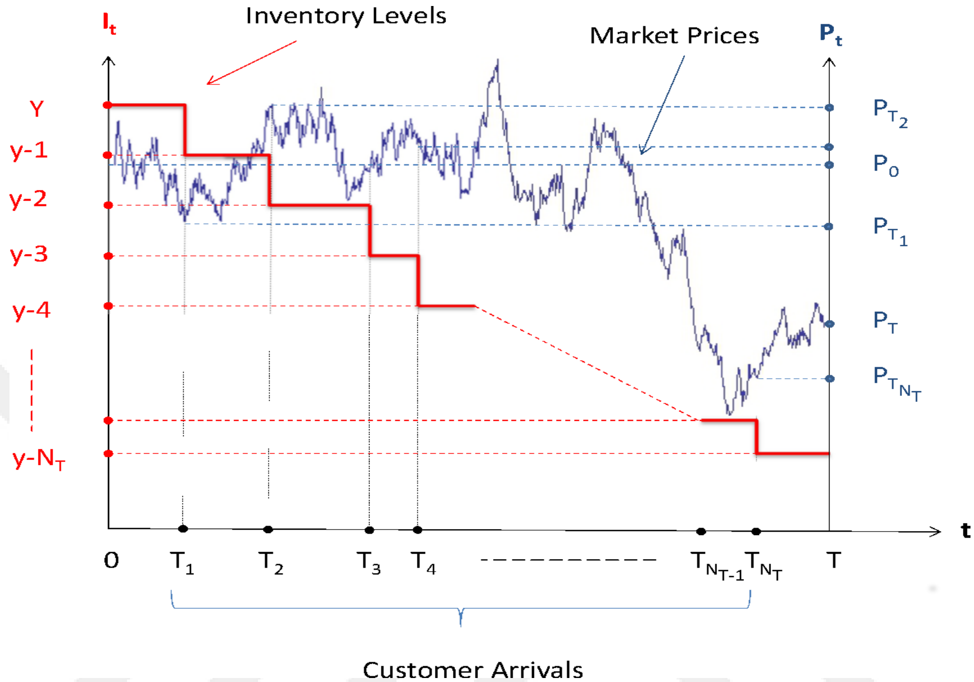

Figure 3.1: An overview of the inventory system.

We define the following additional notation:

x : Inventory level at the beginning of a period

y : Inventory level after ordering

p : Observed price at the beginning of a period

r : Interest rate per unit time

γ : Discount factor for a period $\left(\gamma = e^{-rT}\right)$

$h(p)$: Inventory holding cost per unit as a function of initial market price p,

$b(p)$: Backorder (or lost-sale) cost per unit as a function of initial market price p

We assume that for each period, unit inventory holding and unit backorder (or lost-sale) costs are general nonnegative functions of initial price for that period. The reasoning for the holding cost to depend on the initial price is clear as it consists of physical storage costs as well as the opportunity costs for the inventory investments. Since we assume that market prices represent the purchase prices, it is natural that unit inventory holding cost for the product is a function of initial price p. For the backorder or lost-sale costs, the reasoning is not as straightforward as the holding cost.

Usually, backorder costs depend on the selling price of the item and the length of the backorder period for any customer, etc. However, since the distribution of both selling prices and customer demand processes are determined by the initial market price in our models, we can simply use an approximate value for the backorder cost $b(p)$ where it is a general nonnegative function of initial market price. Note that typical constant unit holding and backorder costs are just special cases where $h(p) = h$ and $b(p) = b$ for some constants h and b.

3.2 Backorder Case

In this model, we allow backordering customer demand in case of inventory shortage. We assume that in case of a backorder, the selling price is set and paid at the time of customer arrival rather than at the time of actual product delivery. In other words, any unsatisfied customer is charged at the time of arrival. We assume that the backordered demand is satisfied at the beginning of the next period. At every period, the decision maker observes the current price p and inventory level x to make an ordering decision which maximizes the expected total discounted profits. Since we assume that the unit demands arrive according to a doubly stochastic Poisson process which is modulated by a Markovian price process and the length of intervals are the same, the probability distribution of number of sales in any interval only depend on the initial price at the beginning of that period. That is, they are conditionally independent. In addition, given the initial price P_0, the probability distribution of total demand until time t is a Poisson random variable with random mean measure, i.e.,

$$P\{N_t = k \mid P_0\} = E\left[\frac{e^{-M_t}M_t^k}{k!} \mid P_0\right]$$

where

$$M_t = \int_0^t \lambda(P_s)\, ds$$

is the expected number of arrivals until time t given market prices. Note that in case each customer demands one unit of the item, expected total demand during time interval $[0, t]$ is given by

$$E\left[N_t|P_0\right] = E\left[M_t|P_0\right] = \int_0^t E\left[\lambda\left(P_s\right)|P_0\right] ds.$$

Expected Discounted Revenue

For both backorder and lost-sales models, total revenue in each period is calculated by summing the revenue from each sale. However, based on the particular backorder setting we investigate, the item is sold to each arriving customer regardless of stock availability and each backordered customer yields a backorder and repurchase cost. Let us define the total discounted revenue collected in time interval $[0, t]$ for any $t > 0$ as

$$R_t = \sum_{n=1}^{N_t} e^{-rT_n} f\left(P_{T_n}\right). \tag{3.1}$$

Note that $f\left(P_{T_n}\right)$ is the selling price for the nth customer who arrived since the beginning of the period and e^{-rT_n} is to discount the unit revenue to the beginning of the period. Summation is performed until the arrival of last customer, i.e., N_tth customer where N_t is the total number of customers who arrived by time t. In this chapter, we assume that the decision maker is risk-neutral and aims to maximize the expected total discounted profit. For this, let us first define expected total discounted revenue during $[0, t]$ as a function of initial price by

$$r_t\left(p\right) = E\left[R_t \mid P_0 = p\right]. \tag{3.2}$$

Note that the expectation in (3.1) is taken with respect to the random components; number of arrivals N_t, arrival time vector $\overline{T}$ (the former can be obtained from the latter) and market prices between $[0, t]$, i.e., $\{P_t; t \in [0, t]\}$. The expected discounted revenue function in (3.2) can be computed as follows

$$\begin{aligned} r_t\left(p\right) &= E\left[R_t \mid P_0 = p\right] \\ &= \int_0^t e^{-rs} E\left[f\left(P_s\right)\lambda\left(P_s\right)|P_0 = p\right] ds. \end{aligned} \tag{3.3}$$

The derivation is given in Appendix. Note that in case λ is constant, i.e., customers arrive according to a regular Poisson process with constant rate, then

$$r_t(p) = \lambda t \overline{f}_t(p)$$

where

$$\overline{f}_t(p) = \int_0^t e^{-rs} E\left[f(P_s) \,|\, P_0 = p\right] ds$$

is the average discounted selling price and λt is the expected number of customers arrived by time t.

A similar approach for the total revenue also appears in Grubbström (2010) who considers a single-period problem where demand is modeled as a compound renewal process. He assumes that selling price is constant and customers that arrive according to a renewal process demand a random amount of the product. There is no fixed sales period and items are sold until all inventory is depleted. Although in our model we sum all individual revenues from each arriving customer, our model construction is somewhat different in the sense that we have a finite sales season and selling price is a stochastic process that also modulates the customer arrival process.

Model Dynamics

The dynamics of the backorder model is as follows. At the beginning of any period, if the current inventory level and market price are x and p respectively, and order-up-to level decision is $y \geq x$, (x and y are integers) the immediate expected profit for the period is

$$g(y; x, p) = -p(y - x) + r_T(p) - c(y; p) \tag{3.4}$$

where

$$c(y; p) = E\left[b(p)(N_T - y)^+ + h(p)(y - N_T)^+ \mid P_0 = p\right] \tag{3.5}$$

and we define $x^+ = \max(0, x)$. The first term in (3.4) is the total purchase cost for y units ordered at the initial price p. Second term is the total revenue collected until time T and the last term is the one-period backorder and inventory holding cost function given in (3.5) in which a cost of $b\,(p) \geq 0$ is charged for each unit backordered and a cost of $h\,(p) \geq 0$ is charged for every remaining unit. Note that N_T denotes the number of arrivals during the period and one-period expected profit is independent of the period. This is due to the fact that conditional random prices

$$P_{kT+T_n} \mid P_{kT} \overset{d}{=} P_{T_n} \mid P_0$$

have the same distribution for any period k since market price process is assumed to be Markovian and time-homogeneous This in turn implies that N is also Markovian whose distribution depends only on the initial market price p. Remember that we do not put any restriction on the price process except the Markov property. Since, for now, we assume that each demand is of size 1, the state space for inventory level in backorder case is $\mathbb{Z}$, i.e., set of integers.

In the last period, without loss of generality, we assume that all remaining items are lost, i.e., there is no salvage value. However, at the current price, the firm is required to raise the inventory level to zero if it turns out to be negative, meaning that there are backordered customers.

For now, we assume that the decision maker is risk-neutral and aims to maximize the expected total discounted profits. We use dynamic programming to solve this problem to optimality. We define the value function $V_k(x, p)$ as the maximum expected total discounted profit for periods from k to M if the initial inventory is x and price is p. We also define the expected discounted one-period revenue function with initial price p as

$$g(y; p) = -py + r_T\,(p) - c\,(y; p)\,. \tag{3.6}$$

Then, the dynamic programming equation (DPE) is

$$V_k(x, p) = \max_{y \geq x} G_k(y; p) + px \tag{3.7}$$

where

$$\Psi_k(y; p) = E\left[V_{k+1}(y - N_T, P_T) | P_0 = p\right] \tag{3.8}$$

and

$$G_k(y; p) = g(y; p) + \gamma \Psi_k(y; p). \tag{3.9}$$

Note that $\Psi_k(y;p)$ is the expected discounted total future profits for the remaining periods. Since we allow backorders, the inventory level for the next period upon ordering decision y is $y - N_T$, which in fact can be negative. For any period k, the risk-neutral firm aims to maximize $G_k(y;p)$, which is the sum of the expected one-period profit, $g(y;p)$, and expected discounted future profits, $\gamma\Psi(y;p)$, resulting from the ordering decision. Additionally, since we assume that the seller serves all arriving customers, the revenue term is independent of decision variable y. Since there is no salvaging, for each (x,p) pair the terminal value function is

$$V_{M+1}(x,p) = -px^-$$

where $x^- = (-x)^+$.

Optimal Ordering Policy

Now we present the structural properties of $G_k(y;p)$ and the form of the optimal policy. In the following discussion, $\triangle f(x) = f(x+1) - f(x)$ represents the forward difference of a discrete function f.

Theorem 3.1 *For $0 \leq k \leq M$, $V_k(x,p)$ is concave in x and $G_k(y;p)$ is concave in y for every p and a price-dependent-base-stock policy is optimal, i.e., there exists a base-stock level $S_k(p)$ for each period k such that if the inventory level is less than the base-stock level, it is optimal to raise the inventory up to $S_k(p)$; otherwise, it is optimal to order nothing. Moreover, optimal base-stock level for period k is given by*

$$S_k(p) = \inf\left\{y \geq 0 : P\{N_T \leq y \mid P_0 = p\} \geq \frac{-p + b(p) + \gamma \triangle \Psi_k(y;p)}{b(p) + h(p)}\right\}. \quad (3.10)$$

Proof. We proceed by induction. First note that the terminal value function $V_{M+1}(x,p)$ is concave in x for each p. Now assume that $V_{k+1}(x;p)$ is concave in x for some $k \leq M$. Then, $\Psi_k(y;p)$ given in (3.8) is concave in y by the linearity of expectation. Note also that for each p, one-period expected profit $g(y;p)$ in (3.6) is concave in y since $-py$ is linear and $b(p) \geq 0$, $h(p) \geq 0$ ensure that $-E[b(p)(N_T - y)^+ + h(p)(y - N_T)^+ \mid P_0 = p]$ is concave. This makes $G_k(y;p)$ given in (3.9) and consequently $V_k(x,p)$ concave functions of y and x, respectively. By induction argument, $V_k(x,p)$ and $G_k(y;p)$ are concave for each period k. Clearly, concavity of $G_k(y;p)$ ensures that the optimality of a base-stock policy with a base-stock level of $S_k(p)$ which is the maximizer of $G_k(y;p)$. To calculate the base-stock level for period k, we can apply the first-order optimality condition on $G_k(y;p)$. More specifically,

$$\begin{aligned} S_k(p) &= \inf\{y \geq 0 : \triangle G_k(y,p) \leq 0\} \\ &= \inf\{y \geq 0 : \triangle g(y;p) + \gamma \triangle \Psi_k(y;p) \leq 0\} \\ &= \inf\{y \geq 0 : -p + b(p) P\{N_T \geq y+1 |\ P_0 = p\} \\ &\quad -h(p) P\{N_T \leq y |\ P_0 = p\} + \gamma \triangle \Psi_k(y;p) \leq 0\} \\ &= \inf\left\{ y \geq 0 : P\{N_T \leq y |\ P_0 = p\} \geq \frac{-p + b(p) + \gamma \triangle \Psi_k(y;p)}{b(p) + h(p)} \right\} \end{aligned}$$

which is (3.10). ■

We have established that a price-dependent-base-stock-type policy is optimal. This is consistent with similar models with discrete dynamics such as Chen et al. (2007). Next, we present the more explicit single-period solution. First, we define expected discounted price process as

$$z_t(p) = E\left[e^{-rt} P_t | P_0 = p\right].$$

Corollary 3.1 *The optimal base-stock level at period M is given by*

$$S_M(p) = \inf\left\{ y \geq 0 : E\left[(b(p) + h(p) + \gamma P_T) 1_{\{N_T \leq y\}} \mid P_0 = p\right] \geq -p + b(p) + z_T(p) \right\}. \tag{3.11}$$

Proof. Note that because of the terminal value function $V_{M+1}(x,p)$ and (3.8),

$$\begin{aligned} \gamma \triangle \Psi_M(y;p) &= -\gamma \triangle E\left[P_T (N_T - y)^+ \mid P_0 = p\right] \\ &= \gamma E\left[P_T (N_T - y)^+ \mid P_0 = p\right] - \gamma E\left[P_T (N_T - y - 1)^+ \mid P_0 = p\right] \\ &= \gamma E\left[P_T \left[(N_T - y)^+ - (N_T - y - 1)^+\right] \mid P_0 = p\right] \\ &= \gamma E\left[P_T 1_{\{N_T \geq y+1\}} \mid P_0 = p\right] = \gamma E\left[P_T \left(1 - 1_{\{N_T \leq y\}}\right) \mid P_0 = p\right] \\ &= z_T(p) - \gamma E\left[P_T 1_{\{N_T \leq y\}} \mid P_0 = p\right]. \end{aligned} \tag{3.12}$$

Substituting (3.12) in (3.10) for $k = M$ yields (3.11). ■

Note that if $-p + b(p) + z_T(p) \leq 0$, the optimal base-stock level will be $S_M(p) = 0$. Although it does not affect the concavity of the expected profit function, it is reasonable to assume that $-p+b(p)+z_T(p) \geq 0$. $-p+b(p)+z_T(p)$ can economically be interpreted as the expected cost of ordering one less unit. If it is negative, it is optimal to order zero.

We have an explicit formula for the base-stock level of the last period. Therefore, we can analyze the behavior of the optimal base-stock level $S_M(p)$ as a function of initial price p. It is clear that the stochastic behavior of the market prices conditional on the initial price and the behavior of the deterministic rate function $\lambda(.)$ play a key role. We make the following three assumptions in which the first two are very plausible for a real-life inventory system and the third can be justified in the context of the specific model setup.

Assumption 3.1 *P_t stochastically increases in the initial price $P_0 = p$.*

Assumption 3.2 *$\lambda(.)$ is a decreasing function.*

Assumption 3.3 *$-p + b(p) + z_T(p)$is decreasing in p.*

Theorem 3.2 *If assumptions 3.1, 3.2 and 3.3 hold, $S_M(p)$ is decreasing in initial price p.*

Proof. Consider the characterization of $S_M(p)$ in (3.11). Note that increasing the initial price stochastically increases the prices P_t between $[0, T]$ by Assumption 3.1. This consequently decreases the number of sales N_T stochastically by Assumption 3.2. This in turn implies that the left-hand side of the infimum, increases in p. By Assumption 3.3, $-p + b + z_T(p)$ is decreasing which makes $S_M(p)$ decreasing in p. ■

Assumption 3.1 requires that the future prices are stochastically higher if the initial price is higher, which is highly intuitive and satisfied by the most practical stochastic price processes. For instance, let us assume that price follows a geometric Brownian motion process:

$$P_t = P_0 e^{vt+\sigma W_t}$$

with drift v and volatility σ where W_t is a Wiener process with $E[W_t] = 0$ and $Var(W_t) = t$. Then, Assumption 3.1 trivially holds.

Assumption 3.2, on the other hand, requires that the deterministic rate function $\lambda(.)$ is a decreasing function of price. Although there may be cases that violate this assumption in volatile markets as explained before, it is a very common assumption in the literature that the customer demand decreases as the price increases. We only need this assumption to show the monotonicity of $S_M(p)$. Price-dependent base-stock policy is an optimal ordering policy regardless of the structure of $\lambda(.)$.

Assumption 3.3 requires that sum of the expected discounted price increase until time T and the unit backorder cost is decreasing in the initial price. Note that we can interpret both $b(p)$ and $z_T(p) - p$ as the loss from ordering one less unit. The latter is due to the difference between two successive ordering prices (discounted) while the former is by the definition of backorder cost. Therefore, Assumption 3.3 essentially implies that total loss from ordering one less unit should be lower for higher initial market prices. If $-p + b(p) + z_T(p)$ does not decrease in initial price p, one can find cases where optimal base-stock level does not decrease as initial price increases.

The result in Theorem 3.2 can also be proved by showing that $\triangle G_M(y,p)$ is decreasing in p, i.e., $G_M(y,p)$ is submodular under Assumptions 3.1, 3.2 and 3.3. However, in either approach, a conclusion can not be drawn for intermediate periods $k < M$. This is also consistent with the findings of Kalymon (1971).

3.3 Model with Lost-Sales

In this section, we explore the lost sales case where we assume that any arriving customer that can not find an available item is lost. This case is more challenging than the backorder case because the expected revenue now depends on the ordering policy. To our knowledge, few results on the structure of the optimal policy exist for the lost sales case with price-dependent demand even for simpler models.

In analogy with the backorder model, let us write the expected total revenue during a period as a function of initial price p and order-up-to decision y as

$$\begin{aligned} r(y;p) &= E\left[\sum_{n=1}^{N_T \wedge y} e^{-rT_n} f\left(P_{T_n}\right) \mid P_0 = p\right] \\ &= \sum_{n=1}^{y} E\left[e^{-rT_n} f\left(P_{T_n}\right) 1_{\{T_n \leq T\}} \mid P_0 = p\right] \end{aligned} \tag{3.13}$$

where $a \wedge b = \min(a, b)$. Note that only the revenue term is different than the previous model by which we now collect revenues until the firm runs out of inventory, i.e., until the arrival of $(N_T \wedge y)$th customer. The total expected discounted one-period profit can be written similarly as

$$g(y;p) = -py + r(y;p) - c\left(y;p\right). \tag{3.14}$$

We write the dynamic programming equation for period k as in (3.7) where $G_k(y;p)$ is given in (3.9) and with a slight change in the future expected profit which is given as

$$\Psi_k(y;p) = E\left[V_{k+1}((y - N_T)^+, P_T)|P_0 = p\right].$$

Since there is no backordering, the inventory level can not be negative in the next period. It should be zero if the demand turns out to be more than the total inventory in the current period.

As in the backorder case, we assume that the salvage price is zero. Therefore, the terminal value function for the lost-sale model is

$$V_{M+1}(x, p) = 0. \tag{3.15}$$

Optimal Ordering Policy

In this section, we present the structural properties of $G_k(y;p)$ and the form of the optimal policy. Note that we can use the transformations

$$(y - N_T)^+ = y - \sum_{n=1}^{y} 1_{\{T_n \leq T\}} \tag{3.16}$$

and

$$(N_T - y)^+ = N_T - y + (y - N_T)^+ = N_T - \sum_{n=1}^{y} 1_{\{T_n \le T\}}. \tag{3.17}$$

Moreover, trivially,

$$y = \sum_{n=1}^{y} 1.$$

Then, by using (3.16) and (3.17), (3.14) becomes

$$\begin{aligned} g(y;p) = &\sum_{n=1}^{y} E\left[1_{\{T_n \le T\}}\left(e^{-rT_n}\alpha P_{T_n} + b + h\right) - p - h \mid P_0 = p\right] \\ &- b\,(p)\,E\left[N_T \mid P_0 = p\right]. \end{aligned} \tag{3.18}$$

One-period expected discounted profit function $g(y;p)$ consists of a finite sum where the upper limit of the summation is the decision variable y. and a constant. Clearly, the behavior of this function is directly determined by the behavior of the inner terms.

In the following discussion, the terms decreasing and increasing refer to weak monotonicity.

Assumption 3.4 $E\left[1_{\{T_n \le T\}}\left(e^{-rT_n} f\left(P_{T_n}\right) + b\,(p) + h\,(p)\right) \mid P_0 = p\right]$ *is decreasing in* n.

Theorem 3.3 *Under Assumption* $(3.4)\,, G_k(y;p)$ *is concave in* y *and* $V_k(x;p)$ *is concave in* x *for every* p *and a base-stock policy is optimal, i.e., there exists a base-stock level* $S_k(p)$ *for each period* k *such that if the inventory level is less than the base-stock level, it is optimal to raise the inventory up to* $S_k(p)$; *otherwise, it is optimal to order nothing. Moreover, optimal base-stock level for period* k *is given by*

$$\begin{aligned} S_k(p) = \inf\Big\{&y \ge 0 : P\left\{N_T \le y \mid P_0 = p\right\} \\ &\ge \frac{-p + b\,(p) + E\left[1_{\{T_{y+1} \le T\}} e^{-rT_{y+1}} f\left(P_{T_{y+1}}\right) \mid P_0 = p\right] + \gamma \bigtriangleup \Psi_k(y;p)}{b\,(p) + h\,(p)}\Big\}. \end{aligned} \tag{3.19}$$

Proof. We prove the result by induction. First note that terminal value function $V_{M+1}(x,p)$ is trivially concave. Now assume that for any $k \leq M$, $V_{k+1}(x,p)$ is concave. Note that forward differences of the one-period profit function in (3.18) is

$$\triangle g(y,p) = E\left[1_{\{T_{y+1}\leq T\}}\left(e^{-rT_{y+1}}f\left(P_{T_{y+1}}\right)+b(p)+h(p)\right)-p-h(p)\mid P_0=p\right]$$

which is also decreasing in y under Assumption (3.4). This makes $g(y,p)$ concave in y since $bE[N_T|P_0=p]$ is a constant. Since $V_{k+1}(x,p)$ is concave by induction, $\Psi_k(y;p)$ is concave which makes $G_k(y,p)$ concave. This in turn implies that $V_k(x,p) = \max_{y\geq x} G_k(y;p)+px$ is concave in x. By induction, it is true that $V_k(x,p)$ and $G_k(y;p)$ are concave for all periods k and initial price p which suggests the existence of an optimal price-dependent base-stock type policy for this inventory model. Similar to the backorder model, optimal base-stock level for any period k can be found by analyzing the forward difference of $G_k(y;p)$. More specifically,

$$\begin{aligned}
S_k(p) &= \inf\{y\geq 0 : \triangle G_k(y;p)\leq 0\} \\
&= \inf\Big\{y\geq 0 : E\left[1_{\{T_{y+1}\leq T\}}\left(e^{-rT_{Y+1}}f\left(P_{T_{y+1}}\right)+b(p)+h(p)\right)\mid P_0=p\right] \\
&\qquad -p-h(p)+\gamma\triangle\Psi_k(y;p)\leq 0\Big\} \\
&= \inf\Big\{y\geq 0 : E\left[1_{\{T_{y+1}\leq T\}}\mid P_0=p\right] \\
&\qquad \leq \frac{p+h(p)-\gamma\triangle\Psi_k(y;p)-E\left[1_{\{T_{y+1}\leq T\}}e^{-rT_{y+1}}f\left(P_{T_{y+1}}\right)\mid P_0=p\right]}{b(p)+h(p)}\Big\} \\
&= \inf\Big\{y\geq 0 : P\{N_T\geq y+1\mid P_0=p\} \\
&\qquad \leq \frac{p+h(p)-\gamma\triangle\Psi_k(y;p)-E\left[1_{\{T_{y+1}\leq T\}}e^{-rT_{y+1}}f\left(P_{T_{y+1}}\right)\mid P_0=p\right]}{b(p)+h(p)}\Big\} \\
&= \inf\Big\{y\geq 0 : P\{N_T\leq y\mid P_0=p\} \\
&\qquad \geq \frac{-p+b(p)+E\left[1_{\{T_{y+1}\leq T\}}e^{-rT_{y+1}}f\left(P_{T_{y+1}}\right)\mid P_0=p\right]+\gamma\triangle\Psi_k(y;p)}{b(p)+h(p)}\Big\}.
\end{aligned}$$

■

For the single-period problem, the optimal order quantity is

$$S_M(p) = \inf\{y\geq 0 : P\{N_T\leq y\mid P_0=p\}$$

$$\geq \frac{-p + b(p) + E\left[1_{\{T_{y+1} \leq T\}} e^{-rT_{y+1}} f\left(P_{T_{y+1}}\right) \mid P_0 = p\right]}{b(p) + h(p)} \Bigg\}. \qquad (3.20)$$

Note that Assumption 3.4 is the necessary condition for $g(y,p)$ to be concave. A sufficient condition, on the other hand, is the case of expected discounted price $z_t(p)$ being decreasing in time. We give the motivation in the following result.

Proposition 3.1 *If the expected discounted price $z_t(p)$ given initial price p is decreasing in t, then $E\left[1_{\{T_n \leq T\}}\left(e^{-rT_n} f(P_{T_n}) + b(p) + h(p)\right) \mid P_0 = p\right]$ is decreasing in n.*

Proof. Note that $(b(p) + h(p)) E\left[1_{\{T_n \leq T\}} \mid P_0 = p\right]$ is decreasing in n as arrival times T_n's form an increasing sequence which makes $1_{\{T_n \leq T\}}$ decreasing. Now define

$$\varphi(t,p) = E\left[e^{-rt} f(P_t) 1_{\{t \leq T\}} \mid P_0 = p\right]$$

Note that if $z_t(p)$ is decreasing in t, $\varphi(t,p)$ is decreasing in t as $1_{\{t \leq T\}}$ is decreasing in t. Now, we can write

$$E\left[e^{-rT_n} f(P_{T_n}) 1_{\{T_n \leq T\}} \mid P_0 = p\right] = E\left[\varphi(T_n, p)\right]$$

which is decreasing in n since T_n is increasing in n. ■

Proposition 3.1 is very easy to verify for most price processes. For instance, for the geometric Brownian motion process given earlier, the expected discounted price at time t is $z_t(p) = pe^{\left(\mu + \frac{1}{2}\sigma^2 - r\right)t}$. Observe that if $\mu + \frac{1}{2}\sigma^2 - r \leq 0$, then expected discounted price is nonincreasing in time and Assumption 3.4 is satisfied.

