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M.Sc. THESIS

DIFFERENCE SCHEMES FOR TELEGRAPH EQUATIONS

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FOREWORD

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July 2017 Kadriye Tuba TÜRKCAN

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LIST OF SYMBOLS AND ABBREVIATIONS

Symbol	**Explanation**
A	: Operator
A, B, C, D, E, V	: Coefficient Matrices of Difference Schemes
D(A)	: Domain of A
F	: Fourier Transform
H	: Hilbert Space
L	: Laplace Transform
N, C	: Constant Coefficients
τ	: Grid Step Size With Respect to Time Variable
h	: Grid Step Size With Respect To Space Variable
φ, ψ, f_k	: Given Functions
$O(\tau), O(h)$	: Truncation Error
u(t,x)	: Unknown Function
u_n^k	: Numerical Representation of u(t,x)

Abbreviation	**Explanation**
BVP	: Boundary Value Problem
CPU	: Central Processing Unit
IVP	: Initial Value Problem
PDE	: Partial Differential Equation
PDQM	: Polinomial Differential Quadrate Method
PDO	: Positive Definite Operator
RMS	: Root Mean Square
VIM	: Variational Iteration Method

ÖZET

YÜKSEK LİSANS TEZİ

TELEGRAF DENKLEMLERİ'NİN FARK ŞEMALARI

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Bu tezde, aşağıdaki Cauchy problemi

$$\begin{cases} \dfrac{d^2 u(t)}{dt^2} + \alpha \dfrac{du(t)}{dt} + Au(t) = f(t), 0 \le t \le T, \\ u(0) = \varphi,\ u_t(0,x) = \psi(x), 0 \le x \le l, \end{cases} \tag{1}$$

bir H Hilbert uzayında özeşlenik (simetrik) pozitif tanımlı A operatörlü telegraf denklemi için ele alınmıştır. Bu problemin çözümü bulunmuş ve bu formül için kararlılık kestirimleri gösterilmiştir. Problem (1)'in yaklaşık çözümünün birinci, ikinci ve üçüncü derece kararlı fark şemaları kurulmuştur. Oluşturulan fark şemalarının çözümünün kararlılık kestirimleri gösterilmiştir. Nümerik çözümleri bulmak için bir örnek problem ele alınmıştır.

Temmuz 2017, 83 sayfa.

Anahtar kelimeler: Telegraf Denklemleri, Fark Şemaları, Kararlılık, Hilbert Uzayı

SUMMARY

M.Sc. THESIS

DIFFERENCE SCHEMES FOR TELEGRAPH EQUATIONS

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Institute of Graduate Studies in Science and Engineering

Department of Mathematics

Supervisor : Assoc. Prof. Dr. Kadri Ulaş AKAY

In this thesis, the following Cauchy Problem

$$\begin{cases} \dfrac{d^2\, u(t)}{dt^2} + \alpha\dfrac{du(t)}{dt} + Au(t) = f(t), 0 \leq t \leq T, \\ u(0) = \varphi,\, u_t(0,x) = \psi(x), 0 \leq x \leq l, \end{cases} \qquad (1)$$

for a telegraph equation with a self adjoint (symmetric) positive definite operator A in H is considered. The result of the above problem is obtained and the consistency conjectures on this formula are presented. So as to the approximate result of problem (1) , first, second and third order of precision variation charts are constructed. For the solutions of these variation charts, stability estimates are presented. A test problem is considered to find the numerical results.

July 2017, 83 pages.

Keywords: Telegraph Equations, Difference Schemes, Stability, Hilbert Space

1. INTRODUCTION

Many problems in engineering and science, like fluid dynamics, elasticity, wave propagation, materials science etc., result in hyperbolic partial differential equations. One of the most commonly used hyperbolic PDEs for modeling real life problems is telegraph equation. In 1876, studying on coaxial marine telegraph cables, Heaviside investigated the telegraph equation which describes the current and voltage on an electric power transmission line. This equation describes facts in a vast array of fields, such as excitons, the conduction of impulses inside the nerves and muscles, the diffusion of pressure waves in blood flow.

In the literature, the solution of telegraph equation has been drawn attention in recent years. For instance, Banasiak and Mika (1998) analyzed singularly perturbed telegraph equations and applied the results to the random walk theory. Jordan and Puri (1999) modeled the distribution of analog and digital signals through media, using the one dimensional telegraph equation. A three-level implicit difference schema was developed by Mohanty (2004), on the linear hyperbolic PDE. Shokri and Dehghan (2008) used Kansa's Method, i.e. radial based function method, on a numerical schema in order to solve the one-dimensional hyperbolic telegraph equation. Biazar et al. (2009) applied VIM for approximately solving the telegraph equation. Jiwari et al. (2012) offered a numerical method using PDQM to find the formula for two-dimensional sine-Gordon equation. Luo and Du (2013) presented a fourth degree technique for the result of telegraph equation, with the help of Hermite interpolation.

As to the computational analysis of telegram equalities, stability estimates of solution has drawn a good deal of attention. Operator theory is a very important and effective approach for studying on stability of approximate solutions of PDEs. Sobolevskii and Ashyralyev (2004) constructed and investigated new high order difference schemes for approximating solutions of regular and singular perturbation BVPs for PDEs. For the analysis of higher order variation charts of the IVPs for hyperbolic PDEs, the stability estimates are established, based on the spectral representations of symmetric positive definite operators in a Hilbert space. In this way, we will be able to study the stability of simple difference schemes for various partial differential equations. For approximately solving nonlocal BVPs for hyperbolic equations, Aggez and Ashyralyev

(2004) constructed first and second-level of precision variation charts; and established the consistency conjectures. Ashyralyev and Ozdemir (2007) studied nonlocal BVP on hyperbolic-parabolic equations and constructed consistency conjectures. They worked along with symmetric positive definite operators in Hilbert space H. Similarly, Ashyralyev and Koksal (2007) constructed second-level of precision variation chart for the approximate solution of the IVP for hyperbolic equation. Also, to the result of the variation chart, they found the consistency conjectures. Koksal and Ashyralyev (2008) developed the stable numerical schemes for the result of wave equation within non-homogeneous cylindrical shells and presented stability estimates.

Furthermore, Emir (2011) constructed a second-order difference scheme and calculated numerical results of telegraph equations in transmission lines, with modified difference scheme. Modanli and Ashyralyev (2015) developed first and second-level variation chart as to the approximate formula of the Cauchy problem to the telegraph equations and established the stability estimates in Hilbert space. Numerical solutions of a telegraph equation with nonlocal boundary conditions were also calculated by these difference schemes. Finally, Ashyralyev et al. (2016) constructed a third-level of precision difference chart for the approximate solution of the Cauchy problem for telegraph equations.

In this study, the Cauchy problem

$$\begin{cases} \dfrac{d^2u(t)}{dt^2} + \alpha \dfrac{du(t)}{dt} + Au(t) = f(t), 0 \le t \le T, \\ \\ u(0) = \varphi, u_t(0) = \psi \end{cases} \tag{1.1}$$

for a telegraph equation is considered with a self-adjoint positive definite operator $A,$ in a Hilbert space H. Here $A \ge \delta I$ and $\delta > 0, \alpha > 0$ and

$$\delta > \frac{\alpha^2}{4}.$$

The primary object of this reseach is to examine the difference charts on approximately solving above problem. It is known that PDEs can be solved analytically using different methods. In this thesis, three different methods, namely Fourier Series, Fourier Transform and Laplace Transform methods, are used to solve three different problems for telegraph equations. However, analytic methods can be used only with constant coefficients. In this thesis, we will use numerical methods for solving PDEs with dependent coefficients. Therefore, here, in this work, first, second and third level of precision variation charts of above problem are constructed. Then, using the operator approach, the consistency conjectures are presented. To show the accuracy of the difference schemes, a test example is solved numerically. In MATLAB implementation, the method is illustrated by numerical experiments.

Various IVPs for telegraph equations can be converted to the IVP in a Hilbert space H with self-adjoint positive definite operator $A.$ This work considers the following Cauchy problem

$$\begin{cases} \dfrac{d^2u(t)}{dt^2} + \alpha \dfrac{du(t)}{dt} + Au(t) = f(t), 0 \le t \le T, \\ u(0) = \varphi, u_t(0) = \psi \end{cases}$$

for telegraph equations as the main problem with self-adjoint positive definite operator A in a Hilbert space H . Here $A \ge \delta I$ and $\delta > 0, \alpha > 0$ and

$$\delta > \frac{\alpha^2}{4}.$$

Let us briefly describe the contents of the various sections of this thesis.

First chapter gives a brief history of the methods used in the literature to solve telegraph equations.

Second chapter, section 2.1contains the main theorem on the stability of problem (1.1). Three different telegraph equations are solved by using Fourier Series, Laplace Transform and Fourier Transform methods. The solution of the abstract problem (1.1) is established. Also, stability estimates for this solution are presented.

Section 2.2consists of stable difference schemes for the approximate solutions of the problem (1.1). First, second, and third order of accuracy difference schemes are constructed. The stability estimates are presented.

Section 2.3is devoted to numerical results. The proposed difference schemes are applied to a test problem. A comparison of the first, second and third-order of accuracy difference schemes is presented based on the numerical results. A MATLAB program is given to illustrate that the third order of accuracy difference scheme is more accurate than the second and the first ones. Figures and tables are included.

The last chapters include results and conclusion. A brief summary and discussion of this thesis is made. A comparison of the first, second and third order of accuracy difference schemes, based on the numerical results, is presented.

2. MATERIALS AND METHODS

2.1.MOTIVATION OF THE PROBLEM

In this thesis, the following Cauchy problem for telegraph equations is considered as the main problem:

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha \frac{\partial u(t,x)}{\partial t} + u(t,x) = \frac{\partial^2 u(t,x)}{\partial x^2} + f(t,x), 0 < t < T, 0 < x < l, \\ u(0,x) = \varphi, u_t(0,x) = \psi, 0 \le x \le l, \\ u(t,0) = u(t,l), 0 \le t \le T. \end{cases} \tag{2.1.1}$$

In this chapter, three examples of Cauchy problem for telegraph equations are solved analytically using Fourier Series, Laplace Transform and Fourier Transform Methods. Then, using operator approach the abstract Cauchy problem for telegraph equation will be studied. The solution of this problem is obtained, and applying the operator approach, the stability estimates for the solution of this problem are presented.

In the following sections, numerical solutions of the abstract Cauchy problem are found. The first, second and third order of accuracy difference schemes for the solution of the abstract Cauchy problem are constructed and the stability estimates are presented. Using these difference schemes, a test problem for a telegraph partial differential equation is solved numerically. Numerical computations are done with the help of MATLAB. Finally, a comparison of the first, second and third order of accuracy difference schemes is presented.

First, we consider the following Cauchy problem for telegraph equations

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha \frac{\partial u(t,x)}{\partial t} + u(t,x) = \frac{\partial^2 u(t,x)}{\partial x^2} + f(t,x), 0 < t < T, 0 < x < l, \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x), 0 \le x \le l, \\ u(t,0) = u(t,l), 0 \le t \le T. \end{cases}$$

This can be solved analytically using different methods. Now, three examples will be illustrated for Fourier Series, Fourier Transform and Laplace Transform Methods.

2.1.1. Examples

Example 2.1.1. *Consider the IVP*

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} + \dfrac{\partial u(t,x)}{\partial t} + u(t,x) = \dfrac{\partial^2 u(t,x)}{\partial x^2} + f(t,x), 0 < t < 1, 0 < x < \pi, \\ f(t,x) = 7\exp(-2t)\sin(2x), 0 < t < 1, 0 < x < \pi, \\ u(0,x) = \sin(2x), u_t(0,x) = -2\sin(2x), 0 \le x \le \pi, \\ u(t,0) = u(t,\pi), 0 \le t \le 1, \end{cases} \tag{2.1.2}$$

for a telegraph equation. For solving the problem (2.1.2), we use the Fourier series method. We search for a solution as

$$u(t,x) = \sum_{n=1}^{\infty} A_n(t)\sin(nx). \text{ Then}$$

$$u_{tt} + u_t + u - u_{xx} = \sum_{n=1}^{\infty} A_n''(t)\sin(nx) + \sum_{n=1}^{\infty} A_n'(t)\sin(nx) + \sum_{n=1}^{\infty} A_n(t)\sin(nx) + \sum_{n=1}^{\infty} A_n(t) n^2 \sin(nx)$$

$$= 7\exp(-2t)\sin 2x.$$

and

$$\sum_{n=1}^{\infty} A_n(0)\sin(nx) = \sin 2x, \sum_{n=1}^{\infty} A_n'(0)\sin(nx) = -2\sin 2x.$$

Equating the coefficients of $\sin(nx)$ for $n = 1,2,...,$ we get

$$\begin{cases} A_2''(t) + A_2'(t) + 5A_2(t) = 7\exp(-2t)\sin(2x) \\ A_2(0) = 1, A_2'(0) = -2 \end{cases}$$

when $n = 2$, and

$$\begin{cases} A_n''(t) + A_n'(t) + (n^2+1)A_n(t) = 0, 0 < t < 1, \\ A_n(0) = 0, A_n'(0) = 0 \end{cases}$$

when $n \neq 2$.

Assume $n \neq 2$. Thus

$$A_n''(t) + A_n'(t) + (n^2+1)A_n(t) = 0$$

and the characteristic equation of this differential equation is

$$k^2 + k + (n^2+1) = 0$$

Solving the equation, we get the following roots

$$k_{1,2} = \frac{-1 \pm \sqrt{1-4(n^2+1)}}{2} = -\frac{1}{2} \pm \frac{\sqrt{-4n^2-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{4n^2+3}}{2} i$$

Thus, we obtain the following general solution

$$A_n(t) = C_1 e^{-\frac{1}{2}t} \cos \frac{\sqrt{4n^2+3}}{2}(nt) + C_2 e^{-\frac{1}{2}t} \sin \frac{\sqrt{4n^2+3}}{2}(nt).$$

Substituting the conditions

$$A_n(0) = 0, A_n'(0) = 0$$

into the equation, we obtain $C_1 = C_2 = 0$. So, $A_n(t) = 0$, for all $n \neq 2$.

Assume $n = 2$. Then,

$$A_2''(t) + A_2'(t) + 5A_2(t) = 7\exp(-2t).$$

Then the general solution is

$$A_2(t) = A_2^c(t) + A_2^p(t).$$

Here, $A_2^c(t)$ is the complementary solution and $A_2^p(t)$ is the particular solution. $A_2^c(t)$ is the solution of the homogeneous differential equation

$$\left(A_2^c(t)\right)'' + \left(A_2^c(t)\right)' + 5A_2^c(t) = 0.$$

The characteristic equation of this differential equation is

$$k^2 + k + 5 = 0;$$

and the roots of this equation are

$$k_{1,2} = \frac{-1 \pm \sqrt{1-20}}{2} = \frac{-1}{2} \pm \frac{\sqrt{19}}{2} i.$$

So, we obtain

$$A_2^c(t) = C_1 e^{-\frac{1}{2}t} \cos\frac{\sqrt{19}}{2} t + C_2 e^{-\frac{1}{2}t} \sin\frac{\sqrt{19}}{2} t.$$

Now, for the particular solution let $A_2^p(t) = ae^{-2t}$. Differentiating and putting it into the differential equation, we get

$$(4a - 2a + 5a)e^{-2t} = 7e^{-2t}$$

Equating the coefficients of e^{-2t}, we get $a = 1$. It follows that

$$A_2^p(t) = \exp(-2t).$$

Therefore,

$$A_2(t) = C_1 e^{-\frac{1}{2}t} \cos\frac{\sqrt{19}}{2} t + C_2 e^{-\frac{1}{2}t} \sin\frac{\sqrt{19}}{2} t + e^{-2t}.$$

Using conditions $A_2(0) = 1$ and $A_2'(0) = -2$, we get $C_1 = C_2 = 0$. Then, $A_2(t) = \exp(-2t)$.

Therefore,

$$u(t,x) = A_2(t)\sin(2x) = \exp(-2t)\sin 2x.$$

Example 2.1.2. Now, we will apply the Laplace transformation method.

