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BOUNDED AND COMPACT LINEAR OPERATORS ON GENERAL MIXED NORM SPACES

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BOUNDED AND COMPACT LINEAR OPERATORS ON GENERAL MIXED NORM SPACES

by

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APPROVAL PAGE

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ABSTRACT

We study some topological properties of the spaces of sequences that are strongly Cesàro bounded, convergent and convergent to zero, of order $\alpha > 0$ and index $p \geq 1$. By using our software we obtain graphical representations of their surface energy functions. Then we determine their β–duals and the shapes of corresponding Wulff's crystals. Furthermore we characterize some new classes of matrix transformations on them. Finally, we find out identities and estimates for the Hausdorff measure of noncompactness of the matrix operators in those classes, and characterize the corresponding classes of compact matrix operators.

Keywords: Strong summability and boundedness, BK spaces, β-duals, matrix transformations, Hausdorff measure of noncompactness, compact operators, visualization, Wulffs crystals.

GENELLEŞTİRİLMİŞ BİRLEŞİK NORMLU UZAYLAR ÜZERİNDEKİ SINIRLI VE KOMPAKT OPERATÖRLER

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ÖZ

$\alpha > 0$ mertebeli ve $p \geq 1$ indeksli kuvvetli Cesàro sınırlı, yakınsak ve 0'a yakınsak dizi uzaylarının bazı topolojik özelliklerini belirledik. Kendi yazılımımızı kullanarak yüzey enerji fonksiyonlarının grafiksel gösterimlerini elde ettik. Daha sonra, bu uzayların $\beta-$duallerini ve onlara karşılık gelen Wulff kristallerinin şekillerini bulduk. Ayrıca, bu dizi uzayları üzerinde bazı yeni matris dönüşümlerinin sınıflarını karakterize ettik. Son olarak, bu sınıflar üzerindeki matris operatörlerinin Hausdorff kompakt olmama ölçümleri için özdeşlikler ve hesaplamalar bulup, onlara karşılık gelen kompakt matris operatör sınıflarını karakterize ettik.

Keywords: Kuvvetli toplanabilme ve sınırlılık, BK uzayları, β-dualleri, matris dönüşümleri, Hausdorff kompakt olmama ölçümü, kompakt operatörler, görüntüleme, Wulff kristalleri.

To my family

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SYMBOL/ABBREVIATION

$\mathbb{C}$	set of complex numbers
$\mathbb{R}$	set of real numbers
$\mathbb{Z}$	set of integers
$\mathbb{N}$	set of natural numbers
$\mathbb{N}_0$	set of natural numbers including 0
w	set of all sequences with complex entries
ϕ	set of finite sequences
ℓ_∞	space of bounded sequences
c	space of convergent sequences
c_0	space of null sequences
ℓ_p	space of absolutely p-summable sequences
bs	space of bounded series
cs	space of convergent series

CHAPTER 1

INTRODUCTION

Strong Cesàro summability of order $\alpha > 0$ with index $p > 0$, denoted by $[C_\alpha]^p$, was defined and studied by Hyslop (Hyslop, 1952), and further studied and generalized by Borwein (Borwein, 1960). The extension of summability $[C_\alpha]^p$ to the case $\alpha = 0$ is referred to as strong convergence of index p; matrix transformations on the spaces $[C_0]^p$ were characterized in (Kuttner and Thorpe, 1979). The definition of strong convergence of index $p = 1$ was extended by Mòricz (Mòricz, 1989) to Λ–strong convergence, denoted by $c(\Lambda)$. Spaces of Λ–strongly convergent sequences and related spaces, their dual spaces and matrix transformations on them were studied in detail in (Malkowsky, 1995); (Malkowsky, 2013). The results of those papers were generalized to the case $p > 1$ in (Malkowsky, 2002); (Malkowsky et al., 2004).

In this thesis we establish some fundamental topological properties of the spaces of sequences that are strongly Cesàro bounded, convergent and convergent to zero, of order $\alpha > 0$ and index $p \geq 1$, determine their β–duals, and characterize some classes of matrix transformations and compact matrix operators on them. We consider these spaces as the domains of the Cesàro matrix $C_{\alpha-1}$ in the spaces of Maddox defined in (Maddox, 1968).

Our results are complementary to those in (Kuttner and Thorpe, 1979) and no estimates of the Hausdorff measure of noncompactness and characterizations of compact matrix operators have been established on the spaces of sequences that are strongly Cesàro bounded, convergent and convergent to zero, of order $\alpha > 0$ and index $p \geq 1$. We use the theories of BK spaces and measures of noncompactness, in particular, the Hausdorff measure of noncompactness, and techniques from (Malkowsky and Rakočević, 2007); (Malkowsky and Rakočević, 2000a);

(Djolović and Malkowsky, 2008); (Başar and Malkowsky, 2011) in the proofs.

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex entries, X and Y be subsets of ω and $x \in \omega$. We write $A_n = (a_{nk})_{k=0}^{\infty}$ for the sequence in the nth row of the matrix A and $A^k = (a_{nk})_{n=0}^{\infty}$ for the sequence in the kth column of A. If each of the series $A_n x = \sum_{k=0}^{\infty} a_{nk} x_k$ converges, then the sequence $Ax = (A_n x)_{n=0}^{\infty}$ is called the **A-transform of x**. The sets $X_A = \{x \in w : Ax \in X\}$ and $M(X,Y) = \{a \in w : a \cdot x = (a_k x_k)_{k=0}^{\infty} \in Y \text{ for all } x \in X\}$ are called the **matrix domain of** A **in** X and the **multiplier space of** X **in** Y, respectively; in particular, $X^{\beta} = M(X, cs)$ and $X^{\gamma} = M(X, bs)$ are called the $\beta-$ and $\gamma-$**duals of** X. Also, (X,Y) is the class of all matrices A such that $X \subset Y_A$; so $A \in (X,Y)$ if and only if $A_n \in X^{\beta}$ for all $n \in \mathbb{N}_0$ and $Ax \in Y$ for all $x \in X$. Let e and $e^{(k)}$ $(k = 0, 1, ...)$ be the sequences with $e_n = 1$ for all $n \in \mathbb{N}_0$, and $e_k^{(k)} = 1$ and $e_n^{(k)} = 0$ for $n \neq k$.

Let $\delta \in \mathbb{R}$. Then the numbers $A_n^{\delta} = \binom{n+\delta}{n}$ for $n = 0, 1, \ldots$ are called the n**th Cesàro coefficients of order** δ. For $\alpha > -1$ the **Cesàro matrix** $C_{\alpha} = (a_{nk})_{n,k=0}^{\infty}$ **of order** α is defined by

$$a_{nk} = \begin{cases} \dfrac{A_{n-k}^{\alpha-1}}{A_n^{\alpha}} & (0 \leq k \leq n) \\ 0 & (k > n) \end{cases} \qquad (n = 0, 1, \ldots);$$

and the n**th** C_{α} **mean of a sequence** $x = (x_k)_{k=0}^{\infty}$ is defined by

$$\sigma_n^{\alpha}(x) = \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} x_k.$$

Let ξ be a complex number. Then the sequence $x = (x_k)_{k=0}^{\infty}$ is said to be **summable** C_{α} **to** ξ if $\lim_{n\to\infty} \sigma_n^{\alpha}(x) = \xi$ for $\alpha > 0$, it is said to be **strongly summable** C_{α} **to zero**, **strongly summable** C_{α} **to** ξ (Hyslop, 1952), and **strongly bounded** C_{α}**, with index** $p > 0$, respectively, if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left|\sigma_k^{\alpha-1}(x)\right|^p = 0, \tag{1.1}$$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left|\sigma_k^{\alpha-1}(x) - \xi\right|^p = 0 \tag{1.2}$$

and

$$\sup_n \frac{1}{n+1} \sum_{k=0}^{n} \left|\sigma_k^{\alpha-1}(x)\right|^p < \infty. \tag{1.3}$$

We write $[C_\alpha]_0^p$, $[C_\alpha]^p$ or $[C_\alpha]_\infty^p$ for the sets of all sequences $x \in \omega$ for which (1.1), (1.2) or (1.3) are satisfied, respectively. In the special case of $\alpha = 1$, we obviously obtain $[C_1]_0^p = w_0^p$, $[C_1]^p = w^p$ and $[C_1]_\infty^p = w_\infty^p$, respectively, the sets of all sequences that are strongly summable C_1 to zero, strongly summable C_1 and strongly bounded C_1, with index p (Maddox, 1968). So we have $[C_\alpha]_0^p = (w_0^p)_{C_{\alpha-1}}$ and $[C_\alpha]_\infty^p = (w_\infty^p)_{C_{\alpha-1}}$, and since $\sigma_n^{\alpha-1}(x) - \xi = \sigma_n^{\alpha-1}(x - \xi \cdot e)$ for all $n \in \mathbb{N}_0$, we also obtain $[C_\alpha]^p = (w^p)_{C_{\alpha-1}}$, that is, $[C_\alpha]_0^p$, $[C_\alpha]^p$ and $[C_\alpha]_\infty^p$ are the matrix domains of the triangles $C_{\alpha-1}$ in w_0^p, w^p and w_∞^p and $[C_\alpha]^p = [C_\alpha]_0^p \oplus e$.

A matrix $T = (t_{nk})_{n=0}^\infty$ is called a **triangle** if $t_{nn} \neq 0$ for all $n \in \mathbb{N}_0$ and $t_{nk} = 0$ for $k > n$ and also every triangle has a unique inverse S which is also a triangle, and $T(Sx) = (TS)x = x$ for all $x \in \omega$ (Wilansky, 1984, Theorem 1.4.8) and (Cooke, 1950, Remark 22 (a)). So the inverse matrix $S^{\alpha-1} = (s_{nk})_{n,k=0}^\infty$ of the Cesàro matrix $C_{\alpha-1}$ of order $\alpha > 0$ is given by

$$s_{nk} = \begin{cases} A_{n-k}^{-\alpha} A_k^{\alpha-1} & (0 \leq k \leq n) \\ 0 & (k > n) \end{cases} \quad (n = 0, 1, \ldots). \tag{1.4}$$

We denote by $R^{\alpha-1} = (r_{n,k})_{n,k=0}^\infty$ the transpose of the matrix $S^{\alpha-1}$.

There are six chapters in this thesis.

Chapter 2 deals with the general theory of FK, BK, and AK spaces and measures of noncompactness. Most of the results of this chapter can be found in (Wilansky, 1984) and (Malkowsky and Rakočević, 2000a).

In Chapter 3 we investigate some topological properties of the spaces $[C_\alpha]_0^p$, $[C_\alpha]^p$ or $[C_\alpha]_\infty^p$ by using blocking technique and determine their β–duals. Also we visualise the norm and the dual norm on these spaces as potential surface and Wulff's crystal for different parameters.

Chapter 4 deals with the characterization of some classes of matrix transformations and the norms of operators defined by the matrices in those classes.

In Chapter 5 we investigate Hausdorff measure of noncompactness of matrix operators in classes we studied in chapter 4.

Chapter 6 is devoted to a conclusion.

CHAPTER 2

THE GENERAL THEORY

2.1 FK, BK, AND AK SPACES

In this section we introduce briefly the theory of FK spaces which plays an important role in the characterisation of matrix transformations between sequence spaces. We start with some definitions.

Definition 1. Let X be a linear space and d a metric on X. Then (X, d), or X for short, is said to be a **linear metric space** if the algebraic operations on X are continuous functions. A complete linear metric space is called a **Fréchet space**.

A linear metric space has an algebraic structure by linearity, however a topological structure by the metric. The continuity of algebraic operations in a linear metric space X means that if $d(x_n, x) \to 0$, $d(y_n, y) \to 0$ and $\lambda_n \to \lambda$ as $n \to \infty$, then $d(x_n + y_n, x + y) \to 0$ and $d(\lambda_n x_n, \lambda x) \to 0$ as $n \to \infty$.

Theorem 2.1.1. *(Maddox, 1970, Exercise2, p.86) The set w is a Fréchet space with respect to the metric d_w defined by*

$$d_\omega(x, y) = \sum \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|} \text{ for all } x, y \in \omega. \tag{2.1}$$

Furthermore convergence in (w, d_w) and coordinatewise convergence are equivalent, that is $x^{(n)} \to x$ as $n \to \infty$ in (w, d_w) if and only if $x_k^{(n)} \to x_k$ as $n \to \infty$ for every k.

Definition 2. A **topological space** $(X, \mathcal{T})$ is a pair consisting of a non-empty set X and a class $\mathcal{T}$ of subsets of X satisfying the following axioms:

($\mathcal{T}1$) The empty set $\emptyset$ and X are in $\mathcal{T}$,

($\mathcal{T}2$) Any union (countable or uncountable) of sets in $\mathcal{T}$ is in $\mathcal{T}$,

($\mathcal{T}3$) The intersection of any finite number of sets in $\mathcal{T}$ is in $\mathcal{T}$.

The sets of $\mathcal{T}$ are called open sets and $\mathcal{T}$ is called a topology for X.

Definition 3. A topological space $(X, \mathcal{T})$ is called **Hausdorff** if and only if, for any x, y in X with $x \neq y$, there exists two disjoint open sets, one containing x and the other containing y.

Definition 4. Let H be a linear space and a Hausdorff space. An **FH space** is a Fréchet space X such that X is a subspace of H and the topology of X is stronger than the topology of H on X.

Definition 5. A subset X of ω is said to be an **FK space** if it is a Fréchet space with continuous coordinates $P_n : X \to \mathbb{C}$ $(n = 0, 1, \dots)$, where $P_n(x) = x_n$ for all $x = (x_k)_{k=0}^{\infty} \in X$. In other words, an **FK space** is an FH space with $H = w$. An FK space is said to be a **BK space** if its metric is given by a norm. An FK space $X \supset \phi$ is said to have **AK**, or be an AK space, if every sequence $x = (x_k)_{k=0}^{\infty} \in X$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$ (Wilansky, 1984, 4.2.13). A sequence $(b_n)_{n=0}^{\infty}$ in a linear metric space X is called a **Schauder basis** if, for every $x \in X$ there exists a unique sequence $(\lambda_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} \lambda_n b_n$.

Theorem 2.1.2. *(Wilansky, 1984, Theorem 4.2.2) Let X be a Fréchet space, Y be an FH space and $f : X \to Y$ be linear. Then, $f : X \to H$ is continuous if and only if $f : X \to Y$ is continuous.*

Proof. Let $\mathcal{T}_X$, $\mathcal{T}_Y$ and $\mathcal{T}_H$ denote the topologies on X, Y and of H on Y. First, we assume that $f : X \to (Y, \mathcal{T}_Y)$ is continuous. Since Y is an FH space, we have $\mathcal{T}_H \subset \mathcal{T}_Y$, and so $f : X \to (Y, \mathcal{T}_H)$ is continuous. Conversely, we assume that $f : X \to (Y, \mathcal{T}_H)$ is continuous. Then it has closed graph by the closed graph lemma (see appendix B.0.3). Since Y is an FH space, we again have $\mathcal{T}_H \subset \mathcal{T}_Y$, and so $f : X \to (Y, \mathcal{T}_Y)$ has closed graph. Consequently $f : X \to (Y, \mathcal{T}_Y)$ is continuous by the closed graph theorem (see appendix B.0.4). ∎

Corollary 2.1.3. *(Wilansky, 1984, Corollary 4.2.3) Let X be a Fréchet space, Y be an FK space, $f : X \to Y$ be linear, and the coordinates $P_n : X \to \mathbb{C}$ $(n = 0, 1, \dots)$*

be defined by $P_n(x) = x_n$ *for all* $x \in X$*. If* $P_n \circ f : X \to \mathbb{C}$ *is continuous for every* n*, then* $f : X \to Y$ *is continuous.*

Proof. Since convergence and coordinatewise convergence are equivalent in ω by Theorem 2.1.1, the continuity of $P_n \circ f : X \to \mathbb{C}$ for all n implies the continuity of $f : X \to \omega$, hence of $f : X \to Y$ by Theorem 2.1.2. ∎

Remark 2.1.1. (Malkowsky and Rakočević, 2000a, Remark 1.16) Let $X \supset \phi$ be an FK space and $a \in w$. If the series $\sum_{k=0}^{\infty} a_k x_k$ converges for all $x \in X$, then the linear functional f_a defined by $f_a(x) = \sum_{k=0}^{\infty} a_k x_k$ for all $x \in X$ is continuous.

Proof. We define linear functionals $f_a^{[n]} : X \to \mathbb{C}$ for all $n \in \mathbb{N}_0$ by $f_a^{[n]}(x) = \sum_{k=0}^{n} a_k x_k$ for all $x \in X$. Since X is an FK space and $f_a^{[n]}$ is a finite linear combination of coordinates, we have $f_a^{[n]} \in X'$ for all n. By hypothesis, the limits $f_a(x) = \lim_{n\to\infty} f_a^{[n]}(x)$ exist for all $x \in X$, hence $f_a \in X'$ by the Banach-Steinhaus theorem (see appendix B.0.5). ∎

Theorem 2.1.4. *(Wilansky, 1984, Theorem 4.2.8) Any matrix map between* FK *spaces is continuous.*

Proof. Let X and Y be FK spaces, $A \in (X, Y)$ and $f_A : X \to Y$ be defined by $f_A(x) = Ax$ for all $x \in X$. Since the maps $P_n \circ f_A : X \to \mathbb{C}$ are continuous for all n by Remark 2.1.1, $f_A : X \to Y$ is continuous by Corollary 2.1.3. ∎

2.2 MEASURES OF NONCOMPACTNESS

In the previous section we notice that matrix transformations between FK spaces are continuous. To characterize the classes of compact matrix transformations we apply the Hausdorff measure of noncompactness. For this reason in this section we will give the axiomatic introduction to measures of noncompactness on bounded sets in complete metric spaces with their most important properties.

