Global Behavior of Solutions of Nonlinear Dissipative Equations of Nonclassical Types

by

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A Dissertation Submitted to the
Graduate School of Sciences and Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Doctor of Philosophy

in

Computational Sciences and Engineering

KOÇ ÜNİVERSİTESİ

January, 2020

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Koç University

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to my beloved mother

ABSTRACT

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In this thesis, we investigate the global behavior of solutions of initial-boundary value problems (IBVP) for nonlinear dissipative equations. We particularly focus on two problems in hydrodynamics: IBVP for Burgers' original model of turbulence (*original Burgers' equations*), and IBVP for Burgers' equation with nonlocal nonlinearity. Motivated by the studies in finite-dimensional asymptotic behavior of dissipative equations, we prove the stabilization of these equations by using finitely many controllers, such as finitely many Fourier modes, finitely many volume elements and finitely many nodal values. We also prove that the asymptotic behavior of solutions of original Burgers' equations can be completely determined by finite number of determining modes. Additionally, we show the existence, uniqueness and stability of the solutions of the inverse source problem for both equations. We show that, under proper assumptions, the solutions of the inverse source problem tends to a particular stationary state solution of the direct problem, and the unknown source term tends to zero as time goes to infinity. Finally, we perform numerical experiments to verify the validity of our theoretical findings on the finite-parameter feedback control problems for original Burgers' equations.

ÖZETÇE

Doğrusal olmayan disipatif denklemlerin çözümlerinin global davranışı
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Ocak, 2020

Bu tez çalışmasında, doğrusal olmayan disipatif başlangıç-sınır değer problemlerinin çözümlerinin global davranışını inceliyoruz. Özellikle, hidrodinamik konuları ile ilgili olan iki problemi ele alıyoruz. Bu problemler; orijinal Burgers denklemleri ve lokal ve doğrusal olmayan terimli Burgers denklemi. Disipatif denklemlerin sonlu-boyutlu asimptotik davranışları ile ilgili çalışmalardan ilham alarak; sonlu sayıda kontrol terimi kullanarak, her iki problemin kararlılığını gösteriyoruz. Sonlu sayıdaki kontrol terimlerine; sonlu sayıda Fourier modları, sonlu sayıda hacim elemanları ve sonlu sayıda düğüm noktaları örnek olarak verilebilir. Ayrıca, orijinal Burgers denklemlerinin çözümlerinin asimptotik davranışlarının, sonlu sayıda belirleyen modlar kullanarak tamamen anlaşılabileceğini kanıtlıyoruz. Bunlara ek olarak, her iki denklem için ters kaynak problemlerinin çözümlerinin varlığını, tekliğini ve kararlılığını gösteriyoruz. Uygun koşullar altında, ters kaynak problemlerinin çözümlerinin, direkt problemin durgun durum çözümlerine yaklaştığını gösteriyoruz. Ayrıca, bilinmeyen kaynak teriminin de, zaman sonsuza giderken, sıfıra yaklaştığını kanıtlıyoruz. Son olarak, orijinal Burgers denklemleri için oluşturduğumuz sonlu sayıda parametreli geribildirimli denetim problemleri ile ilgili teorik çalışmalarımızın geçerliliğini doğrulayan sayısal çalışmalar uyguluyoruz.

ACKNOWLEDGMENTS

First of all, I would like to express my sincere gratitude to my advisors Prof. Dr. Varga Kalantarov and Assoc. Prof. Dr. Emre Mengi for their invaluable guidance and continuous support for my PhD study and research. Thanks to their guidance, I have learned how to be a good researcher and academician. I admire their endless energy, motivation and passion for mathematics and teaching. I am grateful for their support.

Besides my advisors, I also thank my thesis committee Prof. Dr. Metin Muradoğlu, Prof. Dr. Hüsnü Ata Erbay, Prof. Dr. Attila Aşkar and Prof. Dr. Mansur İsgenderoğlu for valuable suggestions and comments. I would like to express my special thanks to Prof. Dr. Attila Aşkar for his encouragement, inspiration and guidance with wise advices throughout my PhD education. I feel so lucky to have been a teaching assistant for his courses. He has taught so much about teaching and life. I would like to acknowledge Prof. Dr. Edriss S. Titi for his valuable suggestions for my research.

I would like to thank Koç University for financially supporting me with scholarship and travel grant, providing complete academic materials for my education and research, and for various scientific, social and cultural events that helped expand my horizon.

I would like to thank my friends, Çiğdem, Oğuz, Fatih, Enis, Didem and Barış for their lifelong friendship and amusement. I will always miss our breakfasts, lunches and all the fun we have had together. I would like to also thank my colleagues: Aryan, Aslı, Begüm, Berkay, Cem, Elif & Yasin, Fırtına, Murad, Murat Can, Nida, Onur, Ruhat, Sedanur, Yasin and especially İhsan for his kindness; and promising undergraduates: Duygu, İpek, Rızacan. Thanks to their encouragement, emotional support and helpfulness, I have overcome the difficulties. I would like to thank Ayşe Abla for our nice conversations and her sincere wishes for me. Furthermore, I would like to thank Bilgesu for invaluable suggestions for my future research. Additionally, I would like to thank Melike, Gamze and Göksu for their friendship and our good memories at the conferences abroad. I also would like to thank my oldest friends Müslüme, Esranur, Selda and Elif for their emotional support and best wishes.

I would like to thank my mother and father for their unconditional love and support. Especially, I would like to express my gratitude to my mother. She has always encouraged and supported me to be a persevering, patient and independent person. I would like to thank my parents-in-law and brother-in-law for their support and encouragement. I also thank my brother and cousins for having great time together in holidays. I am thankful to all my family members, especially my grandmother "Kadriye Sultan", for believing and wishing me to achieve my goals. I would like to express my gratitude to my grandparents who passed away in the first two years of my PhD education.

Finally, I would like to thank my husband and my lifelong friend Hakan for his endless love and support. Throughout this work, he has motivated and encouraged me to do my best. I am grateful to him for living with me in a harmony and appreciate his efforts to build a beautiful future for our family.

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NOMENCLATURE

Notation	Meaning
Ω	domain in $\mathbb{R}^n$
$L^p(\Omega)$	the Banach space of all functions measurable in Ω with the norm
	$\|u\|_{L^p(\Omega)} := \left(\int_\Omega \|u(x)\|^p dx\right)^{\frac{1}{p}} < \infty$
	for $1 \leq p < \infty$
$L^2(\Omega)$	the Hilbert space with the norm defined as in $L^p(\Omega)$ for $p = 2$ and
	inner product $(u, v)_{L(\Omega)} = \int_\Omega u(x)v(x)dx$
$L^\infty(\Omega)$	the Banach space of all functions measurable in Ω with the norm
	$\|u\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}_{x\in\Omega} \|u(x)\| < \infty$
$H^1(\Omega)$	the set of all functions with their first weak derivatives from $L^2(\Omega)$
$C_0^\infty(\Omega)$	the test functions: the set of all functions
	which are infinitely differentiable and have compact support in Ω
$H_0^1(\Omega)$	the Sobolev space which is the completion $C_0^\infty(\Omega)$
$H^{-1}(\Omega)$	the dual space of $H_0^1(\Omega)$
$(\cdot, \cdot)$	the inner product in L^2 space
$\\|\cdot\\|$	the norm in L^2 space
$\\|v(t)\\|$	the L^2-norm of a function $v(x, t)$ with respect to
	the space variable x
λ_k	the k-th eigenvalue of the Sturm-Liouville operator
	under the homogenous Dirichlet boundary conditions
w_k	the k-th eigenfunction of the Sturm-Liouville operator
	under the homogenous Dirichlet boundary conditions
a.e.	almost everwhere

Chapter 1

INTRODUCTION

The turbulence motion of the fluids is studied by many researchers. One of the significant studies in hydrodynamics was conducted by J. M. Burgers [4] in 1939. In order to analyze the behavior of the hydrodynamic equations for the turbulent flow in a channel between parallel walls, Burgers [4] has introduced a nonlinear ODE-ODE system

$$\begin{cases} \frac{dU}{dt} = P - \nu U - v^2, \\ \frac{dv}{dt} = Uv - \nu v, \end{cases} \tag{1.0.1}$$

where U and v depend on only the time variable t, and denote the velocities of the mean motion (or primary motion) and turbulent motion (or secondary motion), respectively. The constant $P > 0$ denotes the external force or pressure, and $\nu > 0$ denotes the kinematic viscosity. This coupled nonlinear ODE system shows the occurrence of a *laminar* and a *turbulent* solution (or motion). The laminar solution means that there is no secondary solution v and the turbulent solution means that the secondary solution v is nonzero. J. M. Burgers [4] shows that the stationary solution (when both $\frac{dU}{dt}$ and $\frac{dv}{dt}$ are zero)

$$U = \frac{P}{\nu}, \quad v = 0, \tag{1.0.2}$$

is stable when the external force $P < \nu^2$ (or $U < \nu$). We observe that this stationary solution (1.0.2) is a laminar solution since there is no secondary solution, i.e., $v = 0$ and the primary solution U is proportional to the external force P. When $P = \nu^2$, the stationary solution is unstable. In the case that $P > \nu^2$, Burgers showed that there are two other stationary solutions

$$U = \nu, \quad v = \pm\sqrt{P - \nu^2}, \tag{1.0.3}$$

which are stable. In this case, the stationary solutions (1.0.3) are turbulent solutions and the laminar solution (the first stationary solution) (1.0.2) is no longer stable. These results were important; however, in [4] the author states that the system (1.0.1) does not give sufficient information about the complexity and the spatial pattern of the turbulent fluid motion since the secondary solutions in (1.0.3) are independent of time and the function v does depend on a spatial variable. In 1948, J. M. Burgers [5] introduced the following coupled ODE-PDE system which we call *original Burgers' equations*

$$\begin{cases} \partial_t v(y,t) = \frac{1}{b}U(t)v(y,t) + \nu\partial_x^2 v(y,t) - 2v(y,t)\partial_x v(y,t), \\ bU'(t) = P - \frac{\nu}{b}U(t) - \frac{1}{b}\int_0^b v^2(y,t)dy. \end{cases} \tag{1.0.4}$$

This system describes the motion of a fluid in a straight channel with parallel walls. The width of this channel is b. As in the system (1.0.1), the constants $P > 0$ and $\nu > 0$ are the external force and the kinematic viscosity and $U(t)$ and $v(y,t)$ are the velocities of mean and turbulent motion, respectively. In (1.0.4) the turbulent motion $v(y,t)$ now depends on the space variable y (the coordinate in the direction of the cross dimension of the channel) and extends from 0 to b and becomes zero at the boundary values 0 and b. In [5], J. M. Burgers analyzed the stability of the stationary solutions of the system in (1.0.4) and find the spectrum of the stationary solutions. In later years, systems defined by equations similar to the ones in (1.0.4) are studied. C. O. Horgan and W. E. Olmstead [31] considered the initial boundary value problem for the dimensionless form of the system in (1.0.4) under homogeneous Dirichlet's boundary conditions and proved asymptotic stability of the stationary solution $u = 0, v = 0$ of the system. T. Dlotko devoted the papers [21], [20] and [22] to the problems of existence and uniqueness of solutions, and the stability of the stationary solution of the initial boundary value problems for the dimensionless version of the system in (1.0.4) with the space variable x in the interval $[0, \pi]$ under the homogeneous Dirichlet boundary conditions $v(0,t) = 0 = v(\pi,t)$. In the paper [19], T. Dlotko proved the existence and uniqueness of the solution of the initial boundary value problem for the two-dimensional version of the (1.0.4). In [24], A. Eden proved the existence of an exponential attractor of the semigroup generated by the initial boundary value problem for the dimensionless version of the system (1.0.4). The existence of inertial manifolds for Burgers' original mathematical model system of turbulence is investigated in the paper [33].

One of the simplifications of the system in (1.0.4) is the viscous Burgers' equation with nonlocal nonlinearities

$$\partial_t v - \nu\partial_x^2 v + 2v\partial_x v - Rv + kv\int_0^1 v^2 dx = h, \quad x \in (0,1), t > 0, \tag{1.0.5}$$

where $R > 0$ and $k > 0$ are positive constants and $h \in L^2(\mathbb{R}^+, L^2(0,1))$. This equation is first introduced in the book [23]. In [18], K. Deng et. al discussed the asymptotic behavior and the global existence of solutions of the unstable Burgers' equation with nonlocal term, i.e., equation (1.0.5) with $k < 0$. The authors showed that the solutions blow up in a finite time under certain conditions on the initial data. For the results on stability of stationary state solutions and existence of finite-dimensional attractors for semigroups generated by initial boundary value problems for this equation, we refer to [54], [9], [50] and the references therein.

1.1 Finite-Parameter Feedback Stabilization

The feedback stabilization by finite-dimensional controllers of nonlinear dissipative PDEs has been shown in [2], [10], [1], [40], [38], [39], [45]. The pioneering study of Azouani and Titi [1] is based on the idea that the finite-dimensional asymptotic behavior is sufficient for designing feedback controls for most dissipative systems. They introduced a finite-parameter feedback control scheme for stabilizing the solutions of a one-dimensional Chafee-Infante reaction-diffusion system with cubic nonlinearity in the form

$$\partial_t u - \nu\partial_x^2 u - \alpha u + u^3 = 0, \tag{1.1.1}$$

under the homogeneous Neumann boundary conditions

$$\partial_x u(0,t) = \partial_x u(L,t) = 0.$$

The idea in [1] is to choose an interpolant operator as a feedback controller. This interpolant operator $I_h : H^1(0,L) \to L^2(0,1)$ is a general linear map with the property

$$\|\psi - I_h(\psi)\| \le ch\|\psi\|_{H^1(0,L)}, \quad \forall \psi \in H^1(0,L),$$

where $c > 0$ is a constant and $h = \frac{L}{N}$ with $N > 1$ denoting an integer that changes depending on the type of the interpolant operator. Below we provide the three examples of interpolant operators introduced in [1]:

1. *The projection onto first N Fourier modes:*
   For a periodic function $\psi \in H^1(0,L)$, the interpolant operator based on finitely many Fourier modes defined as

   $$I_h(\psi) := \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos\frac{k\pi x}{L} + \sum_{k=1}^{N} b_k \sin\frac{k\pi x}{L}, \quad h = \frac{L}{N},$$

   and the Fourier coefficients defined as

   $$a_k = \frac{2}{L}\int_0^L \psi(x)\cos\frac{k\pi x}{L}dx, \quad b_k = \frac{2}{L}\int_0^L \psi(x)\sin\frac{k\pi x}{L}dx.$$

2. *Finite volume elements:*
   For a function $\psi \in H^1(0,L)$, the interpolant operator based on finitely many volume elements or local spatial averages, defined as

   $$I_h(\psi) := \sum_{k=1}^{N} \bar{\psi}_k \chi_{J_k}(x),$$

   where $J_k = \big[(k-1)\frac{L}{N}, k\frac{L}{N}\big)$ for $k = 1, ..., N-1$ and $J_N = \big[(N-1)\frac{L}{N}, L\big]$, $\chi_{J_k}(x)$ is the characteristic function for the interval J_k for $k = 1, ..., N$ defined as follows:

   $$\chi_{J_k}(x) := \begin{cases} 1, & x \in J_k, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\bar{\psi}_k = \frac{1}{|J_k|}\int_{J_k}\psi(x)dx, \quad k = 1, ..., N.$$

3. *Finite nodal values:*

   For a function $\psi \in H^1(0, L)$, the interpolant operator based on finitely many nodal values defined as

$$I_h(\psi) = \sum_{k=1}^{N}\psi(x_k)\chi_{J_k}(x),$$

   where $x_k \in J_k = \left[(k-1)\frac{L}{N}, k\frac{L}{N}\right]$ for $k = 1, ..., N$ and $\chi_{J_k}(x)$ is as before.

E. Lunasin and E. S. Titi [45] investigate the global stabilization of solutions of one-dimensional Kuramoto-Sivashinsky equation

$$\partial_t u + \partial_x^4 u + \partial_x^2 u + u\partial_x u = 0, \tag{1.1.2}$$

based on the finite-parameter feedback control scheme introduced in [1]. Additionally, [45] verify their results by numerical studies for the Chafee-Infante (1.1.1) and Kuramoto-Sivashinsky (1.1.2) equations.

V. Kalantarov and E. S. Titi, in the papers [38] and [39], use the finite-parameter feedback control in order to stabilize the solutions of $3D$ Navier-Stokes-Voight equations

$$\partial_t u - \nu\Delta u - \alpha^2\Delta\partial_t v + (v\cdot\nabla)u + \nabla p = h, \quad \nabla\cdot u = 0, \quad x \in \Omega, t > 0, \tag{1.1.3}$$

and damped nonlinear dispersive equations and some of their modifications, for example

$$\partial_t^2 u - \Delta u + bg(\partial_t u) - \alpha u + f(u) = h(x), \quad x \in \Omega, \quad t > 0, \tag{1.1.4}$$

where $\Omega \subset \mathbb{R}^3$.

One of the recent studies in the feedback control by using finite number of determining parameters is the paper of J. Kalantarova and T. Ozsari [40]. They show that the solutions of complex Ginzburg-Landau equation

$$\partial_t u - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^p u - \gamma u = 0, \quad x \in \Omega, t > 0 \tag{1.1.5}$$

can be stabilized globally, where $\Omega \subset \mathbb{R}^n$.
Motivated by the papers [1, 38–40, 45], we propose various finite-parameter feedback control problems and show the exponential stabilization of solutions of the systems (1.0.4) and (1.0.5).

1.2 Finite Dimensional Asymptotic Behavior

Finite dimensional asymptotic behaviour of dissipative systems has been a significant research subject for the last four decades. First inspiring and rigorous studies were performed by Foias and Prodi [27] in 1967 and Ladyzhenskaya [44] in 1975. They prove that asymptotic behavior of solutions to the initial boundary value problem for the $2D$ Navier–Stokes equations is determined by the asymptotic behavior of the first N Fourier modes for sufficiently large N. Their ideas lead to further studies about finite number of determining parameters such as determining nodes, local volume averages, degrees of freedom and functionals (elements).
Related to the determining modes, we refer to the papers [26] and [37]. The authors in these paper find an upper bound on the number of determining modes for the determining modes for the $2D$ Navier-Stokes equations, $3D$ Navier-Stokes-Voight equations and structurally damped nonlinear wave equations.
C. Foias and R. Temam [28] prove that the large time behavior of the solution of the $2D$ or $3D$ Navier-Stokes equations can be determined by a set of finite number of nodal values or nodes. C. Foias and E.S. Titi [29] show the relation between the concepts of determining nodes, finite difference schemes and finite volumes for the $1D$ Kuramoto-Sivashinsky equation. I. Kukavica [42] finds an upper bound on the number of determining nodes for the Gingzburg-Landau equation. Later, D.A. Jones and E.S. Titi in [34–36] give an upper bound on the number of the determining modes, determining nodes and determining volume elements for the $2D$ Navier-Stokes equations under the periodic boundary conditions. We refer to the papers [15, 16] for the studies on the degrees of freedom and to the papers [3, 11–14] for the determining functionals.
Motivated by these studies and references therein, we show that asymptotic behavior

of the solutions of the initial-boundary value problem for (1.0.4) can be determined by a finite number of determining modes.

1.3 Inverse Source Problems

There are various types of inverse problems which have been intensively studied in numerous branches of mathematical physics [48]. One type of inverse problems is the *inverse source problem.* In this type of problem, there is an unknown source function in the equation. An inverse source problem can be uniquely solvable if there is an extra condition (e.g. integral or final overdetermination condition) on the solution of the direct problem and if this extra condition satisfies some compatibility conditions on the initial data of the problem. We refer to the books [32, 47] and the papers [52, 53] for the existence, uniqueness and stability of the solutions of various types of inverse source problems.

By motivated these studies, we construct the following inverse source problems for (1.0.4)

$$
\begin{aligned}
&\partial_t v = Uv + \nu\partial_x^2 v - 2v\partial_x v + f(t)w(x),\\
&U'(t) = R - \nu U(t) - \int_0^1 v^2(x,t)dx,\\
&U(0) = U_0, \quad v(x,0) = v_0(x),\\
&v(0,t) = v(1,t) = 0,\\
&\int_0^1 v(x,t)w(x)dx = \phi(t),
\end{aligned}
$$

and for (1.0.5)

$$
\begin{aligned}
&\partial_t v - \nu\partial_x^2 v + 2v\partial_x v - Rv + kv\int_0^1 v^2 dx = h(x,t) + f(t)w(x),\\
&v(0,t) = v(1,t) = 0,\\
&v(x,0) = v_0(x),\\
&\int_0^1 v(x,t)w(x)dx = \phi(t).
\end{aligned}
$$

Here, $w(x)$, $\phi(t)$ and $h(x,t)$ are given functions and $f(t)$ is an unknown function that we seek. Note that we use the same notations for the unknown source function

f and overdetermination function ϕ in both problems. However, we note that these functions are not necessarily the same. We refer to Chapter 4 for further detail. We prove the existence and uniqueness of the solutions for both inverse source problems above. Additionally, we find the necessary conditions on the functions w, ϕ and the problem parameters R and ν, and prove the stability of the solutions of those problems.

This thesis is devoted to understand the global behavior of the solutions of the various nonlinear dissipative equations. We focus on the stabilization problems for the original Burgers' equations (1.0.4) and the viscous Burgers' equation with nonlocal nonlinearity (1.0.5). In the rest of this chapter, in Section 1.4 we present some inequalities and significant theorem and lemmas that we utilize throughout the thesis. In Chapter 2, we consider finite-parameter feedback stabilization problems. We study feedback control problems for original Burgers' equations and viscous Burgers' equation with nonlocal nonlinearity based on finitely many parameters such as *finitely many Fourier modes*, *finitely many volume elements* and *finitely many nodal values*. In Chapter 3, we prove the existence of determining modes for original Burgers' equations. In Chapter 4, we prove the existence and uniqueness of the solution of the inverse source problems which are constructed for original Burgers' equations (1.0.4) and Burgers' equation with nonlocal nonlinearity (1.0.5). Additionally, we analyze the stability of the solutions for both problems. In Chapter 5, we present our numerical experiments for the finite-parameter feedback stabilization problems. Finally, in Chapter 6 we summarize our results and describe the future work.

1.4 Preliminaries

In this section, we present preliminary inequalities, lemmas and theorems that we utilized throughout the thesis.

1.4.1 Useful Inequalities

1. **Hölder Inequality**

   Assume that $p \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then

$uv \in L^1(\Omega)$ and

$$\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)}. \tag{1.4.1}$$

2. **1D Gagliardo-Nirenberg interpolation inequality**([39])

   For all $u \in H^2(0,1) \cap H_0^1(0,1)$, the following inequality holds

   $$\|u^{(j)}\|_{L^p(0,1)} \leq \beta\|u\|^{1-\theta}\|u^{(m)}\|^{\theta}, \tag{1.4.2}$$

   where $p \geq 2$, $m = 1, 2$, $\frac{j}{m} \leq \theta \leq 1$, $\theta = \frac{1}{m}\left(\frac{1}{2} + j - \frac{1}{p}\right)$ and $\beta > 0$ is a constant.

3. **Gronwall's Inequality (Differential form)**

   Let $b(t)$ and $f(t)$ be continuous functions for $t \geq \alpha$ and let $u(t)$ be a differentiable function for $t \geq \alpha$. Suppose

   $$u'(t) \leq b(t)u(t) + f(t), \quad t \geq \alpha,$$
   $$u(\alpha) \leq u_0.$$

   Then, for $t \geq \alpha$

   $$u(t) \leq u_0 \exp\left(\int_\alpha^t b(s)ds\right) + \int_\alpha^t f(s)\exp\left(\int_s^t b(\tau)d\tau\right)ds. \tag{1.4.3}$$

4. **Gronwall's inequality (Integral form)**

   Let $u(t)$ and $v(t)$ be nonnegative continuous functions for $t \geq 0$, and satisfy the following inequality:

   $$u(t) \leq C + \int_0^t u(r)v(r)dr, \quad \forall t \geq 0,$$

   where $C \geq 0$. Then

   $$u(t) \leq C\exp\left(\int_0^t v(r)dr\right), \quad \forall t \geq 0. \tag{1.4.4}$$

5. **Young's inequality**

   Let p and q be positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for any nonnegative real numbers a and b the following inequality holds:

   $$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.4.5}$$

6. **Young's Inequality with ε:**

   Let p and q be positive real numbers such that $\frac{1}{p}+\frac{1}{q}=1$ and a and b be nonnegative real numbers. Then for all $\varepsilon>0$, the following inequality holds:

$$ab \leq \frac{\varepsilon}{p}a^p + \frac{1}{q\varepsilon^{\frac{q}{p}}}b^q. \tag{1.4.6}$$

7. **Poincaré-Friedrichs inequality**

   For all $u \in H_0^1$, the following inequality holds

$$\|u\|^2 \leq \lambda_1^{-1}\|u'\|^2, \tag{1.4.7}$$

   where λ_1 is the first eigenvalue of the operator $-\frac{d^2}{dx^2}$ under the homogeneous Dirichlet boundary conditions.

8. **Poincaré-Friedrichs-type inequality**

   For all $u \in H_0^1$, the following inequality holds

$$\sum_{k=N+1}^{\infty} |(u, w_k)|^2 \leq \lambda_{N+1}^{-1}\|u'\|^2, \tag{1.4.8}$$

   where λ_{N+1} is the $(N+1)$th eigenvalue of the operator $-\frac{d^2}{dx^2}$ under the homogeneous Dirichlet boundary conditions.

9. **Sobolev inequality**

   For all $u \in H_0^1$, the following inequality holds

$$\|u\|_{L^\infty}^2 \leq c_0\|u'\|^2, \tag{1.4.9}$$

   where $c_0>0$ is a positive constant.

10. **Monotonicity Inequality**

    If $p \geq 2$, then there exists a number $d_0(p,n)$ such that for each $a, b \in \mathbb{R}^n$ the following inequality holds true

$$(|a|^{p-2}a - |b|^{p-2}b, a-b) \geq d_0|a-b|^p. \tag{1.4.10}$$

1.4.2 Useful Lemmas and Theorems

Lemma 1.4.1 ([51]). *Let V, H, V^* be three Hilbert spaces such that $V \subset H \equiv H^* \subset V^*$, where V^* and H^* are dual spaces of V and H, respectively. If $u \in L^2(0,T;V)$ and $u_t \in L^2(0,T;V^*)$, then the following equality holds true:*

$$\frac{d}{dt}\|u(t)\|^2 = 2\langle u_t, u\rangle_H.$$

Lemma 1.4.2. *([25]) Assume that $\alpha(t)$ and $\beta(t)$ are locally integrable real-valued functions on $[0,\infty)$ that satisfy for some $T > 0$ the following conditions*

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha(\tau)d\tau = \gamma > 0, \tag{1.4.11}$$

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha^-(\tau)d\tau = \Gamma < \infty, \tag{1.4.12}$$

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \beta^+(\tau)d\tau = 0, \tag{1.4.13}$$

where $\alpha^- = \max\{-\alpha, 0\}$ and $\beta^+ = \max\{\beta, 0\}$. Suppose that $\phi(t)$ is an absolutely continuous nonnegative function on $[0,\infty)$ that satisfies the inequality a.e. on $[0,\infty)$

$$\frac{d\phi(t)}{dt} + \alpha(t)\phi(t) \le \beta(t).$$

Then $\phi(t) \to 0$ as $t \to \infty$.

Lemma 1.4.3 ([49]). *Assume that $u \in \dot{H}^1_p(0,L)$, where $\dot{H}^1_p(0,L)$ is the completion of $C^\infty_p(0,L) = \{u \in C^\infty(0,L)|$ u is periodic with period $L\}$ with $\int_0^L u(x)dx = 0$, then*

$$\int_0^L |u(x)|^2 dx \le \left(\frac{L^2}{4\pi^2}\right)\int_0^L |u'(x)|^2 dx.$$

Lemma 1.4.4 ([1]). *Assume that $u \in H^1(0,1)$. Then the following inequality holds*

$$\left\| u - \sum_{k=1}^N \bar{u}_k \chi_{J_k} \right\| \le h\|u'\|, \tag{1.4.14}$$

where $h = \frac{1}{N}$, $J_k = [(k-1)\frac{1}{N}, k\frac{1}{N})$, for $k = 1,2,...,N-1$, $J_N = [\frac{N-1}{N}, 1]$, χ_{J_k} is the characteristic function of the interval J_k and

$$\bar{u}_k = \frac{1}{|J_k|}\int_{J_k} u(x)dx.$$

Proof. Since $\sum_{k=1}^{N} \chi_{J_k}(x) = 1$ we can write that $u(x) = \sum_{k=1}^{N} u(x)\chi_{J_k}(x)$. Thus, we obtain that

$$\left\| u - \sum_{k=1}^{N} \bar{u}_k \chi_{J_k} \right\|^2 = \int_0^1 \left(\sum_{k=1}^{N} (u(x) - \bar{u}_k)\chi_{J_k}(x) \right)^2 dx$$
$$= \int_0^1 \sum_{k,l=1}^{N} (u(x) - \bar{u}_k)(u(x) - \bar{u}_l)\chi_{J_k}(x)\chi_{J_l}(x)dx.$$

Since $\chi_{J_k}(x)\chi_{J_l}(x) = \chi_{J_k}(x)\delta_{kl}$, we have from the previous equation that

$$\left\| u - \sum_{k=1}^{N} \bar{u}_k \chi_{J_k} \right\|^2 = \sum_{k=1}^{N} \int_{J_k} (u(x) - \bar{u}_k)^2 dx \tag{1.4.15}$$

Since $\int_{J_k} (u(x) - \bar{u}_k)dx = 0$ and $u(x) \in H^1(0, L)$, we can apply Lemma 1.4.3 in equality 1.4.15. We obtain:

$$\int_{J_k} (u(x) - \bar{u}_k)^2 dx \leq \left(\frac{h}{2\pi} \right)^2 \int_{J_k} (u'(x))^2 dx. \tag{1.4.16}$$

By summing inequality (1.4.16) from $k = 1$ to $k = N$, we get:

$$\left\| u - \sum_{k=1}^{N} \bar{u}_k \chi_{J_k} \right\|^2 \leq \left(\frac{h}{2\pi} \right)^2 \sum_{k=1}^{N} \int_{J_k} (u'(x))^2 dx$$
$$= \left(\frac{h}{2\pi} \right)^2 \int_0^L (u'(x))^2 dx \leq h^2 \|u'\|^2.$$

Hence, this proves the lemma.