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} + \dfrac{\partial u(t,x)}{\partial t} + u(t,x) = \dfrac{\partial^2 u}{\partial x^2} + f(t,x), 0 < t < 1, 0 < x < \infty, \\ f(t,x) = -\exp -2(t+x), 0 < t < 1, 0 \le x \le \infty, \\ u(0,x) = \exp(-2x), u_t(0,x) = -2\exp(-2x), 0 \le x \le \infty, \\ u(t,0) = \exp(-2t), u_x(t,0) = -2\exp(-2t), 0 \le t \le 1. \end{cases} \tag{2.1.3}$$

We denote

$$\mathbf{L}\{u(t,x)\} = W(t,s).$$

Then

$$\mathbf{L}\{u_{tt}(t,x)\} = W_{tt}(t,s), \mathbf{L}\{u_t(t,x)\} = W_t(t,s),$$

$$\mathbf{L}\{u_{xx}(t,x)\} = s^2 W(t,s) - su(t,0) - u_x(t,0)$$

$$= s^2 W(t,s) + \exp(-2t)(2-s),$$

$$\mathbf{L}\{-\exp(-2(t+x))\} = -\frac{\exp(-2t)}{s+2}.$$

Using boundary conditions and these equations, we obtain

$$\begin{cases} W_{tt}(t,s) + W_t(t,s) + (1-s^2)W(t,s) = \exp(-2t)(2-s) - \dfrac{\exp(-2t)}{s+2}, 0 < t < 1, \\ W(0,s) = \mathbf{L}\{\exp(-2x)\} = \dfrac{1}{s+2}, W_t(0,s) = \mathbf{L}\{-2\exp(-2x)\} = \dfrac{-2}{s+2}. \end{cases}$$

To find the solution, $W(t,s)$ will be separated into two parts

$$W(t,s) = W^c(t,s) + W^p(t,s).$$

Here $W^c(t,s)$ is the solution of the homogeneous equation

$$W_{tt}(t,s) + W_t(t,s) + (1-s^2)W(t,s) = 0.$$

$W^p(t,s) = A(s)\exp(-2t)$ is the solution of the nonhomogeneous equation

$$W_{tt}(t,s) + W_t(t,s) + (1-s^2)W(t,s) = \left(2 - s - \frac{1}{s+2}\right)\exp(-2t) = \frac{3-s^2}{s+2}\exp(-2t).$$

The characteristic equation for this differential equation is

$$r^2 + r + (1-s^2) = 0.$$

The roots of this equation are

$$r_{1,2} = \frac{-1 \pm \sqrt{1-4(1-s^2)}}{2} = -\frac{1}{2} \pm \frac{\sqrt{4s^2-3}}{2}.$$

So, we find the following general solution

$$W^c(t,s) = c_1 e^{-\frac{1}{2}t + \frac{\sqrt{4s^2-3}}{2}t} + c_2 e^{-\frac{1}{2}t - \frac{\sqrt{4s^2-3}}{2}t},$$

for the homogeneous problem; and for the particular solution,

$$\left(W^p(t,s)\right)' = -2A(s)\exp(-2t), \left(W^p(t,s)\right)'' = 4A(s)\exp(-2t).$$

Putting these into the differential equation, we get

$$4A(s)\exp(-2t) - 2A(s)\exp(-2t) + (1-s^2)A(s)\exp(-2t) = \frac{3-s^2}{s+2}\exp(-2t).$$

Equating the coefficients of $\exp(-2t)$, we get $A(s) = \dfrac{1}{s+2}$. From that, it follows

$$W^{p}(t,s)=\frac{\exp(-2t)}{s+2}.$$

Therefore,

$$W(t,s)=c_1e^{-\frac{1}{2}t+\frac{\sqrt{4s^2-3}}{2}t}+c_2e^{-\frac{1}{2}t-\frac{\sqrt{4s^2-3}}{2}t}+\frac{\exp(-2t)}{s+2}.$$

Now, using the boundary conditions (2.1.3) which are transformed to

$$W(0,s)=\frac{1}{s+2} \text{ and } W_t(0,s)=-\frac{2}{s+2},$$

we obtain $c_1=c_2=0.$ So, $W(t,s)=\exp(-2t)\frac{1}{s+2}.$

Finally, we will take the inverse Laplace transform to obtain

$$u(t,x)=\mathbf{L}^{-1}\{W(t,s)\}=\mathbf{L}^{-1}\left\{\exp(-2t)\frac{1}{s+2}\right\}=\exp(-2t)\mathbf{L}^{-1}\left\{\frac{1}{s+2}\right\}=e^{-2t-2x}.$$

Hence, the solution of (2.1.3) is $u(t,x)=e^{-2t-2x}.$

Example 2.1.3.*The last example is an initial value problem solved by using Fourier transform method.*

$$\begin{cases}\dfrac{\partial^2u(t,x)}{\partial t^2}+\dfrac{\partial u(t,x)}{\partial t}-\dfrac{\partial^2u(t,x)}{\partial x^2}+u(t,x)=f(t,x), 0<t<1, -\infty<x<\infty,\\ f(t,x)=\left(5-16x^2\right)\exp-(t+2x^2), 0<t<1, -\infty<x<\infty,\\ u(0,x)=e^{-2x^2}, u_t(0,x)=-e^{-2x^2}, -\infty<x<\infty.\end{cases} \tag{2.1.4}$$

Let $V(t,s)=\mathbf{F}\{u(t,x)\}.$ Further, we have

$$\mathbf{F}\{u_{tt}(t,x)\}=V_{tt}(t,s), \mathbf{F}\{u_t(t,x)\}=V_t(t,s), \mathbf{F}\{u_{xx}(t,x)\}=-s^2V(t,s),$$

$$\mathbf{F}\left\{\left(16x^2-4\right)\exp\left(-2x^2\right)\right\}=\mathbf{F}\left\{\left(\exp\left(-2x^2\right)\right)''\right\}=-s^2\mathbf{F}\left\{\exp\left(-2x^2\right)\right\}.$$

Then, the Fourier transform of the differential equation in (2.1.4) is taken to obtain

$$\begin{cases} V_{tt}(t,s)+V_t(t,s)+\left(s^2+1\right)V(t,s)=\left(s^2+1\right)\exp(-t)\mathbf{F}\left\{\exp(-2x^2)\right\}, \\ 0<t<1,-\infty<x<\infty, \\ V(0,s)=\mathbf{F}\left\{e^{-2x^2}\right\}, V_t(0,s)=-\mathbf{F}\left\{e^{-2x^2}\right\}, -\infty<x<\infty. \end{cases}$$

To solve it, $V(t,s)$ is separated into two parts $V(t,s)=V^c(t,s)+V^p(t,s)$. Here $V^c(t,s)$ is the complementary solution and $V^p(t,s)$ is the particular solution. For the homogeneous equation

$$V_{tt}(t,s)+V_t(t,s)+\left(s^2+1\right)V(t,s)=0,$$

$V^c(t,s)$ is the solution; and for the following nonhomogeneous equation, we write $V^p(t,s)=E(s)\exp(-t)$ as the solution of

$$V_{tt}(t,s)+V_t(t,s)+\left(s^2+1\right)V(t,s)=\left(s^2+1\right)\exp(-t)\mathbf{F}\left\{\exp(-2x^2)\right\}.$$

Therefore,

$$V^c(t,s)=C_1e^{-\frac{1}{2}t}\cos\frac{\sqrt{4s^2+3}}{2}(st)+C_2e^{-\frac{1}{2}t}\sin\frac{\sqrt{4s^2+3}}{2}(st)$$

and for the particular solution,

$$\left(V^p\right)'(t,s)=-E(s)\exp(-t), \left(V^p\right)''(t,s)=E(s)\exp(-t).$$

So differentiating and substituting these into the differential equation, we get

$$E(s)\exp(-t)-E(s)\exp(-t)+\left(s^2+1\right)E(s)\exp(-t)=\left(s^2+1\right)\exp(-t)\mathbf{F}\left\{e^{-2x^2}\right\}.$$

Solving it, we can write $E(s)=\mathbf{F}\left\{e^{-2x^2}\right\}$ and $V^p(t,s)=\mathbf{F}\left\{e^{-2x^2}\right\}\exp(-t)$. Therefore,

$$V(t,s)=C_1e^{-\frac{1}{2}t}\cos\frac{\sqrt{4s^2+3}}{2}(st)+C_2e^{-\frac{1}{2}t}\sin\frac{\sqrt{4s^2+3}}{2}(st)+\mathbf{F}\left\{e^{-2x^2}\right\}\exp(-t).$$

Now, using the boundary conditions in (2.1.4) which are transformed to

$$V(0,s)=\mathbf{F}\left\{e^{-2x^2}\right\},V_t(0,s)=-\mathbf{F}\left\{e^{-2x^2}\right\},$$

we get $C_1=C_2=0$. So,

$$V(t,s)=\exp(-t)\mathbf{F}\left\{e^{-2x^2}\right\}.$$

Finally, the inverse Fourier transform is taken to arrive at the solution of the problem (2.1.4) as

$$u(t,x)=\mathbf{F}^{-1}\left\{\mathbf{F}\left\{e^{-2x^2}\right\}\exp(-t)\right\}=\exp\left(-2x^2-t\right).$$

These were the examples for the solution of problems for telegraph partial differential equations.

2.1.2. The Main Problem

Now, we consider the main problem

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2}+\alpha\dfrac{\partial u(t,x)}{\partial t}+u(t,x)=\dfrac{\partial^2 u(t,x)}{\partial x^2}+f(t,x), 0<t<T, 0<x<l, \\ u(0,x)=\varphi(x), u_t(0,x)=\psi(x), 0\le x\le l, \\ u(t,0)=u(t,l), 0\le t\le T. \end{cases}$$

To find the numerical solution of this problem, we reduce it to the abstract Cauchy problem for telegraph ordinary differential equations. We will introduce the differential operator A defined by formula

$$Av(x)=-v_{xx}(x)+v(x)\big|_{x=x_0},$$

with domain

$$D(A)=\{v(x): v_{xx}(x)\in C[0,l], v(l)=v(0)\},$$

in a Hilbert space H, where $H=C([0,T]\times[0,l])$.

Let $\varphi=\varphi(x)$ and $\psi=\psi(x)$ be elements of $D(A)$. Let $f(t)=f(t,x)$ be known abstract function defined on $[0,T]$ with values on $C[0,l]$ and $u(t)=u(t,x)$ be an unknown abstract function defined on $[0,T]$ with values on $C[0,l]$. Then, we denote

$$u(t)=u(t,x)\big|_{x=x_0}, \frac{du(t)}{dt}=\frac{\partial u(t,x)}{\partial t}\bigg|_{x=x_0} \text{ and } \frac{d^2u(t)}{dt^2}=\frac{\partial^2 u(t,x)}{\partial t^2}\bigg|_{x=x_0}.$$

It is clear that problem (2.1.1) can be reduced to the following abstract Cauchy problem for the ordinary telegraph differential equations:

$$\begin{cases} \dfrac{d^2u(t)}{dt^2}+\alpha\dfrac{du(t)}{dt}+Au(t)=f(t), 0\le t\le T, \\ u(0)=\varphi_0, u'(0)=\psi_0, \end{cases} \tag{2.1.5}$$

with the self-adjoint positive definite operator A in a Hilbert space H. Here, $A\ge\delta I$, $\delta>0$, $\alpha>0$ and

$$\delta>\frac{\alpha^2}{4}. \tag{2.1.6}$$

A self-adjoint linear operator B on a Hilbert space H is called *square root* of A if $B^2=A$; and if $B\ge 0$, B is called a *positive square root* of A and is denoted by $B=A^{1/2}$.

As it was stated by Ashyralyev and Sobolevski (2004) that:

"To call a function $u(t)$ as a solution of problem (2.1.5), the terms below should be met:

i. $u(t)$ is twice continuously differentiable in the segment $[0,T]$.

ii. $u(t)$ belongs to $D(A)$ for all $t\in[0,T]$, and the function $Au(t)$ is continuous on $[0,T]$.

iii. $u(t)$ satisfies the equations and initial conditions (2.1.5). If the function is both continuous and continuously differentiable on $[0,T]$, $\varphi \in D(A)$ and $\psi \in D\left(A^{\frac{1}{2}}\right)$."

Using the approach given by Fattorini (1985), Ashyralyev and Sobolevski (2004) and Modanli (2015) we can write:

"Let $\{C(t), t \geq 0\}$ be a strongly continuous cosine operator-function defined by the formula

$$C(t) = \frac{e^{itR^{1/2}} + e^{-itR^{1/2}}}{2}.$$

Thus, the definition of the sine operator-function $S(t)$ gives

$$S(t)u = \int_0^t C(s)ds$$

and

$$S(t) = R^{-1/2}\frac{e^{itR^{1/2}} - e^{-itR^{1/2}}}{2i}."$$

In this equation $R = A - \frac{\alpha^2}{4}I$. The cosine operator-function theory is explained in detail by Piskarev and Shaw (1997), and Fattorini (1985).

Under the assumption (2.1.6), the formula for the solution of the problem (2.1.5) is obtained. Clearly, the one and only mild solution to problem (2.1.5) for telegraph equation is

$$u(t) = e^{-\frac{\alpha}{2}t}C(t)u(0) + \frac{\alpha}{2}e^{-\frac{\alpha}{2}t}S(t)u(0) + e^{-\frac{\alpha}{2}t}S(t)\psi + \int_0^t e^{-\frac{\alpha}{2}(t-y)}S(t-y)f(y)dy \tag{2.1.7}$$

As it is given by Sobolevski and Ashyralyev (2004), (2.1.5) can be written as the following IVP:

$$\begin{cases} u'(t) + \frac{\alpha}{2}u(t) + iR^{\frac{1}{2}}u(t) = y(t), (0 \leq t \leq T), u(0) = u_0, u'(0) = u_0', \\ y'(t) + \frac{\alpha}{2}y(t) + iR^{\frac{1}{2}}y(t) = f(t). \end{cases}$$

Integration of the above equations gives

$$\begin{cases} u(t) = e^{-\left(\frac{\alpha}{2}+iR^{\frac{1}{2}}\right)t} u(0) + \int_0^t e^{-\left(\frac{\alpha}{2}+iR^{\frac{1}{2}}\right)(t-s)} y(s)ds, \\ y(t) = e^{-\left(\frac{\alpha}{2}-iR^{\frac{1}{2}}\right)t} y(0) + \int_0^t e^{-\left(\frac{\alpha}{2}-iR^{\frac{1}{2}}\right)(t-s)} f(s)ds \end{cases}$$

or equivalently,

$$\begin{cases} u(t) = e^{-\left(\frac{\alpha}{2}+iR^{\frac{1}{2}}\right)t} u(0) + \int_0^t e^{-\left(\frac{\alpha}{2}+iR^{\frac{1}{2}}\right)(t-s)} e^{-\left(\frac{\alpha}{2}-iR^{\frac{1}{2}}\right)s} y(0)ds \\ \quad + \int_0^t e^{-\left(\frac{\alpha}{2}+iR^{\frac{1}{2}}\right)(t-s)} \int_0^s e^{-\left(\frac{\alpha}{2}-iR^{\frac{1}{2}}\right)(s-p)} f(p)dpds. \end{cases}$$

The initial condition $y(0) = u'(0) + \left(\frac{\alpha}{2} + iR^{\frac{1}{2}}\right)u(0)$ is subtituted into the equation; and the equation becomes

$$u(t) = e^{-\left(\frac{\alpha}{2}+iR^{\frac{1}{2}}\right)t} u(0) + \int_0^t e^{-\left(\frac{\alpha}{2}+iR^{\frac{1}{2}}\right)(t-s)} e^{-\left(\frac{\alpha}{2}-iR^{\frac{1}{2}}\right)s} ds\left(u'(0) + \left(\frac{\alpha}{2} + iR^{\frac{1}{2}}\right)u(0)\right)$$

$$+ \int_0^t e^{-\left(\frac{\alpha}{2}+iR^{\frac{1}{2}}\right)(t-s)} \int_0^s e^{-\left(\frac{\alpha}{2}-iR^{\frac{1}{2}}\right)(s-p)} f(p)dpds.$$

By interchanging the order of integration, the equation can be written as

$$u(t)=\left[e^{-\left(\frac{\alpha}{2}+iR^{\frac{1}{2}}\right)t}+\left(\frac{\alpha}{2}+iR^{\frac{1}{2}}\right)\int_0^t e^{-\left(\frac{\alpha}{2}+iR^{\frac{1}{2}}\right)(t-s)}e^{-\left(\frac{\alpha}{2}-iR^{\frac{1}{2}}\right)s}ds\right]u(0)$$

$$+\int_0^t e^{-\left(\frac{\alpha}{2}+iR^{\frac{1}{2}}\right)(t-s)}e^{-\left(\frac{\alpha}{2}-iR^{\frac{1}{2}}\right)s}dsu'(0)+\int_0^t e^{-\frac{\alpha}{2}(t-s)}R^{-\frac{1}{2}}\frac{e^{i(t-s)R^{\frac{1}{2}}}-e^{-i(t-s)R^{\frac{1}{2}}}}{2i}f(s)ds$$

$$=e^{-\frac{\alpha}{2}t}\left[\frac{e^{itR^{\frac{1}{2}}}-e^{-itR^{\frac{1}{2}}}}{2}+\frac{\alpha}{2}R^{-\frac{1}{2}}\frac{e^{itR^{\frac{1}{2}}}-e^{-itR^{\frac{1}{2}}}}{2i}\right]u(0)+e^{-\frac{\alpha}{2}t}\left[R^{-\frac{1}{2}}\frac{e^{itR^{\frac{1}{2}}}-e^{-itR^{\frac{1}{2}}}}{2i}\right]u'(0)$$

$$+\int_0^t e^{-\frac{\alpha}{2}(t-s)}R^{-\frac{1}{2}}\frac{e^{i(t-s)R^{\frac{1}{2}}}-e^{-i(t-s)R^{\frac{1}{2}}}}{2i}f(s)ds.$$

Eventually, from the definitions of $C(t),S(t)$ and $R^{\frac{1}{2}}$, the formula (2.1.7) is obtained.

Lemma 2.1. The inequalities (Ashyralyev and Sobolevskii, 2004)

$$\begin{cases}\|C(t)\|_{H\to H}\le 1,\left\|R^{1/2}S(t)\right\|_{H\to H}\le 1,\left|e^{-\frac{\alpha}{2}t}\right|\le 1,\\ \left\|A^{-1/2}\varphi\right\|_H\le\frac{1}{\sqrt{\delta}}\|\varphi\|_H,\left\|A^{1/2}R^{-1/2}\right\|_{H\to H}\le N(\delta)\end{cases}\qquad(2.1.8)$$

hold, where N is a positive constant.

Let $C(H)$ be the space of continuous H-valued functions $\varphi(t)$ defined on $[0,T]$ and

$$\|u\|_{C(H)}=\max_{0\le t\le T}\|\varphi(t)\|_H$$

be the norm.