The first measure of noncompactness, denoted by α, defined by Kuratowski (Kuratowski, 1930) in 1930. Then, Darbo (Darbo, 1955) used this measure to prove a generalization of Schauder's fixed point theorem (Darbo, 1972). In 1957, the

Hausdorff measure of noncompactnes was introduced by Goldenstein, Gohberg and Markus (Goldenstein et al., 1957) and later studied by Goldenstein and Markus (Goldenstein and Markus, 1965). There are also other measures of noncompactness defined by several authors. They are studied in detail in the monographs (Akhmerov et al., 1986); (Toledano et al., 1997); (Istrătescu, 1981); (Kuratowski, 1958); (Malkowsky and Rakočević, 2000a). Rather than to investigate each of them, here we give the concept of a measure of noncompactness on bounded sets of a metric space and introduce the Kuratowski and Hausdorff measures of noncompactness.

We need some standard notations.

Let (X, d) be a metric space. Then for any $r > 0$ and $x \in X$, the sets $B(x,r) = \{y \in X : d(x,y) < r\}$, $\overline{B}(x,r) = \{y \in X : d(x,y) \leq r\}$ and $S(x,r) = \{y \in X : d(x,y) = r\}$ are the **open, closed balls** and **sphere, with centre x and radius r**, respectively; in particular, we write $B_X = B(0,1)$, $\overline{B}_X = \overline{B}(0,1)$ and $S_X = S(0,1)$ for the **open** and **closed unit balls**, and **unit sphere in X**. If S and S' are subsets of a metric space (X, d) and $x \in X$, then $d(x, S) = \inf\{d(x,s) : s \in S\}$, $d(S, S') = \inf\{d(s,s') : s \in S, s' \in S'\}$ and $diam(S) = \sup\{d(s,\tilde{s}) : s, \tilde{s} \in S\}$ are called the **distance of x and S, distance of S and S'** and **diameter of S**, respectively. We denote the set of all nonempty and bounded subsets of a metric space (X, d) by $\mathcal{M}_X$.

Now we recall some useful definitions.

Let M and S be subsets of a metric space (X, d) and $\epsilon > 0$, then the set S is called **ϵ-net** of M if for any $x \in M$ there exists $s \in S$, such that $d(x,s) < \epsilon$. If the set S is finite then the ϵ-net S of M is called **finite ϵ-net of M**. The set M is said to be **totally bounded** if it has a finite ϵ-net for every $\epsilon > 0$. A subset M of a metric space X is **compact** if every sequence (x_n) in M has a convergent subsequence with its limit in M and the set M is **relatively compact** if the closure $\bar{M}$ of M is compact.

If X and Y are Banach spaces and $L : X \to Y$ is a linear operator, then by $\mathcal{B}(X,Y)$ we denote the set of all bounded linear operators from X to Y and L is said to be **compact** or **completely continuous**, if its domain is all of X and, for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y.

First we consider the concept of a measure of noncompactness of bounded sets in complete metric space.

Definition 6. Let (X,d) be a complete metric space. A set function

$$\phi : \mathcal{M}_X \to [0,+\infty]$$

is called a **measure of noncompactness on** X if it satisfies the following conditions

- **(MNC.1)** $\phi(M) = 0$ if and only if M is a relatively compact set
  (regularity)
- **(MNC.2)** $\phi(M) = \phi(\overline{M})$, for all $M \in \mathcal{M}_X$
  (invariance under closure)
- **(MNC.3)** $\phi(M_1 \cup M_2) = \max\{\phi(M_1), \phi(M_2)\}$, for all $M_1, M_2 \in \mathcal{M}_X$
  (semi-additivity).

The number $\phi(M)$ is called the **measure of noncompactness of the set M**.

Any measure of noncompactness satisfies the following fundamental properties that are immediate consequences of Definition 6.

Lemma 2.2.1. *(Toledano et al., 1997, p. 19) Let* ϕ *be a measure of noncompactness on a complete metric space* (X,d)*. Then* ϕ *satisfies the following properties:*

1. $M_1 \subset M$ *implies* $\phi(M_1) \leq \phi(M)$
   (monotonicity)
2. $\phi(M_1 \cap M_2) \leq \min\{\phi(M_1), \phi(M_2)\}$*, for all* $M_1, M_2 \in \mathcal{M}_X$
3. *If* M *is finite then* $\phi(M) = 0$
   (non-singularity).
4. *Generalized Cantor's intersection theorem: If* $\{M_n\}$ *is a decreasing sequence of nonempty, closed and bounded subsets of* X *and* $\lim_{n\to\infty} \phi(M_n) = 0$*, then the intersection* M_∞ *of all* M_n *is nonempty and compact.*

Now we give the definition of the Kuratowski measure of noncompactness.

Definition 7. Let (X, d) be metric space, and $Q \in \mathcal{M}_X$. Then the **Kuratowski measure of noncompactness of** Q, denoted by $\alpha(Q)$, is the infimum of the set of all numbers $\epsilon > 0$ such that Q can be covered by a finite number of sets with diameters less than ϵ, that is,

$$\alpha(Q) = \inf\{\epsilon > 0 : Q \subset \bigcup_{i=1}^{n} S_i, S_i \subset X, diam(S_i) < \epsilon \, (i = 1, 2, ..., n; n \in \mathbb{N})\}. \quad (2.2)$$

Remark 2.2.1. (a) The Kuratowski measure of noncompactness is a measure of noncompactness in the sense of Definition 7, that is, it satisfies the axioms of regularity, invariance under closure and semi–additivity (Malkowsky and Rakočević, 2000a, Lemma 2.6).

(b) It is obvious that

$$\alpha(Q) \leq diam(Q) \text{ for each bounded subset } Q \text{ of } X. \quad (2.3)$$

It turns out that in an infinite–dimensional normed space the Kuratowski measure of noncompactness of the unit ball is equal to its diameter, that is equality holds in (2.2) for $Q = B_X$.

Theorem 2.2.2. *(Furi and Vignoli, 1970) or (Nussbaum, 1971) Let X be an infinite–dimensional normed space. Then $\alpha(B_X) = 2$.*

Lemma 2.2.3. *(Malkowsky and Rakočević, 2000a, Lemma 2.6) Let M, M_1 and M_2 be bounded subsets of a complete metric space (X, d). Then,*

$$\alpha(M) = 0 \textit{ if and only if } \overline{M} \textit{ is compact }, \quad (2.4)$$

$$\alpha(M) = \alpha(\overline{M}), \quad (2.5)$$

$$M_1 \subset M_2 \textit{ implies } \alpha(M_1) \leq \alpha(M_2), \quad (2.6)$$

$$\alpha(M_1 \cup M_2) = \max\{\alpha(M_1), \alpha(M_2)\}, \quad (2.7)$$

$$\alpha(M_1 \cap M_2) \leq \min\{\alpha(M_1), \alpha(M_2)\}. \quad (2.8)$$

Lemma 2.2.4. *(Darbo, 1955) Let X be a normed space, and $M, M_1, M_2 \in M_X$.*

Then we have

$$\alpha(tM) = |t|\alpha(M) \text{ for any number } t \text{ and } M \in \mathcal{M}_X \text{ (homogeneity)}, \tag{2.9}$$

$$\alpha(M_1 + M_2) \leq \alpha(M_1) + \alpha(M_2), \text{ forall } M_1, M_2 \in \mathcal{M}_X \text{ (subadditivity)}, \tag{2.10}$$

$$\alpha(x_0+M) = \alpha(M) \text{ for any } x_0 \in X \text{ and } M \in \mathcal{M}_X \text{ (translation invariance)}. \tag{2.11}$$

Now we give the definition of the Hausdorff measure of noncompactness.

Definition 8. Let (X, d) be metric space, and $Q \in \mathcal{M}_X$. Then the **Hausdorff measure of noncompactness of the set** Q, denoted by $\chi(Q)$, is the infimum of the set of all real numbers $\epsilon > 0$ such that Q can be covered by a finite number of balls with radii less than ϵ, that is,

$$\chi(Q) = \inf\{\epsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i), x_i \in X, r_i < \epsilon\, (i = 1, 2, ..., n; n \in \mathbb{N})\}. \tag{2.12}$$

The function χ is called **Hausdorff measure of noncompactness**.

Remark 2.2.2. According to the definition of the Hausdorff measure of noncompactness of a set Q, the centres of the balls which cover Q need not to be in Q. So, (2.12) can equivalently be stated as follows:

$$\chi(Q) = \inf\{\epsilon > 0 : Q \text{ has a finite } \epsilon\text{-net in } X\}. \tag{2.13}$$

Lemma 2.2.5. *(Malkowsky and Rakočević, 2000a, Lemma 2.11) Let M, M_1 and M_2 be bounded subsets of a complete metric space (X, d). Then,*

$$\chi(M) = 0 \text{ if and only if } \overline{M} \text{ is compact }, \tag{2.14}$$

$$\chi(M) = \chi(\overline{M}), \tag{2.15}$$

$$M_1 \subset M_2 \text{ implies } \chi(M_1) \leq \chi(M_2), \tag{2.16}$$

$$\chi(M_1 \cup M_2) = \max \chi(M_1), \chi(M_2), \tag{2.17}$$

$$\chi(M_1 \cap M_2) \leq \min \chi(M_1), \chi(M_2). \tag{2.18}$$

Lemma 2.2.6. *(Malkowsky and Rakočević, 2000a, Theorem 2.12) Let X be a normed space, and $M, M_1, M_2 \in M_X$. Then we have*

$$\chi(tM) = |t|\chi(M) \text{ for any number } t \text{ and } M \in \mathcal{M}_X \text{ (homogeneity)}, \tag{2.19}$$

$$\chi(M_1 + M_2) \leq \chi(M_1) + \chi(M_2), \text{ forall } M_1, M_2 \in \mathcal{M}_X \text{ (subadditivity)}, \tag{2.20}$$

$$\chi(x_0 + M) = \chi(M) \text{ for any } x_0 \in X \text{ and } M \in \mathcal{M}_X \text{ (translation invariance)}. \tag{2.21}$$

It turns out that the Hausdorff measure of noncompactness of the unit ball in an infinite dimensional normed space is equal to its radius.

Theorem 2.2.7. *(Malkowsky and Rakočević, 2000a, Theorem 2.14) Let X be an infinite–dimensional normed space. Then $\chi(\overline{B}_X) = 1$.*

The next theorem shows that the Hausdorff and Kuratowski measures are equivalent in the sense of (2.22) below.

Theorem 2.2.8. *(Toledano et al., 1997, Remark 3.2) Let (X, d) be metric space, and $Q \in \mathcal{M}_X$. Then*

$$\chi(Q) \leq \alpha(Q) \leq 2\chi(Q). \tag{2.22}$$

CHAPTER 3

MIXED NORM SPACES

3.1 TOPOLOGICAL STRUCTURES

In this part, we establish some important topological properties of the spaces $[C_\alpha]_0^p$, $[C_\alpha]^p$, and $[C_\alpha]_\infty^p$ for $\alpha > 0$ and $p \geq 1$. We write $\sum_0 = \sum_{k=0}^1$, $\max_0 = \max_{0\leq k\leq 1}$, and $\sum_\nu = \sum_{k=2^\nu}^{2^{\nu+1}-1}$ and $\max_\nu = \max_{2^\nu \leq k \leq 2^{\nu+1}-1}$ for $\nu \geq 1$.

Proposition 3.1.1. *Let $\alpha > 0$ and $p \geq 1$.*
(a) The sets $[C_\alpha]_0^p$, $[C_\alpha]^p$ and $[C_\alpha]_\infty^p$ are BK spaces with respect to

$$\|x\|_{[C_\alpha]_\infty^p} = \sup_\nu \left(\frac{1}{2^\nu} \sum_v |\sigma_k^{\alpha-1}(x)|^p \right)^{1/p} = \sup_\nu \left(\frac{1}{2^\nu} \sum_v \left| \frac{1}{A_k^{\alpha-1}} \sum_{j=0}^k A_{k-j}^{\alpha-2} x_k \right|^p \right)^{1/p}, \tag{3.1}$$

$[C_\alpha]_0^p$ is a closed subset of $[C_\alpha]^p$ and $[C_\alpha]^p$ is a closed subset of $[C_\alpha]_\infty^p$.
(b) For each $n \in \mathbb{N}_0$, we put $c^{(n)} = (c_k^{(n)})_{k=0}^\infty = S^{\alpha-1}e^{(n)}$, that is,

$$c_k^{(n)} = \begin{cases} 0 & (0 \leq k \leq n-1) \\ A_{k-n}^{-\alpha} A_n^{\alpha-1} & (k \geq n). \end{cases}$$

Then every sequence $x = (x_k)_{k=0}^\infty \in [C_\alpha]_0^p$ has a unique representation

$$x = \sum_{n=0}^\infty \sigma_n^{\alpha-1}(x) c^{(n)}. \tag{3.2}$$

Every sequence $x = (x_k)_{k=0}^\infty \in [C_\alpha]^p$ has a unique representation

$$x = \xi \cdot e + \sum_{n=0}^\infty \sigma_n^{\alpha-1}(x - \xi \cdot e) c^{(n)}, \tag{3.3}$$

where ξ is the unique complex number such that $x - \xi \cdot e \in [C_\alpha]_0^p$.

Proof. (a) By (Malkowsky and Rakočević, 2000a, Proposition 3.44), the sets w_0^p, w^p and w_∞^p are BK spaces with respect to

$$\|x\|_{w_\infty^p} = \sup_\nu \left(\frac{1}{2^\nu} \sum_v |x_k|^p \right)^{1/p},$$

w_0^p is a closed subspace of w^p and w^p is a closed subspace of w_∞^p. Hence the sets $[C_\alpha]_0^p$, $[C_\alpha]^p$ and $[C_\alpha]_\infty^p$ are BK spaces with respect to the norm $\|\cdot\|_{[C_\alpha]_\infty^p}$ defined in (3.1) by (Wilansky, 1984, Theorem 4.3.12), and $[C_\alpha]_0^p$ is a closed subset of $[C_\alpha]^p$ and $[C_\alpha]^p$ is a closed subset of $[C_\alpha]_\infty^p$ by (Wilansky, 1984, Theorem 4.3.14).

(b) Since w_0^p has AK by (Malkowsky and Rakočević, 2000a, Proposition 3.44), the representation of $x = (x_k)_{k=0}^\infty \in [C_\alpha]_0^p$ in (3.2) is an immediate consequence of (Jarrah and Malkowsky, 2003, Corollary 2.5 (a)) and (1.4).
We observed above that $[C_\alpha]^p = [C_\alpha]_0^p \oplus e$, and so every $x = (x_k)_{k=0}^\infty \in [C_\alpha]^p$ has a unique representation as in (3.3) by (Jarrah and Malkowsky, 2003, Corollary 2.5 (b)). ∎

Remark 3.1.1. (a) Since w_∞^p has no Schauder basis by (Djolović and Malkowsky, 2012, Lemma 1.1), $[C_\alpha]_\infty^p$ has no Schauder basis by (Jarrah and Malkowsky, 2003, Remark 2.4).
(b) We have $[C_\alpha]^p = (w_0^p \oplus e)_{C_{\alpha-1}}$ by definition, and so it follows from (Jarrah and Malkowsky, 2003, Corollary 2.5 (c)) that every sequence $x \in [C_\alpha]^p$ has a unique representation

$$x = \xi \cdot c^{(-1)} + \sum_{n=0}^\infty \left(\sigma_n^{\alpha-1}(x) - \xi \right) c^{(n)} \tag{3.4}$$

where the sequences $c^{(n)}$ $(n = 0, 1, \dots)$ are defined as in Proposition 3.1.1 and the sequence $c^{(-1)} = (c_k^{(-1)})_{k=0}^\infty$ is given by

$$c_k^{(-1)} = \sum_{j=0}^k A_{k-j}^{-\alpha} A_j^{\alpha-1} = A_k^0 = 1 \text{ for } k = 0, 1, \dots,$$

hence $c^{(-1)} = e$. Since $\sigma_n^{\alpha-1}(x) - \xi = \sigma_n^{\alpha-1}(x - \xi \cdot e)$ for all $n \in \mathbb{N}_0$, the representations in (3.4) and (3.3) are identical.

Remark 3.1.2. We can visualise the norm defined by (3.1) using our software MV-Graphics. Due to the sequences $(x_k)_{k=0}^{\infty}$ have infinitely many dimensions which is impossible to represent in the computer, we identify them with three–dimensional vectors. Any three dimension can be chosen as coordinates, then all other coordinates have to be zero. We represent our norm as a potential surface which is a surface energy function in crystallography.