□

Lemma 1.4.5 ([1]). *Assume that $u \in H^1(0, 1)$. Then we have the following inequality*

$$\left\| u - \sum_{k=1}^{N} u(x_k)\chi_{J_k} \right\| \leq h\|u'\|, \tag{1.4.17}$$

where $h = \frac{1}{N}$, $x_k \in J_k$, J_k and χ_{J_k} as defined in Lemma 1.4.4.

Proof. By using similar procedures in the proof of Lemma 1.4.4, we can write that

$$\left\| u - \sum_{k=1}^{N} u(x_k)\chi_{J_k} \right\|^2 = \int_0^L \sum_{k=1}^{N} (u(x) - u(x_k))^2 \chi_{J_k}(x) dx$$

$$= \sum_{k=1}^{N} \int_{J_k} (u(x) - u(x_k))^2 dx = \sum_{k=1}^{N} \int_{J_k} \left(\int_{x_k}^{x} u'(y) dy \right)^2 dxr$$

$$\leq \sum_{k=1}^{N} \int_{J_k} \left(\int_{J_k} |u'(y)| dy \right)^2 dx \leq h \sum_{k=1}^{N} \left(\int_{J_k} |u'(y)| dy \right)^2. \quad (1.4.18)$$

By applying Cauchy-Schwarz Inequality on the right-hand side of inequality (1.4.18), we obtain:

$$\left\| u - \sum_{k=1}^{N} u(x_k)\chi_{J_k} \right\|^2 \leq h \sum_{k=1}^{N} \int_{J_k} 1 dy \int_{J_k} |u'(y)|^2 dy$$

$$= h^2 \sum_{k=1}^{N} \int_{J_k} |u'(y)|^2 dy \leq h^2 \|u'\|^2.$$

Hence, this proves the lemma.

□

Chapter 2

FEEDBACK STABILIZATION PROBLEMS

This chapter is devoted to investigation of the long time behavior of the solutions of the feedback control problem for the original Burgers' equations (1.0.4) and Burgers' equation with nonlocal nonlinearity (1.0.5). We propose various finite-parameter feedback control problems and show the exponential stabilization of solutions of the systems (1.0.4) and (1.0.5). More precisely, in Section 2.1, we show

1. L^2 and H^1-stabilization of any solution to system (1.0.4), under homogeneous Dirichlet boundary conditions, with the feedback controllers based on finitely many Fourier modes, general interpolant operator and finitely many volume elements.

In Section 2.2, we extend

1. the L^2 and H^1-stabilization result of Section 2.1 also to system (1.0.5) when the feedback control based on finitely many Fourier modes,

2. L^2-stabilization result to system (1.0.5) when the feedback control involves finitely many volume elements and finitely many nodal values.

The results in Sections 2.1.1, 2.2.1 and 2.2.2 are submitted as S. Gumus and V. K. Kalantarov, "Finite-parameter feedback stabilization of original Burgers' equations and Burgers' equation with nonlocal nonlinearities" in [30].

2.1 Original Burgers' Equations

In this section, we consider the following initial-boundary value problem for original Burgers' equations

$$\begin{cases} \partial_t v = Uv + \nu\partial_x^2 v - 2v\partial_x v, & (2.1.1) \\ U'(t) = R - \nu U(t) - \displaystyle\int_0^1 v^2 dx, & (2.1.2) \\ v(0,t) = v(1,t) = 0, \quad U(0) = U_0, \quad v(x,0) = v_0(x), & (2.1.3) \end{cases}$$

where $(x,t) \in [0,1] \times [0,\infty)$.

Definition 2.1.1 ([21]). A pair of functions $[v, U]$ is called a weak solution of the problem (2.1.1)-(2.1.3) if

1. U is absolutely continuous on the interval $[0,T]$ for all $T > 0$ and satisfies

$$U(t) = U_0 + \int_0^t (R - \nu U(\tau) - \|v(\tau)\|^2) d\tau, \quad \text{a.e. in } [0,T],$$

2. $v \in L^2(0,T; H_0^1(0,1)) \cap L^\infty(0,T; L^2(0,1))$ and $\partial_t v \in L^2(0,T; L^2(0,1))$ satisfy

$$(\partial_t v, \eta) + \nu(\partial_x v, \eta') + 2(v\partial_x v, \eta) = (Uv, \eta), \quad \forall \eta \in H_0^1(0,1),$$

3. $v(0,x) = v_0(x) \in L^2(0,1)$.

Before analyzing the behavior of the solution of the feedback control system, first let us find the uniform estimates on the solutions of the problem (2.1.1)-(2.1.3). We present the uniform estimates in the following lemma.

Lemma 2.1.2. *There exist positive numbers T_1 and T_2 depending on initial data and R only such that the following inequalities hold:*

$$\|v(t)\|^2 + |U(t)|^2 \le M_1, \quad \forall t > T_1, \quad \|\partial_x v(t)\|^2 \le M_2, \quad \forall t > T_2,$$

where M_1 and M_2 are positive constants, depending on the initial data $|U_0|$, $\|v_0\|$ and the constants R, ν and λ_1 which is the first eigenvalue of the Sturm-Liouville operator under the homogeneous Dirichlet boundary conditions.

Proof. We multiply (2.1.1) by v in $L^2(0,1)$ and (2.1.2) by U. By adding the resulting equations, we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\|v(t)\|^2+|U(t)|^2\right]+\nu\|\partial_x v(t)\|^2+\nu|U(t)|^2=RU(t)\le\frac{1}{2\nu}R^2+\frac{\nu}{2}|U(t)|^2. \quad (2.1.4)$$

Thanks to the Poincaré-Friedrichs inequality (1.4.7), we have

$$\frac{d}{dt}[\|v(t)\|^2+|U(t)|^2]+d_0[\|v(t)\|^2+|U(t)|^2]\le\frac{R^2}{\nu},$$

where $d_0=\nu\min\{1,2\lambda_1\}$. This inequality implies that

$$\|v(t)\|^2+|U(t)|^2\le[\|v_0\|^2+|U_0|^2]e^{-d_0t}+\frac{R^2}{\nu d_0}(1-e^{-d_0t}).$$

Hence, there exists a number $T_1>0$ such that

$$\|v(t)\|^2+|U(t)|^2\le M_1:=\frac{2R^2}{\nu d_0},\quad \forall t\ge T_1. \quad (2.1.5)$$

In order to get the estimate for $\|\partial_x v(t)\|^2$ first we multiply (2.1.2) by $-\partial_x^2 v$ in $L^2(0,1)$ leading us to

$$\frac{1}{2}\frac{d}{dt}\|\partial_x v(t)\|^2-U(t)\|\partial_x v(t)\|^2+\nu\|\partial_x^2 v(t)\|^2+\int_0^1(\partial_x v(t))^3dx=0. \quad (2.1.6)$$

Employing the Gagliardo-Nirenberg inequality (1.4.2) $\|(\partial_x v)\|_{L^3}\le\beta\|v\|^{\frac{5}{12}}\|\partial_x^2 v\|^{\frac{7}{12}}$ and the Young's inequality (1.4.6) with $\varepsilon=\frac{2}{7},\ p=\frac{8}{7}$ we get

$$\left|\int_0^1(\partial_x v(t))^3dx\right|\le\frac{\nu}{4}\|\partial_x^2 v(t)\|^2+\beta^{24}\nu^{-7}7^7 2^{-10}\|v(t)\|^{10}. \quad (2.1.7)$$

On the other hand,

$$|U(t)|\|\partial_x v(t)\|^2\le|U(t)|\|\partial_x v(t)\|\|\partial_x^2 v(t)\|\le\frac{\nu}{4}\|\partial_x^2 v\|^2+\frac{1}{\nu}|U(t)|^2\|v(t)\|^2. \quad (2.1.8)$$

By using the estimates (2.1.7) and (2.1.8) and Poincaré-Friedrichs inequality (1.4.7) from (2.1.6) we obtain that

$$\frac{d}{dt}\|\partial_x v(t)\|^2+\lambda_1\nu\|\partial_x v(t)\|^2\le\frac{1}{\nu}|U(t)|^2\|v(t)\|^2+\beta^{24}\nu^{-7}7^7 2^{-9}\|v(t)\|^{10}. \quad (2.1.9)$$

Due to the estimate (2.1.5), there exists $T_2>0$ such that

$$\|\partial_x v(t)\|^2\le M_2,\quad \forall t\ge T_2. \quad (2.1.10)$$

□

2.1.1 Finitely Many Fourier Modes

We propose the feedback control system for the problem (2.1.1)-(2.1.3).

$$\begin{cases} \partial_t \tilde{v} = \tilde{U}\tilde{v} + \nu \partial_x^2 \tilde{v} - 2\tilde{v}\partial_x \tilde{v} - \mu \sum_{k=1}^{N} (\tilde{v} - v, w_k) w_k, & (2.1.11) \\ \tilde{U}'(t) = R - \nu \tilde{U}(t) - \int_0^1 \tilde{v}^2(x,t) dx, & (2.1.12) \\ \tilde{v}(0,t) = \tilde{v}(1,t) = 0, \quad \tilde{U}(0) = \tilde{U}_0, \quad \tilde{v}(x,0) = \tilde{v}_0(x), & (2.1.13) \end{cases}$$

where $(x,t) \in [0,1] \times [0,\infty)$, $\mu > 0$ is control parameter and $N \in \mathbb{Z}^+$ is the number of Fourier modes. The Fourier modes are $w_1, ..., w_N$'s which are orthonormal (in $L^2(0,1)$-sense) eigenfunctions of the operator $-\partial_x^2$ under the homogeneous Dirichlet boundary conditions. They are explicitly given by $w_k(x) = \sqrt{2}\sin(k\pi x)$, $k = 1, 2, ..., N$.

We find uniform estimates also for the solution of this feedback control problem (2.1.11)-(2.1.13). Let us show these uniform estimates in the following lemma.

Lemma 2.1.3. *There exist positive numbers T_3 and T_4 such that the following inequalities hold:*

$$|\tilde{U}(t)|^2 + \|\tilde{v}(t)\|^2 \leq M_3, \quad \forall t > T_3, \quad \|\partial_x \tilde{v}(t)\|^2 \leq M_4, \quad \forall t > T_4, \tag{2.1.14}$$

where M_3 and M_4 are positive constants, depending on $|\tilde{U}_0|$ and $\|\tilde{v}_0\|$, R, ν, μ, λ_1, and the uniform bounds M_1, M_2 in Lemma 2.1.2.

Proof. In order to obtain bounds for $\|\tilde{v}(t)\|$ and $|\tilde{U}(t)|$, we multiply equation (2.1.11) by $\tilde{v}$ in $L^2(0,1)$ and (2.1.12) by $\tilde{U}$. By adding the resulting equations we get that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left[\|\tilde{v}(t)\|^2 + |\tilde{U}(t)|^2\right] + \nu|\tilde{U}(t)|^2 + \nu\|\partial_x \tilde{v}(t)\|^2 &= R\tilde{U}(t) \\ - \mu \sum_{k=1}^{N} |(\tilde{v}, w_k)|^2 + \mu \sum_{k=1}^{N} (\tilde{v}, w_k)(v, w_k). & \quad (2.1.15) \end{aligned}$$

Thanks to Young's inequality (1.4.6) and Poincaré-Friedrichs inequality (1.4.7) and the estimate (2.1.5) in Lemma 2.1.2, we obtain the inequality

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left[\|\tilde{v}(t)\|^2 + |\tilde{U}(t)|^2\right] + \frac{\nu}{2}|\tilde{U}(t)|^2 + \lambda_1 \nu \|\tilde{v}(t)\|^2 &\leq \frac{R^2}{2\nu} + \frac{\mu}{4}\sum_{k=1}^{N}(v, w_k)^2 \\ &\leq \frac{R^2}{2\nu} + \frac{\mu}{4}M_1, \quad \forall t \geq T_1. \quad (2.1.16) \end{aligned}$$

Integrating (2.1.16) over the interval (t, T_1), we get the following estimate

$$\|\tilde{v}(t)\|^2 + |\tilde{U}(t)|^2 \leq (\|\tilde{v}(T_1)\|^2 + |\tilde{U}(T_1)|^2)e^{-d_1(t-T_1)} + \frac{1}{d_1}\left(\frac{R^2}{\nu} + \frac{\mu}{2}M_1\right)\left(1 - e^{-d_1(t-T_1)}\right),$$

where $d_1 = \nu \min\{1, 2\lambda_1\}$. Hence, there exists $T_3 > T_1$ such that

$$\|\tilde{v}(t)\|^2 + |\tilde{U}(t)|^2 \leq M_3 := \frac{2R^2}{\nu d_1} + \frac{\mu}{d_1}M_1, \quad \forall t \geq T_3. \tag{2.1.17}$$

Next, we multiply the equation (2.1.12) by $-\partial_x^2 \tilde{v}$ in $L^2(0,1)$ and after simple manipulations we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_x \tilde{v}(t)\|^2 - \tilde{U}(t)\|\partial_x \tilde{v}(t)\|^2 + \nu\|\partial_x^2 \tilde{v}(t)\|^2 + \int_0^1 (\partial_x \tilde{v}(t))^3 dx = -\mu \sum_{k=1}^{N} \lambda_k (\tilde{v}, w_k)^2 + \mu \sum_{k=1}^{N} \lambda_k (\tilde{v}, w_k)(v, w_k) \leq \frac{\mu}{4} \sum_{k=1}^{N} \lambda_k (v, w_k)^2.$$

Employing the inequalities (2.1.7), (2.1.8), we deduce the analog of the inequality in (2.1.9) for $\tilde{v}$, that is

$$\frac{d}{dt}\|\partial_x \tilde{v}(t)\|^2 + \lambda_1 \nu \|\partial_x \tilde{v}(t)\|^2 \leq \frac{1}{\nu}|U(t)|^2 \|\tilde{v}(t)\|^2 + \beta^{24}\nu^{-7}7^7 2^{-9}\|\tilde{v}(t)\|^{10} + \frac{\mu}{2}\|\partial_x v(t)\|^2.$$

Thanks to the estimates (2.1.5), (2.1.10) and (2.1.17) we conclude from the last inequality that

$$\|\partial_x \tilde{v}(t)\|^2 \leq M_4, \quad \forall t \geq T_4 \geq T_3. \tag{2.1.18}$$

Hence, this proves the lemma. □

Global Stabilization in L2-norm

In this part, we prove our main result on the stabilization of the feedback control system (2.1.11)-(2.1.13).

Theorem 2.1.4. *Assume that μ and N are large enough such that*

$$M_5 \leq \mu, \quad \text{and } \lambda_{N+1}^{-1} M_5 \leq \frac{\nu}{4}, \tag{2.1.19}$$

where

$$M_5 := \sqrt{M_3} + \frac{1}{2\nu}M_3^2 + \frac{3}{4}\nu^{-\frac{1}{3}}\beta^{8/3}(\sqrt{M_4} + 2\sqrt{M_2})^{4/3}, \tag{2.1.20}$$

M_2, M_3 and M_4 are the constants in the uniform estimates in (2.1.10), (2.1.17) *and* (2.1.18)*, respectively, and $\beta > 0$ is the constant from Gagliardo-Nirenberg inequality* (1.4.2)*. Then there exists $t_0 > 0$ such that for all $t \geq t_0$ the following inequality holds true*

$$\begin{aligned} &\|\tilde{v}(t) - v(t)\|^2 + |\tilde{U}(t) - U(t)|^2 \\ &\qquad \leq (\|v(t_0) - \tilde{v}(t_0)\|^2 + |U(t_0) - \tilde{U}(t_0)|^2)e^{-d_2(t-t_0)}, \end{aligned} \tag{2.1.21}$$

where $d_2 = \nu \min\{1, \lambda_1\}$.

Proof. By letting $z := \tilde{v} - v$ and $W := \tilde{U} - U$, we see that $[z, W]$ is a solution of the system

$$\begin{cases} \partial_t z = \tilde{U}z + Wv + \nu\partial_x^2 z - 2(\tilde{v}\partial_x z + z\partial_x v) - \mu\sum_{k=1}^{N}(z, w_k)w_k, & (2.1.22) \\ W'(t) = -\nu W(t) - \int_0^1 (\tilde{v}^2(x,t) - v^2(x,t))dx, & (2.1.23) \\ z(0,t) = z(1,t) = 0, \quad W(0) = \tilde{U}_0 - U_0, \quad z(x,0) = \tilde{v}_0 - v_0. & (2.1.24) \end{cases}$$

In order to find *a priori* estimates, we multiply equation (2.1.22) by z in $L^2(0,1)$ and (2.1.23) with W. Then we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|z(t)\|^2 + \nu\|\partial_x z(t)\|^2 &= W(t)(v(t), z(t)) + \tilde{U}(t)\|z(t)\|^2 \\ &\quad - \mu\sum_{k=1}^{N}(z, w_k)^2 + (z^2(t), \partial_x\tilde{v}(t) - 2\partial_x v(t)), \end{aligned} \tag{2.1.25}$$

and

$$\frac{1}{2}\frac{d}{dt}|W(t)|^2 + \nu|W(t)|^2 = -W(t)(z, \tilde{v} + v). \tag{2.1.26}$$

Adding these equations (2.1.25) and (2.1.26), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left[\|z(t)\|^2 + |W(t)|^2\right] + \nu\|\partial_x z(t)\|^2 + \nu|W(t)|^2 &= \tilde{U}(t)\|z(t)\|^2 \\ - W(t)(z, \tilde{v}) - \mu\sum_{k=1}^{N}(z, w_k)^2 + (z^2(t), \partial_x\tilde{v}(t) - 2\partial_x v(t)). \end{aligned} \tag{2.1.27}$$

Thanks to Young's inequality (1.4.6), we can bound the second term on the right-hand side of equation (2.1.27) as follows:

$$|W(t)(z,\tilde{v})| \leq \frac{\nu}{2}|W(t)|^2 + \frac{1}{2\nu}\|\tilde{v}(t)\|^2\|z(t)\|^2. \tag{2.1.28}$$

By using the Gagliardo-Nirenberg inequality (1.4.2) and Young's inequality (1.4.6), we have the following estimate for the last term on the right hand side of (2.1.27):

$$\begin{aligned} |(z^2(t), \partial_x\tilde{v}(t) - 2\partial_x v(t))| &\leq \|z(t)\|_{L^4}^2 \left(\|\partial_x\tilde{v}(t)\| + 2\|\partial_x v(t)\|\right) \\ &\leq \frac{\nu}{4}\|\partial_x z(t)\|^2 + \frac{3}{4}\nu^{-\frac{1}{3}}\beta^{8/3}\left(\|\partial_x\tilde{v}(t)\| + 2\|\partial_x v(t)\|\right)^{\frac{4}{3}}\|z(t)\|^2. \end{aligned} \tag{2.1.29}$$

Employing the estimates (2.1.28) and (2.1.29) in (2.1.27), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|z(t)\|^2 + |W(t)|^2\right] + \frac{\nu}{2}|W(t)|^2 + \frac{3\nu}{4}\|\partial_x z(t)\|^2 \leq -\mu\sum_{k=1}^{N}|(z,w_k)|^2 \\ &\left(|\tilde{U}(t)| + \frac{1}{2\nu}\|\tilde{v}(t)\|^2 + \frac{3}{4}\nu^{-\frac{1}{3}}\beta^{8/3}\left(\|\partial_x\tilde{v}(t)\| + 2\|\partial_x v(t)\|\right)^{\frac{4}{3}}\right)\|z(t)\|^2. \end{aligned} \tag{2.1.30}$$

Thanks to the uniform estimates (2.1.10), (2.1.16) and (2.1.18), we infer from (2.1.30) the inequality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|z(t)\|^2 + |W(t)|^2\right] + \frac{\nu}{2}|W(t)|^2 + \frac{3\nu}{4}\|\partial_x z(t)\|^2 \\ &\qquad\leq M_5\|z(t)\|^2 - \mu\sum_{k=1}^{N}|(z,w_k)|^2, \quad \forall t \geq T_5, \end{aligned}$$

where
$M_5 := \sqrt{M_3} + \frac{1}{2\nu}M_3^2 + \frac{3}{4}\nu^{-\frac{1}{3}}\beta^{8/3}(\sqrt{M_4} + 2\sqrt{M_2})^{4/3}$,
and $T_5 := \max\{T_1, T_2, T_3, T_4\}$. Assume now that μ and N are large enough, in particular they satisfy

$$M_5 \leq \mu, \quad \lambda_{N+1}^{-1}M_5 \leq \frac{\nu}{4}.$$

By using these assumptions, inequality (1.4.8) and the Poincaré-Friedrichs inequality (1.4.7), we get

$$\frac{d}{dt}\left[\|z(t)\|^2 + |W(t)|^2\right] + \nu\lambda_1\|z(t)\|^2 + \nu|W(t)|^2 \leq 0, \quad \forall t \geq t_0 \geq T_5. \tag{2.1.31}$$

Finally, by integrating the last inequality we deduce

$$\|z(t)\|^2 + |W(t)|^2 \leq (\|z(t_0)\|^2 + |W(t_0)|^2)e^{-d_2(t-t_0)}, \quad \forall t \geq t_0 \geq T_5, \tag{2.1.32}$$

where $d_2 = \nu \min\{1, \lambda_1\}$. Hence, we proved the theorem.

□

Global Stabilization in H1-norm

In this part, we prove the H^1-stabilization of the solution of (2.1.11)-(2.1.13).

Theorem 2.1.5. *Assume that μ and N are large enough such that*

$$\mu \geq M_6, \text{ and } \lambda_{N+1}^{-1} M_6 \leq \frac{\nu}{4}, \tag{2.1.33}$$

where

$$M_6 := \frac{4c_0^2}{\nu} M_4 + \frac{1}{\nu}\left(4c_0^2 + \frac{1}{2}\right) M_2 + \sqrt{M_3}. \tag{2.1.34}$$

The constants M_2, M_3 and M_4 are bounds in the uniform estimates (2.1.10), (2.1.17) *and* (2.1.18), *respectively and $c_0 > 0$ is a constant from the Sobolev inequality* (1.4.9). *Then there exists a positive number t_1 such that for all $t \geq t_1$ the following inequality holds true*

$$\begin{aligned} &\frac{1}{2}\|\partial_x \tilde{v}(t) - \partial_x v(t)\|^2 + \|\tilde{v}(t) - v(t)\|^2 + |\tilde{U}(t) - U(t)|^2 \\ &\quad \leq \left[\frac{1}{2}\|\partial_x \tilde{v}(t_1) - \partial_x v(t_1)\|^2 + \|\tilde{v}(t_1) - v(t_1)\|^2 + |\tilde{U}(t_1) - U(t_1)|^2\right] e^{-d_3(t-t_1)}, \end{aligned}$$

where $d_3 = \frac{\nu}{2}\min\left\{1, \frac{\lambda_1}{2}\right\}$.

Proof. We multiply (2.1.22) by $-\partial_x^2 z$ in $L^2(0,1)$ to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\partial_x z(t)\|^2 + \nu\|\partial_x^2 z(t)\|^2 &= \tilde{U}(t)\|\partial_x z(t)\|^2 + W(t)(v, -\partial_x^2 z) \\ &\quad + 2(\tilde{v}\partial_x z + z\partial_x v, \partial_x^2 z) - \mu\sum_{k=1}^{N}(z, w_k)(w_k, -\partial_x^2 z). \end{aligned} \tag{2.1.35}$$

Employing Young's inequality (1.4.6), the second term on the right-hand side of (2.1.35) can be estimated as

$$\begin{aligned}|W(t)(v,-\partial_x^2 z)| &\leq |W(t)||(\partial_x v,\partial_x z)|\\ &\leq \frac{\nu}{2}|W(t)|^2+\frac{1}{2\nu}\|\partial_x v(t)\|^2\|\partial_x z(t)\|^2. \end{aligned} \tag{2.1.36}$$

Thanks to the Sobolev inequality (1.4.9), we obtain the following estimate for the third term on the right-hand side of the equation (2.1.35):

$$\begin{aligned}2|(\tilde{v}\partial_x z+z\partial_x v,\partial_x^2 z)| &\leq 2c_0\|\tilde{v}(t)\|_{L^\infty(0,1)}\|\partial_x z(t)\|\|\partial_x^2 z(t)\|\\ &+2c_0\|z(t)\|_{L^\infty(0,1)}\|\partial_x v(t)\|\|\partial_x^2 z(t)\|\\ &\leq \frac{\nu}{2}\|\partial_x^2 z(t)\|^2+\frac{4c_0^2}{\nu}\left(\|\partial_x\tilde{v}(t)\|^2+\|v(t)\|^2\right)\|\partial_x z(t)\|^2. \end{aligned} \tag{2.1.37}$$

We rewrite the last term of (2.1.35) as follows:

$$-\mu\sum_{k=1}^{N}(z,w_k)(w_k,-\partial_x^2 z)=-\mu\sum_{k=1}^{N}\lambda_k(z,w_k)^2. \tag{2.1.38}$$

Now, by employing the estimates in (2.1.36), (2.1.37) and (2.1.38) in (2.1.35), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_x z(t)\|^2+\frac{\nu}{2}\|\partial_x^2 z(t)\|^2 &\leq \frac{\nu}{2}|W(t)|^2\\ &+\left(\frac{4c_0^2}{\nu}\left(\|\partial_x\tilde{v}(t)\|^2+\|v(t)\|^2\right)+\frac{1}{2\nu}\|\partial_x v(t)\|^2+|\tilde{U}(t)|\right)\|\partial_x z(t)\|^2\\ &-\mu\sum_{k=1}^{N}\lambda_k(z,w_k)^2, \end{aligned} \tag{2.1.39}$$

where $c_0>0$ is the constant in the Sobolev inequality (1.4.9). By employing the uniform estimates (2.1.10), (2.1.17) and (2.1.18), there must exists a $T_5:=\max\{T_1,T_2,T_3,T_4\}$ such that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_x z(t)\|^2+\frac{\nu}{2}\|\partial_x^2 z(t)\|^2 &\leq \frac{\nu}{2}|W(t)|^2\\ &+M_6\|\partial_x z(t)\|^2-\mu\sum_{k=1}^{N}\lambda_k(z,w_k)^2,\quad \forall t\geq T_5, \end{aligned} \tag{2.1.40}$$

where $M_6 := \frac{4c_0^2}{\nu} M_4 + \frac{1}{\nu}\left(4c_0^2 + \frac{1}{2}\right) M_2 + \sqrt{M_3}$.

We add the inequalities (2.1.40) and (2.1.31) and get

$$\frac{d}{dt}\left[\frac{1}{2}\|\partial_x z(t)\|^2 + \|z(t)\|^2 + |W(t)|^2\right] + \frac{\nu}{2}\|\partial_x^2 z(t)\|^2 + \lambda_1 \nu \|z(t)\|^2 + \frac{\nu}{2}|W(t)|^2$$
$$\leq M_6 \|\partial_x z(t)\|^2 - \mu \sum_{k=1}^{N} \lambda_k (z, w_k)^2, \quad \forall t \geq T_5, \quad (2.1.41)$$

It follows from the Poincaré-Friedrichs type inequality (1.4.8) that

$$\begin{aligned} \|\partial_x z(t)\|^2 &= \sum_{k=1}^{N} \lambda_k (z, w_k)^2 + \sum_{k=N+1}^{\infty} \lambda_k (z, w_k)^2 \\ &\leq \sum_{k=1}^{N} \lambda_k (z, w_k)^2 + \lambda_{N+1}^{-1} \|\partial_x^2 z(t)\|^2. \end{aligned} \quad (2.1.42)$$

Thus (2.1.41) implies

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\|\partial_x z(t)\|^2 + \|z(t)\|^2 + |W(t)|^2\right] \\ &+ \left(\frac{\nu}{2} - M_6 \lambda_{N+1}^{-1}\right) \|\partial_x^2 z(t)\|^2 + \lambda_1 \nu \|z(t)\|^2 + \frac{\nu}{2}|W(t)|^2 \\ &\leq -(\mu - M_6) \sum_{k=1}^{N} \lambda_k (z, w_k)^2, \quad \forall t \geq T_5. \end{aligned} \quad (2.1.43)$$

We assume that μ and N are large enough, in particular they satisfy

$$\mu \geq M_6, \quad \lambda_{N+1}^{-1} M_6 \leq \frac{\nu}{4}.$$

Then from (2.1.43), we have that

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\|\partial_x z(t)\|^2 + \|z(t)\|^2 + |W(t)|^2\right] \\ &+ \frac{\nu}{4}\|\partial_x^2 z(t)\|^2 + \lambda_1 \nu \|z(t)\|^2 + \frac{\nu}{2}|W(t)|^2 \leq 0, \quad \forall t \geq T_5. \end{aligned}$$

Employing the Poincaré-Friedrichs inequality (1.4.7) in the last inequality, we deduce

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\|\partial_x z(t)\|^2 + \|z(t)\|^2 + |W(t)|^2\right] \\ &+ d_3 \left[\frac{1}{2}\|\partial_x z(t)\|^2 + \|z(t)\|^2 + |W(t)|^2\right] \leq 0, \quad \forall t \geq t_1 \geq T_5, \end{aligned}$$

where $d_3 = \frac{\nu}{2}\min\left\{1, \frac{\lambda_1}{2}\right\}$. This in turn gives rise to

$$\frac{1}{2}\|\partial_x z(t)\|^2 + \|z(t)\|^2 + |W(t)|^2 \\ \leq \left[\frac{1}{2}\|\partial_x z(t_1)\|^2 + \|z(t_1)\|^2 + |W(t_1)|^2\right] e^{-d_3(t-t_1)}, \quad \forall t \geq t_1 \geq T_5,$$

hence proves the theorem. □

2.1.2 General Interpolant Operator

In this part, we will show that our stabilization results for the feedback control problem is valid also with the general interpolant operator. We propose the following feedback control problem for original Burgers' equation.

$$\begin{cases} \partial_t\tilde{v} = \tilde{U}\tilde{v} + \nu\partial_x^2\tilde{v} - 2\tilde{v}\partial_x\tilde{v} - \mu(I_h(\tilde{v}) - I_h(v)), & (2.1.44) \\ \tilde{U}'(t) = R - \nu\tilde{U}(t) - \|\tilde{v}(t)\|^2, & (2.1.45) \\ \tilde{v}(0,t) = \tilde{v}(1,t) = 0, \quad \tilde{U}(0) = \tilde{U}_0, \quad \tilde{v}(x,0) = \tilde{v}_0(x), & (2.1.46) \end{cases}$$

where $h = \frac{1}{N}$ and $I_h : H^1(0,1) \to L^2(0,1)$ is a general linear map which approximates the inclusion map $I : H^1(0,1) \to L^2(0,1)$ with error on the order of h. Here, I satisfies the inequality

$$\|u - I_h(u)\| \leq ch\|\partial_x u\| \leq ch\|u\|_{H^1(0,1)}, \quad \forall u \in H^1(0,1), \tag{2.1.47}$$

for some positive constant $c > 0$. We refer to the paper of Azouani and Titi [1] for the proof of this inequality (2.1.47).