For the well-posedness of a problem, there should be a unique solution for each set of data and continuous dependence of the solution on the data. Here, using the approach given by Sobolevskii and Ashyralyev (2004), and Modanli (2015), we will illustrate that Cauchy problems for telegraph equations are stable. Now, with this approach the following main theorem on continuous dependence of the solution on the given data will be proved.

2.1.3. Stability Estimates

Theorem 2. 1.1. Let the terms (2.1.6) and (2.1.8) be satisfied; and $\varphi \in D(A),\ \psi \in D\left(A^{\frac{1}{2}}\right)$ and $f(t)$ be a continuously differentiable function on $[0,T]$. Then, problem (2.1.5) owns a unique solution and the stability inequalities below are satisfied:

$$\max_{0\le t\le T}\|u(t)\|_H \le N(\alpha,\delta)\left[\|\varphi\|_H + \left\|A^{-1/2}\psi\right\|_H + \max_{0\le t\le T}\left\|A^{-1/2}f(t)\right\|_H\right], \tag{2.1.9}$$

$$\max_{0\le t\le T}\left\|\frac{du(t)}{dt}\right\|_H + \max_{0\le t\le T}\left\|A^{1/2}u(t)\right\|_H \le N(\alpha,\delta)\left[\left\|A^{1/2}\varphi\right\|_H + \|\psi\|_H + \max_{0\le t\le T}\|f(t)\|_H\right], \tag{2.1.10}$$

$$\max_{0\le t\le T}\left\|\frac{d^2u(t)}{dt^2}\right\|_H + \max_{0\le t\le T}\|Au(t)\|_H \le N(\alpha,\delta)\left[\|A\varphi\|_H + \left\|A^{1/2}\psi\right\|_H + \|f(0)\|_H + \max_{0\le t\le T}\left\|f'(t)\right\|_H dt\right]. \tag{2.1.11}$$

Here $N(\alpha,\delta)$ independent of φ,ψ and $f(t),\ t\in[0,T]$.

Proof. **(**Modanli, 2015). Using the formula (2.1.7), $A \ge \delta I$ and the estimates (2.1.8), the inequalities below can be written:

$$\|u(t)\|_H \le \|C(t)\|_{H\to H} e^{-\frac{\alpha}{2}t}\|\varphi\|_H + \left\|R^{\frac{1}{2}}S(t)\right\|_{H\to H}\left\|A^{1/2}R^{-\frac{1}{2}}\right\|_{H\to H}\left|\frac{\alpha}{2e^{\frac{\alpha}{2}t}}\right|\left\|A^{-1/2}\varphi\right\|_H$$

$$+\left\|R^{\frac{1}{2}}S(t)\right\|_{H\to H}\left\|A^{1/2}R^{-\frac{1}{2}}\right\|_{H\to H} e^{-\frac{\alpha}{2}t}\left\|A^{-1/2}\psi\right\|_H$$

$$+\int_0^t \left\| R^{\frac{1}{2}} S(t-s) \right\|_{H\to H} \left\| A^{1/2} R^{-\frac{1}{2}} \right\|_{H\to H} \left\| A^{-1/2} f(s) \right\|_H ds$$

$$\leq N_1(\alpha,\delta)\left[\|\varphi\|_H + \left\| A^{-1/2}\psi \right\|_H + \max_{0\leq t\leq T} \left\| A^{-1/2} f(t) \right\|_H \right]$$

per $t\in[0,T]$. So, we get

$$\max_{0\leq t\leq T} \|u(t)\|_H \leq N_1(\alpha,\delta)\left[\|\varphi\|_H + \left\| A^{-1/2}\psi \right\|_H + \max_{0\leq t\leq T} \left\| A^{-1/2} f(t) \right\|_H \right].$$

Multiplying both sides of the formula (2.1.7) by $A^{1/2}$ and applying the estimates (2.1.8) in the same way, we get

$$\left\| A^{1/2}u(t) \right\|_H \leq \|C(t)\|_{H\to H} e^{-\frac{\alpha}{2}t} \left\| A^{1/2}\varphi \right\|_H + \left\| R^{\frac{1}{2}} S(t) \right\|_{H\to H} \left\| A^{1/2} R^{-\frac{1}{2}} \right\|_{H\to H} \left| \frac{\alpha}{2e^{\frac{\alpha}{2}t}} \right| \|\varphi\|_H$$

$$+\left\| R^{\frac{1}{2}} S(t) \right\|_{H\to H} \left\| A^{1/2} R^{-\frac{1}{2}} \right\|_{H\to H} e^{-\frac{\alpha}{2}t} \|\psi\|_H$$

$$+\int_0^t \left\| A^{1/2} R^{-\frac{1}{2}} \right\|_{H\to H} \left\| R^{\frac{1}{2}} S(t-s) \right\|_{H\to H} \| f(s)\|_H ds$$

$$\leq N_2(\alpha,\delta)\left[\left\| A^{1/2}\varphi \right\|_H + \|\psi\|_H + \max_{0\leq t\leq T} \| f(t)\|_H \right]$$

for any $t\in[0,T]$. So,

$$\max_{0\leq t\leq T} \left\| A^{1/2}u(t) \right\|_H \leq N_2(\alpha,\delta)\left[\left\| A^{1/2}\varphi \right\|_H + \|\psi\|_H + \max_{0\leq t\leq T} \| f(t)\|_H \right].$$

Now, we estimate $\|Au(t)\|_H$. Multiplying the formula (2.1.7) by A and with the help of an integration by parts, we have

$$Au(t)e^{\frac{\alpha}{2}t} = C(t)A\varphi + \frac{\alpha}{2}A^{\frac{1}{2}}S(t)A^{\frac{1}{2}}\varphi + \frac{\alpha}{2}A^{\frac{1}{2}}S(t)A^{\frac{1}{2}}\psi$$

$$+AR^{-1}\left[e^{\frac{\alpha}{2}t}f(t) - C(t)f(0) - \int_0^t e^{\frac{\alpha}{2}s}C(t-y)\left[\frac{\alpha}{2}f(y) + f'(y)\right]dy\right].$$

Estimates (2.1.8) and this formula gives

$$\|Au(t)\|_H \le \|C(t)\|_{H\to H}\left|e^{-\frac{\alpha}{2}t}\right|\|A\varphi\|_H + \left\|R^{\frac{1}{2}}S(t)\right\|_{H\to H}\left\|A^{1/2}R^{-\frac{1}{2}}\right\|_{H\to H}\left|\frac{\alpha}{2e^{\frac{\alpha}{2}t}}\right|\left\|A^{1/2}\varphi\right\|_H$$

$$+\left\|R^{\frac{1}{2}}S(t)\right\|_{H\to H}\left\|A^{1/2}R^{-\frac{1}{2}}\right\|_{H\to H}\left|e^{-\frac{\alpha}{2}t}\right|\left\|A^{1/2}\psi\right\|_H$$

$$+\left\|AR^{-1}\right\|_{H\to H}\left[\|f(t)\|_H + \|C(t)\|_{H\to H}\|f(0)\|_H\right]$$

$$+\left\|AR^{-1}\right\|_{H\to H}\int_0^t e^{-\frac{\alpha}{2}(t-z)}\|C(t-y)\|_{H\to H}\left[\frac{\alpha}{2}\|f(y)\|_H + \|f'(y)\|_H\right]dy$$

$$\le N_3(\alpha,\delta)\left[\|A\varphi\|_H + \left\|A^{1/2}\psi\right\|_H + \|f(0)\|_H + \max_{0\le t\le T}\|f'(t)\|_H\right]$$

for any $t\in[0,T]$. So, we obtain

$$\max_{0\le t\le T}\|Au(t)\|_H \le N_3(\alpha,\delta)\left[\|A\varphi\|_H + \left\|A^{1/2}\psi\right\|_H + \|f(0)\|_H + \max_{0\le t\le T}\|f'(t)\|_H\right].$$

This final estimate and triangle inequality leads to the estimate for $\max\limits_{0\le t\le T}\left\|\frac{d^2u}{dt^2}\right\|_H$. So, the proof is completed.

2.2. STATEMENT OF THE PROBLEM

In this section, the following abstract Cauchy problem for telegraph equations

$$\begin{cases} \dfrac{d^2u(t)}{dt^2} + \alpha \dfrac{du(t)}{dt} + Au(t) = f(t), 0 \le t \le T, \\ u(0) = \varphi, u'(0) = \psi, \end{cases} \tag{2.2.1}$$

with the self-adjoint positive definite operator A in a Hilbert space H, is taken into consideration. Stable two-step first, second and third order of accuracy difference schemes in t for the solution of this problem are constructed using finite difference method.

Using the approach given by Modanli (2014), the first and second order of accuracy difference schemes in t are constructed. For the third order of accuracy difference scheme in t, Taylor's decomposition on three points, which was given by Yildirim (2011), is used. In addition to these, using the operator approach constructed by Ashyralyev and Sobolevski (2004), the stability estimates are presented.

In the next section, the proposed difference schemes are applied to a test problem to show the accuracy and the efficiency of the numerical method. The Numerical results are obtained using the difference schemes and the difference formulas with the help of MATLAB.

Now, the first, second and third order of accuracy difference schemes in t will be constructed. To construct the two-step difference schemes in t for the approximate solution of this problem, the following sample grid interval is taken on the segment $[0,T]$:

$$[0,T] = \{t_k = k\tau, k = 0,1,..., N, N\tau = T\}.$$

2.2.1. First-order Difference Scheme

We take the IVP (2.2.1) into consideration. Putting $t = t_{k+1}$ and using difference formulas

$$\frac{u(t_{k+1}) - 2u(t_k) + u(t_{k-1})}{\tau^2} - u''(t_{k+1}) = O(\tau),$$

$$\frac{u(t_{k+1})-u(t_k)}{\tau}-u^{'}(t_{k+1})=O(\tau)$$

and

$$\frac{u(\tau)-u(0)}{\tau}-u^{'}(0)=O(\tau),$$

where $u^{''}(t)=\frac{d^2u(t)}{dt^2}$ and $u^{'}(t)=\frac{du(t)}{dt}$, we obtain the first order of accuracy difference scheme. For the rest of our work, we use the notation $u_k=u(t_k)$. Substituting the approximations

$$\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} \text{ and } \frac{u_{k+1}-u_k}{\tau}$$

for $u^{''}(t)$ and $u^{'}(t)$, respectively, we have the following approximation for the telegraph equation in (2.2.1):

$$\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\frac{u_{k+1}-u_k}{\tau}+Au_{k+1}=f_k,\ f_k=f(t_{k+1}), 1\le k\le N-1.$$

The approximation of the first initial condition $u(0)=\varphi$ is $u_0=\varphi$. The approximation for the second initial condition $u^{'}(0)=\psi$ is

$$\frac{u(\tau)-u(0)}{\tau}-u^{'}(0)=O(\tau).$$

Omission of the small term $O(\tau)$ gives $\frac{u_1-u_0}{\tau}=\psi$. Then, for the solution of problem (2.2.1), we have

$$\begin{cases}\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\frac{u_{k+1}-u_k}{\tau}+Au_{k+1}=f_k,\\ 1\le k\le N-1, N\tau=T,\\ u_0=\varphi, \frac{u_1-u_0}{\tau}=\psi\end{cases} \qquad (2.2.2)$$

as the first-order of approximation in t two-step difference scheme (Modanli, 2014).

Now, using the lemma below and the stability estimates given by Ashyralyev and Sobolevskii (2004) and Modanli (2014), the stability estimates for the solution of problem (2.2.1) will be presented.

Lemma 2.2.1. (Ashyralyev and Sobolevskii, 2004) The following estimates hold:

$$\begin{cases} \|q\|_{H\to H} \le \dfrac{1}{1+\dfrac{\alpha\tau}{2}}, \|\tilde{q}\|_{H\to H} \le \dfrac{1}{1+\dfrac{\alpha\tau}{2}}, \\ \left\|\tau R^{\frac{1}{2}} q\right\|_{H\to H} \le 1, \left\|\tau R^{\frac{1}{2}} \tilde{q}\right\|_{H\to H} \le 1, \end{cases} \tag{2.2.3}$$

where

$$q = \left(\left(1+\frac{\alpha\tau}{2}\right)I - i\tau R^{\frac{1}{2}}\right)^{-1}, \tilde{q} = \left(\left(1+\frac{\alpha\tau}{2}\right)I + i\tau R^{\frac{1}{2}}\right)^{-1}.$$

Theorem 2.2.1. Assume that $\varphi \in D(A), \psi \in D\left(A^{\frac{1}{2}}\right)$ and the assumption (2.1.6) holds. Then, the following stability estimates should be met for the solution of the difference scheme (2.2.2)

$$\max_{1\le k\le N}\|u_k\|_H \le C(\alpha,\delta)\left\{\max_{1\le k\le N-1}\left\|A^{-1/2} f_k\right\|_H + \left\|A^{-1/2}\psi\right\|_H + \|\varphi\|_H\right\}, \tag{2.2.4}$$

$$\max_{1\le k\le N}\left\|A^{1/2}u_k\right\|_H \le C(\alpha,\delta)\left\{\max_{1\le k\le N-1}\|f_k\|_H + \|\psi\|_H + \left\|A^{1/2}\varphi\right\|_H\right\}, \tag{2.2.5}$$

$$\max_{1\le k\le N}\|Au_k\|_H \le C(\alpha,\delta)\left\{\max_{2\le k\le N-1}\left\|\frac{1}{\tau}(f_k - f_{k-1})\right\|_H + \|f_1\|_H + \left\|A^{1/2}\psi\right\|_H + \|A\varphi\|_H\right\}. \tag{2.2.6}$$

Here $C(\alpha,\delta)$ is independent of φ, ψ, τ and $f_k, 1\le k\le N-1$ (Modanli, 2014).

Theorem 2.2.1 is proved by Modanli (2014) with the help of estimates (2.2.3), triangle inequality, and the following formula

$$u_k = q\tilde{q}(\tilde{q}-q)^{-1}\left[q^{k-1}-\tilde{q}^{k-1}\right]u_0 + (\tilde{q}-q)^{-1}\left(\tilde{q}^k - q^k\right)u_1 + \sum_{s=1}^{k-1} q\tilde{q}(\tilde{q}-q)^{-1}\left[\tilde{q}^{k-s}-q^{k-s}\right]\tau^2 f_s,$$

$$2 \le k \le N, \qquad (2.2.7)$$

which was constructed by Ashyralyev and Sobolovskii (2004).

2.2.2. Second-order Difference Scheme

Now, putting $t = t_k$, we use the finite difference formulas

$$\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} - u_k'' = O(\tau^2),$$

$$\frac{u_{k+1}-u_{k-1}}{2\tau} - u_k' = O(\tau^2)$$

for the construction of second-order approximation in t two-step difference schemes for the solution of IVP (2.2.1). Further, we have

$$\frac{u(\tau)-u(0)}{\tau} = u'(0) + \frac{\tau}{2}u''(0) + O(\tau^2).$$

The approximation of the first initial condition $u(0)=\varphi$ is $u_0 = \varphi.$ The approximation of the second initial condition $u'(0)=\psi$ is

$$\left(I + \frac{\tau^2 A}{2}\right)\frac{u(\tau)-u(0)}{\tau} = u'(0) + \frac{\tau}{2}u''(0) + O(\tau^2).$$

From the above expression, it follows that

$$\frac{u(\tau)-u(0)}{\tau} + \frac{\tau}{2}\left(Au(\tau) - Au(0)\right) = \frac{u_1-u_0}{\tau} + \frac{\tau}{2}Au_1 - \frac{\tau}{2}Au_0,$$

where $Au = -u_{xx} + u.$ Here $u = u(t,x)$. Then, the right hand side of the equation becomes

$$u'(0) + \frac{\tau}{2}u''(0) + O(\tau^2) = \psi + \frac{\tau}{2}\left(-\alpha\psi + f(0) - Au_0\right).$$

Cancellation and omission of the small terms $O(\tau^2)$ lead to

$$\frac{u_1 - u_0}{\tau} = \frac{\tau}{2}\left(f(0) - Au_1 - \alpha\psi\right) + \psi.$$

Then, for the solution of the problem (2.2.1), two step second order of accuracy in t difference schemes (Mahmut, 2014)

$$\begin{cases} \dfrac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} + \alpha\dfrac{u_{k+1} - u_{k-1}}{2\tau} + \dfrac{A}{2}\left(u_{k+1} + u_{k-1}\right) = f_k, \\ f_k = f(t_k), 1 \le k \le N-1, \\ u_0 = \varphi, \dfrac{u_1 - u_0}{\tau} = \dfrac{\tau}{2}\left(f_0 - Au_1 - \alpha\psi\right) + \psi,\ f_0 = f(0), \end{cases} \tag{2.2.8}$$

$$\begin{cases} \dfrac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} + \alpha\dfrac{u_{k+1} - u_{k-1}}{2\tau} + \dfrac{A}{2}u_k + \dfrac{A}{4}\left(u_{k+1} + u_{k-1}\right) = f_{k,} \\ f_k = f(t_k), 1 \le k \le N-1, \\ u_0 = \varphi, \dfrac{u_1 - u_0}{\tau} = \dfrac{\tau}{2}\left(f_0 - Au_1 - \alpha\psi\right) + \psi,\ f_0 = f(0) \end{cases} \tag{2.2.9}$$

are attained.

Now, the theorem on the stability of these difference schemes is given based on the operator approach (Ashyralyev and Sobolevski, 2004).