Example 3.1.1. For visualisation of the norm $\|\cdot\|_{[C_\alpha]^p_\infty}$ defined in (3.1) we work out computations and successively obtain

for $\nu = 0$

$$\begin{aligned}
\left(\frac{1}{2^0}\sum_{0}\left|\frac{1}{A_k^{\alpha-1}}\sum_{j=0}^{k}A_{k-j}^{\alpha-2}x_j\right|^p\right)^{\frac{1}{p}} &= \left(\frac{1}{2^0}\sum_{k=0}^{1}\left|\frac{1}{A_k^{\alpha-1}}\sum_{j=0}^{k}A_{k-j}^{\alpha-2}x_j\right|^p\right)^{\frac{1}{p}} \\
&= \left(\left|\frac{1}{A_0^{\alpha-1}}\sum_{j=0}^{0}A_{0-j}^{\alpha-2}x_j\right|^p + \left|\frac{1}{A_1^{\alpha-1}}\sum_{j=0}^{1}A_{1-j}^{\alpha-2}x_j\right|^p\right)^{\frac{1}{p}} \\
&= \left(|x_0|^p + \left|\frac{1}{\alpha}(A_1^{\alpha-2}x_0 + A_0^{\alpha-2}x_1)\right|^p\right)^{\frac{1}{p}} \\
&= \left(|x_0|^p + \left|\frac{1}{\alpha}((\alpha-1)x_0 + x_1)\right|^p\right)^{\frac{1}{p}}
\end{aligned}$$

for $\nu = 1$,

$$\begin{aligned}
&\left(\frac{1}{2^1}\sum_{k=2^1}^{2^2-1}\left|\frac{1}{A_k^{\alpha-1}}\sum_{j=0}^{k}A_{k-j}^{\alpha-2}x_j\right|^p\right)^{\frac{1}{p}} = \left(\frac{1}{2}\sum_{k=2}^{3}\left|\frac{1}{A_k^{\alpha-1}}\sum_{j=0}^{k}A_{k-j}^{\alpha-2}x_j\right|^p\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{2}\left[\left|\frac{1}{A_2^{\alpha-1}}\sum_{j=0}^{2}A_{2-j}^{\alpha-2}x_j\right|^p + \left|\frac{1}{A_3^{\alpha-1}}\sum_{j=0}^{3}A_{3-j}^{\alpha-2}x_j\right|^p\right]\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{2}\left[\left|\frac{2}{(\alpha+1)\alpha}(A_2^{\alpha-2}x_0 + A_1^{\alpha-2}x_1 + A_0^{\alpha-2}x_2)\right|^p\right.\right. \\
&\qquad \left.\left. + \left|\frac{6}{(\alpha+2)(\alpha+1)\alpha}(A_3^{\alpha-2}x_0 + A_2^{\alpha-2}x_1 + A_1^{\alpha-2}x_2 + A_0^{\alpha-2}x_3)\right|^p\right]\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{2}\left[\left|\frac{2}{(\alpha+1)\alpha}\left(\frac{\alpha(\alpha-1)}{2}x_0 + (\alpha-1)x_1 + x_2\right)\right|^p\right.\right. \\
&\qquad + \left|\frac{6}{(\alpha+2)(\alpha+1)\alpha}\left(\frac{(\alpha+1)\alpha(\alpha-1)}{6}x_0\right.\right. \\
&\qquad \left.\left.\left.\left. + \frac{\alpha(\alpha-1)}{2}x_1 + (\alpha-1)x_2 + x_3\right)\right|^p\right]\right)^{\frac{1}{p}}
\end{aligned}$$

$$= \Bigg(\frac{1}{2} \Bigg[\bigg| \frac{(\alpha-1)}{(\alpha+1)} x_0 + \frac{2(\alpha-1)}{\alpha(\alpha+1)} x_1 + \frac{2}{(\alpha+1)\alpha} x_2 \bigg|^p + \bigg| \frac{\alpha-1}{\alpha+2} x_0 + \frac{3(\alpha-1)}{(\alpha+2)(\alpha+1)} x_1 + \frac{6(\alpha-1)}{(\alpha+2)(\alpha+1)\alpha} x_2 + \frac{6}{(\alpha+2)(\alpha+1)\alpha} x_3 \bigg|^p \Bigg] \Bigg)^{\frac{1}{p}};$$

for $\nu = 2$,

$$\left(\frac{1}{2^2} \sum_{k=2^2}^{2^3-1} \left| \frac{1}{A_k^{\alpha-1}} \sum_{j=0}^{k} A_{k-j}^{\alpha-2} x_j \right|^p \right)^{\frac{1}{p}} = \left(\frac{1}{4} \sum_{k=4}^{7} \left| \frac{1}{A_k^{\alpha-1}} \sum_{j=0}^{k} A_{k-j}^{\alpha-2} x_j \right|^p \right)^{\frac{1}{p}}$$

$$= \Bigg(\frac{1}{4} \Bigg[\bigg| \frac{1}{A_4^{\alpha-1}} \sum_{j=0}^{4} A_{4-j}^{\alpha-2} x_j \bigg|^p + \bigg| \frac{1}{A_5^{\alpha-1}} \sum_{j=0}^{5} A_{5-j}^{\alpha-2} x_j \bigg|^p + \bigg| \frac{1}{A_6^{\alpha-1}} \sum_{j=0}^{6} A_{6-j}^{\alpha-2} x_j \bigg|^p + \bigg| \frac{1}{A_7^{\alpha-1}} \sum_{j=0}^{7} A_{7-j}^{\alpha-2} x_j \bigg|^p \Bigg] \Bigg)^{\frac{1}{p}}$$

$$= \Bigg(\frac{1}{4} \Bigg[\bigg| \frac{4! \cdot (A_4^{\alpha-2} x_0 + A_3^{\alpha-2} x_1 + A_2^{\alpha-2} x_2 + A_1^{\alpha-2} x_3 + A_0^{\alpha-2} x_4)}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} \bigg|^p + \bigg| \frac{5! \cdot (\alpha+3)(\alpha+4)(A_5^{\alpha-2} x_0 + A_4^{\alpha-2} x_1 + A_3^{\alpha-2} x_2 + A_2^{\alpha-2} x_3 + ...)}{\alpha(\alpha+1)(\alpha+2)} \bigg|^p + \bigg| \frac{6! \cdot (A_6^{\alpha-2} x_0 + A_5^{\alpha-2} x_1 + A_4^{\alpha-2} x_2 + A_3^{\alpha-2} x_3 + ...)}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)} \bigg|^p + \bigg| \frac{7! \cdot (A_7^{\alpha-2} x_0 + A_6^{\alpha-2} x_1 + A_5^{\alpha-2} x_2 + A_4^{\alpha-2} x_3 + ...)}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)(\alpha+6)} \bigg|^p \Bigg] \Bigg)^{\frac{1}{p}}$$

$$= \Bigg(\frac{1}{4} \Bigg[\bigg| \frac{4!}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} \bigg(\frac{(\alpha+2)(\alpha+1)\alpha(\alpha-1)}{4!} x_0 + \frac{(\alpha+1)\alpha(\alpha-1)}{3!} x_1 + \frac{\alpha(\alpha-1)}{2!} x_2 + (\alpha-1) x_3 + x_4 \bigg) \bigg|^p$$
$$+ \bigg| \frac{5!}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} \bigg(\frac{(\alpha+3)(\alpha+2)(\alpha+1)\alpha(\alpha-1)}{5!} x_0 + \frac{(\alpha+2)(\alpha+1)\alpha(\alpha-1)}{4!} x_1 + \frac{(\alpha+1)\alpha(\alpha-1)}{3!} x_2 + \frac{\alpha(\alpha-1)}{2!} x_3 + ... \bigg) \bigg|^p$$
$$+ \bigg| \frac{6!}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)} \bigg(\frac{(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)\alpha(\alpha-1)}{6!} x_0 + \frac{(\alpha+3)(\alpha+2)(\alpha+1)\alpha(\alpha-1)}{5!} x_1 + \frac{(\alpha+2)(\alpha+1)\alpha(\alpha-1)}{4!} x_2 + \frac{(\alpha+1)\alpha(\alpha-1)}{3!} x_3 + ... \bigg) \bigg|^p$$

$$
\begin{aligned}
&+\Bigg|\frac{7!}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)(\alpha+6)} \\
&\qquad\qquad\Bigg(\frac{(\alpha+5)(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)\alpha(\alpha-1)}{7!}x_0 \\
&\qquad\qquad+\ \frac{(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)\alpha(\alpha-1)}{6!}x_1 \\
&\qquad\qquad+\ \frac{(\alpha+3)(\alpha+2)(\alpha+1)\alpha(\alpha-1)}{5!}x_2 \\
&\qquad\qquad+\ \frac{(\alpha+2)(\alpha+1)\alpha(\alpha-1)}{4!}x_3+\ldots\Bigg)\Bigg|^p\Bigg]\Bigg)^{\frac{1}{p}} \\
=&\Bigg(\frac{1}{4}\Bigg[\Bigg|\frac{\alpha-1}{\alpha+3}x_0+\frac{4(\alpha-1)}{(\alpha+3)(\alpha+2)}x_1+\frac{12(\alpha-1)}{(\alpha+3)(\alpha+2)(\alpha+1)}x_2 \\
&\qquad\qquad+\ \frac{24(\alpha-1)}{(\alpha+3)(\alpha+2)(\alpha+1)\alpha}x_3+x_4\Bigg|^p \\
&\quad+\ \Bigg|\frac{\alpha-1}{\alpha+4}x_0+\frac{5(\alpha-1)}{(\alpha+4)(\alpha+3)}x_1+\frac{20(\alpha-1)}{(\alpha+4)(\alpha+3)(\alpha+2)}x_2 \\
&\qquad\qquad+\ \frac{60(\alpha-1)}{(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)}x_3+\ldots\Bigg|^p \\
&\quad+\ \Bigg|\frac{\alpha-1}{\alpha+5}x_0+\frac{6(\alpha-1)}{(\alpha+5)(\alpha+4)}x_1+\frac{30(\alpha-1)}{(\alpha+5)(\alpha+4)(\alpha+3)}x_2 \\
&\qquad\qquad+\ \frac{120(\alpha-1)}{(\alpha+5)(\alpha+4)(\alpha+3)(\alpha+2)}x_3+\ldots\Bigg|^p \\
&\quad+\ \Bigg|\frac{\alpha-1}{\alpha+6}x_0+\frac{7(\alpha-1)}{(\alpha+6)(\alpha+5)}x_1+\frac{42(\alpha-1)}{(\alpha+6)(\alpha+5)(\alpha+4)}x_2 \\
&\qquad\qquad+\ \frac{210(\alpha-1)}{(\alpha+6)(\alpha+5)(\alpha+4)(\alpha+3)}x_3+\ldots\Bigg|^p\Bigg]\Bigg)^{\frac{1}{p}} \\
=&\Bigg(\frac{|\alpha-1|^p}{4}\Bigg[\frac{1}{|\alpha+3|^p}|x_0+\frac{4}{\alpha+2}x_1+\frac{12}{(\alpha+2)(\alpha+1)}x_2 \\
&\qquad\qquad+\ \frac{24}{(\alpha+2)(\alpha+1)\alpha}x_3+x_4\Bigg|^p \\
&\quad+\ \frac{1}{|\alpha+4|^p}|x_0+\frac{5}{\alpha+3}x_1+\frac{20}{(\alpha+3)(\alpha+2)}x_2 \\
&\qquad\qquad+\ \frac{60}{(\alpha+3)(\alpha+2)(\alpha+1)}x_3+\ldots\Bigg|^p \\
&\quad+\ \frac{1}{|\alpha+5|^p}\Bigg|x_0+\frac{6}{\alpha+4}x_1+\frac{30}{(\alpha+4)(\alpha+3)}x_2 \\
&\qquad\qquad+\ \frac{120}{(\alpha+4)(\alpha+3)(\alpha+2)}x_3+\ldots\Bigg|^p \\
&\quad\frac{1}{|\alpha+6|^p}\Bigg|x_0+\frac{7}{\alpha+5}x_1+\frac{42}{(\alpha+5)(\alpha+4)}x_2
\end{aligned}
$$

$$+ \frac{210}{(\alpha+5)(\alpha+4)(\alpha+3)}x_3 + ...\Big|^p\Big]\Big)^{\frac{1}{p}};$$

for $\nu = 3$,

$$\left(\frac{1}{2^3}\sum_{k=2^3}^{2^4-1}\left|\frac{1}{A_k^{\alpha-1}}\sum_{j=0}^{k}A_{k-j}^{\alpha-2}x_j\right|^p\right)^{\frac{1}{p}} = \Big(\frac{|\alpha-1|^p}{8}\Big[\frac{1}{|\alpha+7|^p}\Big|x_0 + \frac{8}{\alpha+6}x_1$$
$$+ \frac{56}{(\alpha+6)(\alpha+5)}x_2 + \frac{336}{(\alpha+6)(\alpha+5)(\alpha+4)}x_3 + ...\Big|^p$$
$$+ \frac{1}{|\alpha+8|^p}\Big|x_0 + \frac{9}{\alpha+7}x_1 + \frac{72}{(\alpha+7)(\alpha+6)}x_2$$
$$+ \frac{504}{(\alpha+7)(\alpha+6)(\alpha+5)}x_3 + ...\Big|^p$$
$$+ \frac{1}{|\alpha+9|^p}\Big|x_0 + \frac{10}{\alpha+8}x_1 + \frac{90}{(\alpha+8)(\alpha+7)}x_2$$
$$+ \frac{720}{(\alpha+8)(\alpha+7)(\alpha+6)}x_3 + ...\Big|^p$$
$$+ \frac{1}{|\alpha+10|^p}\Big|x_0 + \frac{11}{\alpha+9}x_1 + \frac{110}{(\alpha+9)(\alpha+8)}x_2$$
$$+ \frac{990}{(\alpha+9)(\alpha+8)(\alpha+7)}x_3 + ...\Big|^p$$
$$+ \frac{1}{|\alpha+11|^p}\Big|x_0 + \frac{12}{\alpha+10}x_1 + \frac{132}{(\alpha+10)(\alpha+9)}x_2$$
$$+ \frac{1320}{(\alpha+10)(\alpha+9)(\alpha+8)}x_3 + ...\Big|^p$$
$$+ \frac{1}{|\alpha+12|^p}\Big|x_0 + \frac{13}{\alpha+11}x_1 + \frac{156}{(\alpha+11)(\alpha+10)}x_2$$
$$+ \frac{1716}{(\alpha+11)(\alpha+10)(\alpha+9)}x_3 + ...\Big|^p$$
$$+ \frac{1}{|\alpha+13|^p}\Big|x_0 + \frac{14}{\alpha+12}x_1 + \frac{182}{(\alpha+12)(\alpha+11)}x_2$$
$$+ \frac{2184}{(\alpha+12)(\alpha+11)(\alpha+10)}x_3 + ...\Big|^p$$
$$+ \frac{1}{|\alpha+14|^p}\Big|x_0 + \frac{15}{\alpha+13}x_1 + \frac{210}{(\alpha+13)(\alpha+12)}x_2$$
$$+ \frac{2730}{(\alpha+13)(\alpha+12)(\alpha+11)}x_3 + ...\Big|^p\Big]\Big)^{\frac{1}{p}}$$

Now we choose three coordinates x_0, x_1, x_2 to be different than zero and the rest is equal to zero to project the norm $\|\cdot\|_{[C_\alpha]_\infty^p}$ on (x_0, x_1, x_2).

Hence

$$\|x\|_{[C_\alpha]^p_\infty} = \sup \left\{ \left(\frac{1}{2^0} \sum_0 \left| \frac{1}{A_k^{\alpha-1}} \sum_{j=0}^{k} A_{k-j}^{\alpha-2} x_k \right|^p \right)^{1/p}, \right.$$

$$\left(\frac{1}{2^1} \sum_1 \left| \frac{1}{A_k^{\alpha-1}} \sum_{j=0}^{k} A_{k-j}^{\alpha-2} x_k \right|^p \right)^{1/p},$$

$$\left(\frac{1}{2^2} \sum_2 \left| \frac{1}{A_k^{\alpha-1}} \sum_{j=0}^{k} A_{k-j}^{\alpha-2} x_k \right|^p \right)^{1/p},$$

$$\left. \left(\frac{1}{2^3} \sum_3 \left| \frac{1}{A_k^{\alpha-1}} \sum_{j=0}^{k} A_{k-j}^{\alpha-2} x_k \right|^p \right)^{1/p}, \dots \right\}.$$

Then we obtain the following figures for different values of p and α.

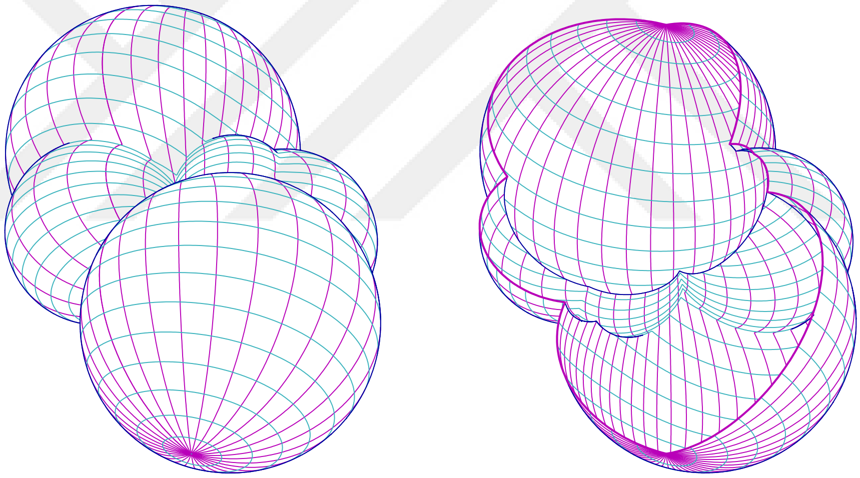

Figure 3.1 Norm $\|\cdot\|_{[C_\alpha]^p_\infty}$ for $p = 1$ and $\alpha = 0.09$ as a potential surface.

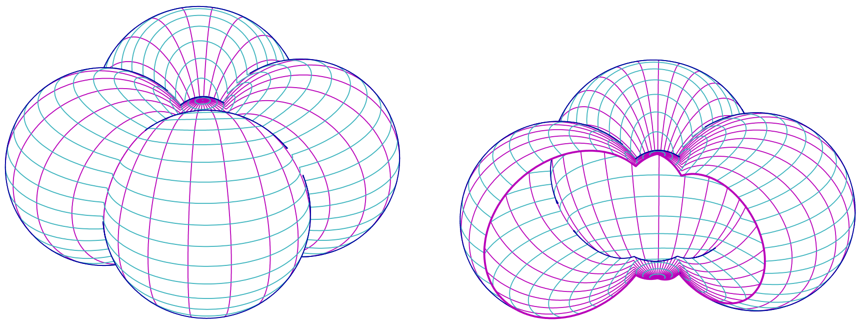

Figure 3.2 Norm $\|\cdot\|_{[C_\alpha]^p_\infty}$ for $p = 1$ and $\alpha = 1.05$ as a potential surface.