Unfortunately, the situation may be more complicated when Assumption 3.4 does not hold and its violation may lead to non-base-stock situations even in very simple cases. For instance, consider the following case where $f(p) = 2p$, $b = h = 0$ and $\lambda(t) = 40$, i.e., customer arrivals are Poisson, independent of the prices. Also assume that the price process is deterministic but is a function of time, such that

$$P_t = \begin{cases} 50 - 80t & 0 \leq t < 0.5 \\ 80t - 30 & 0.5 \leq t \leq 1 \end{cases}.$$

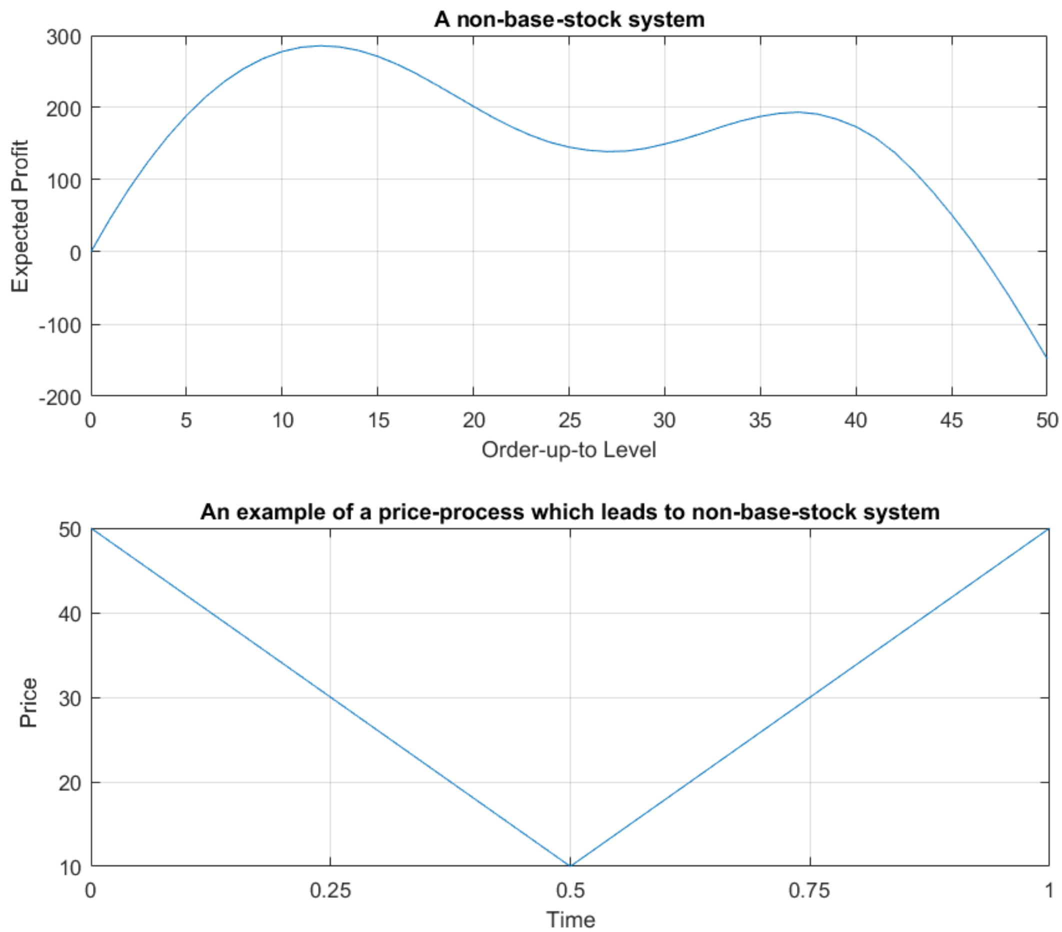

Figure 3.2: A price process that leads to a non-base-stock system.

This simple price process yields a non-base-stock system as observed in Figure 3.2 which plots the expected total profits as a function of order quantity. We observe two critical points, which are local maxima $y^{(1)} = 12$ and $y^{(2)} = 37$. The optimal policy in this case is to order $Q = 12 - x$ units when $0 \leq x < 12$, to order nothing when $12 \leq x \leq 20$, to order $Q = 37 - x$ units when $21 \leq x < 37$ and to order nothing when $x \geq 37$. It turns out that the revenue function may lose properties such as concavity (and even quasiconcavity) if the prices exhibit changing or upward patterns.

Note that we investigated optimal ordering policies for several rather general models. In the next part, we assume a specific form for the price and demand processes and derive some managerial analysis.

Special Case: Demand is Independent of Prices

For the lost-sale model, we consider a single-period special case to derive some closed form characterizations. Assume that customer arrival process is independent of stochastic price movements and let N be a Poisson process with rate λ. Also assume that P is a geometric Brownian motion process with a mean price process given by

$$E\left[P_t \mid P_0\right] = P_0 e^{\mu t}, \quad \text{for } t \geq 0$$

where $\mu \in \mathbb{R}$. For the sake of simplicity, we also assume that inventory holding and backorder costs as well as the interest rate is zero, i.e., $h(p) = b(p) = r = 0$. Assume also that selling price function is proportional to the prevailing prices and $f(p) = \alpha p$ where $\alpha \geq 1$ and $\mu < \lambda$. Then, we have the following result.

Corollary 3.2 *Optimal base-stock level is given by*

$$y^*(p) = \inf\left\{ n \geq 0 : P\left\{\overline{N}_T \leq n\right\} \geq 1 - \frac{1}{\alpha\left(\frac{\lambda}{\lambda-\mu}\right)^{n+1}} \right\}$$

where $\overline{N}_T \sim \textit{Poisson}\left((\lambda-\mu)T\right)$.

Proof. Note that by (3.20), optimal base-stock level is given by

$$\begin{aligned} y^*(p) &= \inf\left\{n \geq 0 : E\left[1_{\{T_{n+1}\leq T\}}\alpha P_{T_{n+1}} | P_0 = p\right] - p \leq 0\right\} \\ &= \inf\left\{n \geq 0 : E\left[1_{\{T_{n+1}\leq T\}}\alpha p e^{\mu T_{n+1}} | P_0 = p\right] - p \leq 0\right\}. \end{aligned} \tag{3.21}$$

Assume that $\overline{T}_n \sim \textit{Erlang}(n, \lambda-\mu)$ and $\overline{N}_T \sim \textit{Poisson}\left((\lambda-\mu)T\right)$. Since $T_{n+1} \sim \textit{Erlang}(n+1, \lambda)$, (3.21) can be written as

$$y^*(p) = \inf\left\{ n \geq 0 : \alpha p \int_{[0,T]} \frac{\lambda^{n+1} t^n e^{-\lambda t}}{n!} e^{\mu t} dt - p \leq 0 \right\}$$

$$= \inf \left\{ n \geq 0 : \alpha p \left(\frac{\lambda}{\lambda - \mu} \right)^{n+1} \int_{[0,T]} \frac{(\lambda - \mu)^{n+1} t^n e^{-(\lambda-\mu)t}}{n!} dt - p \leq 0 \right\}$$
$$= \inf \left\{ n \geq 0 : \alpha p \left(\frac{\lambda}{\lambda - \mu} \right)^{n+1} P\left\{ \overline{T}_{n+1} \leq T \right\} - p \leq 0 \right\}$$
$$= \inf \left\{ n \geq 0 : \alpha p \left(\frac{\lambda}{\lambda - \mu} \right)^{n+1} P\left\{ \overline{N}_T \geq n+1 \right\} - p \leq 0 \right\}$$
$$= \inf \left\{ n \geq 0 : P\left\{ \overline{N}_T \leq n \right\} \geq 1 - \frac{1}{\alpha \left(\frac{\lambda}{\lambda - \mu} \right)^{n+1}} \right\}. \tag{3.22}$$

■

Note that if $\mu = 0$, i.e., the price process is a martingale, then (3.22) reduces to

$$y^* (p) = \inf \left\{ n \geq 0 : P\left\{ N_T \leq n \right\} \geq \frac{\alpha - 1}{\alpha} \right\}$$

where N_T ~Poisson(λT).

We remark here that if $\mu \leq 0$, i.e., expected discounted price process is nonincreasing, Assumption 3.4 is satisfied by Proposition 3.1. Observe that if $\mu < 0$, then both left-hand and right-hand side of (3.22) will be increasing functions of n which does not guarantee the unimodality of the objective function.

Some Managerial Insights

- Note that as retail markup α increases, optimal base-stock level increases. For the special case where customer arrivals are independent of selling prices, it is straightforward that increasing the potential revenue from a sale increases the optimal order quantity. Note that this result may not be valid in the general model where customer arrival rate depends on the prevailing random prices. For instance, in the case of decreasing rate function λ, the behavior of $\alpha p \lambda (\alpha p)$ becomes important. We perform an in-depth analysis in Section 4 where markup α is also a decision variable.

- Assume that $\mu < 0$. Then, it is evident from (3.21) that as μ decreases, optimal base-stock level decreases, which is intuitive in the sense that as selling prices are decreasing and customers keep arriving at the same rate, it is optimal to reduce the ordering amounts.

3.4 Partially Backorder Setting

Backorder and lost-sale models that we examined in Section 3.2 and 3.3, respectively, can be considered as two extreme cases for the firm's operations. The reason is that in the backorder case we analyzed, all unsatisfied customers are assumed to accept backordering with probability 1 and to pay the prevailing selling price. In the lost-sale case, on the other hand, each unsatisfied customer is assumed to be lost completely. Since we explicitly use the arrival times and selling prices in the revenue calculation, we can easily extend these two extreme cases to other models where there are partial backorders and to another case where backordered customer pays the price at the time of next replenishment. These extensions are straightforward combinations of these models. In the former case, for example, we can write the total revenue as a summation similar to (3.3), but each summation term is multiplied with a probability of backorder. In the latter case where backordered customers agrees to pay the price at the end of the sales period, we can write the expected total discounted revenue as

$$r(y;p) = E\left[\sum_{n=1}^{N_T \wedge y} e^{-rT_n} f\left(P_{T_n}\right) + \left(y - N_T\right)^+ e^{-rT} f\left(P_T\right) \mid P_0 = p\right]$$

which can be treated similar to the lost-sale model.

In the next section, we relax the unit demand assumption and analyze a compound Poisson demand case.

3.5 Compound-Poisson Demand Case

So far, we assumed that each arriving customer demands a unit of the product. In this section, we extend this model to a case where each arriving customer requires random amounts of the product independent of the arrival process. Stochastic amounts of each demand forms an independent and identically distributed random sequence

$\{X_n; n \geq 1\}$ which are drawn from a continuous distribution having a cumulative distribution function F. Customer arrival process is the same as in the models previously analyzed in which it is a doubly stochastic Poisson process modulated by the market price process P. We again assume that the decision maker sets the order-up-to levels at the beginning of each period and as customers arrive, the selling price is determined according to price process P.

In compound-Poisson demand model, more interesting case is the lost sales case. Since the revenue terms will be independent of the ordering decision, the backorder case will be very similar to the previous backorder model with unit demands. Therefore, we start our analysis with the lost sale case. In particular, we assume that at any period, if the last arriving customer's demand exceeds on-hand inventory, the customer is partially satisfied. The remaining part of this sale along with future sales in that period are assumed to be lost forever. Finally, we assume that the customers will always require a positive amount, i.e., there is no possibility that they will require nothing. To this end, we define the following new notation. We let

$$D_n = \sum_{k=1}^{n} X_k$$

to denote the cumulative demand including the nth customer. We additionally define

$$N(y) = \inf\{n \geq 1; D_n \geq y\}$$

to denote the order of the last customer who makes a purchase (full or partial) for y units to be depleted. Observe that this quantity is independent of the period length T. It basically indicates how many customers should arrive for the current inventory to be sold. Note that in case of unit demand (i.e., $X_n = 1, \forall n \geq 1$), $N(y) = y$.

With the introduction of new notations, the expected total discounted profit in kth period given initial price p is given in (3.9) where the one-period expected profit now becomes

$$g(y;p) = -py + r(y;p) - c(y;p) \tag{3.23}$$

where the expected total discounted revenue is

$$r(y;p) = E\left[\sum_{n=1}^{N(y)-1} e^{-rT_n} X_n f\left(P_{T_n}\right) 1_{\{T_n \le T\}} \right. \\ \left. + e^{-rT_{N(y)}}\left(y - D_{N(y)-1}\right) f\left(P_{T_{N(y)}}\right) 1_{\left\{T_{N(y)} \le T\right\}} | P_0 = p\right] \quad (3.24)$$

and expected total inventory-related costs (lost-sale and holding) is

$$c\,(y;p) = E\left[b\,(p)\,(D_{N_T} - y)^+ + h\,(p)\,(y - D_{N_T})^+ \mid P_0 = p\right]. \quad (3.25)$$

The first summation inside the expectation in (3.24) is the total discounted revenue collected from fully satisfied customers during the period. The subsequent term, on the other hand, is the revenue collected from the possibly-last customer who is partially satisfied. Note also that, the only distinction between inventory-related expected costs between unit demand (3.5) and compound Poisson demand case (3.25) is that instead of N_T, we now write D_{N_T} to denote total amount of demand in a period. Here we also remark that in case of unit demand, (3.24) reduces to (3.13) since $N(y) = D_{N(y)} = y$.

As in the previous models, we denote future profits as

$$\Psi_k(y;p) = E\left[V_{k+1}((y - D_{N_T})^+, P_T) | P_0 = p\right]$$

and the value function for period k and the boundary condition as (3.7) and (3.15), respectively.

We now analyze the structural properties of $G_k\,(y;p)$ to find the structure of the optimal ordering policy. However, it is difficult to perform a probabilistic analysis as in the unit demand case since there are additional random variables such as $N(y)$ and X_n. We proceed with a sample path analysis on $G_k\,(y;p)$. For now, consider $N(y)$, T_n and X_n as the realizations of these random variables. Observe that the firm will only be able to sell an additional infinitesimal amount dy when $T_{N(y)} \le T$, since otherwise, the last customer will arrive after this period (although we might have a positive amount of inventory). This is due to the definition of $N(y)$. If $T_{N(y)} \le T$, the firm sells dy units with a total revenue of $dye^{-rT_{N(y)}}f\left(P_{T_{N(y)}}\right)$. Therefore, we can write the marginal expected revenue as

$$\begin{aligned} r'(y;p) &= \lim_{dy\downarrow 0}\frac{r(y+dy;p)-r(y;p)}{dy} \\ &= E\left[e^{-rT_{N(y)}}f\left(P_{T_{N(y)}}\right)1_{\left\{T_{N(y)}\le T\right\}}|P_0=p\right]. \end{aligned} \tag{3.26}$$

Note that in this analysis, the possibility that $D_{N(y)}$ is exactly y is ruled out. However, this is not an issue since $P\left\{D_{N(y)}=y\right\}=0$ as X_n's are assumed to be continuous random variables and, by definition of $N(y)$, the last customer is always partially satisfied. We remark that in the unit demand lost-sale case, profit-to-go function for any period is concave under Assumption (3.4). For the compound Poisson demand case, we also need a condition to ensure concavity.

Assumption 3.5 *The expected discounted price $z_t(p)$ is decreasing in t.*

Theorem 3.4 *$G_k(y;p)$ is concave in y and $V_k(x;p)$ is concave in x and a base-stock policy is optimal, i.e., there exists a base-stock level $S_k(p)$ for period k such that if the inventory level is less than the base-stock level, it is optimal to raise the inventory up to $S_k(p)$; otherwise, it is optimal to order nothing. Moreover, optimal base-stock level for period k is*

$$\begin{aligned} S_k(p) = \inf\{y\ge 0: P\{D_{N_T}\le y \mid P_0=p\} \\ \ge \frac{-p+b(p)+E\left[1_{\left\{T_{N(y)}\le T\right\}}f\left(P_{T_{N(y)}}\right)e^{-rT_{N(y)}}|P_0=p\right]+\gamma\Psi_k'(y;p)}{b(p)+h(p)}\right\}. \end{aligned} \tag{3.27}$$

Proof. First note that the terminal value function $V_{M+1}(x,p)=0$ is trivially concave. Now assume that for some $k<M$, $V_{k+1}(x,p)$ is concave. Then, $\gamma\Psi_k(y;p)$ is concave. Note also that $N(y)$ is increasing in y and the same reasoning as in Proposition (3.1) applies here; that is, (3.26) is decreasing if the expected discounted price is a decreasing function of time, i.e., if $z_t(p)$ is decreasing in t. Therefore, $r(y;p)$ given in (3.24) is concave. Moreover, since both $h(p)$ and $b(p)$ are positive parameters, it is

clear that $-E\left[b(p)(D_{N_T}-y)^+ + h(p)\left(y-D_{N_T}\right)^+ |P_0=p\right]$ is also a concave function and the one-period profit function $g\left(y;p\right)$ given in (3.23) is concave. Since both $g\left(y;p\right)$ and $\gamma\Psi_k(y;p)$ are concave, so is $G_k(y,p)$. This in turn makes $V_k(x,p)$ given in (3.7) concave. By induction, $V_k(x,p)$ is concave for all k. Then, it is clear that $G_k(y,p)$ is concave for all k. To characterize the optimal base-stock levels, consider the first-order optimality condition for $G_k\left(y;p\right)$,

$$\begin{aligned} G_k'\left(y;p\right) &= -p + r'(y;p) + b\left(p\right)P\left\{D_{N_T} > y \mid P_0 = p\right\} \\ &\quad - h\left(p\right)\left\{D_{N_T} \leq y \mid P_0 = p\right\} + \gamma\Psi_k'(y;p) \\ &= -p + b\left(p\right) + \gamma\Psi_k'(y;p) + E\left[e^{-rT_{N(y)}}\alpha P_{T_{N(y)}} 1_{\left\{T_{N(y)}\leq T\right\}}|P_0 = p\right] \\ &\quad - \left(h\left(p\right) + b\left(p\right)\right)P\left\{D_{N_T} \leq y \mid P_0 = p\right\} \\ &= 0 \end{aligned}$$

which can also be written as

$$P\left\{D_{N_T} \leq y \mid P_0 = p\right\} = \frac{-p + b\left(p\right) + E\left[e^{-rT_n}\alpha P_{T_{N(y)}} 1_{\left\{T_{N(y)}\leq T\right\}}\right] + \gamma\Psi_k'(y;p)}{b\left(p\right) + h\left(p\right)}. \tag{3.28}$$

However, since the distribution of D_{N_T} has a mass at $y=0$, (3.28) should be corrected as (3.27). Note that if $X_k = 1$ for all k, then (3.27) reduces to (3.19) since $N\left(y\right) = y$ and $D_{N_T} = N_T$. ■

In the case of complete backordering, the extension to compound Poisson demand is much simpler. As before, the revenue terms do not depend on the order-up-to decision y. Therefore, the analysis for this extension will be exactly the same as in Section 3.2 when we replace N_T with D_{N_T} and take expectations accordingly.

3.6 Fixed Ordering Cost Case

In previous sections, only variable unit purchase costs were incorporated in the profit function. However, it is well-known that independent of the order size, a prevalent fixed cost may be incurred for each order. This may be a fixed cost of using a vehicle of transportation for procurement, etc. Previous analysis can be extended to the case where there is a fixed order cost of $K > 0$ for each positive order amount. In this case, the value function for period k becomes

$$V_k(x,p) = G_k^*(x,p) + px \tag{3.29}$$

where

$$G_k^*(x,p) = \max\left\{G_k(x;p),\ \max_{y\geq x} G_k(y;p) - K\right\}.$$

The first function inside maximum operator refers to not ordering. In the last period, we assume that there is no fixed cost and the terminal value function is again given by $V_M(x,p) = -K - px^-$ for the backorder case and $V_M(x,p) = 0$ for the lost-sale case. Existence of a fixed order cost fundamentally changes the structure of the problem since we do not necessarily have concave profit functions as in the linear order cost case. Therefore, a base-stock policy is not usually suboptimal for this case. For this problem, the profit-to-go function that is being maximized at each period is $K-$concave.

Theorem 3.5 *$G_k(y;p)$ is K-concave for any initial price p and a price-dependent (s,S) policy is optimal, i.e., there exists $s_k(p) \leq S_k(p)$ such that whenever the inventory level x is below $s_k(p)$, it is optimal to order up to $S_k(p)$; otherwise it is optimal not to order. The optimal order-up-to level is given by* (3.10) *and* (3.19) *for the backorder and lost-sale cases, respectively and the reorder level is given by*

$$s_k(p) = \inf\left\{x \geq 0 : G_k(x,p) \geq G_k(S_k(p),p) - K\right\}.$$

Proof. Note that the proof is valid for both backorder and lost-sale cases under Assumption 3.4 for the latter. Let us proceed with the backorder case. Note that $V_M(x,p) = -K - px^-$ is K-concave. Now assume that $V_{k+1}(x,p)$ is K-concave in x for some $k \leq M-1$. Then $\Psi_k(y;p)$ given in (3.8) is K-concave which makes $\gamma\Psi_k(y;p)$ $\gamma K-$concave which in turn makes $G_k(y;p)$ in (3.9) K-concave since $g(y;p)$ is concave. Then $G_k^*(x;p)$ is also $K-$concave in x and this leads to $V_k(x,p)$ being $K-$concave (Porteus (2002)). This clearly implies that a price-dependent (s,S) policy is optimal.

■

3.7 Some Relevant Price Processes

In this section, we review some of the important financial price processes that are used to model the movements of financial instruments, commodities, exchange rates, etc. One of the most important financial price models is the geometric Brownian motion. In this model, the stock price at time t is given by the following stochastic differential equation

$$dS_t = \mu dt + \sigma dW_t$$

where W is a Wiener process, μ and σ are the drift and volatility terms. This model is the basis for Black-Scholes option pricing formulas and due to its simplicity, the calculations with this process are relatively easy and lead to closed-form solutions (see Baxter and Rennie (1996)).

Another well-known price model is the Ornstein-Uhlenbeck process where the prices follow

$$dS_t = -\kappa (\mu - S_t) dt + \sigma dW_t.$$

In this model, the prices tend to revert to their long-term mean μ with a degree of volatility σ and a reversion rate parameter κ. This model is particularly useful when one models commodity price processes as they are known to exhibit some mean-reversion (Baxter and Rennie (1996)). A more specialized model developed by Schwartz and Smith (2000), on the other hand, uses both of the above models to represent the commodity price movements by taking into account both long and short-term behaviors. In the short-term, the commodity prices show mean-reversion properties whereas in the long term they revert to an equilibrium. In particular, it is assumed that market prices follow

$$P_t = e^{\chi_t + \xi_t} \tag{3.30}$$

where χ_t is an Ornstein-Uhlenbeck process

$$d\chi_t = -\kappa \chi_t dt + \sigma_\chi dW_t^{(\chi)}$$

which models the short-term deviations by reverting towards zero. On the other hand, long-term equilibrium level ξ_t is a Brownian motion process

$$d\xi_t = \mu_\xi dt + \sigma_\xi dW_t^{(\xi)}.$$

Moreover, $W_t^{(\chi)}$ and $W_t^{(\xi)}$ are correlated Wiener processes with a correlation coefficient of ρ, i.e.,

$$dW_t^{(\chi)} dW_t^{(\xi)} = \rho dt$$

(see Schwartz and Smith (2000)).

In our numerical setup, we use a risk-neutral probability measure that makes the price process given in (3.30) a martingale to test the effect of price volatilities on the optimal expected profits and optimal controls. This is particularly interesting since changing the volatility related parameters of a martingale price process does not change its expected values in time, which we desire in order to capture the sole effect of volatility. To find a risk-neutral version of (3.30), we first define two independent Brownian motions W^1, W^2 and equivalently write

$$W_t^{(\xi)} = W_t^1$$

and

$$W_t^{(\chi)} = \rho W_t^1 + \sqrt{1-\rho^2} W_t^2.$$

Applying Ito's formula to P_t one can find that

$$dP_t = \left(-\kappa \chi_t + \mu_\xi + \sigma_\chi + \sigma_\chi \sigma_\xi \rho + \frac{\sigma_\chi^2}{2} + \frac{\sigma_\xi^2}{2} \right) P_t dt$$
$$+ (\sigma_\xi + \sigma_\chi \rho) P_t dW_t^1 + \sigma_\chi \sqrt{1-\rho^2} P_t dW_t^2.$$

Cameron-Martin-Girsanov Theorem for n-factor models state that there exists a probability measure $\mathcal{Q}$ such that

$$dP_t = \sigma_1 P_t dW_t^{(1)} + \sigma_2 P_t dW_t^{(2)} \tag{3.31}$$

is a martingale where

$$\sigma_1 = (\sigma_\xi + \sigma_\chi \rho)$$

and

$$\sigma_2 = \sigma_\chi \sqrt{1-\rho^2}$$

and $W_t^{(1)}, W_t^{(2)}$ are two independent Brownian motions with respect to Q (see, for example, Baxter and Rennie (1996)). Note that analytical Ito's solution for (3.31) is

$$P_t = P_0 e^{-1/2\left(\sigma_1^2+\sigma_2^2\right)t+\sigma_1 W_t^{(1)}+\sigma_2 W_t^{(2)}} \tag{3.32}$$

In our model, we will use (3.32) as our market price process.

The next section gives a numerical example and a sensitivity analysis on the price parameters for the lost-sale model that is analyzed in Section 3.3.

3.8 Numerical Analysis

In this section, we conduct a numerical analysis on the lost-sale model and aim to investigate the effect of some parameters (especially price related parameters) on the value function. For the demand process, we use three different rate functions, namely exponential, normal and piecewise linear rate functions. The exponential rate function is assumed to have the form

$$\lambda_E(p) = \overline{\lambda}_E e^{-\theta\alpha p} \tag{3.33}$$

where θ is a sensitivity parameter for the arriving customers. Note that this sort of a rate function applies to the cases where individual customer arrivals form an independent Poisson process with rate $\overline{\lambda}_E$ and arriving customers have i.i.d. reservation prices which are exponentially distributed random variables with parameter θ. Similarly, the normal rate function is assumed to have the form

$$\lambda_N(p) = \overline{\lambda}_N \left(1-\Phi\left(\frac{\alpha p-\mu_N}{\sigma_N}\right)\right) \tag{3.34}$$

where Φ is the cumulative distribution function of the standard normal random variable and μ_N and σ_N are the mean and standard deviation, respectively. Finally, the piecewise linear rate function is of the form

$$\lambda_L(p) = (A - B\alpha p)^+ \tag{3.35}$$

where A represents a potential arrival rate and B represents the customer sensitivity. We use these functions to test the effect of different rate functions to price changes on the optimal expected profits.

For the price process P, we use the risk-neutral model in (3.31) and employ a simulation approach to estimate the one-period expected profits since a direct analytical approach is challenging in the lost sale model for this price process. Steps of this Monte-Carlo simulation is as follows:

Simulation of Price and Arrival Paths

To simulate the price process given in (3.31) , we use $n = 100$ equally-spaced discretization of each unit of time, i.e., the interval $[0, 1]$. In addition, we use $N = 2000$ as the replication number which is statistically significant and does not lead to large computing times. Here are the steps to simulate price and arrival processes.

- Using the incremental independence and Gaussian properties of Wiener processes, we first generate N $Normal(0, 1/n)$ random variables for both $W_t^{(1)}$ and $W_t^{(2)}$. This is due to the fact that for $k = 1/n$

  $$W_{tk} - W_{t(k-1)} \sim Normal(0, 1/n).$$

  Cumulative sum of these incremental realizations gives random paths for Wiener processes $W_t^{(1)}$ and $W_t^{(2)}$. We then use these Wiener realizations in the analytical Ito's solution given in (3.32) to generate the desired price process.