Theorem 2.2.2. Assume that $\varphi \in D(A)$, $\psi \in D\left(A^{\frac{1}{2}}\right)$ assumption (2.1.6) holds . Then, the following stability estimates hold for the solution of difference schemes (2.2.8) and (2.2.9):

$$\max_{1\le k\le N}\|u_k\|_H \le C(\alpha,\delta)\left\{\max_{0\le k\le N-1}\left\|A^{-1/2}f_k\right\|_H+\left\|A^{-1/2}\psi\right\|_H+\|\varphi\|_H\right\},$$

$$\max_{1\le k\le N}\left\|A^{1/2}u_k\right\|_H \le C(\alpha,\delta)\left\{\max_{0\le k\le N-1}\|f_k\|_H+\|\psi\|_H+\left\|A^{1/2}\varphi\right\|_H\right\},$$

$$\max_{1\le k\le N}\|Au_k\|_H \le C(\alpha,\delta)\left\{\max_{1\le k\le N-1}\left\|\frac{1}{\tau}(f_k-f_{k-1})\right\|_H+\|f_0\|_H+\left\|A^{1/2}\psi\right\|_H+\|A\varphi\|_H\right\}.$$

Here $C(\alpha,\delta)$ is free from $\psi,\varphi,\tau,$ and $f_k,\ 1\le k\le N-1$ (Modanli, 2015).

2.2.3. Third-order Difference Scheme

Finally, for the approximate solution of the problem (2.2.1), we apply following Taylor's decomposition on three points given by Yildirim (2011):

$$u_{k+1}-2u_k+u_{k-1}-\frac{2}{3}\tau^2u_k^{''}-\frac{\tau^2}{6}\left[u_{k+1}^{''}+u_{k+1}^{''}\right]+\frac{1}{12}\tau^4u_{k+1}^{(4)}=O(\tau^5) \tag{2.2.10}$$

and

$$u_{k+1}-u_{k-1}=2\tau\left\{\frac{2}{3}u_k^{'}+\frac{1}{6}\left[u_{k+1}^{'}+u_{k-1}^{'}\right]\right\}=O(\tau^5) \tag{2.2.11}$$

to construct the third order of accuracy difference scheme.

First, we write the equation in (2.2.1) as the following way and taking the third and fourth derivative, we have

$$u^{''}(t)=-\alpha u^{'}(t)-Au(t)+f(t),$$

$$u^{'''}(t)=\left(\alpha^2-A\right)u^{'}(t)+\alpha Au(t)-\alpha f(t)+f^{'}(t),$$

$$u^{(4)}(t)=\left[\left(-\alpha^3+2\alpha A\right)u^{'}(t)-\left(\alpha^2-A\right)Au(t)\right]+\left(\alpha^2-A\right)f(t)-\alpha f^{'}(t)+f^{''}(t).$$

With the help of (2.2.10) and (2.2.11), substituting

$$\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} \quad \text{and} \quad \frac{u_{k+1}-u_{k-1}}{2\tau}$$

for $u_k^{''}$ and $u_k^{'}$, respectively, into the differential equation in (2.2.1), we will have the following expression as the third order approximation of the differential equation (2.2.1):

$$\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\frac{u_{k+1}-u_{k-1}}{2\tau}-\frac{2}{3}\left[u_k^{''}+\alpha u_k^{'}\right]-\frac{1}{6}\left[u_{k+1}^{''}+\alpha u_{k+1}^{'}+u_{k-1}^{''}+\alpha u_{k-1}^{'}\right]$$

$$+\frac{\tau^2}{12}\left[\left(-\alpha^3+2\alpha A\right)\tau^{-1}\left(u_{k+1}-u_k\right)-\left(\alpha^2-A\right)Au_{k+1}+\left(\alpha^2-A\right)f_{k+1}-\alpha f_{k+1}^{'}+f_{k+1}^{''}\right]$$

$$=O(\tau^3), 1\le k\le N-1.$$

Omission of the small term and substitution of the values for the derivatives of $u(t)$ into the above expression result in

$$\begin{cases} \dfrac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\dfrac{u_{k+1}-u_{k-1}}{2\tau}+\dfrac{2}{3}Au_k+\dfrac{1}{6}A\left(u_{k+1}+u_{k-1}\right) \\ +\dfrac{\tau^2}{12}\left[\left(-\alpha^3+2\alpha A\right)\dfrac{u_{k+1}-u_k}{\tau}-\left(\alpha^2-A\right)Au_{k+1}\right]=f_k, \\ f_k=\dfrac{2}{3}f(t_k)+\dfrac{1}{6}\left[f(t_{k+1})+f(t_{k-1})\right] \\ -\dfrac{\tau^2}{12}\left[\left(\alpha^2-A\right)f(t_{k+1})-\alpha f^{'}(t_{k+1})+f^{''}(t_{k+1})\right], 1\le k\le N-1. \end{cases} \tag{2.2.12}$$

From now on, we use the notation f_k for the summation of all the functions with $f(t_k)$ in the differential equation in (2.2.1). So, (2.2.12) is the third order approximation for the differential equation in (2.2.1).

Now, we construct the third order approximations for the initial conditions in (2.2.1). The approximation of the first initial condition $u(0)=\varphi$ is

$$u_0 = \varphi. \tag{2.2.13}$$

The approximation of the second initial condition $u'(0)=\psi$ is

$$\left(I+\frac{\tau^2}{2}A\right)\frac{u(\tau)-u(0)}{\tau}=\left(I+\frac{\tau^2}{2}A\right)u'(0)+\frac{\tau}{2}u''(0)+\frac{\tau^2}{6}u'''(0)+O(\tau^3).$$

Substituting the values of the derivatives of $u(0)$ into the above expression and omitting the small term $O(\tau^3)$, we have

$$\left(I+\frac{\tau^2}{2}A\right)\frac{u_1-u_0}{\tau}=\left(I+\frac{\tau^2}{2}A\right)\psi+\frac{\tau}{2}\left[-\alpha\psi-A\varphi+f(0)\right]+\frac{\tau^2}{6}\left[\alpha^2\psi+\alpha A\varphi-f(0)+f'(0)\right]. \tag{2.2.14}$$

Hence, the approximations (2.2.12), (2.2.13) and (2.2.14) lead to the following third-order of approximation in t two-step difference scheme (Ashyralyev et al., 2016)

$$\begin{cases} \dfrac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\dfrac{u_{k+1}-u_{k-1}}{2\tau}+\dfrac{2}{3}Au_k+\dfrac{1}{6}A(u_{k+1}+u_{k-1}) \\ +\dfrac{\tau^2}{12}\left[\left(-\alpha^3+2\alpha A\right)\dfrac{u_{k+1}-u_k}{\tau}-\left(\alpha^2-A\right)Au_{k+1}\right]=f_k, \\ f_k=\dfrac{2}{3}f(t_k)+\dfrac{1}{6}\left[f(t_{k+1})+f(t_{k-1})\right] \\ -\dfrac{\tau^2}{12}\left[\left(\alpha^2-A\right)f(t_{k+1})-\alpha f'(t_{k+1})+f''(t_{k+1})\right], 1\le k\le N-1, \\ u_0=\varphi, \\ \left(I+\dfrac{\tau^2}{2}A\right)\dfrac{u_1-u_0}{\tau}=\left(I+\dfrac{\tau^2}{2}A\right)\psi+\dfrac{\tau}{2}\left[-\alpha\psi-A\varphi\right]+\dfrac{\tau^2}{6}\left[\alpha^2\psi+\alpha A\varphi-A\psi\right]+f_0, \\ f_0=\dfrac{\tau}{2}f(0)-\dfrac{\tau^2}{6}\left[f(0)-f'(0)\right] \end{cases} \tag{2.2.15}$$

for the solution of problem (2.2.1).

The stability estimates for the solution of this difference scheme are given by the following theorem:

Theorem 2.2.3. Assume that $\varphi \in D(A),\ \psi \in D\left(A^{\frac{1}{2}}\right)$ and the assumption (2.1.6) holds. Then, the following stability estimates hold on behalf of the formula of difference scheme (2.2.15):

$$\|u_k\|_H \le C\left\{\sum_{s=0}^{N-1}\|A^{-1/2} f_s\|_H \tau + \|A^{-1/2}\psi\|_H + \|\varphi\|_H\right\}, k = 0,2,3,..., N,$$

$$\|u_1\|_H \le C\{\|\tau\psi\|_H + \|\varphi\|_H + \|f_0\|_H\},$$

$$\|A^{1/2}u_k\|_H \le C\left\{\sum_{s=0}^{N-1}\|f_s\|_H \tau + \|\psi\|_H + \|A^{1/2}\varphi\|_H\right\}, k = 0,2,3,..., N,$$

$$\|A^{1/2}u_1\|_H \le C\{\|\tau A^{1/2}\psi\|_H + \|A^{1/2}\varphi\|_H + \|A^{1/2} f_0\|_H\},$$

$$\|Au_k\|_H \le C\left\{\sum_{s=1}^{N-1}\|f_s - f_{s-1}\|_H + \|f_0\|_H + \|A^{1/2}\psi\|_H + \|A\varphi\|_H\right\}, k = 0,2,3,...N,$$

$$\|Au_1\|_H \le C\{\|\tau A\psi\|_H + \|A\varphi\|_H + \|Af_0\|_H\},$$

where C is independent of $f_s, 1 \le s \le N-1, \psi$ and φ (Ashyralyev et al., 2016).

In this section, stable first, second and third order of accuracy difference schemes have been constructed for the solution of problem (2.2.1). For the solutions of these difference schemes, stability estimates have been presented. In the following section, these difference schemes will be applied to a test problem for a telegraph equation in order to obtain the numerical results.

2.3. NUMERICAL RESULTS

In this section, a test problem for a telegraph equation is considered. The difference schemes proposed in section 2.2 are applied to this trial problem in order to indicate the certainty and efficiency of numerical method. Difference equations with matrix coefficients are obtained as a result of calculations. To solve these difference equations, iterative method is applied. Numerical computations are carried out with the help of MATLAB. The errors are analyzed by looking at graphs and tables. In the end, a comparison among first, second and third order of accuracy difference schemes is made.

2.3.1. A Test Problem

For numerical results, the following IVP

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2}+\dfrac{\partial u(t,x)}{\partial t}+u(t,x)=\dfrac{\partial^2 u(t,x)}{\partial x^2}+7\exp(-2t)\sin(2x), \\ 0<t<1, 0<x<\pi, \\ u(0,x)=\sin(2x), u_t(0,x)=-2\sin(2x), 0\le x\le \pi, \\ u(t,0)=u(t,\pi)=0, 0\le t\le 1 \end{cases} \tag{2.3.1}$$

for a telegraph equation is considered. The problem has the following exact solution:

$$u(t,x)=\exp(-2t)\sin(2x).$$

First, second and third order of accuracy difference schemes are applied to the approximate formulas of the IVP (2.3.1). Iterative method is applied to solve the difference equations with matrix coefficients.

First, the set $[0,1]_\tau \times [0,\pi]_h$ of a group of mesh points

$$[0,1]_{\tau} \times [0,\pi]_{h} = (t_k, x_n) : \left\{ \begin{matrix} t_k = k\tau, \ 1 \le k \le N-1, \ \ N\tau = 1, \\ x_n = nh, \ 1 \le n \le M-1, Mh = \pi \end{matrix} \right\}$$

is considered, dependent on the small parameters τ and h.

2.3.2. The First Order of Accuracy Difference Scheme

For the approximate solution of problem (2.3.1), we perform the following difference formula

$$\frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1})}{h^2} - u''(x_n) = O(h^2) \tag{2.3.2}$$

and the difference scheme (2.2.2). The formula (2.3.2) and the difference scheme (2.2.2) result in

$$\begin{cases} \dfrac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + \dfrac{u_n^{k+1} - u_n^k}{\tau} + u_n^{k+1} - \dfrac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} = 7\exp(-2t_k)\sin(2x_n), \\ 1 \le k \le N-1, 1 \le n \le M-1 \end{cases} \tag{2.3.3}$$

as the approximation of the equation in (2.3.1). The approximation of the first initial condition $u(0,x) = \sin(2x)$ is

$$u_n^0 = \sin(2x_n). \tag{2.3.4}$$

The approximation of the second initial condition $u_t(0,x) = -2\sin(2x)$ is obtained applying the formula in difference scheme (2.2.2) as

$$u_n^1 = (1 - 2\tau)\sin 2x_n. \tag{2.3.5}$$

The approximations (2.3.3), (2.3.4) and (2.3.5) give the following first-order of accuracy in t and the second-order of accuracy in x difference scheme:

$$\begin{cases} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+\dfrac{u_n^{k+1}-u_n^k}{\tau}+u_n^{k+1}-\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}=7\exp(-2t_k)\sin(2x_n), \\ 1\le k\le N-1, 1\le n\le M-1, \\ u_n^0=\sin(2x_n), u_n^1=(1-2\tau)\sin 2x_n, 0\le n\le M, \\ u_0^{k+1}=u_M^{k+1}=0, -1\le k\le N-1. \end{cases} \tag{2.3.6}$$

Note that, here and in the future, u_n^k represents $u(t_k, x_n)$ in the equation. We obtain $(N+1)\times(M+1)$ system of linear equations in (2.3.6). Before writing these in matrix form, in order to make the calculations easier one will write the frame again as

$$\begin{cases} \left(-\dfrac{1}{h^2}\right)u_{n-1}^{k+1}+\left(\dfrac{1}{\tau^2}+\dfrac{1}{\tau}+1+\dfrac{2}{h^2}\right)u_n^{k+1}+\left(-\dfrac{1}{h^2}\right)u_{n+1}^{k+1}+ \\ +\left(\dfrac{-2}{\tau^2}-\dfrac{1}{\tau}\right)u_n^k+\left(\dfrac{1}{\tau^2}\right)u_n^{k-1}=f(t_k,x_n)=\varphi_n^k, \\ \varphi_n^k=7\exp(-2t_k)\sin(2x_n), x_n=nh, tk=k\tau, \\ 1\le k\le N-1, 1\le n\le M-1, \\ u_n^0=\sin(2x_n), u_n^1=(1-2\tau)\sin(2x_n), x_n=nh, 0\le n\le M, \\ u_0^{k+1}-u_M^{k+1}=0, -1\le k\le N-1. \end{cases} \tag{2.3.7}$$

Hereby, each distinct coefficient will be denoted by letters a, b, c and d as follows

$$a=\frac{1}{h^2},\ b=\frac{1}{\tau^2}+\frac{1}{\tau}+1+\frac{2}{h^2}, c=\frac{-2}{\tau^2}-\frac{1}{\tau}, d=\frac{1}{\tau^2}.$$

Then, the equation in the system (2.3.3) can be written as system of second order difference equation with matrix coefficients as

$$AU^{k+1}+BU^{k}+CU^{k-1}=D\varphi^{k},1\le k\le N-1,$$

where

$$A=\begin{bmatrix}1&0&0&0&0&\cdots&0&0&0&0\\ a&b&a&0&0&\cdots&0&0&0&0\\ 0&a&b&a&0&\cdots&0&0&0&0\\ 0&0&a&b&a&\cdots&0&0&0&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\cdots&\vdots&\vdots&\vdots&\vdots\\ 0&0&0&0&0&\cdots&a&0&0&0\\ 0&0&0&0&0&\cdots&b&a&0&0\\ 0&0&0&0&0&\cdots&a&b&a&0\\ 0&0&0&0&0&\cdots&0&a&b&a\\ 0&0&0&0&0&\cdots&0&0&0&1\end{bmatrix}_{(M+1)\times(M+1)},$$

$$B=\begin{bmatrix}0&0&0&\cdots&0&0&0\\ 0&c&0&\cdots&0&0&0\\ 0&0&c&\cdots&0&0&0\\ \vdots&\vdots&\vdots&\cdots&\vdots&\vdots&\vdots\\ 0&0&0&\cdots&c&0&0\\ 0&0&0&\cdots&0&c&0\\ 0&0&0&\cdots&0&0&0\end{bmatrix}_{(M+1)\times(M+1)},\ C=\begin{bmatrix}0&0&0&\cdots&0&0&0\\ 0&d&0&\cdots&0&0&0\\ 0&0&d&\cdots&0&0&0\\ \vdots&\vdots&\vdots&\cdots&\vdots&\vdots&\vdots\\ 0&0&0&\cdots&d&0&0\\ 0&0&0&\cdots&0&d&0\\ 0&0&0&\cdots&0&0&0\end{bmatrix}_{(M+1)\times(M+1)},$$

$$D=\begin{bmatrix}1&0&0&\cdots&0&0&0\\ 0&1&0&\cdots&0&0&0\\ 0&0&1&\cdots&0&0&0\\ \vdots&\vdots&\vdots&\cdots&\vdots&\vdots&\vdots\\ 0&0&0&\cdots&1&0&0\\ 0&0&0&\cdots&0&1&0\\ 0&0&0&\cdots&0&0&1\end{bmatrix}_{(M+1)\times(M+1)},\ \varphi^{k}=\begin{bmatrix}\varphi_0^k\\ \varphi_1^k\\ \vdots\\ \varphi_M^k\end{bmatrix}_{(M+1)\times 1},$$

$$U^s = \begin{bmatrix} U_0^s \\ U_1^s \\ \vdots \\ U_{M-1}^s \\ U_M^s \end{bmatrix}_{(M+1)\times(1)}, s = k-1, k, k+1$$

and

$$\varphi_n^k = \begin{cases} 0, n = 0, \\ f(t_k, x_n) = 7\exp(-2t_k)\sin(2x_n), 1 \le n \le M-1, \\ 0, n = M. \end{cases}$$

Then, (2.3.6) can be written as

$$\begin{cases} AU^{k+1} + BU^k + CU^{k-1} = D\varphi^k, 1 \le k \le N-1, \\ U_n^0 = \sin(2x_n), U_n^1 = (1-2\tau)\sin(2x_n). \end{cases}$$