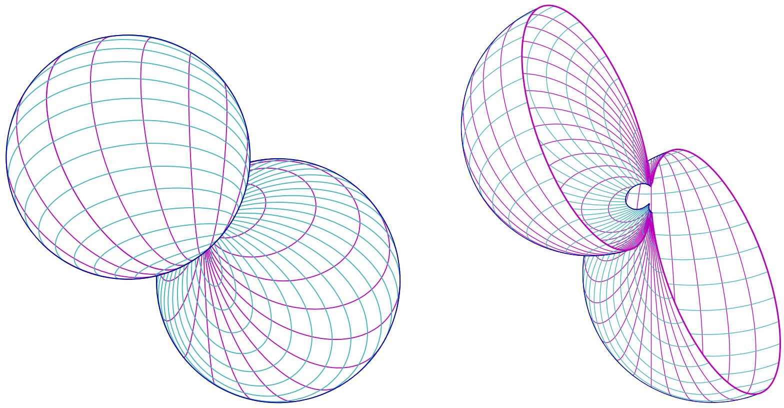

Figure 3.3 Norm $\|\cdot\|_{[C\alpha]^p_\infty}$ for $p = 1$ and $\alpha = 15$ as a potential surface.

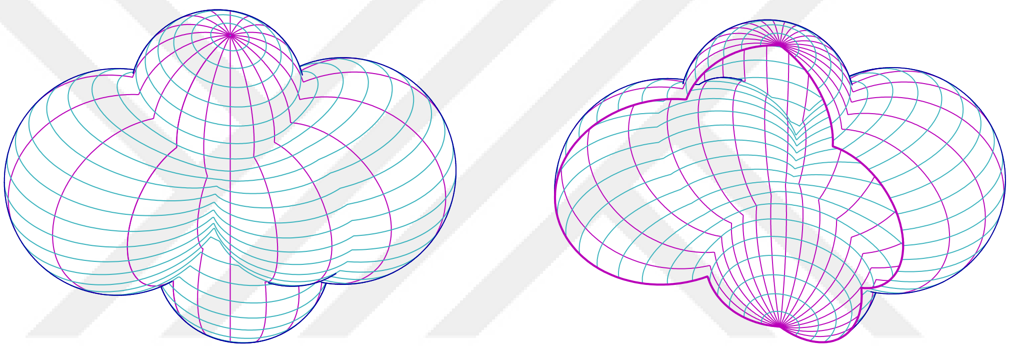

Figure 3.4 Norm $\|\cdot\|_{[C\alpha]^p_\infty}$ for $p = 1.01$ and $\alpha = 0.5$ as a potential surface.

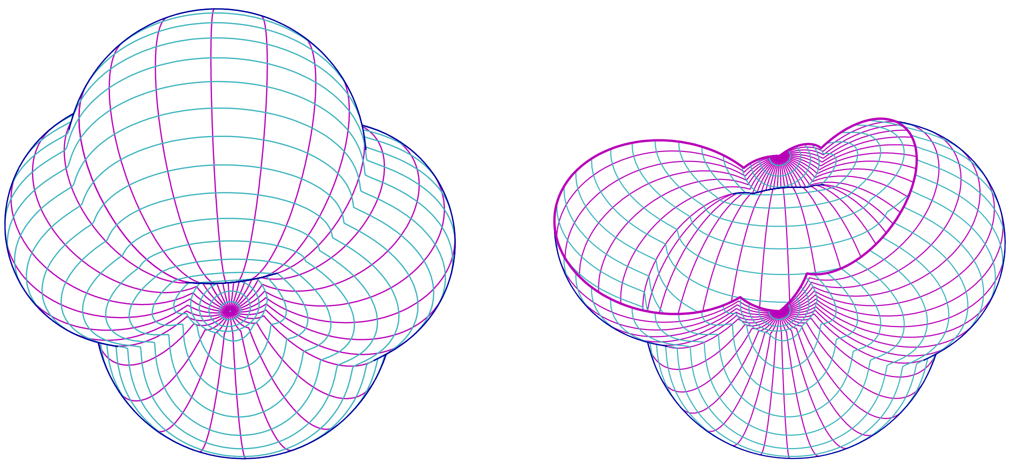

Figure 3.5 Norm $\|\cdot\|_{[C\alpha]^p_\infty}$ for $p = 1.2$ and $\alpha = 1.2$ as a potential surface.

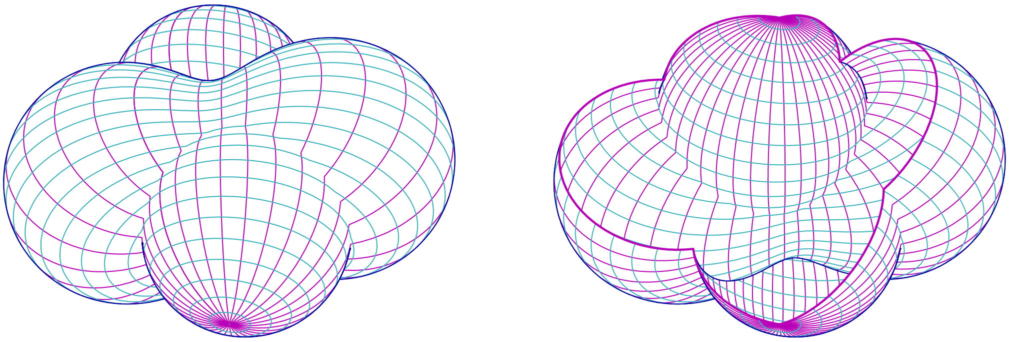

Figure 3.6 Norm $\|\cdot\|_{[C_\alpha]^p_\infty}$ for $p = 2$ and $\alpha = 0.6$ as a potential surface.

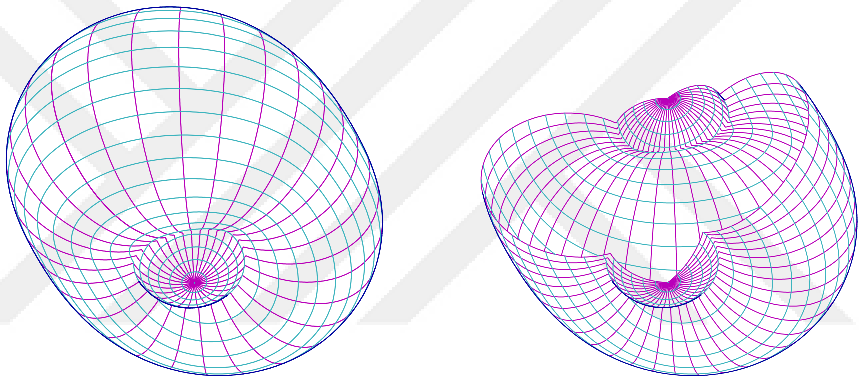

Figure 3.7 Norm $\|\cdot\|_{[C_\alpha]^p_\infty}$ for $p = 2$ and $\alpha = 1.2$ as a potential surface.

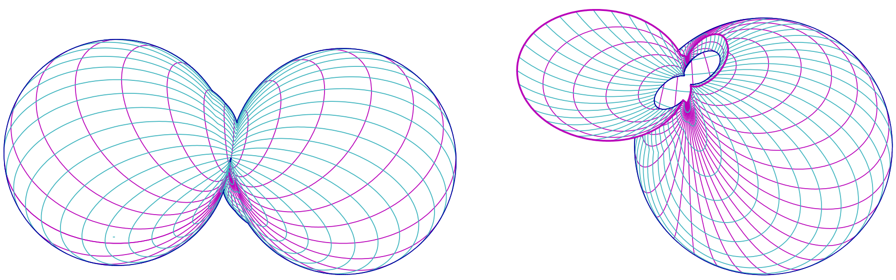

Figure 3.8 Norm $\|\cdot\|_{[C_\alpha]^p_\infty}$ for $p = 2$ and $\alpha = 8$ as a potential surface.

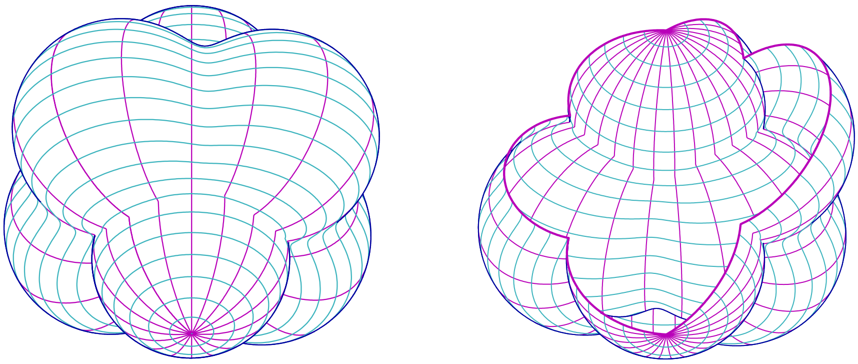

Figure 3.9 Norm $\|\cdot\|_{[C_\alpha]^p_\infty}$ for $p = 10$ and $\alpha = 1.02$ as a potential surface.

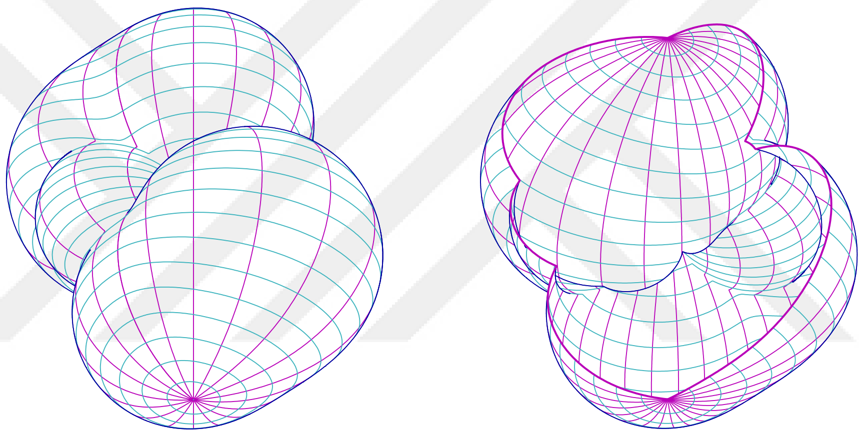

Figure 3.10 Norm $\|\cdot\|_{[C_\alpha]^p_\infty}$ for $p = 10$ and $\alpha = 0.15$ as a potential surface.

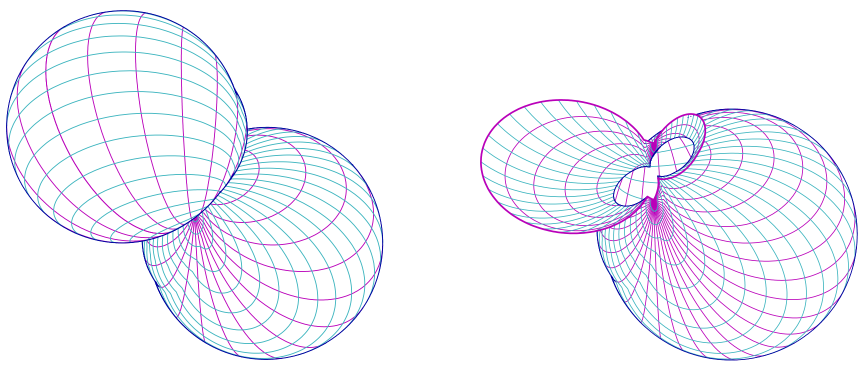

Figure 3.11 Norm $\|\cdot\|_{[C_\alpha]^p_\infty}$ for $p = 10$ and $\alpha = 10$ as a potential surface.

3.2 THE BLOCKING TECHNIQUE

We recall that we observed in Chapter 1 that the sets $[C_\alpha]_0^p$, $[C_\alpha]^p$ and $[C_\alpha]_\infty^p$ can be considered as the domains of the Cesàro matrix $C_{\alpha-1}$ in the sets

$$w_0^p = \left\{ x \in w : \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} |x_k|^p = 0 \right\},\ w^p = w_0^p \oplus e$$

and

$$w_\infty^p = \left\{ x \in w : \sup_n \frac{1}{n+1} \sum_{k=0}^{n} |x_k|^p < \infty \right\}$$

of sequences that are strongly summable to zero, strongly summable and strongly bounded, respectively, by the Cesáro method of order 1, with index $p \geq 1$. The proof of Proposition 3.1.1 uses the so–called block norm $\|\cdot\|_{w_\infty^p}$ instead of the natural norm $\|\cdot\|'_{w_\infty^p}$ with

$$\|x\|'_{w_\infty^p} = \sup_n \left(\frac{1}{n+1} \sum_{k=0}^{n} |x_k|^p \right)^{1/p},$$

the so–called section norm. It is well known that the block and section norms are equivalent on each of the spaces w_0^p, w^p and w_∞^p ((Maddox, 1968) or (Malkowsky, 1995, Theorem 1)). The block norm is used for technical reasons in the determination of the duals of the spaces w_0^p, w^p and w_∞^p, and the characterization of matrix transformations on those sequence spaces.

This approach is a special case of the so–called blocking technique which has important applications in various parts of analysis, in particular, in the study of sequence and function spaces, the study of operators between such spaces, and in the classical inequalities.

In these theories, expressions of the form

$$\sum_{n=0}^{\infty} \left(a_n \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} \right)^q \quad \text{for } 1 \leq p, q < \infty \tag{3.5}$$

and

$$\sup_n \left(a_n \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} \right) \quad \text{for } 1 \leq p < \infty \tag{3.6}$$

play an important role, and also in Hardy's famous inequality which states that, for any $p > 0$, there is a constant $K > 0$ such that

$$\sum_{n=0}^{N} \left(\frac{1}{n+1} \sum_{k=0}^{n} x_k \right)^p \leq K \sum_{n=0}^{N} x_n^p$$

for all $N \in \mathbb{N}_0$ and for all nonnegative real numbers $x_0, x_1, \ldots, x_N$.

The analysis of expressions as in (3.5) and (3.6) which are called a norm in section form can be extremely difficult.

In many cases it is helpful to suitably renorm those expressions, by using so–called norms in block form

$$\sum_{\nu=0}^{\infty} \left(\frac{1}{2^{\nu\alpha}} \left(\sum_{k \in I_\nu}^{\infty} |x_k|^p \right)^{1/p} \right)^q \text{ and } \sup_n \left(\frac{1}{2^{\nu\alpha}} \left(\sum_{k \in I_\nu}^{\infty} |x_k|^p \right)^{1/p} \right), \tag{3.7}$$

where the intervals I_ν form a partition of $\mathbb{N}_0$ into disjoint intervals. The most common partition is that into dyadic blocks $[2^\nu, 2^{\nu+1}-1]$. Such renorming is referred to as the blocking technique; it is of great practical importance, since the analysis of norms in block form is much simpler; in many aspects they behave like the familiar ℓ_p norms.

The renorming of the spaces w_0^p, w^p and w_∞^p is a simple example for the blocking technique.

In 1974, Jagers (Jagers, 1974) studied the sequence spaces

$$ces_p = \left\{ x \in w : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p < \infty \right\} \text{ for } p \geq 1,$$

which are Banach spaces with the section norm

$$\|x\|'_{ces_p} = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p}.$$

We observe that Hardy's inequality immediately implies the set inclusion $\ell_p \subset ces_p$. It can be found in (Grosse-Erdmann, 1998) that an equivalent norm on ces_p is the dyadic block norm

$$\|x\|_{ces_p} = \left(\sum_{\nu=0}^{\infty} \frac{1}{2^{\nu(p-1)}} \left(\sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k| \right)^p \right)^{1/p}.$$

This is one more example of an application of the blocking technique.

In 1969, Hedlund (Hedlund, 1969) introduced the mixed norm spaces

$$\ell(p,q) = \left\{ x \in w : \sum_{\nu=0}^{\infty} \left(\sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k|^p \right)^{q/p} < \infty \right\}$$

which in many aspects behave like the classical ℓ_p spaces. It is clear that the Cesàro sequence space ces_p is a weighted $\ell(1,p)$ space.

A comprehensive study of the blocking technique and its applications to the theory of sequence spaces can be found in the monograph (Grosse-Erdmann, 1998).

3.3 DUAL SPACES

Here, we determine the β–duals of the spaces $[C_\alpha]_0^p$, $[C_\alpha]^p$ and $[C_\alpha]_\infty^p$ and make use of the blocking technique. We write

$$\mathcal{M}_p = \{a \in w : \|a\|_{\mathcal{M}_p} < \infty\}, \text{ where}$$

$$\|a\|_{\mathcal{M}_p} = \begin{cases} \sum\limits_{\nu=0}^{\infty} 2^\nu \cdot \max_v |a_k| & (p = 1) \\ \sum\limits_{\nu=0}^{\infty} 2^{\nu/p} \left(\sum_v |a_k|^q\right)^{1/q} & \left(1 < p < \infty; q = \dfrac{p}{p-1}\right) \end{cases}.$$

Let $a \in \omega$ and $X \subset \omega$ be a linear metric space. Then we write

$$\|a\|_{X,\delta}^* = \sup \left\{ \left| \sum_{k=0}^{\infty} a_k x_k \right| : x \in \overline{B}(0,\delta) \right\} \text{ for } \delta > 0$$

provided the right hand side is defined and finite which is the case whenever X is an FK space and $a \in X^\beta$ (Wilansky, 1984, Theorem 7.2.9); if X is a BK space then we write

$$\|a\|^* = \|a\|_X^* = \sup\left\{\left|\sum_{k=0}^{\infty} a_k x_k\right| : x \in \overline{B}_X\right\}.$$

The following results are known for $1 \le p < \infty$ (Malkowsky and Rakočević, 2000b, Lemma 1)

$$(w_0^p)^\beta = (w^p)^\beta = (w_\infty^p)^\beta = \mathcal{M}_p \text{ and } \|a\|^* = \|a\|_{\mathcal{M}_p} \text{ on } \mathcal{M}_p. \tag{3.8}$$

Lemma 3.3.1. *(Malkowsky and Rakočević, 2007, Lemma 3.1) Let X be an FK space with AK and $Z = X_T$. We write $R = S^t$ for the transpose of S. Then we have*

$$(X_T)^\beta \subset (X^\beta)_R.$$

The following result involves (Malkowsky and Rakočević, 2007, Theorem 3.2) and an improvement of it.

