ISTANBUL TECHNICAL UNIVERSITY ★ INFORMATICS INSTITUTE

SENSITIVITY ANALYSIS OF EXPECTED SHORTFALL BY MEANS OF A SECOND-ORDER APPROXIMATION

M.Sc. THESIS

Güven Gül POLAT

Computational Science and Engineering

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Thesis Advisor: Prof. Dr. Burç ÜLENGİN

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ BİLİŞİM ENSTİTÜSÜ

İKİNCİ DERECE YAKLAŞTIRIM YOLUYLA BEKLENEN KAYIP HASSASLIK ANALİZİ

YÜKSEK LİSANS TEZİ

Güven Gül POLAT
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Hesaplamalı Bilim ve Mühendislik

Hesaplamalı Bilim ve Mühendislik

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HAZİRAN 2012

Güven Gül Polat, a **M.Sc.** student of ITU **Informatics Institute 702081006**, successfully defended the **thesis** entitled **"SENSITIVITY ANALYSIS OF EXPECTED SHORTFALL BY MEANS OF A SECOND-ORDER APPROXIMATION"**, which she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

Thesis Advisor :	**Prof. Dr. Burç ÜLENGİN** **Istanbul Technical University**	....
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Date of Submission : 4 May 2012
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To my family,

FOREWORD

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ABBREVIATIONS

ES	:	Expected Shortfall
EWMA	:	Exponentially weighted moving average
GARCH	:	Generalized autoregressive conditional heteroskedasticity
ISE	:	İstanbul Stock Exchange
MES	:	Marginal Expected Shortfall
MKL	:	Math Kernel Library
MLE	:	Maximum likelihood estimation
MPI	:	Message Passing Interface
MT19937	:	Mersenne Twester pseudorandom number generator
pdf	:	Probability density function
UHeM	:	National Center for High Performance Computing of Turkey
VaR	:	Value at Risk

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SENSITIVITY ANALYSIS OF EXPECTED SHORTFALL BY MEANS OF A SECOND-ORDER APPROXIMATION

SUMMARY

Financial institutions require risk quantification in order to set a capital reserve to cover adverse market movements. Expected Shortfall (ES) is a coherent risk measure that serves the purpose of inferring market risk. Since it is described as the average loss beyond a specified threshold, ES represents a protective attitude. Identifying the overall ES for a position with multiple risk factors is achieved by a weighted aggregation of the rate of returns on the underlying risk factors. On the other hand, this relation does not exactly hold in terms of geometric (logarithmic) return. Because geometric return is the one that is more adequate to work with in the context of market relations and risk measurement than arithmetic return, a second-order approximation can be considered over weighted combination to increase accuracy in the estimation of ES. Particularly, handling each risk factor separately attract attention by the financial crisis of 2007-2009 to better understand factor contribution to the overall risk. Sensitivity analysis of the approximations mentioned is therefore performed via first derivatives of ES with respect to position allocation.

Risk refers to the variation of the future value of a position because of market fluctuations. A typical implementation of the Monte Carlo method involves simulating repeatedly from possible future events by enabling the use of the best available models of financial markets. In this study, the estimation of ES and its sensitivity is accordingly based on Monte Carlo simulation utilizing parallel computing techniques due to its computational cost with a relatively slow convergence rate. Totally, in addition to the increase in the accuracy of the estimation by a higher order approximation, it is demonstrated that the acceleration of the simulation by a parallel execution on a distributed memory system.

İKİNCİ DERECE YAKLAŞTIRIM YOLUYLA BEKLENEN KAYIP HASSASLIK ANALİZİ

ÖZET

Risk terimi, ekonomik, politik, sosyal ve teknolojik konularda yaygın bir kullanıma sahiptir. Genel olarak risk, özel bir hareketle ilişkili olan kayıp veya hasarın gerçekleşme ihtimalidir. Finansal olarak ise ters piyasa hareketlerinden etkilenmemek için ayrılan sermaye rezervini ifade eder. Finansal piyasalarda beklenen kayıp arttıkça beklenen kazanç da artmaktadır. Bu durum finansal kurumların aktif bir şekilde risk almasına yol açmaktadır. Risk yönetiminin buradaki rolü, ters piyasa hareketleri yüzünden oluşabilecek kayıp miktarını belirlemek amacıyla risk tayini yapmaktır. Risk yönetimi sistemleri, birden fazla risk faktöründen oluşan pozisyon için tüm riski tayin eden bütünsel çözümler içerir. Ayrıca, risk faktörlerini ve aralarındaki etkileşimi anlamaya çalışır.

Piyasa riski, finansal varlıkların değerindeki ters hareketler nedeniyle ortaya çıkan bir risk türüdür. Riske Maruz Değer (RMD), kavramsal basitliği, hesaplama kolaylığı ve hazır uygulanabilirliği sayesinde standartlaşan bir risk ölçüm tekniğidir. RMD, belirli bir güven düzeyinde elde tutma süresi boyunca olası en büyük kayıp olarak tanımlanır. RMD'den başka piyasa riski tayinine hizmet eden Beklenen Kayıp (BK) ise tutarlı bir risk ölçüm tekniğidir. BK, belirli bir eşik değerin ötesindeki ortalama kayıp olarak tanımlandığından koruyucu bir tutum sergilemektedir. Bu eşik değer çoğunlukla RMD seviyesi olarak belirlenir. BK, RMD'nin barındırdığı yetersizlikleri ortadan kaldıran özelliklere sahiptir:

- RMD ötesindeki kayıp hakkında bilgi vermesi
- birikimli pozisyon riskinin risk faktörlerinin birikimli riskinden küçük olması
- daha genel stokastik şartlarda geçerliliğin sağlanması.

Varlıkların BK tayininde kullanılan getiri oranı iki şekilde hesaplanabilir: aritmetik ve geometrik (logaritmik) getiri. Birden fazla risk faktörü içeren bir pozisyon için getiri oranı, ilgili risk faktörlerinin getiri oranlarının ağırlıklı ortalaması alınarak elde edilir. Bu ilişki aritmetik getiri göz önüne alındığında tam olarak sağlanırken geometrik getiri söz konusu olduğunda sadece yaklaştırım olarak kalmaktadır. Öte yandan geometrik getiri, piyasa ilişkileri ve risk ölçümü bağlamında çalışmak için aritmetik getiriden daha elverişlidir:

- Varlık fiyatlarının eksi değer almasını engeller.
- Çok dönem getiri hesabı için tek dönem getirilerinin toplanması örneğindeki gibi hesaplama kolaylığı sağlar.

Bu durumda BK tahmininde doğruluğu arttırmak amacıyla ağırlıklı birleşim yerine stratejik varlık dağılımı için türetilen ikinci derece yaklaştırım kullanılabilir. Önerilen ikinci derece yaklaştırımda geometrik getirilerin ağırlıklı birleşimi terimine, faktörler arasındaki kovaryansa ve faktörlerin ağırlığına dayalı terimler eklenmiştir. Matematiksel gösterilimin yanında yaklaştırımla ilgili finansal notlar:

- Varlık fiyatlarının geometrik Brownian hareketini izlediği sürekli zamanda tam olarak tutmaktadır.
- Kısa zaman aralıkları için daha kesin sonuçlar vermektedir.
- Yüksek derecede olduğundan teorik olarak bakışımsızlık ve sivrilik etkilerini yansıtmaktadır.
- Ağırlıklı birleşim ile aradaki fark, yüksek volatilite dönemlerinde büyüyebilmektedir.