The system written in matrix form results in a second order difference formula in terms of k with matrix coefficients. Now, we will apply iterative method to get the solution of this difference equation. Solving it, we get

$$U^{k+1} = -A^{-1}BU^k - A^{-1}CU^{k-1} + A^{-1}D\varphi^k, 1 \le k \le N-1.$$

2.3.3. The Second Order of Accuracy Difference Scheme

Here, the second order of accuracy difference scheme (2.2.9) is applied to problem (2.3.1). Using the difference formula

$$\frac{u(x_{n+1}) - 2u(x_n) + u(x_{n-1})}{h^2} - u''(x_n) = O(h^2)$$

and the difference scheme (2.2.9), we get

$$\begin{cases} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+\dfrac{u_n^{k+1}-u_n^{k-1}}{2\tau}-\dfrac{1}{2}\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}-\dfrac{1}{4}\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} \\ -\dfrac{1}{4}\dfrac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}+\dfrac{1}{2}u_n^k+\dfrac{1}{4}\left(u_n^{k+1}+u_n^{k-1}\right)= f\left(t_k,x_n\right), \\ f\left(t_k,x_n\right)=7\exp(-2t_k)\sin(2x_n), x_n=nh, t_k=k\tau \end{cases} \tag{2.3.8}$$

for the approximation of the differential equation in (2.3.1). The approximation of the first initial condition $u(0,x)=\sin(2x)$ is

$$u_n^0=\sin(2x_n) \tag{2.3.9}$$

and the approximation of the second initial condition $u_t(0,x)=-2\sin(2x)$ is

$$\frac{u_n^1-u_n^0}{\tau}-\frac{\tau}{2}\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2}+\frac{\tau}{2}u_n^1=\frac{9\tau-4}{2}\sin(2x_n), 1\le n\le M-1 \tag{2.3.10}$$

as a result of the application of difference scheme (2.2.9) into the initial conditions in (2.3.1). The approximations (2.3.8), (2.3.9) and (2.3.10) result in the second order of accuracy difference chart with respect to t

$$\begin{cases} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+\dfrac{u_n^{k+1}-u_n^{k-1}}{2\tau}-\dfrac{1}{2}\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}-\dfrac{1}{4}\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} \\ -\dfrac{1}{4}\dfrac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}+\dfrac{1}{2}u_n^k+\dfrac{1}{4}\left(u_n^{k+1}+u_n^{k-1}\right)= f\left(t_k,x_n\right), \\ f\left(t_k,x_n\right)=7\exp(-2t_k)\sin(2x_n), x_n=nh, t_k=k\tau, \\ 1\le k\le N-1, 1\le n\le M-1, \\ u_n^0=\sin(2x_n),\ x_n=nh, \\ \dfrac{u_n^1-u_n^0}{\tau}-\dfrac{\tau}{2}\dfrac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2}+\dfrac{\tau}{2}u_n^1=\dfrac{9\tau-4}{2}\sin(2x_n), x_n=nh, 1\le n\le M-1, \\ u_0^{k+1}=u_M^{k+1}=0, -1\le k\le N-1. \end{cases} \tag{2.3.11}$$

There are $(N+1)\times(M+1)$ system of linear equations in (2.3.11). Hereby, one may type this frame as follows:

$$\begin{cases}\left(-\dfrac{1}{4h^2}\right)u_{n-1}^{k+1}+\left(\dfrac{1}{\tau^2}+\dfrac{1}{2\tau}+\dfrac{1}{4}+\dfrac{1}{2h^2}\right)u_n^{k+1}+\left(-\dfrac{1}{4h^2}\right)u_{n+1}^{k+1}+\left(-\dfrac{1}{2h^2}\right)u_{n-1}^{k}\\ +\left(-\dfrac{2}{\tau^2}+\dfrac{1}{2}+\dfrac{1}{h^2}\right)u_n^k+\left(-\dfrac{1}{2h^2}\right)u_{n+1}^k+\left(-\dfrac{1}{4h^2}\right)u_{n-1}^{k-1}+\left(\dfrac{1}{\tau^2}-\dfrac{1}{2\tau}+\dfrac{1}{4}+\dfrac{1}{2h^2}\right)u_n^{k-1}\\ +\left(-\dfrac{1}{4h^2}\right)u_{n+1}^{k-1}=f\left(t_k,x_n\right),\ f\left(t_k,x_n\right)=7\exp\left(-2t_k\right)\sin\left(2x_n\right),\\ x_n=nh, t_k=k\tau, 1\le k\le N-1, 1\le n\le M-1,\\ u_n^0=\sin(2x_n), x_n=nh,\\ \dfrac{u_n^1-u_n^0}{\tau}-\dfrac{\tau}{2}\dfrac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2}+\dfrac{\tau}{2}u_n^1=\dfrac{9\tau-4}{2}\sin\left(2x_n\right), 1\le n\le M-1,\\ u_0^{k+1}=u_M^{k+1}=0, -1\le k\le N-1.\end{cases} \tag{2.3.12}$$

Now, we denote each specific coefficient with a,b,c,d and e as follows

$$a=-\frac{1}{4h^2}, b=\frac{1}{\tau^2}+\frac{1}{2\tau}+\frac{1}{4}+\frac{1}{2h^2}, c=-\frac{1}{2h^2}, d=-\frac{2}{\tau^2}+\frac{1}{2}+\frac{1}{h^2}, e=\frac{1}{\tau^2}-\frac{1}{2\tau}+\frac{1}{4}+\frac{1}{2h^2}.$$

Writing the system in (2.3.12) as a second order difference equation with matrix coefficients of $U^s, s=k-1,k,k+1,$ we get

$$AU^{k+1}+BU^k+CU^{k-1}=D\varphi^k, 1\le k\le N-1,$$

where

$$A=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ a & b & a & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & a & b & a & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & a & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & b & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & a & b & a & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & a & b & a \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ c & d & c & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & c & d & c & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & c & d & c & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & d & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & c & d & c & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & c & d & c \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$C=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ a & e & a & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & a & e & a & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & e & a & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & e & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & a & e & a & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & a & e & a \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$D=\begin{bmatrix}1&0&0&\cdots&0&0&0\\0&1&0&\cdots&0&0&0\\0&0&1&\cdots&0&0&0\\\vdots&\vdots&\vdots&\cdots&\vdots&\vdots&\vdots\\0&0&0&\cdots&1&0&0\\0&0&0&\cdots&0&1&0\\0&0&0&\cdots&0&0&1\end{bmatrix}_{(M+1)\times(M+1)},$$

$$\varphi^k=\begin{bmatrix}\varphi_0^k\\\varphi_1^k\\\vdots\\\varphi_M^k\end{bmatrix}_{(M+1)\times 1},$$

$$U^s=\begin{bmatrix}U_0^s\\U_1^s\\\vdots\\U_{M-1}^s\\U_M^s\end{bmatrix}_{(M+1)\times(1)}, s=k-1,k,k+1$$

and

$$\varphi_n^k=\begin{cases}0,\ n=0,\\ f(t_k,x_n)=7\exp(-2t_k)\sin(2x_n), 1\le n\le M-1,\\ 0,\ n=M.\end{cases}$$

Then, (2.3.12) can be written as

$$\begin{cases}AU^{k+1}+BU^k+CU^{k-1}=D\varphi^k, 1\le k\le N-1,\\ U_n^0=\sin(2x_n).\end{cases}$$

To solve this difference equation in terms of k with matrix coefficients, we use iterative method.

Solving it, we get

$$U^{k+1} = -A^{-1}BU^{k} - A^{-1}CU^{k-1} + A^{-1}D\varphi^{k}, 1 \le k \le N-1.$$

For the initial conditions

$$\begin{cases} \dfrac{u_n^1 - u_n^0}{\tau} - \dfrac{\tau}{2}\dfrac{u_{n+1}^1 - 2u_n^1 + u_{n-1}^1}{h^2} + \dfrac{\tau}{2}u_n^1 = \dfrac{9\tau-4}{2}\sin(2x_n), 1 \le n \le M-1 \\ u_n^0 = \sin(2x_n) \end{cases} \tag{2.3.13}$$

in (2.3.12), again, this $(N+1)\times(M+1)$ frame of linear equations may be rewritten in the form

$$\left(-\frac{\tau}{2h^2}\right)U_{n-1}^1 + \left(\frac{1}{\tau} + \frac{\tau}{h^2} + \frac{\tau}{2}\right)U_n^1 + \left(-\frac{\tau}{2h^2}\right)U_{n+1}^1 = \left(\frac{1}{\tau}\right)U_n^0 + \psi_n, \tag{2.3.14}$$

where $\psi_n = \begin{cases} 0, n = 0, \\ \dfrac{9\tau-4}{2}\sin(2x_n), 1 \le n \le M-1, \\ 0, n = M. \end{cases}$

Now, the distinct coefficients of this equation are represented as

$$j = -\frac{\tau}{2h^2}, p = \frac{1}{\tau} + \frac{\tau}{h^2} + \frac{\tau}{2}.$$

Writing the system in (2.3.14) as a difference equation with matrix coefficients, we get

$$EU^1 = VU^0 + \tilde{\psi}.$$

Here,

$$E=\begin{bmatrix}1&0&0&0&0&\dots&0&0&0&0\\ j&p&j&0&0&\dots&0&0&0&0\\ 0&j&p&j&0&\dots&0&0&0&0\\ 0&0&j&p&j&\dots&0&0&0&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\dots&\vdots&\vdots&\vdots&\vdots\\ 0&0&0&0&0&\dots&j&0&0&0\\ 0&0&0&0&0&\dots&p&j&0&0\\ 0&0&0&0&0&\dots&j&p&j&0\\ 0&0&0&0&0&\dots&0&j&p&j\\ 0&0&0&0&0&\dots&0&0&0&1\end{bmatrix}_{(M+1)\times(M+1)},$$

$$V=\begin{bmatrix}\frac{1}{\tau}&0&0&\dots&0&0\\ 0&\frac{1}{\tau}&0&\dots&0&0\\ 0&0&\frac{1}{\tau}&\dots&0&0\\ \vdots&\vdots&\vdots&\dots&\vdots&\vdots\\ 0&0&0&\dots&\frac{1}{\tau}&0\\ 0&0&0&\dots&0&\frac{1}{\tau}\end{bmatrix}_{(M+1)\times(M+1)}, U^s=\begin{bmatrix}U_0^s\\ U_1^s\\ \vdots\\ U_{M-1}^s\\ U_M^s\end{bmatrix}_{(M+1)\times(1)}, s=0,1$$

and $\tilde{\psi}=\begin{bmatrix}\psi_0\\ \psi_1\\ \vdots\\ \psi_M\end{bmatrix}_{(M+1)\times 1}$.

Then, (2.3.13) can be written as

$$\begin{cases}EU^1=VU^0+\tilde{\psi}\\ U_n^0=\sin(2x_n).\end{cases}$$

Solving it, we get

$$U^1=E^{-1}VU^0+E^{-1}\tilde{\psi}.$$

Therefore, to solve the system in (2.3.12), the procedure

$$\begin{cases} U^{k+1} = -A^{-1}BU^{k} - A^{-1}CU^{k-1} + A^{-1}D\varphi^{k}, 1 \le k \le N-1, \\ U_{n}^{0} = \sin(2x_{n}), U^{1} = E^{-1}VU^{0} + E^{-1}\tilde{\psi}, x_{n} = nh, 1 \le n \le M-1 \end{cases}$$

will be applied.

2.3.4. The Third Order of Accuracy Difference Scheme

Now, we apply the difference scheme (2.2.15) on the problem (2.3.1). To do that, we use the following difference formulas (Özgür, 2011):

$$\frac{u(x_{n+1}) - 2u(x_{n}) + u(x_{n-1})}{h^{2}} - u''(x_{n}) = O(h^{2}),$$

$$\frac{u(x_{n+2}) - 4u(x_{n+1}) + 6u(x_{n}) - 4u(x_{n-1}) + u(x_{n-2})}{h^{4}} - u^{(iv)}(x_{n}) = O(h^{2}), \tag{2.3.15}$$

$$\frac{2u(0) - 5u(h) + 4u(2h) - u(3h)}{h^{2}} - u''(0) = O(h^{2}),$$

$$\frac{2u(1) - 5u(1-h) + 4u(1-2h) - u(1-3h)}{h^{2}} - u''(1) = O(h^{2}).$$

First, we need to consider the following approximation of the differential equation in the difference scheme (2.2.15):

$$\frac{u_{k+1} - 2u_{k} + u_{k-1}}{\tau^{2}} + \alpha\frac{u_{k+1} - u_{k-1}}{2\tau} + \frac{2}{3}Au_{k} + \frac{1}{6}A(u_{k+1} + u_{k-1}) +$$

$$+\frac{\tau^{2}}{12}\left[\left(-\alpha^{3} + 2\alpha A\right)\frac{u_{k+1} - u_{k}}{\tau} - \left(\alpha^{2} - A\right)Au_{k+1}\right] = f_{k}.$$

Since $\alpha = 1$ for problem (2.3.1), writing this equation in a proper way, we obtain

$$\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\frac{u_{k+1}-u_{k-1}}{2\tau}+\frac{2}{3}Au_k+\frac{1}{6}A(u_{k+1}+u_{k-1}) \tag{2.3.16}$$

$$+\frac{\tau^2}{12}\left[-\frac{u_{k+1}-u_k}{\tau}+\frac{2A}{\tau}(u_{k+1}-u_k)-Au_{k+1}+A^2u_{k+1}\right]=f_k.$$

To find the approximation of this equation, we need to substitute Au and A^2u into it. We know that A has the form

$$Au=-u_{xx}+u.$$

Calculating $A^2u,$ we obtain

$$A^2u=A(Au)=A(-u_{xx}+u)=-(-u_{xx}+u)_{xx}+(-u_{xx}+u)=u_{xxxx}-2u_{xx}+u.$$

Substituting the values for Au and A^2u into (2.3.16), the equation becomes

$$\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\frac{u_{k+1}-u_{k-1}}{2\tau}+\frac{\tau^2}{12}(u_{xxxx})_{k+1}+\frac{\tau-4}{6}(u_{xx})_k \tag{2.3.17}$$

$$-\left(\frac{1}{6}+\frac{\tau^2+2\tau}{12}\right)(u_{xx})_{k+1}-\frac{1}{6}(u_{xx})_{k-1}+\frac{8-\tau}{12}u_k+\frac{\tau+2}{12}u_{k+1}+\frac{1}{6}u_{k-1}=f_k.$$

Now, we need to use the approximation for f_k, which was given in the difference scheme (2.2.15) as

$$f_k=\frac{2}{3}f(t_k)+\frac{1}{6}[f(t_{k+1})+f(t_{k-1})]-\frac{\tau^2}{12}\left[(\alpha^2-A)f(t_{k+1})-\alpha f'(t_{k+1})+f''(t_{k+1})\right].$$

Writing $\alpha=1$ and $Af=-f_{xx}+f$ in the equation, we get

$$f_k=\frac{2}{3}f(t_k)+\frac{1}{6}[f(t_{k+1})+f(t_{k-1})]-\frac{\tau^2}{12}(f_{xx}-f_t+f_{tt})_{k+1}. \tag{2.3.18}$$

Here, taking derivatives of f with respect to t and x, we get

$$f = 7e^{-2t}\sin 2x$$

$$f_{xx} = -28e^{-2t}\sin 2x$$

$$f_t = -14e^{-2t}\sin 2x$$

$$f_{tt} = 28e^{-2t}\sin 2x.$$

Substitution of these into (2.3.18) leads to

$$f_k = \left[\frac{2}{3} + \frac{1-\tau^2}{6}e^{-2\tau} + \frac{1}{6}e^{2\tau}\right]7\exp(-2t_k)\sin(2x_n).$$

Therefore, (2.3.17) becomes

$$\begin{cases} \dfrac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + \dfrac{u_{k+1}-u_{k-1}}{2\tau} + \dfrac{\tau^2}{12}(u_{xxxx})_{k+1} + \dfrac{\tau-4}{6}(u_{xx})_k \\ -\left(\dfrac{1}{6} + \dfrac{\tau^2+2\tau}{12}\right)(u_{xx})_{k+1} - \dfrac{1}{6}(u_{xx})_{k-1} + \dfrac{8-\tau}{12}u_k + \dfrac{\tau+2}{12}u_{k+1} + \dfrac{1}{6}u_{k-1} = f_k, \\ f_k = \left[\dfrac{2}{3} + \dfrac{1-\tau^2}{6}e^{-2\tau} + \dfrac{1}{6}e^{2\tau}\right]7\exp(-2t_k)\sin(2x_n) \end{cases}$$

Applying the difference formulas in (2.3.15) to this equation, we get

$$\begin{cases} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} + \dfrac{u_n^{k+1}-u_n^{k-1}}{2\tau} + \dfrac{\tau^2}{12}\dfrac{u_{n+2}^{k+1}-4u_{n+1}^{k+1}+6u_n^{k+1}-4u_{n-1}^{k+1}+u_{n-2}^{k+1}}{h^4} + \dfrac{\tau-4}{6}\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} \\ -\left(\dfrac{1}{6} + \dfrac{\tau^2+2\tau}{12}\right)\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} - \dfrac{1}{6}\dfrac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2} + \dfrac{8-\tau}{12}u_n^k + \dfrac{\tau+2}{12}u_n^{k+1} \\ +\dfrac{1}{6}u_n^{k-1} = f(t_k, x_n),\ f(t_k, x_n) = \left[\dfrac{2}{3} + \dfrac{1-\tau^2}{6}e^{-2\tau} + \dfrac{1}{6}e^{2\tau}\right]7\exp(-2t_k)\sin(2x_n), \\ x_n = nh, t_k = k\tau, 1 \le k \le N-1, 2 \le n \le M-2 \end{cases} \tag{2.3.19}$$

as the approximation for the differential equation in (2.3.1).