Proposition 3.3.2. *a) Let X be an FK space with AK, T be a triangle, S be its inverse and $R = S^t$ be the transpose of S. Then $a \in (X_T)^\beta$ if and only if*

$$a \in (X^\beta)_R \text{ and } W = (w_{mk})_{m,k=0}^{\infty} \in (X, c_0) \tag{3.9}$$

where

$$w_{mk} = \begin{cases} \sum_{j=m}^{\infty} a_j s_{jk} & (0 \le k \le m) \\ 0 & (k > m) \end{cases} \quad (m = 0, 1, \dots);$$

moreover, if $a \in (X_T)^\beta$ then we have

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} (R_k a)(T_k x) \text{ for all } x \in X_T. \tag{3.10}$$

b) The statement in part a) also holds when $W \in (X, \ell_\infty)$.

Proof. a) First we show that $a \in Z^\beta$ implies that the conditions in (3.9) hold. We write $Z = X_T$ and assume that $a \in Z^\beta$. Then it follows by Lemma 3.3.1 that $Ra \in X^\beta$, hence the series $\sum_{j=n}^{\infty} a_j s_{jk}$ converge for all n and k, that is, the matrix W is defined. Furthermore, we have

$$\begin{aligned}\sum_{k=0}^{n}(R_k a)(T_k z) - \sum_{k=0}^{n}\omega_{nk}T_k z &= \sum_{k=0}^{n}(R_k a - \omega_{nk})T_k z\\ \sum_{k=0}^{n}(\sum_{j=k}^{\infty} a_j s_{jk} - \sum_{j=n}^{\infty} a_j s_{jk})T_k z &= \sum_{k=0}^{n}(\sum_{j=k}^{n-1} a_j s_{jk})T_k z\\ \sum_{j=k}^{n-1}(a_j \sum_{k=0}^{j+1} s_{jk}T_k z) &= \sum_{j=0}^{n-1} a_j S_j(Tz) = \sum_{j=0}^{n-1} a_j z_j,\end{aligned}$$

that is,

$$\sum_{j=0}^{n-1} a_j z_j = \sum_{k=0}^{n}(R_k a)(T_k z) - W_n(Tz) \text{ for all } n \text{ and all } z \in Z. \tag{3.11}$$

Let $x \in X$ be given. Then we have $z = Sx \in Z$, and so $a \in Z^\beta$ and $a \in (X^\beta)_R$. This implies $Wx = W(Tz) \in c$ by (3.11). Since $x \in X$ was arbitrary, we have $W \in (X, c) \subset (X, \ell_\infty)$. Furthermore, since $R_k a = \sum_{j=\ell}^{\infty} a_j s_{jk}$ exists for each k, we have

$$\lim_{n\to\infty}\omega_{nk} = \lim_{n\to\infty}\sum_{j=n}^{\infty} a_j s_{jk} = 0 \text{ for each } k. \tag{3.12}$$

Now $W \in (X, \ell_\infty)$ and (3.12) imply $W \in (X, c_0)$ by (Wilansky, 1984, 8.3.6, p 123). Now we show that if $a \in Z^\beta$ then (3.10) holds. Let $a \in Z^\beta$. Then the conditions (3.9) hold, and so (3.10) follows from (3.11). Finally, we show that the conditions in (3.9) are satisfied. If $z \in Z$ then $x = Tz \in X$, and so $a.z \in cs$ by (3.11), that is, $a \in Z^\beta$.

b) Since $(X, c_0) \subset (X, \ell_\infty)$, it remains to show that $a \in (X^\beta)_R$ and $W \in (X, \ell_\infty)$ imply $a \in (X_T)^\beta$. We assume $a \in (X^\beta)_R$ and $W \in (X, \ell_\infty)$. It follows from $Ra \in X^\beta$ that w_{mk} exists for each m and k, and $\lim_{m\to\infty} w_{mk} = 0$ for each k. Since X has AK, this and $W \in (X, \ell_\infty)$ together imply $W \in (X, c_0)$ by (Wilansky, 1984, 8.3.6). Finally $a \in (X^\beta)_R$ and $W \in (X, c_0)$ together imply $a \in (X_T)^\beta$ by (Malkowsky and Rakočević, 2007, Theorem 3.2).

∎

Now we determine the γ–duals. The following general result holds.

Proposition 3.3.3. *Let X be an FK space. Then we have $a \in (X_T)^\gamma$ if and only if*

$$\sup_n \|C_n(a;T)\|^*_{X,\delta} < \infty \text{ for some } \delta > 0, \tag{3.13}$$

where the matrix $C(a;T) = (c_{nk})_{n,k=0}^\infty$ is given by

$$c_{nk} = \begin{cases} \sum_{j=k}^{n} a_j s_{jk} & (0 \le k \le n) \\ 0 & (k > n) \end{cases} \quad (n = 0, 1, \dots):$$

Proof. We write $Z = X_T$, and define the matrix $B = B(a;T) = (b_{nk})_{n,k=0}^\infty$ by $b_{nk} = a_n s_{nk}$ for $(0 \le k \le n)$ and $b_{nk} = 0$ for $k > n$ $(n = 0, 1, \dots)$. Since $x \in X$ if and only if $z = Sx \in Z$, and

$$a_n z_n = a_n(S_n x) = a_n \sum_{k=0}^{n} s_{nk} x_k = \sum_{k=0}^{n} b_{nk} x_k = B_n x \text{ for all } n,$$

we obtain $a \in Z^\gamma$ if and only if $B \in (X, bs)$, and this is the case by (Malkowsky and Rakočević, 2000a, Theorem 3.8) if and only if $C(a;T) \in (X, \ell_\infty)$, that is, by (Malkowsky and Rakočević, 2000a, Theorem 1.23 (b)), if and only if the condition in (3.13) is satisfied. ∎

Now we determine the β–duals of the spaces $[C_\alpha]_0^p$, $[C_\alpha]^p$ and $[C_\alpha]_\infty^p$.

Theorem 3.3.4. *Let $p \ge 1$ and $\alpha > 0$. Then we have*
(a) $a \in ([C_\alpha]_0^p)^\beta$ if and only if $a \in (\mathcal{M}_p)_{R^{\alpha-1}}$ and $W \in (w_0^p, \ell_\infty)$;
(b) $a \in ([C_\alpha]^p)^\beta$ if and only if $a \in (\mathcal{M}_p)_{R^{\alpha-1}}$ and $W \in (w^p, c)$;
(c) $a \in ([C_\alpha]_\infty^p)^\beta$ if and only if $a \in (\mathcal{M}_p)_{R^{\alpha-1}}$ and $W \in (w_\infty^p, c_0)$.
(d) If $X = w_0^p$ or $X = w_\infty^p$ and $a \in (X_{C_{\alpha-1}})^\beta$, then

$$\sum_{k=0}^\infty a_k z_k = \sum_{k=0}^\infty (R_k^{\alpha-1} a)(\sigma_k^{\alpha-1}(x)) \text{ for all } x \in X_{C_{\alpha-1}};$$

$$\text{also } \|a\|^*_{X_{C_{\alpha-1}}} = \|R_k^{\alpha-1} a\|_{\mathcal{M}_p}. \tag{3.14}$$

If $a \in ([C_\alpha]^p)^\beta$ then

$$\sum_{k=0}^\infty a_k z_k = \sum_{k=0}^\infty (R_k^{\alpha-1} a)(\sigma_k^{\alpha-1}(x)) - \xi\rho \text{ with } \xi \text{ from (1.2) and } \rho = \lim_{m\to\infty} W_m e; \tag{3.15}$$

also

$$\|a\|^*_{[C_\alpha]^p} = |\rho| + \|R^{\alpha-1}a\|_{\mathcal{M}_p} \text{ for all } a \in ([C_\alpha]^p)^\beta. \tag{3.16}$$

(e) We have $a \in ([C_\alpha]^p_0)^\gamma$ *if and only if*

$$\sup_n \|C_n(a; C_{\alpha-1})\|_{\mathcal{M}_p} < \infty \tag{3.17}$$

and $([C_\alpha]^p_0)^\gamma = ([C_\alpha]^p)^\gamma = ([C_\alpha]^p_\infty)^\gamma$.

Proof. (a) This is an immediate consequence of Proposition 3.3.3.

(b), (c) Parts (b) and (c) follow from (Başar et al., 2008, Lemma 3.1 (b), (c)).

(d) The identities in (3.14), (3.15) and (3.16) follow from (Başar et al., 2008, (3.11), (3.12), (3.9) and (3.13)).

(e) Since w^p_0 is a BK space, we have by (3.13) and (3.8) that $a \in ([C_\alpha]^p_0)^\gamma$ if and only if (3.17) holds; also, by (3.8) $\|\cdot\|^*_X = \|\cdot\|_{\mathcal{M}_p}$ for $X = w^p_0, w^p, w^p_\infty$, hence $([C_\alpha]^p_0)^\gamma = ([C_\alpha]^p)^\gamma = ([C_\alpha]^p_\infty)^\gamma$. ∎

Remark 3.3.1. It is useful to state the explicit formulas in the previous theorem. We obtain from (1.4) that

$$R^{\alpha-1}_k a = \sum_{j=k}^\infty A^{-\alpha}_{j-k} A^{\alpha-1}_k a_j \text{ for } k = 0, 1, \dots,$$

hence $a \in (\mathcal{M}_p)_{R^{\alpha-1}}$ if and only if

$$\left\|R^{\alpha-1}a\right\|_{\mathcal{M}_p} = \begin{cases} \sum\limits_{\nu=0}^\infty 2^\nu \max_v \left|\sum\limits_{j=k}^\infty A^{-\alpha}_{j-k} A^{\alpha-1}_k a_j\right| < \infty & (p = 1) \\ \sum\limits_{\nu=0}^\infty 2^{\nu/p} \left(\sum_v \left|\sum\limits_{j=k}^\infty A^{-\alpha}_{j-k} A^{\alpha-1}_k a_j\right|^q\right)^{1/q} < \infty & \left(p > 1; q = \dfrac{p}{p-1}\right). \end{cases} \tag{3.18}$$

Furthermore, we have

$$w_{mk} = \begin{cases} \sum\limits_{j=m}^\infty A^{-\alpha}_{j-k} A^{\alpha-1}_k a_j & (0 \le k \le m) \\ 0 & (k > m) \end{cases} \quad (m = 0, 1, \dots),$$

and, by (Başar et al., 2008, Theorem 2.4 **1.**), $W \in (w_0^p, \ell_\infty)$ if and only if

$$\sup_m \|W_m\|_{\mathcal{M}_p} < \infty. \tag{3.19}$$

If $m \in \mathbb{N}_0$ is given, let $\nu(m)$ denote the uniquely defined integer with $2^{\nu(m)} \le m \le 2^{\nu(m)+1} - 1$. Then it follows that

$$\|W_m\|_{\mathcal{M}_p} = \begin{cases} \sum_{\nu=0}^{\nu(m)-1} 2^\nu \max_v \left| \sum_{j=m}^{\infty} A_{j-k}^{-\alpha} A_k^{\alpha-1} a_j \right| + 2^{\nu(m)} \max_{2^{\nu(m)} \le k \le m} \left| \sum_{j=m}^{\infty} A_{j-k}^{-\alpha} A_k^{\alpha-1} a_j \right| \\ \quad (p = 1); \\ \sum_{\nu=0}^{\nu(m)-1} 2^{\nu/p} \left(\sum_v \left| \sum_{j=m}^{\infty} A_{j-k}^{-\alpha} A_k^{\alpha-1} a_j \right|^q \right)^{1/q} \\ \qquad\qquad + 2^{\nu(m)/p} \left(\sum_{k=2^{\nu(m)}}^{m} \left| \sum_{j=m}^{\infty} A_{j-k}^{-\alpha} A_k^{\alpha-1} a_j \right|^q \right)^{1/q} \\ \quad \left(p > 1; q = \dfrac{p}{p-1} \right). \end{cases}$$

We have by (Başar et al., 2008, Theorem 2.4 **7.**) that $W \in (w^p, c)$ if and only if (3.19) holds,

$$\rho = \lim_{m\to\infty} W_m e = \lim_{m\to\infty} \sum_{k=0}^{m} w_{mk} = \lim_{m\to\infty} \sum_{k=0}^{m} \sum_{j=m}^{\infty} A_{j-k}^{-\alpha} A_k^{\alpha-1} a_j \text{ exists} \tag{3.20}$$

and $\lim_{m\to\infty} w_{mk} = \gamma_k$ exists for each k, which is redundant since the convergence of $R_k^{\alpha-1} a$ for each k implies $\lim_{m\to\infty} w_{mk} = \gamma_k = 0$ for each k.
Finally, it follows from (Başar et al., 2008, Theorem 2.4 **2.**) that $W \in (w_\infty^p, c_0)$ if and only if $\lim_{m\to\infty} \|W_m\|_{\mathcal{M}_p} = 0$.

There is an alternative way to determine the β–duals.

Remark 3.3.2. It follows from (Malkowsky and Rakočević, 2007, Lemmas 3.7 and 3.8) that if X is an FK space with AK then $a \in (X_T)^\beta$ if and only if $C(a;T) \in (X, c)$ where $C(a;T)$ is the matrix defined in Proposition 3.3.3. We also have $a \in ((w_\infty^p)_T)^\beta$ or $a \in ((w^p)_T)^\beta$ if and only if $C(a; C_{\alpha-1}) \in (X, c)$ by (Malkowsky and Rakočević, 2007, Lemma 3.7) and (Başar et al., 2008, Lemma 3.1 (c) or (b)), respectively.

Remark 3.3.3. We apply our results to crystallography. We use Wulff's principle (Wulff, 1901) which allows us to determine the shape of crystals from our norms.

Theorem 3.3.5 (Wulff's principle)**.** *Let ∂B^n denote the unit sphere in $\mathbb{R}^{n+1}$, and $F : \partial B^n \to \mathbb{R}$ be a surface energy function. The set $PM = \{\vec{x} = F(\vec{e})\vec{e} \in \mathbb{R}^{n+1} : \vec{e} \in \partial B^n\}$ can be considered as a natural representation of F. For every $\vec{e} \in \partial B^n$, let $E_{\vec{e}}$ denote the hyperplane orthogonal to $\vec{e}$ and through the point P with position vector $\vec{p} = F(\vec{e})\vec{e}$, and $H_{\vec{e}}$ be the half space which contains the origin 0 and has the boundary $E_{\vec{e}} = \partial H_{\vec{e}}$. Then, the crystal C_F which has F as its surface energy function is uniquely determined and given by $C_F = \bigcap_{\vec{e}\in\partial B^n} H_{\vec{e}} = \bigcap_{\vec{e}\in\partial B^n}\{\vec{x} : \vec{x} \bullet \vec{e} \leq F(\vec{e})\}$.*

The principles of Wulff's construction of crystals were studied in (Malkowsky and Veličković, 2012) and it was proved that if surface energy function is equal to a norm then the boundary of the corresponding Wulff's crystal is given by the dual norm. Also, we have

Corollary 3.3.6. *(Malkowsky and Veličković, 2012, Corollary 5.5) Let $\|\cdot\|$ be a norm on $\mathbb{R}^{n+1}$ and, for each $\vec{w} \in \partial B^n$, let $\phi_{\vec{w}} : \mathbb{R}^{n+1} \to \mathbb{R}$ be defined by $\phi_{\vec{w}}(x) = \vec{w} \bullet \vec{x} = \sum_{k=1}^{n+1} w_k x_k$ $(\vec{x} \in \mathbb{R}^{n+1})$. Then, the boundary $\partial C_{\|\cdot\|}$ of Wulff's crystal corresponding to $\|\cdot\|$ is given by*

$$\partial C_{\|\cdot\|} = \left\{\vec{x} = \frac{1}{\|\phi_e\|^*} \cdot \vec{e} \in \mathbb{R}^{n+1} : \vec{e} \in \partial B^n\right\}, \tag{3.21}$$

where $\|\phi_{\vec{e}}\|^*$ is the norm of the functional $\phi_{\vec{e}}$, that is, the dual norm of $\|\cdot\|$.