2007-2009 finansal kriziyle birlikte her risk faktörünü ayrıca ele almak, pozisyonun bütün riskine neden olan katkıyı daha iyi anlamak adına dikkat çekmeye başlamıştır. Bu durumda da bahsedilen yaklaştırımların hassaslık analizi, faktör paylarına göre BK birinci türevleri alınarak yapılır. Önerilen ikinci derece yaklaştırımın hassaslık analizinde aşağıdaki öğeler yer almaktadır:

- eşik değere koşullu bağlılık
- faktör ağırlığı
- faktörler arası kovaryans, dolayısıyla korelasyon.

Risk, piyasa dalgalanmaları nedeniyle pozisyonun gelecek değerinde meydana gelen değişimlerle ilgilidir. Monte Carlo yönteminin tipik bir uygulaması, finansal piyasalarda erişilebilir en iyi modellerin kullanımını mümkün kılarak olası gelecek olguların defalarca benzetimini içerir. Buna bağlı olarak çalışmada BK ve hassaslığının tahmini, Monte Carlo benzetimine dayanmaktadır. Çalışmada, varlık fiyatlarının izleyeceği model olarak logaritmik fiyat farklarına dayanan geometrik Brownian hareketi seçilidir. Dolayısıyla olasılık dağılımı normal dağılım şeklinde özelleşir. Her bir risk faktörü için risk faktörleri arasındaki korelasyonu dikkate alan rassal sayı üretiminin ardından seçilen fiyat modeli kullanılarak getiri hesabı yapılır. Son aşamada, ağırlıklı birleşime ve önerilen ikinci derece yaklaştırıma göre iki farklı BK ölçümü yapılabilir.

İMKB100 endeksi, İstanbul Menkul Kıymetler Borsası (İMKB) hisse senedi piyasasında temel endeks olarak kullanılır. IMKB100 endeksi içerisinden 80 adet firmaya ait hisse senetleri kullanılarak yaklaştırımların etkisi örneklenmektedir. Finansal kriz nedeniyle ortaya çıkan sapmaları vurgulamak amacıyla 01.07.2008-02.07.2009 tarih aralığı ele alınmaktadır. Öncelikle, ilgili dönem içerisinde ağırlıklı birleşim ve ikinci derece yaklaştırım kullanılarak elde edilen iki farklı günlük geometrik getiri sonucu, pozisyonun gerçekleşmiş günlük geometrik getirisiyle karşılaştırılmaktadır. Burada, ikinci derece yaklaştırımın pozisyonun gerçekleşmiş günlük geometrik getirisine daha çok yakınsadığı gösterilmektedir. Sonrasında, ilgili dönemin son günü itibariyle günlük BK ölçümü ve hassaslık analizi yapılarak risk hesabı üzerindeki etkiler incelenmektedir. Sonuçta ikinci derece yaklaştırım, ağırlıklı birleşimden daha düşük BK değerleri üretmektedir. Risk faktörlerinin hassaslıkları da ikinci derece yaklaştırımda çoğunlukla daha düşük değerler almaktadır.

Geometrik getiri ve risk hesaplamaları C programlama dili kullanılarak yapılmaktadır. Monte Carlo yönteminin nispeten yavaş bir yakınsaklık derecesine sahip hesaplama yükü nedeniyle dağınık bellekli mimari sisteminde paralel hesaplama tekniklerinden faydalanılmaktadır. Çeşitli performans kriterleri aracılığıyla risk faktörü sayısı kadar işlemci kullanılarak hızlanma sağlandığı gösterilmektedir.

Toplamda, daha yüksek derece bir yaklaştırım yoluyla tahmin doğrulunun arttırılmasına ek olarak dağınık bellekli mimari sisteminde bir paralel hesaplama ile benzetimdeki hızlanma vurgulanmaktadır.

1. INTRODUCTION

The term risk is widely utilized within the literature on economic, political, social and technological subjects (Cheng et al, 2004). In a general sense, risk is expressed as a chance of injury or loss related to a specified action (Elliott and Miao, 2009). Financially, risk is a capital reserve to cover adverse market movements. In other words, risk means random profit or loss of a position. It can be positive (profit) or negative (loss) (Cheng et al, 2004). The major sources of loss in financial institutions are typically identified as market risk, credit risk and operational risk. Market risk refers to the losses because of adverse movements in the value of financial assets. Credit risk results from being unwilling or unable of the counterparties to fulfill contractual obligations which are due. Operational risk is that of incurring losses due to failed or inadequate internal processes, systems and people or to external events. Since these categories often interact with each other, any classification is arbitrary to some extent (Jorion, 2007).

In financial markets, there is generally no so-called "free lunch" which is another way of saying no profit without risk. This leads financial institutions to actively take on risks. The role of financial risk management is thereby to measure and manage these risks through various methods such as diversification, hedging, or repackaging and transferring back to markets (Eberlein et al, 2007). Particularly, regulators and supervisory authorities require each financial institution to determine the capital reserve amount via risk management methods in order to prevent bankruptcy if large losses occur (Eberlein et al, 2007; Lan, 2010).

Risk management frameworks and systems include integrated solutions whose goal is to assess the overall risk for a position of multiple risk factors (Eberlein et al, 2007). The philosophy of a risk management framework is to try to understand the individiual risk factors and the interaction among each other, and to quantify the overall risk (Constantinescu, 2011). The quantification is generally performed by modeling the uncertain payoff as a random variable, and enables a certain functional is applied to. Such functionals are defined as risk measures (Föllmer and Schied, 2008). An adequate risk measure must be responsive in uncertain market conditions, and be reflective of the latest available information in a non-independent and identically distributed framework (Scaillet, 2004).

Besides quantification, the decomposition of risk is presented as a useful risk management tool in practice, e.g. selecting risk factors that achieve the best risk-return trade-off, allocating capital to individual risk factors, or transfer pricing (Yamai and Yoshiba, 2002a; Acharya et al, 2009). Furthermore, the Global Financial Crisis of 2007-2009 has motivated academic research and supervisory policy agenda to better understand risk contribution to the whole in order to capture systemic risk. Following Acharya et al. (2010) and Brownlees and Engle (2010), systemic risk contribution of each financial institution depends on its expected loss in a systemic crisis and its degree of leverage. While the degree of leverage is readily available, loss contribution requires to be estimated using appropriate time series methods.

The overall risk of a market and its sensitivity to each risk factor thus become vital to the survival of financial institutions. A coherent risk measure that serves such a purpose is Expected Shortfall (ES). ES is the loss conditional on the return being equal to or less than a threshold and the return on a portfolio is given by the weighted combination of the underlying equities returns in terms of arithmetic return. Hereby, sensitivity analysis is performed with respect to portfolio allocation. Since it is more adequate to work with geometric (logarithmic) returns in risk assessment and weighted combination equation is only approximately achieved in this case, a second-order approximation is considered for the portfolio geometric return.