Now, using the difference scheme (2.2.15), we will obtain the approximations for the initial conditions in problem (2.3.1).The approximation for the first initial condition $u(0,x)=\sin(2x)$ is

$$u_n^0=\sin(2x_n). \tag{2.3.20}$$

For the second initial condition $u_t(0,x)=-2\sin(2x)$, we will use the following approximation

$$\left(I+\frac{\tau^2}{2}A\right)\frac{u_1-u_0}{\tau}=\left(I+\frac{\tau^2}{2}A\right)\psi+\frac{\tau}{2}\left[-\alpha\psi-A\varphi+f(0)\right]+\frac{\tau^2}{6}\left[\alpha^2\psi+\alpha A\varphi-f(0)+f'(0)\right],$$

which was given in (2.2.15). Substituting $\alpha=1$ and

$$A\varphi=-\varphi_{xx}+\varphi,\ A\psi=-\psi_{xx}+\psi,$$

we have

$$\frac{u_1-u_0}{\tau}+\frac{\tau}{2}Au_1-\frac{\tau}{2}Au_0=\left(I-\frac{\tau}{2}+3\frac{\tau^2}{6}\right)\psi-\frac{2}{6}\tau^2\psi_{xx}+\left(-\frac{\tau}{2}+\frac{\tau^2}{6}\right)(-\varphi_{xx}+\varphi)$$

$$+\left(\frac{\tau}{2}-\frac{\tau^2}{6}\right)f(0)+\frac{\tau^2}{6}f'(0).$$

Simplifying the difference formula in (2.3.15) and using

$$\varphi=\sin 2x,\ \ \varphi_{xx}=-4\sin x,\ \ \psi=-2\sin 2x,\ \ \psi_{xx}=8\sin 2x,$$

$f(0)$ and $f'(0)$ into the above equation result in the approximation for the second initial condition:

$$\frac{u_n^1-u_n^0}{\tau}-\frac{\tau}{2}\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2}+\frac{\tau}{2}u_n^1=\left[-2+\frac{9\tau}{2}-\frac{38\tau^2}{6}\right]\sin(2x_n). \tag{2.3.21}$$

The approximations (2.3.19), (2.3.20), (2.3.21) and also (2.3.15) lead to the following third-order of accuracy difference scheme with respect to t:

$$\begin{cases}\dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+\dfrac{u_n^{k+1}-u_n^{k-1}}{2\tau}+\dfrac{\tau^2}{12}\dfrac{u_{n+2}^{k+1}-4u_{n+1}^{k+1}+6u_n^{k+1}-4u_{n-1}^{k+1}+u_{n-2}^{k+1}}{h^4}+\dfrac{\tau-4}{6}\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}\\ -\left(\dfrac{1}{6}+\dfrac{\tau^2+2\tau}{12}\right)\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}-\dfrac{1}{6}\dfrac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}+\dfrac{8-\tau}{12}u_n^k+\dfrac{\tau+2}{12}u_n^{k+1}\\ +\dfrac{1}{6}u_n^{k-1}=f\left(t_k,x_n\right), f\left(t_k,x_n\right)=\left[\dfrac{2}{3}+\dfrac{1-\tau^2}{6}e^{-2\tau}+\dfrac{1}{6}e^{2\tau}\right]7\exp(-2t_k)\sin(2x_n),\\ x_n=nh,t_k=k\tau,1\le k\le N-1,2\le n\le M-2,\\ u_n^0=\sin(2x_n),\ x_n=nh,\\ \dfrac{u_n^1-u_n^0}{\tau}-\dfrac{\tau}{2}\dfrac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2}+\dfrac{\tau}{2}u_n^1=\left[-2+\dfrac{9\tau}{2}-\dfrac{38\tau^2}{6}\right]\sin\left(2x_n\right),\\ x_n=nh,1\le n\le M-1,\\ u_0^{k+1}=u_M^{k+1}=0,-1\le k\le N-1,\\ 2u_0^{k+1}-5u_1^{k+1}+4u_2^{k+1}-u_3^{k+1}=0,2u_M^{k+1}-5u_{M-1}^{k+1}+4u_{M-2}^{k+1}-u_{M-3}^{k+1}=0,\\ -1\le k\le N-1.\end{cases}$$

(2.3.22)

Using the same approach as the previous difference schemes, we will write the $(N+1)\times(M+1)$ frame of linear equations in (2.3.22), in the matrix form. Before that, we denote

$$A=\frac{\tau-4}{6},B=\frac{1}{6}+\frac{\tau^2+2\tau}{12},C=\frac{8-\tau}{12},D=\frac{2+\tau}{12}.$$

Then, we will write the system again as the coefficients of u^{k+1},u^k and u^{k-1}:

$$
\begin{cases}
\left(\frac{\tau^2}{12h^4}\right)u_{n-2}^{k+1}+\left(-\frac{\tau^2}{3h^4}-\frac{B}{h^2}\right)u_{n-1}^{k+1}+\left(\frac{1}{\tau^2}+\frac{1}{2\tau}+\frac{\tau^2}{2h^4}+\frac{2B}{h^2}+D\right)u_n^{k+1} \\
+\left(-\frac{\tau^2}{3h^4}-\frac{B}{h^2}\right)u_{n+1}^{k+1}+\left(\frac{\tau^2}{12h^4}\right)u_{n+2}^{k+1}+\left(\frac{A}{h^2}\right)u_{n-1}^{k}+\left(-\frac{2}{\tau^2}-\frac{2A}{h^2}+C\right)u_n^{k} \\
+\left(\frac{A}{h^2}\right)u_{n+1}^{k}+\left(-\frac{1}{6h^2}\right)u_{n-1}^{k-1}+\left(\frac{1}{\tau^2}-\frac{1}{2\tau}+\frac{1}{3h^2}+\frac{1}{6}\right)u_n^{k-1}+\left(-\frac{1}{6h^2}\right)u_{n+1}^{k-1}=f(t_k,x_n), \\
f(t_k,x_n)=\left[\frac{2}{3}+\frac{1-\tau^2}{6}e^{-2\tau}+\frac{1}{6}e^{2\tau}\right]7\exp(-2t_k)\sin(2x_n),\ x_n=nh, t_k=k\tau, \\
1\le k\le N-1, 2\le n\le M-2, \\
u_n^0=\sin(2x_n),\ x_n=nh, \\
\frac{u_n^1-u_n^0}{\tau}-\frac{\tau}{2}\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2}+\frac{\tau}{2}u_n^1=\left[-2+\frac{9\tau}{2}-\frac{38\tau^2}{6}\right]\sin(2x_n), \\
x_n=nh, 0\le n\le M, \\
u_0^{k+1}=u_M^{k+1}=0, -1\le k\le N-1. \\
2u_0^{k+1}-5u_1^{k+1}+4u_2^{k+1}-u_3^{k+1}=0, 2u_M^{k+1}-5u_{M-1}^{k+1}+4u_{M-2}^{k+1}-u_{M-3}^{k+1}=0, \\
-1\le k\le N-1.
\end{cases}
\tag{2.3.23}
$$

Here, we denote each distinct coefficient of the differential equation in (2.3.23) with a, b, c, d, e, f, g as follows:

$$a=\frac{\tau^2}{12h^4},\ b=-\frac{\tau^2}{3h^4}-\frac{B}{h^2},\ c=\frac{1}{\tau^2}+\frac{1}{2\tau}+\frac{\tau^2}{2h^4}+\frac{2B}{h^2}+D,$$

$$d=\frac{A}{h^2},\ e=-\frac{2}{\tau^2}-\frac{2A}{h^2}+C,\ f=-\frac{1}{6h^2},\ g=\frac{1}{\tau^2}-\frac{1}{2\tau}+\frac{1}{3h^2}+\frac{1}{6}.$$

Then, we will write the system as the difference equation

$$AU^{k+1} + BU^{k} + CU^{k-1} = D\varphi^{k}, 1 \le k \le N-1,$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 2 & -5 & 4 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ a & b & c & b & a & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & a & b & c & b & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & b & c & b & a & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & a & b & c & b & a \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 4 & -5 & 2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & d & e & d & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & d & e & d & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & e & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & d & e & d & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & f & g & f & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & f & g & f & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & g & f & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & f & g & f & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$D=\begin{bmatrix}1&0&0&\dots&0&0\\0&1&0&\dots&0&0\\0&0&1&\dots&0&0\\\vdots&\vdots&\vdots&\dots&\vdots&\vdots\\0&0&0&\dots&1&0\\0&0&0&\dots&0&1\end{bmatrix}_{(M+1)\times(M+1)},$$

$$\varphi^k=\begin{bmatrix}\varphi_0^k\\\varphi_1^k\\\vdots\\\varphi_M^k\end{bmatrix}_{(M+1)\times 1},$$

$$U^s=\begin{bmatrix}U_0^s\\U_1^s\\\vdots\\U_{M-1}^s\\U_M^s\end{bmatrix}_{(M+1)\times(1)},s=k-1,k,k+1$$

and

$$\varphi_n^k=\begin{cases}0,n=0,\\0,n=1,\\\left[\frac{2}{3}+\frac{1-\tau^2}{6}e^{-2\tau}+\frac{1}{6}e^{2\tau}\right]7\exp(-2t_k)\sin(2x_n),2\le n\le M-2,\\0,n=M-1,\\0,n=M.\end{cases}$$

Then, (2.3.23) can be written as

$$\begin{cases} AU^{k+1}+BU^{k}+CU^{k-1}=D\varphi^{k}, 1\le k\le N-1, \\ U_n^0=\sin(2x_n). \end{cases}$$

Iterative method is applied to solve this second order difference equation in terms of k with matrix coefficients. Solving it, we get

$$U^{k+1}=-A^{-1}BU^{k}-A^{-1}CU^{k-1}+A^{-1}D\varphi^{k}, 1\le k\le N-1.$$

Hereby, the $(N+1)\times(M+1)$ frame of linear equations, for initial conditions

$$\begin{cases} \frac{u_n^1-u_n^0}{\tau}-\frac{\tau}{2}\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2}+\frac{\tau}{2}u_n^1=\left[-2+\frac{9\tau}{2}-\frac{38\tau^2}{6}\right]\sin(2x_n), 1\le n\le M-1, \\ u_n^0=\sin(2x_n), x_n=nh \end{cases} \tag{2.3.24}$$

in (2.3.23) will be written again as

$$\left(-\frac{\tau}{2h^2}\right)U_{n-1}^1+\left(\frac{1}{\tau}+\frac{\tau}{h^2}+\frac{\tau}{2}\right)U_n^1+\left(-\frac{\tau}{2h^2}\right)U_{n+1}^1=\left(\frac{1}{\tau}\right)U_n^0+\psi_n.$$

We denote

$$j=-\frac{\tau}{2h^2}, p=\frac{1}{\tau}+\frac{\tau}{h^2}+\frac{\tau}{2},$$

$$\psi_n=\begin{cases} 0, n=0, \\ \left[-2+\frac{9\tau}{2}-\frac{38}{6}\tau^2\right]\sin(2x_n), 1\le n\le M-1, \\ 0, n=M, \end{cases}$$

$$\tilde{\psi} = \begin{bmatrix} \psi_0 \\ \psi_1 \\ \cdots \\ \psi_M \end{bmatrix}_{(M+1)\times 1},$$

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ j & p & j & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & j & p & j & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & j & p & j & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & p & j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & j & p & j & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & j & p & j \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$V = \begin{bmatrix} \frac{1}{\tau} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{\tau} & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{1}{\tau} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{\tau} & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{\tau} \end{bmatrix}_{(M+1)\times(M+1)},$$

$$U^s = \begin{bmatrix} U_0^s \\ U_1^s \\ \vdots \\ U_{M-1}^s \\ U_M^s \end{bmatrix}_{(M+1)\times(1)}, s = 0,1.$$

So, (2.3.24) becomes

$$\begin{cases} EU^1 = V\,U^0 + \tilde{\psi} \\ U_n^0 = \sin(2x_n). \end{cases}$$

Solving it, we get

$$U^1 = E^{-1}VU^0 + E^{-1}\tilde{\psi}.$$

Therefore, the difference equation is solved by the application of

$$\begin{cases} U^{k+1} = -A^{-1}BU^k - A^{-1}CU^{k-1} + A^{-1}D\varphi^k, 1 \le k \le N-1, \\ U_n^0 = \sin(2x_n), U^1 = E^{-1}VU^0 + E^{-1}\tilde{\psi}, x_n = nh, 1 \le n \le M-1. \end{cases}$$

So far, we have found the solutions of the systems of equations obtained by the application of first, second and third order difference schemes in section 2.2 on problem (2.3.1). In the next section, we analyze these results.

2.3.5. Error Examination

The numerical results of the exact solution and difference schemes are analyzed in this section. To get the solutions for the difference schemes (2.3.6), (2.3.11) and (2.3.22), MATLAB is used. The following RMS formula is applied to calculate the errors:

$$E_0 = \max_{0 \le k \le N} \left(\sum_{n=0}^{M} \left| u(t_k, x_n) - u_n^k \right|^2 h \right)^{\frac{1}{2}}.$$

Here, the exact solution is represented by $u(t_k, x_n)$ and the numerical solution at (t_k, x_n) is represented by u_n^k, and N is the number of steps with respect to time and M is the number of steps with respect to space. In Figure 2.1, the graph of the exact solution is given from different perspectives. The space-time graphs of numerical solutions for the first, second and third degree of accuracy difference schemes are given in Figures 2.2, 2.3 and 2.4, respectively.