Theorem 2.1.6. *Assume that μ and N satisfy the following inequalities*

$$\mu > M_7, \quad N > \sqrt{\frac{2\mu c^2}{\nu}}, \tag{2.1.48}$$

where

$$M_7 := 2\sqrt{M_1} + \frac{1}{\nu}M_1 + \frac{3}{2}\beta^{\frac{4}{3}}\nu^{-\frac{1}{3}}M_2^{\frac{2}{3}}, \tag{2.1.49}$$

M_1, M_2 are constants satisfying (2.1.5), (2.1.10) *and $c > 0$ is the constant from the inequality* (2.1.47). *Suppose also that $I_h : H^1(0,1) \to L^2(0,1)$ is a linear map*

satisfying (2.1.47). *Then there exists* $t_0 > 0$ *such that*

$$\|\tilde{v}(t) - v(t)\|^2 + |\tilde{U}(t) - U(t)|^2 \\ \leq e^{-d_4(t-t_0)}[\|\tilde{v}(t_0) - v(t_0)\|^2 + |\tilde{U}(t_0) - U(t_0)|^2], \quad \forall t \geq t_0,$$

where $d_4 := \nu \min\{1, \lambda_1\}$.

Proof. As in the previous sections, we let $z := \tilde{v} - v$ and $W := \tilde{U} - U$, and see that $[z, W]$ is a solution of the following problem

$$\begin{cases} \partial_t z = (W+U)z + Wv + \nu\partial_x^2 z - 2(z\partial_x z + v\partial_x z + z\partial_x v) - \mu I_h(z) & (2.1.50) \\ W'(t) = -\nu W(t) - \|z(t)\|^2 - 2(z, v), & (2.1.51) \\ z(0,t) = z(1,t) = 0, \quad W(0) = \tilde{U}_0 - U_0, \quad z(x,0) = \tilde{v}_0 - v_0. & (2.1.52) \end{cases}$$

Here, we remark that we have expressed some terms in (2.1.50) and (2.1.51) as follows:

$$\tilde{U}\tilde{v} - Uv = \tilde{U}z + Wv = (W+U)z + Wv,$$
$$2\left(\tilde{v}\partial_x\tilde{v} - v\partial_x v\right) = 2(\tilde{v}\partial_x z + z\partial_x v) = 2\left[(z+v)\partial_x z + z\partial_x v\right],$$

and

$$\|\tilde{v}(t)\|^2 - \|v(t)\|^2 = \int_0^1 \tilde{v}^2(x,t) - v^2(x,t)dx = \int_0^1 z(x,t)(\tilde{v}(x,t) + v(x,t))dx \\ = \int_0^1 z(x,t)\left[z(x,t) + 2v(x,t)\right]dx = \|z(t)\|^2 + 2(z,v).$$

We multiply (2.1.50) by z in $L^2(0,1)$ and (2.1.51) by W and obtain

$$\frac{1}{2}\|z(t)\|^2 + \nu\|\partial_x z(t)\|^2 = W(t)\|z(t)\|^2 + U(t)\|z(t)\|^2 + W(t)(v,z) \\ - 2(z\partial_x z, z) - 2\left[(v\partial_x z + z\partial_x v, z)\right] - \mu(I_h(z), z), \quad (2.1.53)$$

as well as

$$\frac{1}{2}|W(t)|^2 + \nu|W(t)|^2 = -W(t)\|z(t)\|^2 - 2W(t)(z,v). \quad (2.1.54)$$

Here, we note that $2(z\partial_x z, z) = 0$. Furthermore, we get the following equality by using integration by parts $-2\left[(v\partial_x z + z\partial_x v, z)\right] = -(\partial_x v, z^2)$.

We add equations (2.1.53) and (2.1.54) and obtain that

$$\frac{1}{2}\frac{d}{dt}\left[\|z(t)\|^2 + |W(t)|^2\right] + \nu\|\partial_x z(t)\|^2 + \nu|W(t)|^2$$
$$= U(t)\|z(t)\|^2 - W(t)(z,v) - (\partial_x v, z^2) - \mu(I_h(z), z). \quad (2.1.55)$$

By employing Young's inequality (1.4.6), we estimate the second term on the right-hand side as

$$|W(t)(z,v)| \leq |W(t)|\|v(t)\|\|z(t)\| \leq \frac{\nu}{2}|W(t)|^2 + \frac{1}{2\nu}\|v(t)\|^2\|z(t)\|^2. \quad (2.1.56)$$

Thanks to Gagliardo-Nirenberg inequality (1.4.2) and Young's inequality (1.4.6), we estimate the third term on the right-hand side as

$$|(\partial_x v, z^2)| \leq \|\partial_x v(t)\|\|z(t)\|^2_{L^4(0,1)} \leq \beta\|\partial_x v(t)\|\|z(t)\|^{\frac{3}{2}}\|\partial_x z(t)\|^{\frac{1}{2}}$$
$$\leq \frac{\nu}{4}\|\partial_x z(t)\|^2 + \frac{3}{4}\beta^{\frac{4}{3}}\nu^{-\frac{1}{3}}\|\partial_x v(t)\|^{\frac{4}{3}}\|z(t)\|^2. \quad (2.1.57)$$

Employing the property of interpolant operator I (2.1.47) and Young's inequality (1.4.6), we can rewrite the last term on the right-hand side of (2.1.55) as

$$-\mu(I_h(z), z) = -\mu(I_h(z) - z, z) - \mu\|z(t)\|^2 \leq \mu\|z - I_h(z)\|\|z(t)\| - \mu\|z(t)\|^2$$
$$\leq \mu ch\|\partial_x z(t)\|\|z(t)\| - \mu\|z(t)\|^2$$
$$\leq \frac{\nu}{4}\|\partial_x z(t)\|^2 + \frac{\mu^2c^2h^2}{\nu}\|z(t)\|^2 - \mu\|z(t)\|^2. \quad (2.1.58)$$

By plugging (2.1.56), (2.1.57) and (2.1.58) in (2.1.55), we obtain

$$\frac{d}{dt}\left[\|z(t)\|^2 + |W(t)|^2\right] + \nu\|\partial_x z(t)\|^2 + \nu|W(t)|^2$$
$$\leq \left[2|U(t)| + \frac{1}{\nu}\|v(t)\|^2 + \frac{3}{2}\beta^{\frac{4}{3}}\nu^{-\frac{1}{3}}\|\partial_x v(t)\|^{\frac{4}{3}} - 2\mu\right]\|z(t)\|^2 + \frac{2\mu^2c^2h^2}{\nu}\|z(t)\|^2.$$

Due to the uniform estimates in (2.1.5) and (2.1.10), we deduce that there exists $t_0 \geq T_2 \geq T_1$ such that the following inequality holds:

$$\frac{d}{dt}\left[\|z(t)\|^2 + |W(t)|^2\right] + \nu\|\partial_x z(t)\|^2 + \nu|W(t)|^2$$
$$\leq -(2\mu - M_7)\|z(t)\|^2 + \frac{2\mu^2c^2h^2}{\nu}\|z(t)\|^2, \quad \forall t_0 \geq T_2, \quad (2.1.59)$$

where $M_7 := 2\sqrt{M_1} + \frac{1}{\nu}M_1 + \frac{3}{2}\beta^{\frac{4}{3}}\nu^{-\frac{1}{3}}M_2^{\frac{2}{3}}$. Now, we assume that μ and N are large enough, in particular

$$\frac{2\mu^2c^2h^2}{\nu} < \mu, \quad \text{i.e. } N > \sqrt{\frac{2\mu c^2}{\nu}}, \tag{2.1.60}$$

since $h = \frac{1}{N}$. Then we have from the previous inequality that

$$\frac{d}{dt}\left[\|z(t)\|^2 + |W(t)|^2\right] + \nu\|\partial_x z(t)\|^2 + \nu|W(t)|^2 + (\mu - M_7)\|z(t)\|^2 \leq 0, \quad \forall t_0 \geq T_2.$$

We assume also that $\mu > M_7$ and use the Poincaré-Friedrichs inequality (1.4.7) to obtain that

$$\frac{d}{dt}\left[\|z(t)\|^2 + |W(t)|^2\right] + d_4\left[\|\partial_x z(t)\|^2 + |W(t)|^2\right] \leq 0, \quad \forall t_0 \geq T_2,$$

where $d_4 := \nu\min\{1, \lambda_1\}$. Hence,

$$\begin{aligned}&\|\tilde{v}(t) - v(t)\|^2 + |\tilde{U}(t) - U(t)|^2 \\ &\qquad \leq e^{-d_4(t-t_0)}[\|\tilde{v}(t_0) - v(t_0)\|^2 + |\tilde{U}(t_0) - U(t_0)|^2], \quad \forall t \geq t_0 \geq T_2.\end{aligned}$$

□

2.1.3 Finitely Many Volume Elements

Here, we consider another type of feedback stabilization problem for the original Burgers' equations (2.1.1)-(2.1.3). In this feedback control system, the control operator is based on the finite volume elements

$$\begin{cases} \partial_t\tilde{v} = \tilde{U}\tilde{v} + \nu\partial_x^2\tilde{v} - 2\tilde{v}\partial_x\tilde{v} - \mu\displaystyle\sum_{k=1}^{N}(\bar{\tilde{v}}_k - \bar{v}_k)\chi_{J_k}(x), & (2.1.61)\\ \tilde{U}'(t) = R - \nu\tilde{U}(t) - \|\tilde{v}(t)\|^2, & (2.1.62)\\ \tilde{v}(0,t) = \tilde{v}(1,t) = 0, \quad \tilde{U}(0) = \tilde{U}_0, \quad \tilde{v}(x,0) = \tilde{v}_0(x), & (2.1.63)\end{cases}$$

where $J_k = [(k-1)\frac{1}{N}, k\frac{1}{N})$, for $k = 1, 2, ..., N-1$, $J_N = [\frac{N-1}{N}, 1]$, χ_{J_k} is the characteristic function of the interval J_k and

$$\bar{\tilde{v}}_k := \frac{1}{|J_k|}\int_{J_k}\tilde{v}(x,t)dx, \quad \bar{v}_k := \frac{1}{|J_k|}\int_{J_k}v(x,t)dx.$$

We state our main result in the following theorem

Theorem 2.1.7. *Assume that the parameters $\mu > 0$ and integer $N > 0$ satisfy*

$$\mu > M_7, \quad N > \sqrt{\frac{2\mu c^2}{\nu}},$$

where M_7 is as defined in (2.1.49) *and $c > 0$ is constant in* (1.4.14)*. There there exists $t_0 > 0$ such that*

$$\|\tilde{v}(t) - v(t)\|^2 + |\tilde{U}(t) - U(t)|^2 \\ \leq e^{-d_4(t-t_0)} \left[\|\tilde{v}(t_0) - v(t_0)\|^2 + |\tilde{U}(t_0) - U(t_0)|^2\right], \quad \forall t \geq t_0,$$

holds, where $d_4 := \nu \min\{1, \lambda_1\}$.

Proof. We apply the similar procedures as in Section 2.1.2. Let $z = \tilde{v} - v$ and $W = \tilde{U} - U$ and see that (z, W) is a solution of the following initial-boundary value problem

$$\begin{cases} \partial_t z = \tilde{U} z + W v + \nu \partial_x^2 z - 2(\tilde{v}\partial_x z + z\partial_x v) - \mu \sum_{k=1}^{N} \bar{z}_k \chi_{J_k}(x), & (2.1.64) \\ W' = -\nu W - \|\tilde{v}(t)\|^2 + \|v(t)\|^2, & (2.1.65) \\ z(0,t) = z(1,t) = 0, \quad W(0) = \tilde{U}_0 - U_0, \quad z(x,0) = \tilde{v}_0(x) - v_0(x). & (2.1.66) \end{cases}$$

We multiply (2.1.64) by z in $L^2(0,1)$ and (2.1.65) by W and obtain

$$\frac{1}{2}\frac{d}{dt}\left[\|z(t)\|^2 + |W(t)|^2\right] + \nu\|\partial_x z(t)\|^2 + \nu|W(t)|^2 \\ = U(t)\|z(t)\|^2 - W(t)(z, v) - (\partial_x v, z^2) - \mu \int_0^1 \sum_{k=1}^{N} \bar{z}_k \chi_{J_k}(x) z dx. \quad (2.1.67)$$

An upper bound for the last term on the right-hand side of (2.1.67) by using (1.4.14) and Young's inequality (1.4.6) is given by

$$-\mu \int_0^1 \sum_{k=1}^{N} \bar{z}_k \chi_{J_k}(x) z dx = -\mu \int_0^1 \left(\sum_{k=1}^{N} \bar{z}_k \chi_{J_k}(x) - z\right) z dx - \mu\|z(t)\|^2 \\ \leq \mu \left\| z - \sum_{k=1}^{N} \bar{z}_k \chi_{J_k}(x) \right\| \|z(t)\| - \mu\|z(t)\|^2 \leq \mu c h \|\partial_x z(t)\| \|z(t)\| - \mu\|z(t)\|^2 \\ \leq \frac{\nu}{4}\|\partial_x z(t)\|^2 \frac{\mu^2 c^2 h^2}{\nu} - \mu\|z(t)\|^2. \quad (2.1.68)$$

Due to (2.1.56), (2.1.57) and (2.1.68), we obtain the following inequality from (2.1.67):

$$\frac{d}{dt}\left[\|z(t)\|^2 + |W(t)|^2\right] + \nu\|\partial_x z(t)\|^2 + \nu|W(t)|^2$$
$$\leq \left[2|U(t)| + \frac{1}{\nu}\|v(t)\|^2 + \frac{3}{2}\beta^{\frac{4}{3}}\nu^{-\frac{1}{3}}\|\partial_x v(t)\|^{\frac{4}{3}} - 2\mu\right]\|z(t)\|^2 + \frac{2\mu^2c^2h^2}{\nu}\|z(t)\|^2.$$

The rest of the proof is exactly the same as the proof of Theorem 2.1.7. Utilizing the uniform estimates (2.1.5) and (2.1.10), and assumption (2.1.60) and $\mu > M_7$, we obtain that

$$\frac{d}{dt}\left[\|z(t)\|^2 + |W(t)|^2\right] + d_4\left[\|\partial_x z(t)\|^2 + |W(t)|^2\right] \leq 0, \quad \forall t_0 \geq T_2,$$

where $d_4 := \nu\min\{1, \lambda_1\}$. This in turn leads us to

$$\|\tilde{v}(t) - v(t)\|^2 + |\tilde{U}(t) - U(t)|^2$$
$$\leq e^{-d_4(t-t_0)}[\|\tilde{v}(t_0) - v(t_0)\|^2 + |\tilde{U}(t_0) - U(t_0)|^2], \quad \forall t \geq t_0 \geq T_2.$$

□

2.2 Burgers' Equation with Nonlocal Nonlinearity

In this section we study the feedback stabilization problem of Burgers' equation with nonlocal nonlinearity (1.0.5). We consider the following initial-boundary value problem

$$\begin{cases} \partial_t v - \nu\partial_x^2 v + 2v\partial_x v - Rv + kv\int_0^1 v^2 dx = h, & (2.2.1)\\ v(0,t) = v(1,t) = 0, \quad v(x,0) = v_0(x), & (2.2.2)\end{cases}$$

where $(x,t) \in [0,1] \times [0,\infty)$, $h \in L^2(\mathbb{R}^+; L^2(0,1))$ is a given source term, $\nu > 0$, $k > 0$ and $R > 0$ are given numbers. We analyze the feedback stabilization problem for (2.2.1)-(2.2.2) with the feedback control operator based on *finitely many Fourier modes, finite volume elements* and *finitely many nodal values.* First, we state the definition of a weak solution of this problem.

Definition 2.2.1. A function v is called a weak solution of the problem (2.2.1)-(2.2.2) if $v \in L^2(0,T;H_0^1(0,1)) \cap L^\infty(0,T;L^2(0,1))$ and $\partial_t v \in L^2(0,T;L^2(0,1))$ satisfy the following equality $\forall \eta \in H_0^1(0,1)$

$$(\partial_t v, \eta) + \nu(\partial_x v, \eta') + 2(v\partial_x v, \eta) - R(v,\eta) + k\|v(t)\|^2(v,\eta) = (h,\eta).$$

2.2.1 Finitely Many Fourier Modes

We propose the following feedback stabilization problem

$$\begin{cases} \partial_t u - \nu\partial_x^2 u + 2u\partial_x u - Ru + k\|u\|^2 u = h - \mu \displaystyle\sum_{k=1}^{N}(u-v,w_k)w_k, & (2.2.3) \\ u(0,t) = u(1,t) = 0, \quad u(x,0) = u_0(x), & (2.2.4) \end{cases}$$

where $(x,t) \in [0,1] \times [0,\infty)$. Before we show the L^2 stabilization, we prove the following lemma:

Lemma 2.2.2. *There exist a positive number $T^* > 0$ such that the following inequalities hold:*

$$\|v(t)\|^2 \leq H_1, \quad \|\partial_x v(t)\|^2 \leq H_2, \quad \|u(t)\|^2 \leq H_3, \quad \|\partial_x u(t)\|^2 \leq H_4, \quad \forall t \geq T^*,$$

where $H_1, H_2, H_3, H_4 > 0$ are constants.

Proof. We multiply (2.2.1) by v in $L^2(0,1)$ to get first energy equation

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 + \nu\|\partial_x v(t)\|^2 - R\|v(t)\|^2 + k\|v(t)\|^4 = (h,v). \tag{2.2.5}$$

Utilizing the inequalities

$$|(h,v)| \leq \frac{\nu}{2}\|\partial_x v\|^2 + \frac{1}{2\lambda_1\nu}\|h\|^2, \quad R\|v\|^2 \leq k\|v\|^4 + \frac{1}{4k}R^2,$$

we obtain from (2.2.5) the following inequality

$$\frac{d}{dt}\|v(t)\|^2 + \nu\lambda_1\|v(t)\|^2 \leq \frac{1}{2k}R^2 + \frac{1}{\lambda_1\nu}\|h(t)\|^2.$$

From this inequality we get the estimate

$$\|v(t)\|^2 \leq \|v_0\|^2 e^{-\nu\lambda_1 t} + \frac{R^2}{2\lambda_1\nu k} + \frac{1}{\lambda_1\nu}\int_0^t \|h(\tau)\|^2 d\tau$$

which implies that

$$\|v(t)\|^2 \le H_1, \quad \forall t \ge T_1, \quad H_0 := \int_0^\infty \|h(t)\|^2 dt, \tag{2.2.6}$$

where H_1 depends only on R, k, ν, λ_1 and H_0. Next, we multiply (2.2.1) by $-\partial_x^2 v$ in $L^2(0,1)$ and obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_x v\|^2 + \nu\|\partial_x^2 v\|^2 + \int_0^1 (\partial_x v)^3 dx - R\|\partial_x v\|^2 + k\|v\|^2\|\partial_x v\|^2 = -(h, \partial_x^2 v). \tag{2.2.7}$$

Here, we again use Young's inequality (1.4.6) to obtain that

$$\begin{aligned} R\|\partial_x v\|^2 \le R\|v\|\|\partial_x^2 v\| \le \frac{\nu}{8}\|\partial_x^2 v\|^2 + \frac{2R^2}{\nu}\|v\|^2, \\ |(h, \partial_x^2 v)| \le \frac{\nu}{8}\|\partial_x^2 v\|^2 + \frac{2}{\nu}\|h\|^2. \end{aligned} \tag{2.2.8}$$

Employing the Gagliardo-Nirenberg inequality (1.4.2) and the Young's inequality (1.4.6) with $\varepsilon = \frac{2\nu}{7}$, $p = \frac{8}{7}$ we get

$$\left|\int_0^1 (\partial_x v(t))^3 dx\right| \le \frac{\nu}{4}\|\partial_x^2 v(t)\|^2 + \beta^{24} 7^7 2^{-10} \nu^{-7} \|v(t)\|^{10}. \tag{2.2.9}$$

Utilizing (2.2.9), two inequalities in (2.2.8) and the Poincaré-Friedrichs inequality (1.4.7), we get from (2.2.7)

$$\frac{d}{dt}\|\partial_x v(t)\|^2 + \nu\lambda_1\|\partial_x v(t)\|^2 \le \frac{4}{\nu}\|h(t)\|^2 + \frac{4R^2}{\nu}\|v(t)\|^2 + \beta^{24} 7^7 2^{-9} \nu^{-7}\|v(t)\|^{10},$$

where $\beta > 0$ is a constant of Gagliardo-Nirenberg inequality (1.4.2). Due to (2.2.6) by integrating this inequality, we get the estimate

$$\|\partial_x v(t)\|^2 \le H_2, \quad \forall t \ge T_2 \ge T_1, \tag{2.2.10}$$

where H_2 depends on R, ν, β, H_0 and H_1.

Let us obtain estimates for solutions of the controlled system (2.2.3)-(2.2.4). Multiplication of (2.2.3) by u in $L^2(0,1)$ gives:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu\|\partial_x u(t)\|^2 - R\|u(t)\|^2 + k\|u(t)\|^4 \\ &\qquad = -\mu\sum_{k=1}^N (u, w_k)^2 + \mu\sum_{k=1}^N (v, w_k)(u, w_k) + (h, u). \end{aligned} \tag{2.2.11}$$

By using the Young's inequality (1.4.6) we obtain the following inequalities

$$|(h,u)| \leq \|u\|^2 + \frac{1}{4}\|h\|^2 \leq \frac{k}{2}\|u\|^4 + \frac{1}{2k} + \frac{1}{4}\|h\|^2, \tag{2.2.12}$$

$$R\|u\|^2 \leq \frac{k}{2}\|u\|^4 + \frac{1}{2k}R^2, \tag{2.2.13}$$

$$\mu\sum_{k=1}^{N}(v,w_k)(u,w_k) \leq \mu\sum_{k=1}^{N}(u,w_k)^2 + \frac{\mu}{4}\sum_{k=1}^{N}(v,w_k)^2$$

$$\leq \mu\sum_{k=1}^{N}(u,w_k)^2 + \frac{\mu}{4}\|v(t)\|^2. \tag{2.2.14}$$

Employing (2.2.12)-(2.2.14) in (2.2.11)

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu\|\partial_x u(t)\|^2 \leq \frac{\mu}{4}\|v(t)\|^2 + \frac{1}{4}\|h(t)\|^2 + \frac{1}{2k}(1+R^2). \tag{2.2.15}$$

By integrating (2.2.15) we deduce from that

$$\|u(t)\|^2 \leq H_3,\ \forall t \geq T_3 > T_2, \tag{2.2.16}$$

where H_3 depends on R, k, ν, μ, H_0 and H_1.

Next, we multiply (2.2.3) by $-\partial_x^2 u$ in $L^2(0,1)$

$$\frac{1}{2}\frac{d}{dt}\|\partial_x u(t)\|^2 + \nu\|\partial_x^2 u(t)\|^2 + \int_0^1(\partial_x u)^3dx - R\|\partial_x u(t)\|^2 + k\|u(t)\|^2\|\partial_x u(t)\|^2$$

$$= -\mu\sum_{k=1}^{N}\lambda_k(u,w_k)^2 + \mu\sum_{k=1}^{N}\lambda_k(v,w_k)(u,w_k) - (h,\partial_x^2 u). \tag{2.2.17}$$

Thanks to the inequalities (2.2.9) and (2.2.8) employed for the term $\partial_x^2 u$ and the inequality

$$\mu\Big|\sum_{k=1}^{N}\lambda_k(v,w_k)(u,w_k)\Big| \leq \mu\sum_{k=1}^{N}\lambda_k(u,w_k)^2 + \frac{\mu}{4}\sum_{k=1}^{N}\lambda_k(v,w_k)^2,$$

(2.2.17) implies

$$\frac{1}{2}\frac{d}{dt}\|\partial_x u\|^2 + \frac{\nu}{2}\|\partial_x^2 u\|^2 \leq \beta^{24}7^7 2^{-10}\nu^{-7}\|u\|^{10} + 4R^2\|u\|^2 + 4\|h\|^2 + \frac{\mu}{4}\|\partial_x v\|^2.$$

Integrating the last inequality we obtain the next bound for solutions of the problem

$$\|\partial_x u(t)\|^2 \leq H_4, \quad \forall t \geq T_4 > T_3. \tag{2.2.18}$$

Hence, there exists a positive number T^* such that $T^* = \max\{T_1, T_2, T_3, T_4\}$ which satisfies the assertion of the lemma. □

Stabilization in L2-norm

We state our main result about the L^2-stabilization of the problem (2.2.1)-(2.2.2) in the following theorem.

Theorem 2.2.3. *Suppose that $\xi > \frac{\lambda_1\nu}{2}$ is an arbitrary number, N and μ are so large that*

$$\frac{\nu}{2} > \lambda_{N+1}^{-1}(\xi - \frac{\lambda_1\nu}{2} + 2A_0 + 2R), \quad \mu > \frac{1}{2}\xi - \frac{\lambda_1\nu}{4} + R + A_0,$$

where A_0 is defined in (2.2.25). *Then each solution of the problem* (2.2.1)-(2.2.2) *under the homogeneous Dirichlet's boundary conditions is approaching the solution of the problem* (2.2.3)-(2.2.4) *with an exponential rate $e^{-\xi t}$ in $L^2(0,1)$ sense.*

Proof. We denote $z = v - u$ and see that z is a solution of the following problem

$$\begin{cases} \partial_t z - \nu\partial_x^2 z + 2z\partial_x u + 2v\partial_x z - Rz + k\|u\|^2 u - k\|v\|^2 v \\ = -\mu\sum_{k=1}^{N}(z, w_k)w_k, & (2.2.19) \\ z(0,t) = z(1,t) = 0, & (2.2.20) \\ z(x,0) = u_0(x) - v_0(x), & (2.2.21) \end{cases}$$

where $(x,t) \in [0,1] \times [0,\infty)$.

Multiplying the equation (2.2.19) by z in $L^2(0,1)$ and using the monotonicity inequality (1.4.10) we get

$$\frac{1}{2}\frac{d}{dt}\|z\|^2 + \nu\|\partial_x z\|^2 + 2(z^2, \partial_x u) - (z^2, \partial_x v) - R\|z\|^2 = -\mu\sum_{k=1}^{N}(z, w_k)^2. \quad (2.2.22)$$

Then we estimate the third and fourth terms on the left hand side of (2.2.22) by using Gagliardo-Nirenberg inequality (1.4.2) and Young's inequality (1.4.6):

$$\begin{aligned} 2|(z^2, \partial_x u)| &\le 2\beta\|z(t)\|_{L^4(0,1)}^2\|\partial_x u(t)\| \\ &\le \frac{\nu}{4}\|\partial_x z\|^2 + \frac{3}{4}(2\beta)^{\frac{4}{3}}\nu^{-\frac{1}{3}}\|z\|^2\|\partial_x u\|^{\frac{4}{3}}. \end{aligned} \quad (2.2.23)$$

and

$$|(z^2,\partial_x v)| \le \|z\|_{L^4}^2\|\partial_x v\| \le \beta^2\|z\|^{\frac{3}{2}}\|\partial_x z\|^{\frac{1}{2}}\|\partial_x v\|$$
$$\le \frac{\nu}{4}\|\partial_x z\|^2 + \frac{3}{4}\beta^{\frac{4}{3}}\nu^{-\frac{1}{3}}\|z\|^2\|\partial_x v\|^{\frac{4}{3}}, \qquad (2.2.24)$$

Thus due to (2.2.10) and (2.2.18) there exists $T_5 \ge T_4$ such that

$$|(z^2,\partial_x v)| + 2|(z^2,\partial_x u)| \le \frac{\nu}{2}\|\partial_x z\|^2 + A_0\|z\|^2, \quad \forall t \ge T_5,$$

where

$$A_0 = \frac{3}{4}(2\beta)^{\frac{4}{3}}\nu^{-\frac{1}{3}}H_4^{2/3} + \frac{3}{4}(\beta)^{\frac{4}{3}}\nu^{-\frac{1}{3}}H_2^{2/3}. \qquad (2.2.25)$$

Employing the last inequality we get from (2.2.22) that

$$\frac{d}{dt}\|z(t)\|^2 + \nu\|\partial_x z(t)\|^2 - 2(A_0+R)\|z(t)\|^2 = -2\mu\sum_{k=1}^{N}(z,w_k)^2. \qquad (2.2.26)$$

Next, we multiply (2.2.26) by $e^{\sigma t}$ with $\sigma = \xi - \frac{\lambda_1\nu}{2}$ and rewrite the obtained relation in the form

$$\frac{d}{dt}\left(e^{\sigma t}\|z(t)\|^2\right) + e^{\sigma t}\nu\|\partial_x z(t)\|^2 - (\sigma + 2A_0 + 2R)e^{\sigma t}\|z(t)\|^2$$
$$= -2\mu e^{\sigma t}\sum_{k=1}^{N}(z,w_k)^2.$$

The last equality we write in the following form

$$\frac{d}{dt}\left(e^{\sigma t}\|z(t)\|^2\right) + e^{\sigma t}\nu\|\partial_x z(t)\|^2 + [2\mu - (\sigma + 2A_0 + 2R)]e^{\sigma t}\sum_{k=1}^{N}(z,w_k)^2$$
$$- (\sigma + 2A_0 + 2R)e^{\sigma t}\sum_{k=N+1}^{\infty}(z,w_k)^2 = 0. \qquad (2.2.27)$$

Since $\sum_{k=N+1}^{\infty}(z,w_k)^2 \le \lambda_{N+1}^{-1}\|\partial_x z(t)\|^2$, we can choose N so large that

$$\frac{\nu}{2} > (\sigma + 2A_0 + 2R)\lambda_{N+1}^{-1} \quad \text{and} \quad \mu \ge \frac{\sigma}{2} + R + A_0,$$

and deduce from (2.2.27) the inequality

$$\frac{d}{dt}\left(e^{\sigma t}\|z(t)\|^2\right) + \frac{\nu}{2}e^{\sigma t}\|\partial_x z(t)\|^2 \le 0, \ \forall t \ge t_0 \ge T_5, \qquad (2.2.28)$$

which implies that $\|z(t)\|^2 \le e^{-\xi(t-t_0)}\|z(t_0)\|^2$. □

Stabilization in H1-norm

In this subsection, we state our result on the $H1$-stabilization of the problem (2.2.1)-(2.2.2).