An identical practice over XU100 index is tackled in this study to illustrate the impact of the approximations. XU100 is designed by Istanbul Stock Exchange (ISE) as the basic index for ISE stock market. It consists of one hundred stocks which are selected among the stocks of companies listed on National Market and the stocks of real estate investment trusts and venture capital investment trusts listed on Corporate Products Market. The time interval between 01.07.2008 and 03.07.2009 is observed by highlighting the deviations due to the financial crisis.

Section 2 gives definitions which are basis for market risk and represents the speriorities of ES against the standard market risk measure Value at Risk (VaR). Following, it mathematically demonstrates a portfolio ES and its sensitivity to individual risk factors by means of a weighted combination and a second-order approximation. Despite its computational cost with a relatively slow convergence rate, Monte Carlo simulation is an attractive methodology for precise estimates due to its capability of modeling the behavior of possible future events. Section 3 clearly describes the estimation methods of ES on a portfolio, and focuses on the stages of a financial Monte Carlo simulation. It is shown over the implementation in Section 4 that the accuracy is increased by the second-order approximation and the computational complexity is reduced utilizing parallel computing techniques. Finally, section 5 briefly sums up, and makes recommendations for future research.

2. MARKET RISK

Market risk is the risk of adverse deviations in the value of financial instruments because of market movements during the time interval needed for liquidating the transactions. The period of liquidation is vital to the assessment of those adverse deviations. Whether it gets longer, the potential worst-case loss is higher due to the fact that market volatility tends to increase over longer horizons (Bessis, 2010).

On the other hand, it is rationale to limit market risk to the liquidation period since liquidating instruments or hedging their future changes of value is possible at any time. The liquidation period varies with the types of instruments, e.g. one day for foreign exchanges, and much longer for exotic derivatives. In particular, ragulatory identifies rules to set the liquidation period (Bessis, 2010).

Foreign exchange rates, interest rates and stock prices are the three typical forms that reflect market risk. Currency risk refers to the losses due to changes in exchange rates. Interest rate risk is the risk of decrease in net interest income through the changes of interest rates. Equity risk designates the losses that result from stock market dynamics (Sevil, 2001).

2.1 Market Risk Measures

Market risk was previously considered as a correcting factor of expected return. Such primitive measures were convenient for an immediate order of all preferences. Then, variance was proposed by Markowitz in order to measure the risk related to the return on assets and utilized until the standard risk measure, VaR, was introduced (Cheng et al, 2004).

2.1.1 VaR: lacking subadditivity

VaR was referred in the late 1980s by major financial firms for risk assessment of their trading portfolios. Following, J.P. Morgan, one of the world's leading global investment banks, presented VaR as a standard risk measure in 1994 (Linsmeier and Pearson, 2000). It is now widely applied by other financial institutions, nonfinancial corporations and institutional investors due to its conceptual simplicity, computational facility, and ready applicability (Yamai and Yoshiba, 2002b).

VaR is defined as "possible maximum loss over a given holding period within a fixed confidence level". Mathematically, VaR at the $100(1-\alpha)$ percent confidence level is the lower 100α percentile of the return distribution.

$$VaR_{\alpha} = -inf\{x|P[X \leq x] > \alpha\} \qquad \textbf{(2.1)}$$

where X is the return of a specified portfolio. If $inf\{x|A\}$ is the lower limit of x given event A, $inf\{x|P[X \leq x] > \alpha\}$ denotes the lower 100α percentile of return distribution (Yamai and Yoshiba, 2002b).

Despite its popularity in practice, VaR has the drawbacks:

- conveying no information about the extent of loss beyond the VaR level
- lacking subadditivity and thereby discouraging diversification (Artzner et al, 1999)
- probable violation of second order stochastic dominance and so of risk aversion in the traditional sense (Rau-Bredow, 2004).

This criticism has lead to a search for more appropriate alternatives. Accordingly in 1999, Artzner et al. introduced axioms on risk measures and showed that these axioms should be achieved by any risk measure that is to be used for effective risk regulation or management.

2.1.2 Coherent risk measure

Artzner et al. (1999) stated four axioms and called a risk measure satisfying these axioms as coherent. Denoting by p a coherent risk measure for random variables X and Y, the four axioms that have to hold are:

1. Translation invariance: $p(X+k) = p(X) - k$, for all $X \in \mathcal{G}$ and all real numbers k.
2. Subadditivity: $p(X+Y) \leq p(X) + p(Y)$, for all $X, Y \in \mathcal{G}$.
3. Positive homogeneity: $p(\lambda X) = \lambda p(X)$, for all $\lambda \geq 0$ and all $X \in \mathcal{G}$.
4. Monotonicity: $p(Y) \leq p(X)$, for all $X, Y \in \mathcal{G}$ with $X \leq Y$.

where Ω is the set of states of nature and assumed to be finite, and $\mathscr{G}$ is the set of all risks, namely the set of all real-valued functions on Ω.

Translation invariance indicates that the addition of a sure amount k to the initial position X reduces the risk $p(X)$, the cash needed to make the position acceptable, by k. It is clear that

$$p(X + p(X)) = p(X) - p(X) = 0 \tag{2.2}$$

adding $p(X)$, the cash needed for the measured risk, to the position X causes a neutral position (Dowd, 2005; Cheng et al, 2004).

Subadditivity ensures the risk from the cumulative position $X+Y$ is smaller than the cumulative risks $p(X) + p(Y)$ (Jadhav et al, 2009). It reflects an expectation how a risk measure bahaves under the composition or addition of positions. It also presents motivation for portfolio diversification (Jadhav et al, 2009; Göb, 2011).

Subadditivity reports $p(\lambda X) = \lambda p(X)$ for all $\lambda \geq 0$ and all $X \in \mathscr{G}$. Positive homogeneity imposes this axiom by providing proportion to the risk of a position with its scale or size (Dowd, 2005; Cheng et al, 2004).

Through monotonicity, final net worth that have the relation $X \leq Y$ should obviously generate the opposite relation in terms of their risks $p(X) \geq p(Y)$ (Cheng et al, 2004).

Any risk measure that fails to serve some of the axioms will perform paradoxical results because of wrong evaluation of relative risks (Acerbi et al, 2008). It is proved that VaR is not always subadditive even though it assures axioms translation invariance, positive homogeneity and monotonicity. Thus, VaR is not a coherent risk mesure (Artzner et al, 1999).

2.1.3 ES: leading coherency

A coherent alternative risk measure to VaR is given by ES. ES is defined as

$$ES_{\alpha} = -E[X|X \leq C] \tag{2.3}$$

where X is the return of a specified portfolio and C is a known threshold, generally the VaR at a specified confidence level. $E[X|X \leq C]$ accordingly denotes the expected value of X which is conditional on being equal to or less than a given threshold C.

Alleviating the first drawback of VaR, it measures the average loss beyond the VaR level, i.e. the average loss in the worst α cases (Yamai and Yoshiba, 2002a; Caporin and De Magistris, 2011; Rau-Bredow, 2004). Secondly, in addition to the axioms translation invariance, positive homogeneity and monotonocity, it is shown that ES fulfills the subadditivity axiom which ensures its coherence as a risk measure (Artzner et al, 1999; Yamai and Yoshiba, 2002a). In this way, ES fulfills the property of convexity which enables efficient decomposition and optimization (Yamai and Yoshiba, 2002a; Dowd, 2005; Föllmer and Schied, 2008). Last, ES is valid under more general conditions than VaR. Particularly, an ES based decision is consistent with expected utility maximization in a second order stochastic dominance while a VaR based decision is only in first order stochastic dominance (Dowd, 2005).