- Remember that conditional on the price path, doubly stochastic process of customer arrivals reduces to an ordinary nonhomogeneous Poisson process. Therefore, for each sample path of market prices P, we generate a nonhomogeneous Poisson arrival stream. More specifically, we utilize the thinning algorithm of Lewis and Shedler (1979). According to thinning algorithm, for each price path, we find the maximum realized intensity $\lambda_{\max}$ and generate a Poisson process with this maximum rate. At each arrival time, we additionally generate an independent uniform random variable

$$U \sim Uniform(0, 1)$$

and accept the arrival if $U < \Lambda_U / \lambda_{\max}$ where Λ_U is the realized intensity at time U, i.e., $\Lambda_U = \lambda(P_U)$ for the particular rate function. This way, the stream of accepted arrivals form a nonhomogeneous Poisson arrival vector.

Note that using these price and arrival time realizations, expected revenue and profit functions can easily be computed. To solve the dynamic programs outlined in earlier sections, we also use a simulation approach along with a state space reduction approximation which are explained next.

Dynamic Programming Approximation

Since we use a two-factor geometric Brownian process to model the market price process, unbounded state space, i.e., $\mathbb{R}^+$ is very problematic for solving the dynamic programming equations to optimality. To overcome this problem, we use another discretization for the price state to compute value functions. In particular, at any period, we equally discretize possible price realizations to find price states. Moreover, we increase the number of possible discretized states as we proceed in periods. The steps of this approximation are as follows:

- For the first period $k = 1$, we find maximum and minimum values in realizations of P_T and divide the corresponding interval into $l = 100$ equal intervals whose middle points are assumed to be price states. Then, the probability distribution is also evident since as each price realization as a result of simulation falls into a particular interval.

- For any intermediate period k we use the same logic. However, since the gap between minimum and maximum price realizations increase, we now divide the corresponding interval into kl equal intervals. In this approach, one can think of state space for random prices as an horizontal and increasing cone.

Numerical Setup and Sensitivity Results

Throughout this numerical analysis, we take initial price $P_0 = 100$, selling price function $f(p) = \alpha p$ with $\alpha = 4$, fixed holding cost $h = 5$, fixed lost-sale cost $b = 20$ and the interest rate $r = 0$. For the demand rate functions, we use $A = 380$, $B = 0.8$ for the linear case, $\lambda_E = 160, \theta = 0.0025$ for the exponential case and finally $\lambda_N = 120$ and mean and standard deviation of the normal distribution as $\mu_N = 400$ and $\sigma_N = 100$, respectively. Note that each rate function gives the same result for $p = P_0 = 100$. However, each of them have different robustness to changes in price which will be important in the following sensitivity analysis.

For the lost-sale model, we first analyzed how the optimal expected profits change with respect to the magnitude of price volatility. Note that, since the price process given in (3.32) is a martingale, i.e., constant in expectation, altering the values of σ_ξ and σ_χ increases only the volatility of within-period prices. We take $M = 4$, i.e, we solve a 4-period dynamic programming recursion and with $\rho = 0.3$, $\sigma_\xi = 0.05$, we change the value of σ_χ from 0 to 0.2. As observed in 3.3, for each rate function in (3.33), (3.34) and (3.35) we observe that the optimal expected profits decrease as the price volatility increases. This suggests that price volatilities are undesirable for the firm. There are also differences in the magnitude of the effect of volatility for these rate functions. Clearly, this is due to the robustness of these functions to price changes. However, for each of the three rate functions, we observed the negative effect of increased volatility on the optimal expected profits. This observation holds for the vast majority of the cases with plausible parameter values. Only in some extreme cases where, for instance, a more volatile price process leads to much higher arrival rates, the optimal expected profits increase. In terms of optimal base-stock levels, on the other hand, we do not particularly observe any monotonicity with respect to price volatility.

A similar sensitivity analysis can also be conducted to observe the effect of correlation parameter ρ. We observe that as the value of ρ increases, the value of optimal expected profits decrease. This is again due to the fact that a higher ρ means a more

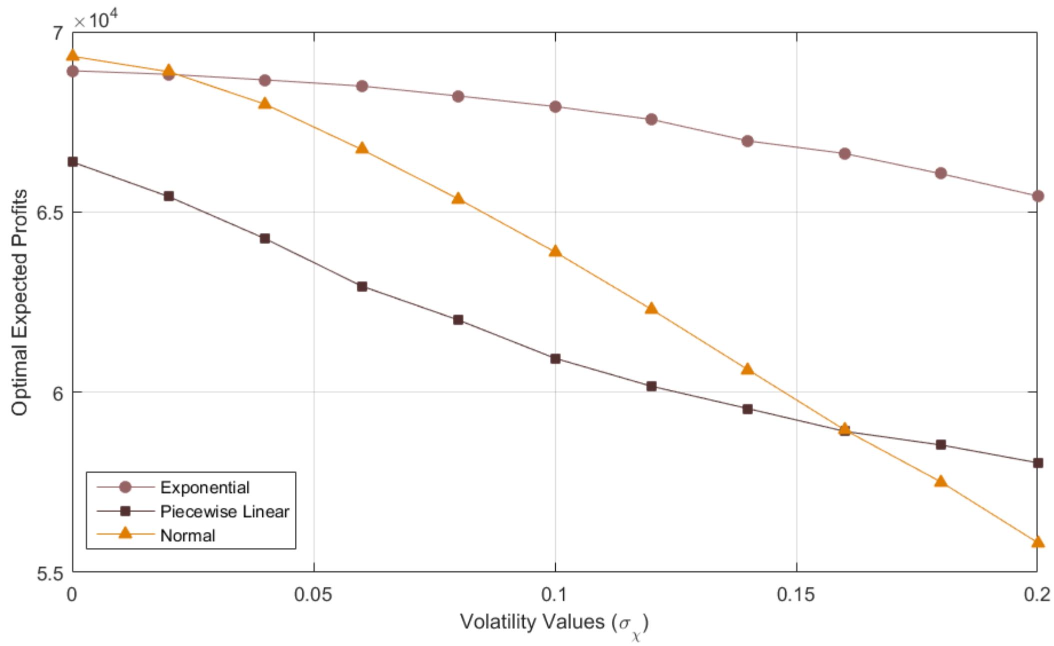

Figure 3.3: Effect of price volatility on optimal expected profits.

volatile price process. More specifically, the variance of P_t given in (3.32) is

$$Var\left(P_t\right) = P_0^2\left(e^{\sigma_\xi^2 + 2\sigma_\xi\sigma_\chi\rho + \sigma_\chi^2} - 1\right)$$

which is an increasing function of ρ.

In another numerical setup, we compare our proposed model with a deterministic approximation model to test the effectiveness of modeling price fluctuations explicitly. In particular, we take the model in Kalymon (1971) as benchmark, where prices are constant within sales periods, however they are still random with the same probability distribution at the end (and beginning) of each period. We again use the price process given in (3.31) with $\rho = 0.3$ and $\sigma_\xi = 0.05$. Note that, since we use a martingale price process, the expected prices do not differ from the initial price in time. For a given volatility level σ_χ, we find the optimal base-stock levels for both models and use them in the proposed model that considers within-period price fluctuations to make a consistent comparison between resulting expected profits. We use the piecewise linear rate function given in (3.35) as the rate process and assume that $B = 0.8$.

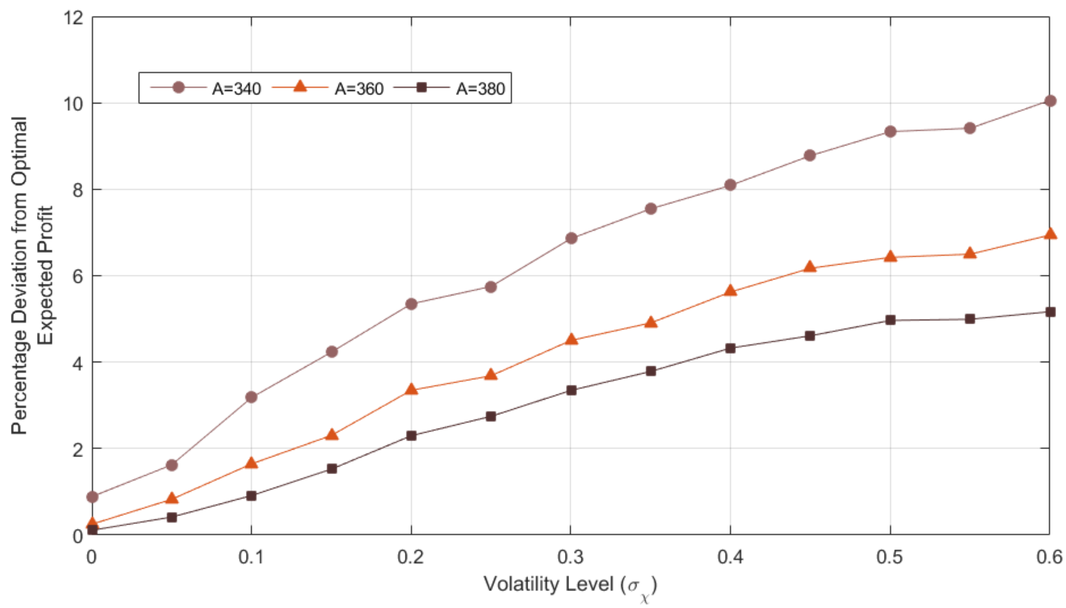

Figure 3.4: Deviation from optimal results when approximate model is used.

Other parameters are assumed as $P_0 = 100$, $\alpha = 4$, $b = 20$, $h = 5$, $T = 1$. For three potential customer arrival rates ($A = 340$, $A = 360$, $A = 380$), Figure 3.4 shows the percentage deviation from optimal expected profits for different volatility (σ_χ) levels.

Figure 3.4 shows that as prices get more volatile, then the benefit of using the proposed model that explicitly considers within-period price fluctuations increases. We also note that the benefit of using the proposed model greatly increases when the potential arrival rate A decreases since a lower A implies an arrival rate process which is more prone to price increases. In other words, if A is low, then there will be more occurrences with zero arrival rate if price are more volatile.

Although it is intuitive, we also remark here that as period length T increases, then the gap between deterministic approximation and the proposed model increases. This can be observed in Figure 3.5 which plots the percentage deviation from optimal expected profit with respect to changes in within-period length when second approximate model is used. This is for the single-period model and the period length is increased from $T = 0.6$ to $T = 3$ for three different volatility levels.

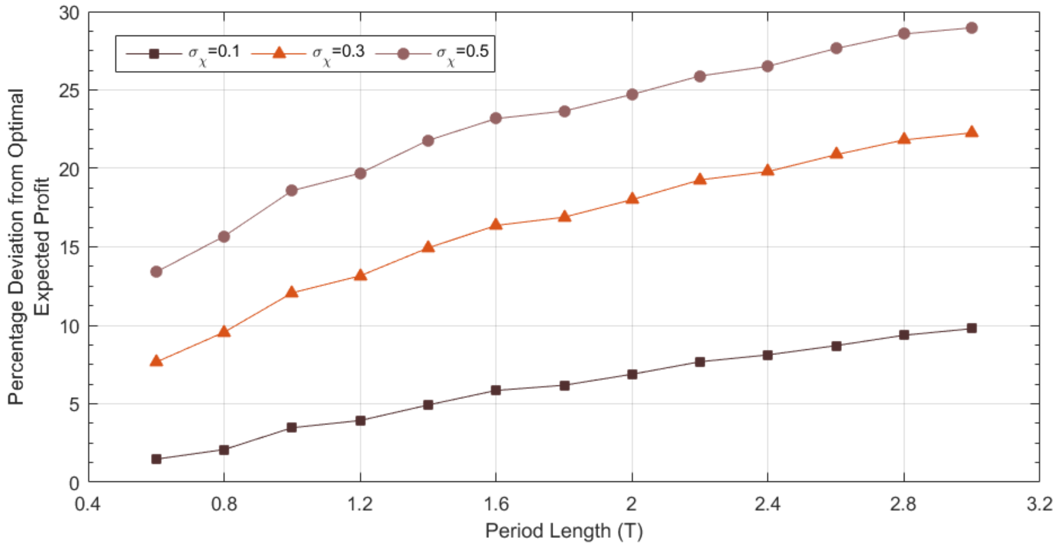

Figure 3.5: Effect of period length on the gap with the approximate model.

We also observe that as the number of periods increase, the percentage deviation from optimal expected profits decreases. This is also intuitive as the decision maker has additional opportunities to react to price changes as the number of ordering periods are higher. The effect of the number of periods can be observed in Figure 3.6 which plots the percentage deviation from optimal expected profit when the benchmark model is used as an approximation with respect to number of periods.

3.9 Summary

In this chapter, we analyzed an inventory management problem where purchase and selling prices are described by a continuous-time stochastic price process which also influences the customer demand. In contrast with most of the existing literature, within each period demand arrives continuously and is influenced by the continuous price process. In this setting, sales revenues depend on individual arrival times of demands and not only on total accumulated demand. We show that for the backorder case, price-dependent base-stock policies are optimal under standard assumptions.

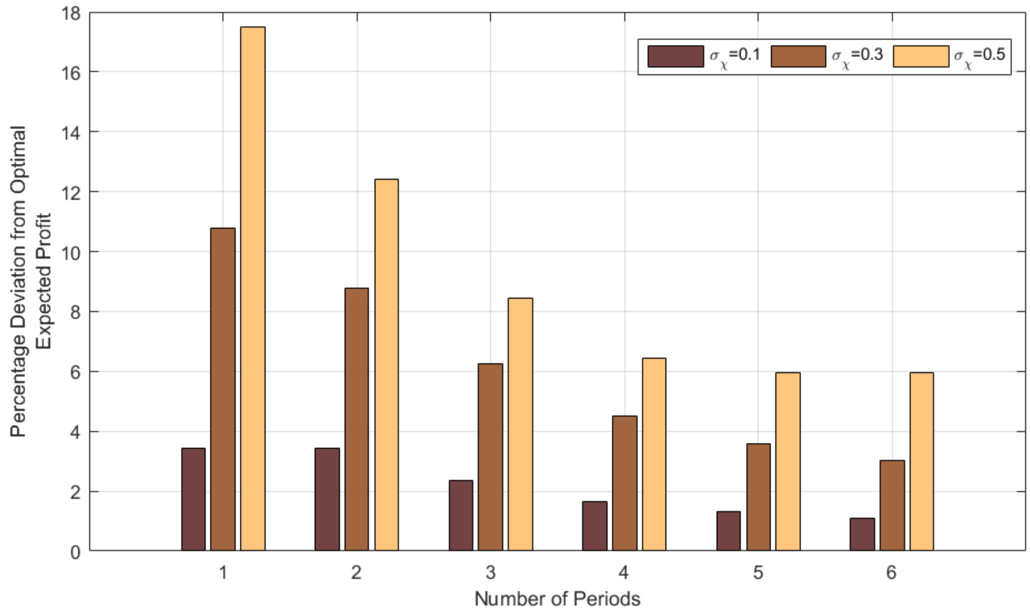

Figure 3.6: Effect of number of periods on the gap with the approximate model.

This implies that in any ordering period, the firm's order-up-to decision is only affected from observed market price, which stochastically describes the evolution of upcoming prices and customer arrivals. Moreover, this also extends to the more challenging lost sales and compound Poisson cases under additional plausible conditions. A violation of these may lead to non-base-stock environment even for very simple price cases. In a numerical setting where market prices are modeled as a two-factor price process, it is observed that price volatility has a significant effect on expected profits and optimal inventory policies. In particular, if expected future prices remain the same, increased volatility leads to smaller expected profits. In addition, these examples illustrate that modeling within-period price fluctuations in contrast with available models in existing literature is more advantageous as the level of price volatility increases. This is observed by comparing the proposed model and several approximate models that ignore price fluctuations. It is also observed that as period lengths increase or number of ordering opportunities decrease, then the advantage of explicit modeling increases.

Chapter 4

MARKUP PRICING IN THE PRESENCE OF PRICE FLUCTUATIONS

Besides a successful inventory management policy, pricing is one of the most effective tools that a firm has in order to increase its revenues. The impact of a successful pricing strategy lies in its effect on sales as price is one of most critical determinants of customer demand. By effectively controlling the demand, firms also have the potential to create a more efficient supply chain due to decreased variability and better management of the mismatch between supply and demand. Even more value can be created by integrating pricing decisions with inventory, production and distribution decisions. As much as it is essential for manufacturing firms to coordinate these decisions rather than employing a decentralized approach, integration of pricing and inventory decisions has the potential of thoroughly increasing supply chain effectiveness (Chan et al. (2004)).

In classical inventory-pricing models, firms are assumed to control the sales prices freely. However, for some firms, the nature of the inventory item may prevent them from controlling the selling prices fully. An example is for firms selling and/or producing commodity-based products whose underlying prices are market-determined and prevailing input price volatilities pass to customers to an extent. If one sells a commodity-based item, for instance, material, labor and overhead costs as well as prevailing commodity prices need to be considered in determining the selling price to stay profitable. Although it may be argued that profit margins are only based on marginal costs (more specifically, wholesale price at the time of inventory replenishment), the decision maker can not overlook prevailing market prices at the time of customer demand due to high competition. On the other hand, accounting for volatilities in pricing decisions is rather overlooked in the literature.

In this chapter, we further delve into backorder model analyzed in Section 3.2 and investigate how altering the selling prices affects firm's profitability. In particular, we now assume that the firm sells a rather specialized product to its customers and has the power to influence the sales prices. With this model, we contribute to the literature by incorporating the effect of fluctuating commodity prices into inventory and pricing decisions. Unlike traditional models that use a demand function that has an error term and a deterministic part which changes with respect to pricing decision, we assume that the firm sets a proportional markup for constantly changing market prices. Again, customer arrivals are modeled as a process that is modulated by the stochastic price process as well as the markup decision. In this section, we also investigate how the level of fluctuations in market prices affect optimal performance measures.

4.1 The Model

As in Chapter 3, we again assume that stochastic evolution of the market prices is given by the process $P = \{P_t\ ; t \geq 0\}$ with state space $\mathbb{R}^+ = (0, \infty)$. For this section, we do not necessarily assume that P is Markovian and time-stationary. However, we assume that P has continuous sample paths. We use a particular multiplicative form for the selling price function f and assume that the selling price for the item at any time t is merely αP_t where $\alpha > 1$ is the firm's markup. This implies that the selling prices are both determined by the stochastic price process and markup. The firm sets the proportional markup α and ordering decision y at the beginning of sales horizon $t = 0$. Note that unlike traditional pricing models in the literature, in this model the firm cannot fully determine the selling price. To a degree, the firm has to pass price fluctuations to the customer, but has the freedom to influence it via a proportional constant. This is an appropriate setting if the firm sells products where inherent price fluctuations somehow pass to the customer which is typical in jewelry, or gold retailers. In addition, this setting applies to the non-price-taker firms which sell exclusive products so that they have more freedom to control markups.

For this model, to account for random prices and their affects on customer demand, we again assume that individual customers arrive according to a doubly-stochastic Poisson process with a stochastic intensity measure $\Lambda^{\alpha} = \{\Lambda_t^{\alpha} = \lambda(\alpha P_t); t \geq 0\}$ where $\lambda(.)$ is a nonnegative deterministic function of random selling price. Note that we changed the notation for stochastic intensity process to Λ^{α} to denote its connection to markup control. For the operational setting, we use the compound Poisson model analyzed in Section 3.5 where at each arrival, customers demand a random amount of product independent of the price process. Remember that we let $X = \{X_n; n \geq 1\}$ denote the stochastic individual demands where we assume that each X_n is positive, independent and identically distributed with common expectation μ.

We also denote the customer arrival process as $N^{\alpha} = \{N_t^{\alpha}; t \geq 0\}$ where superscript α denotes its connection to our control variable α. We additionally define D_n as the cumulative demand by nth customers so that

$$D_n = \sum_{k=1}^{n} X_k.$$

With this notation, total random demand during sales season is $D_{N_T^{\alpha}}$.

We assume a backorder setting and assume that if there is not enough on-hand inventory, newly arrived customers are charged at the prevailing selling price and satisfied at time $t = T$. This case is applicable to situations where the firm sells exclusive products such that arriving customers may be unable to find elsewhere. Jewelry stores, for instance, usually take orders for diamond rings etc. to be supplied later, yet their selling prices are determined considering the current market prices of diamond and gold at the time of customer order, not the market prices at the time of delivery. To keep the model simpler, we do not assume any physical holding cost or salvage revenue in our analysis, i.e., $h = 0$, yet by discounting all future cash flows, we are capturing the opportunity costs associated with the firm's capital investment. We assume that the firm incurs a penalty cost of b for each unit of backordered demand. The firm needs to raise the inventory level up to zero by purchasing at the market price P_T at time T in case of backorders during the sales season. .

We assume that $b+E[P_T]-P_0>0$ where P_0 is the initial price at time 0, which states that underage cost, i.e., cost of ordering one less unit is positive. If this is not satisfied, then the firm does not order at all and simply backorders each arriving customer. The objective of the firm is to maximize the expected total discounted profit by simultaneously setting order-up-to level and proportional markup.

Now, let R_T^α denote the total random revenue until time T, which is generated by summing individual revenues from sales. More specifically,

$$R_T^\alpha=\sum_{n=1}^{N_T^\alpha} e^{-rT_n}\alpha P_{T_n} X_n$$

where T_n denotes the arrival time of nth customer and, as stated before, N_T^α is the total number of individual customers arrived by time T when markup is α. Also, $\alpha P_{T_n} X_n$ is the random revenue obtained from the nth customer. We also discount all individual revenues to time 0 by multiplying them with e^{-rT_n} where r is the interest rate per unit time. The risk-neutral firm is concerned with the expected revenue until time T as a function of retail markup α, which we denote as

$$r_T(\alpha)=E\left[R_T^\alpha\right].$$

In a compact form, expected total sales revenues until time T can be written as

$$r_T(\alpha)=\mu\int_0^T e^{-rt}E\left[\alpha P_t\lambda\left(\alpha P_t\right)\right]dt \tag{4.1}$$

where $\mu=E\left[X_n\right]$. Note that this is a special case of expected total revenue given in (3.3) with $f\left(p\right)=\alpha p$ and $E\left[X_n\right]=\mu$ whose derivation is given in Appendix. We will also frequently use the probability distribution of N_T^α although it does not appear in (4.1). But first let us define $\mathcal{P}=\{P_t;t\in[0,T]\}$ as the random prices in the sales horizon $[0,T]$. With this notation, N_T^α, total number of individual customers who arrived during $[0,T]$ is Poisson with conditional random mean

$$M_T^\alpha=E\left[N_T^\alpha\mid\mathcal{P}\right]=\int_0^T\lambda(\alpha P_t)dt. \tag{4.2}$$

Expected total sales, on the other hand, is

$$d_T(\alpha) = E\left[D_{N_T^\alpha}\right] = E\left[\sum_{n=1}^{N_T^\alpha} X_n\right] = \mu E\left[M_T^\alpha\right]$$
$$= \mu \int_0^T E\left[\lambda(\alpha P_t)\right] dt. \tag{4.3}$$

Assuming that there is no initial inventory, we write the expected total profit as a function of markup α and order-up-to level y as

$$g(y, \alpha) = -P_0 y + r_T(\alpha) - E\left[(b + P_T)\left(D_{N_T^\alpha} - y\right)^+\right] \tag{4.4}$$

where the first term denotes the total purchase cost, the second term denotes the expected total discounted revenue and the last term denotes the backorder and re-purchase costs. The objective of the decision maker is to solve

$$\max_{\alpha > 0, y \geq 0} g(y, \alpha)$$

by choosing a proportional markup $\alpha \in (0, \infty)$ and an order-up-to level $y \in [0, \infty)$ Next section characterizes the form of the optimal inventory-markup pricing policy.

4.2 Optimal Inventory Control & Markup Pricing

In this section, we analyze the behavior of the expected profit function $g(y, \alpha)$ with respect to y and α and corresponding optimal inventory and markup pricing strategies. We begin by analyzing the optimal inventory policy for a fixed markup decision α. Note that for the backorder case, assuming a general sales-price function and a unit-demand setting, we already characterized the form of optimal ordering policies in Section 3.2. On the other hand, we only presented results and derivations for the lost-sale compound Poisson case in Section 3.5. Below, we give the result for the backorder and compound-Poisson demand case.

Optimal Inventory Policy for a Given Markup

In the following sections, we will use $\phi(y;\alpha) = P\left\{D_{N_T^\alpha} = y\right\}$ and $\Phi(y;\alpha) = P\left\{D_{N_T^\alpha} < y\right\}$ to denote the probability density and cumulative distribution functions of $D_{N_T^\alpha}$ evaluated at y, respectively.

Theorem 4.1 *Given markup α, $g(y,\alpha)$ is concave in order-up-to level y and a base-stock policy is optimal, i.e., it is optimal to order up to the optimal base-stock level $y^*(\alpha)$ if initial inventory is less than $y^*(\alpha)$; otherwise, it is optimal to order nothing. The optimal base-stock level is given by*

$$y^*(\alpha) = \inf\left\{y \geq 0 : E\left[(b+P_T)\,1_{\left\{D_{N_T^\alpha} \leq y\right\}}\right] \geq b + E[P_T] - P_0\right\}. \tag{4.5}$$

Proof. The first and second-order derivatives of $g(y,a)$ with respect to y is given by

$$\begin{aligned} g_y(y,\alpha) &= -P_0 + E\left[(b+P_T)\,1_{\left\{D_{N_T^\alpha} \geq y\right\}}\right] \\ &= -P_0 + b + E[P_T] - E\left[(b+P_T)\,1_{\left\{D_{N_T^\alpha} < y\right\}}\right] \end{aligned} \tag{4.6}$$

and

$$g_{yy}(y,\alpha) = -E\left[(b+P_T)\frac{\partial}{\partial y}E\left[1_{\left\{D_{N_T^\alpha} < y\right\}} \mid \mathcal{P}\right]\right] \tag{4.7}$$

$$= -E\left[(b+P_T)\frac{\partial}{\partial y}\Phi(y;\alpha|\mathcal{P})\right] \tag{4.8}$$

$$= -E\left[(b+P_T)\,\phi(y;\alpha|\mathcal{P})\right]. \tag{4.9}$$

Observe that (4.7) is negative for all y and α. Since expected profit is concave, a base-stock policy is optimal. For each markup level α, the optimal base-stock level is the maximizer of $g(y,\alpha)$ which is found by,

$$\begin{aligned} y^*(\alpha) &= \inf\left\{y : -P_0 + b + E[P_T] - E\left[(b+P_T)\,1_{\left\{D_{N_T^\alpha} < y\right\}}\right] \leq 0\right\} \\ &= \inf\left\{y : E\left[(b+P_T)\,1_{\left\{D_{N_T^\alpha} \leq y\right\}}\right] \geq b + E[P_T] - P_0\right\}. \end{aligned}$$

■

Concavity of the objective function ensures that a base-stock inventory policy is optimal given markup and optimal base-stock level is given by (4.5). In the next section, we analyze the behavior of expected total profit function with respect to the markup for fixed inventory level.

Optimal Markup for a Given Inventory Level

In this section, we make the following reasonable assumptions and prove a series of lemmas that will lead to our main characterization. Analogues of these assumptions in models without price volatilities are quite common in pricing literature, Ziya et al. (2004).

Assumption 4.1 $\lambda(x)$ *is convex decreasing.*

Assumption 4.2 $x\lambda(x)$ *is concave.*

The next two lemmas establish that the expected total revenue is concave and expected total sales is convex in markup level.

Lemma 4.1 *The expected total discounted revenue* $r_T(\alpha)$ *is concave in markup* α.