Theorem 2.2.4. *Assume that $\xi > \frac{\lambda_1 \nu}{2}$ is an arbitrary number and μ and N are so large that*

$$\mu > \frac{1}{2}\sigma + Q, \quad \frac{\nu}{2} \geq \lambda_{N+1}^{-1}(\sigma + 2Q),$$

where

$$Q := \frac{4c_0^2}{\nu}(H_4 + H_3) + k\lambda_1^{-\frac{1}{2}}(\sqrt{H_1} + \sqrt{H_3})\sqrt{H_2} + R. \tag{2.2.29}$$

Then there exists $t_0 > 0$ such that the solution of the problem (2.2.3)-(2.2.4) *approaches the solution of* (2.2.1)-(2.2.2) *with an exponential rate $e^{-\xi t}$ in $H^1(0,1)$ sense. In other words, the following inequality holds true*

$$\|\partial_x \tilde{v}(t) - \partial_x v(t)\|^2 \leq e^{-\xi(t-t_0)}\|\partial_x \tilde{v}(t_0) - \partial_x v(t_0)\|^2, \quad \forall t \geq t_0.$$

Here, the constants H_1, H_2, H_3 and H_4 are bounds for uniform estimates in Lemma 2.2.2 and $c_0 > 0$ is the constant of the Sobolev inequality (1.4.9).

Proof. We multiply the equation (2.2.19) by $-\partial_x^2 z$ and we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_x z(t)\|^2 + \nu\|\partial_x^2 z(t)\|^2 &= 2(z\partial_x u + v\partial_x z, \partial_x^2 z) + R\|\partial_x z(t)\|^2 \\ &\quad + k(\|u(t)\|^2 u - \|v(t)\|^2 v, \partial_x^2 z) - \mu\sum_{k=1}^{N}\lambda_k (z, w_k)^2. \end{aligned} \tag{2.2.30}$$

The third term on the right-hand side of the last equality can be rewritten and estimated by using Poincaré-Friedrichs inequality (1.4.7) as follows:

$$\begin{aligned} k(\|u(t)\|^2 u - \|v(t)\|^2 v, \partial_x^2 z) &= k\|u(t)\|^2 (z, \partial_x^2 z) + k(\|u\|^2 - \|v(t)\|^2)(v, \partial_x^2 z) \\ &= -k\|u(t)\|^2\|\partial_x z(t)\|^2 - k(u+v, z)(\partial_x v, \partial_x z), \\ &\leq k|(u+v, z)||(\partial_x v, \partial_x z)|, \\ &\leq k\lambda_1^{-\frac{1}{2}}\left(\|u(t)\| + \|v(t)\|\right)\|\partial_x v(t)\|\|\partial_x z(t)\|^2. \end{aligned} \tag{2.2.31}$$

Similar to (2.1.37), we have the estimate

$$2|(z\partial_x u + v\partial_x z, \partial_x^2 z)| \tag{2.2.32}$$

$$\leq \frac{\nu}{2}\|\partial_x^2 z(t)\|^2 + \frac{4c_0^2}{\nu}(\|\partial_x u(t)\|^2 + \|\partial_x v(t)\|^2)\|\partial_x z(t)\|^2, \tag{2.2.33}$$

where $c_0 > 0$ is a constant of the Sobolev inequality. Employing the estimates (2.2.31) and (2.2.32) in (2.2.30) obtain that

$$\frac{1}{2}\frac{d}{dt}\|\partial_x z(t)\|^2 + \frac{\nu}{2}\|\partial_x^2 z(t)\|^2 \leq \left(\frac{4c_0^2}{\nu}(\|\partial_x u(t)\|^2 + \|\partial_x v(t)\|^2)\right)\|\partial_x z(t)\|^2$$
$$+ \left(k\lambda_1^{-\frac{1}{2}}(\|u(t)\| + \|v(t)\|)\|\partial_x v(t)\| + R\right)\|\partial_x z(t)\|^2$$
$$- \mu\sum_{k=1}^{N}\lambda_k(z, w_k)^2. \tag{2.2.34}$$

Employing the uniform estimates (2.2.6), (2.2.10), (2.2.16) and (2.2.18) in (2.2.34), we deduce that there exists $T_5 \geq T_4$ such that the following inequality holds for all $t \geq T_5$

$$\frac{1}{2}\frac{d}{dt}\|\partial_x z(t)\|^2 + \frac{\nu}{2}\|\partial_x^2 z(t)\|^2 \leq Q\|\partial_x z(t)\|^2 - \mu\sum_{k=1}^{N}\lambda_k(z, w_k)^2, \tag{2.2.35}$$

where $Q := \frac{4c_0^2}{\nu}(H_4 + H_3) + k\lambda_1^{-\frac{1}{2}}(\sqrt{H_1} + \sqrt{H_3})\sqrt{H_2} + R$. We multiply the last inequality by $2e^{\sigma t}$ with $\sigma := \xi - \frac{\nu\lambda_1}{2} > 0$ and get that

$$\frac{d}{dt}\left(e^{\sigma t}\|\partial_x z(t)\|^2\right) - (\sigma + 2Q)e^{\sigma t}\|\partial_x z(t)\|^2 + \nu e^{\sigma t}\|\partial_x^2 z(t)\|^2$$
$$\leq -2\mu\sum_{k=1}^{N}\lambda_k(z, w_k)^2, \quad \forall t \geq T_5.$$

Thanks to (2.1.42) from the last inequality we obtain that

$$\frac{d}{dt}\left(e^{\sigma t}\|\partial_x z(t)\|^2\right) + (\nu - \lambda_{N+1}^{-1}(\sigma + 2Q))e^{\sigma t}\|\partial_x^2 z(t)\|^2$$
$$\leq -(2\mu - \sigma - 2Q)\sum_{k=1}^{N}\lambda_k(z, w_k)^2, \quad \forall t \geq T_5.$$

We assume that μ and N are so large that

$$\mu > \frac{1}{2}\sigma + Q, \quad \frac{\nu}{2} \geq \lambda_{N+1}^{-1}(\sigma + 2Q),$$

Thus, we obtain that

$$\frac{d}{dt}\left(e^{\sigma t}\|\partial_x z(t)\|^2\right) + \frac{\lambda_1 \nu}{2} e^{\sigma t}\|\partial_x z(t)\|^2 \leq 0, \quad \forall t \geq t_0 \geq T_5. \tag{2.2.36}$$

Hence, (2.2.36) implies that

$$\|\partial_x z(t)\|^2 \leq e^{-(\sigma+\frac{\lambda_1\nu}{2})(t-t_0)}\|\partial_x z(t_0)\|^2, \quad \forall t \geq t_0 \geq T_5.$$

□

2.2.2 Finitely Many Volume Elements

In this section, we take a control operator based on finitely many volume elements and propose the following feedback control problem for (2.2.1)-(2.2.2):

$$\begin{cases} \partial_t u - \nu\partial_x^2 u + 2u\partial_x u - Ru + ku\|u\|^2 = h(x,t) \\ -\mu \displaystyle\sum_{k=1}^{N}(\bar{u}_k - \bar{v}_k)\chi_{J_k}(x), & (2.2.37) \\ u(0,t) = u(1,t) = 0, \quad u(x,0) = u_0(x), & (2.2.38) \end{cases}$$

where $J_k = [(k-1)\frac{1}{N}, k\frac{1}{N})$, for $k = 1, 2, ..., N-1$, $J_N = [\frac{N-1}{N}, 1]$, χ_{J_k} is the characteristic function of the interval J_k and

$$\bar{u}_k = \frac{1}{|J_k|}\int_{J_k}\tilde{v}(x,t)dx, \quad \bar{v}_k = \frac{1}{|J_k|}\int_{J_k} v(x,t)dx.$$

Theorem 2.2.5. *Let $\sigma > 0$ be an arbitrary number. Assume that N and μ are so large enough that*

$$\mu > \frac{1}{2}(\sigma + A_1), \quad N^2 \geq \frac{4\mu^2}{\nu^2\lambda_1}.$$

where A_1 is defined in (2.2.45)*. Then each solution of the problem* (2.2.37)-(2.2.38) *is approaching the solution of* (2.2.1)-(2.2.2) *with an exponential rate in $L^2(0,1)$ sense.*

Proof. By letting $z = u - v$, we see that z is a solution of the following problem

$$\begin{cases} \partial_t z - \nu\partial_x^2 z + 2z\partial_x z + 2v\partial_x z + 2z\partial_x v - Rz + ku\|u\|^2 - k\|v\|^2 v \\ = -\mu \displaystyle\sum_{k=1}^{N}(\bar{z}_k)\chi_{J_k}(x), & (2.2.39) \\ z(0,t) = z(1,t) = 0, \quad z(x,0) = u_0(x) - v_0(x), & (2.2.40) \end{cases}$$

where $\bar{z}_k = \frac{1}{|J_k|}\int_{J_k} z(x,t)dx$. Multiplying (2.2.39) by z in $L^2(0,1)$ and using the fact that $2(z\partial_x z, z) = 0$ and (1.4.10), we obtain that

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|^2 + \nu\|\partial_x z(t)\|^2 + 2(v\partial_x z + z\partial_x v, z) - R\|z(t)\|^2 \leq -\mu\left(\sum_{k=1}^{N}\bar{z}_k\chi_{J_k}(x), z\right). \quad (2.2.41)$$

By using integration by parts, Gagliardo-Nirenberg inequality (1.4.2) and Young's inequality (1.4.6), we estimate the third term on the left-hand side of (2.2.41)

$$\begin{aligned} 2(v\partial_x z + z\partial_x v, z) &= (\partial_x v, z^2) \leq \|\partial_x v(t)\|\|z(t)\|^2_{L^4(0,1)} \\ &\leq \beta\|\partial_x v(t)\|\|z(t)\|^{\frac{3}{2}}\|\partial_x z(t)\|^{\frac{1}{2}} \\ &\leq \frac{\nu}{4}\|\partial_x z(t)\|^2 + \frac{3}{4}\beta^{\frac{4}{3}}\nu^{-\frac{1}{3}}\|\partial_x v(t)\|^{\frac{4}{3}}\|z(t)\|^2. \end{aligned} \quad (2.2.42)$$

We rewrite and estimate the term on the right-hand side of (2.2.41) by using (1.4.14) and Young's inequality (1.4.6) as follows:

$$\begin{aligned} -\mu\left(\sum_{k=1}^{N}\bar{z}_k\chi_{J_k}(x), z\right) &= -\mu\left(\sum_{k=1}^{N}\bar{z}_k\chi_{J_k}(x) - z, z\right) - \mu\|z(t)\|^2 \\ &\leq \mu\left\|z - \sum_{k=1}^{N}\bar{z}_k\chi_{J_k}(x)\right\|\|z(t)\| - \mu\|z(t)\|^2 \\ &\leq \mu h\|\partial_x z(t)\|\|z(t)\| - \mu\|z(t)\|^2 \\ &\leq \frac{\nu}{4}\|\partial_x z(t)\|^2 + \frac{\mu^2 h^2}{\nu}\|z(t)\|^2. \end{aligned} \quad (2.2.43)$$

Here, the constant $h = \frac{1}{N}$ comes from (1.4.14). Thanks to (2.2.42) and (2.2.43) from (2.2.41) we get that

$$\frac{d}{dt}\|z(t)\|^2 + \nu\|\partial_x z(t)\|^2 \leq \left(2R + \frac{3}{2}\beta^{\frac{4}{3}}\nu^{-\frac{1}{3}}\|\partial_x v(t)\|^{\frac{4}{3}} - 2\mu\right)\|z(t)\|^2 + \frac{2\mu^2 h^2}{\nu}\|z(t)\|^2. \quad (2.2.44)$$

By utilizing the uniform estimate (2.2.10) for $\|\partial_x v(t)\|$ and multiplying (2.2.44) by $e^{\sigma t}$ with an arbitrary $\sigma > 0$, we obtain that

$$\frac{d}{dt}\left(e^{\sigma t}\|z(t)\|^2\right) + \nu e^{\sigma t}\|\partial_x z(t)\|^2 + (2\mu - \sigma - A_1)\, e^{\sigma t}\|z(t)\|^2 \leq \frac{2\mu^2 h^2}{\nu}e^{\sigma t}\|z(t)\|^2, \quad \forall t \geq t_0 \geq T_2,$$

where

$$A_1 := 2R + \frac{3}{2}\beta^{\frac{4}{3}}\nu^{-\frac{1}{3}}H_2^{\frac{2}{3}}. \tag{2.2.45}$$

Utilizing Poincaré-Friedrichs inequality (1.4.7), we have from the last inequality that

$$\frac{d}{dt}\left(e^{\sigma t}\|z(t)\|^2\right) + \left(\nu\lambda_1 - \frac{2\mu^2 h^2}{\nu}\right)e^{\sigma t}\|\partial_x z(t)\|^2 + (2\mu - \sigma - A_1)\, e^{\sigma t}\|z(t)\|^2 \le 0, \quad \forall t \ge t_0 \ge T_2, \tag{2.2.46}$$

We assume that

$$\mu \ge \frac{\sigma}{2} + \frac{A_1}{2}, \quad \nu\lambda_1 \ge \frac{4\mu^2 h^2}{\nu}, \tag{2.2.47}$$

i.e. by substituting $h = \frac{1}{N}$ we assume that

$$N^2 \ge \frac{4\mu^2}{\nu^2\lambda_1}.$$

Thus, we obtain from (2.2.46) that

$$\frac{d}{dt}\left(e^{\sigma t}\|z(t)\|^2\right) + \frac{\nu\lambda_1}{2}e^{\sigma t}\|z(t)\|^2 \le 0, \quad \forall t \ge t_0 \ge T_2.$$

Hence, this implies that

$$\|z(t)\|^2 \le \|z(t_0)\|^2 e^{-(\sigma + \frac{\nu\lambda_1}{2})(t-t_0)}, \quad \forall t \ge t_0 \ge T_2.$$

$\square$

2.2.3 Finitely Many Nodal Values

In this section, we take a control operator based on finitely many nodal values and propose the following feedback control problem for (2.2.1)-(2.2.2):

$$\begin{cases} \partial_t u - \nu\partial_x^2 u + 2u\partial_x u - Ru + ku\|u\|^2 = h(x,t) \\ -\mu \displaystyle\sum_{k=1}^{N}(u(x_k) - v(x_k))\chi_{J_k}(x), & (2.2.48) \\ u(0,t) = u(1,t) = 0, \quad u(x,0) = u_0(x), & (2.2.49) \end{cases}$$

where J_k and χ_{J_k} are as defined in the previous section and $x_k \in J_k$ are arbitrary points in the intervals J_k for $k = 1, 2, ..., N$.

Theorem 2.2.6. *Assume that N and μ are so large enough that*

$$\mu > \frac{1}{2}(\sigma + A_1), \quad N^2 \geq \frac{4\mu^2}{\nu^2 \lambda_1},$$

where A_1 is defined in (2.2.45). *Then each solution of the problem* (2.2.48)-(2.2.49) *is approaching the solution of* (2.2.1)-(2.2.2) *with an exponential rate in $L^2(0,1)$ sense.*

Proof. We see that $z := u - v$ is a solution of the following problem:

$$\begin{cases} \partial_t z - \nu \partial_x^2 z + 2z\partial_x z + 2v\partial_x z + 2z\partial_x v - Rz + ku\|u\|^2 - k\|v\|^2 v \\ == -\mu \sum_{k=1}^{N} z(x_k)\chi_{J_k}(x), & (2.2.50) \\ z(0,t) = z(1,t) = 0, \quad z(x,0) = u_0(x) - v_0(x), & (2.2.51) \end{cases}$$

Multiplying (2.2.50) by z in $L^2(0,1)$ and using the fact that $2(z\partial_x z, z) = 0$ and (1.4.10) $k(u\|u\|^2 - \|v\|^2 v, u - v) \geq 0$, we obtain that

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|^2 + \nu\|\partial_x z(t)\|^2 + 2(v\partial_x z + z\partial_x v, z) - R\|z(t)\|^2 \leq -\mu \left(\sum_{k=1}^{N} z(x_k)\chi_{j_k}(x), z \right). \quad (2.2.52)$$

We estimate the third term on the left-hand side and last term on the right-hand side of (2.2.52) similarly as in (2.2.42) and (2.2.43). Thus, we obtain from (2.2.52) the following inequality

$$\frac{d}{dt}\|z(t)\|^2 + \nu\|\partial_x z(t)\|^2 \leq \left(2R + \frac{3}{2}\beta^{\frac{4}{3}}\nu^{-\frac{1}{3}}\|\partial_x v(t)\|^{\frac{4}{3}} - 2\mu \right) \|z(t)\|^2 + \frac{2\mu^2 h^2}{\nu}\|z(t)\|^2. \quad (2.2.53)$$

By utilizing the uniform estimate (2.2.10) for $\|\partial_x v(t)\|$ and multiplying (2.2.53) by $e^{\sigma t}$ with an arbitrary $\sigma > 0$, we obtain that

$$\frac{d}{dt}\left(e^{\sigma t}\|z(t)\|^2\right) + \nu e^{\sigma t}\|\partial_x z(t)\|^2 + (2\mu - \sigma - A_1)\, e^{\sigma t}\|z(t)\|^2 \leq \frac{2\mu^2 h^2}{\nu} e^{\sigma t}\|z(t)\|^2, \quad \forall t \geq t_0 \geq T_2,$$

where A_1 is the same constant as defined in (2.2.45). Thanks to Poincaré-Friedrichs inequality (1.4.7), from the previous inequality we get that

$$\frac{d}{dt}\left(e^{\sigma t}\|z(t)\|^2\right) + \left(\nu\lambda_1 - \frac{2\mu^2 h^2}{\nu}\right) e^{\sigma t}\|z(t)\|^2 \\ + (2\mu - \sigma - A_1)\, e^{\sigma t}\|z(t)\|^2 \leq 0, \quad \forall t \geq t_0 \geq T_2, \quad (2.2.54)$$

We assume that μ and N are sufficiently large such that

$$\mu > \frac{1}{2}(\sigma + A_1), \quad \nu\lambda_1 \geq \frac{4\mu^2 h^2}{\nu}.$$

i.e. the second assumption can be written as follows:

$$N^2 \geq \frac{4\mu^2}{\nu^2\lambda_1}.$$

Thus, we obtain from (2.2.54) that

$$\frac{d}{dt}\left(e^{\sigma t}\|z(t)\|^2\right) + \frac{\nu\lambda_1}{2} e^{\sigma t}\|z(t)\|^2 \leq 0, \quad \forall t \geq t_0 \geq T_2. \quad (2.2.55)$$

Hence, this implies that

$$\|z(t)\|^2 \leq \|z(t_0)\|^2 e^{-(\sigma + \frac{\nu\lambda_1}{2})(t - t_0)}, \quad \forall t \geq t_0 \geq T_2.$$

□

Chapter 3

DETERMINING MODES

In this chapter, we give an estimate for the number of determining modes for problem (2.1.1)-(2.1.3). We consider original Burgers' equations (2.1.1)-(2.1.2) with a forcing term $h(x,t) \in L^\infty(0,\infty; L^2(0,1))$.

$$\begin{cases} \partial_t v = Uv + \nu\partial_x^2 v - 2v\partial_x v + h(x,t), \\ U'(t) = R - U(t) - \|v(t)\|^2. \end{cases}$$

We define the following Hilbert spaces:

$$H = L^2(0,1) \times \mathbb{R}^+, \quad V = H_0^1(0,1) \times \mathbb{R}^+. \tag{3.0.1}$$

Let $u_i = [v_i, U_i]$, $i = 1, 2$ be in the spaces H and V. We define the inner products on H and V as follows:

$$(u_1, u_2)_H = (v_1, v_2) + U_1U_2, \quad (u_1, u_2)_V = (\partial_x v_1, \partial_x v_2) + U_1U_2. \tag{3.0.2}$$

The norms on these spaces are given by

$$\|u(t)\|_H^2 = \|v(t)\|^2 + U^2(t), \quad \|u(t)\|_V^2 = \|\partial_x v(t)\|^2 + U^2(t). \tag{3.0.3}$$

Thanks to the Poincaré-Friedrichs inequality (1.4.7), we obtain that

$$\|u(t)\|_H^2 \leq \left(\pi^{-2}\|v(t)\|^2 + U^2(t)\right) \leq \|u(t)\|_V^2. \tag{3.0.4}$$

We formulate our problem as the following evolution equation

$$\begin{cases} \frac{du}{dt} + \nu Au + B(u,u) = f, \\ u(0) = u_0, \end{cases} \tag{3.0.5}$$

where $u_0 = [v_0, U_0]$ and $f = [h(x,t), R]$ in H. The linear operator A is defined as follows:

$$A = \begin{bmatrix} -\partial_x^2 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.0.6}$$

Observe that A is a positive, self-adjoint operator with the domain $\mathcal{D}(A) = V \cap H^2(0,1) \times \mathbb{R}$ and it has a compact inverse. By using the definitions (3.0.2), (3.0.3), we get

$$(Au, u)_H = (-\partial_x^2 v, v) + U^2(t) = \|\partial_x v(t)\|^2 + U^2(t) = \|u(t)\|_V^2. \tag{3.0.7}$$

For $u_1, u_2 \in V$, the nonlinear term $B : V \times V \to H$ is defined as follows:

$$B(u_1, u_2) = \begin{bmatrix} -U_2 v_1 + \frac{4}{3} v_1 \partial_x v_2 + \frac{2}{3} v_2 \partial_x v_1 \\ (v_1, v_2) \end{bmatrix}. \tag{3.0.8}$$

We see that B is a bilinear and continuous map. For $u_1, u_2, u_3 \in V$, we have

$$(B(u_1, u_2), u_2)_H = 0, \tag{3.0.9}$$

$$|(B(u_1, u_2), u_3)_H| \leq c_1 \|u_1\|_H^{\frac{1}{2}} \|u_1\|_V^{\frac{1}{2}} \|u_2\|_V \|u_3\|_H^{\frac{1}{2}} \|u_3\|_V^{\frac{1}{2}}, \tag{3.0.10}$$

$$\|B(u_1, u_2)\|_H \leq c_2 \|u_1\|_V \|u_2\|_V, \tag{3.0.11}$$

where $c_1 > 0$ and $c_2 > 0$ are positive constants.

Now, we find energy estimates for the equation (3.0.5). Let us multiply (3.0.5) with $u = [v, U]$ in H. Thanks to the equations (3.0.7) and (3.0.9), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_H^2 + \nu \|u(t)\|_V^2 = (f, u)_H. \tag{3.0.12}$$

By using (3.0.4) and Young's inequality (1.4.6), we estimate the term on the right-hand side of the last equation as follows:

$$\begin{aligned} |(f(t), u)_H| &\leq \|f(t)\|_H \|u(t)\|_H \leq \|f(t)\|_H \|u(t)\|_V \\ &\leq \frac{1}{2\nu}\|f(t)\|_H^2 + \frac{\nu}{2}\|u(t)\|_V^2. \end{aligned}$$

Utilizing this estimate from (3.0.12) we obtain the following inequality

$$\frac{d}{dt}\|u(t)\|_H^2 + \nu \|u(t)\|_V^2 \leq \frac{1}{\nu}\|f(t)\|_H^2. \tag{3.0.13}$$

Using the inequality (3.0.4) and Gronwall's inequality (1.4.3), from (3.0.13) we deduce

$$\|u(t)\|_H^2 \leq \|u_0\|_H^2 \exp(-\nu t) + \frac{1}{\nu}\int_0^t \|f(\tau)\|_H^2 \exp(-\nu(t-\tau))d\tau. \tag{3.0.14}$$

Assume that

$$\limsup_{t\to\infty} \|f(t)\|_H \leq F, \tag{3.0.15}$$

where $F > 0$ is a constant. From the inequality (3.0.14), we have that

$$\|u(t)\|_H^2 \leq \|u_0\|_H^2 \exp(-\nu t) + \frac{F^2}{\nu^2}\left(1 - \exp(-\nu t)\right). \tag{3.0.16}$$

Thus, for all u_0 there exists a t_0 which depends on u_0 such that

$$\|u(t)\|_H^2 \leq \frac{2F^2}{\nu^2} =: D_1, \quad \forall t \geq t_0. \tag{3.0.17}$$

Now, let us multiply (3.0.5) with Au in H.

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_V^2 + \nu\|Au(t)\|_H^2 + (B(u,u), Au)_H = (f, Au)_H. \tag{3.0.18}$$

Let us estimate the last term on the left-hand side of the previous equation.

$$\begin{aligned}(B(u,u), Au)_H &= \left(\begin{bmatrix} -Uv + 2v\partial_x v \\ \|v(t)\|^2 \end{bmatrix}, \begin{bmatrix} -\partial_x^2 v \\ U \end{bmatrix} \right)_H \\ &= U(t)(v, \partial_x^2 v) - 2\int_0^1 v\partial_x v \partial_x^2 v dx + U(t)\|v(t)\|^2 \\ &\leq |U(t)|\|v(t)\|\|\partial_x^2 v(t)\| + \|\partial_x v(t)\|_{L^3(0,1)}^3 + |U(t)|\|v(t)\|^2. \end{aligned} \tag{3.0.19}$$

Thanks to Young's inequality (1.4.6), we estimate the first term on the right-hand side of the last inequality as follows:

$$|U(t)|\|v(t)\|\|\partial_x^2 v(t)\| \leq \frac{\nu}{4}\|\partial_x^2 v(t)\|^2 + \frac{1}{\nu}|U(t)|^2\|v(t)\|^2. \tag{3.0.20}$$

Employing Gagliardo-Nirenberg inequality (1.4.2) and Young's inequality (1.4.6) with $\varepsilon = \frac{2\nu}{7}$, $p = \frac{8}{7}$, we have

$$\begin{aligned}\|\partial_x v(t)\|_{L^3(0,1)}^3 &\leq \beta^3 \|v(t)\|^{\frac{5}{4}} \|\partial_x^2 v(t)\|^{\frac{7}{4}} \\ &\leq \frac{\nu}{4}\|\partial_x^2 v(t)\|^2 + \beta^{24} 7^7 2^{-10} \nu^{-7} \|v(t)\|^{10}. \end{aligned} \tag{3.0.21}$$

Plugging the estimates (3.0.20) and (3.0.21) into (3.0.19) and utilizing the estimate (3.0.17), we obtain that

$$
\begin{aligned}
(B(u,u),Au)_H &\le \frac{\nu}{2}\|\partial_x^2 v(t)\|^2 + \frac{1}{\nu}|U(t)|^2\|v(t)\|^2 + \beta^{24}7^7 2^{-10}\nu^{-7}\|v(t)\|^{10} \\
&\le \frac{\nu}{2}\|\partial_x^2 v(t)\|^2 + \frac{1}{\nu}M_1^2 + \beta^{24}7^7 2^{-10}\nu^{-7}M_1^5, \quad \forall t \ge t_0. \quad (3.0.22)
\end{aligned}
$$

Also, we estimate the term on the right-hand side of the equation (3.0.18)

$$
|(f,Au)_H| \le \|f(t)\|_H\|Au(t)\|_H \le \frac{\nu}{4}\|Au(t)\|_H^2 + \frac{1}{\nu}\|f(t)\|_H^2. \quad (3.0.23)
$$

Thanks to the estimates (3.0.22) and (3.0.23) from (3.0.18), we have

$$
\begin{aligned}
&\frac{d}{dt}\|u(t)\|_V^2 + \frac{\nu}{2}\|Au(t)\|_H^2 \\
&\qquad \le \frac{2}{\nu}\|f(t)\|_H^2 + \frac{1}{\nu}D_1^2 + \beta^{24}7^7 2^{-10}\nu^{-7}D_1^5, \quad \forall t \ge t_0. \quad (3.0.24)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\|Au(t)\|_H^2 &= \|\partial_x^2 v(t)\|^2 + U^2(t) \ge \pi\|\partial_x v(t)\|^2 + U^2(t) \\
&\ge b^{-4}\left(\pi\|\partial_x v(t)\|^2 + U^2(t)\right) \\
&\ge \pi\left(\|\partial_x v(t)\|^2 + U^2(t)\right) = \pi\|u(t)\|_V^2. \quad (3.0.25)
\end{aligned}
$$

Using (3.0.15) and (3.0.25), (3.0.24) implies $\forall t \ge t_0$ that

$$
\frac{d}{dt}\|u(t)\|_V^2 + \frac{\nu\pi}{2}\|u(t)\|_V^2 \le \frac{2F^2}{\nu} + \frac{1}{\nu}D_1^2 + \beta^{24}7^7 2^{-10}\nu^{-7}D_1^5. \quad (3.0.26)
$$

Thanks to Gronwall's inequality (1.4.3), we deduce that

$$
\|u(t)\|_V^2 \le \|u(t_0)\|_V^2 \exp\left(-\frac{\nu\pi}{2}t\right) + \frac{2D_2}{\nu\pi}\exp\left(-\frac{\nu\pi}{2}(t-t_0)\right), \quad \forall t \ge t_0,
$$

where

$$
D_2 = \frac{2F^2}{\nu} + \frac{1}{\nu}D_1^2 + \beta^{24}7^7 2^{-10}\nu^{-7}D_1^5. \quad (3.0.27)
$$

Hence,

$$
\|u(t)\|_V^2 \le \frac{4D_2}{\nu\pi}, \quad \forall t \ge t_0. \quad (3.0.28)
$$

Now, we integrate the inequality (3.0.13) over the interval $(t, t+T)$.

$$\nu \int_t^{t+T} \|u(\tau)\|_V^2 d\tau \leq \|u(t)\|_H^2 + \frac{1}{\nu} \int_t^{t+T} \|f(\tau)\|^2 d\tau. \tag{3.0.29}$$

Employing (3.0.15) and (3.0.17), we obtain from (3.0.29)

$$\nu \int_t^{t+T} \|u(\tau)\|_V^2 d\tau \leq \frac{2F^2}{\nu^2} + \frac{b^2 F^2}{\nu} T, \quad \forall t \geq t_0.$$

This inequality can be rewritten as follows:

$$\frac{1}{T} \int_t^{t+T} \|u(\tau)\|_V^2 d\tau \leq \frac{2F^2}{\nu^3 T} + \frac{b^2 F^2}{\nu^2}.$$

For T large enough, we have that

$$\frac{1}{T} \int_t^{t+T} \|u(\tau)\|_V^2 d\tau \leq \frac{2F^2}{\nu^2}.$$

Hence,

$$\limsup_{t\to\infty} \frac{1}{T} \int_t^{t+T} \|u(\tau)\|_V^2 d\tau \leq \frac{2F^2}{\nu^2}. \tag{3.0.30}$$

We will utilize the estimate (3.0.30) in the following section.