Still, VaR is widely used for economic capital calculation due to its conceptual simplicity. The economic capital calculated via VaR at the $100(1-\alpha)$ percent confidence level relates to the capital needed to keep the default probability below 100α percent. Thus, the default probability can be controlled by risk practitioners through the use of VaR for risk management (Yamai and Yoshiba, 2002b).

On the other hand, ES that is by definition higher than VaR refers to a more conservative performance in economic capital calculation. Besides, as a natural remedy for the deficiencies of VaR, which is not a coherent risk measure in general, ES attract attention from VaR through risk management practice (Yamai and Yoshiba, 2002b).

2.2 Portfolio ES

The rate of return on a portfolio composed of n specific assets at time t is given by

$$\begin{aligned} R_{p,t} &= \mathbf{w}_t^{\mathrm{T}} \mathbf{R}_t \\ &= \sum_{i=1}^{n} w_{i,t} R_{i,t} \end{aligned} \tag{2.4}$$

where $\mathbf{w}_t$ is the n-dimensional vector of weights, $\mathbf{R}_t$ the n-dimensional vector of arithmetic returns with the elements $w_{i,t}$ and $R_{i,t}$, respectively. There is the obvious constraint $\sum_{i=1}^{n} w_{i,t} = 1$. The arithmetic return of asset i is equal to

$$\begin{aligned} R_{i,t} &= (S_{i,t} - S_{i,t-1})/S_{i,t-1} \\ &= S_{i,t}/S_{i,t-1} - 1 \end{aligned} \tag{2.5}$$

where $S_{i,t}$ is the asset price of asset i at time t (Penza and Bansal, 2001). Accordingly,

$$w_{i,t} = \frac{S_{i,t-1}h_{i,t-1}}{\sum_{i=1}^{n} S_{i,t-1}h_{i,t-1}} \tag{2.6}$$

where $h_{i,t}$ is the total number of assets of asset i at time t.

Since ES can be represented in terms of rate of return, the ES on a portfolio composed of n specific assets based on arithmetic return framework is given by

$$\begin{aligned} ES_\alpha &= -E\left[\mathbf{w}^{\mathrm{T}}\mathbf{R} | R_p \leq C\right] \\ &= -E\left[\sum_{i=1}^{n} w_i R_i | R_p \leq C\right] \end{aligned} \tag{2.7}$$

However, it is more adequate to work with geometric returns in the context of market relations and risk measurement due to:

- guaranteeing that asset prices can never become negative
- enabling much easier calculations such as the sum of the one-period geometric returns for multiple-period geometric return (Dowd, 1998).

The geometric return on asset i is defined as

$$\begin{aligned} r_{i,t} &= \ln S_{i,t} - \ln S_{i,t-1} \\ &= \ln(S_{i,t}/S_{i,t-1}) \\ &= \ln(1 + R_{i,t}) \end{aligned} \tag{2.8}$$

More generally,

$$\text{Geometric return} = \ln(1 + \text{Arithmetic return}) \tag{2.9}$$

When geometric price differences are considered, the weighted combination of the underlying asset returns does not exactly hold (Campbell et al, 2002; Caporin and De

Magistris, 2011).

$$
\begin{aligned}
r_{p,t,1} = \ln(1+R_{p,t}) &= \ln(1+\sum_{i=1}^{n} w_{i,t}R_{i,t}) \\
&\neq \sum_{i=1}^{n} w_{i,t}(\ln S_{i,t} - \ln S_{i,t-1}) \\
&= \sum_{i=1}^{n} w_{i,t}\ln(1+R_{i,t}) \\
&= \sum_{i=1}^{n} w_{i,t}r_{i,t} = \mathbf{w}_t^{\mathrm{T}}\mathbf{r}_t
\end{aligned} \quad \textbf{(2.10)}
$$

where $\mathbf{r}_t$ is the n-dimensional vector of geometric returns with the elements $r_{i,t}$.

This relation was studied by Campbell and Viceira (1999) and Campbell et al. (2002) in strategic asset allocation framework, and the following approximation for the geometric return on a portfolio was derived:

$$
\begin{aligned}
r_{p,t,2} &= \mathbf{w}_t^{\mathrm{T}}\mathbf{r}_t + \frac{1}{2}\mathbf{w}_t^{\mathrm{T}}(\mathrm{diag}(\Omega_t) - \Omega_t\mathbf{w}_t) \\
&= \sum_{i=1}^{n} w_{i,t}r_{i,t} + \frac{1}{2}\sum_{i=1}^{n} w_{i,t}\Omega_{ii,t} - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Omega_{ij,t}w_{i,t}w_{j,t}
\end{aligned} \quad \textbf{(2.11)}
$$

where Ω_t the covariance matrix of assets geometric returns with the elements $\Omega_{ij,t}$ and $\mathrm{diag}(\Omega_t)$ is the vector containing the diagonal elements of the covariance matrix.

The covariance of random variables X and Y with means μ_X and μ_Y is measured by

$$
\mathrm{cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] \quad \textbf{(2.12)}
$$

or equivalently,

$$
\mathrm{cov}(X,Y) = E(XY) - \mu_X\mu_Y \quad \textbf{(2.13)}
$$

The second description occurs from the distributive property of expected value (Larsen and Marx, 1981). Expected value is for the number of most recent observations the number of which must be high enough to generate reasonable covariance estimates, on the other hand low enough to respond to the latest market events.

The notion of covariance links variance and correlation. While variance is a measure of volatility, correlation indicates the extent to which two series X and Y move together. A correlation coefficient lies in the interval $[-1,+1]$, and takes the value $+1$ if movement in line is exact, 0 if there is no link, -1 if movement in line but in the opposite direction is exact (Luenberger, 1998). The dimensionless correlation coefficient $\mathrm{corr}(X,Y)$ of X and Y is accordingly derived by normalizing the covariance

$$\mathrm{corr}(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y} \tag{2.14}$$

The two key features of covariance are

- $\mathrm{cov}(X,X) = \mathrm{var}(X)$ generalizing the concept of variance since

$$\mathrm{var}(X) = E\left[(X - \mu_X)^2\right] \tag{2.15}$$

- $\mathrm{cov}(X,Y) = 0$ if X and Y are independent (Larsen and Marx, 1981).

A covariance matrix Ω is symmetric with the elements Ω_{ij} those are covariances between pairs of n random variables denoted by $X_1, X_2, \ldots, X_n$. The elements on the main diagonal can be described as the variances of each variable since $\mathrm{cov}(X_i, X_i) = \mathrm{var}(X_i)$ (Tabachnick and Fidell, 2007).

$$\Omega_{ij} = \mathrm{cov}(X_i, X_j) = E\left[(X_i - \mu_i)(X_j - \mu_j)\right] \tag{2.16}$$

Since it is desired to evaluate covariance matrix at each time step in the second-order approximation procedure, three methodologies are introduced.