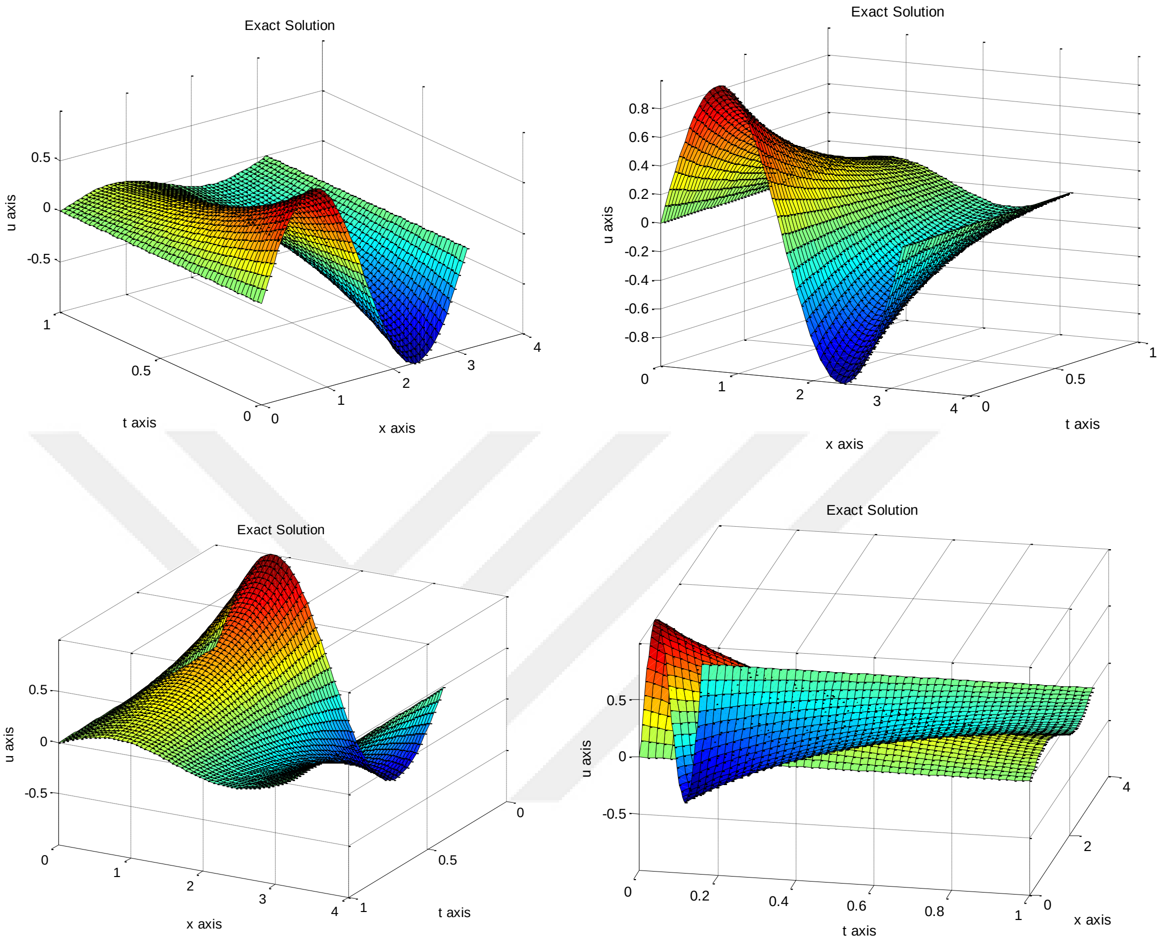

Figure 2.1: The Exact Solution

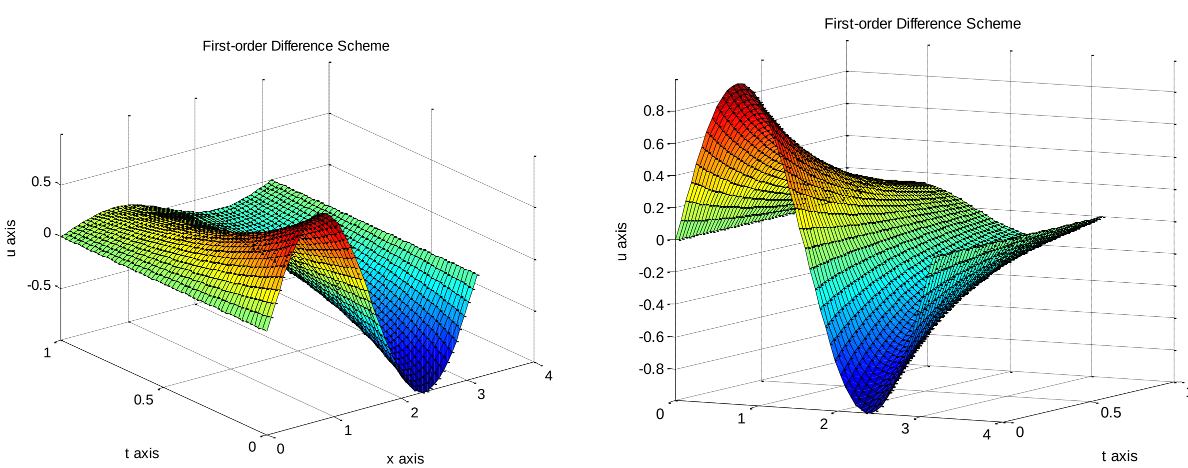

Figure 2.2: First-Order Difference Scheme

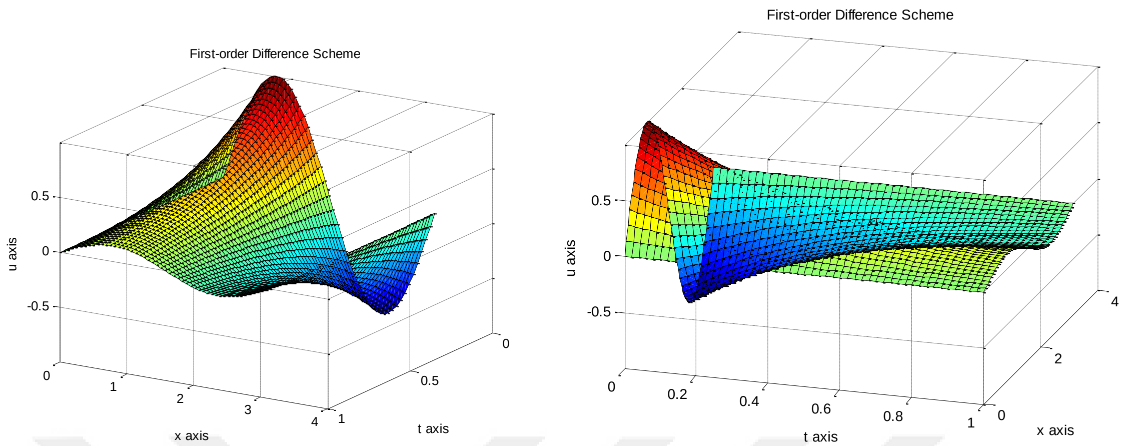

Figure 2.2 (continued): First-Order Difference Scheme

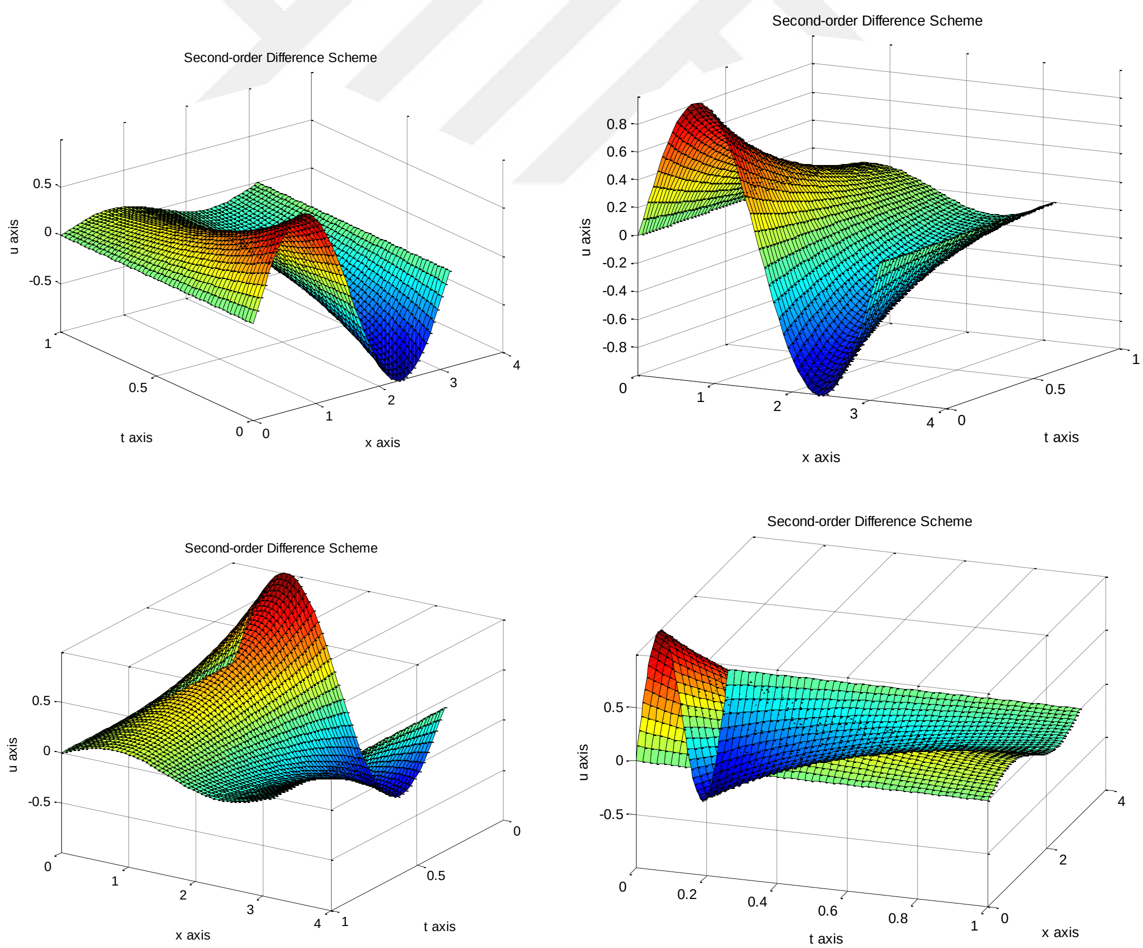

Figure 2.3: Second-Order Difference Scheme

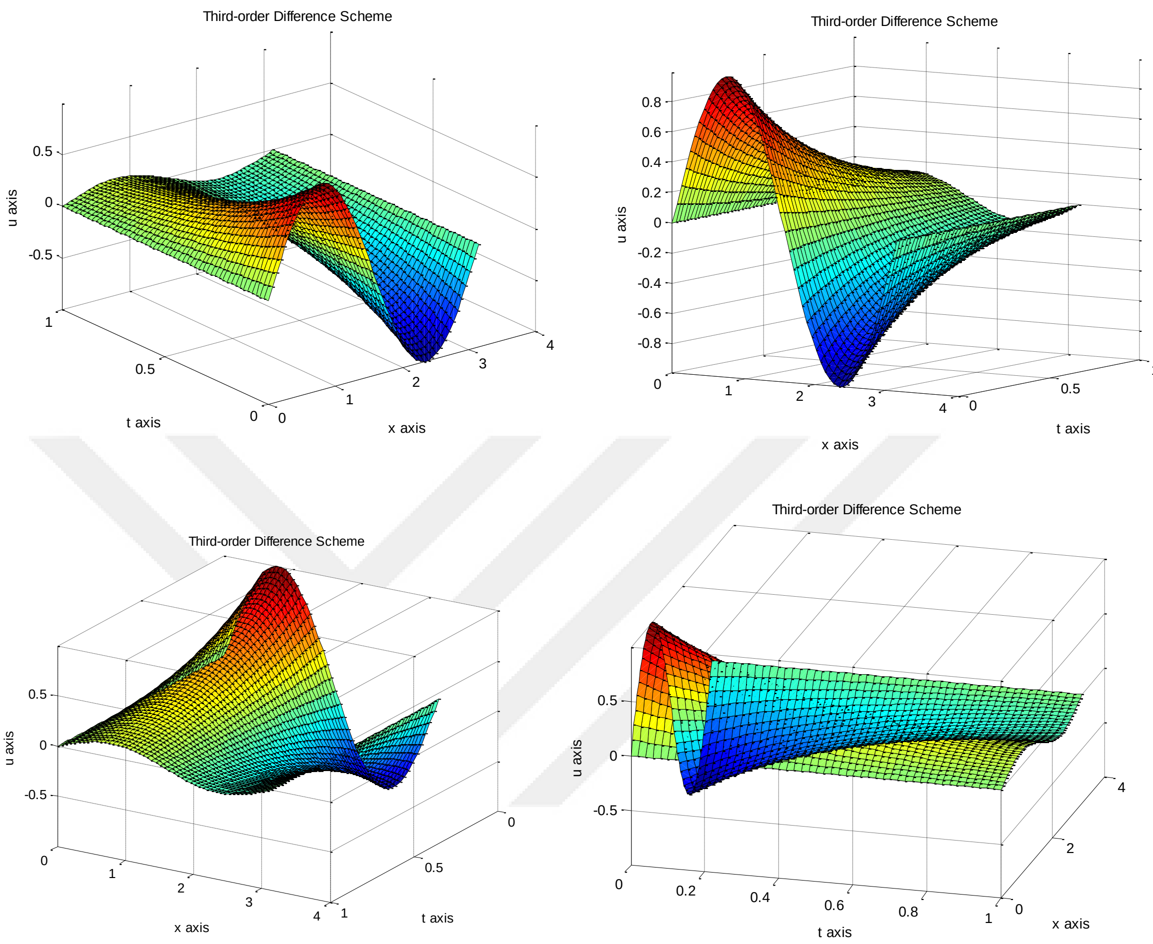

Figure 2.4: Third-Order Difference Scheme

By looking at the graphs, it is almost impossible to see the difference between the definite solution and the numeric results of given problem although the graphs are given from different perspectives. Therefore, as to the precise analogy of exact and numerical solutions; and further as to the analogy of three different difference schemes, errors are to be calculated. Error analysis is illustrated in Table 2.1, for $N = M = 50$ and 100; and in Table 2.2, for various values of N and M.

Table 2.1: Error Examination

$N = M$		50	100
First Order	Errors	0.0109829013	0.0057957026
	CPU	0	0.09375
Second Order	Errors	0.0014344666	0.0003586775
	CPU	0	0.265625
Third Order	Errors	0.0007630991	0.0001931353
	CPU	0	0.265625

Table 2.2: Analogy of errors

First-order	**N, M ($N \simeq M^2$)**	100, 10	200,14	400, 20
	Error	0.0242	0.0126	0.0063
Second-order	**N, M (N=M)**	30, 30	60, 60	120, 120
	Error	0.004	0.000996	0.000249
Third-order	**N, M ($N^3 \simeq M^2$)**	10, 35	20, 90	40, 253
	Error	0.0011	0.000129	0.0000159

It can be seen from Tables 2.1 and 2.2 that, the accuracy of the numerical results by the third-order of accuracy difference scheme is greater than those obtained by the second-order. Also, the accuracy of the second-order difference scheme is greater compared to the first. Moreover, it is observed from Table 2.2 that if the amount of N is multiplied by two for the first-degree, second-degree and third-degree difference charts, the errors of the numerical results decreased by 2, 4 and 8 times, respectively (Ashyralyev et al., 2016).

3. RESULTS

In the first chapter, a brief history is given on the solution of Telegraph equations. Three different problems are solved analytically using Fourier Series, Laplace Transform and Fourier Transform Methods.

In the second chapter, the following Cauchy Problem

$$\begin{cases} \dfrac{d^2u(t)}{dt^2}+\alpha\dfrac{du(t)}{dt}+Au(t)=f(t), 0\le t\le T, \\ \\ u(0)=\varphi, u^{'}(0)=\psi \end{cases} \tag{3.1}$$

for a telegraph equation is considered. It is in a Hilbert space H, with a given self-adjoint positive definite operator A. Main theorem on stability of (3.1) is given.

Section 2.2 is devoted to the study of first, second and third order difference charts as to the approximate formula of the Cauchy problem (3.1). Difference schemes are obtained by using Taylor's decomposition on three points, which was given by Ashyralyev and Sobolevski (2004).

In section 2.3, computational results are given. The proposed difference schemes are applied to a test problem. The difference schemes are converted to difference equation systems with matrix coefficients. Iterative method is applied to solve these systems. Numerical computations are done with the help of MATLAB. Computed results are given in Error Analysis. The errors are computed by the RMS formula for the accurate comparison of the difference schemes. Tables and figures are included.

Finally, a comparison of the difference schemes is presented on the basis of numerical analysis. One can deduce from graphs and tables that, the numerical outcomes by the third level of precision difference chart is clearer than the second or the first order.

4. DISCUSSION

In this thesis, the investigation of the numerical solutions of a Cauchy problem for the second-order telegraph PDEs is performed. Abstract Cauchy problem

$$\begin{cases} \dfrac{d^2u(t)}{dt^2} + \alpha \dfrac{du(t)}{dt} + Au(t) = f(t), 0 \le t \le T, \\ u(0) = \varphi, u'(0) = \psi \end{cases} \tag{4.1}$$

for the telegraph equation was considered.

For approximate solution of problem (4.1), first level of precision difference chart

$$\begin{cases} \dfrac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} + \alpha \dfrac{u_{k+1} - u_k}{\tau} + Au_{k+1} = f_k, \\ f_k = f(t_{k+1}), 1 \le k \le N-1, N\tau = T, \\ u_0 = \varphi, \dfrac{u_1 - u_0}{\tau} = \psi \end{cases}$$

is constructed.

The second-level of precision difference charts

$$\begin{cases} \dfrac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} + \alpha \dfrac{u_{k+1} - u_{k-1}}{2\tau} + \dfrac{A}{2}(u_{k+1} + u_{k-1}) = f_k, \\ f_k = f(t_k), 1 \le k \le N-1, \\ u_0 = \varphi, \dfrac{u_1 - u_0}{\tau} = \dfrac{\tau}{2}(f_0 - Au_1 - \alpha\psi) + \psi, f_0 = f(0), \end{cases}$$

$$\begin{cases} \dfrac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\dfrac{u_{k+1}-u_{k-1}}{2\tau}+\dfrac{A}{2}u_k+\dfrac{A}{4}\left(u_{k+1}+u_{k-1}\right)=f_{k,} \\ f_k=f\left(t_k\right), 1\le k\le N-1, \\ u_0=\varphi, \dfrac{u_1-u_0}{\tau}=\dfrac{\tau}{2}\left(f_0-Au_1-\alpha\psi\right)+\psi,\ f_0=f(0) \end{cases}$$

are constructed for the approximate formula of the problem (4.1).

The third-level of precision difference chart

$$\begin{cases} \dfrac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\dfrac{u_{k+1}-u_{k-1}}{2\tau}+\dfrac{2}{3}Au_k+\dfrac{1}{6}A\left(u_{k+1}+u_{k-1}\right) \\ +\dfrac{\tau^2}{12}\left[\left(-\alpha^3+2\alpha A\right)\dfrac{u_{k+1}-u_k}{\tau}-\left(\alpha^2-A\right)Au_{k-1}\right]=f_k, \\ f_k=\dfrac{2}{3}f\left(t_k\right)+\dfrac{1}{6}\left[f\left(t_{k+1}\right)+f\left(t_{k-1}\right)\right] \\ -\dfrac{\tau^2}{12}\left[\left(\alpha^2-A\right)f\left(t_{k+1}\right)-\alpha f'\left(t_{k+1}\right)+f''\left(t_{k+1}\right)\right], 1\le k\le N-1, \\ u_0=\varphi, \\ \left(I+\dfrac{\tau^2}{2}A\right)\dfrac{u_1-u_0}{\tau}=\left(I+\dfrac{\tau^2}{2}A\right)\psi+\dfrac{\tau}{2}\left[-\alpha\psi-A\varphi\right]+\dfrac{\tau^2}{6}\left[\alpha^2\psi+\alpha A\varphi-A\psi\right]+f_0, \\ f_0=\dfrac{\tau}{2}f(0)-\dfrac{\tau^2}{6}\left[f(0)-f'(0)\right] \end{cases}$$

is constructed for approximately solving the problem (4.1).

Proposed difference schemes are applied to a test problem. For numerical results, an IVP over telegraph PDE is considered. MATLAB is used for obtaining the numerical results. The comparison of the difference schemes demonstrates that third-order of accuracy difference scheme gives more precise results than second-order or the first-order of accuracy difference schemes.

5. CONCLUSION AND RECOMMENDATIONS

This thesis is primarily dedicated to computational analysis of a Cauchy problem for the second-order telegraph equations. Our main goal in this work is deriving an approximation for the solution of telegraph equations with Dirichlet conditions. The abstract Cauchy problem

$$\begin{cases} \dfrac{d^2u(t)}{dt^2} + \alpha \dfrac{du(t)}{dt} + Au(t) = f(t), 0 \le t \le T, \\ u(0) = \varphi, u'(0) = \psi \end{cases} \tag{5.1}$$

for telegraph equations is taken into consideration. The following original results are obtained.