Example 3.3.1. Finally, we visualise β–duals of $[C_\alpha]^p_\infty$. We consider the dual norm $\|\cdot\|_{M_p}$ defined by (3.18) and obtain

for $k = 0$

$$\begin{aligned}
R_0^{\alpha-1}x &= \sum_{j=0}^{\infty} A_{j-0}^{-\alpha} A_0^{\alpha-1} x_j \\
&= A_0^{-\alpha}x_0 + A_1^{-\alpha}x_1 + A_2^{-\alpha}x_2 + A_3^{-\alpha}x_3 + \dots \\
&= x_0 + (1-\alpha)x_1 + \frac{(2-\alpha)(1-\alpha)}{2}x_2 \\
&\qquad\qquad + \frac{(3-\alpha)(2-\alpha)(1-\alpha)}{6}x_3 + \dots \\
&= x_0 - (\alpha-1)x_1 + \frac{(\alpha-2)(\alpha-1)}{2}x_2 \\
&\qquad\qquad - \frac{(\alpha-3)(\alpha-2)(\alpha-1)}{6}x_3 + \dots;
\end{aligned}$$

for $k=1$

$$\begin{aligned}
R_1^{\alpha-1}x &= \sum_{j=1}^{\infty} A_{j-1}^{-\alpha} A_1^{\alpha-1} x_j \\
&= \alpha\left[A_0^{-\alpha}x_1 + A_1^{-\alpha}x_2 + A_2^{-\alpha}x_3 + \dots\right] \\
&= \alpha\left[x_1 + (1-\alpha)x_2 + \frac{(2-\alpha)(1-\alpha)}{2}x_3 + \dots\right] \\
&= \alpha\left[x_1 - (\alpha-1)x_2 + \frac{(\alpha-2)(\alpha-1)}{2}x_3 + \dots\right];
\end{aligned}$$

for $k=2$

$$\begin{aligned}
R_2^{\alpha-1}x &= \sum_{j=2}^{\infty} A_{j-2}^{-\alpha} A_2^{\alpha-1} x_j \\
&= \frac{\alpha(\alpha+1)}{2}\left[A_0^{-\alpha}x_2 + A_1^{-\alpha}x_3 + \dots\right] \\
&= \frac{\alpha(\alpha+1)}{2}\left[x_2 + (1-\alpha)x_3 + \dots\right] \\
&= \frac{\alpha(\alpha+1)}{2}\left[x_2 - (\alpha-1)x_3 + \dots\right];
\end{aligned}$$

for $k=3$

$$\begin{aligned}
R_3^{\alpha-1}x &= \sum_{j=3}^{\infty} A_{j-3}^{-\alpha} A_3^{\alpha-1} x_j \\
&= \frac{(\alpha+2)(\alpha+1)\alpha}{6}\left[A_0^{-\alpha}x_3 + \dots\right] \\
&= \frac{(\alpha+2)(\alpha+1)\alpha}{6}[x_3 + \dots].
\end{aligned}$$

Now we choose three dimensions x_0, x_1, x_2 to project the norm $\|\cdot\|_{M_p}$ on (x_0, x_1, x_2). If $p=1$, then we have

- for $\nu = 0$,

$$\begin{aligned}
2^{\nu}\max_{\nu}|R_k^{\alpha-1}x| = 2^0 \max_{0}|R_k^{\alpha-1}x| &= \max_{0\le k\le 1}|R_k^{\alpha-1}x| \\
&= \max\left\{R_0^{\alpha-1}x, R_1^{\alpha-1}x\right\} \\
&= \max\left\{\left|x_0 - (\alpha-1)x_1 + \frac{(\alpha-2)(\alpha-1)}{2}x_2\right|,\right. \\
&\qquad \left.|\alpha\left[x_1 - (\alpha-1)x_2\right]|\right\};
\end{aligned}$$

- for $\nu = 1$,

$$\begin{aligned} 2^{\nu} \max_{\nu} |R_k^{\alpha-1}x| &= 2^1 \max_{2^1 \leq k \leq 2^2-1} |R_k^{\alpha-1}x| \\ &= 2 \max \left\{ |R_2^{\alpha-1}x|, |R_3^{\alpha-1}x| \right\} \\ &= 2 \max \left\{ |\frac{\alpha(\alpha+1)}{2}x_2|, 0 \right\} = |\alpha(\alpha+1)x_2| \end{aligned}$$

- for $\nu = 2$,

$$2^{\nu} \max_{\nu} |R_k^{\alpha-1}x| = 2^2 \max_{2^2 \leq k \leq 2^3-1} |R_k^{\alpha-1}x| = 0.$$

Hence

$$\begin{aligned} \|R^{\alpha-1}x\|_{M_1} &= \sum_{\nu=0}^{\infty} 2^{\nu} \max_{\nu} |R_k^{\alpha-1}x| \\ &= \max \left\{ \left| x_0 - (\alpha-1)x_1 + \frac{(\alpha-2)(\alpha-1)}{2}x_2 \right| \right. \\ &\qquad \left. + |\alpha[x_1 - (\alpha-1)x_2]| + |\alpha(\alpha+1)x_2| \right\}. \end{aligned}$$

If $p > 1$,

- for $\nu = 0$,

$$\begin{aligned} 2^{\frac{\nu}{p}} \left(\sum_{\nu} \left|R_k^{\alpha-1}x\right|^q \right)^{\frac{1}{q}} &= 2^{\frac{0}{p}} \left(\sum_{0} |R_k^{\alpha-1}x|^q \right)^{\frac{1}{q}} = \left(\sum_{0} |R_k^{\alpha-1}x|^q \right)^{\frac{1}{q}} \\ &= \left(\left|R_0^{\alpha-1}x\right|^q + \left|R_1^{\alpha-1}x\right|^q \right)^{\frac{1}{q}} \\ &= \left(\left| x_0 - (\alpha-1)x_1 + \frac{(\alpha-2)(\alpha-1)}{2}x_2 \right|^q \right. \\ &\qquad \left. + |\alpha\left[x_1 - (\alpha-1)x_2\right]|^q \right)^{\frac{1}{q}} \end{aligned}$$

- for $\nu = 1$,

$$\begin{aligned} 2^{\frac{\nu}{p}} \left(\sum_{\nu} \left|R_k^{\alpha-1}\right|^q \right)^{\frac{1}{q}} &= 2^{\frac{1}{p}} \left(\sum_{k=2^1}^{2^2-1} \left|R_k^{\alpha-1}x\right|^q \right)^{\frac{1}{q}} \\ &= 2^{\frac{1}{p}} \left(\left|R_2^{\alpha-1}x|^q\right| + |R_3^{\alpha-1}x|^q| \right)^{\frac{1}{q}} \\ &= 2^{\frac{1}{p}} |\frac{\alpha(\alpha+1)}{2}x_2| = 2^{\frac{1}{p}-1}|\alpha(\alpha+1)x_2|. \end{aligned}$$

Hence

$$\begin{aligned}
\|R^{\alpha-1}x\|_{M_p} &= \sum_{\nu=0}^{\infty} 2^{\frac{\nu}{p}} \left(\sum_{\nu} |R_k^{\alpha-1}x|^q \right)^{\frac{1}{q}} = \sum_{\nu=0}^{1} 2^{\frac{\nu}{p}} \left(\sum_{\nu} |R_k^{\alpha-1}x|^q \right)^{\frac{1}{q}} \\
&= \left(\left| x_0 - (\alpha-1)x_1 + \frac{(\alpha-2)(\alpha-1)}{2} x_2 \right|^q \right. \\
&\left. + |\alpha\left[x_1 - (\alpha-1)x_2\right]|^q \right)^{\frac{1}{q}} + 2^{\frac{1}{p}-1} |\alpha(\alpha+1)x_2|.
\end{aligned}$$

Then we obtain the following figures for different values of p and α.

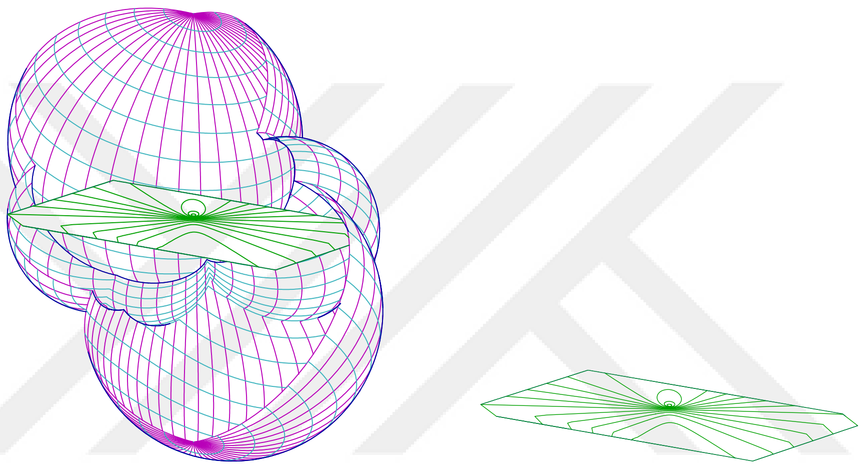

Figure 3.12 Norm $\|\cdot\|_{[C_\alpha]^p_\infty}$ and the appropriate crystal for $p = 1$ and $\alpha = 0.09$ as a potential surface.

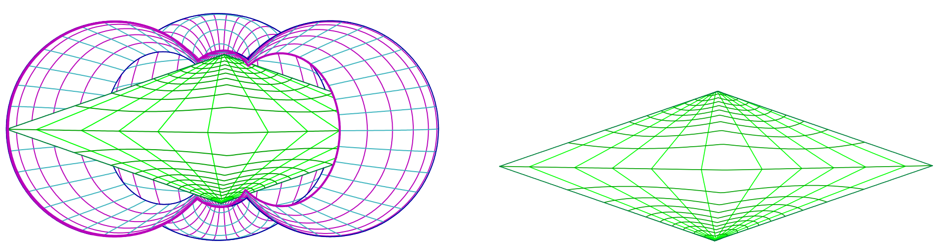

Figure 3.13 Norm $\|\cdot\|_{[C_\alpha]^p_\infty}$ and the appropriate crystal for $p = 1$ and $\alpha = 1.05$ as a potential surface.

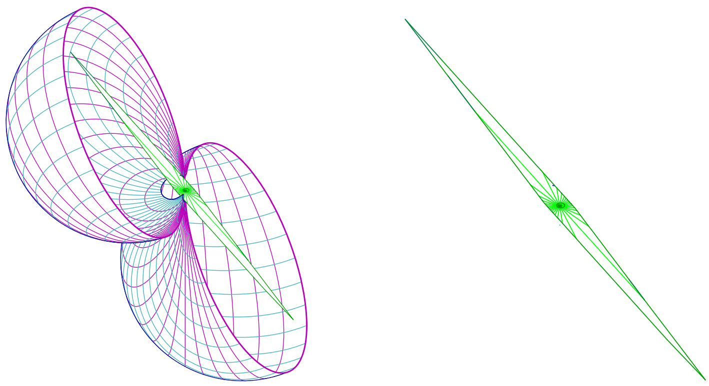

Figure 3.14 Norm $\| \cdot \|_{[C_\alpha]^p_\infty}$ and the appropriate crystal for $p = 1$ and $\alpha = 15$ as a potential surface.

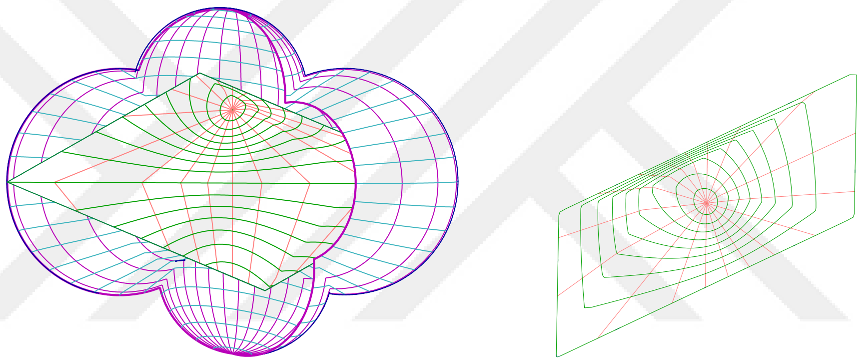

Figure 3.15 Norm $\| \cdot \|_{[C_\alpha]^p_\infty}$ and the appropriate crystal for $p = 1.01$ and $\alpha = 0.5$ as a potential surface.

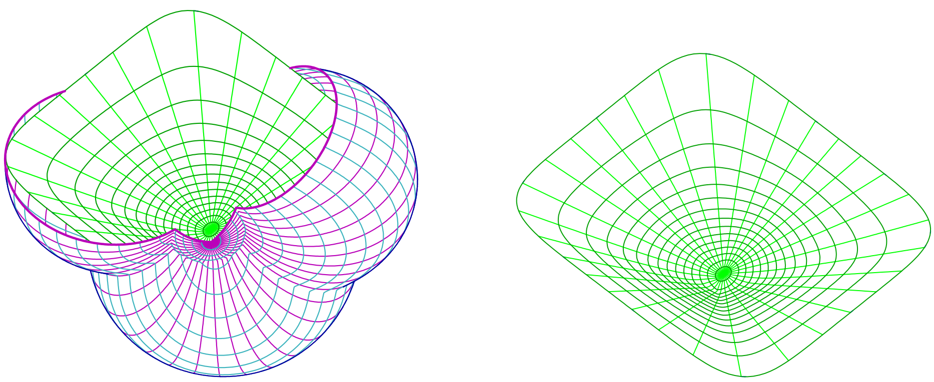

Figure 3.16 Norm $\| \cdot \|_{[C_\alpha]^p_\infty}$ and the appropriate crystal for $p = 1.2$ and $\alpha = 1.2$ as a potential surface.

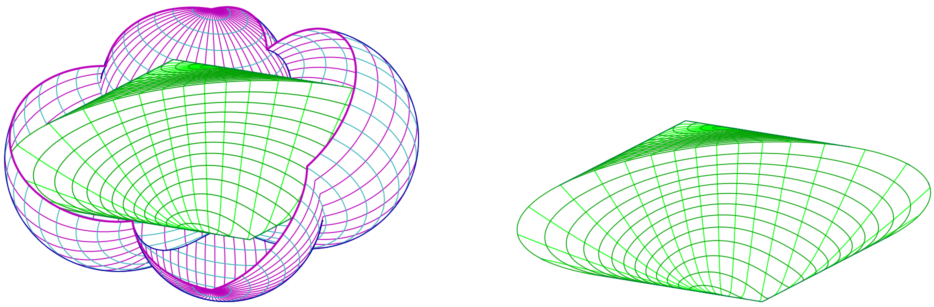

Figure 3.17 Norm $\|\cdot\|_{[C_\alpha]^p_\infty}$ and the appropriate crystal for $p = 2$ and $\alpha = 0.6$ as a potential surface.

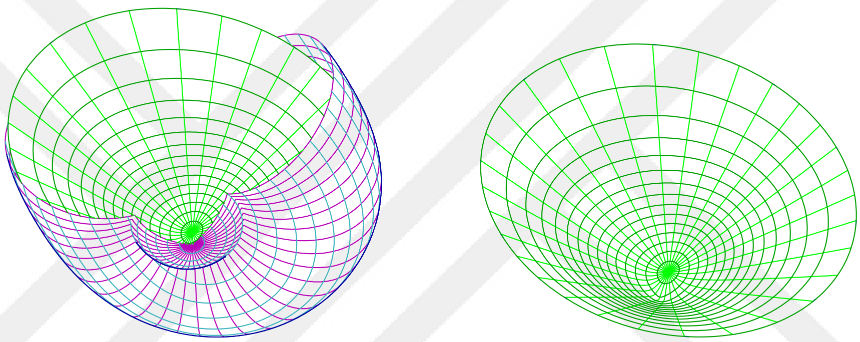

Figure 3.18 Norm $\|\cdot\|_{[C_\alpha]^p_\infty}$ and the appropriate crystal for $p = 2$ and $\alpha = 1.2$ as a potential surface.

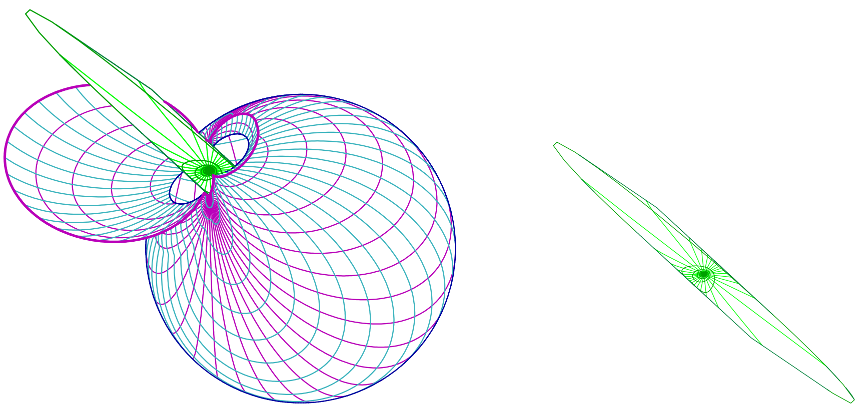

Figure 3.19 Norm $\|\cdot\|_{[C_\alpha]^p_\infty}$ and the appropriate crystal for $p = 2$ and $\alpha = 8$ as a potential surface.

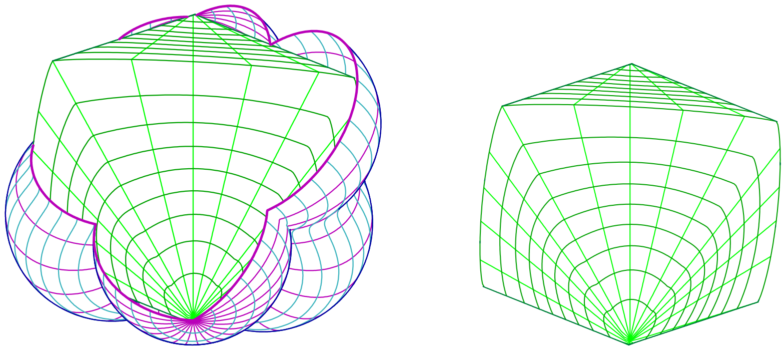

Figure 3.20 Norm $\|\cdot\|_{[C_\alpha]^p_\infty}$ and the appropriate crystal for $p = 10$ and $\alpha = 1.02$ as a potential surface.

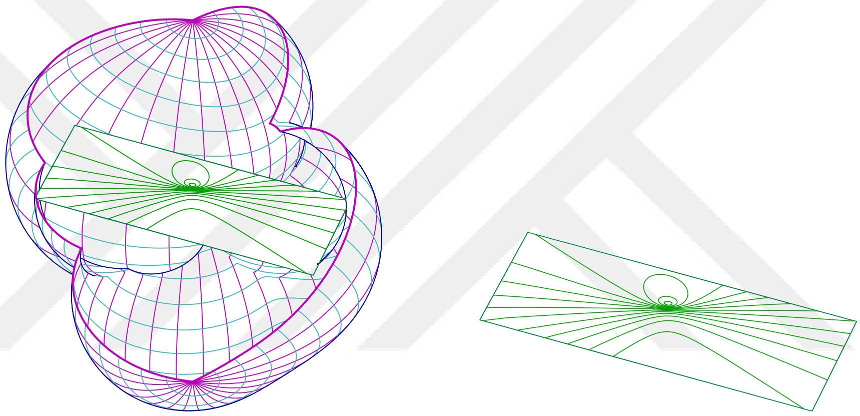

Figure 3.21 Norm $\|\cdot\|_{[C_\alpha]^p_\infty}$ and the appropriate crystal for $p = 10$ and $\alpha = 0.15$ as a potential surface.