3.1 Asymptotically Determining Modes

In the this section, we give the estimates for the number of determining modes for the Original Burgers' equations. Let P_m denote the $L^2(0,b)$ orthogonal projection from H onto the m-dimensional subspace $H_m = span\{w_1, w_2, ..., w_m\}$. We set $Q_m = I - P_m$. Assume that u and y are two solutions of the following evolution equations which represent the original Burgers' equations with different forcing terms f and g and different initial data, respectively

$$\frac{du}{dt} + \nu Au + B(u,u) = f, \quad u(0) = u_0, \tag{3.1.1}$$

$$\frac{dy}{dt} + \nu Ay + B(y,y) = g, \quad y(0) = y_0. \tag{3.1.2}$$

Definition 3.1.1 (See [25, 37]). A set of first m modes, $\{w_1, w_2, ..., w_m\}$ associated with the projection P_m is called *asymptotically determining modes* in H if the following asymptotic behavior of the forcing terms and the projections

$$\lim_{t\to\infty} \|f(t) - g(t)\|_H = 0 \text{ and } \lim_{t\to\infty} \|P_m(u(t)) - P_m(y(t))\|_H = 0 \tag{3.1.3}$$

imply the asymptotic behavior of the solutions

$$\lim_{t\to\infty} \|u(t) - y(t)\|_H = 0.$$

In the following theorem, we give our main result about the estimation of the number of the asymptotic determining modes.

Theorem 3.1.2. *Assume that the following conditions are satisfied*

$$\|f(t)\|_H \leq F < \infty, \quad \forall t > 0, \qquad \lim_{t\to\infty} \|f(t) - g(t)\|_H = 0,$$
$$\lim_{t\to\infty} \|P_m(u(t)) - P_m(y(t))\|_H = 0.$$

Then the first m eigenfunctions, $\{w_1, w_2, ..., w_m\}$, of the Sturm-Liouville operator under the homogeneous Dirichlet boundary conditions are asymptotically determining modes for the original Burgers' equations represented by the abstract evolution equation (3.1.1)*, provided that m is large enough such that*

$$\lambda_{m+1} > \frac{2C^2F^2}{\nu^4},$$

where $C > 0$ is a constant comes from the properties (3.0.10) *and* (3.0.11) *of the map B.*

Proof. Let $w = u - y$. From the equations (3.1.1) and (3.1.2) we see that w satisfies the following equation

$$\partial_t w + \nu A w + B(w, u) + B(y, w) = f(t) - g(t). \tag{3.1.4}$$

Let $p(t) = P_m w(t)$ and $q(t) = Q_m w(t)$ denote the $L^2(0, 1)$ orthogonal projection of w from H onto H_m and from H onto $H_m^{\perp}$, the complement of H_m, respectively. Then we have

$$\frac{1}{2}\frac{d}{dt}\|q(t)\|_H^2 + \nu\|q(t)\|_V^2 + (B(w, u), q) + (B(y, w), q) = (f(t) - g(t), q(t))_H. \tag{3.1.5}$$

Note that since B is a bilinear map and $w = p + q$, we can rewrite last two terms on the left-hand side of the equation (3.1.5) as follows.

$$(B(w, u), q) = (B(p, u), q) + (B(q, u), q),$$
$$(B(y, w), q) = (B(y, p), q) + (B(y, q), q).$$

Utilizing the properties of B (3.0.9), (3.0.10) and (3.0.11), we estimate these terms

$$|(B(p,u),q)| \leq C_1\|p(t)\|_H^{\frac{1}{2}}\|p(t)\|_V^{\frac{1}{2}}\|u(t)\|_H^{\frac{1}{2}}\|u(t)\|_V^{\frac{1}{2}}\|q(t)\|_V, \tag{3.1.6}$$

$$|(B(q,u),q)| \leq C_2\|u(t)\|_V\|q(t)\|_H\|q(t)\|_V$$

$$\leq \frac{C_2^2}{2\nu}\|u(t)\|_V^2\|q(t)\|_H^2 + \frac{\nu}{2}\|q\|_V^2, \tag{3.1.7}$$

$$|(B(y,p),q)| \leq C_3\|y(t)\|_H^{\frac{1}{2}}\|y(t)\|_V^{\frac{1}{2}}\|p(t)\|_H^{\frac{1}{2}}\|p(t)\|_V^{\frac{1}{2}}\|q(t)\|_V, \tag{3.1.8}$$

$$|(B(y,q),q)| = 0, \tag{3.1.9}$$

where C_1, C_2 and C_3 are positive constants. By using the equation (3.0.7) and estimates (3.1.6)-(3.1.9) from (3.1.4) we obtain

$$\frac{d}{dt}\|q(t)\|_H^2 + \nu\|q(t)\|_V^2 - \frac{C_2^2}{\nu}\|u(t)\|_V^2\|q(t)\|_H^2 \leq \beta(t), \tag{3.1.10}$$

where

$$\beta(t) := 2\|p(t)\|_H^{\frac{1}{2}}\|p(t)\|_V^{\frac{1}{2}}\|q(t)\|_V\left(C_1\|u(t)\|_H^{\frac{1}{2}}\|u(t)\|_V^{\frac{1}{2}} + C_3\|y(t)\|_H^{\frac{1}{2}}\|y(t)\|_V^{\frac{1}{2}}\right).$$

Applying Poincaré-Friedrichs type inequality $\|q(t)\|_V^2 \geq \lambda_{m+1}\|q(t)\|_H^2$, we have from (3.1.10)

$$\frac{d}{dt}\|q(t)\|_H^2 + \alpha(t)\|q(t)\|_H^2 \leq \beta(t), \tag{3.1.11}$$

where $\alpha(t) = \nu\lambda_{m+1} - \frac{C_2^2}{\nu}\|u(t)\|_V^2$.

Now, let us check the conditions of Lemma 1.4.2. From the estimate (3.0.30) we have

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|u(\tau)\|_V^2 d\tau \leq \frac{2F^2}{\nu^2}.$$

Thus, $\alpha(t) = \nu\lambda_{m+1} - \frac{C_2^2}{\nu}\|u(t)\|_V^2$ satisfy the condition (1.4.11)

$$\liminf_{t\to\infty}\frac{1}{T}\int_t^{t+T}\alpha(\tau)d\tau \geq \nu\lambda_{m+1} - \frac{C_2^2}{\nu}\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\|u(\tau)\|_V^2 d\tau$$

$$\geq \nu\lambda_{m+1} - \frac{2C_2^2F^2}{\nu^3} > 0,$$

provided that there exists a $T > 0$ large enough such that

$$\lambda_{m+1} > \frac{2C_2^2b^2F^2}{\nu^4}. \tag{3.1.12}$$

We have shown the uniform estimates for $\|u(t)\|_H$ and $\|u(t)\|_V$ in (3.0.17) and (3.0.28). One can derive uniform estimates for $\|y(t)\|_H$ and $\|y(t)\|_V$ similarly. Thus, using these uniform estimates and the fact that $\|P_m(u(t)) - P_m(y(t))\| \to 0$ as $t \to \infty$, we see that

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha^-(\tau)d\tau < \infty, \quad \liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \beta^+(\tau)d\tau = 0.$$

Thus, Lemma 1.4.2 implies thats $\|q(t)\|_H \to 0$ as $t \to \infty$. Hence, we obtain that $\lim_{t\to\infty} \|u(t) - y(t)\|_H = 0$. $\square$

Chapter 4

INVERSE SOURCE PROBLEMS

This chapter is devoted to the study existence, uniqueness and stability of solutions of the inverse source problems for original Burgers' equations and Burgers' equation with nonlocal nonlinearity.

4.1 Inverse Problem for Original Burgers' Equations

In this section, we analyze the existence, uniqueness and stability of the solutions of an inverse source problem for original Burgers' equations (2.1.1)-(2.1.3). We formulate the inverse source problem as follows:

$$\begin{cases} \partial_t v = Uv + \nu\partial_x^2 v - 2v\partial_x v + f(t)w(x), & (4.1.1) \\ U'(t) = R - \nu U(t) - \int_0^1 v^2(x,t)dx, & (4.1.2) \\ U(0) = U_0, \quad v(x,0) = v_0(x), \quad v(0,t) = v(1,t) = 0, & (4.1.3) \\ \int_0^1 v(x,t)w(x)dx = \phi(t), & (4.1.4) \end{cases}$$

where $(x,t) \in [0,1] \times [0,\infty)$, the functions w and ϕ are given and f is unknown source function.

Definition 4.1.1. A set of functions $\{v, U, f\}$ is called a weak solution of the problem (4.1.1)-(4.1.4) if $v \in L^2(0,T;H_0^1(0,1)) \cap L^\infty(0,T;L^2(0,1))$, $U \in L^\infty(0,T)$, $f \in L^2(0,T)$ and satisfy the following equations for each $T > 0$:

$$\frac{d}{dt}\int_0^1 v(x,t)\eta(x)dx - U(t)\int_0^1 v(x,t)\eta(x)dx + \nu\int_0^1 \partial_x v(x,t)\eta'(x)dx$$
$$+ 2\int_0^1 v(x,t)\partial_x v(x,t)\eta(x)dx = f(t)\int_0^1 w(x)\eta(x)dx, \quad \forall \eta \in H_0^1(0,1), \quad (4.1.5)$$

$$U(t) = U_0 + \int_0^t (R - \nu U(\tau) - \|v(\tau)\|^2)d\tau. \quad (4.1.6)$$

In order to show well-posedness of (4.1.1)-(4.1.4) we need the following conditions:

$$w \in H^2(0,1) \cap H_0^1(0,1), \quad \|w\| = K_1, \quad \|w'\| \leq K_2, \tag{4.1.7}$$

$$v_0(x) \in L^2(0,1), \quad \phi(t) \in H^1(0,T), \quad \forall T > 0, \tag{4.1.8}$$

$$\int_0^1 v_0(x)w(x)dx = \phi(0), \tag{4.1.9}$$

where K_1 and K_2 are positive constants.

4.1.1 Existence and Uniqueness

In this section, we show that the existence and uniqueness of the solution of the inverse source problem (4.1.1)-(4.1.4). We use similar procedures to prove the existence and uniqueness of the solution as in [47] and [52].

We define an operator $A : L^2(0,T) \to L^2(0,T)$ such that

$$(Af)(t) = \frac{1}{K_1^2}\left[\phi'(t) - U(t)(v,w) + \nu(\partial_x v, w') + 2(v\partial_x v, w)\right]. \tag{4.1.10}$$

We claim that under some conditions, the solvability of the following operator equation

$$Af = f, \tag{4.1.11}$$

related with the solvability of the problem (4.1.1)-(4.1.4). In the following theorem we state this assertion.

Theorem 4.1.2. *Assume that the conditions* (4.1.7)-(4.1.8) *hold. Then the following statements are true:*
(i) If the problem (4.1.1)-(4.1.4) *has a solution, then so does* (4.1.11).
(ii) If (4.1.11) *has a solution and the compatibility condition* (4.1.9) *holds, then there exists a solution of* (4.1.1)-(4.1.4).

Proof. (i) Assume that (4.1.1)-(4.1.4) has a solution denoted by $\{v, U, f\}$. By the definition of solution, v satisfies (4.1.5). We choose $\eta(x) = w(x)$ and substitute into (4.1.5) and employ the integral overdetermination condition (4.1.4) and (4.1.7). Then we get that

$$\phi'(t) - U(t)(v,w) + \nu(\partial_x v, w') + 2(v\partial_x, w) = f(t)K_1^2. \tag{4.1.12}$$

The left-hand side of (4.1.12) is equal to $K_1^2(Af)(t)$. Since $K_1 > 0$, we obtain that $(Af)(t) = f(t)$. Hence, f is a solution of (4.1.11).

(ii) Assume that (4.1.11) has a solution, denoted by f, which belongs to $L^2(0,T)$. Let us substitute this f into (4.1.1). Then by (4.1.1)-(4.1.3) we obtain a direct initial-boundary value problem. We know that this problem has a unique solution $\{v, U\}$. We refer to the paper of T. Dlotko [21] for the proof of the existence and uniqueness of the solution of original Burgers' equations. Now, we need only to show that v satisfies the integral overdetermination condition (4.1.4). We substitute $\eta(x) = w(x)$ into (4.1.5) and use (4.1.7). Then we obtain that

$$\begin{aligned}\frac{d}{dt}\int_0^1 v(x,t)w(x)dx - U(t)\int_0^1 v(x,t)w(x)dx + \nu\int_0^1 \partial_x v(x,t)w'(x)dx \\ - 2\int_0^1 v(x,t)\partial_x v(x,t)w(x)dx = f(t)K_1^2.\end{aligned} \tag{4.1.13}$$

Since f is a solution of (4.1.11), we can write that

$$\phi'(t) - U(t)(v,w) + \nu(\partial_x v, w') - 2(v\partial_x v, w) = f(t)K_1^2. \tag{4.1.14}$$

By subtracting (4.1.14) from (4.1.13) and using (4.1.9), we obtain that

$$E'(t) = 0, \quad E(0) = 0, \tag{4.1.15}$$

where $E(t) := \int_0^1 v(x,t)w(x)dx - \phi(t)$.

Since the solution of the Cauchy problem (4.1.15) is $E(t) \equiv 0$ for all $t \geq 0$, we get that $\int_0^1 v(x,t)w(x)dx = \phi(t)$. □

Now, we define a ball

$$\mathcal{D} := \left\{ f \in L^2(0,T) : \quad \left\| f - \frac{\phi'(t)}{K_1^2} \right\| \leq r \right\}. \tag{4.1.16}$$

We prove the unique solvability of the operator equation (4.1.10) in the following theorem.

Theorem 4.1.3. *Let $w \in H^2 \cap H_0^1(0,1)$, $\|w\| = K_1$ and $\phi \in H^1(0,T)$ for all $T > 0$. Assume that the operator A maps $\mathcal{D}$ into itself. Then there exists a positive integer k such that A^k is a contraction mapping in the ball $\mathcal{D}$.*

Proof. The proof of this theorem requires some preliminary calculations. We prove the theorem in three steps as follows:

1. *A priori* estimates for the terms v and U corresponding to a source term f.
2. An upper bound for the terms $\|v_1(t) - v_2(t)\|$ and $|U_1(t) - U_2(t)|$.
3. A^k is a conraction.

Step 1: *A Priori* Estimates

In this part, we find *a priori* estimates for the functions v and U. We multiply (4.1.1) by v in $L^2(0,1)$ and (4.1.2) by U, add the resultant equations and get

$$\frac{1}{2}\frac{d}{dt}[\|v(t)\|^2 + |U(t)|^2] + \nu\|\partial_x v(t)\|^2 + \nu|U(t)|^2 = f(t)(w,v) + RU(t). \quad (4.1.17)$$

We estimate the terms on the right-hand side of (4.1.17) by utilizing Young's (1.4.5) and Poincaré-Friedrichs inequality (1.4.7)

$$RU(t) \leq \frac{R^2}{2\nu} + \frac{\nu}{2}|U(t)|^2, \quad (4.1.18)$$

$$|f(t)(w,v)| \leq |f(t)|K_1\|v(t)\| \leq \frac{K_1^2|f(t)|^2}{2\nu\lambda_1} + \frac{\nu}{2}\|\partial_x v(t)\|^2. \quad (4.1.19)$$

Thus, we obtain from (4.1.17)

$$\frac{d}{dt}[\|v(t)\|^2 + |U(t)|^2] + \nu\|\partial_x v(t)\|^2 + \nu|U(t)|^2 \leq \frac{R^2}{\nu} + \frac{K_1^2|f(t)|^2}{\nu\lambda_1}. \quad (4.1.20)$$

We cancel the last two terms on the left-hand side of (4.1.17) and integrate the obtained inequality over the interval $(0,t)$. Thus, we get that

$$\|v(t)\|^2 + |U(t)|^2 \leq \left(\|v_0\|^2 + |U_0|^2\right) + \frac{R^2 t}{\nu} + \frac{K_1^2}{\nu\lambda_1}\int_0^t |f(\tau)|^2 d\tau.$$

Hence,

$$\sup_{t\in[0,T]} \|v(t)\|^2 + \sup_{t\in[0,T]} |U(t)|^2 \leq 2\left(\|v_0\|^2 + |U_0|^2\right) + \frac{2R^2 T}{\nu} + \frac{2K_1^2}{\nu\lambda_1}\|f\|^2_{L^2(0,T)}. \quad (4.1.21)$$

By integrating (4.1.20) over the interval $(0,T)$ we obtain that

$$\nu \int_0^T \|\partial_x v(t)\|^2 dt + \nu \int_0^T |U(t)|^2 dt \le \frac{R^2 T}{\nu} + \frac{K_1^2}{\nu\lambda_1}\|f\|^2_{L^2(0,T)}. \tag{4.1.22}$$

Step 2: An upper bound for differences of two solutions

Let f_1 and f_2 be in $\mathcal{D}$ associated with $\{v_1, U_1\}$ and $\{v_2, U_2\}$, respectively. Then we have

$$\partial_t v_1 = U_1 v_1 + \nu \partial_x^2 v_1 - 2 v_1 \partial_x v_1 + f_1(t) w(x), \tag{4.1.23}$$

$$U_1'(t) = R - \nu U_1(t) - \|v_1(t)\|^2, \tag{4.1.24}$$

and

$$\partial_t v_2 = U_2 v_2 + \nu \partial_x^2 v_2 - 2 v_2 \partial_x v_2 + f_2(t) w(x), \tag{4.1.25}$$

$$U_2'(t) = R - \nu U_2(t) - \|v_2(t)\|^2, \tag{4.1.26}$$

under the same initial and boundary conditions

$$U_1(0) = U_0 = U_2(0), \tag{4.1.27}$$

$$v_1(x,0) = v_0(x) = v_2(x,0), \tag{4.1.28}$$

$$v_1(0,t) = v_1(1,t) = 0 = v_2(0,t) = v_2(1,t). \tag{4.1.29}$$

We aim to find an upper bound for $\|v_1(t) - v_2(t)\|$ and $|U_1(t) - U_2(t)|$ which is related to $\|f_1 - f_2\|_{L^2(0,T)}$. In order to make calculations easier, we let $z := v_1 - v_2$ and $W := U_1 - U_2$. By subtracting (4.1.25) from (4.1.23) and (4.1.26) from (4.1.24), we obtain that

$$\begin{aligned}\partial_t z = {} & Wz + U_1 z + W v_1 + \nu \partial_x^2 z - 2(z\partial_x z + v_1 \partial_x z + z \partial_x v_1) \\ & + (f_1 - f_2) w(x), \end{aligned} \tag{4.1.30}$$

$$W'(t) = -\nu W(t) - \|z(t)\|^2 - 2(z, v_1). \tag{4.1.31}$$

We multiply (4.1.30) by z in $L^2(0,1)$ and (4.1.31) by W and add the obtain relations to get that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|z(t)\|^2 + |W(t)|^2\right] + \nu\|\partial_x z(t)\|^2 + \nu |W(t)|^2 \\ &\quad = U_1(t)\|z(t)\|^2 - W(t)(z, v_1) + 2(v_1 \partial_x z, z) + (f_1(t) - f_2(t))(w, z). \end{aligned} \tag{4.1.32}$$

We estimate the terms on the right-hand side of (4.1.32) by employing the Young's (1.4.6), Sobolev (1.4.9) and Poincaré-Friedrichs inequality (1.4.7) as

$$|W(t)(z, v_1)| \le \frac{\nu}{2}|W(t)|^2 + \frac{1}{2\nu}\|v_1\|^2\|z(t)\|^2, \tag{4.1.33}$$

$$\begin{aligned} 2|(v_1\partial_x z, z)| &\le 2\|v_1(t)\|_{L^\infty(0,1)}\|\partial_x z(t)\|\|z(t)\| \le 2c_0\|\partial_x v_1(t)\|\|\partial_x z(t)\|\|z(t)\| \\ &\le \frac{\nu}{4}\|\partial_x z(t)\|^2 + \frac{4c_0}{\nu}\|\partial_x v_1(t)\|^2\|z(t)\|^2, \end{aligned} \tag{4.1.34}$$

and

$$\begin{aligned} &|(f_1(t) - f_2(t))(w, z)| \le K_1|f_1(t) - f_2(t)|\|z(t)\| \\ &\le K_1\lambda_1^{-\frac{1}{2}}|f_1(t) - f_2(t)|\|\partial_x z(t)\| \le \frac{\nu}{4}\|\partial_x z(t)\|^2 + \frac{K_1^2}{\nu\lambda_1}|f_1(t) - f_2(t)|^2. \end{aligned} \tag{4.1.35}$$

Thanks to these estimates we get from (4.1.32) that

$$\begin{aligned} &\frac{d}{dt}\left[\|z(t)\|^2 + |W(t)|^2\right] + \nu\|\partial_x z(t)\|^2 + \nu|W(t)|^2 \\ &\le \left[2|U_1(t)| + \frac{8c_0}{\nu}\|\partial_x v_1(t)\|^2\right]\|z(t)\|^2 + \frac{2K_1^2}{\nu\lambda_1}|f_1(t) - f_2(t)|^2 \\ &\le \theta(t)\left[\|z(t)\|^2 + |W(t)|^2\right] + \frac{2K_1^2}{\nu\lambda_1}|f_1(t) - f_2(t)|^2, \end{aligned} \tag{4.1.36}$$

where

$$\theta(t) := 2|U_1(t)| + \frac{8c_0}{\nu}\|\partial_x v_1(t)\|^2. \tag{4.1.37}$$

We employ Gronwall's inequality (1.4.3) for (4.1.36) and obtain that

$$\begin{aligned} \|z(t)\|^2 + |W(t)|^2 &\le \frac{2K_1^2}{\nu\lambda_1}\int_0^t |f_1(s) - f_2(s)|^2\left(\exp\int_s^t \theta(\tau)d\tau\right)ds \\ &\le \frac{2K_1^2}{\nu\lambda_1}\left(\exp\int_0^t \theta(\tau)d\tau\right)\int_0^t |f_1(s) - f_2(s)|^2 ds. \end{aligned}$$

Here, we note that $\|z(0)\| = \|v_1(0) - v_2(0)\| = 0$ and $W_0 = U_1(0) - U_2(0) = 0$ by (4.1.27) and(4.1.28).

Hence, the last inequality implies that

$$\begin{aligned} &\|v_1(t) - v_2(t)\|^2 + |U_1(t) - U_2(t)|^2 \\ &\le \frac{2K_1^2}{\nu\lambda_1}\int_0^t |f_1(s) - f_2(s)|^2\left(\exp\int_s^t \theta(\tau)d\tau\right)ds \\ &\le \frac{2K_1^2}{\nu\lambda_1}\left(\exp\int_0^t \theta(\tau)d\tau\right)\int_0^t |f_1(s) - f_2(s)|^2 ds. \end{aligned} \tag{4.1.38}$$

As a final calculation in this step, we estimate the integral of $\theta(t)$ over the interval $(0,T)$. By definition (4.1.37) and adapting *a priori* estimates (4.1.21) and (4.1.22) for the terms U_1 and v_1, we obtain that

$$\begin{aligned}
&\int_0^T \theta(s)ds = 2\int_0^T |U_1(s)|ds + \frac{8c_0}{\nu}\int_0^T \|\partial_x v_1(s)\|^2 ds \\
&\leq 2T \sup_{t\in[0,T]} |U_1(t)| + \frac{8c_0}{\nu}\left(\frac{R^2T}{\nu^2} + \frac{K_1^2}{\nu^2\lambda_1}\|f_1\|^2_{L^2(0,T)}\right) \\
&\leq 2T\left(2\left(\|v_0\|^2 + |U_0|^2\right) + \frac{2R^2T}{\nu} + \frac{2K_1^2}{\nu\lambda_1}\|f_1\|^2_{L^2(0,T)}\right) \\
&+ \frac{8c_0}{\nu}\left(\frac{R^2T}{\nu^2} + \frac{K_1^2}{\nu^2\lambda_1}\|f_1\|^2_{L^2(0,T)}\right) \\
&\leq \theta_1 := 4T(\|v_0\|^2 + |U_0|^2) + \frac{4R^2T}{\nu}\left(T + \frac{2c_0}{\nu^2}\right) \\
&\qquad + \frac{4K_1^2}{\nu\lambda_1}\left(T + \frac{2c_0}{\nu^2}\right)\|f_1\|^2_{L^2(0,T)}, \quad (4.1.39)
\end{aligned}$$

where $\theta_1 > 0$ is a constant.

Step 3: A^k is a contraction

We find an upper bound for $\|Af_1 - Af_2\|_{L^2(0,t)}$. Note that the notations $z : v_1 - v_2$ and $W := U_1 - U_2$ are used in this part, as well. By using the definition (4.1.10) of A, we have that

$$\begin{aligned}
&|Af_1(t) - Af_2(t)| \\
&= \frac{1}{K_1^2}\big| - U_1(v_1, w) + U_2(v_2, w) + \nu(\partial_x z, w') + 2(v_1\partial_x v_1 - v_2\partial_x v_2, w)\big| \\
&\leq \frac{1}{K_1^2}\left[K_1|U_1(t)|\|z(t)\| + K_1|W(t)|\|v_2(t)\| + \nu\|z(t)\|\|w''\|\right] \\
&+ \frac{1}{K_1^2}\|w\|_{L^\infty(0,1)}\|z(t)\|(\|v_1(t)\| + \|v_2(t)\|) \\
&\leq \frac{1}{K_1^2}\left[\psi(t)\|z(t)\| + K_1\|v_2(t)\|\|W(t)\|\right] \\
&\qquad \leq \frac{1}{K_1^2}\left[\psi(t)\|z(t)\| + \left(\frac{K_1^2}{2} + \frac{1}{2}\|v_2(t)\|^2\right)\|W(t)\|\right] \quad (4.1.40)
\end{aligned}$$

where

$$\psi(t) := K_1|U_1(t)| + \nu\|w''\| + \|w\|_{L^\infty(0,1)}(\|v_1(t)\| + \|v_2(t)\|). \quad (4.1.41)$$

Utilizing Young's inequality (1.4.5), Sobolev inequality (1.4.9) and the estimate (4.1.21) for U_1, v_1 and v_2, we find an estimate for $\psi(t)$

$$\begin{aligned}\psi(t) &\le \frac{1}{2}K_1^2 + \frac{1}{2}|U_1(t)|^2 + \nu\|w''\| + \frac{c_0^2}{2}\|w''\|^2 + 2\|v_1(t)\|^2 + 2\|v_2(t)\|^2\\ &\le \frac{1}{2}K_1^2 + \frac{1}{2}\sup_{t\in[0,T]}|U_1(t)|^2 + \left(\nu + \frac{c_0^2}{2}\right)\|w''\|\\ &+ 2\sup_{t\in[0,T]}\|v_1(t)\|^2 + 2\sup_{t\in[0,T]}\|v_2(t)\|^2\\ &\le \frac{1}{2}K_1^2 + \left(\nu + \frac{c_0^2}{2}\right)\|w''\|\\ &+ 9\left(\|v_0\|^2 + |U_0|^2 + \frac{R^2T}{\nu}\right) + \frac{K_1^2}{\nu\lambda_1}\left(5\|f_1\|^2_{L^2(0,T)} + 4\|f_2\|^2_{L^2(0,T)}\right)\\ &\le 9\left(\|v_0\|^2 + |U_0|^2 + \frac{R^2T}{\nu} + \frac{K_1^2}{\nu\lambda_1}\tilde{r}^2\right) =: \delta_1, \end{aligned} \tag{4.1.42}$$

where $\delta_1 > 0$ is a constant and since f_1 and f_2 are in $\mathcal{D}$, we have

$$\|f_1\|_{L^2(0,T)} \le r + \frac{1}{K_1^2}\|\phi'\|_{L^2(0,T)} =: \tilde{r}, \quad \|f_2\|_{L^2(0,T)} \le \tilde{r}.$$

Similarly, we estimate the last term on the right-hand side of (4.1.40) as follows:

$$\frac{K_1^2}{2} + \frac{1}{2}\|v_2(t)\|^2 \le \delta_2 := \frac{K_1^2}{2} + \|v_0\|^2 + |U_0|^2 + \frac{R^2T}{\nu} + \frac{K_1^2}{\nu\lambda_1}\|f_2\|^2_{L^2(0,T)} \tag{4.1.43}$$

where $\delta_2 > 0$ is a constant. Therefore, we have from (4.1.40) that

$$\begin{aligned}|Af_1(t) - Af_2(t)| &\le \frac{1}{K_1^2}\left[\delta_1\|v_1(t) - v_2(t)\| + \delta_2\|U_1(t) - U_2(t)\|\right]\\ &\le \delta_3\left[\|v_1(t) - v_2(t)\| + |U_1(t) - U_2(t)\|\right], \end{aligned} \tag{4.1.44}$$

where $\delta_3 := \frac{1}{K_1^2}\max\{\delta_1, \delta_2\}$. Thanks to the estimates (4.1.38) and (4.1.39) we get that

$$\begin{aligned}\|Af_1 - Af_2\|^2_{L^2(0,t)} &= \int_0^t |Af_1(s) - Af_2(s)|^2 ds\\ &\le \delta_3^2\int_0^t \left[\|v_1(s) - v_2(s)\| + |U_1(s) - U_2(s)\|\right]^2 ds\\ &\le 2\delta_3^2\int_0^t \|v_1(s) - v_2(s)\|^2 + |U_1(s) - U_2(s)\|^2 ds\\ &\le \frac{4\delta_3^2K_1^2}{\nu\lambda_1}\exp\left(\int_0^s \theta(\tau)d\tau\right)\int_0^t\int_0^s |f_1(\tau) - f_2(\tau)|^2 d\tau ds\\ &\le \delta_4\int_0^t \|f_1 - f_2\|^2_{L^2(0,s)}ds, \quad \forall t \in [0,T]. \end{aligned} \tag{4.1.45}$$

where $\delta_4 := \frac{4\delta_3^2 K_1^2}{\nu\lambda_1} e^{\theta_1}$ does not depend on t. We know that A maps $\mathcal{D}$ to itself by the assumption in the theorem. Observe that

$$\|A^2 f_1 - A^2 f_2\|^2_{L^2(0,T)} \leq \delta_4 \int_0^T \|Af_1 - Af_2\|^2_{L^2(0,t)} dt$$
$$\leq \delta_4 \int_0^T \delta \int_0^t \|f_1 - f_2\|_{L^2(0,s)} ds dt \leq \delta_4^2 \|f_1 - f_2\|_{L^2(0,T)} \int_0^T \int_0^t ds dt$$
$$\leq \delta_4^2 \|f_1 - f_2\|_{L^2(0,T)} \int_0^T t dt \leq \frac{\delta_4^2 T^2}{2} \|f_1 - f_2\|_{L^2(0,T)}. \quad (4.1.46)$$

Thus, we can define the k-th degree of the operator A for any $k \in \mathbb{N}^+$. From the inequality (4.1.46), we have the following estimate for A^k:

$$\|A^k f_1 - A^k f_2\|^2_{L^2(0,T)} \leq \delta_4 \int_0^T \|A^{k-1} f_1 - A^{k-1} f_2\|^2_{L^2(0,t_1)} dt_1$$
$$\leq \delta_4^2 \int_0^T \int_0^{t_1} \|A^{k-2} f_1 A^{k-2} f_2\|^2_{L^2(0,t_2)} dt_2 dt_1$$
$$\leq \delta_4^k \|f_1 - f_2\|^2_{L^2(0,T)} \int_0^T \int_0^{t_1} \dots \int_0^{t_{k-1}} dt_k \dots dt_2 dt_1$$
$$\leq \frac{\delta_4^k T^k}{k!} \|f_1 - f_2\|^2_{L^2(0,T)}. \quad (4.1.47)$$

Since the term $\left(\frac{\delta_4^k T^k}{k!}\right)^{\frac{1}{2}}$ tends to 0 as $k \to \infty$, we can find an integer $k_0 > 0$ such that

$$\left(\frac{\delta_4^{k_0} T^{k_0}}{k_0!}\right)^{\frac{1}{2}} \leq 1.$$

Hence, by choosing $k = k_0$, we prove that $A^k : \mathcal{D} \to \mathcal{D}$ is a contraction operator. □

In order to complete the unique solvability of (4.1.11) we need to show that the operator A maps the closed ball $\mathcal{D}$ into itself.