1. Historical estimation is the most straightforward approach that assumes that the covariance matrix is constant over time. After a window size is specified, volatilities and covariances are estimated simultaneously. On the other hand, this approach has a deficiency of being strictly accurate only if the "true" covariance matrix is obtained. Such a condition is never satisfied in practice.
2. The deficiency of historical estimation approach is alleviated by evaluating covariance matrix utilizing multivariate exponentially weighted moving average (EWMA) method. EWMA is a particular case of the equally weighted moving average method in the form of

$$\Omega_t = \lambda \Omega_{t-1} + (1 - \lambda) r_{t-1}^{\mathrm{T}} r_{t-1} \tag{2.17}$$

   where λ is a constant decay term between 0 and 1. The lower the value of λ is, the higher the weight of the recent observations are. Due to the fact that it accommodates changing volatilities and covariances over time, EWMA method is more flexible than historical estimation.

It is desired that each volatility and covariance has its own specific decay factor to achieve the best fit for separate estimates. However, a large number of different λ values can be difficult to handle in addition to no guarantee to perform a positive definite covariance matrix estimate. Such considerations led to choose a single decay factor. Accordingly, JP Morgan suggests to use EWMA model with $\lambda = 0.94$ for daily estimates.

3. Generalized autoregressive conditional heteroskedasticity (GARCH) models are generally better than EWMA method in forecasting the future level of covariances. GARCH (1, 1) model for updating a covariance matrix is

$$\Omega_t = \omega + \alpha X_{t-1} Y_{t-1} + \beta \Omega_{t-1} \qquad \textbf{(2.18)}$$

and the long-term average covariance is $\omega/(1-\alpha-\beta)$ where ω, α and β are the parameters to be estimated. Despite its preference over EWMA method, the number of parameters are so large that cause a bottleneck. Furthermore, it is shown that the covariance matrices obtained via EWMA method are sometimes the best when the matrices are used for risk assessment (Dowd 2005; Hull, 2003).

The proposed approximation exactly holds in continuous time where asset prices follow a geometric Brownian motion and is highly accurate for short time intervals (Campbell et al, 2002). The notes about the equation are (Caporin and De Magistris, 2011):

- A covariance and weights based term is added to weighted combination of single geometric returns.
- It can be illustrated as a second order approximation of the portfolio geometric return by means of assets geometric returns.

In addition, the following remarks are emphasized through the analysis of Equation (2.11) (Caporin and De Magistris, 2011):

- The portfolio geometric return depends on the assets geometric returns and on their covariances together.

- Higher order approximations theoretically include the effect of assymmetry (due to co-skewness matrices) and peakedness (due to leptokurtosis by means of co-kurtosis matrices) which are concerned when large deviations from normality occur.
- Most relevant, the difference between the geometric return aggregation and Equation (2.11) may enlarge in the presence of high volatility phases.

Finally, the ES on a portfolio composed of n specific assets based on geometric return framework is approximated by

1. first-order:

$$\begin{aligned} ES_{\alpha,1} &= -E\left[\mathbf{w}^{\mathrm{T}}\mathbf{r}|r_{p,1} \le C\right] \\ &= -E\left[\sum_{i=1}^{n} w_i r_i | r_{p,1} \le C\right] \end{aligned} \tag{2.19}$$

2. second-order:

$$\begin{aligned} ES_{\alpha,2} &= -E\left[\mathbf{w}^{\mathrm{T}}\mathbf{r} + \frac{1}{2}\mathbf{w}^{\mathrm{T}}(\mathrm{diag}(\Omega) - \Omega\mathbf{w})|r_{p,2} \le C\right] \\ &= -E\left[\sum_{i=1}^{n} w_i r_i + \frac{1}{2}\sum_{i=1}^{n} w_i \Omega_{ii} - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Omega_{ij} w_i w_j | r_{p,2} \le C\right] \end{aligned} \tag{2.20}$$

2.2.1 Sensitivity analysis

According to Acharya et al. (2010) and Brownlees and Engle (2010), the sensitivity of ES on a portfolio with respect to portfolio allocation is measured via Marginal Expected Shortfall (MES).

Based on the notation in the relation (2.10), MES is defined as the partial derivative of ES at the $100(1-\alpha)$ percent confidence level to w_k and is indicated as a conditional expectation (Yamai and Yoshiba, 2002a).

$$\begin{aligned} MES_{\alpha,1,k} = \frac{\partial ES_{\alpha,1}}{\partial w_k} &= \frac{\partial}{\partial w_k}\left(-E\left[\sum_{i=1}^{n} w_i r_i | r_{p,1} \le C\right]\right) \\ &= \frac{\partial}{\partial w_k}\left(-\sum_{i=1}^{n} w_i E[r_i | r_{p,1} \le C]\right) \\ &= -\sum_{i=1}^{n}\frac{\partial w_i}{\partial w_k} E[r_i | r_{p,1} \le C] \end{aligned} \tag{2.21}$$

The partial derivative form produces Kronecker delta denoted by δ_{ik}

$$\frac{\partial w_i}{\partial w_k} = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases} \equiv \delta_{ik} \tag{2.22}$$

and MES by means of a first-order approximation is

$$\begin{aligned} MES_{\alpha,1,k} &= -\sum_{i=1}^{n} \delta_{ik} E[r_i | r_{p,1} \leq C] \\ &= -E[r_k | r_{p,1} \leq C] \end{aligned} \tag{2.23}$$

MES implies how a particular asset risk reflects to the portfolio's overall risk. In other words, MES is the expectation of a particular asset loss when the portfolio itself is in its left tail (Acharya et al, 2010).

Since MES derives from the assumption in Equation (2.21), it is suggested to consider the proposed second-order approximation for the relation between the portfolio geometric return and the geometric returns of individual assets.

$$\begin{aligned} MES_{\alpha,2,k} = \frac{\partial ES_{\alpha,1}}{\partial w_k} &= \frac{\partial}{\partial w_k}\left(-E\left[\sum_{i=1}^{n} w_i r_i + \frac{1}{2}\sum_{i=1}^{n} w_i \Omega_{ii} - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \Omega_{ij} w_i w_j | r_{p,2} \leq C\right]\right) \\ &= \frac{\partial}{\partial w_k}\left(-\sum_{i=1}^{n} w_i E[r_i | r_{p,2} \leq C]\right) - \frac{\partial}{\partial w_k}\left(\frac{1}{2}\sum_{i=1}^{n} w_i E[\Omega_{ii} | r_{p,2} \leq C]\right) \\ &\quad + \frac{\partial}{\partial w_k}\left(\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j E[\Omega_{ij} | r_{p,2} \leq C]\right) \\ &= -\sum_{i=1}^{n} \frac{\partial w_i}{\partial w_k} E[r_i | r_{p,2} \leq C] - \frac{1}{2}\sum_{i=1}^{n} \frac{\partial w_i}{\partial w_k} E[\Omega_{ii} | r_{p,2} \leq C] \\ &\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial w_k}(w_i w_j) E[\Omega_{ij} | r_{p,2} \leq C] \end{aligned} \tag{2.24}$$

The partial derivative form in the third term of the above equation is obtained by the product rule formula which is used to find the derivatives of products of two or more functions.