Proof. Note that since $x\lambda(x)$ is concave, $\alpha P_t \lambda(\alpha P_t)$ is concave in α for each P_t which makes $e^{-rt}E[\alpha P_t \lambda(\alpha P_t)]$, hence $r_T(\alpha)$ given in (4.1) is concave in α. ■

Lemma 4.2 *The expected total demand (sales)* $d_T(\alpha)$ *is convex decreasing in markup* α.

Proof. Since $\lambda(.)$ is convex decreasing, $\lambda(\alpha P_t)$ is convex decreasing for each P_t which makes $E[\lambda(\alpha P_t)]$, hence $d_T(\alpha)$ given in (4.3) convex decreasing in α. ■

From now on, we also use the notation

$$\triangle_k E\left[(y - D_k)^+\right] = E\left[(y - D_{k+1})^+ - (y - D_k)^+\right]$$

and

$$\triangle_k^2 E\left[(y - D_k)^+\right] = E\left[(y - D_{k+2})^+ - 2(y - D_{k+1})^+ + (y - D_k)^+\right]$$

to denote the first and second-order forward differences. We will also make use of the following lemma in forthcoming analysis.

Lemma 4.3 *$E\left[(y-D_k)^+\right]$ and $E\left[(D_k-y)^+\right]$ are integer convex in k.*

Proof. For any discrete function to be integer convex, the second-order forward differences should be positive. Note that,

$$\begin{aligned}\triangle_k E\left[(y-D_k)^+\right] &= E\left[(y-D_{k+1})^+ - (y-D_k)^+\right] \\ &= E\left[(y-D_k-X_{k+1})^+ - (y-D_k)^+\right].\end{aligned}$$

Using

$$(a-b)^+ = a - min\{a,b\}$$

for any $a, b \in \mathbb{R}$, we can write

$$\triangle_k E\left[(y-D_k)^+\right] = -E\left[\min\left\{X_{k+1}, (y-D_k)^+\right\}\right]. \tag{4.10}$$

As k increases, $-E\left[\min\left\{X_{k+1}, (y-D_k)^+\right\}\right]$ increases so that the second-order difference with respect to k is nonnegative, i.e., $\triangle_k^2 E\left[(y-D_k)^+\right] \geq 0$. Similarly,

$$E\left[(D_k-y)^+\right] = E\left[D_k - y + (y-D_k)^+\right]$$

is also integer convex in k. ■

In the following characterizations, $(M_T^\alpha)' = \frac{\partial}{\partial\alpha} M_T^\alpha$, $(M_T^\alpha)" = \frac{\partial^2}{\partial\alpha^2} M_T^\alpha$, $r_T'(\alpha) = \frac{\partial}{\partial\alpha} r_T^\alpha$ and $r_T''(\alpha) = \frac{\partial^2}{\partial\alpha^2} r_T^\alpha$ denote first and second order derivatives of M_T^α and r_T^α given in (4.2) and (4.1) respectively.

Theorem 4.2 *Assume that y is fixed. Then $g(y,\alpha)$ is concave in α and optimal markup $\alpha^*(y)$ is found by solving*

$$r_T'(\alpha) - \mu E\left[(b+P_T)(M_T^\alpha)'\right] + E\left[(b+P_T)(M_T^\alpha)' \min\left\{\left(y-D_{N_T^\alpha}\right)^+, X_{N_T^\alpha+1}\right\}\right] = 0. \tag{4.11}$$

Proof. It is shown in the Appendix that the first and second-order partial derivatives of $g(y,a)$ with respect to α are given by

$$\begin{aligned} g_\alpha(y,\alpha) &= r_T'(\alpha) - \mu E\left[(b+P_T)(M_T^\alpha)'\right] \\ &\quad + E\left[(b+P_T)(M_T^\alpha)'\min\left\{\left(y-D_{N_T^\alpha}\right)^+, X_{N_T^\alpha+1}\right\}\right] \end{aligned}$$

and

$$\begin{aligned} g_{\alpha\alpha}(y,\alpha) &= r_T''(\alpha) - \mu E\left[(b+P_T)(M_T^\alpha)''\right] - E\left[(b+P_T)(M_T^\alpha)'' E\left[\triangle\left(y-D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]\right] \\ &\quad - E\left[(b+P_T)\left((M_T^\alpha)'\right)^2 E\left[\triangle^2\left(y-D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]\right], \end{aligned}$$

respectively. By Lemma 4.1 and Lemma 4.2, $r_T''(\alpha)<0$ and $(M_T^\alpha)''>0$. Additionally, by Lemma 4.3, the last term is also negative. Observe also that,

$$E\left[\triangle\left(y-D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right] = -E\left[\min\left\{X_{N_T^\alpha+1}, \left(y-D_{N_T^\alpha}\right)^+\right\} \mid \mathcal{P}\right] \geq -E\left[X_{N_T^\alpha+1}\right] = -\mu.$$

Then the following inequality holds:

$$\begin{aligned} g_{\alpha\alpha}(y,\alpha) &\leq r_T''(\alpha) - \mu E\left[(b+P_T)(M_T^\alpha)''\right] + \mu E\left[(b+P_T)(M_T^\alpha)''\right] \\ &\quad - E\left[(b+P_T)\left((M_T^\alpha)'\right)^2 E\left[\triangle^2\left(y-D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]\right] \\ &= r_T''(\alpha) - E\left[(b+P_T)\left((M_T^\alpha)'\right)^2 E\left[\triangle^2\left(y-D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]\right] \leq 0. \end{aligned}$$

The last inequality is due to Lemma 4.3 and $r_T''(\alpha)\leq 0$. Since $g_{\alpha\alpha}(y,\alpha)\leq 0$, expected total profit is concave in markup α for each inventory level y. We can find the optimal markup by setting the first partial derivative with respect to α equal to zero, i.e., $g_\alpha(y,\alpha)=0$, which is given in (4.11). ■

In Theorem 4.1 and Theorem 4.2, we characterize the optimality equations for controls y and α. The decision maker simultaneously solves equations (4.5) and (4.11) to find the optimal controls. Next proposition explains that the expected profit function is submodular in two control variables which leads to several monotonicity properties for the optimal controls.

Proposition 4.1 *$g(y,a)$ is submodular. Consequently, optimal inventory level $y^*(\alpha)$ is decreasing in α and optimal markup $\alpha^*(y)$ is decreasing in y.*

Proof. We show the submodularity of $g(y,\alpha)$ by showing that $g_{y\alpha}(y,\alpha) < 0$. Note that

$$g_{y\alpha}(y,\alpha) = -E\left[(b+P_T)(M_T^\alpha)'\left(P\left\{D_{N_T^\alpha}+1<y \mid \mathcal{P}\right\} - P\left\{D_{N_T^\alpha}<y \mid \mathcal{P}\right\}\right)\right]$$

as shown in the Appendix. Note that since $(M_T^\alpha)' \leq 0$ and $P\left\{D_{N_T^\alpha+1} \leq y \mid \mathcal{P}\right\} \leq P\left\{D_{N_T^\alpha} \leq y \mid \mathcal{P}\right\}$, the derivative of the expected profit function with respect to each variable is negative, i.e., $g_{y\alpha}(y,\alpha) < 0$. Moreover, observe that in (4.5), as α increases $1_{\left\{D_{N_T^\alpha}<y\right\}}$ increases for fixed y which results in lower $y^*(\alpha)$. Similarly, since $g_{y\alpha}(y,\alpha) < 0$, $g_\alpha(y,\alpha)$ is lower for higher values of y that is $g_\alpha(y,\alpha)$ is decreasing in y. Then, it is clear that $\alpha^*(y)$ is lower for higher values of y. ■

In this section, we explicitly characterize the optimality equations for the two controls, markup and inventory level, and prove that optimal decisions are decreasing functions of each other. In other words, if the firm has more inventory on hand, for instance, he should charge a lower markup, which is highly intuitive. Similarly, for higher values of markup, if given, the firm needs to hold less inventory as less number of customers are expected to arrive during the sales season. In the next section, we theoretically analyze the effect of volatile market prices on the expected revenues, sales and profits.

4.3 The Effect of Price Variability on Expected Profit

In our model, we used a general stochastic price process where we assume that it has continuous and nonnegative price paths. In this section we analyze how the optimal expected profits change with respect to the variability of this price process. We use convex ordering of random variables and stochastic processes in our analysis. The following definitions are from Müller and Stoyan (2002).

Definition 4.3 *Let X and Y denote two generic random variables. X is said to precede Y in convex order (increasing convex order, decreasing convex order) if $E[f(X)] \leq E[f(Y)]$ for all convex (increasing convex, decreasing convex) functions f, i.e.,*

$$X \underset{cx(icx,dcx)}{\leq} Y \Leftrightarrow E[f(X)] \leq E[f(Y)]$$

for all convex (increasing convex, decreasing convex) functions f.

Similar to convex ordering of random variables, the definition of convex ordering of stochastic processes is the following.

Definition 4.4 *Let $X = \{X_t; t \geq 0\}$ and $Y = \{Y_t; t \geq 0\}$ denote two stochastic processes. Then,*

$$X \underset{cx,icx,dcx}{\leq} Y \Leftrightarrow E\left[f\left(X_t\right)\right] \leq E\left[f\left(Y_t\right)\right]$$

for all $t \geq 0$ and for all convex (increasing convex, decreasing convex) functions f.

In other words, two stochastic processes are said to be convexly ordered if random values at each time are convexly ordered. Convex orders are generally used to order random variables in terms of their variabilities. A property of the convex orders is the following.

Remark 4.5 *If $X \underset{cx}{\leq} Y$, then $E[X] = E[Y]$ and $Var(X) \leq Var(Y)$. That is, convexly ordered random variables are also ordered in magnitude of their variances although their mean are the same.*

In the rest of this chapter, we will use two market price processes, namely, $P^{(1)} = \left\{P_t^{(1)}; t \geq 0\right\}$ and $P^{(2)} = \left\{P_t^{(2)}; t \geq 0\right\}$ to compare the expected revenues, profits and sales that are previously examined. Analogously, we denote their corresponding rate processes (intensity measures) as $\Lambda^{(1)}$ and $\Lambda^{(2)}$ where $\Lambda^{(i)} = \left\{\Lambda_t^{(i)} = \lambda\left(\alpha P_t^{(i)}\right); t \geq 0\right\}$ for $i = 1, 2$ and corresponding counting measures as $N^{(i)} = \left\{N_t^{(i)}; t \geq 0\right\}$. Let us denote $r_T^{(i)}(\alpha)$, $d_T^{(1)}(\alpha)$ and $g^{(i)}(y, \alpha)$ as the expected revenue, expected sales and expect profit functions under market price process $P^{(i)}$ for $i = 1, 2$.

Next two lemmas show that a more variable price process leads to lower expected revenues and higher expected sales.

Proposition 4.2 *If $P^{(1)} \underset{cx}{\leq} P^{(2)}$, then $r_T^{(1)}(\alpha) \geq r_T^{(2)}(\alpha)$ for each $\alpha \in \mathbb{R}^+$.*

Proof. Note that since $\alpha P_t \lambda(\alpha P_t)$ is a concave function of P_t by Assumption 4.2, $P_t^{(1)} \underset{cx}{\leq} P_t^{(2)}$ implies

$$E\left[\alpha P_t^{(1)} \lambda\left(\alpha P_t^{(1)}\right)\right] \geq E\left[\alpha P_t^{(2)} \lambda\left(\alpha P_t^{(2)}\right)\right].$$

Then it is easy to see that

$$r_T^{(1)}(\alpha) = \mu \int_0^T e^{-rt} E\left[\alpha P_t^{(1)} \lambda\left(\alpha P_t^{(1)}\right)\right] dt \geq \mu \int_0^T e^{-rt} E\left[\alpha P_t^{(2)} \lambda\left(\alpha P_t^{(2)}\right)\right] dt = r_T^{(2)}(\alpha).$$

■

Proposition 4.3 *If $P^{(1)} \underset{cx}{\leq} P^{(2)}$, then $d_T^{(1)}(\alpha) \leq d_T^{(2)}(\alpha)$ for each $\alpha \in \mathbb{R}^+$.*

Proof. Similar to the proof of Proposition 4.2, since $\lambda(\alpha P_t)$ is a convex function of P_t by Assumption 4.1, $P_t^{(1)} \underset{cx}{\leq} P_t^{(2)}$ implies

$$E\left[\lambda\left(\alpha P_t^{(1)}\right)\right] \leq E\left[\lambda\left(\alpha P_t^{(2)}\right)\right].$$

Then it is easy to see that

$$d_T^{(1)}(\alpha) = \mu \int_0^T E\left[\lambda\left(\alpha P_t^{(1)}\right)\right] dt \leq \mu \int_0^T E\left[\lambda\left(\alpha P_t^{(2)}\right)\right] dt = d_T^{(2)}(\alpha).$$

■

Next proposition proves that if the market price processes are convexly ordered, so are their rate processes.

Proposition 4.4 *If $P^{(1)} \underset{cx}{\leq} P^{(2)}$, then $\Lambda^{(1)} \underset{icx}{\leq} \Lambda^{(2)}$.*

Proof. Let $t \geq 0$ be fixed and f be an increasing convex function. Then, note that $f(\lambda)$ is convex where it can be shown

$$f(\lambda)' = \lambda' f'(\lambda)$$

and

$$f(\lambda)'' = \lambda'' f'(\lambda) + (\lambda')^2 f''(\lambda) \geq 0$$

since λ is convex by Assumption 4.1 and $f' \geq 0$ by the assumption. Since $f(\lambda)$ is convex,

$$E\left[f\left(\Lambda_t^{(1)}\right)\right] = E\left[f\left(\lambda\left(\alpha P_t^{(1)}\right)\right)\right] \leq E\left[f\left(\lambda\left(\alpha P_t^{(2)}\right)\right)\right] = E\left[f\left(\Lambda_t^{(2)}\right)\right].$$

Since this is true for any convex increasing f and $t \geq 0$, $\Lambda^{(1)} \underset{icx}{\leq} \Lambda^{(2)}$. ■

We use the following proposition from Błaszczyszyn and Yogeshwaran (2009) that links the convex ordering of intensity measure of a doubly-stochastic Poisson process to the counting measure.

Proposition 4.5 $\Lambda^{(1)} \underset{cx,icx,dcx}{\leq} \Lambda^{(2)}$ *implies* $N^{(1)} \underset{cx,icx,dcx}{\leq} N^{(2)}$.

Proof. See Błaszczyszyn and Yogeshwaran (2009). ■

Note that by Proposition 4.4 and Proposition 4.5, one can straightforwardly assert the following corollary.

Corollary 4.1 $P^{(1)} \underset{cx}{\leq} P^{(2)}$ *implies* $N^{(1)} \underset{icx}{\leq} N^{(2)}$.

Proof. Follows from Proposition 4.4 and Proposition 4.5. ■

Lemma 4.4 $P^{(1)} \underset{cx}{\leq} P^{(2)}$ *implies* $E\left[P_T^{(1)}(D_{N_T^{\alpha(1)}} - y)^+\right] \leq E\left[P_T^{(2)}(D_{N_T^{\alpha(2)}} - y)^+\right]$.

Proof. Assume that $P^{(1)} \underset{cx}{\leq} P^{(2)}$. Then by Corollary 4.1, $N^{(1)} \underset{icx}{\leq} N^{(2)}$. Together with this relationship, we will use the relationship between convex ordering of conditional random variables and convex ordering of their unconditional counterparts. It is known that if two random variables are convex ordered, so are their conditional counterparts, (Leskelä et al. (2017)). Then, we can write

$$\begin{aligned} P^{(1)} \underset{cx}{\leq} P^{(2)} &\Rightarrow \left\{P_t^{(1)}; t \in [0,T)\right\} \mid P_T \underset{cx}{\leq} \left\{P_t^{(2)}; t \in [0,T)\right\} \mid P_T \\ &\Rightarrow N_T^{(1)} \mid P_T \underset{icx}{\leq} N_T^{(2)} \mid P_T. \end{aligned}$$

Now assume that P_T is given. Then,

$$E\left[(D_{N_T^{\alpha(1)}} - y)^+ \mid P_T\right] = E\left[E\left[(D_{N_T^{\alpha(1)}} - y)^+ \mid N_T^{\alpha(1)}\right] \mid P_T\right]$$

$$\leq E\left[E\left[(D_{N_T^{\alpha(2)}} - y)^+ \mid N_T^{\alpha(2)}\right] \mid P_T\right]$$
$$= E\left[(D_{N_T^{\alpha(2)}} - y)^+ \mid P_T\right].$$

This is due to the fact that $E\left[(D_{N_T^\alpha} - y)^+ \mid N_T^\alpha\right]$ is an increasing convex function of N_T^α by Lemma 4.3. From here, it follows that

$$\begin{aligned} E\left[P_T^{(1)}(D_{N_T^{\alpha(1)}} - y)^+\right] &= E\left[P_T^{(1)} E\left[(D_{N_T^{\alpha(1)}} - y)^+ \mid P_T^{(1)}\right]\right] \\ &\leq E\left[P_T^{(2)} E\left[(D_{N_T^{\alpha(2)}} - y)^+ \mid P_T^{(2)}\right]\right] \\ &= E\left[P_T^{(2)}(D_{N_T^{\alpha(2)}} - y)^+\right]. \end{aligned}$$

■

The following theorem uses the previous lemmas and is the main result of this section. It indicates that a more variable price process leads to lower expected profits.

Theorem 4.6 *If $P^{(1)} \underset{cx}{\leq} P^{(2)}$, then $g^{(1)}(y,\alpha) \geq g^{(2)}(y,\alpha)$ for all y and α.*

Proof. To prove that expected profits are ordered, we first write $g\,(y,\alpha)$ given in (4.4) as

$$g(y,\alpha) = -p_0 y + r_T\,(\alpha) - E\left[(b + P_T)\,(D_{N_T^\alpha} - y)^+\right].$$

Then the result is clear by Proposition 4.2 and Lemma 4.4. ■

Corollary 4.2 *Let $g^{*(i)} = \max\limits_{\alpha>0,y\geq 0} g^{(i)}(y;\alpha)$ denote the optimal expected profit under market price process $P^{(i)}$. If $P_t^{(1)} \underset{cx}{\leq} P_t^{(2)}$, then $g^{(1)*} \geq g^{(2)*}$.*

Proof. Let $(y^{*(i)}, \alpha^{*(i)})$ be the optimal controls for the model with price process $P^{(i)}$. Then

$$g^{(1)}(y^{*(1)}, \alpha^{*(1)}) \geq g^{(1)}(y^{*(2)}, \alpha^{*(2)}) \geq g^{(2)}(y^{*(2)}, \alpha^{*(2)}).$$

■

Corollary 4.2 suggests that as the price variability increases the optimal expected profit decreases.

4.4 Summary

This chapter investigates a single-period, joint inventory-pricing problem of a firm that faces stochastic price volatilities. In particular, a continuous underlying random price process affects the selling prices, which consequently alters the customer arrival process. Besides continuous market price fluctuations, the firm is not a price-taker and is able to control a multiplicative sales markup to influence customer arrivals and effective selling prices in order to maximize expected profits. This setting applies to the cases where the firm sells rather exclusive products such that it has the power to control his markup freely. This kind of a pricing problem is not addressed in the existing literature in which most typical models use selling price as the control variable. Assuming that the firm has a fixed amount of stock on hand, we prove that the expected profit function is concave in markup level and explicitly characterize the optimality condition. Studying monotonicity properties of the expected profit function, we are able to show that as the inventory level increases, the optimal markup decreases. Similarly, we show that as the markup increases, optimal base-stock level decreases. Last, we analyze how the variability of the price process affects the optimal expected revenues and profits. We utilize the concept of convex ordering of random processes and find that as the prices become more variable, optimal expected revenues and profits decrease. This intuitively highlights that price variability is disadvantageous for the firm which also supports the numerical findings of Chapter 3.

Chapter 5

FINANCIAL HEDGING OF SYSTEMS WITH RANDOMLY FLUCTUATING PRICES

Employing successful inventory and pricing strategies are great operational hedges for firms to mitigate their risks. However, there are always other undesired uncertainties which may put these firms into financial distress if not well managed. Financial hedging is one of the most effective ways to confront such risk exposures for risk-averse firms. A financial hedge is an investment position taken in a financial market to offset the risks of another investment. For instance, a multinational firm that sells to foreign markets may use currency options to mitigate its exchange rate risk on sales revenue. A manufacturer that needs to procure its raw material from a volatile commodity market, on the other hand, may use available futures written on the price of that commodity to fix its procurement cost against undesirable fluctuations. Moreover, it is known that through risk reduction, financial hedging increases a firm's value (Froot et al. (1993)).

Various approaches were proposed to incorporate the risk sensitivity of the decision maker in supply chain operations. These approaches include well-known expected utility and mean-variance (MV) formulations, maximizing satisficing probability and downside risk methods such as value-at-risk (VaR) and conditional value-at-risk (CVaR). In inventory management literature, in particular, these methods are based on adjusting the ordering policy so as to reduce the risks at the expense of expected profit. Financial hedging, on the other hand, is the practice of risk reduction through external investments in financial markets. In case of a significant correlation between the random operational cash flow of the firm and a financial index or security, a firm may reduce the risk of its random cash flow for any operational inventory policy using the right hedge (Gaur and Seshadri (2005), Caldentey and Haugh (2006)).

In this chapter, we take a risk-sensitive approach and analyze various financial hedging policies to minimize the risk associated with previously analyzed cash flows. More specifically we investigate optimal static and dynamic financial hedging strategies utilizing a minimum-variance hedging framework, which leads to useful characterizations and implementable solutions.

Next, we briefly review the minimum-variance framework and subsequently give details about dynamic applications for a price-fluctuating inventory setting.

5.1 Minimum-Variance Hedging

In this part, in order to illustrate minimum-variance hedging framework, we review the application for some basic operational settings. Assume that a firm involves in some operational activities and has a random return of X at time T. Assume also that there is an investment opportunity (such as a traded financial instrument) that yields a unit random return Y at time T which has some correlation with X. To keep the firm only focusing on operational activities, let us also assume that the price of the financial instrument is such that the expected net return from this investment is zero (the frequently made martingale assumption). Under the latter assumption, the only reason to make such a financial investment is to exploit the correlation between X and Y to better manage the overall financial risk.

A reasonable risk management objective is to minimize the total variance of the profit. The firm needs to decide how much to invest in Y to minimize the variance of the total cash flow at T

$$\min_a \; Var(X + aY).$$

We can write

$$Var(X + aY) = Var(X) + a^2 Var(Y) + 2aCov(X, Y).$$

This is a convex function in a and the minimizing value of a is given by:

$$a^* = -\frac{Cov(X, Y)}{Var(Y)}. \tag{5.1}$$

We can then characterize the reduction in variance between the unhedged cash flow X and the hedged cash flow $X + a^*Y$ as

$$\Delta = Var(X) - Var(X + a^*Y).$$

After some simplification

$$\Delta = \frac{Cov(X,Y)^2}{Var(Y)} = \rho^2_{X,Y} Var(X)$$

and the relative reduction with respect to the unhedged cash flow is:

$$\Delta_R = \Delta / Var(X) = \rho^2_{X,Y}$$

where $\rho_{X,Y}$ is the correlation coefficient between X and Y.

It is clear from the characterization in (5.1) that if there is a negative correlation between the random operational payoff and external investment yield, then one should buy a^* units of Y. Similarly, in the case of positive correlation, one should shortsell a^* units of Y, if possible, in order to minimize the cash flow variance. Moreover, the relative reduction in variance is directly characterized by $\rho^2_{X,Y}$. Note that as $|\rho_{X,Y}|$ increase, the relative variance reduction increases. At the extreme, a perfect correlation ($\rho_{X,Y} = 1$ or $\rho_{X,Y} = -1$) leads to a 100% reduction in variance.

Another way of looking at a^* is in terms of relative variances is

$$a^* = -\rho_{X,Y} \frac{\sigma_X}{\sigma_Y} \tag{5.2}$$

where σ_X and σ_Y are standard deviations of X and Y, respectively. The characterization given in (5.2) clearly exhibits the relationship between optimal hedge and the riskiness of operational and external investments as well as their degree of correlation.

In the next section, we utilize the minimum-variance concept outlined in this part for the operational settings of previous chapters. That is for inventory models which involve both demand uncertainties and continuous random price fluctuations. Since the main randomness is due to a market price process which affect purchase and selling prices as well as customer arrival times, we will assume that there exists financial securities which are correlated with this price process. We then analyze the relationship between this random price process, operational decisions and optimal hedges in both single and multi-period settings.

5.2 Minimum-Variance Hedging for Inventory Models with Demand and Price Uncertainty

As in previous chapters, we consider operational cash flows that are dependent on operational decisions y, a random input price process P and a random customer demand process N (which may depend on P). This leads to a random operational cash flow $CF(y, N, P)$. Typical operations analysis then looks for the value of y that maximizes $E[CF(y, N, P)]$ or some other measure that may take into account risk considerations. In Chapter 3, assuming that the firm is risk-neutral, we had analyzed the optimal value of y that maximizes $E[CF(y, N, P)]$. Let us now assume that there are some financial securities whose prices are denoted by S which are correlated with the price process P. Let us also denote by θ the investment strategy (the amounts to invest in each of the securities) and by $G(\theta, S)$ the payoff from the financial portfolio constructed via investment strategy θ. It is by now well established that there may be benefits in investing in such a portfolio to hedge the operational cash flow. Let us denote by HCF the hedged cash flow:

$$HCF(\theta, y, N, P, S) = CF(y, N, P) + G(\theta, S).$$

This chapter focuses on the following variance minimization problem for a given operational policy y:

$$\min_{\theta} Var(HCF(\theta, y, N, P, S))$$

This formulation has some nice properties that may be appealing conceptually. Intrinsically, the financial hedge, quantified by θ is dependent on the operational decision y but the operational decision which may depend on other longer term factors does not depend on the financial portfolio. This underlines the fact that the operation is the main focus and know-how of the firm and the financial hedge is a support to the operations and may be provided separately if operational parameters are shared. We can then further specialize to explore different trade-offs. In particular, one consistent benchmark is to take operational decisions that maximize the expected unhedged cash flows $E[CF(y, N, P)]$ and find the corresponding optimal financial hedge.

Another nice feature of the minimum-variance formulation is that it leads to a structured optimization problem whose solution can be obtained explicitly even for complicated price and demand processes as we show in the coming subsections. Moreover, in Section 5.3, we show that, under this formulation, it is possible to obtain a dynamic programming formulation for multi-period inventory problems and compute its solution in a tractable manner. While other multi-period inventory models that consider financial hedging (such as Kouvelis et al. (2013), Kouvelis et al. (2015)) do not allow carrying inventory from one period to the next, our formulation enables handling inventory carrying.

It is useful to contrast the minimum-variance approach with the well-known mean-variance optimization objective that investigates the trade-offs between the expected payoff and its variance. For a complete understanding of the mean-variance type risk trade-off, one needs to trace the efficient frontier of non-dominated solutions. By definition, the minimum variance approach yields minimum variances for each operational policy. If the operational policy space is small and structured (as in the single ordering opportunity case where a base-stock policy is optimal), one can easily trace the efficient frontier numerically by searching over this space for all minimum variance policies. We present such an example in Section 5.4. Even if the policy space is complicated (as in multi-period models where the optimal ordering policy is a base-stock policy with a different target level at each period), the operational decision may take into account a smaller subset of plausible policies for which the efficient frontier can be traced numerically.