Theorem 4.1.4. *Assume that* $\delta_0 < r$ *where*

$$\delta_0 = \frac{\sqrt{T}}{K_1^2} \left[\left(2(\|v_0\|^2 + |U_0|^2) + \frac{2R^2 T}{\nu} + \frac{2K_1^2}{\nu\lambda_1} \tilde{r}^2 \right) \left(\frac{K_1^2}{2} + (1 + \|w'\|_{L^\infty(0,1)}) \right) \right]$$
$$+ \frac{\sqrt{T}}{K_1^2} \frac{\nu^2}{2} \|w''\|^2, \quad (4.1.48)$$

and r *is the radius of the ball* $\mathcal{D}$ *which is defined in* (4.1.16) *and* $\tilde{r} = r + \frac{1}{K_1^2} \|\phi'\|_{L^2(0,T)}$. *Then the operator* A *which is defined in* (4.1.10) *maps the closed ball* $\mathcal{D}$ *into itself.*

Proof. Let f be an arbitrary function in $\mathcal{D}$, i.e. $\|f\|_{L^2(0,T)} \leq \tilde{r}$. Then

$$\left\|Af - \frac{\phi'}{K_1^2}\right\|_{L^2(0,T)}^2 = \int_0^T \left|Af(t) - \frac{\phi'(t)}{K_1^2}\right|^2 dt$$
$$= \frac{1}{K_1^4}\int_0^T \left|-U(t)(v,w) + \nu(\partial_x v, w') + 2(v\partial_x v, w')\right|^2 dt. \quad (4.1.49)$$

By using Young's inequality (1.4.5) and *a priori* estimate (4.1.21)

$$\begin{aligned}
&|-U(t)(v,w) + \nu(\partial_x v, w') + 2(v\partial_x v, w')| \\
&\leq |U(t)|\|v(t)\|K_1 + \nu\|v(t)\|\|w''\| + \|w'\|_{L^\infty(0,1)}\|v(t)\|^2 \\
&\leq \frac{K_1^2}{2}|U(t)|^2 + (1 + \|w'\|_{L^\infty(0,1)})\|v(t)\|^2 + \frac{\nu^2}{2}\|w''\|^2 \\
&\leq \left(\frac{K_1^2}{2} + (1 + \|w'\|_{L^\infty(0,1)})\right)\left(2(\|v_0\|^2 + |U_0|^2) + \frac{2R^2T}{\nu} + \frac{2K_1^2}{\nu\lambda_1}\tilde{r}^2\right) \\
&\qquad + \frac{\nu^2}{2}\|w''\|^2 =: D,
\end{aligned}$$

Thus, we have

$$\left\|Af - \frac{\phi'}{K_1^2}\right\|_{L^2(0,T)} \leq \frac{\sqrt{T}}{K_1^2} D = \delta_0. \quad (4.1.50)$$

Since $\delta_0 \leq r$ by the assumption of the theorem, we see from (4.1.50) that A maps $\mathcal{D}$ into itself.

□

Theorem 4.1.5. *Let $w \in H^2 \cap H_0^1(0,1)$, $\|w\| = K_1$ and $\phi \in H^1(0,T)$ for all $T > 0$. We assume that the compatibility condition* (4.1.9) *and the bound condition* (4.1.48) *for the radius of the closed ball $\mathcal{D}$ defined in* (4.1.16) *hold. Then there exists a solution $\{v, U, f\}$ for the inverse problem* (4.1.1)-(4.1.4) *with $f \in \mathcal{D}$, and this solution is unique.*

Proof. Let us prove the first statement. By Theorem 4.1.4, we know that the operator A maps $\mathcal{D}$ into itself and by Theorem 4.1.3 A^k is a contraction map on $\mathcal{D}$. Then from the contraction mapping principle, the nonlinear operator A has unique fixed point in $\mathcal{D}$. This means that the nonlinear operator equation (4.1.11) has a unique solution. By Theorem (4.1.2) we know that the inverse problem (4.1.1)-(4.1.4) has also a solution.

Let us prove that this solution is unique. Assume that $\{v_1, U_1, f_1\}$ and $\{v_2, U_2, f_2\}$ be two distinct solutions of (4.1.1)-(4.1.4) with both f_1 and f_2 are in $\mathcal{D}$. If f_1 and f_2 are equal a.e. in $[0, T]$, then by the uniqueness of the solution of the direct problem, we have that $\{v_1, U_1\}$ and $\{v_2, U_2\}$ are equal a.e. in $[0, 1] \times [0, T]$ and $[0, T]$. If f_1 and f_2 are distinct a.e. in $[0, T]$, then by Theorem 4.1.2, we know that f_1 and f_2 are both solutions of the operator equation (4.1.11). However, we have already proven that $Af = f$ has unique solution in $\mathcal{D}$. Hence, there are no distinct solutions $\{v_1, U_1, f_1\}$ and $\{v_2, U_2, f_2\}$ of (4.1.1)-(4.1.4).

□

4.1.2 Stability

In this section, we aim to analyze the asymptotic behavior of the solution of (4.1.1)-(4.1.4). Particularly, we show that the solution (v, U) of (4.1.1)-(4.1.4) tends to the stationary state solution of (2.1.1)-(2.1.3) and f tends to zero as time goes to infinity. We present our result in the following theorem:

Theorem 4.1.6. *Assume that the conditions* (4.1.7)-(4.1.9) *hold and the following limit relations are satisfied*

$$\alpha := \nu\lambda_1^{-1} - \frac{2R}{\nu} > 0, \quad \lim_{t\to\infty} |\phi(t)| = 0, \quad \lim_{t\to\infty} |\phi'(t)| = 0.$$

Then the solution $\{v, U, f\}$ *of* (4.1.1)-(4.1.4) *satisfy the following relations:*

$$\lim_{t\to\infty} \left[\|v(t)\|^2 + \left| U(t) - \frac{R}{\nu} \right|^2 \right] = 0,$$

$$\lim_{t\to\infty} \int_t^{t+1} \|\partial_x v(\tau)\|^2 d\tau = 0, \qquad \lim_{t\to\infty} |f(t)| = 0.$$

Proof. Let $Y(t) = U(t) - \frac{R}{\nu}$ and rewrite (4.1.1)-(4.1.4) as follows:

$$\begin{cases} \partial_t v = \left(Y(t) + \dfrac{R}{\nu} \right) v + \nu\partial_x^2 v - 2v\partial_x v + f(t)w(x), & (4.1.51) \\ Y'(t) = -\nu Y(t) - \|v(t)\|^2, & (4.1.52) \\ Y(0) = U_0 - \dfrac{R}{\nu}, \quad v(x, 0) = v_0(x), \quad v(0, t) = v(1, t) = 0, & (4.1.53) \\ \displaystyle\int_0^1 v(x, t)w(x)dx = \phi(t), & (4.1.54) \end{cases}$$

Multiplying (4.1.51) by w in $L^2(0,1)$ we obtain

$$f(t) = \frac{1}{K_1^2}\left[\phi'(t) - \left(Y(t) + \frac{R}{\nu}\right)\phi(t) - \nu(\partial_x^2 v, w) - 2(v\partial_x v, w)\right]. \tag{4.1.55}$$

We plug f into (4.1.51) and multiply the equation by v in $L^2(0,1)$.

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 + \nu\|\partial_x v(t)\|^2 = \left(Y(t) + \frac{R}{\nu}\right)\|v(t)\|^2 + K_1^{-2}\phi'(t)\phi(t)$$
$$- \left(Y(t) + \frac{R}{\nu}\right)K_1^{-2}\phi^2(t) - \nu K_1^{-2}(\partial_x^2 v, w)\phi(t) - 2K_1^{-2}(v\partial_x v, w)\phi(t). \tag{4.1.56}$$

Multiplying (4.1.52) by Y we obtain the following equation

$$\frac{1}{2}\frac{d}{dt}|Y(t)|^2 + \nu|Y(t)|^2 = -Y(t)\|v(t)\|^2. \tag{4.1.57}$$

We add (4.1.56) and (4.1.57) and get

$$\frac{1}{2}\frac{d}{dt}\left[\|v(t)\|^2 + |Y(t)|^2\right] + \nu\|\partial_x v(t)\|^2 + \nu|Y(t)|^2 = \frac{R}{\nu}\|v(t)\|^2 + K_1^{-2}\phi'(t)\phi(t)$$
$$- \left(Y(t) + \frac{R}{\nu}\right)K_1^{-2}\phi^2(t) - \nu K_1^{-2}(\partial_x^2 v, w)\phi(t) - 2K_1^{-2}(v\partial_x v, w)\phi(t). \tag{4.1.58}$$

Employing Young's inequality (1.4.6) and Poincaré-Friedrichs inequality (1.4.7), we estimate the terms on the right-hand side of the last inequality as follows:

$$K_1^{-2}\phi'(t)\phi(t) \le K_1^{-2}\frac{|\phi'(t)|^2}{2} + K_1^{-2}\frac{|\phi(t)|^2}{2}, \tag{4.1.59}$$

$$K_1^{-2}|Y(t)|\phi^2(t) \le \frac{\nu}{2}|Y(t)|^2 + \nu^{-1}K_1^{-4}|\phi(t)|^4, \tag{4.1.60}$$

$$\nu K_1^{-2}|(\partial_x v, w')||\phi(t)| \le \frac{\nu}{4}\|\partial_x v(t)\|^2 + \nu K_1^{-4}K_2^2|\phi(t)|^2, \tag{4.1.61}$$

and

$$2K_1^{-2}|(v\partial_x v, w)\phi(t)| \le K_1^{-2}|(v^2, w')||\phi(t)| \le K_1^{-2}\|v(t)\|_{L^4}^2\|w'\||\phi(t)|$$
$$\le \beta K_1^{-2}K_2\|v(t)\|^{\frac{3}{2}}\|\partial_x v(t)\|^{\frac{1}{2}}|\phi(t)| \le \beta K_1^{-2}K_2\lambda_1^{-\frac{3}{4}}\|v(t)\|\|\partial_x v(t)\||\phi(t)|$$
$$\le \frac{\nu}{4}\|\partial_x v(t)\|^2 + \nu^{-1}\beta^2\lambda_1^{-\frac{3}{2}}K_1^{-4}K_2^2|\phi(t)|^2\|v(t)\|^2, \tag{4.1.62}$$

where $\beta > 0$ is a constant in Gagliardo-Nirenberg inequality (1.4.2). Thanks to the estimates (4.1.59)-(4.1.62) and Poincaré-Friedrichs inequality (1.4.7), we obtain from (4.1.58) the following inequality

$$\begin{aligned}&\frac{d}{dt}\left[\|v(t)\|^2+|Y(t)|^2\right]+\nu\|\partial_x v(t)\|^2+\nu|Y(t)|^2\\&\le\frac{2R}{\nu}\lambda_1^{-1}\|\partial_x v(t)\|^2+2\nu^{-1}\beta^2\lambda_1^{-\frac{5}{2}}K_1^{-4}K_2^2|\phi(t)|^2\|\partial_x v(t)\|^2\\&\qquad+K_1^{-2}|\phi'(t)|^2+K_1^{-2}|\phi(t)|^2\left(1+2\nu^{-1}K_1^{-2}|\phi(t)|^2+2\nu K_1^{-4}K_2^2\right).\end{aligned}\tag{4.1.63}$$

We assume that

$$\alpha:=\nu-\frac{2R}{\nu}\lambda_1^{-1}>0.\tag{4.1.64}$$

Since $\lim_{t\to\infty}|\phi(t)|^2=0$, we can find time $t_0>0$ such that

$$\gamma:=\alpha-2\nu^{-1}\beta^2\lambda_1^{-\frac{5}{2}}K_1^{-4}K_2^2|\phi(t_0)|^2>0.\tag{4.1.65}$$

Thus, we obtain the following inequality

$$\frac{d}{dt}\left[\|v(t)\|^2+|Y(t)|^2\right]+\gamma\|\partial_x v(t)\|^2+\nu|Y(t)|^2\le M_1(t),\quad\forall t>0,\tag{4.1.66}$$

where

$$M_1(t):=K_1^{-2}|\phi'(t)|^2+K_1^{-2}|\phi(t)|^2\left(1+2\nu K_1^{-4}K_2^2\right)+2\nu^{-1}K_1^{-4}|\phi(t)|^4.\tag{4.1.67}$$

Hence, we obtain

$$\begin{aligned}\|v(t)\|^2+|Y(t)|^2\le{}&\exp(-d_0(t-t_0))\left[\|v(t_0)\|^2+|Y(t_0)|^2\right]\\&+\int_{t_0}^{t}\exp(-d_0(t-s))M_1(s)ds,\quad\forall t\ge t_0,\end{aligned}\tag{4.1.68}$$

where $d_0=\min\{\gamma,\nu\}$. This implies that

$$\lim_{t\to\infty}\left(\|v(t)\|^2+|Y(t)|^2\right)=\lim_{t\to\infty}\left(\|v(t)\|^2+\left|U(t)-\frac{R}{\nu}\right|^2\right)=0.\tag{4.1.69}$$

By integrating (4.1.66) over $(t,t+1)$, we obtain

$$\gamma\int_t^{t+1}\|\partial_x v(\tau)\|^2d\tau\le\left[\|v(t)\|^2+|Y(t)|^2\right]+\int_t^{t+1}M_1(\tau)d\tau.\tag{4.1.70}$$

We note that the condition $\lim_{t\to\infty}|\phi(t)|=0$ (similarly ϕ', h and v) implies that

$$\lim_{t\to\infty}\int_t^{t+1}|\phi(s)|ds=0.$$

Thus we get that $\lim_{t\to\infty}\int_t^{t+1} M_1(s)ds = 0$.

Hence, we obtain from (4.1.70) that

$$\lim_{t\to\infty}\int_t^{t+1}\|\partial_x v(\tau)\|^2 d\tau = 0. \tag{4.1.71}$$

Let us show the last limit relation in the theorem. From (4.1.55) we obtain

$$\begin{aligned}|f(t)| \le\ & \frac{1}{K_1^2}\left[|\phi'(t)| + \left|Y(t)+\frac{R}{\nu}\right||\phi(t)|\right]\\ &+\frac{1}{K_1^2}\left[\nu K_2\|\partial_x v(t)\| + K_2\|v(t)\|^{\frac{3}{4}}\|\partial_x v(t)\|^{\frac{1}{4}}\right].\end{aligned}$$

Thanks to the limit relations (4.1.69) and (4.1.71), we obtain that $\lim_{t\to\infty}|f(t)| = 0$.

□

4.2 Inverse Source Problem for Burgers' Equation with Nonlocal Nonlinearity

In this section, we analyze the existence, uniqueness and stability of the solutions of the inverse source problem for (1.0.5). We propose the following inverse source problem

$$\begin{cases}\partial_t v - \nu\partial_x^2 v + 2v\partial_x v - Rv + kv\displaystyle\int_0^1 v^2 dx = h(x,t) + f(t)w(x), & (4.2.1)\\ v(0,t) = v(1,t) = 0, \quad v(x,0) = v_0(x), & (4.2.2)\\ \displaystyle\int_0^1 v(x,t)w(x)dx = \phi(t), & (4.2.3)\end{cases}$$

where w and ϕ are given and f is unknown. We assume that the conditions (4.1.7)-(4.1.9) hold.

We define a solution of this problem as follows:

Definition 4.2.1. A pair of functions $\{v, f\}$ is called a weak solution of (4.2.1)-(4.2.3) such that $f \in L^2(0,T)$ and $v \in L^2(0,T;H_0^1(0,1)) \cap L^\infty(0,T;L^2(0,1))$ satisfy the following equality for all $\eta \in H_0^1(0,1)$:

$$\frac{d}{dt}\int_0^1 v(x,t)\eta(x)dx + \nu\int_0^1 \partial_x v(x,t)\eta'(x)dx + 2\int_0^1 v(x,t)\partial_x v(x,t)\eta(x)dx$$
$$- R\int_0^1 v(x,t)\eta(x)dx + k\|v(t)\|^2\int_0^1 v(x,t)\eta(x)dx$$
$$= \int_0^1 h(x,t)\eta(x)dx + f(t)\int_0^1 w(x)\eta(x)dx, \quad \forall t \in [0,T]. \quad (4.2.4)$$

4.2.1 Existence and Uniqueness

In this section, we show the existence and uniqueness of solution of the problem (4.2.1)-(4.2.3).

First, we define a nonlinear operator $A : L^2(0,T) \to L^2(0,T)$ such that

$$(Af)(t) = \frac{1}{K_1^2}\left[\phi'(t) - R\phi(t)\right]$$
$$+ \frac{1}{K_1^2}\left[\nu(\partial_x v, w') + 2(v\partial_x v, w) + k\|v(t)\|^2(v,w) - (h,w)\right], \quad (4.2.5)$$

where $K_1 = \|w\|$ as in (4.1.7). In order to prove the solvability of the inverse source problem (4.2.1)-(4.2.3), we need to prove the solvability of the following operator equation:

$$Af = f. \quad (4.2.6)$$

In the following theorem, we show that the solvability of (4.2.1)-(4.2.3) is related to the solvability of (4.2.6) and vice versa.

Theorem 4.2.2. *Assume that the conditions* (4.1.7)-(4.1.8) *hold. Then the following assertions are true:*
(i) If (4.2.1)-(4.2.3) *is solvable, then so is* (4.2.6).
(ii) If (4.2.6) *is solvable and the compatibility condition* (4.1.9) *holds, then there exists a solution of* (4.2.1)-(4.2.3).

Proof. (i) Assume that the problem (4.2.1)-(4.2.3) has a solution, let say $\{v, f\}$.

Then (4.2.4) holds. We substitute $\eta(x) = w(x)$ into (4.2.4), and obtain

$$\begin{aligned}\frac{d}{dt}\int_0^1 v(x,t)w(x)dx + \nu\int_0^1 \partial_x v(x,t)w'(x)dx + 2\int_0^1 v(x,t)\partial_x v(x,t)w(x)dx \\ - R\int_0^1 v(x,t)w(x)dx + k\|v(t)\|^2\int_0^1 v(x,t)w(x)dx \\ = \int_0^1 h(x,t)wdx + f(t)\|w\|^2. \qquad (4.2.7)\end{aligned}$$

By using the integral overdetermination condition (4.2.3) and $\|w\| = K_1$ in (4.1.7), we have

$$\begin{aligned}\phi'(t) + \nu(\partial_x v, w') + 2(\partial_x v, w) - R\phi(t) + k\|v(t)\|^2(v,w) - (h,w) \\ = f(t)K_1^2. \qquad (4.2.8)\end{aligned}$$

From the definition of the operator A in (4.2.5), we see that the left-hand side of (4.2.8) is equal to $(Af)K_1^2$. Since $K_1 > 0$, from (4.2.8) we have that $Af = f$. Hence, f is a solution of (4.2.6).

(ii) We assume that (4.2.6) has a solution in $L^2(0,T)$. We denote this solution by f and substitute into the equation (4.2.1). Since f is known, we obtain a direct problem (4.2.1)-(4.2.2). The existence of a solution of this problem can be proven by Galerkin method. Let us denote this solution by v. Now, we need only to show that v satisfy also the integral overdetermination condition (4.2.3). We substitute again $\eta = w$ in (4.2.4) and obtain (4.2.7). Since f is a solution of (4.2.6) by using (4.2.5), we can write the equation (4.2.8). We subtract the equation (4.2.8) from (4.2.7) and obtain

$$\frac{d}{dt}E(t) - RE(t) = 0, \qquad (4.2.9)$$

where $E(t) := \int_0^1 v(x,t)w(x)dx - \phi(t)$.

Since the compatibility condition (4.1.9) holds, we have initial data $E(0) = 0$ for (4.2.9). Hence, the solution of this Cauchy problem is

$$E(t) = E(0)\exp(-Rt) = 0, \quad \forall t > 0.$$

This implies that $\int_0^1 v(x,t)w(x)dx = \phi(t)$. □

Now, we proceed with the unique solvability of the operator equation (4.2.6). We define a ball $\mathcal{D}$ as follows:

$$\mathcal{D} = \left\{ f \in L^2(0,T) : \quad \left\| f - \frac{1}{K_1^2}(\phi'(t) - R\phi(t)) \right\|_{L^2(0,T)} \leq r_1 \right\}. \tag{4.2.10}$$

Next, we prove that $A : \mathcal{D} \to \mathcal{D}$ is a contraction mapping in the following theorem.

Theorem 4.2.3. *Let $w \in H^2 \cap H_0^1(0,1)$, $\|w\| = K_1$ and $\phi \in H^1(0,T)$ for all $T > 0$. Assume that A maps the ball $\mathcal{D}$ defined in* (4.2.10) *into itself. Then there exists an integer $s > 0$ such that A^s is a contraction mapping in the ball $\mathcal{D}$.*

Proof. We prove this theorem in three steps as in the proof of Theorem 4.1.3

1. *A priori* estimates for the term v corresponding to a source term f.

2. An upper bound for the term $\|v_1(t) - v_2(t)\|$.

3. A^s is contraction.

Step 1: *A Priori* Estimates

In this part, we find *a priori* estimates for the solution of (4.2.1)-(4.2.3). We multiply (4.2.1) by v in $L^2(0,1)$ and get that

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 + \nu\|\partial_x v(t)\|^2 - R\|v(t)\|^2 + k\|v(t)\|^4$$
$$= (h,v) + f(t)(w,v). \tag{4.2.11}$$

Thanks to Young's inequality (1.4.6) and Poincaré-Friedrichs inequality (1.4.7), we obtain

$$\frac{d}{dt}\|v(t)\|^2 + \nu\|\partial_x v(t)\|^2 \leq \frac{R^2}{2k} + \frac{2}{\nu\lambda_1}\|h(t)\|^2 + \frac{2K_1^2}{\nu\lambda_1}|f(t)|^2.$$

We integrate the last inequality over $(0,t)$ and we get

$$\|v(t)\|^2 + \nu\int_0^t \|\partial_x v(\tau)\|^2 d\tau \leq \|v_0\|^2 + \frac{R^2 t}{2k}$$
$$+ \frac{2}{\nu\lambda_1}\int_0^t \|h(\tau)\|^2 d\tau + \frac{2K_1^2}{\nu\lambda_1}\int_0^t |f(\tau)|^2 d\tau.$$

From this inequality, we can deduce the following estimates:

$$\sup_{t\in[0,T]} \|v(t)\|^2 \leq \|v_0\|^2 + \frac{R^2T}{2k} + \frac{2}{\nu\lambda_1}\|h\|^2_{L^2(0,T;L^2(0,1))} + \frac{2K_1^2}{\nu\lambda_1}\|f\|^2_{L^2(0,T)}, \tag{4.2.12}$$

and

$$\nu\int_0^T \|\partial_x v(\tau)\|^2 d\tau \leq \|v_0\|^2 + \frac{R^2T}{2k} + \frac{2}{\nu\lambda_1}\|h\|^2_{L^2(0,T;L^2(0,1))} + \frac{2K_1^2}{\nu\lambda_1}\|f\|^2_{L^2(0,T)}. \tag{4.2.13}$$

Step 2: An upper bound for differences of two solutions

Let f_1 and f_2 be in the ball $\mathcal{D}$. Then there are v_1 and v_2 corresponding to f_1 and f_2, respectively, such that

$$\partial_t v_1 - \nu\partial_x^2 v_1 + 2v_1\partial_x v_1 - Rv_1 + kv_1\int_0^1 v_1^2 dx = h(x,t) + f_1(t)w(x), \tag{4.2.14}$$

$$\partial_t v_2 - \nu\partial_x^2 v_2 + 2v_2\partial_x v_2 - Rv_2 + kv_2\int_0^1 v_2^2 dx = h(x,t) + f_2(t)w(x). \tag{4.2.15}$$

We subtract (4.2.15) from (4.2.14) and let $z := v_1 - v_2$. Then we obtain

$$\partial_t z - \nu\partial_x^2 z + 2v_1\partial_x z + 2z\partial_x v_2 - Rz + k(v_1\|v_1\|^2 - v_2\|v_2\|^2) = (f_1(t) - f_2(t))w(x). \tag{4.2.16}$$

We multiply (4.2.16) by z in $L^2(0,1)$ and estimate some terms to get the following equality

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|^2 + \nu\|\partial_x z(t)\|^2 + 2(v_1\partial_x z, z) + 2(z\partial_x v_2, z) - R\|z(t)\|^2 + k(v_1\|v_1\|^2 - v_2\|v_2\|^2, z) = (f_1(t) - f_2(t))(w,z). \tag{4.2.17}$$

Thanks to monotonicity inequality (1.4.10) we have that

$$k(v_1\|v_1\|^2 - v_2\|v_2\|^2, z) \geq 0, \tag{4.2.18}$$

where $d_0 > 0$ is a constant. By using Gagliardo-Nirenberg inequality (1.4.2), Poincaré-Friedrichs inequality (1.4.7) and Young's inequality (1.4.6), we get the following estimates:

$$\begin{aligned}2|(v_1\partial_x z, z)| = |-(\partial_x v_1, z^2)| &\le \beta\|z(t)\|^{\frac{3}{2}}\|\partial_x z(t)\|^{\frac{1}{2}}\|\partial_x v_1(t)\| \\ &\le \beta\lambda_1^{-\frac{1}{4}}\|z(t)\|^{\frac{1}{2}}\|\partial_x z(t)\|^{\frac{3}{2}}\|\partial_x v_1(t)\| \\ &\le \frac{\nu}{4}\|\partial_x z(t)\|^2 + \beta^4\lambda_1^{-1}2^{-2}3^3\nu^{-3}\|\partial_x v_1(t)\|^4\|z(t)\|^2, \end{aligned} \tag{4.2.19}$$

$$\begin{aligned}2|(\partial_x v_2, z^2)| \le 2\|\partial_x v_2(t)\|\|z(t)\|^2_{L^4(0,1)} &\le 2\beta\|z(t)\|^{\frac{3}{2}}\|\partial_x z(t)\|^{\frac{1}{2}}\|\partial_x v_2(t)\| \\ &\le 2\beta\lambda_1^{-\frac{1}{4}}\|z(t)\|^{\frac{1}{2}}\|\partial_x z(t)\|^{\frac{3}{2}}\|\partial_x v_2(t)\| \\ &\le \frac{\nu}{4}\|\partial_x z(t)\|^2 + \beta^4\lambda_1^{-1}3^3\nu^{-3}\|\partial_x v_2(t)\|^4\|z(t)\|^2, \end{aligned} \tag{4.2.20}$$

and

$$\begin{aligned}|(f_1(t) - f_2(t))(w, z)| &\le K_1|f_1(t) - f_2(t)|\|z(t)\| \\ &\le \frac{\nu}{4}\|\partial_x z(t)\|^2 + \frac{K_1^2}{\nu\lambda_1}|f_1(t) - f_2(t)|^2. \end{aligned} \tag{4.2.21}$$

Thanks to these estimates (4.2.18)-(4.2.21), from (4.2.17) we obtain that

$$\frac{d}{dt}\|z(t)\|^2 + \frac{\nu}{2}\|\partial_x z(t)\|^2 \le \alpha(t)\|z(t)\|^2 + \frac{2K_1^2}{\nu\lambda_1}|f_1(t) - f_2(t)|^2, \tag{4.2.22}$$

where

$$\alpha(t) := 2R + a_1\|\partial_x v_1(t)\|^4 + a_2\|\partial_x v_2(t)\|^4 > 0, \quad \forall t > 0, \tag{4.2.23}$$

$$a_1 := \beta^4\lambda_1^{-1}2^{-2}3^3\nu^{-3}, \quad a_2 := \beta^4\lambda_1^{-1}3^3\nu^{-3}. \tag{4.2.24}$$

We cancel the positive term $\frac{\nu}{2}\|\partial_x z(t)\|^2$ on the left-hand side of (4.2.22) and employ Gronwall's inequality (1.4.3) to obtain

$$\begin{aligned}\|z(t)\|^2 \le \|z(0)\|^2 \exp\left(\int_0^t \alpha(s)ds\right) \\ + \frac{2K_1^2}{\nu\lambda_1}\int_0^t \exp\left(\int_s^t \alpha(\tau)d\tau\right)|f_1(s) - f_2(s)|^2 ds.\end{aligned}$$