$$\frac{\partial}{\partial w_k}(w_i w_j) = \left(\frac{\partial w_i}{\partial w_k} w_j + w_i \frac{\partial w_j}{\partial w_k}\right) = \delta_{ik} w_j + w_i \delta_{ik} \tag{2.25}$$

MES accordingly becomes

$$\begin{aligned}MES_{\alpha,2,k} =& -\sum_{i=1}^{n}\delta_{ik}E[r_i|r_{p,2}\leq C]-\frac{1}{2}\sum_{i=1}^{n}\delta_{ik}E[\Omega_{ii}|r_{p,2}\leq C]\\&+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\delta_{ik}w_j+w_i\delta_{ik})E[\Omega_{ij}|r_{p,2}\leq C]\\=&-E[r_k|r_{p,2}\leq C]-\frac{1}{2}E[\Omega_{kk}|r_{p,2}\leq C]\\&+\frac{1}{2}\left(\sum_{j=1}^{n}w_jE[\Omega_{kj}|r_{p,2}\leq C]+\sum_{i=1}^{n}w_iE[\Omega_{ik}|r_{p,2}\leq C]\right)\end{aligned} \quad \textbf{(2.26)}$$

Since the covariance matrix is symmetric, MES by means of a second-order approximation:

$$\begin{aligned}MES_{\alpha,2,k} &= -E[r_k|r_{p,2}\leq C]-\frac{1}{2}E[\Omega_{kk}|r_{p,2}\leq C]+\frac{1}{2}\left(2\sum_{i=1}^{n}w_iE[\Omega_{ik}|r_{p,2}\leq C]\right)\\&=-E\left[r_k+\frac{1}{2}\Omega_{kk}-\sum_{i=1}^{n}w_i\Omega_{ik}|r_{p,2}\leq C\right]\end{aligned} \quad \textbf{(2.27)}$$

Thus, three other factors impact on MES of the proposed correction:

- the asset risk conditionally to a threshold
- the portfolio weight on assets
- the covariance, and so the correlations between the specific asset and the other elements of the portfolio (Caporin and De Magistris, 2011).

3. NONPARAMETRIC ESTIMATION OF PORTFOLIO ES

ES is the average of the worst 100α% of losses (Dowd, 2005)

$$ES_\alpha = \frac{1}{\alpha}\int_0^\alpha VaR_p \mathrm{d}p \quad \textbf{(3.1)}$$

Whether the loss distribution is discrete,

$$ES_\alpha = \frac{1}{\alpha}\sum_{p=0}^{\alpha}\left(p^{th}\text{highest loss x probability of} p^{th}\text{highest loss}\right) \quad \textbf{(3.2)}$$

Historical simulation is a nonparametric way of estimating ES. It directly uses the past data as a guide to predict the future value of financial instruments without an assumption of a probability distribution. The main stages of a historical simulation in risk measurement are

Stage 1. Identifying the risk factors

Stage 2. Collecting data of each risk factor over a specified time interval

Stage 3. Calculating the portfolio return value within the interval.

This procedure provides alternative scenarios to the number of movements in the interval which are then ranked to assess the ES (Hull, 2003). The drawback of historical simulation is the excessive reliability on a given set of past data. The larger the data is, the more reliable but more retrospective it makes the analysis (Parasuraman, 2011).

Since risk is associated with the variation of the future value of a position because of market fluctuations, it is better to consider future values only in risk assessment (Artzner et al, 1999). Monte Carlo methods accordingly rely on the behavior of possible future events by enabling the use of the best available models of financial markets (Lan, 2010).

Estimation methods other than Monte Carlo simulation contain simplifications and approximations that cause doubt on the validity of the results (Lan, 2010).

Monte Carlo simulation approach is widely applied inspite of its computational cost with a relatively slow convergence rate (Singla et al, 2008).

A typical implementation of the Monte Carlo method involves simulating repeatedly from random processes to estimate the result (Dowd, 2005). The simulation procedure in risk measurement is

Stage 1. Selecting a model for the price of risk factors

Stage 2. Estimating the probability distribution and parameters

Stage 3. Constructing random paths for each risk factor

Stage 4. Calculating the portfolio return value at the end of the target time period

Stage 5. Repeating stages 3 and 4 enough times to be confident (Jorion, 2007; Dowd, 2005).

These stages generate a distribution of values which can be sorted to infer the ES (Jorion, 2007).

3.1 Portfolio Monte Carlo Simulation

Monte Carlo simulation procedure for a portfolio composed of n specific assets involves simulations of the portfolio value at the end of a specified time period. The differences between the current value and the simulated future value of a portfolio present estimates of the profit or loss over that given time horizon (Singla et al, 2008). The portfolio ES is then simply the appropriate average value of the sorted return estimates. Accordingly, an estimation of MES derives from the average values of the individual risk factors in that particular scenario by serving conditional expectation purpose. Here is a brief example to illustrate the procedure assuming a Monte Carlo simulation of 1000 return paths. For a confidence level 95%, the portfolio ES is the average value of the worst 50 scenarios and the average values of the individual positions in that 50 worst scenarios refer to estimates of MES, i.e. the partial derivatives of ES.

The Monte Carlo simulation stages for the estimation of the ES on a portfolio and the sensitivities of the relative risk factors are described in the following subsections in detail.

3.1.1 Model selection

A model that reflects the behavior of the risk factor prices is selected in primary. Portfolio risk measurement is achieved by assessing the value of a portfolio at the end of a predefined time period. The foremost models for pricing financial instruments over those time horizons are driven by stochastic processes (Singla et al, 2008; Dowd, 2005).

3.1.1.1 Stochastic processes

A variable with a changing value over time in an uncertain way is said to follow a stochastic process. Stochastic processes can be classified through time or variable. On time basis, the kinds of stochastic processes are discrete time where the value of the variable can change at certain fixed points in time, and continuous time where changes can occur at any time. Also on variable basis, there are discrete variable where the underlying variable can take only certain values, and continuous variable where any value in a certain range is possible (Hull, 2003).

A particular stochastic process is the Markov process where the future value of a variable relies only on the current value, the past is irrelevant. It is generally assumed that prices of financial instruments follow a Markov process. The Markov property thereby states that the probability distribution of the price at a future time is independent of its history (Jorion, 2007).

The following stochastic processes are derived from a Markov stochastic process with a higher complexity, respectively.

Wiener process: A Wiener process has the properties for a variable z:

- During a short time interval δt, the change δz is where ε is a random variable with a standardized normal (Gaussian) distribution (i.e., a normal distribution with a mean of zero and a variance of 1.0). δz accordingly has a normal distribution with a mean of zero and a standard deviation of $\sqrt{\delta t}$, or a variance of δt.
- The values of δz for any two separate short periods of time δt are independent enabling a Markov property (Hull, 2003).

Generalized Wiener process: A generalized Wiener process for a variable x which is built from a Wiener process is

$$\delta x = a\delta t + b\varepsilon\sqrt{\delta t} \quad \textbf{(3.3)}$$

where a and b are constants representing expected drift rate and standard deviation, respectively (Jorion, 2007).