Next, we specify the inventory models, corresponding operational policies and the input price, demand and security price processes to consider different cases. The models are also general enough to cover many commonly encountered special cases in the inventory literature (only demand or price uncertainty, periodic vs. continuous demand, etc.).

5.2.1 Single-Period Model with Dynamic Hedging

In this section, we analyze the financial risk hedging of a single-item single-period inventory model with fluctuating prices and randomly arriving demand in continuous time. More specifically, we again assume that the stochastic process $P = \{P_t; t \geq 0\}$ denotes the market prices which are compounded to time T for the product where P_0 represents the unit purchase price at time 0. As in Chapter 3, we assume that the selling price of the item at any time t is given by $f(P_t)$ where f is the firm's general selling price function which is assumed to be positive.

On the operations side, we assume that planning horizon is $[0, T]$ and the firm has a single inventory order opportunity at time 0. We assume that customers arrive according to a stochastic process during the sales period and demand one unit of the item. The sales revenue depends on customer arrival times as well as prevailing selling prices at those times. In case of shortage, arriving customer demands are assumed to be backordered and satisfied at time T. To incorporate the effect of random selling prices on the customer demand, we assume that customers arrive according to a doubly stochastic Poisson process with the intensity process $\Lambda = \{\Lambda_t = \lambda(P_t); t \geq 0\}$ where λ is a nonnegative function. The rate function λ determines the dependency between selling prices and customer arrivals. We use $N = \{N_t : t \geq 0\}$ to denote the customer (or demand) arrival process where N_t is the total number of demand by time t. Here, we also note that throughout this chapter, the prices of all inventory and financial products are assumed to be compounded to time T.

To keep the model general, we assume that the firm incurs a unit holding cost $h(P_T)$ for each unit of unsold items and a total backorder cost of $b(P_T)$ for each unit of backordered items. Note that the backorder cost $b(P_T)$ potentially includes any penalty or goodwill cost as well as the repurchase cost P_T for the backordered demand. A standard example would be $b(P_T) = b + P_T$ where b is a penalty cost and P_T is the repurchase cost at time T. The holding cost, on the other hand, may involve any physical storage or opportunity costs as well as any salvage revenue which may also be random considering the nature of the product that the firm sells. Our results hold for the general functions $h(P_T)$ and $b(P_T)$, and we will assume so throughout this chapter except for some special cases and the numerical analysis in Section 5.4. The operational setting we consider, although quite general, is appropriate for situations where stochastic prices possibly pass to consumers, hence affect both customer demand and retail margins. For this operational model, we can write the total cash flow at time T from operational decisions as

$$CF\left(y,N,P\right)=-P_0 y+\sum_{j=1}^{N_T} f\left(P_{T_j}\right)-\left[b\left(P_T\right)\left(N_T-y\right)^{+}+h\left(P_T\right)\left(y-N_T\right)^{+}\right] \quad (5.3)$$

where y is the order quantity at time 0. This is slightly different from operational setting given in (3.4) and (3.5) , as now backorder and holding costs are also functions of random prices at the end of planning horizon, i.e., P_T. This is to generalize the model to incorporate any repurchasing or salvaging costs in the backorder and holding cost functions. Note that in the multi-period inventory management problem in Chapter 3, these costs were modeled explicitly, so that backorder and holding costs were only functions of observed prices at the beginning of each period. We remark here that as in Chapter 3, the first term in (5.3) is the total purchase cost for y units whereas the second term is the total revenue from sales. It is generated by summing individual revenues for all arriving customers where T_j is the arrival time of jth customer and $f\left(P_{T_j}\right)$ is the selling price at that time. The summation is performed until the N_Tth customer, i.e., the last customer who arrived during the sales period. The last term is the total backorder and holding costs where $(N_T-y)^+$ and $(y-N_T)^+$ denote the total number of shortages and overages, respectively.

It is obvious that the firm is exposed to certain risks in its inventory operations. These risks are mostly related to market prices as they influence both selling prices and customer demand. We propose a dynamic financial hedging strategy to alleviate both price and demand risks for this inventory system. We assume that there are some financial securities available in the market and the firm has certain opportunities to invest in them. In particular, we assume that there are M financial securities which are correlated with the market price process P. We let $S = \left(S^{(1)}, S^{(2)}, ..., S^{(M)}\right)$ denote the price processes for these securities where $S^{(i)} = \left\{S_t^{(i)}; t \geq 0\right\}$. For each $i = 1, .., M$, $S^{(i)}$ represents the price process of security i that is compounded to time T.

We assume that there are prespecified trading times $\mathcal{T} = \{t_0, t_1, t_2, ..., t_{n-1}, t_n\}$ with $0 = t_0 < t_1 < ... < t_{n-1} < t_n = T$ for the firm. In this section, we assume that the firm decides on a single order quantity at the beginning of the selling season and then applies a dynamic hedging strategy based on its stocking decision in order to minimize the variance of total cash flow at the end of selling season. A dynamic hedging strategy uses available information at trading times about the states of market and security prices as well as current inventory level to decide on a financial position for each financial security. We let $\theta = (\theta_0, \theta_1, ..., \theta_{n-1})$ denote a financial hedging strategy where $\theta_k = \left(\theta_k^{(1)}, ..., \theta_k^{(M)}\right)$ is a column vector that represents the financial positions to hold at time t_k for securities $i = 1, .., M$. With this formulation, we can write the final payoff at time T for the financial hedging strategy θ as

$$\begin{aligned} G(\theta, S) &= \sum_{i=1}^{M} \sum_{k=0}^{n-1} \theta_k^{(i)} \left(S_{t_{k+1}}^{(i)} - S_{t_k}^{(i)}\right) \\ &= \sum_{k=0}^{n-1} \theta_k^T \triangle S_k \end{aligned}$$

where $\triangle S_k = S_{t_{k+1}} - S_{t_k}$ is an $M \times 1$ column vector that consists of financial payoffs (compounded to time T) of holding a unit of each security during $[t_k, t_{k+1}]$. We define the total hedged cash flow as the sum of operational and financial profits

$$HCF(\theta, y, N, P, S) = CF(y, N, P) + G(\theta, S). \tag{5.4}$$

In general, we seek financial hedges to minimize the variance of the total cash flow for any ordering policy y. But a plausible objective of the decision maker may be to solve

$$\begin{aligned} \max_{y \geq 0} \quad & E\left[HCF(\theta(y), y, N, P, S)\right] \\ \text{subject to} \quad & \theta(y) = \arg\min_{\theta} Var\left(HCF(\theta, y, N, P, S)\right). \end{aligned}$$

In other words, the problem is to find the order quantity which maximizes the expected total hedged cash flow at time T while using a financial hedging strategy that minimizes the variance. Compared to conventional objective functions in the literature such as mean-variance utility, this particular objective function points to a certain type of risk preference such that it selects the inventory policy that maximizes the mean hedged cash flow while ensuring that the variance is minimized through hedging. Note that mean-variance maximization objectives are of the form $E[CF] - \alpha Var(CF)$ where α is the risk-sensitivity parameter that determines how much weight to be given to risk-minimization compared to mean-return maximization. In our formulation, for any inventory policy, we give complete characterization of hedging policies in a single-period setting. Additionally, in a multi-period setting, we characterize the value function and show that dynamic variance-minimizing hedging policies can be found in an efficient and tractable way. This enables one to implement numerical considerations even for large number of trading periods and consequently try out different operational policies to generate a risk-return schema. In other words, a risk-return efficient frontier can be generated in both static and dynamic settings and this is possible for a very general class of operational settings. We also illustrate this in Section 5.4.

Before we start analyzing the dynamic hedging model in detail we first proceed with a special case where there is only a single trading opportunity at $t = 0$ and the firm is not allowed to revise his portfolio dynamically. We call this the static hedging model. Here, we also remark that the proofs for some of the technical results are provided in the Appendix.

5.2.2 Static Hedging Model

In this part, we assume that $\mathcal{T} = \{t_0 = 0, t_1 = T\}$, i.e., there is only a single trading opportunity initially at time 0. Then, a minimum variance hedge for any order quantity y is found by minimizing the variance of (5.4) , i.e., by solving

$$\theta(y) = \arg\min_{\theta} Var\left(CF(y, N, P) + \theta^T \triangle S\right)$$

where $\triangle S = \triangle S_0 = S_T - S_0$ and $\theta = \theta_0 = \left(\theta_0^{(1)}, ..., \theta_0^{(M)}\right)$. Note that we can rewrite the objective function as

$$\begin{aligned} Var\left(CF\left(y,N,P\right)+\theta^T \triangle S\right) &= Var\left(CF\left(y,N,P\right)\right)+Var\left(\theta^T \triangle S\right) \\ &\quad + 2Cov\left(CF\left(y,N,P\right),\theta^T \triangle S\right) \\ &= Var\left(CF\left(y,N,P\right)\right)+\theta^T C\theta + 2\theta^T \mu\left(y\right) \end{aligned} \tag{5.5}$$

where C is an $M \times M$ covariance matrix with entries

$$C_{ij} = Cov\left(\triangle S^{(i)}, \triangle S^{(j)}\right) = Cov\left(S_T^{(i)}, S_T^{(j)}\right).$$

The last equality is due to the fact that at time $t = 0$, the prices of securities are known with certainty and $S_0^{(i)}$ is constant for all $i = 1, ...M$. Additionally, $\mu\left(y\right)$ is an $M \times 1$ column vector where its ith element is

$$\begin{aligned} \mu_i\left(y\right) &= Cov\left(CF\left(y,N,P\right), \triangle S^{(i)}\right) \\ &= Cov\left(\sum_{j=1}^{N_T} f\left(P_{T_j}\right), S_T^{(i)}\right) - Cov\left(h\left(P_T\right)\left(y-N_T\right)^+, S_T^{(i)}\right) \\ &\quad - Cov\left(b\left(P_T\right)\left(N_T-y\right)^+, S_T^{(i)}\right). \end{aligned} \tag{5.6}$$

Note that if we assume that given price process P, customer arrival process N is independent of S, then we can write the first term in (5.6) as

$$Cov\left(\sum_{j=1}^{N_T} f\left(P_{T_j}\right), S_T^{(i)}\right) = \int_0^T Cov(f\left(P_u\right)\lambda(P_u), S_T^{(i)})du. \tag{5.7}$$

whose derivation is given in the Appendix.

Theorem 5.1 *The objective function in* (5.5) *is convex in* θ *and the minimum-variance hedge for an order quantity* y *is given by*

$$\theta^*\left(y\right) = -C^{-1}\mu\left(y\right). \tag{5.8}$$

Proof. The first-order optimality condition for the objective in (5.5) is obtained by setting the gradient equal to zero so that $2C\theta + 2\mu\left(y\right) = 0$ which gives $\theta^*\left(y\right)$ in (5.8).

Note also that the second order condition is satisfied since the Hessian matrix is $2C$ which is always positive semi-definite since C is a covariance matrix. ■

Theorem 5.1 gives the minimum variance portfolio for any order quantity y. Note that we only need a covariance matrix of security prices and a covariance vector between the random cash flow and security prices to compute the minimum-variance hedge. In the upcoming sections, we also show that with a few changes this structure is preserved. Substituting (5.8) in (5.4), we obtain

$$HCF\left(\theta^{*}\left(y\right),y,N,P,S\right)=CF\left(y,N,P\right)-\mu\left(y\right)^{T}C^{-1}\bigtriangleup S \tag{5.9}$$

since $\left(C^{-1}\right)^{T}=C^{-1}$.

The next theorem characterizes the optimal order quantity that maximizes the expected value of the final cash flow with the minimum-variance hedging portfolio. We first state the following assumption to ensure the uniqueness of optimal order quantity.

Assumption 5.1 *The function*

$$E\left[\left(h\left(P_T\right)+b\left(P_T\right)\right)1_{\{N_T\leq y\}}\right]-Cov\left(\left(h\left(P_T\right)+b\left(P_T\right)\right)1_{\{N_T\leq y\}},S_T\right)^{T}C^{-1}E\left[\triangle S\right]$$

is strictly increasing in y.

As we will see later, Assumption 5.1 is always satisfied if $E\left[\triangle S\right]=0$, i.e., the securities are martingales, and if the demand process N is independent of the market price process.

Theorem 5.2 *Under Assumption 5.1, the optimal order quantity that maximizes* (5.9) *is*

$$\begin{aligned} y^{*}=\inf\Big\{y\geq0: & \,E\left[\left(h\left(P_T\right)+b\left(P_T\right)\right)1_{\{N_T\leq y\}}\right] \\ & -Cov\left(\left(h\left(P_T\right)+b\left(P_T\right)\right)1_{\{N_T\leq y\}},S_T\right)^{T}C^{-1}E\left[\triangle S\right] \\ & \geq -P_0+E\left[b\left(P_T\right)\right]-Cov\left(b\left(P_T\right),S_T\right)^{T}C^{-1}E\left[\triangle S\right]\Big\}. \end{aligned} \tag{5.10}$$

Note that no restriction is put on financial securities for the result given in Theorem 5.2. A very plausible special case for the financial securities is that they do not yield any positive values in expectation. In other words, they can not be used for speculative purposes, rather, they can be used for hedging. Next corollary presents the expectation-maximizing order quantity when $E[\triangle S] = 0$.

Corollary 5.1 *If the securities are assumed to be fairly priced, i.e.,* $E[\triangle S] = 0$, *Assumption 5.1 is always satisfied and the optimal order quantity reduces to the risk-neutral solution*

$$y^* = \inf\left\{y \geq 0 : E\left[\left(h\left(P_T\right) + b\left(P_T\right)\right) 1_{\{N_T \leq y\}}\right] \geq -P_0 + E\left[b\left(P_T\right)\right]\right\}. \tag{5.11}$$

Note that the assumption of $E[\triangle S] = 0$ is common in the finance literature. It implicitly assumes that the financial market itself is complete such that there exists a risk-neutral probability measure that makes the underlying stock price process a martingale. This measure can then be used to find fair prices of financial derivatives of this stock which, in turn, are also martingales. The complete financial market is a natural assumption and it allows the decision makers to use derivative securities for hedging purposes. In the static hedging case, we do not need to impose a complete financial market assumption to characterize both optimal inventory and hedging policies. However, in the dynamic hedging case, we will assume that the financial market itself is complete and the security prices follow martingale price processes in order to ensure the separability of the dynamic programming formulations. Note that although the financial market is complete, our model falls into the world of partially complete markets since there are certain risks (demand related risks) that can not be removed by financial hedging (see Caldentey and Haugh (2006)).

Next, we analyze another special case of the static hedging model where we assume that the market prices do not affect the demand process.

Special Case: Demand process is independent of market prices.

In this section we assume that $N = \{N_t; t \geq 0\}$ is a Poisson process with rate λ independent of the market price process P and security price processes S. The following corollary characterizes the optimal hedging portfolio and order quantity.

Corollary 5.2 *If N is a Poisson process with rate λ independent of P and S, the optimal hedging portfolio is given by $\theta^*(y) = -C^{-1}\mu(y)$ where*

$$\mu(y) = \lambda \int_0^T Cov\left(f(P_t), S_T\right) dt - E\left[(y - N_T)^+\right] Cov\left(h(P_T), S_T\right) - E\left[(N_T - y)^+\right] Cov\left(b(P_T), S_T\right). \tag{5.12}$$

Moreover, the optimal order quantity that maximizes the expected cash flow while using the minimum-variance portfolio $\theta^*(y)$ reduces to

$$y^* = \inf\left\{y \geq 0 : P\{N_T \leq y\} \geq \frac{-P_0 + E[b(P_T)] - Cov\left(b(P_T), S_T\right)^T C^{-1} E[\triangle S]}{E[h(P_T)] + E[b(P_T)] - Cov\left(h(P_T) + b(P_T), S_T\right)^T C^{-1} E[\triangle S]}\right\} \tag{5.13}$$

Observe that (5.10) reduces to (5.13) when the dependency between P and N is removed which then yields the well-known critical-fractile Newsvendor solution is obtained. This is also further refined if we again assume that financial securities do not bring any value in expectation. It is clear that the critical fractile only consists of traditional underage and overage costs in this case. This can be easily observed in the next corollary.

Corollary 5.3 *If all securities are martingales, then $E[\triangle S] = 0$ and the optimal order quantity that maximizes the expected hedged cash flow is*

$$y^* = \inf\left\{y \geq 0 : P\{N_T \leq y\} \geq \frac{-P_0 + E[b(P_T)]}{E[h(P_T)] + E[b(P_T)]}\right\}.$$

Special case: Hedging with a Single Future.

In this section, we analyze another special case where there is a single future written on P_T under the assumption that N is independent of P and S. It is clear that futures are the most widely used and most effective financial contracts in financial risk hedging. This fact is also numerically demonstrated in Section 5.4.

We assume that S is a future on P_T, which implies $S_0 = P_0$, $S_T = P_T$, (Baxter and Rennie (1996)). Then, $\triangle S = P_T - P_0$. Recall that both S and P represents the values that are compounded to time T. Note also that

$$C = Cov(\triangle S, \triangle S) = Var\left(P_T\right). \tag{5.14}$$

To keep the exposition simple, we also define specific holding and backorder functions. In particular, we assume that the firm incurs a constant penalty cost b and a repurchase cost of P_T for each unit of backordered demand, i.e., $b\left(P_T\right) = b + P_T$. Similarly, we assume that the firm incurs a holding cost of h, which may include physical storage costs as well as opportunity costs. Let us further assume that the firm salvages the remaining items at δ fraction of the market price at time T which yields a total holding cost function of $h\left(P_T\right) = h - \delta P_T$. Assume also that $\delta \in [0, 1]$. Note that with these assumptions, the total operational cash flow to be hedged is

$$-P_0 y + \sum_{j=1}^{N_T} f\left(P_{T_j}\right) - \left(h - \delta P_T\right)\left(y - N_T\right)^+ - \left(b + P_T\right)\left(N_T - y\right)^+ .$$

The next corollary characterizes the optimal order quantity and the minimum-variance hedge when there is only the future with terminal time T available for hedging.

Corollary 5.4 *If only a single future contingent on P_T is used and there are no intermediate trading points, the order quantity that maximizes the expected total hedged cash flow is*

$$y^* = \inf\left\{y \geq 0 : P\left\{N_T \leq y\right\} \geq \frac{b}{b + h + (1-\delta)P_0}\right\}. \tag{5.15}$$

and the optimal position on the future for any order quantity y is

$$\theta^* = E\left[\left(N_T - y\right)^+\right] - \delta E[(y - N_T)^+] - \lambda \int_{[0,T]} \beta_t dt$$

where

$$\beta_t = \frac{Cov\left(f\left(P_t\right), P_T\right)}{Var\left(P_T\right)}.$$

Corollary 5.4 characterizes the optimal number of future contracts to be bought or sold in order to minimize the variance of the final cash flow when the order-up-to level is y at time 0. Under the assumption that N is independent of P, it is clear that the three components in θ^* respectively hedge the random repurchase cost $-P_T\left(N_T - y\right)^+$, the random salvage revenue $\delta P_T\left(y - N_T\right)^+$and the total sales revenue $\sum_{j=1}^{N_T} f\left(P_{T_j}\right)$. It is intuitive that the decision maker should take a $E\left[\left(N_T - y\right)^+\right]$ units of long position on the future in order to eliminate the price risk in the repurchase of backordered items. Note that there is still randomness in demand as one can not eliminate the risk associated with $\left(N_T - y\right)^+$ since N is independent of security price movements. This is similar for the salvage risk as well where the firm should take a $\delta E\left[\left(y - N_T\right)^+\right]$ units of short position on the future, to mitigate the price risk. The last term, on the other hand, partially eliminates the price related risks in the operational cash flow. This is due to the fact that only a single future contingent on the market price at time T is used for hedging whereas the revenue term is affected by all realized market prices during $[0, T]$.

Moreover, we can establish that the higher the order level y, the lower $\theta^*(y)$ is. This can be easily observed using the transformation $\left(y - N_T\right)^+ = \left(N_T - y\right)^+ + y - N_T$ and writing

$$\theta^* = (1 - \delta)\, E\left[\left(N_T - y\right)^+\right] - \delta y + \delta\lambda T - \lambda\int_0^T \beta_t dt.$$

Note that the inventory model analyzed in this chapter is rather general in the sense that it involves both selling price and demand uncertainty. Moreover, the selling prices and backorder and holding costs are represented in terms of general functions. This allows us to consider a number of special cases and extensions based on the analyzed model. For instance, if we assume that the selling price function is constant and zero, i.e., if $f = 0$ and both backorder and holding costs are constants such as b and h, then this is a typical example of a single period newsvendor model with only demand uncertainty. More specifically, for this special case the cash flow reduces to

$$CF(y, N, P) = -P_0 y - b(N_T - y)^+ - h(y - N_T)^+$$

where only the demand N_T is affected by the market prices P which are correlated with the financial securities. This is similar to the single period model considered in Gaur and Seshadri (2005) in which the authors specifically assumed that the demand is a linear function of the price of a financial asset.

The newsvendor example was a pure demand uncertainty example. If we assume that both $b(p) = h(p) = 0$ and the inventory decision is $y = 0$, then, as a special case, this can be considered as a make-to-order model with negligible processing times where the cash flow reduces to

$$CF(y, N, P) = \sum_{j=1}^{N_T} f\left(P_{T_j}\right).$$

Note that the manufacturer produces and sells the product to individual customers as they arrive depending on a general selling price function f. If we further assume that $f(p) = \alpha p$, then this is the typical revenue stream of currency exchange offices where the market price process P may represent the movements of dollar currency with respect to euro and α is the percentage commission that the exchange office obtains from each transaction.

In the next section, we generalize the previous analysis by allowing the firm to change its portfolio at each trading time after observing the available information.

5.2.3 Dynamic Hedging Model

In this section, we consider a dynamic hedging model where, similar to the static hedging model, we assume that an inventory decision is made at $t = 0$ and the sales period is between $[0, T]$. However, for this case we assume that security prices $S^{(i)} = \left\{S_t^{(i)}; t \geq 0\right\}$ are Markovian and follow martingale price processes, i.e., $E\left[S_t^{(i)} | \mathcal{F}_u\right] = S_u^{(i)}$ for all i and $u < t$ where $\mathcal{F}_u$ is the filtration until time u. We also assume that given P_t and S_t, $\{P_u; u > t\}$ is independent of S_t and given S_t $\{S_u; u > t\}$ is independent of

P_t. Our aim is to find the minimum-variance dynamic hedging strategy, i.e., for fixed y, the objective is to solve

$$\begin{aligned}\min_{\theta} Var\left(HCF\left(\theta,y,N,P,S\right)\right) = E\left[\left(CF\left(y,N,P\right)+\sum_{k=0}^{n-1}\theta_k^T \triangle S_k\right)^2\right] \\ - E\left[CF\left(y,N,P\right)\right]^2. \qquad (5.16)\end{aligned}$$

Note that the last term is due to each $S^{(i)}$ being a martingale, i.e., $E\left[\triangle S_k\right]=0$ for each k. Then the problem is equivalent to minimizing the second moment of the final hedged cash flow, i.e.,

$$\min_{\theta} E\left[\left(CF\left(y,N,P\right)+\sum_{k=0}^{n-1}\theta_k^T \triangle S_k\right)^2\right]. \qquad (5.17)$$

Then, objective function given in (5.17) is separable in terms of dynamic programming, i.e., it can be solved by backward induction. First, let us define the total demand between t_1 and t_2 for any $t_1 < t_2$ as $N_{[t_1,t_2]}$. Then the total demand during the kth period is $N_{[t_k,t_{k+1}]}$. Similarly, we define the total revenue from sales during $[t_k, t_{k+1}]$ as

$$R_{[t_k,t_{k+1}]} = \sum_{j=1}^{N_{[t_k,t_{k+1}]}} f\left(P_{T_j+t_k}\right)$$

where T_j denotes the arrival time of jth customer after time t_k. Observe that the total revenue during $[0, T]$ is

$$\sum_{k=0}^{n-1} R_{[t_k,t_{k+1}]} = R_{[t_0,t_n]} = \sum_{j=1}^{N_T} f\left(P_{T_j}\right).$$

We also need another state to keep track of the current level of the profit. We define the wealth at the beginning of period $k+1$ as

$$W_{k+1} = W_k + R_{[t_k,t_{k+1}]} + \theta_k^T \triangle S_k$$

where we suppose that the initial wealth is $W_0 = 0$. Note that the total wealth at the beginning of period $k+1$ is the sum of wealth at the beginning of period k and operational and financial gains during $[t_k, t_{k+1}]$. On the other hand, inventory level at the beginning of period $k+1$ is

$$X_{k+1} = X_k - N_{[t_k,t_{k+1}]} \tag{5.18}$$

with initial condition $X_0 = y$ where y is the order-up-to decision at time 0. Observe that we can write the objective function given in (5.17) as

$$E\left[\left(CF(y,N,P) + \sum_{k=0}^{n-1} \theta_k^T \triangle S_k\right)^2\right] = E\left[\left(W_n - \left[b\left(P_{t_n}\right)\left(-X_n\right)^+ + h\left(P_{t_n}\right)X_n^+\right]\right)^2\right].$$

We construct the dynamic programming formulation by the equation

$$\begin{aligned} V_k(x,w,p,s) &= \min_{\theta_k} E\left[V_{k+1}\left(X_{k+1}, W_{k+1}, P_{t_{k+1}}, S_{t_{k+1}}\right)\right. \\ &\qquad \left. | X_k = x, W_k = w, P_{t_k} = p, S_{t_k} = s\right] \\ &= \min_{\theta_k} E\left[V_{k+1}\left(x - N_{[t_k,t_{k+1}]}, w + R_{[t_k,t_{k+1}]} + \theta_k^T \triangle S_k, P_{t_{k+1}}, S_{t_{k+1}}\right)\right. \\ &\qquad \left. | P_{t_k} = p, S_{t_k} = s\right] \end{aligned} \tag{5.19}$$

where the boundary condition is

$$V_n(x,w,p,s) = \left(w - b(p)(-x)^+ - h(p)x^+\right)^2.$$

Note that at each trading period the decision maker observes the current inventory level, wealth, market price and security prices to construct a portfolio in order to minimize the second moment of the final payoff.

Minimum Variance Hedging Strategy.

The following theorem characterizes the optimal hedging policy and the form of the value function. First let us define, similar to the previous model, a covariance matrix $C_k(s)$ for period k with elements

$$C_k(s)_{ij} = Cov\left(S_{t_{k+1}}^{(i)}, S_{t_{k+1}}^{(j)} \mid S_{t_k}^{(i)} = s^{(i)}, S_{t_k}^{(j)} = s^{(j)}\right) \tag{5.20}$$

and a covariance vector $\mu_k(x,p,s)$ with elements

$$\mu_k(x,p,s)_j = Cov\left(R_{[t_k,t_n]} - b\left(P_{t_n}\right)\left(N_{[t_k,t_n]} - x\right)^+ - h\left(P_{t_n}\right)\left(x - N_{[t_k,t_n]}\right)^+,\right.$$

$$S^{(j)}_{t_{k+1}} \mid P_{t_k} = p, S^{(j)}_{t_k} = s^{(j)}\Big) \tag{5.21}$$

where $R_{[t_k,t_n]}$ is the total revenue from period k to the last period. Furthermore, let us define

$$g_k(x,w,p) = E\Big[\Big(w + R_{[t_k,t_n]} - b(P_{t_n})\left(N_{[t_k,t_n]} - x\right)^+ \\ -h(P_{t_n})\left(x - N_{[t_k,t_n]}\right)^+\Big)^2 \mid P_{t_k} = p\Big] \tag{5.22}$$

which denotes the second moment of the operational cash flows from period k to the last period given the wealth, inventory and market price levels w, x and p. Last, we define the following recursion

$$h_k(x,p,s) = -\mu_k(x,p,s)^T C_k(s)^{-1}\mu_k(x,p,s) \\ + E\left[h_{k+1}\left(x - N_{[t_k,t_{k+1}]}, P_{t_{k+1}}, S_{t_{k+1}}\right) \mid P_{t_k} = p, S_{t_k} = s\right]$$

for $k = n-1, n-2, ..., 0$ with the terminal condition $h_n(x,p,s) = 0$. These functions will be used in characterization of the optimal hedging policy for a given order-up-to level y. Although the recursion is complicated, we show that there is a surprisingly nice characterization of the value function and the optimal portfolio which is given in the next theorem.