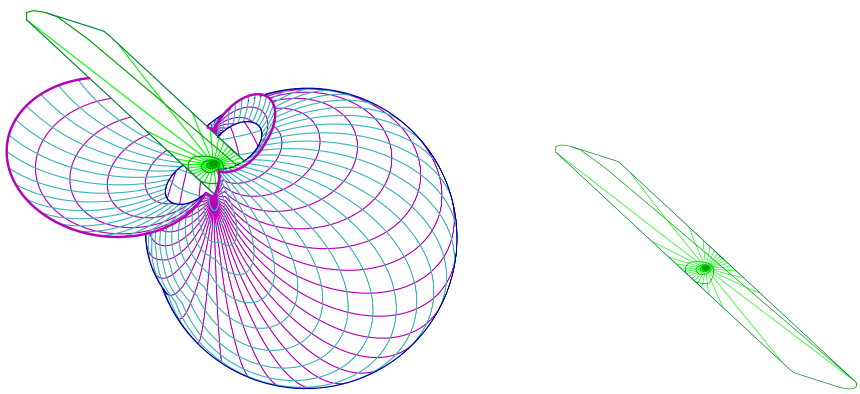

Figure 3.22 Norm $\|\cdot\|_{[C_\alpha]^p_\infty}$ and the appropriate crystal for $p = 10$ and $\alpha = 10$ as a potential surface.

CHAPTER 4

CHARACTERISATIONS OF BOUNDED LINEAR OPERATORS ON SOME GENERAL MIXED NORM SPACES

In this chapter, we characterize the classes (X, Y) where X is any of the spaces $[C_\alpha]_0^p$, $[C_\alpha]^p$ or $[C_\alpha]_\infty^p$ and Y is any of the spaces ℓ_∞, c or c_0. We also determine the norms of the operators defined by the matrices in those classes.

Let $A = (a_{nk})_{n,k=0}^\infty$ be an infinite matrix. We define the matrices $\hat{A} = (\hat{a}_{nk})_{n,k=0}^\infty$ and $W^{(n)} = (w_{mk}^{(n)})_{m,k=0}^\infty$ for $n = 0, 1, \dots$ by

$$\hat{a}_{nk} = R_k^{\alpha-1} A_n = \sum_{j=k}^\infty A_{j-k}^{-\alpha} A_k^{\alpha-1} a_{nj} \text{ for } n, k = 0, 1, \dots$$

and

$$w_{mk}^{(n)} = \begin{cases} \sum\limits_{j=m}^\infty A_{j-k}^{-\alpha} A_k^{\alpha-1} a_{nj} & (0 \leq k \leq m) \\ 0 & (k > m) \end{cases} \qquad (m = 0, 1, \dots).$$

The following result is an immediate consequence of (Başar et al., 2008, Theorem 4.2) with $T = C_{\alpha-1}$.

Theorem 4.0.7. *The necessary and sufficient conditions for the entries of $A \in (X_{C_{\alpha-1}}, Y)$ when $X \in w_0^p, w^p, w_\infty^p$ and $Y = \{\ell_\infty, c_0, c\}$ can be read from the following table*

Table 4.1

To \ From	$[C_\alpha]^p_\infty$	$[C_\alpha]^p_0$	$[C_\alpha]^p$
ℓ_∞	**1.**	**2.**	**3.**
c_0	**4.**	**5.**	**6.**
c	**7.**	**8.**	**9.**

where

1. (1.1$^+$) $\sup_n \|\hat{A}_n\|_{\mathcal{M}_p} < \infty$ *and* (1.2$^+$) $\lim_{m\to\infty} \|W_m^{(n)}\|_{\mathcal{M}_p} = 0$ *for all* n

2. (1.1$^+$) *and* (2.1$^+$) $\sup_m \|W_m^{(n)}\|_{\mathcal{M}_p} < \infty$ *for all* n

3. (1.1$^+$), (2.1$^+$), (3.1$^+$) $\rho^{(n)} = \lim_{m\to\infty} W_m^{(n)} e$ *exists for each* n *and*

(3.2$^+$) $\sup_n \left| \hat{A}_n e - \rho^{(n)} \right| < \infty$

4. (1.2$^+$) *and* (4.1$^+$) $\lim_{n\to\infty} \|\hat{A}_n\|_{\mathcal{M}_p} = 0$

5. (1.2$^+$), (2.1$^+$) *and* (5.1$^+$) $\lim_{n\to\infty} \hat{a}_{nk} = 0$ *for all* k

6. (1.1$^+$), (2.1$^+$), (3.1$^+$), (5.1$^+$) *and* (6.1$^+$) $\lim_{n\to\infty} \left(\hat{A}_n e - \rho^{(n)} \right) = 0$

7. (1.1$^+$), (7.1$^+$) $\hat{\alpha}_k = \lim_{n\to\infty} \hat{a}_{nk}$ *exists for all* k

(7.2$^+$) $(\hat{\alpha}_k), \hat{A}_n \in \mathcal{M}_p$ *for all* n *and*

(7.3$^+$) $\lim_{n\to\infty} \|\hat{A}_n - (\hat{\alpha}_k)\|_{\mathcal{M}_p} = 0$

8. (1.1$^+$), (2.1$^+$) *and* (7.1$^+$)

9. (1.1$^+$), (2.1$^+$), (3.1$^+$), (7.1$^+$) *and* (9.1$^+$) $\lim_{n\to\infty} \left(\hat{A}_n e - \rho^{(n)} \right) = \beta$ *exists.*

Remark 4.0.4. We note that by (3.20) and the definition of $\hat{A}$ and $W^{(n)}$

$$\hat{A}_n e - \rho^{(n)} = \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} A_{j-k}^{-\alpha} A_k^{\alpha-1} a_{nj} - \lim_{m\to\infty} \sum_{k=0}^{m} \sum_{j=m}^{\infty} A_{j-k}^{-\alpha} A_k^{\alpha-1} a_{nj},$$

$$\hat{\alpha}_k = \lim_{n\to\infty} \sum_{j=k}^{\infty} A_{j-k}^{-\alpha} A_k^{\alpha-1} a_{nj} \text{ for each } k$$

and, for $n = 0, 1, \ldots,$

$$\left\|\hat{A}_n - (\hat{\alpha}_k)\right\|_{\mathcal{M}_p} = \begin{cases} \sum_{\nu=0}^{\infty} 2^{\nu} \max_v |\hat{a}_{nk} - \hat{\alpha}_k| & (p=1) \\ \sum_{\nu=0}^{\infty} 2^{\nu/p} \left(\sum_v |\hat{a}_{nk} - \hat{\alpha}_k|^p\right)^{1/p} & \left(p > 1; q = \dfrac{p}{p-1}\right). \end{cases}$$

Now we determine the norms of the operators associated with the matrices in the classes of Theorem 4.0.7.

Lemma 4.0.8. *a) The statement of Proposition 3.3.2 also holds when* $X = \omega_0^p$ *or* $X = \omega_\infty^p$.

b) (Başar et al., 2008, Theorem 3.2) If $a \in \{(\omega^p)_T\}^\beta$ *then we have for all* $z \in \omega^p(T)$

$$\sum_{k=0}^{\infty} a_k z_k = \sum_{k=0}^{\infty} ((R_k a)(T_k z) - \xi\rho \tag{4.1}$$

where $\xi = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} |x_k|^p$ *and* $\rho = \lim_{n\to\infty} \|W_n\|_{M_p}$.

Proof. a) Since $X = \omega_0^p$ is an FK space with AK, we only prove the statement for $X = \omega_\infty^p$. Let $X = \omega_\infty^p$. We have to show that $W \in (\omega_\infty^p, c)$ implies $W \in (\omega_\infty^p, c_0)$. If $W \in (\omega_\infty^p, c)$ then it follows

$$\|W_n\|_{M_p} \text{ converges uniformly in } n. \tag{4.2}$$

But, in Part a) of the proof of Proposition 3.3.2, we also have $\lim_{n\to\infty} \omega_{nk} = 0$ for each k. This and (4.2) imply

$$\lim_{n\to\infty} \|W_n\|_{M_p} = 0.$$

From this we obtain $W \in (\omega_\infty^p, c_0)$.

b) Let $a \in \{(\omega^p)_T\}^\beta$ and $z \in (\omega^p)_T$ be given. Then we have $x = Tz \in \omega^p$ and $\xi = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} |x_k|^p$ exists. Hence there is $x^{(0)} \in \omega_0^p$ such that $x = x^{(0)} + \xi e$. We put $z^{(0)} = Sx^{(0)}$. Then it follows that $z^{(0)} \in (\omega_0^p)_T$ and $z = Sx = S(x^{(0)} + \xi e) = z^{(0)} + \xi Se$ and we obtain

$$\sum_{k=0}^{n-1} a_k z_k = \sum_{k=0}^{n} (R_k a)(T_k z) - W_n(T(z^{(0)} + \xi Se))$$

$$= \sum_{k=0}^{n}(R_k a)(T_k z) - W_n(Tz^{(0)}) - \xi W_n e \text{ for all } n.$$

The first term in the last equality converges as $n \to \infty$ since $Ra \in M_p$ by Corollary 3.3.4. The second term in the last equality tends to zero as $n \to \infty$ since $a \in \{(\omega^p)_T\}^\beta \subset \{(\omega_0^p)_{(T)}\}^\beta$ implies $W \in (\omega_0^p, c_0)$ by Corollary 3.3.4. Finally we also have $W \in (\omega^p, c)$ by Corollary 3.3.4 and this implies $\rho = \lim_{n\to\infty} W_n e$ exists. Now the identity in (4.1) follows.

∎

Remark 4.0.5. a) If $a \in X^\beta$ where $X = [C_\alpha]_0^p$ or $X = [C_\alpha]_\infty^p$ then we have for all $z \in X$

$$\sum_{k=0}^{\infty} a_k z_k = \sum_{k=0}^{\infty}((R^{\alpha-1})_k a)((C_{\alpha-1})_k z). \tag{4.3}$$

b) If $a \in \{[C_\alpha]^p\}^\beta$ then we have for all $z \in [C_\alpha]^p$

$$\sum_{k=0}^{\infty} a_k z_k = \sum_{k=0}^{\infty}((R^{\alpha-1})_k a)((C_{\alpha-1})_k z) - \xi\rho \tag{4.4}$$

where $\xi = \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n}|x_k|^p$ and $\rho = \lim_{n\to\infty}\sum_{k=0}^{n}\omega_{nk}$.

Now we determine the norms $\|a\|_{[C_\alpha]_\infty^p}$, $\|a\|_{[C_\alpha]_0^p}$ and $\|a\|_{[C_\alpha]^p}$.

Proposition 4.0.9. *We have*

a) $\|a\|^*_{[C_\alpha]_\infty^p} = \|a\|^*_{[C_\alpha]_0^p} = \|R^{\alpha-1}a\|_{M_p}$ *for all* $a \in \{[C_\alpha]_\infty^p\}^\beta, \{[C_\alpha]_0^p\}^\beta$.

b) $\|a\|^*_{[C_\alpha]^p} = \|R^{\alpha-1}a\|_{M_p} + |\rho|$ *for all* $a \in \{[C_\alpha]^p\}^\beta$
where $\rho = lim_{n\to\infty}\sum_{k=0}^{n}\omega_{nk}$.

Proof. a) Let $a \in \{[C_\alpha]_0^p\}^\beta$. Then it follows from part a) of Remark 4.0.5 that $R_{\alpha-1}a \in M_p$ and (3.10) holds. Since $z \in [C_\alpha]_0^p$ if and only if $x = C_{\alpha-1}z \in \omega_0^p$, and $\|z\|_{[C_\alpha]_0^p} = \|x\|_{M_p}$ by (Wilansky, 1984, Theorem 4.3.12, p.63), the right-hand side of (3.10) defines a functional $f \in \omega_0^{p*}$ with its norm $\|f\| = \|R^{\alpha-1}a\|_{M_p}$ and $\|a\|^*_{[C_\alpha]_0^p} = \|f\|$ by the definition of the norm $\|\cdot\|^*_{[C_\alpha]_0^p}$.

b) Let $a \in \{[C_\alpha]^p\}^\beta$. Then it follows from part b) of Remark 4.0.5 that $R^{\alpha-1}a \in M_p$ and (4.1) holds. Since $z \in [C_\alpha]^p$ if and only if $x = C_{\alpha-1}z \in \omega^p$, and

$\|z\|_{[C_\alpha]^p} = \|x\|_{M_p}$ by (Wilansky, 1984, Theorem 4.3.12, p.63), the right-hand side of (4.1) defines a functional $f \in \omega^{p*}$ with its norm $\|f\| = \|R^{\alpha-1}a\|_{M_p} + |\rho|$ and $\|a\|^*_{[C_\alpha]^p} = \|f\|$ by the definition of the norm $\|\cdot\|^*_{[C_\alpha]^p}$.

∎

Lemma 4.0.10. *(Malkowsky and Rakočević, 2007, Theorem 3.6) Let X and Y be BK spaces and X have AK or $X = \ell_\infty$. If $A \in (X_T, Y)$, then we have*

$$\|L_A\| = \|L_{\hat{A}}\| \tag{4.5}$$

where $\hat{A}$ is the matrix with the rows $\hat{A}_n = R(A_n)$ for $n = 0, 1, ...$

Lemma 4.0.11. *(Malkowsky and Rakočević, 2000a, Theorem 1.23) Let X be a BK space and Y be any of the spaces ℓ_∞, c, c_0. If $A \in (X, Y)$ then*

$$\|L_A\| = \|A\|_{(X,\infty)} = \sup_n \|A_n\|^*_X < \infty. \tag{4.6}$$

Theorem 4.0.12. *Let Y be any of the spaces ℓ_∞, c or c_0.*
(a) If $A \in (X_T, Y)$, where $X = w_0^p$ or $X = w_\infty^p$, then we have

$$\|L_A\| = \|A\|_{(X_T,\infty)} = \sup_n \|\hat{A}_n\|^*_X = \sup_n \|RA_n\|_{\mathcal{M}_p} \tag{4.7}$$

(b) If $(A \in (w^p)_T, Y)$, then we have

$$\|L_A\| = \|A\|_{((w^p)_T,\infty)} = \sup_n \left(\|RA_n\|_{\mathcal{M}_p} + |\rho^{(n)}|\right), \tag{4.8}$$

where $\rho^{(n)}$ is defined in (3.1^+) in **3.** *of Theorem 4.0.7.*

Proof. (a) Let $X = w_0^p$ or $X = w_\infty^p$.
If $A \in (X_T, Y)$ then it follows from (Malkowsky and Rakočević, 2007, Theorem 3.4 and Remark 3.5(b)) that $\hat{A} \in (X, Y)$ and $Ax = \hat{A}(Tx)$ for all $x \in X_T$. For $X = w_0^p$, it follows from Lemma 4.0.10 that

$$\|L_{\hat{A}}\| = \|L_A\|. \tag{4.9}$$

Since the norms on w_0^p and w_∞^p, and on $(w_0^p)_T$ and $(w_\infty^p)_T$ are the same, the identity in (4.9) also holds for $X = w^p_\infty$. Furthermore, $\hat{A} \in (X, \ell_\infty)$ implies by Lemma 4.0.11, (3.8) and the definition of the matrix $\hat{A}$

$$\|L_{\hat{A}}\| = \sup_n \|\hat{A}_n\|^*_X = \sup_n \|\hat{A}_n\|_{\mathcal{M}_p} = \sup_n \|RA_n\|_{\mathcal{M}_p}.$$

Thus we have shown the identity in (4.7) for $Y = \ell_\infty$. The identity in (4.7) for $Y = c_0$ or $Y = c$ now follows, since $(X_T, Y) \subset (X_T, \ell_\infty)$.

(b) Let $A \in ((w^p)_T, Y)$ where $Y \in \{\ell_\infty, c, c_0\}$. Then it follows by (1.1^+) in Theorem 4.0.7 **3.**, **6.** and **9.** that $\sup_n \|\hat{A}_n\|_{\mathcal{M}_p} < \infty$. Together with (3.2^+) in Theorem 4.0.7 **3.** for $Y = \ell_\infty$, (6.1^+) in Theorem 4.0.7 **6.** for $Y = c_0$ and (9.1^+) in Theorem 4.0.7 **9.** for $Y = c$, we obtain $\rho^{(n)} \in \ell_\infty$ for each n. Therefore the right hand side in (4.8) is defined and finite. Since w^p is a BK space, we have as in Part (a)

$$\|L_A\| = \sup_n \|A_n\|^*_{[C_\alpha]^p}, \tag{4.10}$$

and $A_n \in ([C_\alpha]^p)^\beta$ for all n implies by Lemma 4.0.8 b)

$$\|A_n\|^*_{[C_\alpha]^p} = \|RA_n\|_{\mathcal{M}_p} + |\rho^{(n)}| \text{ for all } n. \tag{4.11}$$

Now (4.8) follows from (4.10) and (4.11). ∎

Corollary 4.0.13. *Let* $Y = \ell_\infty, c, c_0$.

a) If $A \in ([C_\alpha]^p_0, Y)$, *then we have*

$$\|L_A\| = \sup_n \|R_{\alpha-1}A_n\|_{M_p}. \tag{4.12}$$

b) If $A \in ([C_\alpha]^p, Y)$, *then we have*

$$\|L_A\| = \sup_n \left(\|R_{\alpha-1}A_n\|_{M_p} + |\rho|\right) \tag{4.13}$$

where $\rho = \{\rho^{(n)}\}_{n=1}^\infty = \{\lim_{m\to\infty} \sum_{k=1}^m \sum_{j=m}^\infty a_{nj} A^{-\alpha}_{j-k} A^{\alpha-1}_k\}_{n=1}^\infty$.

CHAPTER 5

COMPACT OPERATORS ON SOME GENERAL MIXED NORM SPACES

5.1 HAUSDORFF MEASURE OF NONCOMPACTNESS OF THESE OPERATORS

In this section we investigate Hausdorff measure of noncompactness of operators between Banach spaces.

Definition 9. (Malkowsky and Rakočević, 2000a, Definition 2.24) Let X and Y be Banach spaces and χ_1 and χ_2 be measures of noncompactness on X and Y. Then the operator $L : X \to Y$ is called (χ_1, χ_2)–**bounded** if $L(Q) \in \mathcal{M}_Y$ for every $Q \in \mathcal{M}_X$ and there exist a constant $C > 0$ such that

$$\chi_2(L(Q)) \leq C \cdot \chi_1(Q) \text{ for all } Q \in \mathcal{M}_X; \tag{5.1}$$

if L is (χ_1, χ_2)–bounded then the number

$$\|L\|_{(\chi_1,\chi_2)} = \inf\{C > 0 : (5.1) \text{ holds }\}$$

is called the (χ_1, χ_2)–**measure of noncompactness of** L; we also write $\|L\|_\chi = \|L\|_{(\chi,\chi)}$, for short, and call $\|L\|_\chi$ the **Hausdorff measure of noncompactness of** L.