Ito process: A generalized Wiener process leads to an Ito process with the parameters a and b defined as functions:

$$\delta x = a(x,t)\delta t + b(x,t)\varepsilon\sqrt{\delta t} \quad \textbf{(3.4)}$$

At last, a typical stochastic process for stock prices which is known as geometric Brownian motion is developed

$$\delta S = \mu S\delta t + \sigma S\varepsilon\sqrt{\delta t} \quad \textbf{(3.5)}$$

or

$$\frac{\delta S}{S} = \mu\delta t + \sigma\varepsilon\sqrt{\delta t} \quad \textbf{(3.6)}$$

where δS is the change in the stock price S in a short time interval δt, ε is a random number with a standardized normal distribution. The parameter μ is the expected rate of return per unit of time and σ is the standard deviation of the stock price. Since both of the parameters are assumed to be constant, $\frac{\delta S}{S}$ is normally distributed with mean $\mu\delta t$ and standard deviation $\sigma\sqrt{\delta t}$ which means

$$\frac{\delta S}{S} \sim \phi(\mu\delta t, \sigma\sqrt{\delta t}) \quad \textbf{(3.7)}$$

where $\phi(m,s)$ denotes a normal distribution with mean m and standard deviation s (Hull, 2003). Due to the drift and deviation terms are proportional to the current value of the price S, the process is called geometric.

The key features of geometric Brownian motion are: guiding for the Black-Scholes formula which leads option pricing over time, securing that the stock prices will get positive values.

Despite its convenient implementation for stock prices, the process has the shortcoming: assuming that the price changes have a normal distribution while they may have fatter tails than the normal distribution in practice (Jorion, 2007).

3.1.1.2 Itô's lemma and the log-normal property

A variable x following an Itô process is previously defined as

$$\delta x = a(x,t)\delta t + b(x,t)\varepsilon\sqrt{\delta t} \tag{3.8}$$

Itô's lemma implies that a function G of x, and t follows the process

$$\delta G = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)\delta t + \frac{\partial G}{\partial x}b\delta z \tag{3.9}$$

and thereby an Itô process where $\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2$ is a drift rate and $(\frac{\partial G}{\partial x})b^2$ is a variance rate. Through Itô's lemma, a geometric Brownian motion follows the process which is followed by a function G of S, and t.

$$\delta G = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)\delta t + \frac{\partial G}{\partial S}\sigma S\delta z \tag{3.10}$$

In order to imply log-normal property, the process followed by $\ln S$ is derived applying Itô's lemma.

$$\text{If } G = \ln S,\ \begin{array}{l} \frac{\partial G}{\partial S} = \frac{1}{S} \\ \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2} \\ \frac{\partial G}{\partial t} = 0 \end{array} \tag{3.11}$$

The process followed by G becomes

$$\delta G = \left(\mu - \frac{\sigma^2}{2}\right)\delta t + \sigma\delta z \tag{3.12}$$

so that

$$\ln S(t+\delta t) - \ln S(t) = \left(\mu - \frac{\sigma^2}{2}\right)\delta t + \sigma\varepsilon\sqrt{\delta t} \tag{3.13}$$

or equivalently

$$S(t+\delta t) = S(t)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)\delta t + \sigma\varepsilon\sqrt{\delta t}\right] \tag{3.14}$$

Because it generally provides more accurate results to simulate $\ln S$ rather than S, the equation is utilized to generate random paths in Monte Carlo simulation for the estimation of ES. From Equation (3.12), $G = \ln S$ follows a generalized Wiener process due to the constant variables μ and σ. Since it has a constant drift rate $(\mu - \frac{\sigma^2}{2})$ and a constant variance rate δ^2, the difference in $\ln S$ between time t and a future time $t+\delta t$

is normally distributed with mean $(\mu - \frac{\sigma^2}{2})\delta t$ and standard deviation $\sigma\sqrt{\delta t}$. In other words,

$$\ln S(t+\delta t) - \ln S(t) \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)\delta t, \sigma\sqrt{\delta t}\right] \tag{3.15}$$

or

$$\ln S(t+\delta t) \sim \phi\left[\ln S(t) + \left(\mu - \frac{\sigma^2}{2}\right)\delta t, \sigma\sqrt{\delta t}\right] \tag{3.16}$$

where S_T is the stock price at a future time T, S_0 is the stock price at time zero (Hull, 2003).

3.1.2 Probability distribution specification

The probability distribution and parameters of the predefined model are assessed at the second stage of the procedure. Since the risk factors are decided to follow a geometric Brownian motion, a normal disribution is the one that must be specified.

3.1.2.1 Normal distribution

Normal distribution has a leading role in finance because of sufficiently representing the behaviour of many financial variables, e.g. the daily rate of return in a stock price in geometric Brownian motion. The shape of a normal distribution is like a bell with a center more weighted and tails tapering off to zero. It can be modeled by two parameters, the mean μ expressing the location and the variance σ^2 the dispersion (Jorion, 2007).

A normal distribution for a random variable X with mean μ and variance σ^2 has the following probability density function (pdf)

$$f(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \tag{3.17}$$

where X takes the value x on the domain $x \in (-\infty, \infty)$. A normal distribution with a mean of zero and a variance (or standard deviation) of 1.0 is known as a standard normal distribution with the pdf (Dowd, 2005)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2} \tag{3.18}$$

In order to find the parameters μ and σ, maximum likelihood estimation (MLE) procedure is applied. If $f(\mathbf{x}|\theta)$ denotes the pdf specifying the probability of observing

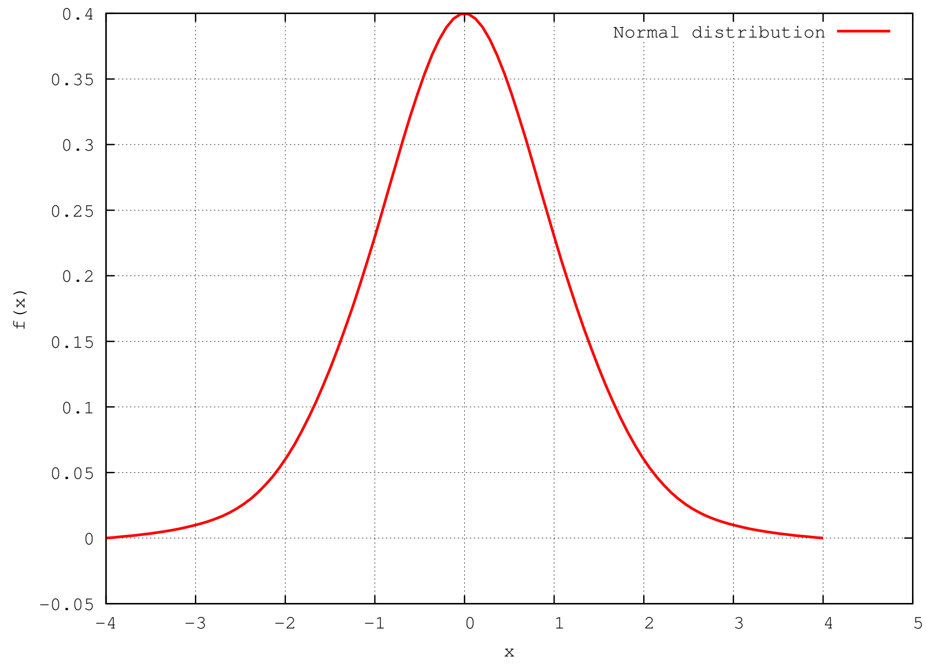

Figure 3.1: Normal distribution.

data vector **x** and the parameter vector θ, the likelihood function is defined by reversing the roles of the data vector **x** and the parameter vector θ.

$$L(\theta|\mathbf{x}) = f(\mathbf{x}|\theta) \tag{3.19}$$

Through the probability theory, the pdf for the independent and identically distributed data $x_1, \ldots, x_n$ given the parameter vector θ can be represented as a multiplication of pdfs for individual observations x_i

$$L(\theta|x_1, \ldots, x_n) = f(x_1, \ldots, x_n|\theta) = f(x_1|\theta) \ldots f(x_n|\theta) = \prod_{i=1}^{n} f(x_i|\theta) \tag{3.20}$$

The principle of MLE is based on searching for the value of the parameter vector θ that maximizes the likelihood function, $L(\theta|\mathbf{x})$. The probability distribution thereby makes the observed data "most likely".