Theorem 5.3 ***(a)*** *The value function at any period k is*

$$V_k(x,w,p,s) = g_k(x,w,p) + h_k(x,p,s).$$

(b) *The minimum-variance portfolio at period k is*

$$\theta^*_k(x,p,s) = -C_k(s)^{-1}\mu_k(x,p,s)$$

Theorem 5.3 characterizes the optimal trading policy for a given initial order y. If the firm wishes to use the inventory level y^* that maximizes the expected total hedged cash flow, it is found by (5.11) and the optimal trading strategy at the initial period is

$$\theta^*_0(x = y^*, p, s) = -C_0(s)^{-1}\mu_0(y^*, p, s).$$

Theorem 5.3-(a) offers a powerful decomposition of variance terms where the leading term depends on wealth state and denotes the second moment of the operational hedged cash flow. The latter term, on the other hand, does not depend on the wealth which is the only state that is affected by hedging decisions at each period. This separation allows us to find the result in 5.3-(b) in an efficient and tractable way which proves to be very useful in numerical implementations. Note that one can further write $\mu_k(x,p,s)_j$ as

$$\begin{aligned}\mu_k(x,p,s)_j = &\int_{t_k}^{T} Cov(f(P_u)\lambda(P_u), S^{(j)}_{t_{k+1}}|P_{t_k}=p, S^{(j)}_{t_k}=s^{(j)})du\\ &- Cov\left(b(P_T)\left(N_{[t_k,t_n]}-x\right)^+, S^{(j)}_{t_{k+1}}|P_{t_k}=p, S^{(j)}_{t_k}=s^{(j)}\right)\\ &\quad - Cov\left(h(P_T)\left(x-N_{[t_k,t_n]}\right)^+, S^{(j)}_{t_{k+1}}|P_{t_k}=p, S^{(j)}_{t_k}=s^{(j)}\right)\end{aligned}$$

using the identity in (5.7).

5.3 Multi-period Inventory Model with Dynamic Hedging

In this section, we generalize the single period inventory model with dynamic financial hedging such that the decision maker may simultaneously revise both inventory level and hedging portfolio at each period. To this end, we slightly modify the cost structure and assume that a penalty cost of b and a holding cost of h are incurred at each period for the backordered and remaining items, respectively. We assume that there is also a clearance opportunity and in the last period, the firm purchases the backordered items at the market price P_{t_n} and sells the remaining ones at δP_{t_n}. Note that both of them are functions of the market price at time T. In financial hedging, we employ a similar analysis by first fixing an ordering strategy, and then finding the minimum-variance hedging policy. Here, we also note that the following analysis is also valid if the frequency of trading is higher than the frequency of inventory replenishment, i.e., between two inventory replenishments, there can also be multiple trading opportunities. However, for exposition, we only consider the case where both financial and inventory decisions are only taken at the beginning of each period simultaneously.

Let us define an admissible ordering policy as a decision rule that is a function of the inventory level, wealth, the product market price and the financial security prices at the beginning of each period where admissible action space is $Y_k(x,w,p,s) = \{y \in \mathbb{N}; y \geq x\}$ where $\mathbb{N}$ is the set of nonnegative integers. Let $y = [y_0, ..., y_{n-1}]$ be any admissible ordering policy. We define X_k as the inventory level at the beginning of time t_k. Different than (5.18), the evolution of the inventory level for period k is

$$X_{k+1} = y_k - N_{[t_k,t_{k+1}]}$$

with the initial condition $X_0 = 0$ since at each period the inventory level is raised up to the level y_k. For notational convenience, let us define the time T compounded value of the cash flow between $[t_i, t_j]$, i.e., between periods i and $i+1$, under ordering policy y as

$$CF^y_{[t_i,t_j]} = \sum_{k=i}^{j-1} CF^y_{[t_k,t_{k+1}]}$$

where

$$CF^y_{[t_k,t_{k+1}]} = -P_{t_k}(y_k - X_k) + R_{[t_k,t_{k+1}]} - b(-X_{k+1})^+ - hX^+_{k+1}$$

for $k \leq n-2$. Note that the terms are purchase cost, sales revenue and backorder and holding cost for period k. For the last period, we define

$$CF^y_{[t_{n-1},t_n]} = -P_{t_{n-1}}(y_{n-1} - X_{n-1}) + R_{[t_{n-1},t_n]} - (b + P_{t_n})(-X_n)^+ - (h - \delta P_{t_n})X_n^+$$

where the only difference from intermediate periods is the inclusion of repurchasing from the market and salvaging terms. With these notations, total operational cash flow at time T is $CF^y_{[t_0,t_n]}$. Similar to (5.4), we can write the total hedged cash flow as

$$HCF(y,\theta,X,P,N,S) = CF^y_{[t_0,t_n]} + \sum_{k=0}^{n-1} \theta_k^T \left(S_{t_{k+1}} - S_{t_k}\right) \tag{5.23}$$

where the last term is the financial profit generated from portfolio decisions. We define the wealth process by

$$W_{k+1} = W_k + CF^y_{[t_k,t_{k+1}]} + \theta_k^T \left(S_{t_{k+1}} - S_{t_k}\right)$$

where we initially suppose $W_0 = 0$. With these formulations, we can rewrite the hedged cash flow in (5.23) as

$$HCF\left(y, \theta, X, P, N, S\right) = W_n.$$

Minimum-Variance Hedging Strategy for a given Ordering Policy

We first use dynamic programming to find the minimum-variance hedging strategy for any admissible ordering policy $y = [y_0, ..., y_{n-1}]$. Since the security prices are assumed to be martingales, similar to the previous models, the objective is to minimize the second moment of the total hedged cash flow. We define the dynamic programming equation under ordering policy y as

$$\begin{aligned} V_k\left(x, w, p, s; y\right) &= \min_{\theta_k} E\left[V_{k+1}\left(X_{k+1}, W_{k+1}, P_{t_{k+1}}, S_{t_{k+1}}; y\right)\right. \\ &\qquad \left. \mid X_k = x, W_k = w, P_{t_k} = p, S_{t_k} = s\right] \\ &= \min_{\theta_k} E\left[V_{k+1}\left(y_k - N_{[t_k, t_{k+1}]}, w + CF^y_{[t_k, t_{k+1}]} + \theta_k^T \triangle S_k, P_{t_{k+1}}, S_{t_{k+1}}; y\right)\right. \\ &\qquad \left. \mid X_k = x, P_{t_k} = p, S_{t_k} = s\right] \end{aligned} \tag{5.24}$$

with the boundary condition

$$V_n\left(x, w, p, s; y\right) = w^2.$$

As in the previous model, the states of the system are defined as inventory level x, wealth level w, market price p and security price vector s. The value function given in (5.24) represents the minimum value of the second moment of the final cash flow at period n given the states at period k and the ordering policy y. The following theorem characterizes the optimal trading strategy for a given admissible ordering policy y. Let us first define

$$g_k\left(x, w, p; y\right) = E\left[\left(w + CF^y_{[t_k, t_n]}\right)^2 \mid X_k = x, P_{t_k} = p\right]$$

which mainly represent the second moment of the operational profits from period k to the last period n. Note that with this definition

$$g_n\left(x, w, p; y\right) = w^2 = V_n\left(x, w, p, s; y\right) \tag{5.25}$$

since $CF_{[t_n,t_n]} = 0$. Let us also define the recursion

$$h_k(x,p,s;y) = -\mu_k(x,p,s;y)^T C_k(s)^{-1} \mu_k(x,p,s;y) + E\left[h_{k+1}\left(X_{k+1}P_{t_{k+1}}, S_{t_{k+1}}; y\right) \mid P_{t_k} = p, S_{t_k} = s\right]$$

for $k = n-1, n-2, ..., 0$ with the terminal condition $h_n(x,p,s;y) = 0$. This will be used in characterizing the optimal trading strategy. Similar to the previous model, $C_k(s)$ is the covariance matrix defined in (5.20). This is the covariance matrix of the security prices for any period $k+1$ given the security prices at period k. Similarly, the covariance vector $\mu_k(x,p,s)$ with elements

$$\mu_k(x,p,s;y)_j = Cov\left(CF^y_{[t_k,t_n]}, S^{(j)}_{t_{k+1}} \mid X_k = x, P_{t_k} = p, S^{(j)}_{t_k} = s^{(j)}\right)$$

represents the covariances between the security prices for period $k+1$ and the random operational cash flow from period k to the last period n given the ordering policy y, initial inventory level x, market price p and security price vector s.

Theorem 5.4 ***(a)*** *The value function at any period k is*

$$V_k(x,w,p,s;y) = g_k(x,w,p;y) + h_k(x,p,s;y).$$

(b) *The minimum-variance portfolio at period k is*

$$\theta^*_k(x,p,s;y) = -C_k(s)^{-1} \mu_k(x,p,s;y). \tag{5.26}$$

As in the single period inventory model, at each period, the decision maker chooses the order-up-to level independent of the portfolio decision. Then he chooses his portfolio. In Section 3.2, we already showed that for each period, a price-dependent base-stock policy is optimal (see Theorem 3.1). The decision maker may use the expectation-maximizing ordering strategy and then based on this, the characterization in (5.26) can be used to minimize the variance dynamically for that policy.

The multi-period inventory model analyzed in this section also covers a selection of inventory models available in the literature. For instance, if we assume that the selling price function is $f = 0$ and there is no repurchasing and salvaging, then this is one of the typical models where there is both demand and purchase price uncertainty. This is very similar to the model considered in Kalymon (1971).

5.4 Numerical Analysis

In this section, we numerically analyze the effect of financial hedging on price and demand risks using particular derivative securities written on the market prices of the inventory item. As in Chapter 3, to model the market prices, we use the two-factor model of commodity prices proposed by Schwartz and Smith (2000), which models both the short-term and long-term properties of prices. In particular, it is again assumed that market prices follow

$$P_t = P_0 e^{-1/2\left(\sigma_1^2+\sigma_2^2\right)t+\sigma_1 W_t^{(1)}+\sigma_2 W_t^{(2)}} \tag{5.27}$$

where $\sigma_1 = (\sigma_\xi + \sigma_\chi \rho)$ and $\sigma_2 = \sigma_\chi \sqrt{1-\rho^2}$ with $W_t^{(1)}, W_t^{(2)}$ being two independent Brownian motions with respect to Q. This is the same setting in Section 3.8. Note that the price process P is a martingale under $\mathcal{Q}$. For the financial securities, we assume that there is a future and a call option with a strike price of K on the value of P_T which the firm may use to hedge his risks. Note that since the market price process is itself a martingale, its derivative securities follow martingale price processes as well. Let us assume that $S^{(1)}$ and $S^{(2)}$ are price processes for the future and call option, respectively. Then, the fair price of the future at time t is

$$S_t^{(1)} = P_t$$

whereas the fair-price of the call option is

$$S_t^{(2)} = E\left[(P_T - K)^+ \mid P_t\right]$$

assuming that both P and S represent values that are compounded to time T.

We assume that $T = 1$ and use $P_0 = 20$, $\sigma_\chi = 0.25, \sigma_\xi = 0.15$ and $\rho = 0.3$ as the financial parameters. Note that since we use a martingale price process for the market prices, $E[P_t] = P_0 = 20$. For the call option, we take the strike price $K = 20$.

For the operational setting, we take the selling price function as $f(p) = \alpha p$ and assume that $\alpha = 2$. The backorder cost function and holding cost functions are assumed to be $b(p) = 4 + p$ and $h(p) = 1$, respectively. For the demand process, we assume that the rate function is piecewise linear in the selling price. That is, the instantaneous arrival rate at time t is

$$\Lambda_t = (A - B\alpha P_t)^+$$

where we assume that $A = 90$ and $B = 0.7$ and αP_t is the selling price at time t. Note that A is a parameter that denotes the potential customer arrival rate whereas B represents the sensitivity of the customers to price changes.

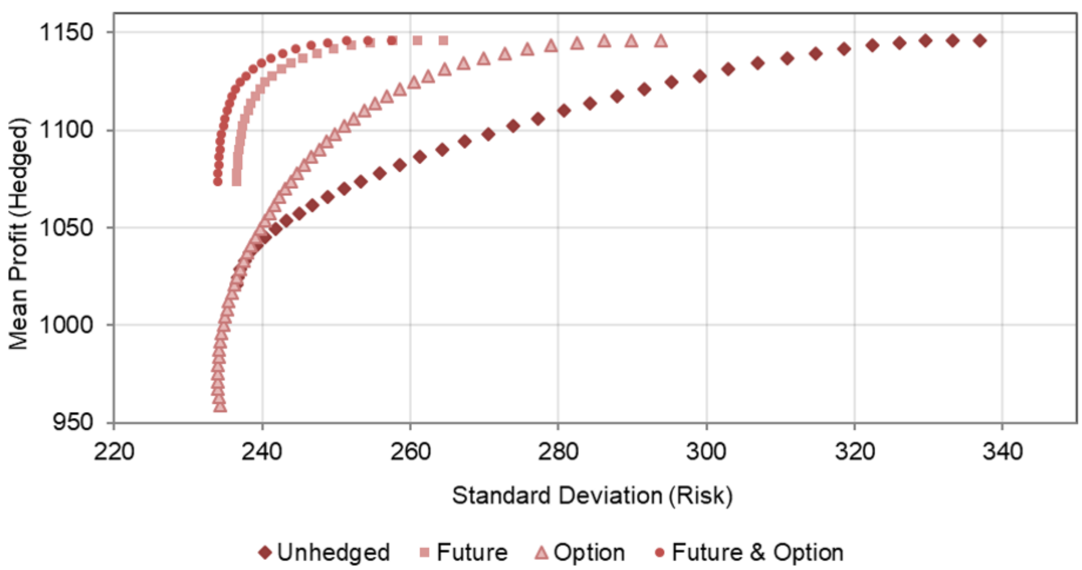

Figure 5.1: Mean-variance efficient frontier.

We use the same simulation setting in Section 3.8, i.e., we first generate a price path using (5.27) and then conditional on these prices, we generate a nonhomogeneous arrival stream using the thinning algorithm. This in turn enables us to generate simulated profit realizations.

In our numerical setting, we compare four different cases of cash flow streams for different scenarios. These cases are the following.

- Case 1: Unhedged cash flow (referred to as “undhedged”)

- Case 2: The hedging portfolio consists of a single future (referred to as “future”)

- Case 3: The hedging portfolio consists of a single call option (referred to as "call option")

- Case 4: The hedging portfolio consists of a future and a call option (referred to as "Call option & future")

We first show that one can easily generate the efficient risk-return frontier numerically to perform a detailed risk-sensitivity analysis for the decision maker. This is possible due to the tractability and efficiency of finding variance-minimizing hedging policies for any given ordering policy. Figure 5.1 exhibits the mean-variance efficient frontier for the single period static hedging case. These curves are generated by computing the optimal hedging policy for varying order quantities. The decision maker then may choose the desired mean-variance level according to his/her risk preference.

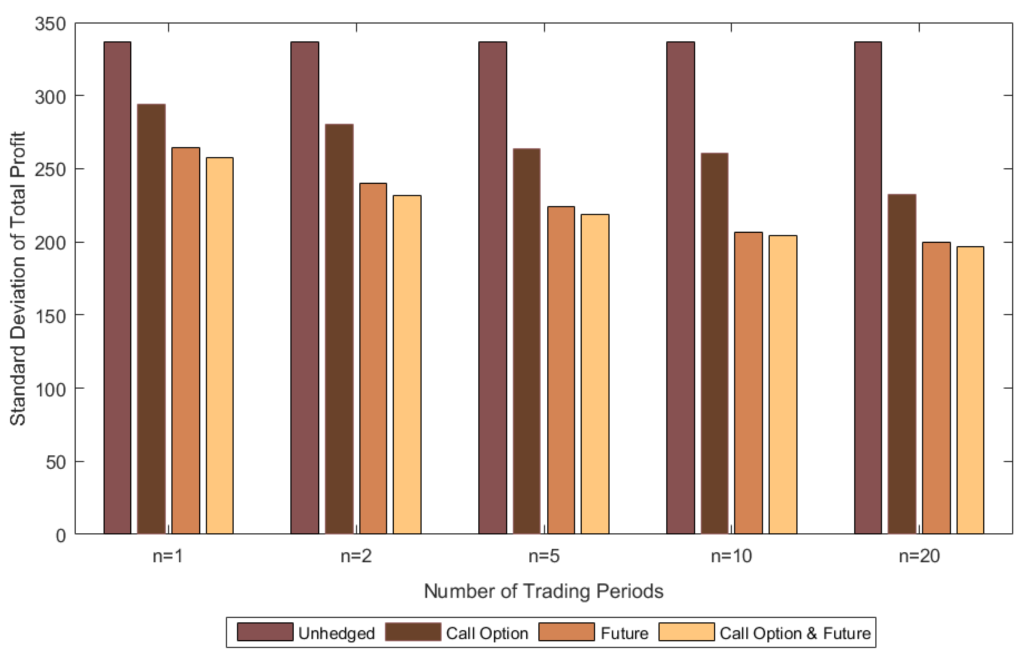

Figure 5.2: Effect of number of trading periods on risk reduction.

The rest of the numerical analysis is performed for the expected profit maximization objective to have consistent comparisons across models. We first test the effect of number of trading periods and different portfolios on the standard deviation of the total profit. Note that the above economic parameters result in an optimal order quantity as 51 with an expected profit of 1145.70 and a standard deviation of 336.97 which are also observed in Figure 5.1. The order quantity which maximizes the expected hedged profit (which is equal to the expected operational profit since the price process is a martingale) is found by using Corollary 5.3. Here one can also observe that if price volatilities are disregarded and it is assumed that market price is constant at $P_t = 20$, then $\lambda_t = 90 - (0.7)(2)(20) = 62$ which yields the optimal order quantity as 54. The difference is due to price volatilities and their effects on operational policies. The effect of price volatility on the optimal order quantity becomes greater as customer sensitivity to price changes increases.

It is clear that even though using the call option and the future does not change the expected profits, they help to reduce the variance. As seen in Figure 5.2, for static hedging ($n = 1$) and dynamic hedging cases with different number of trading times ($n = 2$, $n = 5$, $n = 10$ and $n = 20$) using only the future does a better job in variance reduction than using only the call option. It is obvious that using both of them reduces the variance even further. Note also that as more trading opportunities become available, the more variance reduction takes place. For the case of 20 trading periods, nearly 50% reduction in standard deviation is observed. Another illustrative example can also be seen in Figure 5.3 which plots the histograms of unhedged and hedged cash flows for a number of different trading periods where hedging is performed using both future and call option. Note that as the number of trading periods increase, the distribution of total profits gets narrower and more clustered around the mean compared to unhedged profits which is wider. This implies that extreme values are more likely to occur when the cash flow is not hedged using financial instruments.

Observing Figure 5.3, it is also logical to assert that employing theoretically and computationally tractable minimum-variance framework, one can also improve other risk measures such as semivariance. Note that semivariance is defined as the variance

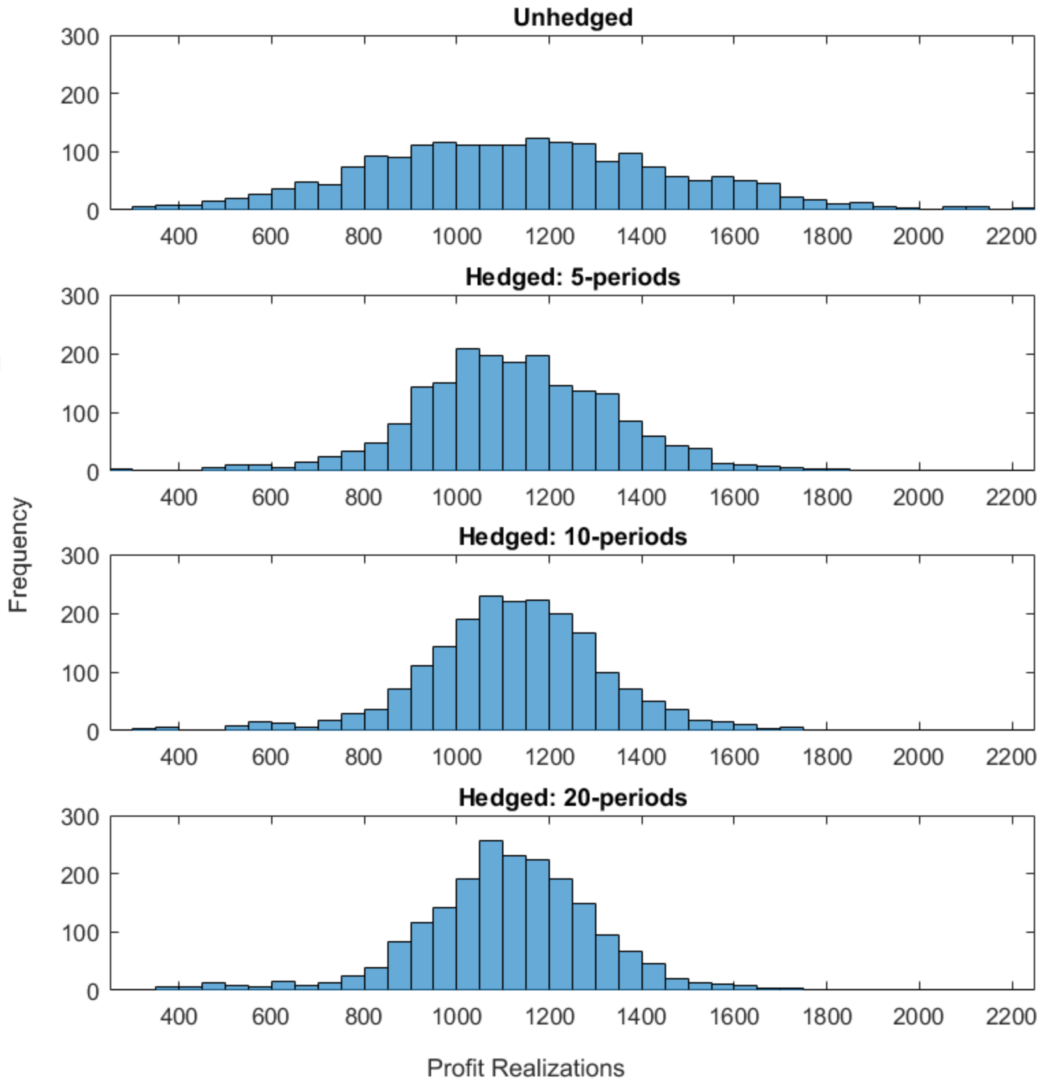

Figure 5.3: Histogram of unhedged and hedged cash flows.

of observations below a target level such as mean. It is clear that for this inventory system, if financial products are used for variance minimization, then the semivariance also decreases as the distribution function of total profit is getting narrower on the left side as well, i.e., the undesirable low profit outcomes.

Knowing the effects of both call option and future leads us to consider different strike prices as well. For the single period model, with the same financial and economic parameters, it turns out that when both securities are used, a strike price of $K = 21$ on the option gives the minimum variance for the final cash flow (expected profits and order quantity are the same). For the case where only a call option is used, it is expected that for $K = 0$ the minimum variance value is achieved as the security is now the same security as the future. As the value of the strike price K gets larger, the minimum variance value decreases. The reason is that for very small values of K the call option practically behaves like a future and this finding is in line with the result in Figure 5.2 where it was observed that the effect of using the future is greater in total risk reduction. Similarly, for large values of K, the call option does not yield any value and is useless in hedging. Consequently, when both a call option and the future are used, as K gets larger the standard deviation decreases until a certain point, and then it starts to increase again which is also observed in Figure 5.4. It is also interesting to observe that one can do quite well in risk reduction by using only a call option with strike price $K = 10$. One should however note that call options with all strike prices may not be available in the market and one may have to choose among those available using the results of Figure 5.4.

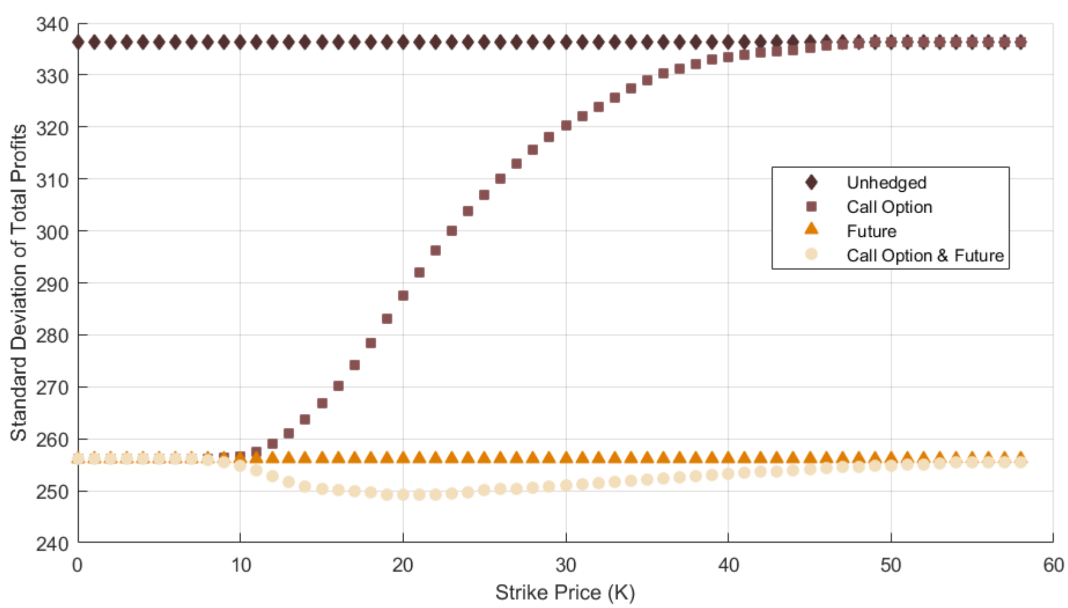

Figure 5.4: Effect of strike price for the call option on risk reduction.

Next, we investigate the effect of varying the customer price sensitivity parameter

B. It turns out that as B increases, the effect of hedging on risk reduction decreases as percentage reduction in standard deviation of the total profit (hence the variance) decreases (Figure 5.5). A larger B results in lower instantaneous arrival rate for each price realization, hence the expected number of customers arriving during each period decreases. This consequently decreases the expected total profit and its maximizing order quantity. At extremely high values of B the effective demand is near-zero and the profits are stable but vanishing.