If X and Y are Banach spaces and $L \in \mathcal{B}(X, Y)$ then the following facts are well known:

$$\|L\|_\chi = \chi(L(S_X)) \quad \text{(Malkowsky and Rakočević, 2000a, Theorem 2.25)} \tag{5.2}$$

and

$$L \text{ is compact if and only if } \|L\|_\chi = 0$$

$$\text{(Malkowsky and Rakočević, 2000a, Corollary 2.26 (2.58)).} \quad (5.3)$$

Lemma 5.1.1. *[Goldenstein, Gohberg, Markus] (Malkowsky and Rakočević, 2000a, Theorem 2.23) Let* X *be a Banach space with a Schauder basis* $(b_n)_{n=1}^{\infty}$*,* $Q \in \mathcal{M}_X$*,* $\mathcal{P}_n : X \to X$ *be the projector onto the linear span of* $b_1, b_2, \dots, b_n$ *and* $\mathcal{R}_n = I - \mathcal{P}_n$ *where* I *is the identity on* X*. Then we have*

$$\frac{1}{a} \limsup_{n\to\infty} \left(\sup_{x\in Q} \|\mathcal{R}_n(x)\| \right) \leq \chi(Q) \leq \limsup_{n\to\infty} \left(\sup_{x\in Q} \|\mathcal{R}_n(x)\| \right), \tag{5.4}$$

where $a = \limsup_n \|\mathcal{R}_n\|$ *denotes the basis constant of* (b_n)*.*

Example 5.1.1. (Malkowsky, 2008, p. 26) Let us consider the basis constant a for the space c. Since every sequence $x = (x_k)_{k=0}^{\infty} \in c$ has a unique representation $x = \xi e + \sum_{k=0}^{\infty} (x_k - \xi) e^{(k)}$ with $\xi = \lim_{k\to\infty} x_k$, we define the projector $\mathcal{P}_n : c \to c$ by $\mathcal{P}_n(x) = \xi e + \sum_{k=0}^{n} (x_k - \xi) e^{(k)}$ and the sequence $\tilde{x} = \mathcal{R}_n(x)$ given by $\tilde{x}_k = 0$ for $0 \leq k \leq n$ and $\tilde{x}_k = x_k - \xi$ for $k \geq n+1$. Hence we have $|\tilde{x}| \leq |x_k| + |\xi| \leq 2\|x\|_\infty$ for all k and $\|\mathcal{R}_n\| \leq 2$.

Now let x be the sequence with $x_{n+1} = -1$ and $x_k = 1$ for $k \neq n+1$. Then $\xi = 1$,$\|x\|_\infty = 1$, and $\|\mathcal{R}_n(x)\|_\infty = 2$, hence $\|\mathcal{R}_n\| = 2$. Therefore $\lim_{n\to\infty} \|\mathcal{R}_n\| = 2$.

Lemma 5.1.2. *(Malkowsky and Rakočević, 2000a, Theorem 2.15) Let* $Q \in \mathcal{M}_X$ *where* $X = \ell_p$ $(1 \leq p < \infty)$ *or* $X = c_0$*. If* $\mathcal{P}_n : X \to X$ *is defined by* $\mathcal{P}_n(x) = x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)}$ $(n = 0, 1, \dots)$ *for all* $x = (x_k)_{k=0}^{\infty} \in X$*, then we have*

$$\chi(Q) = \lim_{n\to\infty} \left(\sup_{x\in Q} \|\mathcal{R}_n(x)\| \right). \tag{5.5}$$

5.2 CHARACTERISATIONS OF COMPACT OPERATORS ON GENERAL MIXED NORM SPACES

In this section, we characterise the classes of compact operators L_A when $A \in (X_T, Y)$ where X is any of the spaces $[C_\alpha]_0^p$, $[C_\alpha]^p$, and $[C_\alpha]_\infty^p$ and Y is any of the spaces c_0 or c. This is achieved by applying the *Hausdorff measure of noncompactness*. We also find out identities and inequalities for the Hausdorff measure of noncompactness of the operators L_A in the cases just mentioned.

Now we establish some inequalities or identities for the Hausdorff measures of noncompactness of operators.

Theorem 5.2.1. *Let X be any of the spaces w_0^p, w^p and w_∞^p, and $Y = c_0$ or $Y = c$. Then estimates for $\|L_A\|_\chi$ when $A \in (X_T, Y)$ can be read from the following table*

Table 5.1

From / To	$(w_0^p)_T$	$(w^p)_T$	$(w_\infty^p)_T$
c_0	**1.**	**2.**	**3.**
c	**4.**	**5.**	**6.**

where

1. *and* **3.**(1.1*) $\|L_A\|_\chi = \lim_{r\to\infty}\left(\sup_{n\geq r}\|\hat{A}_n\|_{\mathcal{M}_p}\right)$

2. (2.1*) $\|L_A\|_\chi = \lim_{r\to\infty}\left(\sup_{n\geq r}\left(\|\hat{A}_n\|_{\mathcal{M}_p} + |\rho^{(n)}|\right)\right)$

where $\rho^{(n)} = \lim_{m\to\infty} W_m^{(n)} e$ *for all* n

4. *and* **6.**(4.1*) $\frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{n\geq r}\left\|\hat{A}_n - (\hat{\alpha}_k)\right\|_{\mathcal{M}_p}\right) \leq \|L_A\|_\chi \leq \lim_{r\to\infty}\left(\sup_{n\geq r}\left\|\hat{A}_n - (\hat{\alpha}_k)\right\|_{\mathcal{M}_p}\right)$

where $\hat{\alpha}_k = \lim_{n\to\infty}\hat{a}_{nk}$ *for each* k

5. (5.1*)
$$\begin{cases} \frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{n\geq r}\left\|\hat{A}_n - (\hat{\alpha}_k)\right\|_{\mathcal{M}_p} + \left|\sum_{k=0}^{\infty}\hat{\alpha}_k - \beta - \rho^{(n)}\right|\right) \leq \|L_A\|_\chi \\ \leq \lim_{r\to\infty}\left(\sup_{n\geq r}\left\|\hat{A}_n - (\hat{\alpha}_k)\right\|_{\mathcal{M}_p} + \left|\sum_{k=0}^{\infty}\hat{\alpha}_k - \beta - \rho^{(n)}\right|\right) \end{cases}$$

$$where\ \beta = \lim_{n\to\infty} (\hat{A}_n e - \rho^{(n)})$$

Proof. **1.** Since w_0^p is a BK space with AK, it follows from (Malkowsky and Rakočević, 2007, Lemma 4.1) that $\|L_A\|_\chi = \|L_{\hat{A}}\|_\chi$, where $\hat{A} \in (w_0^p, \ell_\infty)$. Now (Başar and Malkowsky, 2011, Corollary 3.8) yields the identity in (1.1*).

4. Similarly the identity in (4.1*) follows from (Malkowsky and Rakočević, 2007, Lemma 4.1) and (Başar and Malkowsky, 2011, Corollary 3.6).

5. First $A \in ((w^p)_T, c)$ implies $\hat{A} \in (w_0^p, c)$ and $W^{(n)} \in (w^p, c)$ for all n by (Başar et al., 2008, Lemma 4.1 (b)). So $\hat{\alpha}_k = \lim_{n\to\infty} \hat{a}_{nk}$ exists for all k by (Başar et al., 2008, Theorem 4.2 **7.** (7.1)), and $(\hat{\alpha}_k) \in \mathcal{M}_p$ by (Djolović and Malkowsky, 2008, (3.7)). Also $W^{(n)} \in (w^p, c)$ for all n implies by (Başar et al., 2008, Theorem 4.2 **9.** (3.1) and (9.2)) that the limits

$$\rho^{(n)} = \lim_{n\to\infty} W_m^{(n)} e \text{ and } \beta = \lim_{n\to\infty} \left(\hat{A}_n e - \rho^{(n)}\right) \text{ exist for all } n. \tag{5.6}$$

Now let $x \in (w^p)_T$ be given and ξ be the unique complex number such that $Tx - \xi \cdot e \in w_0^p$. Then we have by (Başar et al., 2008, (4.5)) $Ax = \hat{A}(Tx) - \xi(\rho^{(n)})$, that is,

$$y_n = A_n x = \hat{A}_n(Tx) - \xi\rho^{(n)} = \hat{A}_n(Tx - \xi e) + \xi\left(\hat{A}_n e - \rho^{(n)}\right) \text{ for all } n \tag{5.7}$$

We observe that $\hat{A} \in (w_0^p, c)$ implies $\hat{A}_n \in (w_0^p)^\beta = (w^p)^\beta$ for all n, so $\hat{A}e$ and $\hat{A}Tx$ are defined for all $x \in (w^p)_T$. It follows from $\hat{A} \in (w_0^p, c)$ and $Tx - \xi \cdot e \in w_0^p$ from (Djolović and Malkowsky, 2008, (3.9)) and $(\hat{\alpha}_k) \in \mathcal{M}_p$ that

$$\eta_0 = \lim_{n\to\infty} \hat{A}_n(Tx - \xi \cdot e) = \sum_{k=0}^{\infty} \hat{\alpha}_k(T_k x - \xi) = \sum_{k=0}^{\infty} \hat{\alpha}_k T_k x - \xi \sum_{k=0}^{\infty} \hat{\alpha}_k, \tag{5.8}$$

hence by (5.6) and (5.8)

$$\eta = \lim_{n\to\infty} y_n = \eta_0 + \xi\beta. \tag{5.9}$$

Thus we have by (5.7) and (5.9)

$$y_n - \eta = \sum_{k=0}^{\infty} \hat{a}_{nk} T_k x - \xi\rho^{(n)} - (\eta_0 + \xi\beta) = \sum_{k=0}^{\infty} (\hat{a}_{nk} - \hat{\alpha}_k) T_k x + \xi\left(\sum_{k=0}^{\infty} \hat{\alpha}_k - \beta - \rho^{(n)}\right).$$

Since $\|x\|_{(w_\infty^p)_T} = \|Tx\|_{w_\infty^p}$ for all $x \in (w^p)_T$, we obtain by (4.7) for all r

$$\sup_{x \in S_{(w^p)_T}} \|\mathcal{R}_{r-1}(Ax)\| = \|\mathcal{R}_{r-1}(Ax)\|_{w^p}^* = \sup_{n \geq r}\left(\left\|\hat{A}_n - (\hat{\alpha}_k)\right\|_{\mathcal{M}_p} + \left|\sum_{k=0}^{\infty}\hat{\alpha}_k - \beta - \rho^{(n)}\right|\right)$$

and the estimate in (5.1*) follows from (5.2) and (5.4) since $a = 2$ and the limit exists in (5.4).

2. Let $A \in ((w_0^p)_T, c_0)$. Then we obtain as in the proof of **5.**, $\hat{\alpha}_k = 0$ for all k, $\beta = 0$ and

$$\sup_{x \in S_{(w^p)_T}} \|\mathcal{R}_{r-1}(Ax)\| = \sup_{n \geq r}\left(\left\|\hat{A}_n\right\|_{\mathcal{M}_p} + \left|\rho^{(n)}\right|\right)$$

and the identity in (2.1*) follows from (5.2) and (5.5).

3. and **6.** It follows from (Başar et al., 2008, (4.1)) that $A \in ((w_\infty^p)_T, Y)$ implies $Ax = \hat{A}(Tx)$ for all $x \in (w_\infty^p)_T$, and it follows that $\|L_A\| = \|L_{\hat{A}}\|$. Since $((w_\infty^p)_T, Y) \subset ((w_0^p)_T, Y)$, we obtain (1.1*) in **3.** and (4.1*) in **6.**. ∎

We obtain the following characterizations of compact operators from Theorem 5.2.1.

Corollary 5.2.2. *Let X and Y be any of the spaces of Theorem 5.2.1. Then if $A \in (X_T, Y)$ then the conditions for L_A to be compact can be read from the following table*

Table 5.2

From / To	$(w_0^p)_T$, $(w_\infty^p)_T$	$(w^p)_T$
c_0	**1.**	**2.**
c	**3.**	**4.**

where

1. (1.1**) $\lim_{r\to\infty}\left(\sup_{n\geq r}\|\hat{A}_n\|_{\mathcal{M}_p}\right) = 0$

2. (2.1**) $\lim_{r\to\infty}\left(\sup_{n\geq r}\left(\|\hat{A}_n\|_{\mathcal{M}_p} + |\rho^{(n)}|\right)\right) = 0$

3. (3.1**) $\lim_{r\to\infty}\left(\sup_{n\geq r}\left\|\hat{A}_n - (\hat{\alpha}_k)\right\|_{\mathcal{M}_p}\right) = 0$

$$\textbf{4.}\ (4.1^{**})\ \lim_{r\to\infty}\left(\sup_{n\geq r}\left\|\hat{A}_n-(\hat{\alpha}_k)\right\|_{\mathcal{M}_p}+\left|\sum_{k=0}^{\infty}\hat{\alpha}_k-\beta-\rho^{(n)}\right|\right)=0.$$

Remark 5.2.1. The estimates and identities for the Hausdorff measure of noncompactness of L_A when $A\in([C_\alpha]_0^p,Y),([C_\alpha]^p,Y),([C_\alpha]_\infty^p,Y)$, and the characterizations of the corresponding compact operators are obtained from Theorem 5.2.1 and Corollary 5.2.2 with $T=C_{\alpha-1}$.

CHAPTER 6

CONCLUSION

This thesis is focused on the spaces of sequences that are strongly Cesàro bounded, convergent and convergent to zero, of order $\alpha > 0$ and index $p \geq 1$, denoted by $[C_\alpha]^p_\infty$, $[C_\alpha]^p$ and $[C_\alpha]^p_0$ respectively. The following original results are obtained:

- These spaces are considered as the domains of the Cesàro matrix $C_{\alpha-1}$ in the spaces w^p_∞, w^p and w^p_0.
- Some topological properties of these spaces are investigated and the norm $\|\cdot\|_{[C_\alpha]^p_\infty}$ is visualised for some parameters p and α by using our software MVGraphics.
- $\beta-$ duals of these spaces are determined and visualised by using Wulff's principle.
- The classes of matrix transformations from the spaces $[C_\alpha]^p_0$, $[C_\alpha]^p$ and $[C_\alpha]^p_\infty$ to ℓ_∞, c or c_0 are characterized and the norms of the operators defined by the matrices in these classes are determined.
- Some identities and estimates for the Hausdorff measure of noncompactness of the matrix operators in those classes are established and the corresponding classes of compact matrix operators are characterized.

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APPENDIX A

DECLARATION STATEMENT FOR THE ORIGINALITY OF THE THESIS, FURTHER STUDIES AND PUBLICATIONS FROM THESIS WORK

A.1 DECLARATION STATEMENT FOR THE ORIGINALITY OF THE THESIS

I hereby declare that this thesis comprises my original work. No material in this thesis has been previously published and written by another person, except where due reference is made in the text of the thesis. I further declare that this thesis contains no material which has been submitted for a degree or diploma or other qualifications at any other university.

Signature:

Date: April, 2016

A.2 PUBLICATIONS FROM THESIS WORK

1. Abdullah Alotaibi, Eberhard Malkowsky, Havva Nergiz, "Compact Operators on some Generalized Mixed Norm Spaces", Filomat, Vol. 28, No. 5, Nov. 2014, pp. 1019-1026.

2. Eberhard Malkowsky, Havva Nergiz, "Matrix transformations and compact operators on spaces of strongly Cesaro summable and bounded sequences of order α", Contemporary Analysis and Applied Mathematics, Vol. 3, No. 2, Oct. 2015, pp. 263-279.

APPENDIX B

FUNCTIONAL ANALYSIS

The following results are well known from functional analysis.

Theorem B.0.3. *[Closed graph lemma] (Wilansky, 1964)[Theorem 11.1.1] Any continuous map into a Hausdorff space has closed graph.*

Theorem B.0.4. *[Closed graph theorem] (Wilansky, 1964)[Theorem 11.2.2] If X and Y are Fréchet spaces and $f : X \to Y$ is a closed linear map, then f is continuous.*

Theorem B.0.5. *[Banach–Steinhaus theorem] (Wilansky, 1964)[Corollary 11.2.4] Let (f_n) be a pointwise convergent sequence of linear functionals on a Fréchet space X. Then f is defined by $f(x) = \lim_{n\to\infty} f(x)$ is continuous.*

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PUBLICATIONS

Academic Journals

- Eberhard Malkowsky, Havva Nergiz, " Matrix transformations and compact operators on spaces of strongly Cesaro summable and bounded sequences of order α ", Contemporary Analysis and Applied Mathematics, Vol. 3, No. 2, Oct. 2015, pp. 263-279.
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- Havva Nergiz, Feyzi Başar, " Some Geometric Properties of the Domain of the Double Sequential Band Matrix $B(\widetilde{r},\widetilde{s})$ in the Sequence Space $\ell(p)$ ", Abstract and Applied Analysis, No. 2013, Feb. 2013
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- Havva Nergiz, Feyzi Başar, " Some Topological and Geometric Properties of the Domain of the Double Sequential Band Matrix $B(\widetilde{r},\widetilde{s})$ in the Sequence Space $\ell(p)$", First International Conference on Analysis and Applied Mathematics, Gümüşhane, Oct. 2012, AIP Conference Proceedings, 1470, pp. 163-168
- Havva Nergiz, " The Characterisations of Some Classes of Linear Operators Between Some Sequence Spaces ", International Conference on Applied Analysis and Algebra, Istanbul/Turkey, Jun. 2012,

RESEARCH PROJECTS

Linear Topological Spaces, Operators Between Fréchet Spaces with Applications and Visualisations, TUBITAK, October 2014, scholarship student.