Since $v_1(x,0) = v_0(x) = v_2(x,0)$, we know that $z(x,0) = 0$. Thus, the last inequality implies that

$$\|v_1(t) - v_2(t)\|^2 \le \exp\left(\int_0^t \alpha(s)ds\right)\left[\frac{2K_1^2}{\nu\lambda_1}\int_0^t |f_1(s) - f_2(s)|^2 ds\right]. \tag{4.2.25}$$

Here, by utilizing the calculations in [43] (in Lemma 1 at p.147), we deduce from (4.2.25) that

$$\|v_1(t)-v_2(t)\| \le \exp\left(R+\sqrt{a_1}\int_0^t \|\partial_x v_1(\tau)\|^2 d\tau + \sqrt{a_2}\int_0^t \|\partial_x v_2(\tau)\|^2 d\tau\right)$$
$$\times\left[\sqrt{2}K_1\nu^{-\frac{1}{2}}\lambda_1^{-\frac{1}{2}}\int_0^t |f_1(s)-f_2(s)|ds\right].$$

Thanks to the estimate (4.2.13) we obtain from the last inequality that

$$\|v_1(t)-v_2(t)\| \le \exp\left(R+\frac{\sqrt{a_1}}{\nu}\gamma_1+\frac{\sqrt{a_2}}{\nu}\gamma_2\right)$$
$$\times\left[\sqrt{2}K_1\nu^{-\frac{1}{2}}\lambda_1^{-\frac{1}{2}}\int_0^t |f_1(s)-f_2(s)|ds\right], \tag{4.2.26}$$

where

$$\gamma_1 := \|v_0\|^2+\frac{R^2T}{2k}+\frac{2}{\nu\lambda_1}\|h\|^2_{L^2(0,T;L^2(0,1))}+\frac{2K_1^2}{\nu\lambda_1}\|f_1\|^2_{L^2(0,T)}, \tag{4.2.27}$$

$$\gamma_2 := \|v_0\|^2+\frac{R^2T}{2k}+\frac{2}{\nu\lambda_1}\|h\|^2_{L^2(0,T;L^2(0,1))}+\frac{2K_1^2}{\nu\lambda_1}\|f_2\|^2_{L^2(0,T)}. \tag{4.2.28}$$

Step 3: A^s is a contraction

We have obtained all the estimates which are necessary to prove A^s is a contraction map. Now, we find a bound for $\|Af_1-Af_2\|$. By the definition of A (4.2.5), we have

$$|Af_1(t)-Af_2(t)| \le \frac{1}{K_1^2}\left|\nu(\partial_x z, w')+2(v_1\partial_x v_1, w)-2(v_2\partial_x v_2, w)\right|$$
$$+\frac{1}{K_1^2}\left|k(v_1\|v_1(t)\|^2-v_2\|v_2(t)\|^2, w)\right| \tag{4.2.29}$$

Let us bound the terms on the right-hand side of the (4.2.29). We have

$$\nu|(\partial_x z, w')| = \nu|(z, w'')| \le \nu\|z(t)\|\|w''\|, \tag{4.2.30}$$

$$2|(v_1\partial_x v_1, w)-(v_2\partial_x v_2, w)| = |-(v_1^2-v_2^2, w')|$$
$$\le \sup_{x\in[0,1]}|w'(x)|\|z(t)\|(\|v_1(t)\|+\|v_2(t)\|), \tag{4.2.31}$$

and

$$k|(v_1\|v_1(t)\|^2-v_2\|v_2(t)\|^2, w)|$$
$$= k|\|v_1(t)\|^2(z,w)+(\|v_1(t)\|^2-\|v_2(t)\|^2)(v_2,w)|$$
$$\le kK_1\|z(t)\|\left(\|v_1(t)\|^2+\|v_2(t)\|(\|v_1(t)\|+\|v_2(t)\|)\right)$$
$$\le \frac{3kK_1}{2}\|z(t)\|\left(\|v_1(t)\|^2+\|v_2(t)\|^2\right). \tag{4.2.32}$$

Thus we obtain from (4.2.29) that

$$|Af_1(t) - Af_2(t)| \leq b(t)\|v_1(t) - v_2(t)\|, \tag{4.2.33}$$

where

$$b(t) = \frac{1}{K_1^2}\left[\nu\|w''\| + \sup_{x\in[0,1]}|w'(x)|(\|v_1(t)\| + \|v_2(t)\|)\right] + \frac{1}{K_1^2}\left[\frac{3kK_1}{2}\left(\|v_1(t)\|^2 + \|v_2(t)\|^2\right)\right].$$

Thanks to Young's inequality (1.4.5) and (4.2.12), we estimate

$$b(t) \leq \frac{1}{K_1^2}\left(\nu\|w''\| + \frac{1}{2}\sup_{x\in[0,1]}|w'(x)|^2 + \frac{3(1+kK_1)}{2}b_1\right),$$

where

$$b_1 := 2\|v_0\|^2 + \frac{R^2T}{2k} + \frac{4}{\nu\lambda_1}\|h\|^2_{L^2(0,T;L^2(0,1))} + \frac{2K_1^2}{\nu\lambda_1}\left(\|f_1\|^2_{L^2(0,T)} + \|f_2\|^2_{L^2(0,T)}\right). \tag{4.2.34}$$

Thus, we have

$$b(t) \leq b_2 := \frac{1}{K_1^2}\left(\nu\|w''\| + \frac{1}{2}\sup_{x\in[0,1]}|w'(x)|^2 + \frac{3(1+kK_1)}{2}b_1\right). \tag{4.2.35}$$

Now, we can estimate the following term by using (4.2.26) and (4.2.33)

$$\begin{aligned}\|Af_1 - Af_2\|^2_{L^2(0,t)} &= \int_0^t |Af_1(\eta) - Af_2(\eta)|^2 d\eta \\ &\leq b_2^2\int_0^t \|v_1(\eta) - v_2(\eta)\|^2 d\eta \\ &\leq b_2^2\exp\left(2R + \frac{2\sqrt{a_1}}{\nu}\gamma_1 + \frac{2\sqrt{a_2}}{\nu}\gamma_2\right) \\ &\times \int_0^t\left(\int_0^\eta |f_1(\tau) - f_2(\tau)|d\tau\right)^2 d\eta.\end{aligned} \tag{4.2.36}$$

The last term on the right-hand side of (4.2.36) can be estimated by using Hölder inequality as

$$\begin{aligned}\int_0^t\left(\int_0^\eta |f_1(\tau) - f_2(\tau)|d\tau\right)^2 d\eta &\leq \int_0^t\left(\eta\|f_1 - f_2\|^2_{L^2(0,\eta)}\right)d\eta \\ &\leq T\int_0^t \|f_1 - f_2\|^2_{L^2(0,\eta)}d\eta.\end{aligned} \tag{4.2.37}$$

Since f_1 and f_2 in $\mathcal{D}$, we have that

$$\|f_1\|_{L^2(0,T)} \leq r_1 + \frac{1}{K_1^2}\|\phi'(t) - R\phi(t)\|_{L^2(0,T)} =: r_2, \quad \|f_2\|_{L^2(0,T)} \leq r_2,$$

We estimate (4.2.27) and (4.2.28) by using r_2 as

$$\tilde{\gamma}_1 \leq \|v_0\|^2 + \frac{R^2T}{2k} + \frac{2}{\nu\lambda_1}\|h\|^2_{L^2(0,T;L^2(0,1))} + \frac{2K_1^2}{\nu\lambda_1}r_2^2, \tag{4.2.38}$$

$$\tilde{\gamma}_2 \leq \|v_0\|^2 + \frac{R^2T}{2k} + \frac{2}{\nu\lambda_1}\|h\|^2_{L^2(0,T;L^2(0,1))} + \frac{2K_1^2}{\nu\lambda_1}r_2^2. \tag{4.2.39}$$

Thus, from (4.2.36) we have

$$\|Af_1 - Af_2\|_{L^2(0,t)} \leq \sqrt{b_3}\left(\int_0^t \|f_1 - f_2\|^2_{L^2(0,s)}ds\right)^{\frac{1}{2}}, \quad \forall t \in [0,T], \tag{4.2.40}$$

where

$$b_3 := Tb_2^2 \exp\left(2R + \frac{2\sqrt{a_1}}{\nu}\tilde{\gamma}_1 + \frac{2\sqrt{a_2}}{\nu}\tilde{\gamma}_2\right). \tag{4.2.41}$$

We see that b_3 does not depend on t. We know that A maps $\mathcal{D}$ to itself by the assumption in the theorem. Thus, we can define the s-th degree of the operator A for any $s \in \mathbb{N}^+$. Using similar procedures in (4.1.46) and (4.1.47), from the inequality (4.2.40), we have the following estimate for A^s:

$$\|A^s f_1 - A^s f_2\|_{L^2(0,T)} \leq \left(\frac{b_3^s T^s}{s!}\right)^{\frac{1}{2}} \|f_1 - f_2\|_{L^2(0,T)}. \tag{4.2.42}$$

Since the term $\left(\frac{b_3^s T^s}{s!}\right)^{\frac{1}{2}}$ tends to 0 as $s \to \infty$, we can find an integer $s_0 > 0$ such that

$$\left(\frac{b_3^{s_0} T^{s_0}}{s_0!}\right)^{\frac{1}{2}} \leq 1.$$

Hence, by choosing $s = s_0$, we prove that $A^s : \mathcal{D} \to \mathcal{D}$ is a contraction operator . □

In order to complete the unique solvability of (4.1.11) we need to show that the operator A maps the closed ball $\mathcal{D}$ into itself.

Theorem 4.2.4. *Assume that* $b_0 < r_1$ *where*

$$b_0 = \frac{\sqrt{T}}{K_1^2}\left[\frac{\nu^2\|w''\|^2}{2} + K_1\|h\|_{L^2(0,T;L^2(0,1))}\right]$$
$$+\frac{\sqrt{T}}{K_1^2}\left(\|v_0\|^2 + \frac{R^2T}{2k} + \frac{2}{\nu\lambda_1}\|h\|^2_{L^2(0,T;L^2(0,1))} + \frac{2K_1^2}{\nu\lambda_1}r_2^2\right)\left(2 + \|w'\|_{L^\infty(0,1)}\right)$$
$$+\frac{\sqrt{T}}{K_1^2}\frac{k^2K_1^2}{4}\left(\|v_0\|^2 + \frac{R^2T}{2k} + \frac{2}{\nu\lambda_1}\|h\|^2_{L^2(0,T;L^2(0,1))} + \frac{2K_1^2}{\nu\lambda_1}r_2^2\right)^2, \quad (4.2.43)$$

and $r1$ *is the radius of the ball* $\mathcal{D}$ *which is defined in* (4.2.10) *and* $r_2 = r_1 + \frac{1}{K_1^2}\|\phi'(t) - R\phi(t)\|_{L^2(0,T)}$. *Then the operator* A *defined in* (4.1.10) *maps the closed ball* $\mathcal{D}$ (4.2.10) *into itself.*

Proof. Let f be an arbitrary function in $\mathcal{D}$, i.e., $\|f\|_{L^2(0,T)} \le r_1 + \frac{1}{K_1^2}\|\phi'(t) - R\phi(t)\|_{L^2(0,T)} = r_2$. We estimate the norm of $\left(Af - \frac{1}{K_1^2}(\phi'(t) - R\phi(t))\right)$ as

$$\left\|Af - \frac{1}{K_1^2}(\phi'(t) - R\phi(t))\right\|^2_{L^2(0,T)} = \int_0^T \left|Af(t) - \frac{1}{K_1^2}(\phi'(t) - R\phi(t))\right|^2 dt$$
$$= \frac{1}{K_1^4}\int_0^T \left|\nu(\partial_x v, w') + 2(v\partial_x v, w) + k\|v(t)\|^2(v,w) - (h,w)\right|^2 dt. \quad (4.2.44)$$

We estimate the term in the integral by using Young's inequality (1.4.5) and *a priori* estimate (4.2.12)

$$\begin{aligned}
&|\nu(\partial_x v, w') + 2(v\partial_x v, w) + k\|v(t)\|^2(v,w) - (h,w)| \\
&\le \nu\|v(t)\|\|w''\| + \|w'\|_{L^\infty(0,1)}\|v(t)\|^2 + kK_1\|v(t)\|^3 + K_1\|h(t)\| \\
&\le \frac{\nu^2\|w''\|^2}{2} + \|v(t)\|^2\left(2 + \|w'\|_{L^\infty(0,1)} + \frac{k^2K_1^2}{4}\|v(t)\|^2\right) + K_1\|h\|_{L^2(0,T;L^2(0,1))} \\
&\le \frac{\nu^2\|w''\|^2}{2} + K_1\|h\|_{L^2(0,T;L^2(0,1))} \\
&+ \left(\|v_0\|^2 + \frac{R^2T}{2k} + \frac{2}{\nu\lambda_1}\|h\|^2_{L^2(0,T;L^2(0,1))} + \frac{2K_1^2}{\nu\lambda_1}r_2^2\right)\left(2 + \|w'\|_{L^\infty(0,1)}\right) \\
&+ \frac{k^2K_1^2}{4}\left(\|v_0\|^2 + \frac{R^2T}{2k} + \frac{2}{\nu\lambda_1}\|h\|^2_{L^2(0,T;L^2(0,1))} + \frac{2K_1^2}{\nu\lambda_1}r_2^2\right)^2.
\end{aligned}$$

Hence, from (4.2.44) we obtain that

$$\left\|Af - \frac{1}{K_1^2}(\phi'(t) - R\phi(t))\right\|_{L^2(0,T)} \le b_0. \quad (4.2.45)$$

Since $b_0 \ge r_1$, (4.2.45) implies that the operator A defined in (4.2.5) maps $\mathcal{D}$ (4.2.10) into itself. □

Finally, we state the main theorem in this section. We prove that under some conditions, the solution of the inverse source problem (4.2.1)-(4.2.3) exits and it is unique.

Theorem 4.2.5. *Let $w \in H^2 \cap H_0^1(0,1)$, $\|w\| = K_1$ and $\phi \in H^1(0,T)$ for all $T > 0$. We assume that the compatibility condition* (4.1.9) *and the bound condition* (4.2.43) *for the radius of the closed ball $\mathcal{D}$ defined in* (4.2.10) *hold. Then there exists a solution $\{v, f\}$ for the inverse problem* (4.2.1)-(4.2.3) *with $f \in \mathcal{D}$, and this solution is unique.*

Proof. Let us prove the first statement. By Theorem 4.2.4, we know that the operator A defined in (4.2.5), maps $\mathcal{D}$ into itself and by Theorem 4.2.3 A^k is a contraction map on $\mathcal{D}$. Then from the contraction mapping principle, the nonlinear operator A has unique fixed point in $\mathcal{D}$. This means that the nonlinear operator equation (4.2.6) has a unique solution. By Theorem (4.2.2) we know that the inverse problem (4.2.1)-(4.2.3) has also a solution.

Let us prove that this solution is unique. Assume that $\{v_1, f_1\}$ and $\{v_2, f_2\}$ be two distinct solutions of (4.2.1)-(4.2.3) with both f_1 and f_2 are in $\mathcal{D}$ (4.2.10). If f_1 and f_2 are equal a.e. in $[0,T]$, then by the uniqueness of the solution of the direct problem, we have that v_1 and v_2 are equal a.e. in $[0,1] \times [0,T]$. If f_1 and f_2 are distinct a.e. in $[0,T]$, then by Theorem 4.2.2, we know that f_1 and f_2 are both solutions of the operator equation (4.2.6). However, we have already proven that $Af = f$ has unique solution in $\mathcal{D}$. Hence, there are no distinct solutions $\{v_1, f_1\}$ and $\{v_2, f_2\}$ of (4.2.1)-(4.2.3).

□

4.2.2 Stability

In this section, we present the stabilization result of the problem (4.2.1)-(4.2.3) in the following theorem.

Theorem 4.2.6. *Assume that the conditions* (4.1.7)-(4.1.9) *hold and*

$$d := \frac{\nu}{2} - 2R\lambda_1^{-1} > 0, \quad \lim_{t\to\infty} |\phi(t)| = 0, \quad \lim_{t\to\infty} |\phi'(t)| = 0, \quad \lim_{t\to\infty} \|h(t)\| = 0.$$

Then the solution $\{v, f\}$ *of* (4.2.1)-(4.2.3) *satisfy the following relations:*

$$\lim_{t\to\infty} \|v(t)\|^2 = 0, \quad \lim_{t\to\infty} \int_t^{t+1} \|\partial_x v(\tau)\|^2 d\tau = 0, \quad \lim_{t\to\infty} |f(t)| = 0.$$

Proof. We multiply (4.2.1) by w in $L^2(0,1)$ and obtain f as

$$f(t) = \frac{1}{K_1^2}\left[\phi'(t) - \nu(\partial_x^2, w) + 2(v\partial_x v, w)\right] \\ - \frac{1}{K_1^2}\left[R\phi(t) + k\|v(t)\|^2\phi(t) - (h,w)\right]. \quad (4.2.46)$$

Substituting (4.2.46) into (4.2.1) we get the following equality

$$\partial_t v - \nu\partial_x^2 v + 2v\partial_x v - Rv + kv\int_0^1 v^2 dx = h(x,t) + \frac{w(x)\phi'(t)}{K_1^2} \\ - \frac{w(x)}{K_1^2}\left[\nu(\partial_x^2, w) + 2(v\partial_x v, w) - R\phi(t) + k\|v(t)\|^2\phi(t) - (h,w)\right]. \quad (4.2.47)$$

Multiplying (4.2.47) by v in $L^2(0,1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 + \nu\|\partial_x v(t)\|^2 - R\|v(t)\|^2 + k\|v(t)\|^4 = (h,v) + \frac{\phi'(t)\phi(t)}{K_1^2} \\ - \frac{\phi(t)}{K_1^2}\left[\nu(\partial_x^2 v, w) + 2(v\partial_x v, w) - R\phi(t) + k\|v(t)\|^2\phi(t) - (h,w)\right]. \quad (4.2.48)$$

We estimate the terms on the right-hand side of (4.2.48):

$$|(h,v)| \le \frac{\nu}{4}\|\partial_x v(t)\|^2 + \frac{1}{\nu\lambda_1}\|h(t)\|^2, \quad (4.2.49)$$

$$\frac{k\|v(t)\|^2|\phi(t)|^2}{K_1^2} \le k\|v(t)\|^4 + \frac{|\phi(t)|^4}{4kK_1^4}, \quad (4.2.50)$$

$$\frac{|(h,w)\phi(t)|}{K_1^2} \le \frac{\|h(t)\||\phi(t)|}{K_1} \le \frac{\|h(t)\|^2}{4R} + \frac{R|\phi(t)|^2}{K_1^2}. \quad (4.2.51)$$

Employing the estimates (4.1.59), (4.1.61) and (4.1.62) in the previous part and (4.2.49)-(4.2.51) in (4.2.48), we obtain

$$\frac{d}{dt}\|v(t)\|^2 + \frac{\nu}{2}\|\partial_x v(t)\|^2 \le 2R\|v(t)\|^2 + \|h(t)\|^2\left(\frac{2}{\nu\lambda_1} + \frac{1}{2R}\right) \\ + \frac{|\phi'(t)|^2}{K_1^2} + \frac{|\phi(t)|^2}{K_1^2} + \frac{|\phi(t)|^4}{2kK_1^2}. \quad (4.2.52)$$

We employ Poincaré-Friedrichs inequality (1.4.7) and assume that

$$d := \frac{\nu}{2} - 2R\lambda_1^{-1} > 0. \quad (4.2.53)$$

Thus, we obtain that

$$\frac{d}{dt}\|v(t)\|^2 + d\|\partial_x v(t)\|^2 \leq M_2(t), \tag{4.2.54}$$

where

$$M_2(t) := \|h(t)\|^2\left(\frac{2}{\nu\lambda_1} + \frac{1}{2R}\right) + \frac{|\phi'(t)|^2}{K_1^2} + \frac{|\phi(t)|^2}{K_1^2} + \frac{|\phi(t)|^4}{2kK_1^2}. \tag{4.2.55}$$

Thanks to Poincaré-Friedrichs inequality (1.4.7) and Gronwall's inequality (1.4.3), we obtain

$$\|v(t)\|^2 \leq \|v_0\|^2 \exp(-d\lambda_1 t) + \int_0^t \exp(-d\lambda_1(t-s))M_2(s)ds.$$

Hence, $\lim_{t\to\infty}\|v(t)\|^2 = 0$. By integrating (4.2.54) over $(t, t+1)$, we have

$$d\int_t^{t+1}\|\partial_x v(s)\|^2 ds \leq \int_t^{t+1} M_2(s)ds.$$

This implies that

$$\lim_{t\to\infty}\int_t^{t+1}\|\partial_x v(s)\|^2 ds = 0.$$

Now, let us show the last limit relation. From (4.2.46) we obtain

$$\begin{aligned}|f(t)| \leq \frac{1}{K_1^2}\left[|\phi'(t)| + \nu K_2\|v(t)\| + K_2\|v(t)\|^{\frac{3}{4}}\|\partial_x v(t)\|^{\frac{1}{4}} + R|\phi(t)|\right] \\ + \frac{1}{K_1^2}\left[k\|v(t)\|^2|\phi(t)| + K_1\|h(t)\|\right].\end{aligned}$$

Hence, we have that $\lim_{t\to\infty}|f(t)| = 0$.

□

Chapter 5

NUMERICAL SIMULATIONS

In this chapter, we present our numerical results for the finite-parameter feedback control problem for the original Burgers' equations. The organization of this chapter is as follows:

1. In Section 5.1, we solve numerically the initial-boundary value problem for the original Burgers' equations given by

$$\begin{cases} \partial_t v(x,t) = \dfrac{1}{b}U(t)v(x,t) + \nu\partial_x^2 v(x,t) - 2v(x,t)\partial_x v(x,t), & (5.0.1) \\ U'(t) = \dfrac{P}{b} - \dfrac{\nu}{b^2}U(t) - \dfrac{1}{b^2}\displaystyle\int_0^b v^2(x,t)dx, & (5.0.2) \\ U(0) = U_0, \quad v(x,0) = v_0(x), \quad v(0,t) = v(b,t) = 0, & (5.0.3) \end{cases}$$

and the feedback control problem based on finitely many Fourier modes for (5.0.1)-(5.0.3)

$$\begin{cases} \partial_t \tilde{v}(x,t) = \dfrac{1}{b}\tilde{U}(t)\tilde{v}(x,t) + \nu\partial_x^2 \tilde{v}(x,t) - 2\tilde{v}\partial_x \tilde{v} & \\ -\mu\displaystyle\sum_{k=1}^{M}(\tilde{v}-v, w_k)w_k, & (5.0.4) \\ \tilde{U}'(t) = \dfrac{P}{b} - \dfrac{\nu}{b^2}\tilde{U}(t) - \dfrac{1}{b^2}\displaystyle\int_0^b \tilde{v}^2(x,t)dx, & (5.0.5) \\ \tilde{U}(0) = U_0, \quad \tilde{v}(x,0) = \tilde{v}_0(x), \quad \tilde{v}(0,t) = \tilde{v}(b,t) = 0, & (5.0.6) \end{cases}$$

where $(x,t) \in [0,b] \times [0,\infty)$, the constants $\nu > 0$ and P are given and represent the viscosity parameter and pressure, respectively and $\mu > 0$ is control parameter and $M \in \mathbb{Z}^+$ is the number of Fourier modes that we aim to find. The Fourier modes are $w_1, ..., w_M$, which are orthonormal (in $L^2(0,b)$-sense) eigenfunctions of the operator $-\partial_x^2$ under the homogeneous Dirichlet boundary conditions. They are explicitly given by

$$w_k(x) = \sqrt{\frac{2}{b}}\sin\left(\frac{k\pi}{b}x\right).$$

We denote the standard inner product on $L^2(0,b)$ by $(\cdot,\cdot)$. Thus, we can rewrite the control operator as follows

$$\begin{aligned}\mu\sum_{k=1}^{M}(\tilde{v}-v,w_k)w_k \\ &= \mu\sum_{k=1}^{M}\left(\int_0^b(\tilde{v}(x,t)-v(x,t))\sqrt{\frac{2}{b}}\sin\left(\frac{k\pi}{b}x\right)dx\right)\sqrt{\frac{2}{b}}\sin\left(\frac{k\pi}{b}x\right) \\ &= \frac{2}{b}\mu\sum_{k=1}^{M}\left(\int_0^b(\tilde{v}(x,t)-v(x,t))\sin\left(\frac{k\pi}{b}x\right)dx\right)\sin\left(\frac{k\pi}{b}x\right).\end{aligned}$$

2. In Section 5.2, we test our numerical approach on an example. We consider the dimensionless form of the original Burgers' equations on which we have carried out our theoretical study in the previous chapters. The initial-boundary value problem for the dimensionless form of the original Burgers' equations are given by

$$\begin{cases}\partial_t v(x,t) = U(t)v(x,t) + \partial_x^2 v(x,t) - 2v(x,t)\partial_x v(x,t), & (5.0.7)\\ U'(t) = R - U(t) - \int_0^1 v^2(x,t)dx, & (5.0.8)\\ U(0)=U_0, \quad v(x,0)=v_0(x), \quad v(0,t)=v(b,t)=0, & (5.0.9)\end{cases}$$

where $R=\frac{Pb^2}{\nu^2}$ is Reynold's number. In our numerical experiment, we set the initial conditions as

$$U(0)=1, \quad v(x,0)=\sin(\pi x). \tag{5.0.10}$$

3. In Section 5.3, we verify the validity of our theoretical result in Section 2.1 on a numerical example. We find smallest values of the control parameters μ and M that provide the solution of the controlled problem tends to the prescribed solution of the uncontrolled problem as time goes to infinity.

We present the MATLAB routines and output figures in the publicly open website `https://github.com/serapgumus/FeedbackControlForOriginalBurgers`.

5.1 Numerical Solutions of Original Burgers' Equations and the Feedback Control Problem

In this part, we describe the numerical methods employed. In order to solve the original Burgers' equations numerically, first we rewrite this coupled ODE-PDE system as a single equation. We represent the original Burgers' equations in the form:

$$\frac{du}{dt} = \mathcal{L}u + \mathcal{N}(u), \tag{5.1.1}$$

with the initial condition

$$u_0 = \begin{bmatrix} v_0(x) \\ U_0 \end{bmatrix},$$

where $u = \begin{bmatrix} v \\ U \end{bmatrix}$ and v, U are the solution of the original Burgers' equations (5.0.1)-(5.0.3). Here, $\mathcal{L}$ is the linear part of the problem given by

$$\mathcal{L} = \begin{bmatrix} \nu\partial_x^2 & 0 \\ 0 & -\frac{\nu}{b^2} \end{bmatrix}, \tag{5.1.2}$$

while the nonlinear part $\mathcal{N}(u)$ in (5.1.1) is

$$\mathcal{N}(u) = \begin{bmatrix} \frac{1}{b}Uv - 2v\partial_x v \\ \frac{P}{b} - \frac{1}{b^2}\|v(t)\|^2 \end{bmatrix}.$$

To solve the problem in (5.1.1), we use finite difference methods for the discretization in the space. More specifically, we use central difference for the second derivative and forward difference for the first derivative in order to construct the differentiation matrices. For the time discretization as in the paper by Lunasin and Titi [45], we use the *Exponential time-differencing fourth-order Runge-Kutta* method known as ETDRK4. This method is improved by Kassam and Trefethen [41] so that it is more stable against the rounding error by exploiting the ideas in [17]. Kassam and Trefethen [41] also provide the MATLAB codes based on the ETDRK4 method that solve the initial-boundary value problems involving Kuramoto-Sivashinsky and

Allen-Cahn equations. We modify the codes given in [41] to solve the original Burgers' equations and the feedback control problem associated with original Burgers' equations.

In order to solve the feedback control problem (5.0.4)-(5.0.6), we employ a similar approach. In particular, we represent the feedback control problem as

$$\frac{d\tilde{u}}{dt} = \mathcal{L}\tilde{u} + \hat{\mathcal{N}}(\tilde{u}), \quad \tilde{u}_0 = \begin{bmatrix} \tilde{v}_0(x) \\ \tilde{U}_0 \end{bmatrix}, \tag{5.1.3}$$

where $\tilde{u} = \begin{bmatrix} \tilde{v} \\ \tilde{U} \end{bmatrix}$, and $\tilde{v}$, $\tilde{U}$ denote the solutions of the feedback control problem in (5.0.4)-(5.0.6). The linear part $\mathcal{L}$ is the same as in (5.1.2). In the nonlinear part, we include the feedback control operator giving rise to

$$\hat{\mathcal{N}}(\tilde{u}) = \begin{bmatrix} \frac{1}{b}\tilde{U}\tilde{v} - 2\tilde{v}\partial_x\tilde{v} - \mu I(\tilde{v} - v) \\ \frac{P}{b} - \frac{1}{b^2}\|\tilde{v}(t)\|^2 \end{bmatrix},$$

where

$$I(\tilde{v} - v) = \sum_{k=1}^{M} (\tilde{v} - v, w_k) w_k.$$

We use *Composite Simpson Rule* in order to calculate the integrals in the $L^2(0,b)$-norms of v and $\tilde{v}$ and in the $L^2(0,b)$-inner product in the control operator. The MATLAB function named as **feedOrBurETDRK4.m** finds the solution of the feedback control problem.

Next, we present the numerical simulation results. As mentioned in the previous chapter, our main purpose is to find "best values" for the parameters μ and M, representing the control parameter and the number of Fourier modes, respectively. Here, by as "best values" we mean that the parameter values as small as possible yet that allow the solution of the feedback control problem to approach the prescribed solution as time goes to infinity, i.e., that allow the satisfaction of the condition

$$\lim_{t\to\infty} \|\tilde{v}(t) - v(t)\|^2 + |\tilde{U}(t) - U(t)|^2 = 0. \tag{5.1.4}$$

We use the parameters N, h, ν, R, b and t_{max} which denote the number of grid points in the space discretization, time step, viscosity constant, Reynolds number, the width of the channel i.e. the end point of the space interval domain $[0, b]$ and the right end of the time interval, respectively. The parameter values used in our experiments are indicated in Table 5.1.