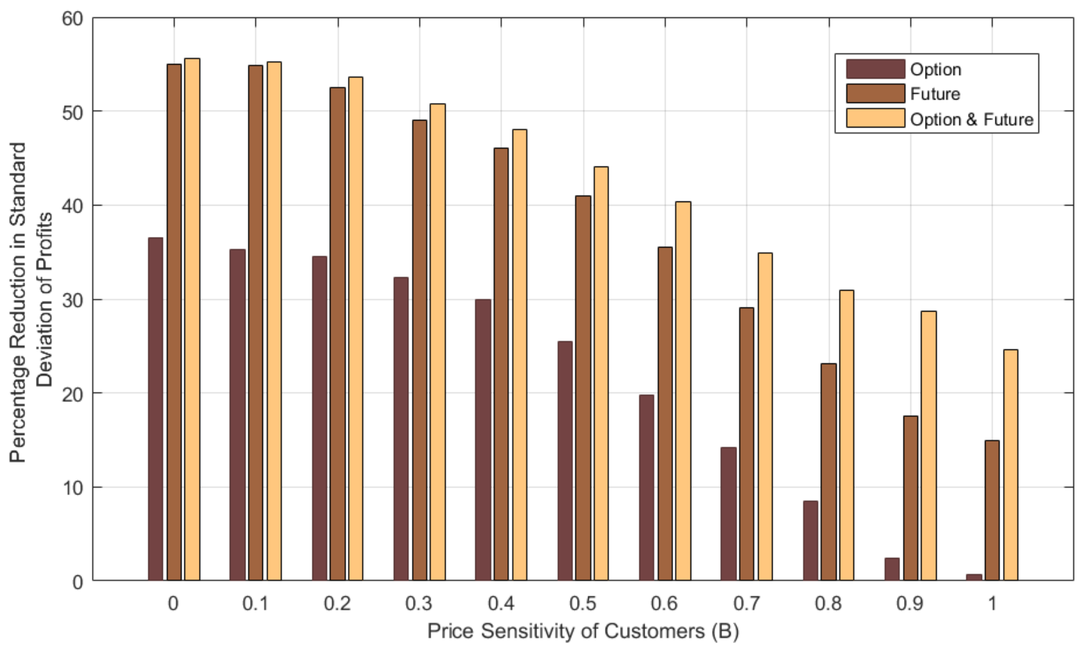

Figure 5.5: Effect of price sensitivity on risk reduction.

Next, we numerically analyze the effect of price volatility on the effect of financial hedging. For this case, we take the number of trading periods $n = 5$ and we assume that $B = 0$, i.e., customers are insensitive to the price changes and they arrive according to a Poisson process with rate $A = 90$ independent of the market price process. While fixing all the economic parameters, we alter the value of short-term volatility σ_χ between 0 and 0.7. Note that as the value of σ_χ changes, the expected prices do not change since they are martingales and since demand is independent of market

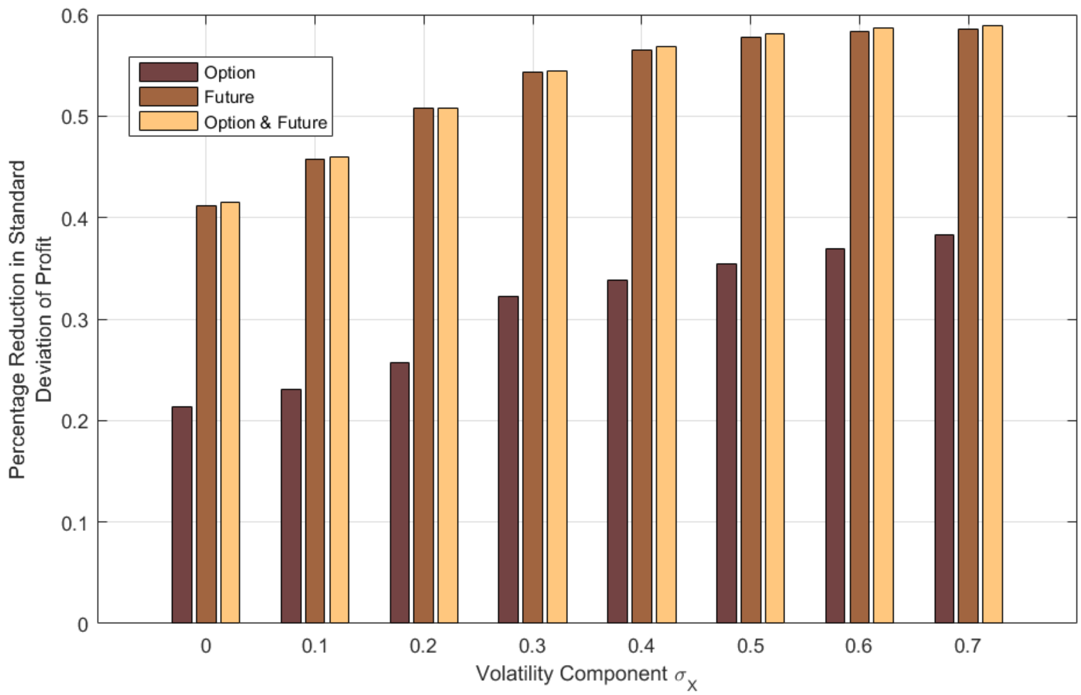

Figure 5.6: Effect of price volatility on risk reduction.

prices, the order quantity that maximizes the expected profit is also unchanged. As seen in Figure 5.6, for this case, we observe that as the price volatility increases, the percentage reduction in standard deviation of the dynamically hedged cash flow also increases.

The effect of backorder costs to the variability of cash flow is rather anticipated. For instance, if the backorder cost increases, the critical fractile and consequently the expectation-maximizing order quantity increase. Although for each inventory decision one obtains a smaller mean profit, variance of the cash flow increases. As a result, increasing backorder costs leads to an increased coefficient of variation in all four cases considered.

5.5 Summary

This chapter addresses the dynamic financial hedging problem of a fairly general inventory model with stochastic demand and prices and a number of financial securities that are correlated with the underlying price process. The risk-averse firm exploits these correlations by taking positions in the financial market and aims to find a hedging strategy that minimizes the variance of the total cash flow for any ordering policy.

In the case where there is a single hedging and inventory ordering opportunity, we characterize the optimal variance-minimizing portfolio. The optimal hedging portfolio consists of a a covariance matrix of security prices and a covariance vector between security prices and random cash flow. We refine the characterization for the special cases of independent demand and price, martingale security prices and the case where there is a single future available in the market.

Second, we analyze the dynamic hedging version of the single period inventory model by assuming that there is a fixed number of trading periods in which the decision maker may revise his initial portfolio using the available information about the market and security prices and current inventory. This is a much more complicated setting but under martingale security prices assumption, we show that the dynamic programming formulation has a separability property and we exploit this to obtain a dynamic minimum-variance portfolio for any ordering policy. One nice feature of our approach is that it leads to numerically implementable solutions.

In the last part, we use simulation to test the effect of financial hedging in risk reduction in a numerical analysis. We use a risk-neutral two-factor price process and observe that using a future does a better job in hedging than a call option. We also observe that as the number of trading periods increases, the amount of variance reduction increases.

Chapter 6

CONCLUDING REMARKS

In this dissertation, we analyzed ordering, pricing and financial hedging policies for a fairly general inventory system with price and demand uncertainties. In particular, we consider an operational setting where purchase and selling prices are described by a continuous-time stochastic price process which, then, also influences the customer demand. In contrast with most of the existing literature, within each period demand arrives continuously and is modulated by the continuous price process. In this setting, sales revenues depend on individual arrival times of demands and not simply on total accumulated demand. This is an appropriate model for consumer demand that is strongly affected by fluctuating market prices that are transferred to the customer. Clearly the existence of this price process has an effect on operational policies such as inventory ordering and even pricing if the firm is allowed to alter the selling price to an extent. It is also clear that such a cash flow exhibits many undesirable uncertainties for a risk-averse firm. Especially, besides random customer demand, fluctuating market prices add another level to the cash flow's overall risk. However, if the underlying price process of the inventory item has some correlation with price evolution of freely-traded financial securities in financial markets, then this clearly presents an opportunity for the firm to alleviate apparent risks.

In Chapter 3, we analyze how fluctuating price environment affect firm's ordering policies. In both backorder and lost-sale settings, we show that price-dependent base-stock policies are optimal under some plausible conditions whose violation has a potential to lead to non-base-stock cases. We also observe through numerical illustrations that price volatility has a significant negative effect on firm's expected profits. In addition, we show that the proposed model has some apparent advantages over approximate models that ignore price volatilities within ordering cycles.

In Chapter 4, we investigate the markup pricing problem of the firm that faces stochastic price volatilities. Although a portion of market-driven prices pass to the customer, the firm also sets a multiplicative markup to control its potential customer demand, hence revenues. This is a different kind of a pricing problem since the existing models use selling price as the control variable and the firm has full control on the price. We characterize the optimal controls for both markup and stock levels, and show their reverse effects on each other. Last, we theoretically prove that price volatilities are disadvantageous for the firm in the special case with convex order stochastic properties.

Finally, in Chapter 5, we assume that the firm has access to a financial market and may invest in potential portfolios of financial products which are correlated with the underlying market price process of the inventory item. In this setting, we assume that after committing to an ordering strategy, the firm tries to minimize the variance of the final cash flow by applying a dynamic trading strategy. This objective assumes that operation is the main focus and know-how of the firm, and finance department supports the firm by reducing the risk with dynamically constructed portfolios. On the other hand, this framework leads to nice and implementable characterizations which make it useful for various complicated operational dynamics. For the inventory setting, we find explicit characterizations of optimal portfolios for various cases and through numerical illustrations, we show the effect and efficiency of financial hedging on risk reduction.

A few other research directions can be considered for the class of models analyzed in this dissertation. First, our assumption of a risk-neutral decision maker can be relaxed. In the backorder model, for instance, the firm has the risk of repurchasing the backordered items later at a higher price. By introducing an appropriate risk-measure, risk-sensitive inventory management in the presence of fluctuating costs and revenues can be examined. While the operational setting in minimum-variance hedging covers many interesting inventory models as special cases, a similar approach is likely to work in other inventory systems that are not covered. In the financial hedging context, one may also be interested in studying the case where security price processes give nonzero expected payoffs. This requires a new dynamic programming formulation as the objective function, the variance of the final cash flow, seems to lose its separability properties. In minimum-variance framework, another consideration would be to introduce a budget constraint on the financial investments as for now, we assume that the decision maker may invest any amount to reduce the variance. Last, instead of variance minimization, one may want to use other risk-measures such as value-at-risk and conditional value-at-risk as the objective functions for the inventory models with continuous price fluctuations as these objectives are also gaining attention in finance literature.

Chapter 7

APPENDIX

Derivation of Expected Total Revenue given in (3.3)

Let $\mathcal{P} = \{P_s : s \in [0,t]\}$. Then,

$$\begin{aligned} r_t(p) &= E\left[\sum_{n=1}^{N_t} e^{-rT_n} f\left(P_{T_n}\right)\right] = E\left[E\left[\sum_{n=1}^{N_t} e^{-rT_n} f\left(P_{T_n}\right) \mid \mathcal{P}\right]\right] \\ &= E\left[\sum_{k=0}^{\infty} P\left\{N_t = k \mid \mathcal{P}\right\} E\left[\sum_{n=1}^{k} e^{-rT_n} f\left(P_{T_n}\right) \mid \mathcal{P},\ N_t = k\right]\right]. \end{aligned}$$

Note that conditioned on $N_t = k$ and $\mathcal{P}$, T_n is the order statistics of k i.i.d. random variables on $[0,t]$ with cumulative distribution function

$$\Phi(s) = \frac{\int_0^s \lambda(P_u)du}{\int_0^t \lambda(P_u)du} \tag{7.1}$$

and probability density function

$$\phi(s) = \Phi'(s) = \frac{\lambda(P_s)}{\int_0^t \lambda(P_u)du}$$

on $0 \leq s \leq t$. Then,

$$r_t(p) = E\left[\sum_{k=0}^{\infty} P\left\{N_t = k \mid \mathcal{P}\right\} kE\left[e^{-r\widetilde{T}} f\left(P_{\widetilde{T}}\right) \mid \mathcal{P}\right]\right]$$

where $\widetilde{T}$ is a random variable with distribution Φ given in (7.1). This implies that

$$\begin{aligned} r_t(p) &= E\left[E\left[N_t \mid \mathcal{P}\right] E\left[e^{-r\widetilde{T}} f\left(P_{\widetilde{T}}\right) \mid \mathcal{P}\right]\right] \\ &= E\left[E\left[N_t \mid \mathcal{P}\right]\left(\frac{\int_0^t e^{-ru} f\left(P_u\right)\lambda(P_u)du}{\int_0^t \lambda(P_u)du}\right)\right]. \end{aligned}$$

Note that

$$E\left[N_t \mid \mathcal{P}\right] = \int_0^t \lambda(P_u)du$$

which implies that

$$r_t\left(p\right) = \int_0^t e^{-ru} E\left[f\left(P_u\right)\lambda(P_u)\right]du.$$

Derivation of Partial Derivatives of $g(y,\alpha)$

Let $\mathcal{P} = \{P_s : s \in [0,T]\}$.

First-order derivative of $g\left(y,\alpha\right)$ with respect to α is

$$g_\alpha\left(y,\alpha\right) = \frac{\partial}{\partial\alpha}E\left[R_T^\alpha - \left(b+P_T\right)D_{N_T^\alpha} - \left(b+P_T\right)\left(y - D_{N_T^\alpha}\right)^+\right].$$

We find

$$\begin{aligned}\frac{\partial}{\partial\alpha}E\left[\left(b+P_T\right)D_{N_T^\alpha}\right] &= \frac{\partial}{\partial\alpha}E\left[E\left[\left(b+P_T\right)D_{N_T^\alpha} \mid \mathcal{P}\right]\right] = E\left[\left(b+P_T\right)\frac{\partial}{\partial\alpha}E\left[D_{N_T^\alpha} \mid \mathcal{P}\right]\right] \\ &= \mu E\left[\left(b+P_T\right)\left(M_T^\alpha\right)'\right]\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial\alpha}E\left[\left(b+P_T\right)\left(y-D_{N_T^\alpha}\right)^+\right] &= \frac{\partial}{\partial\alpha}E\left[E\left[\left(b+P_T\right)\left(y-D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]\right] \\ &= E\left[\left(b+P_T\right)\frac{\partial}{\partial\alpha}E\left[\left(y-D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]\right] \qquad (7.2)\end{aligned}$$

where

$$\begin{aligned}\frac{\partial}{\partial\alpha}E\left[\left(y-D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right] &= \frac{\partial}{\partial\alpha}\sum_{k=0}^{\infty}P\left\{N_T^\alpha = k \mid \mathcal{P}\right\}E\left[\left(y-D_k\right)^+\right] \\ &= \frac{\partial}{\partial\alpha}\sum_{k=0}^{\infty}\left(\frac{e^{-M_T^\alpha}\left(M_T^\alpha\right)^k}{k!}E\left[\left(y-D_k\right)^+\right]\right) \\ &= \sum_{k=0}^{\infty}\left(\left(M_T^\alpha\right)'\left(-\frac{e^{-M_T^\alpha}\left(M_T^\alpha\right)^k}{k!} + \frac{e^{-M_T^\alpha}\left(M_T^\alpha\right)^{k-1}}{(k-1)!}\right)E\left[\left(y-D_k\right)^+\right]\right) \\ &= -\left(M_T^\alpha\right)'\sum_{k=0}^{\infty}P\left\{N_T^\alpha = k \mid \mathcal{P}\right\}E\left[\left(y-D_k\right)^+\right]\end{aligned}$$

$$+\left(M_T^\alpha\right)' \sum_{k=0}^{\infty} P\left\{N_T^\alpha = k \mid \mathcal{P}\right\} E\left[\left(y - D_{k+1}\right)^+\right].$$

We defined earlier that $\triangle_k \left(y - D_k\right)^+ = \left(y - D_{k+1}\right)^+ - \left(y - D_k\right)^+$. This makes

$$\begin{aligned}
\frac{\partial}{\partial \alpha} E\left[\left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right] &= \left(M_T^\alpha\right)' \sum_{k=0}^{\infty} P\left\{N_T^\alpha = k \mid \mathcal{P}\right\} E\left[\triangle_k \left(y - D_k\right)^+\right] \\
&= \left(M_T^\alpha\right)' E\left[\triangle \left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right] \qquad (7.3)
\end{aligned}$$

Finally, we find

$$\begin{aligned}
g_\alpha (y, \alpha) &= r_T' (\alpha) - \mu E\left[\left(b + P_T\right)\left(M_T^\alpha\right)'\right] \\
&\quad - E\left[\left(b + P_T\right)\left(M_T^\alpha\right)' E\left[\triangle \left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]\right] \\
&= r_T' (\alpha) - \mu E\left[\left(b + P_T\right)\left(M_T^\alpha\right)'\right] \\
&\quad - E\left[\left(b + P_T\right)\left(M_T^\alpha\right)' \triangle \left(y - D_{N_T^\alpha}\right)^+\right].
\end{aligned}$$

In another notation, we can write

$$\begin{aligned}
g_\alpha (y, \alpha) = {} & r_T' (\alpha) - \mu E\left[\left(b + P_T\right)\left(M_T^\alpha\right)'\right] \\
& + E\left[\left(b + P_T\right)\left(M_T^\alpha\right)' \min\left\{\left(y - D_{N_T^\alpha}\right)^+, X_{N_T^\alpha + 1}\right\}\right].
\end{aligned}$$

By (7.2), second-order derivative with respect to α is

$$g_{\alpha\alpha} (y, \alpha) = r_T'' (\alpha) - \mu E\left[\left(b + P_T\right)\left(M_T^\alpha\right)''\right] - E\left[\left(b + P_T\right) \frac{\partial^2}{\partial \alpha^2} E\left[\left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]\right]$$

where we can write by (7.3) that

$$\begin{aligned}
\frac{\partial^2}{\partial \alpha^2} E\left[\left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right] = {} & \left(M_T^\alpha\right)'' E\left[\triangle \left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right] \\
& + \left(M_T^\alpha\right)' \frac{\partial}{\partial \alpha} E\left[\triangle \left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right].
\end{aligned}$$

Performing a similar analysis for finding (7.3), one can write

$$\frac{\partial}{\partial \alpha} E\left[\triangle \left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right] = \left(M_T^\alpha\right)' E\left[\triangle^2 \left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]$$

which leads to

$$\frac{\partial^2}{\partial \alpha^2} E\left[\left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right] = \left(M_T^\alpha\right)'' E\left[\triangle \left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]$$

$$+\left((M_T^\alpha)'\right)^2 E\left[\triangle^2\left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]$$

and

$$\begin{aligned}
g_{\alpha\alpha}(y,\alpha) &= r_T''(\alpha) - \mu E\left[(b+P_T)(M_T^\alpha)''\right] \\
&\quad - E\left[(b+P_T)(M_T^\alpha)'' E\left[\triangle\left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]\right] \\
&\quad - E\left[(b+P_T)\left((M_T^\alpha)'\right)^2 E\left[\triangle^2\left(y - D_{N_T^\alpha}\right)^+ \mid \mathcal{P}\right]\right] \\
&= r_T''(\alpha) - \mu E\left[(b+P_T)(M_T^\alpha)''\right] - E\left[(b+P_T)(M_T^\alpha)'' \triangle\left(y - D_{N_T^\alpha}\right)^+\right] \\
&\quad - E\left[(b+P_T)\left((M_T^\alpha)'\right)^2 \triangle^2\left(y - D_{N_T^\alpha}\right)^+\right].
\end{aligned}$$

Finally, partial derivative with respect to each variable is:

$$\begin{aligned}
g_y(y,\alpha) &= -P_0 + E\left[(b+P_T)\,1_{\left\{D_{N_T^\alpha} \geq y\right\}}\right] \\
&= -P_0 + b + E\left[P_T\right] - E\left[(b+P_T)\,1_{\left\{D_{N_T^\alpha} < y\right\}}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial\alpha} g_y(y,\alpha) \\
&= -\frac{\partial}{\partial\alpha} E\left[(b+P_T)\,1_{\left\{D_{N_T^\alpha} < y\right\}}\right] \\
&= -E\left[(b+P_T)\frac{\partial}{\partial\alpha} E\left[1_{\left\{D_{N_T^\alpha} < y\right\}} \mid \mathcal{P}\right]\right] \\
&= -E\left[(b+P_T)\frac{\partial}{\partial\alpha} P\left\{D_{N_T^\alpha} < y \mid \mathcal{P}\right\}\right] \\
&= -E\left[(b+P_T)\frac{\partial}{\partial\alpha}\sum_{k=0}^{\infty}\left(\frac{e^{-M_T^\alpha}\left(M_T^\alpha\right)^k}{k!} F^{(k)}(y)\right)\right] \\
&= -E\left[(b+P_T)(M_T^\alpha)'\left(\sum_{k=0}^{\infty} - \frac{e^{-M_T^\alpha}\left(M_T^\alpha\right)^k}{k!} F^{(k)}(y) + \sum_{k=1}^{\infty}\frac{e^{-M_T^\alpha}\left(M_T^\alpha\right)^{k-1}}{(k-1)!} F^{(k)}(y)\right)\right] \\
&= -E\left[(b+P_T)(M_T^\alpha)'\left(\sum_{k=0}^{\infty} - \frac{e^{-M_T^\alpha}\left(M_T^\alpha\right)^k}{k!} F^{(k)}(y) + \sum_{k=0}^{\infty}\frac{e^{-M_T^\alpha}\left(M_T^\alpha\right)^{k}}{k!} F^{(k+1)}(y)\right)\right] \\
&= -E\left[(b+P_T)(M_T^\alpha)'\left(P\left\{D_{N_T^\alpha} + 1 < y \mid \mathcal{P}\right\} - P\left\{D_{N_T^\alpha} < y \mid \mathcal{P}\right\}\right)\right]
\end{aligned}$$

Proof of (5.7)

Note that similar to the derivation of (3.3), under the assumption that given $\mathcal{P}$, N is independent of S, one can easily find

$$E\left[\sum_{j=1}^{N_T} f\left(P_{T_j}\right) S_T\right] = \int_0^T E\left[f\left(P_u\right)\lambda(P_u) S_T\right] du$$

and

$$E\left[\sum_{j=1}^{N_T} f\left(P_{T_j}\right)\right] = \int_0^T E\left[f\left(P_u\right)\lambda(P_u)\right] du.$$

Then

$$\begin{aligned} Cov\left(\sum_{j=1}^{N_T} f\left(P_{T_j}\right), S_T\right) &= E\left[\sum_{j=1}^{N_T} f\left(P_{T_j}\right) S_T\right] - E\left[\sum_{j=1}^{N_T} f\left(P_{T_j}\right)\right] E\left[S_T\right] \\ &= \int_0^T E\left[f\left(P_u\right)\lambda(P_u) S_T\right] du - \int_0^T E\left[P_u \lambda(P_u)\right] E\left[S_T\right] du \\ &= \int_0^T Cov(f\left(P_u\right)\lambda(P_u), S_T) du \end{aligned}$$

Proof of Theorem 5.2

First note that the first-order forward difference of $\mu\left(y\right)$ given in (5.6) is

$$\begin{aligned} \triangle\mu\left(y\right) &= \mu\left(y+1\right) - \mu\left(y\right) \\ &= Cov\left(b\left(P_T\right)\left(N_T - y\right)^+, S_T\right) - Cov\left(b\left(P_T\right)\left(N_T - y - 1\right)^+, S_T\right) \\ &\quad + Cov\left(h\left(P_T\right)\left(y - N_T\right)^+, S_T\right) - Cov\left(h\left(P_T\right)\left(y - N_T + 1\right)^+, S_T\right) \\ &= Cov\left(b\left(P_T\right) 1_{\{N_T \geq y+1\}}, S_T\right) - Cov\left(h\left(P_T\right) 1_{\{N_T \leq y\}}, S_T\right) \\ &= Cov\left(b\left(P_T\right), S_T\right) - Cov\left(\left(h\left(P_T\right) + b\left(P_T\right)\right) 1_{\{N_T \leq y\}}, S_T\right). \end{aligned}$$

Similarly, first-order difference of $CF\left(y, N, P\right)$ is

$$\begin{aligned} \triangle CF\left(y, N, P\right) &= -P_0 + b\left(P_T\right) 1_{\{N_T \geq y+1\}} - h\left(P_T\right) 1_{\{N_T \leq y\}} \\ &= -P_0 + b\left(P_T\right) - \left(h\left(P_T\right) + b\left(P_T\right)\right) 1_{\{N_T \leq y\}}. \end{aligned}$$

We find the optimal order quantity as

$$\begin{aligned}
y^* &= \inf\left\{y \geq 0 : \triangle E\left[HCF\left(\theta^*\left(y\right), y, N, P, S\right)\right] \leq 0\right\} \\
&= \inf\left\{y \geq 0 : E\left[\triangle CF\left(y, N, P\right) - \triangle\mu\left(y\right)^T C^{-1} \triangle S\right] \leq 0\right\} \\
&= \inf\Big\{y \geq 0 : -P_0 + E\left[b\left(P_T\right)\right] - E\left[\left(h\left(P_T\right) + b\left(P_T\right)\right) 1_{\{N_T \leq y\}}\right] \\
&\qquad -Cov\left(b\left(P_T\right), S_T\right)^T C^{-1} E\left[\triangle S\right] \\
&\qquad\quad +Cov\left(\left(h\left(P_T\right) + b\left(P_T\right)\right) 1_{\{N_T \leq y\}}, S_T\right)^T C^{-1} E\left[\triangle S\right] \leq 0\Big\} \\
&= \inf\Big\{y \geq 0 : E\left[\left(h\left(P_T\right) + b\left(P_T\right)\right) 1_{\{N_T \leq y\}}\right] \\
&\qquad -Cov\left(\left(h\left(P_T\right) + b\left(P_T\right)\right) 1_{\{N_T \leq y\}}, S_T\right)^T C^{-1} E\left[\triangle S\right] \\
&\qquad\quad \geq -P_0 + E\left[b\left(P_T\right)\right] - Cov\left(b\left(P_T\right), S_T\right)^T C^{-1} E\left[\triangle S\right]\Big\}.
\end{aligned}$$

Under Assumption 5.1, the left hand side of the inequality is increasing in y, which ensures that there exists a unique optimal solution since the right hand side is constant.