• Following first-level of precision difference scheme was constructed for approximately solving the problem (5.1):

$$\begin{cases} \dfrac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} + \alpha \dfrac{u_{k+1} - u_k}{\tau} + Au_{k+1} = f_k, \\ f_k = f(t_{k+1}), 1 \le k \le N-1, N\tau = T, \\ u_0 = \varphi, \dfrac{u_1 - u_0}{\tau} = \psi \end{cases}$$

For the solution of this difference scheme, the stability estimates were presented.

• The following second-order of accuracy difference schemes were constructed for approximately solving the problem (5.1):

$$\begin{cases} \dfrac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} + \alpha \dfrac{u_{k+1} - u_{k-1}}{2\tau} + \dfrac{A}{2}\left(u_{k+1} + u_{k-1}\right) = f_k, \\ f_k = f\left(t_k\right), 1 \le k \le N-1, \\ u_0 = \varphi, \dfrac{u_1 - u_0}{\tau} = \dfrac{\tau}{2}\left(f_0 - Au_1 - \alpha\psi\right) + \psi, \; f_0 = f(0), \end{cases}$$

$$\begin{cases} \dfrac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\dfrac{u_{k+1}-u_{k-1}}{2\tau}+\dfrac{A}{2}u_k+\dfrac{A}{4}\left(u_{k+1}+u_{k-1}\right)=f_k, \\ f_k=f\left(t_k\right), 1\le k\le N-1, \\ u_0=\varphi, \dfrac{u_1-u_0}{\tau}=\dfrac{\tau}{2}\left(f_0-Au_1-\alpha\psi\right)+\psi,\ f_0=f(0). \end{cases}$$

For the solution of the foregoing difference schemes, the stability estimates were presented.

• The third-level of precision difference chart below was constructed for the approximate solution of the problem (5.1):

$$\begin{cases} \dfrac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\dfrac{u_{k+1}-u_{k-1}}{2\tau}+\dfrac{2}{3}Au_k+\dfrac{1}{6}A\left(u_{k+1}+u_{k-1}\right) \\ +\dfrac{\tau^2}{12}\left[\left(-\alpha^3+2\alpha A\right)\dfrac{u_{k+1}-u_k}{\tau}-\left(\alpha^2-A\right)Au_{k-1}\right]=f_k, \\ f_k=\dfrac{2}{3}f\left(t_k\right)+\dfrac{1}{6}\left[f\left(t_{k+1}\right)+f\left(t_{k-1}\right)\right] \\ -\dfrac{\tau^2}{12}\left[\left(\alpha^2-A\right)f\left(t_{k+1}\right)-\alpha f'\left(t_{k+1}\right)+f''\left(t_{k+1}\right)\right], 1\le k\le N-1, \\ u_0=\varphi, \\ \left(I+\dfrac{\tau^2}{2}A\right)\dfrac{u_1-u_0}{\tau}=\left(I+\dfrac{\tau^2}{2}A\right)\psi+\dfrac{\tau}{2}\left[-\alpha\psi-A\varphi\right]+\dfrac{\tau^2}{6}\left[\alpha^2\psi+\alpha A\varphi-A\psi\right]+f_0, \\ f_0=\dfrac{\tau}{2}f(0)-\dfrac{\tau^2}{6}\left[f(0)-f'(0)\right]. \end{cases}$$

So as to the result of the above difference chart, consistency conjectures were presented.

• Proposed difference schemes were applied to a trial problem in order to demonstrate the consistency and efficiency of numerical method. For numerical results, the following IVP

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} + u(t,x) = \frac{\partial^2 u(t,x)}{\partial x^2} + 7\exp(-2t)\sin(2x), \\ 0 < t < 1, 0 < x < \pi, \\ u(0,x) = \sin(2x), u_t(0,x) = -2\sin(2x), 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0, 0 \le t \le 1 \end{cases}$$

was considered. The exact and numerical solutions were analyzed. It can be observed from the analogy that third-order of approximation difference scheme gives more certain results than second-order or the first order of accuracy difference schemes.

• MATLAB is used to attain the numerical outcomes. The MATLAB implementation used for the computations is given in Appendix.

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APPENDICES

APPENDIX 1. Programming for the first-order of accuracy difference scheme.

```
function df
N=input ('N gir'); M=input ('M gir');
val=100;
for tt=1:20;
time1=cputime;
x=linspace (0,pi,M+1);
t=linspace (0,1,N+1);
h=pi/M;    tau=1/N;
d=-2/ (tau^2)-1/ (tau);
e=1/ (tau^2);
U (:,1)=sin(2*x);
U (:,2) = (1-2*tau)*sin(2*x);
for   k=2:N;
for n=2:M;
A (n,n-1)=-1/(h^2) ;
A (n,n)=1/ (tau^2) +2/ (h^2) +1/ (tau)+1 ;
A (n,n+1)=-1/ (h^2) ;
B (n,n)=d;   C (n,n)=e;
fii (n,k-1)=ff (t(k), x(n)) ;
end;
A (1,1) =1;    A(M+1,M+1) = 1;
B(M+1,M+1)=0;    C(M+1,M+1)=0;
fii (1,k-1) =0 ;   fii (M+1,k-1)=0;
U( :,k+1) = inv (A) * (-B*U(:,k)-C*U(:,k-1)+fii (:,k-1)) ;
end;
U;
p=transpose (U) ;
%%%%%%%%%%'E. S. OF PDE' %%%%%%%%%%%
for n=1:M+1;
for k=1:N+1;
t=(k-1)*tau;
x=(n-1)*h;
%est (n, k:k) =(1-t+t^2) * (sin (2*x)) ;
est (n, k:k)=exp (-2*t)*(sin(2*x));
%es (n, k:k) = exp (-t^3+t) * (sin (2*x)) ;
%est (n, k:k) = exp (-t^3) * (sin (2*x));
end;
end;
es=transpose (est) ;
%%%%%%%%%% END EXACT SOLUTION %%%%%%%%%%%%%%%%
%%%%%%% ERR ANAL OF GEN SOL OF THE DIFF SCHEME %%%%%%%
```

APPENDIX 1 (continued).

```
for i=1:N-1;
for j=1:M-1;
ftf(i, j)=p(i+1, j+1)-es(i+1, j+1);
end;
end;
fmat1=abs(ftf);
fmat2=fmat1.*fmat1*h;
fmat3=sum(fmat2, 2);
fmat4=fmat3.^(1/2);
sumerror=max(fmat4);
time2=cputime;
if (val> (time2-time1))
val=time2-time1;
end;
end;
sumerror
val
%%%%%%%%%%%%%%%%%%% END OF ERROR ANALYSIS %%%%%%%%
%%%%%%%%%%%%%%%%%GRAPH OF THE SOLUTION %%%%%%%%%%%
U;
est;
[xler,tler]=meshgrid(0:tau:1, 0:h:pi);
table=[est;U]; table(1:2:end,:)=est; table(2:2:end,:)=U;
q=min(min(table));
w=max(max(table));
figure;
surf(tler,xler,est);
%title('EXACT SOLUTION'); set(gca,'ZLim',[q w]);
rotate3d;
%X Label ('x'); YLabel ('t'); ZLabel ('u') ;
figure; surf(tler,xler,U);
%title('First-Order Difference Scheme'); set(gca,'ZLim',[q w]);
rotate3d ;
%X Label ('x'); Y Label ('t'); Z Label ('u');
%%%%%%%%%%%%% END GRAPH %%%%%%%%%%%%%%%%%
function u=g(t,x); %u=1;
u= (1+0*t+0*x) ;
function u=ff (t,x) ;
%u= (2+ 5* (t^2+t+1))*sin (2*x) ;
%u=6*exp (-t) *sin (2*x);
%u=(-6*t+9*t^4+4*(t+x))*exp(-t^3)*sin(2*x) ;
%u=(-2*t+3)*exp(-t^2)*sin(2*x);
u= (2+5* (1+0*t+0*x))*exp (-2*t)*sin (2*x) ;
%u= (2+4* (t+x) * (1-t+t^2)) *sin (2*x) ;
```

APPENDIX 2. Programming for the second order of accuracy difference scheme.

```
function ds
N=input ('N gir') ; M=input ('M gir') ;
%N=40; M=40;
val=100;
for tt=1:20;
time1=cputime;
x=linspace (0,pi,M+1);
t=linspace (0,1,N+1);
h=pi/M; tau=1/N;
U (:,1)=sin (2*x) ;
for k=2:N;
for n=2:M;
A(n,n-1)=-g(t(k),x(n)) / (4*(h^2)) ;
A(n,n)=1/ (2*tau) +1/ (tau^2) +g(t(k), x(n)) / (2*(h^2)) +g (t(k), x(n)) /4;
A(n,n+1)=-g (t(k),x(n)) / (4* (h^2) );
B(n,n-1)=-g (t(k), x(n)) / (2*(h^2)) ;
B(n,n)=-2/ (tau^2) +g (t(k), x(n)) / (h^2)+g(t(k), x(n)) / 2;
B(n,n+1)=-g(t(k),x(n)) / (2* (h^2));
C(n,n-1)=-g (t(k) ,x(n)) / (4* (h^2));
C(n,n)=-1/ (2*tau) +1/ (tau^2) +g (t (k), x(n) ) / (2* (h^2)) +g(t(k), x(n) ) / 4;
C(n,n+1)=-g(t (k),x(n)) / (4* (h^2)) ;
D(n,n-1)=-tau/ (2* (h^2) );
D(n,n) =1/tau+tau/ (h^2) +tau/2;
D(n,n+1)=-tau/ (2* (h^2) ) ;
fii (n,k-1) =ff (t(k), x(n)) ;
ro (n,1)= (-2+ (9/2) *tau)*sin (2*x(n) ) ;
end;
A (1,1) =1 ; A(M+1,M+1) =1 ;
B (M+1 ,M+1) =0;
C (1, :) =0 ; C (M+1, :) =0;
D (1,1)=1; D (M+1,M+1) = 1;
fii (1,k-1) =0;   fii (M+1,k-1) =0;
ro (1,1) =0 ; ro (M+1,1) =0;
U( : , 1);
U( : , 2)=inv(D) * (1/tau) *U(: , 1) +inv (D) *ro( : , 1) ;
U( : , k+1 )=inv (A) * (-B*U (: , k) -C*U ( : , k-1 ) +fii (:, k-1)) ;
end;
U;
p=transpose (U) ;
%%%%%%%%%%'EXCT SOL OF THIS PDE' %%%%%%%%%%%
for n=1:M+1;
for k=1:N+1;
t=(k-1)*tau;
x=(n-1)*h;
```

APPENDIX 2 (continued).

```
%est(n,k:k)=(1-t+t^2) * (sin (2*x) ) ;
est (n, k:k) =exp (-2*t)) * (sin(2*x)) ;
%es (n, k:k) =exp (-t^3+t) * (sin (2*x)) ;
%est(n, k:k) =exp (-t^3) * (sin (2*x)) ;
end;
end;
es=transpose (est) ;
%%%%%%%%%% END EXACT SOLUTION %%%%%%%%%%%%%%%%
%%%%%%% ERR ANAL OF GEN SOL OF THE DIFF SCHEME %%%%%%%
for i=1:N-1;
for j=1:M-1;
ftf(i, j)=p(i+1, j+1)-es(i+1, j+1);
end;
end;
fmat1=abs(ftf);
fmat2=fmat1.*fmat1*h;
fmat3=sum(fmat2, 2);
fmat4=fmat3.^(1/2);
sumerror=max(fmat4);
time2=cputime;
if (val> (time2-time1))
val=time2-time1;
end;
end;
sumerror
val
%%%%%%%%%%%%%% END OF ERROR ANALYSIS %%%%%%%%%%
%%%%%%%%%%%%%%GRAPH OF THE SOLUTION %%%%%%%%%%
U;
est;
[xler,tler]=meshgrid(0:tau:1, 0:h:pi);
table=[est;U]; table(1:2:end,:)=est; table(2:2:end,:)=U;
q=min(min(table));
w=max(max(table));
figure;
surf(tler,xler,est);
%title('EXACT SOLUTION'); set(gca,'ZLim',[q w]);
rotate3d;
X Label ('x'); Y Label ('t'); Z Label ( 'u');
figure; surf(tler,xler,U);
%title('Second-order Difference Scheme'); set(gca,'ZLim',[q w]);
rotate3d ;
%X Label ('x'); Y Label ('t');Z Label ('u');
%%%%%%%%%%% END GRAPH %%%%%%%%%%%%
```

APPENDIX 2 (continued).

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function uu=g(t,x);
uu=1+0*t+0*x;
function u=ff (t,x) ;
u=7*exp (-2*t) *sin (2*x) ;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

APPENDIX 3. Programming for the third-order of accuracy difference scheme.

```
function dt
N=input (‘N gir’) ; M=input (‘M gir’) ;
%N=40; M=40;
val=100;
for tt=1:20;
time1=cputime;
x=linspace (0,pi,M+1);
t=linspace (0,1,N+1);
h=pi/M; tau=1/N;
U (:,1)=sin (2*x) ;
AA=(tau-4) /6;
BB= (1/6) + ((tau^2) + 2*tau) /12;
CC= (8-tau) /12;
DD= (2+tau) /12;
for k=2:N;
for n=3:M-1;
A(n,n-2)=(tau^2) / (12*(h^4));
A(n,n-1)=- (tau^2) / (3*(h^4)) - BB / (h^2);
A(n,n)=1 / (tau^2) + 1/ (2 * tau) + (tau^2) / (2 * (h^4) +2* BB / (h^2) +DD;
A(n, n+1)= -(tau^2) / (3* (h^4))-BB/ (h^2) ;
A(n,n+2)= (tau^2) / (12 * (h^4));
B(n,n-1)=AA / (h^2) ;
B(n,n)=-2/ (tau^2) -2 * AA / (h^2)+ CC;
B(n,n+1)=AA / (h^2);
C(n,n-1)=-1 / (6* (h^2));
C(n,n)=1/ (tau^2) - 1 / (2* tau) + 1 / (3 * (h^2))+1/6;
C(n,n+1)=-1 / (6* (h^2)) ;
end;
for i=2:M;
D(i,i-1)=-tau/ (2* (h^2) );
D(i,i) =1/tau+tau/ (h^2) +tau/2;
D(i,i+1)=-tau/ (2* (h^2) ) ;
end;
for q=3:M-1;
fii (q,k-1) =ff (t(k), x(q)) ;
end;
for w=2:M;
ro (w,1)= (-2+ (9/2) * tau-(38/6) * (tau^2)) *sin (2*x(w) ) ;
end;
A (1,1) =1 ; A(M+1,M+1) =1 ;
A (2,1)=2; A (2,2)=-5; A(2,3) = 4; A(2,4) = -1;
A(M,M-2) =-1; A (M,M-1) =4, A(M,M) =-5; A (M,M+1) =2;
B (M+1 ,M+1) =0;
C (M+1, M+1) =0;
```

APPENDIX 3 (continued).

```
D (1,1)=1; D (M+1,M+1) = 1;
fii (1,k-1) =0;   fii (M+1,k-1) =0;
ro (1,1) =0 ; ro (M+1,1) =0;
ro;
U( : , 1);
U( : , 2)=inv(D) * (1/tau) *U(: , 1) +inv (D) *ro( : , 1) ;
U( : , k+1 )=inv (A) * (-B*U (: , k) -C*U ( : , k-1 ) +fii (:, k-1)) ;
end;
U;
p=transpose (U) ;
%%%%%%%%%%'EXCT SOL OF THIS PDE' %%%%%%%%%%%
for n=1:M+1;
for k=1:N+1;
t=(k-1)*tau;
x=(n-1)*h;
%est(n,k:k)=(1-t+t942) * (sin (2*x) ) ;
est (n, k:k) =exp (-2*t)) * (sin(2*x)) ;
%es (n, k:k) =exp (-t943+t) * (sin (2*x)) ;
%est(n, k:k) =exp (-t943) * (sin (2*x)) ;
end;
end;
es=transpose (est) ;
%%%%%%%%%%% END EXACT SOLUTION %%%%%%%%%%%%%%%%%
%%%%%%%% ERR ANAL OF GEN SOL OF THE DIFF SCHEME %%%%%%%
for i=1:N-1;
for j=1:M-1;
ftf(i, j)=p(i+1, j+1)-es(i+1, j+1);
end;
end;
fmat1=abs(ftf);
fmat2=fmat1.*fmat1*h;
fmat3=sum(fmat2, 2);
fmat4=fmat3.94(1/2);
sumerror=max(fmat4);
time2=cputime;
if (val> (time2-time1))
val=time2-time1;
end;
end;
sumerror
val
%%%%%%%%%%%%%%%%%%% END OF ERROR ANALYSIS %%%%%%%%%
%%%%%%%%%%%%%%%%%GRAPH OF THE SOLUTION %%%%%%%%%%%%
U;
```

APPENDIX 3 (continued).

```
est;
[xler,tler]=meshgrid(0:tau:1, 0:h:pi);
table=[est;U]; table(1:2:end,:)=est; table(2:2:end,:)=U;
q=min(min(table));
w=max(max(table));
figure;
surf(tler,xler,est);
rotate3d;
figure; surf(tler,xler,U);
rotate3d ;
%%%%%%%%%%% END GRAPH %%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
tau=1/N;
function u=ff(t,x);
u=((2/3)+(1/6) * ((2499/2500) *exp (-2*50)+exp(2/50))*7 *exp (-2*t)*sin (2*x) ;
%u=((2/3)+(1/6) * ((9999/10000) *exp (-2*100)+exp(2/100))*7 *exp (-2*t)*sin (2*x) ;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

CURRICULUM VITAE

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