N	h	ν	R	b	t_{max}
64	10	0.250	20	1	100

Table 5.1: The values of the parameters which are used in the numerical experiment in Section 5.1.

Remark 5.1.1. Let us note that instead of taking the pressure term P as an input parameter, we consider Reynold's number R as an input parameter since it gives information about the dynamics of the flow. We know that the pressure can be expressed in terms of Reynold's number as $P = \frac{R\nu^2}{b^2}$.

We take the following initial data

$$U_0 = 0.2, \quad v_0(x) = 0.4\sin(\pi x), \tag{5.1.5}$$

for the initial-boundary problem associated with the original Burgers's equations as in (5.0.1)-(5.0.3), and

$$\tilde{U}_0 = 0.3, \quad \tilde{v}_0(x) = 0.5\sin(\pi x), \tag{5.1.6}$$

for the feedback control problem in (5.0.4)-(5.0.6). In addition to these initial conditions and the fixed parameter values mentioned above, we set

$$\mu = 5 \text{ and } M = 15.$$

It turns out that, with these choices for μ and M, it is possible to make the solution of the feedback control problem to be very close to the prescribed solution at sufficiently large time, e.g., for $t \approx 100$.

In Figures 5.1 and 5.2, we present the solution of the initial-boundary value problem associated with the original Burgers' equation in (5.0.1)-(5.0.3), in other words the uncontrolled problem. Figure 5.1 plots v as a function of time and space, whereas Figure 5.2 plots U as a function of time.

In Figure 5.3, we observe that the solution $\tilde{v}$ of the feedback control problem approaches the solution v of the uncontrolled problem in the $L^2(0, b)$ sense as time increases. Similarly, Figure 5.4 illustrates that the solution $\tilde{U}$ of the feedback control problem approaches U, the solution associated with the uncontrolled problem as time increases.

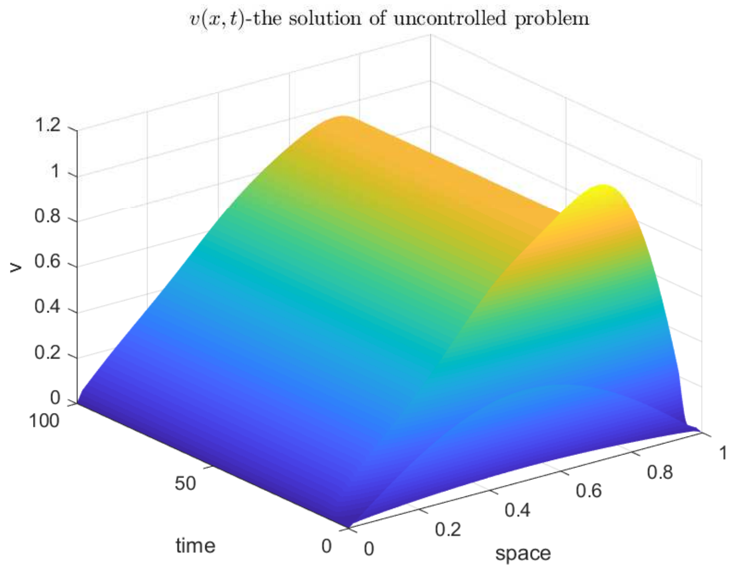

Figure 5.1: The plot of the solution $v(x, t)$ as a function of space (x) and time (t) for the uncontrolled problem (5.0.1)-(5.0.3).

5.2 Test for the Numerical Solution of the Original Burgers' Equations

In this part, we test our numerical approach on the dimensionless form of initial-boundary value problem associated with original Burgers' equations as in (5.0.7)-(5.0.9) and with the initial conditions in (5.0.10). The MATLAB codes for this part is available in the file **testForSolutionOfOriginalBurger.m**.

We adopt the numerical methods presented in Section 5.1, but we use different values for the parameters. We provide these in a table such as Table 5.2.

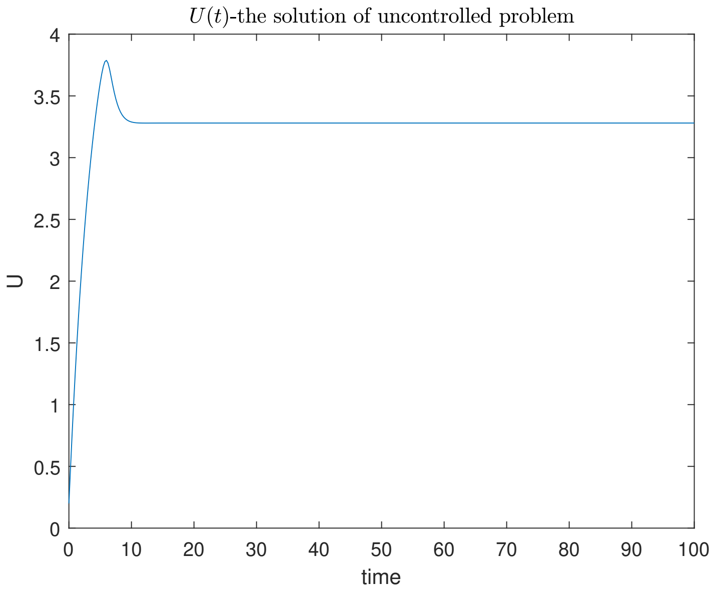

Figure 5.2: The plot of the solution $U(t)$ as a function of time for the uncontrolled problem (5.0.1)-(5.0.3).

N	h	ν	R	b	t_{max}
400	0.01	1	50	1	10

Table 5.2: The values of the parameters which are used in the numerical experiment in Section 5.2.

We add forcing terms $f(x,t)$ and $g(t)$ to equations (5.0.7) and (5.0.8), respectively, leading us to

$$\begin{cases} \partial_t v = U(t)v(x,t) + \partial_x^2 v(x,t) - 2v(x,t)\partial_x v(x,t) + f(x,t), & (5.2.1) \\ U'(t) = R - U(t) - \int_0^1 v^2(x,t)dx + g(t), & (5.2.2) \\ v(0,t) = v(1,t) = 0, \quad U(0) = 1, \quad v(x,0) = \sin(\pi x). & (5.2.3) \end{cases}$$

Let $v(x,t) = e^{-t}\sin(\pi x)$ and $U(t) = e^{-t}$ be an analytic solution for (5.2.1)-(5.2.3). Note that v satisfies the boundary and initial conditions, whereas U satisfies the initial condition. By plugging v and U into the equations (5.2.1) and (5.2.2), we

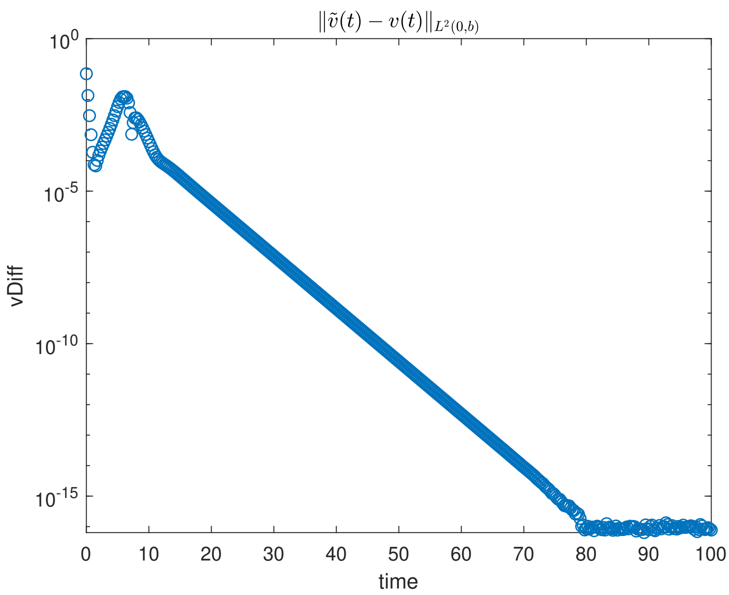

Figure 5.3: The L^2-norm of the difference between the solution $\tilde{v}$ of the feedback control problem and the solution v of the uncontrolled problem.

find

$$f(x,t) = e^{-t}\sin(\pi x)\left(2\pi cos(\pi x) - e^{-t} - 1 + \pi^2\right), \tag{5.2.4}$$

$$g(t) = \frac{1}{2}e^{-2t} - R. \tag{5.2.5}$$

Thus, $v(x,t) = e^{-t}\sin(\pi x)$ and $U(t) = e^{-t}$ are solutions of the (5.2.1) and (5.2.3) with $f(x,t)$, $g(t)$ as in (5.2.4)-(5.2.5).

Figure 5.5 illustrates the numerical approximation for the exact solution $v(x,t)$ by our numerical method, while in Figure 5.6, we plot the exact solution $v(x,t) = e^{-t}\sin(\pi x)$. We also plot the exact solution $U(t)$ and its numerical approximation in Figure 5.7. Here, we see that there is a slight difference between the numerical and exact solution.

Figure 5.8 provides us the plot of the absolute error between $U(t)$ and its numerically computed counterpart. Since solutions start with the same initial condition, initially there is no error. We observe that the absolute error increases up to the 3.6×10^{-3} at time $t = 1$. For $t > 1$, we see that the error decreases.

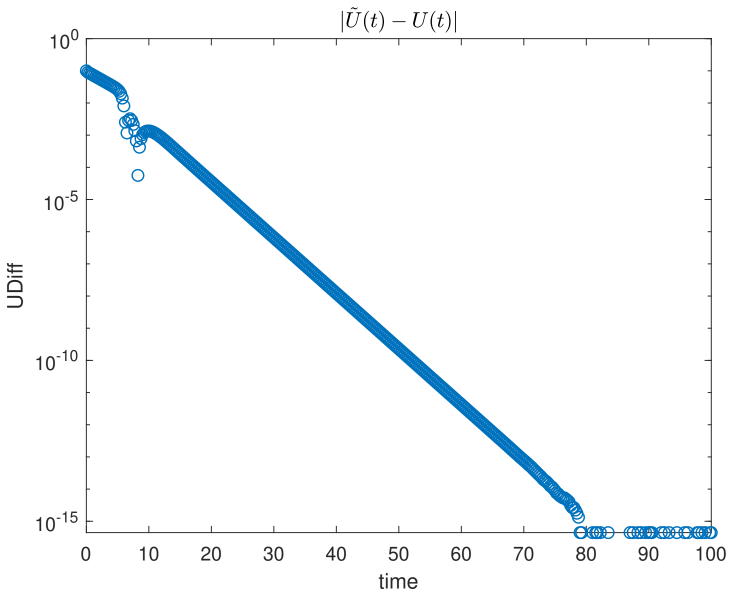

Figure 5.4: The plot of $|\tilde{U}(t) - U(t)|$ as a function of time (t) where $\tilde{U}$, U are the solutions associated with the feedback control problem and uncontrolled problem, respectively.

5.3 *Test Results for Problem Parameters*

In this section, we aim to demonstrate numerically on an example that our theoretical results in Section 2.1.1 are valid. In particular, our purpose is to find the smallest values of the control parameters μ and M that satisfy (5.1.4) up to a prescribed tolerance. We consider problems (5.0.1)-(5.0.3) and (5.0.4)-(5.0.6) under the initial data in (5.1.5) and (5.1.6), respectively. We focus on the change in the L^2-norm of the difference between the solutions, $\tilde{v}$ and v, as one of the problem parameters changes. The MATLAB codes for this section are in the files **testForParameter.m**, **testForControlParameters.m** and **testResults.m**. The MATLAB function **testForParameter.m** takes one parameter out of 8 problem parameters as an input and calculates the relative error between the solutions $\tilde{v}$ and v in the L^2-norm sense, i.e. we calculate

$$v_{error} = \frac{\|\tilde{v}(t) - v(t)\|_{L^2(0,b)}}{\|\tilde{v}(t)\|_{L^2(0,b)}}. \tag{5.3.1}$$

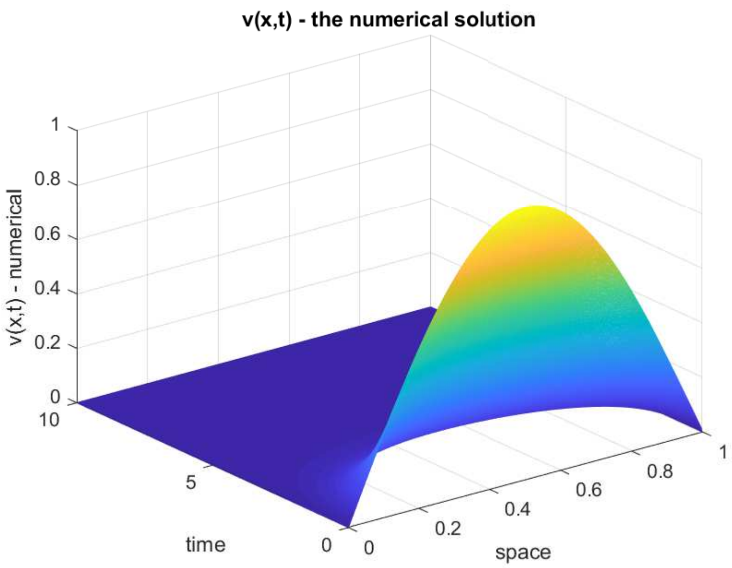

Figure 5.5: Plot of the numerical approximation for the exact solution $v(x,t)$ of (5.2.1)-(5.2.3).

The routine **testResults.m** plots the graph of the relative error as the chosen parameter varies and other parameters take the values in Table 5.3.

μ	M	ν	R	N	b	h	t_{max}
3.5	15	0.25	50	200	1	0.01	10

Table 5.3: The values of the parameters that we use in Section 5.3.

We present the results in Figures 5.9 and 5.10, in particular plot the graph of the relative error between $\tilde{v}$ and v with respect to the problem parameters. We let one of the parameters vary, for instance the control coefficient μ, fix the other 7 parameters in Table 5.3 and calculate the relative error (5.3.1) for various μ values. The plots of the relative error are depicted as a function of the control operator coefficient μ, the Fourier modes in the control operator M, the viscosity parameter ν, and the Reynold's number R in Figure 5.9, and the number of space discretization points

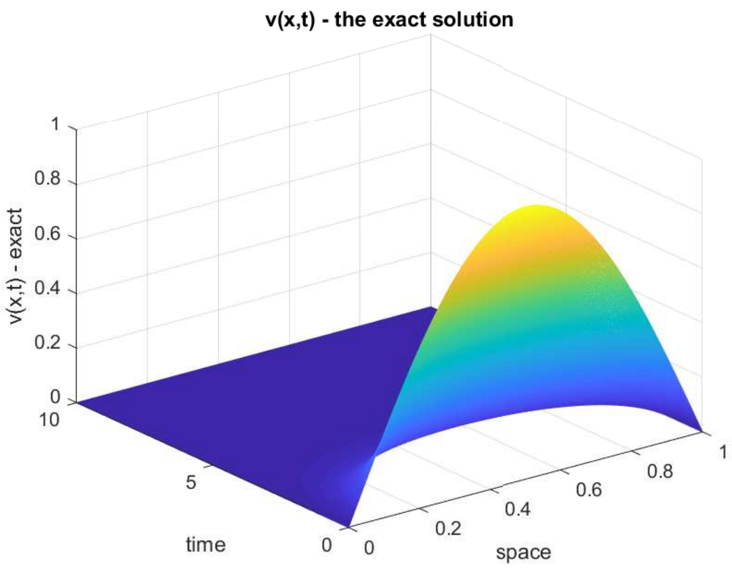

Figure 5.6: Plot of the exact solution $v(x,t)$ of (5.2.1)-(5.2.3).

N, the width of the channel b, the time step h, and the maximum time value t_{max} in Figure 5.10.

Observe that there are some parameter values that the relative error takes the minimum value. In Table 5.4, the minimum value of the relative error is stated together with the corresponding the parameter value. Since some parameters are correlated and affect each other, for example the viscosity term ν and Reynold's number R, choosing all the minimal parameter values as in Table 5.4 do not give us the smallest relative error. In fact, in such a case equations are not well posed. To overcome this issue, we set the parameters except the control parameters μ and M as in Table 5.5. Here, we set the maximal time value to $t_{max} = 37$ since the relative error, attains its minimal value already at $t_{max} = 37$ in Figure 5.10. In our numerical experiment, the admissible values for the control parameters μ are in the interval $(0, 10]$ and for M in the set $\{1, 2, ..., 30\}$. We present the result of this numerical experiment in Table 5.6.

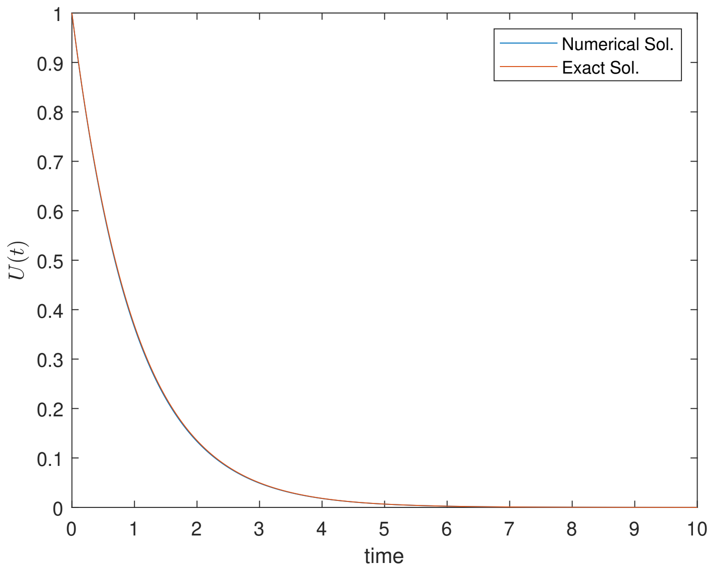

Figure 5.7: Plots of the exact solution $U(t)$ for (5.2.1)-(5.2.3), as well as its numerically computed counterpart.

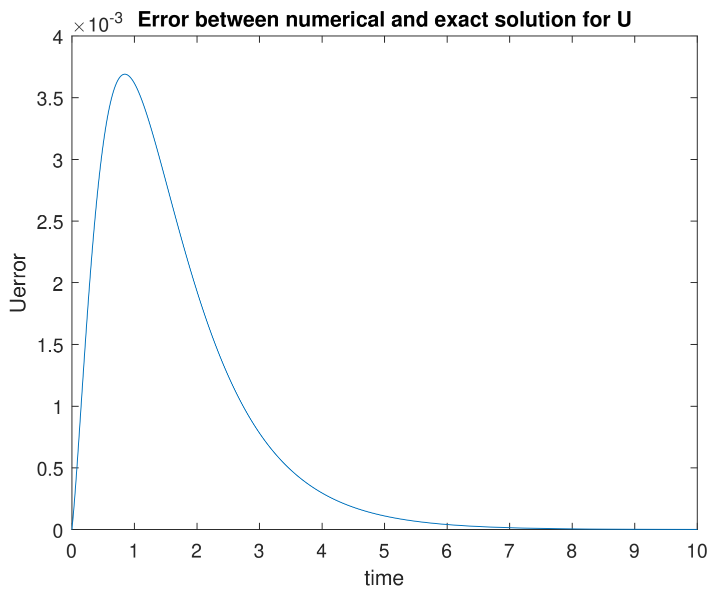

Figure 5.8: The plot of the absolute error $U_{error} = |U_{exact} - U_{num}|$ where $U(t)$ is the exact solution of (5.2.1)-(5.2.3), while $U_{num}(t)$ is its numerically computed counterpart.

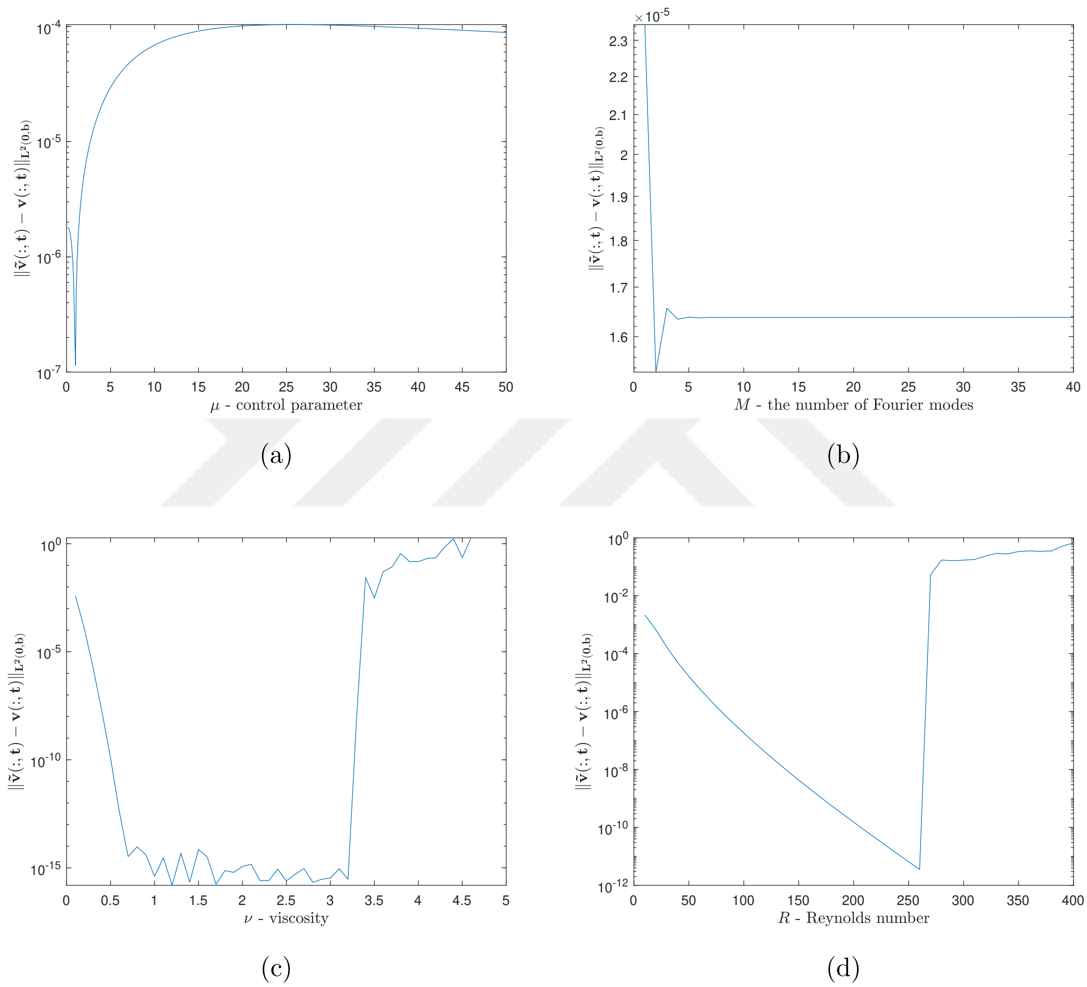

Figure 5.9: The figure depicts how the relative error (5.3.1) varies with respect to the problem parameter; in particular with respect to the control operator coefficient μ, the Fourier modes in the control operator M, the viscosity parameter ν, and the Reynold's number R.

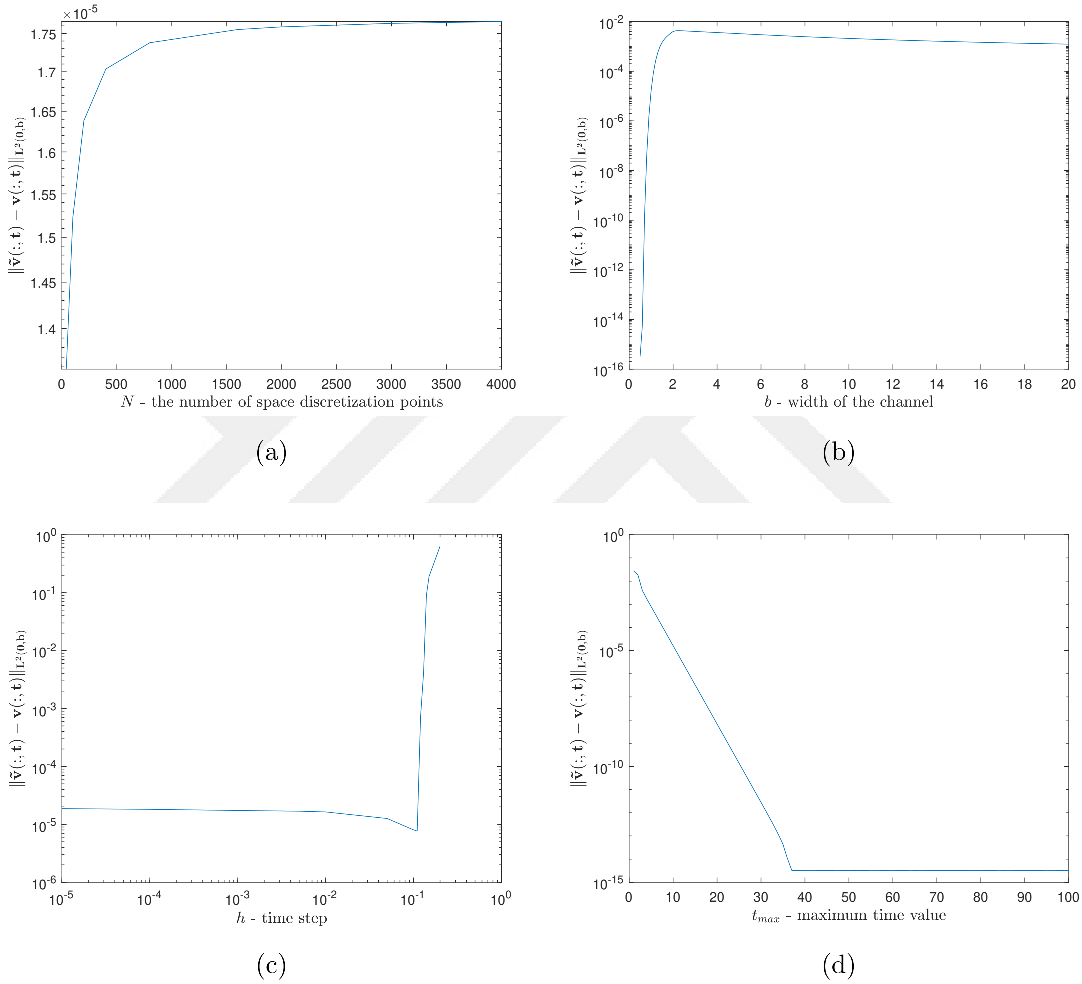

Figure 5.10: The figure depicts how the relative error (5.3.1) varies with respect to the problem parameters; in particular with respect to the number of space discretization points N, the width of the channel b, the time step h, and the maximum time value t_{max}.

Parameter value	$\min v_{error}$
$\mu = 1$	$1.1451e - 07$
$M = 2$	$1.5319e - 05$
$\nu = 1.2$	$1.5479e - 16$
$R = 260$	$3.5177e - 12$
$N = 40$	$1.3577e - 05$
$b = 0.5$	$3.2359e - 16$
$h = 0.1$	$8.0230e - 06$
$t_{max} = 37$	$3.2584e - 15$

Table 5.4: The minimum values of the relative error (5.3.1) with the corresponding parameter values based on the results in Figures 5.9 and 5.10.

ν	R	N	b	h	t_{max}
0.25	50	200	1	0.01	37

Table 5.5: The fixed parameter values that we use in the search of the control parameters μ and M which satisfy (5.1.4).

μ	M	$\min v_{error}$
1	2	$1.6763e - 15$

Table 5.6: The control parameter values μ and M which satisfy (5.1.4) for the solutions of the problems (5.0.1)-(5.0.3) and (5.0.4)-(5.0.6) under the initial data (5.1.5) and (5.1.6) with the predetermined parameter values in Table 5.5.

Chapter 6

CONCLUSION AND FUTURE WORK

In this thesis, we have analyzed the stabilization of various initial-boundary value problems for original Burgers' equations and Burgers' equation with nonlocal nonlinearity.

In Chapter 2, we have focused on the feedback control stabilization problems. First, we have considered the feedback control problem for original Burgers' equations (1.0.4) by using finitely many determining parameters such as finitely many Fourier modes, general interpolant operator and finitely many volume elements. We have shown that under appropriate choices of the control parameters, the solution of the control problem tends to the prescribed solution of the uncontrolled problem with an exponential decay rate as time goes to infinity. For the second equation in (1.0.5), we have shown the global feedback stabilization by employing finitely many Fourier modes, finitely many volume elements and finitely many nodal values. We have also obtained that the stabilization of the solutions of (1.0.5) has an arbitrary exponential decay rate, $e^{-\sigma t}$.

In Chapter 3, we have proved that the asymptotic behavior of the solutions of original Burgers' equations with an additional forcing term (dependent on both space and time variables), can be determined completely by determining modes.

In Chapter 4, we have shown the existence and uniqueness of the solutions and analyzed the stabilization of the inverse source problems constructed for original Burgers' equations (1.0.4) and Burgers' equation with nonlocal nonlinearity (1.0.5). We have found the necessary and sufficient conditions on the given terms, which satisfy the stabilization of the solutions. In addition to the stability analysis, we have also shown the existence and uniqueness of the solutions for both inverse source problems.

Finally, in Chapter 5 we have presented some numerical experiments related to the feedback control stabilization of the original Burgers' equation (1.0.4). We have considered the feedback control problem based on the finitely many Fourier modes. First, we have found the numerical approximate solutions for both uncontrolled and controlled problems and tested our numerical approach on an example. We have compared these solutions and determined the relative error. Moreover, we have analyzed the behavior of the relative error as the problem parameters change.

Future Research

One of the significant subjects of the control theory for PDEs is *boundary control*. Byrnes et al. [6–9] studied a boundary control problem for the classical viscous Burgers' equation with Neumann boundary conditions. They proved that if the initial data is small enough, then the solution tends to zero with an exponential decay rate. Ly et a. [46] studied the same boundary control problem and showed that the results of Byrnes et al. [6–9] are valid for a larger set of initial data. Motivated by these studies, we plan to analyze the global behavior of solutions of non-controlled OBEs (1.0.4) under Neumann boundary conditions and the solutions of the controlled problem with the following boundary control:

$$\begin{cases} \partial_x v(0,t) - k_0 v(0,t) = 0, \\ \partial_x v(b,t) + k_1 v(b,t) = 0, \end{cases} \tag{6.0.1}$$

where $k_0, k_1 > 0$. In addition to theoretical studies, we also plan to perform computational studies and support our theoretical results by numerical experiments.

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