Proof of Corollary 5.2

Note that for the general case, we already established the form of the optimal hedging portfolio by (5.8). Now consider $\mu\left(y\right)$ given in (5.6) . Then since arrival rate is constant

$$\int_0^T Cov(f\left(P_u\right)\lambda(P_u), S_T)du = \lambda\int_0^T Cov\left(f\left(P_t\right), S_T\right)dt$$

and due to independence of N from S and P,

$$Cov\left(h\left(P_T\right)\left(y - N_T\right)^+, S_T\right) = E\left[\left(y - N_T\right)^+\right] Cov\left(h\left(P_T\right), S_T\right)$$

and

$$Cov\left(b\left(P_T\right)\left(N_T - y\right)^+, S_T\right) = E\left[\left(N_T - y\right)^+\right] Cov\left(b\left(P_T\right), S_T\right).$$

This gives

$$\mu\left(y\right) = \lambda\int_0^T Cov\left(f\left(P_t\right), S_T\right)dt - E\left[\left(y - N_T\right)^+\right] Cov\left(h\left(P_T\right), S_T\right)$$

$$- E\left[(N_T - y)^+\right] Cov\left(b\left(P_T\right), S_T\right).$$

Using the optimal order quantity characterization for the general case given in (5.10), we get

$$\begin{aligned}
y^* = \inf &\Big\{y \geq 0 : E\left[\left(h\left(P_T\right) + b\left(P_T\right)\right) 1_{\{N_T \leq y\}}\right] \\
&\quad -Cov\left(\left(h\left(P_T\right) + b\left(P_T\right)\right) 1_{\{N_T \leq y\}}, S_T\right)^T C^{-1} E\left[\triangle S\right] \\
&\quad \geq -P_0 + E\left[b\left(P_T\right)\right] - Cov\left(b\left(P_T\right), \triangle S\right)^T C^{-1} E\left[\triangle S\right]\Big\} \\
= \inf &\left\{y \geq 0 : P\left\{N_T \leq y\right\} E\left[h\left(P_T\right) + b\left(P_T\right)\right]\right. \\
&\quad -P\left\{N_T \leq y\right\} Cov\left(h\left(P_T\right) + b\left(P_T\right), S_T\right)^T C^{-1} E\left[\triangle S\right] \\
&\quad \left.\geq -P_0 + E\left[b\left(P_T\right)\right] - Cov\left(b\left(P_T\right), S_T\right)^T C^{-1} E\left[\triangle S\right]\right\} \\
= \inf &\left\{y \geq 0 : P\left\{N_T \leq y\right\}\right. \\
&\quad \left.\geq \frac{-P_0 + E\left[b\left(P_T\right)\right] - Cov\left(b\left(P_T\right), S_T\right)^T C^{-1} E\left[\triangle S\right]}{E\left[h\left(P_T\right)\right] + E\left[b\left(P_T\right)\right] - Cov\left(h\left(P_T\right) + b\left(P_T\right), S_T\right)^T C^{-1} E\left[\triangle S\right]}\right\}.
\end{aligned}$$

Proof of Corollary 5.4

Note that

$$\begin{aligned}
Cov\left(h\left(P_T\right) + b\left(P_T\right), S_T\right) &= Cov\left(h + b + (1-\delta) P_T, P_T\right) \\
&= (1-\delta) Var\left(P_T\right). \qquad (7.4)
\end{aligned}$$

and

$$Cov\left(b\left(P_T\right), S_T\right) = Cov\left(b + P_T, P_T\right) = Var\left(P_T\right). \qquad (7.5)$$

Putting (5.14), (7.4) and (7.5) in (5.13) gives (5.15).

To compute optimal hedge, we use (5.12), which gives

$$\begin{aligned}
\theta^*(y) &= -C^{-1}\mu(y) \\
&= -\frac{\lambda \int_0^T Cov\left(f\left(P_t\right), P_T\right) dt - E\left[(y - N_T)^+\right] Cov\left(h - \delta P_T, S_T\right)}{Var\left(P_T\right)} \\
&\quad + \frac{E\left[(N_T - y)^+\right] Cov\left(b + P_T, S_T\right)}{Var\left(P_T\right)}
\end{aligned}$$

$$= E\left[(N_T - y)^+\right] - \delta E\left[(y - N_T)^+\right] - \lambda \int_0^T \frac{Cov\left(f\left(P_t\right), P_T\right)}{Var\left(P_T\right)} dt$$

$$= E\left[(N_T - y)^+\right] - \delta E\left[(y - N_T)^+\right] - \lambda \int_0^T \beta_t dt$$

Proof of Theorem 5.3

We use induction. Note that the claim is true for period n since $R_{[t_n,t_n]} = 0$ and $N_{[t_n,t_n]} = 0$. Now assume that

$$V_{k+1}\left(x, w, p, s\right) = g_{k+1}\left(x, w, p\right) + h_{k+1}\left(x, p, s\right).$$

Then, by (5.19)

$$\begin{aligned}
&V_k\left(x, w, p, s\right) \\
&= \min_{\theta_k} E\left[V_{k+1}\left(x - N_{[t_k,t_{k+1}]}, w + R_{[t_k,t_{k+1}]} + \theta_k^T \triangle S_k, P_{t_{k+1}}, S_{t_{k+1}}\right) \mid P_{t_k} = p, S_{t_k} = s\right] \\
&= \min_{\theta_k} E\left[g_{k+1}\left(x - N_{[t_k,t_{k+1}]}, w + R_{[t_k,t_{k+1}]} + \theta_k^T \triangle S_k, P_{t_{k+1}}\right) \mid P_{t_k} = p, S_{t_k} = s\right] \\
&\quad + E\left[h_{k+1}\left(x - N_{[t_k,t_{k+1}]}, P_{t_{k+1}}, S_{t_{k+1}}\right) | P_{t_k} = p, S_{t_k} = s\right].
\end{aligned} \tag{7.6}$$

Note that second term is independent of θ and the problem is equivalent to minimizing the first part. The optimization problem can be written as

$$\begin{aligned}
&\min_{\theta_k} E\left[g_{k+1}\left(x - N_{[t_k,t_{k+1}]}, w + R_{[t_k,t_{k+1}]} + \theta_k^T \triangle S_k, P_{t_{k+1}}\right) | P_{t_k} = p, S_{t_k} = s\right] \\
&= \min_{\theta_k} E\Big[\Big(w + R_{[t_k,t_{k+1}]} + \theta_k^T \triangle S_k + R_{[t_{k+1},t_n]} - b\left(P_{t_n}\right)\left(N_{[t_{k+1},t_n]} + N_{[t_k,t_{k+1}]} - x\right)^+ \\
&\qquad -h\left(P_{t_n}\right)\left(x - N_{[t_k,t_{k+1}]} - N_{[t_{k+1},t_n]}\right)^+\Big)^2 \mid P_{t_k} = p, S_{t_k} = s\Big] \\
&= \min_{\theta_k} \Big\{ E\left[\left(w + R_{[t_k,t_n]} - b\left(P_{t_n}\right)\left(N_{[t_k,t_n]} - x\right)^+ - h\left(P_{t_n}\right)\left(x - N_{[t_k,t_n]}\right)^+\right)^2 | P_{t_k} = p\right] \\
&\qquad + E\left[\left(\theta_k^T \triangle S_k\right)^2 \mid S_{t_k} = s\right] \\
&\qquad +2\theta_k^T E\Big[\left(w + R_{[t_k,t_n]} - b\left(P_{t_n}\right)\left(N_{[t_k,t_n]} - x\right)^+ - h\left(P_{t_n}\right)\left(x - N_{[t_k,t_n]}\right)^+\right) \triangle S_k \\
&\qquad \mid P_{t_k} = p, S_{t_k} = s\Big]\Big\} \\
&= \min_{\theta_k} \left\{g_k\left(x, w, p\right) + Var\left(\theta_k^T S_{t_{k+1}} \mid S_{t_k} = s\right)\right.
\end{aligned}$$

$$+2\theta_k^T Cov\left(R_{[t_k,t_n]} - b\left(P_{t_n}\right)\left(N_{[t_k,t_n]} - x\right)^+ - h\left(P_{t_n}\right)\left(x - N_{[t_k,t_n]}\right)^+, S_{t_{k+1}}\right.$$
$$\left. \mid P_{t_k} = p, S_{t_k} = s)\right\}$$
$$= \min_{\theta_k}\left\{g_k\left(x,w,p\right) + \theta_k^T C_k\left(s\right)\theta_k + 2\theta_k^T \mu_k\left(x,p,s\right)\right\}. \tag{7.7}$$

Note that the preceding equations followed since $E\left[\triangle S_k\right] = 0$ for all trading periods k. Similar to the single-period problem, the optimal portfolio is

$$\begin{aligned}\theta_k^* &= \arg\max_{\theta_k}\left\{g_k\left(x,w,p\right) + \theta_k^T C_k\left(s\right)\theta_k + 2\theta_k^T \mu_k\left(x,p,s\right)\right\} \\ &= -C_k\left(s\right)^{-1}\mu_k\left(x,p,s\right). \end{aligned} \tag{7.8}$$

Putting (7.8) in (5.19) and using (7.6) and (7.7) , we obtain

$$\begin{aligned}V_k\left(x,w,p,s\right) &= g_k\left(x,w,p\right) + \theta_k^{*T} C_k\left(s\right)\theta_k^* + 2\theta_k^{*T}\mu_k\left(x,p,s\right) \\ &\quad + E\left[h_{k+1}\left(x - N_{[t_k,t_{k+1}]}, P_{t_{k+1}}, S_{t_{k+1}}\right)|P_{t_k} = p, S_{t_k} = s\right] \\ &= g_k\left(x,w,p\right) + \left(C_k\left(s\right)^{-1}\mu_k\left(x,p,s\right)\right)^T C_k\left(s\right)\left(C_k\left(s\right)^{-1}\mu_k\left(x,p,s\right)\right) \\ &\quad - 2\left(C_k\left(s\right)^{-1}\mu_k\left(x,p,s\right)\right)^T\mu_k\left(x,p,s\right) \\ &\quad\quad + E\left[h_{k+1}\left(x - N_{[t_k,t_{k+1}]}, P_{t_{k+1}}, S_{t_{k+1}}\right)|P_{t_k} = p, S_{t_k} = s\right] \\ &= g_k\left(x,w,p\right) + \mu_k\left(x,p,s\right)^T C_k\left(s\right)^{-1} C_k\left(s\right) C_k\left(s\right)^{-1}\mu_k\left(x,p,s\right) \\ &\quad - 2\mu_k\left(x,p,s\right)^T C_k\left(s\right)^{-1}\mu_k\left(x,p,s\right) \\ &\quad\quad + E\left[h_{k+1}\left(x - N_{[t_k,t_{k+1}]}, P_{t_{k+1}}, S_{t_{k+1}}\right)|P_{t_k} = p, S_{t_k} = s\right] \\ &= g_k\left(x,w,p\right) - \mu_k\left(x,p,s\right)^T C_k\left(s\right)^{-1}\mu_k\left(x,p,s\right) \\ &\quad + E\left[h_{k+1}\left(x - N_{[t_k,t_{k+1}]}, P_{t_{k+1}}, S_{t_{k+1}}\right)|P_{t_k} = p, S_{t_k} = s\right] \\ &= g_k\left(x,w,p\right) + h_k\left(x,p,s\right). \end{aligned}$$

Proof of Theorem 5.4

We use induction. Note that by (5.25) and $h_n = 0$, the claim (a) is true for period n. Now assume that

$$V_{k+1}\left(x,w,p,s;y\right) = g_{k+1}\left(x,w,p;y\right) + h_{k+1}\left(x,p,s;y\right)$$

for any $k \leq n-2$.Then, by (5.19)

$$\begin{aligned}
&V_k(x,w,p,s;y)\\
&=\min_{\theta_k} E\left[V_{k+1}\left(X_{k+1},W_{k+1},P_{t_{k+1}},S_{t_{k+1}};y\right) \mid X_k=x,W_k=w,P_{t_k}=p,S_{t_k}=s\right]\\
&=\min_{\theta_k}\left\{E\left[g_{k+1}\left(X_{k+1},W_{k+1},P_{t_{k+1}};y\right) \mid X_k=x,W_k=w,P_{t_k}=p,S_{t_k}=s\right]\right.\\
&\quad\left.+E\left[h_{k+1}\left(X_{k+1},P_{t_{k+1}},S_{t_{k+1}};y\right)|X_k=x,P_{t_k}=p,S_{t_k}=s\right]\right\}.
\end{aligned}$$

Note that second part is independent of θ since only the wealth evolution, i.e., W_{k+1} is affected by the portfolio decision. Therefore, optimal θ is found by minimizing the first part. Note that we can write

$$\begin{aligned}
&\min_{\theta_k} E\left[g_{k+1}\left(X_{k+1},W_{k+1},P_{t_{k+1}};y\right) \mid X_k=x,W_k=w,P_{t_k}=p,S_{t_k}=s\right]\\
&=\min_{\theta_k} E\left[g_{k+1}\left(X_{k+1},w+CF^y_{[t_k,t_{k+1}]}+\theta_k^T \triangle S_k,P_{t_{k+1}};y\right)\right.\\
&\qquad\qquad\left.\mid X_k=x,W_k=w,P_{t_k}=p,S_{t_k}=s\right]\\
&=\min_{\theta_k} E\left[\left(w+CF^y_{[t_k,t_{k+1}]}+CF^y_{[t_{k+1},t_n]}+\theta_k^T \triangle S_k\right)^2|X_k=x,P_{t_k}=p,S_{t_k}=s\right]\\
&=\min_{\theta_k} E\left[\left(w+CF^y_{[t_k,t_n]}+\theta_k^T \triangle S_k\right)^2 \mid X_k=x,P_{t_k}=p,S_{t_k}=s\right]
\end{aligned}$$

We now take the square of the expression in the last equality which leads to

$$\begin{aligned}
&\min_{\theta_k}\left\{E\left[\left(w+CF^y_{[t_k,t_n]}\right)^2|X_k=x,P_{t_k}=p\right]+E\left[\left(\theta_k^T \triangle S_k\right)^2 \mid S_{t_k}=s\right]\right.\\
&\qquad\left.+2\theta_k^T E\left[\left(w+CF^y_{[t_k,t_n]}\right)\triangle S_k \mid X_k=x,P_{t_k}=p,S_{t_k}=s\right]\right.\\
&=\min_{\theta_k}\left\{g_k(x,w,p)+Var\left(\theta_k^T S_{t_{k+1}} \mid S_{t_k}=s\right)\right.\\
&\qquad\left.+2\theta_k^T Cov\left(CF^y_{[t_k,t_n]},\ S_{t_{k+1}} \mid X_k=x,P_{t_k}=p,S_{t_k}=s\right)\right\}\\
&=\min_{\theta_k}\left\{g_k(x,w,p)+\theta_k^T C_k(s)\,\theta_k+2\theta_k^T\mu_k(x,p,s)\right\}.
\end{aligned}$$

Similar to the single-period problem, the optimal portfolio is

$$\begin{aligned}
\theta_k^* &= \arg\max_{\theta_k}\left\{g_k(x,w,p)+\theta_k^T C_k(s)\,\theta_k+2\theta_k^T\mu_k(x,p,s)\right\}\\
&=-C_k(s)^{-1}\mu_k(x,p,s).
\end{aligned}$$

Putting θ_k^* in (5.19), we obtain

$$\begin{aligned}
&V_k\left(x,w,p,s;y\right) \\
&= g_k\left(x,w,p;y\right) + \theta_k^{*T} C_k\left(s\right)\theta_k^* + 2\theta_k^{*T}\mu_k\left(x,p,s;y\right) \\
&\quad + E\left[h_{k+1}\left(X_{k+1},P_{t_{k+1}},S_{t_{k+1}};y\right)|P_{t_k}=p,S_{t_k}=s,X_k=x\right] \\
&= g_k\left(x,w,p;y\right) + \left(C_k\left(s\right)^{-1}\mu_k\left(x,p,s;y\right)\right)^T C_k\left(s\right)\left(C_k\left(s\right)^{-1}\mu_k\left(x,p,s;y\right)\right) \\
&\quad - 2\left(C_k\left(s\right)^{-1}\mu_k\left(x,p,s;y\right)\right)^T\mu_k\left(x,p,s;y\right) \\
&\qquad + E\left[h_{k+1}\left(X_{k+1},P_{t_{k+1}},S_{t_{k+1}};y\right)|P_{t_k}=p,S_{t_k}=s,X_k=x\right] \\
&= g_k\left(x,w,p;y\right) + \mu_k\left(x,p,s;y\right)^T C_k\left(s\right)^{-1} C_k\left(s\right) C_k\left(s\right)^{-1}\mu_k\left(x,p,s;y\right) \\
&\quad - 2\mu_k\left(x,p,s;y\right)^T C_k\left(s\right)^{-1}\mu_k\left(x,p,s;y\right) \\
&\qquad + E\left[h_{k+1}\left(X_{k+1},P_{t_{k+1}},S_{t_{k+1}};y\right)|P_{t_k}=p,S_{t_k}=s,X_k=x\right] \\
&= g_k\left(x,w,p;y\right) - \mu_k\left(x,p,s;y\right)^T C_k\left(s\right)^{-1}\mu_k\left(x,p,s;y\right) \\
&\quad + E\left[h_{k+1}\left(X_{k+1},P_{t_{k+1}},S_{t_{k+1}};y\right)|P_{t_k}=p,S_{t_k}=s,X_k=x\right] \\
&= g_k\left(x,w,p;y\right) + h_k\left(x,p,s;y\right).
\end{aligned}$$

BIBLIOGRAPHY

V. Agrawal and S. Seshadri. Impact of uncertainty and risk aversion on price and order quantity in the newsvendor problem. *Manufacturing and Service Operations Management*, 2:410–423, 2000.

S. Ahmed, U. Çakmak, and A. Shapiro. Coherent risk measures in inventory problems. *European Journal of Operational Research*, 182:226–238, 2007.

M. Anvari. Optimality criteria and risk in inventory models: The case of the newsboy problem. *Journal of the Operational Research Society*, 38:625–632, 1987.

P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath. Coherent measures of risk. *Mathematical Finance*, 9(3):203–228, 1999.

S. Banerjee and N. S. Meitei. Effect of declining selling price: profit analysis for a single period inventory model with stochastic demand and lead time. *Journal of the Operational Research Society*, 61(4):696–704, 2009.

M. Baxter and A. Rennie. *Financial Calculus: An introduction to derivative pricing*. Cambridge University Press, 1996.

P. Berling and V. Martínez-de Albéniz. Optimal inventory policies when purchase price and demand are stochastic. *Operations Research*, 59(1):109–124, 2011.

P. Berling and Z. Xie. Approximation algorithms for optimal purchase/inventory policy when purchase price and demand are stochastic. *OR Spectrum*, 36(4):1077–1095, 2014.

O. Berman and J. A. Schnabel. Mean-variance analysis and the single-period inventory problem. *International Journal of Systems Science*, 17:1145–1151, 1986.

B. Błaszczyszyn and D. Yogeshwaran. Directionally convex ordering of random measures, shot noise fields, and some applications to wireless communications. *Advances in Applied Probability*, 623–646, 2009.

M. Bouakiz and M. Sobel. Inventory control with exponential utility criterion. *Operations Research*, 40:603–608, 1992.

R. Caldentey and M. Haugh. Optimal control and hedging of operations in the presence of financial markets. *Mathematics of Operations Research*, 31:285–304, 2006.

L. M. Chan, Z. M. Shen, D. Simchi-Levi, and J. L. Swann. Coordination of pricing and inventory decisions: a survey and classification. In *Handbook of Quantitative Supply Chain Analysis*, 335–392. Springer, 2004.

D. Chen, S. Benjaafar, and W. L. Cooper. On the impact of input price volatility in stochastic inventory systems. 2014. Working Paper.

F. Chen and A. Federgruen. Mean-variance analysis of basic inventory models. *Technical Manuscript*, 2000. Columbia Business School.

X. Chen, M. Sim, D. Simchi-Levi, and P. Sun. Risk aversion in inventory management. *Operations Research*, 55:603–608, 2007.

F. Cheng and S. Sethi. Optimality of state-dependent (s, S) policies in inventory models with markov-modulated demand and lost sales. *Production and Operations Management*, 8(2):183–192, 1999.

J. Chod, N. Rudi, and J. A. Van Mieghem. Operational flexibility and financial hedging: Complements or substitutes? *Management Science*, 56(6):1030–1045, 2010.

T. Choi, D. Li, and H. Yan. Mean-variance analysis for the newsvendor problem. *IEEE Transactions on Systems, Man, and Cybernetics Part A: Systems and Humans*, 38: 1169–1180, 2008.

K. Chung. Risk in inventory models: The case of the newsboy problem, optimality conditions. *The Journal of the Operational Research Society*, 41:173–176, 1990.

D. R. Cox. Some statistical methods connected with series of events. *Journal of the Royal Statistical Society. Series B (Methodological)*, 129–164, 1955.

Q. Ding, L. Dong, and P. Kouvelis. On the integration of production and financial hedging decisions in global markets. *Operations Research*, 55(3):470–489, 2007.

L. Eeckhoudt, C. Gollier, and H. Schlesinger. The risk-averse (and prudent) newsboy. *Management Science*, 41:786–794, 1995.

A. S. Erdem and S. Özekici. Inventory models with random yield in a random environment. *International Journal of Production Economics*, 78(3):239–253, 2002.

E. Erel. The effect of continuous price change in the EOQ. *Omega*, 20(4):523–527, 1992.

A. Federgruen and A. Heching. Combined pricing and inventory control under uncertainty. *Operations research*, 454–475, 1999.

K. A. Froot, D. S. Scharfstein, and J. Stein. Risk management: Coordinating corporate investment and financing policies. *the Journal of Finance*, 48(5):1629–1658, 1993.

G. Gallego and H. Hu. Optimal policies for production/inventory systems with finite capacity and markov-modulated demand and supply processes. *Annals of Operations Research*, 126(1-4):21–41, 2004.

V. Gaur and S. Seshadri. Hedging inventory risk through market instruments. *Manufacturing and Service Operations Management*, 7:103–120, 2005.

S. Gavirneni. Periodic review inventory control with fluctuating purchasing costs. *Operations Research Letters*, 32(4):374–379, 2004.

J.-P. Gayon, I. Talay-Değirmenci, F. Karaesmen, and E. L. Örmeci. Optimal pricing and production policies of a make-to-stock system with fluctuating demand. *Probability in the Engineering and Informational Sciences*, 23(02):205–230, 2009.

A. Goel and G. Gutierrez. Integrating commodity markets in the optimal procurement policies of a stochastic inventory system. *Available at SSRN 930486*, 2006.

K. Golabi. Optimal inventory policies when ordering prices are random. *Operations Research*, 33(3):575–588, 1985.

J. Gotoh and Y. Takano. Newsvendor solutions via conditional value-at-risk minimization. *European Journal of Operations Research*, 179:80–96, 2007.

R. Grubbström. The newsboy problem when customer demand is a compound renewal process. *European Journal of Operational Research*, 203(1):134–142, 2010.

X. Guo, P. Kaminsky, P. Tomecek, and M. Yuen. Optimal spot market inventory strategies in the presence of cost and price risk. *Mathematical Methods of Operations Research*, 73(1):109–137, 2011.

Ç. Haksöz and S. Seshadri. Supply chain operations in the presence of a spot market: a review with discussion. *Journal of the Operational Research Society*, 58(11): 1412–1429, 2007.

M. A. Hariga. Effects of inflation and time-value of money on an inventory model with time-dependent demand rate and shortages. *European Journal of Operational Research*, 81(3):512–520, 1995.

P. Jorion. *Value at risk: the new benchmark for managing financial risk*, volume 2. McGraw-Hill New York, 2007.

B. Kalymon. Stochastic prices in a single item inventory purchasing model. *Operations Research*, 19:1434–1458, 1971.

A. P. Katariya, E. Tekin, and S. Çetinkaya. A capacitated replenishment-liquidation model under contractual and spot markets with stochastic demand. *Probability in the Engineering and Informational Sciences*, 28(02):223–269, 2014.

M. Khouja and S. Park. Optimal lot sizing under continuous price decrease. *Omega*, 31(6):539–545, 2003.

P. Kouvelis, R. Li, and Q. Ding. Managing storable commodity risks: The role of inventory and financial hedge. *Manufacturing & Service Operations Management*, 15(3):507–521, 2013.

P. Kouvelis, Z. Pang, and Q. Ding. Dynamic financial hedging strategies for a storable commodity with demand uncertainty. Working Paper, 2015.

A. Lau and H. Lau. Maximizing the probability of achieving a target profit in a two-product newsboy problem. *Decision Sciences*, 19:392–408, 1988.

H. Lau. The newsboy problem under alternative optimization objectives. *Journal of the Operational Research Society*, 31:525–535, 1980.

L. Leskelä, M. Vihola, et al. Conditional convex orders and measurable martingale couplings. *Bernoulli*, 23(4A):2784–2807, 2017.

P. A. Lewis and G. S. Shedler. Simulation of nonhomogeneous poisson processes by thinning. *Naval Research Logistics Quarterly*, 26(3):403–413, 1979.

J. Li, H. Lau, and A. Lau. A two-product newsboy problem with satisficing objective and independent exponential demands. *IIE transactions*, 23:29–39, 1991.

E. Luciano, L. Peccati, and D. Cifarelli. VaR as a risk measure for multiperiod static inventory models. *International Journal of Production Economics*, 81:375–384, 2003.

H. Markowitz. *Portfolio Selection: Efficient Diversification of Investments*. John Wiley, New York, 1959.

A. Muharremoglu and J. N. Tsitsiklis. A single-unit decomposition approach to multiechelon inventory systems. *Operations Research*, 56(5):1089–1103, 2008.

A. Müller and D. Stoyan. *Comparison methods for stochastic models and risks*, volume 389. Wiley, 2002.

J. Ni, L.-K. Chu, and B. P. Yen. Coordinating operational policy with financial hedging for risk-averse firms. *Omega*, 59:279–289, 2016.

K. Okyay, F. Karaesmen, and S. Özekici. Hedging demand and supply risks in the newsvendor model. *OR Spectrum*, 37(2):475–501, 2014.

S. Özekici and M. Parlar. Inventory models with unreliable suppliers in a random environment. *Annals of Operations Research*, 91:123–136, 1999.

A. Özler, B. Tan, and F. Karaesmen. Multi-product newsvendor problem with value-at-risk considerations. *International Journal of Production Economics*, 117:244–255, 2009.

M. Parlar and Z. Weng. Balancing desirable but conflicting objectives in the newsvendor problem. *IIE Transactions*, 35:131–142, 2003.

N. Petruzzi and M. Dada. Pricing and the newsvendor problem: A review with extensions. *Operations Research*, 47(2):183–194, 1999.

E. L. Porteus. *Foundations of stochastic inventory theory*. Stanford University Press, 2002.

E. Sankarasubramanian and S. Kumaraswamy. Optimal ordering quantity to realize a pre-determined level of profit. *Management Science*, 29:512–514, 1983.

F. Sayın, F. Karaesmen, and S. Özekici. Newsvendor model with random supply and financial hedging: Utility-based approach. *International Journal of Production Economics*, 154:178–189, 2014.

H. Scarf. The optimality of (s, S) policies in the dynamic inventory problem. In K. Arrow, S. Karlin, and P. Suppes, editors, *Mathematical Methods in the Social Sciences*. Stanford University Press, Stanford, CA, 1960.

E. Schwartz and J. Smith. Short-term variations and long-term dynamics in commodity prices. *Management Science*, 46(7):893–911, 2000.

N. Secomandi. Optimal commodity trading with a capacitated storage asset. *Management Science*, 56(3):449–467, 2010.

J. Smith. Evaluating income streams: A decision analysis approach. *Management Science*, 44:1690–1708, 1998.

J.-S. Song and P. Zipkin. Inventory control in a fluctuating demand environment. *Operations Research*, 41(2):351–370, 1993.

M. Tekin and S. Özekici. Mean-variance newsvendor model with random supply and financial hedging. *IIE Transactions*, 47:910–928, 2015.

T. M. Whitin. Inventory control and price theory. *Management Science*, 2(1):61–68, 1955.

J. Wu, J. Li, S. Wang, and T. Cheng. Mean-variance analysis of the newsvendor model with stockout cost. *Omega*, 37:724–730, 2009.

P. C. Yang, H. M. Wee, B. S. Liu, and O. K. Fong. Mitigating hi-tech products risks due to rapid technological innovation. *Omega*, 39(4):456–463, 2011.

C. A. Yano and S. M. Gilbert. Coordinated pricing and production/procurement decisions: A review. *Managing Business Interfaces*, 65–103, 2005.

S. Ziya, H. Ayhan, and R. D. Foley. Relationships among three assumptions in revenue management. *Operations Research*, 52(5):804–809, 2004.