Due to the two functions are monotonically related to each other, MLE is achieved by maximizing the logarithm of the likelihood (log-likelihood) function $\ln L(\theta|\mathbf{x})$ rather than $L(\theta|\mathbf{x})$ for computational convenience in practice.

$$\ln L(\theta|x_1, \ldots, x_n) = \sum_{i=1}^{n} \ln f(x_i|\theta) \tag{3.21}$$

Assuming that the log-likelihood function is differentiable,

$$\frac{\partial \ln L(\theta|\mathbf{x})}{\partial \theta_i} = 0 \tag{3.22}$$

is satisfied at the resulting parameter θ_i for all $i = 1, \ldots, k$ because first derivatives of a continuous differentiable function vanish at extremum points.

Since the first derivative is only adequate to determine $\ln L(\theta|\mathbf{x})$ is a maximum or minimum, an additional condition must be also satistifed. The shape of the log-likelihood function should be convex in the neighborhood of the resulting parameter vector θ to be a maximum. The second derivatives of the log-likelihoods with negative values ensure this convexity property (Myung, 2003).

$$\frac{\partial^2 \ln L(\theta|\mathbf{x})}{\partial \theta_i^2} \leq 0 \tag{3.23}$$

For a normal distribution, the log-likelihood function is

$$\begin{aligned} f(x_1, \ldots, x_n|\mu, \sigma) &= \ln \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2} \\ &= -\frac{1}{2} n \ln(2\pi) - n \ln \sigma - \frac{1}{2} \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right)^2 \end{aligned} \tag{3.24}$$

and accordingly the first derivatives with respect to the parameters μ and σ

$$\frac{\partial \ln f}{\partial \mu} = \frac{\sum_{i=1}^{n}(x_i - \mu)}{\sigma^2} = 0 \tag{3.25}$$

$$\frac{\partial \ln f}{\partial \sigma} = -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{\sigma^3} = 0 \tag{3.26}$$

generates respectively

$$\mu = \frac{\sum_{i=1}^{n} x_i}{n} \tag{3.27}$$

$$\begin{aligned} \sigma &= \sqrt{\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}}, \\ \sigma^2 &= \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n} \end{aligned} \tag{3.28}$$

while the second derivatives are all negative values.

$$\frac{\partial^2 \ln f}{\partial \mu} = -\frac{n}{\sigma^2} < 0 \tag{3.29}$$

$$\begin{aligned} \frac{\partial^2 \ln f}{\partial \sigma^2} &= \frac{n}{\sigma^2} - \frac{3\sum_{i=1}^{n}(x_i - \mu)^2}{\sigma^4} \\ &= \frac{n\sigma^2 - 3\sum_{i=1}^{n}(x_i - \mu)^2}{\sigma^4} = \frac{-2\sum_{i=1}^{n}(x_i - \mu)^2}{\sigma^4} < 0 \end{aligned} \tag{3.30}$$

3.1.3 Random number generation

At the foremost stage of Monte Carlo approach, normally distributed random paths for n separate risk factors are uniformly constructed in order to serve a Weiner process that is followed by such risk factors. When the underlying model distribution is stable, an increase in the number of random paths reduces the relative estimation error (Yamai and Yoshiba, 2002a).

From Itô's lemma, the process followed by $\ln S$ requires risk factors to be uncorrelated, i.e. correlated with the coefficient of zero. Since the correlation circumstance cannot be ensured within multiple risk factors in practice, a random path is designated from ε that considers the correlation between pairs of n risk factors.

The vector of n independent normally distributed random numbers ν is transformed to the one with correlated elements ε via Cholesky factorization of the relative covariance matrix.

3.1.3.1 Cholesky factorization

If Ω is a symmetric positive definite matrix in the vector space of all $n-by-n$ real matrices $\mathbb{R}^{nxn}$, then there is a unique lower triangular matrix $\mathbf{G} \in \mathbb{R}^{nxn}$ with positive diagonal elements, that ensures $\Omega = \mathbf{G}\mathbf{G}^{\mathrm{T}}$. A covariance matrix $\Omega \in \mathbb{R}^{nxn}$ is

- symmetric due to $\Omega = \Omega^{\mathrm{T}}$
- positive definite if $\mathbf{x}\Omega\mathbf{x}_t^{\mathrm{T}} > 0$ for all nonzero $\mathbf{x}$ in the vector space of real n-vectors, $\mathbb{R}^{nxn}$.

In particular, the factorization $\Omega = \mathbf{G}\mathbf{G}^{\mathrm{T}}$ is defined as the Cholesky factorization and $\mathbf{G}$ refers to the Cholesky triangle (Golub and van Loan, 1996).

The vector of correlated random numbers is generated by multiplying the Cholesky triangle by the vector of independent normally distributed random numbers.

$$\varepsilon = \mathbf{G}\nu$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix} = \begin{bmatrix} g_{11} & 0 & 0 & \dots & 0 \\ g_{21} & g_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & g_{n3} & \dots & g_{nn} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \end{bmatrix} \tag{3.31}$$

3.1.4 Return evaluation

The correlated normal random numbers are used to generate random paths according to geometric Brownian motion followed by $\ln S$ using Itô's lemma.

$$S(t+\delta t) = S(t)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)\delta t + \sigma\varepsilon\sqrt{\delta t}\right] \quad \textbf{(3.32)}$$

The parameters μ and σ which are obtained via MLE application to the pdf of a normal distribution in Section 3.1.2.1 are also utilized here.

$$\mu = \frac{\sum_{i=1}^{n} x_i}{n} \quad \textbf{(3.33)}$$

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}}$$

$$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n} \quad \textbf{(3.34)}$$

Therefore, the rate of return values at the end of each short time interval of individual risk factors are readily calculated in this stage. Proper aggregation according to the specified return evaluation, i.e. arithmetic or geometric, is then performed to obtain the rate of return values at the end of the target time period.

3.1.5 Inferring ES and sensitivity analysis

To assess the portfolio rate of return composed of n specific assets at time $t+1$, the rate of return values of separate risk factors are combined according to the equation of

1. first-order

$$r_{p,t+1,1} = \mathbf{w}_{t+1}^{\mathrm{T}}\mathbf{r}_{t+1} \quad \textbf{(3.35)}$$

2. second-order

$$r_{p,t+1,2} = \mathbf{w}_{t+1}^{\mathrm{T}}\mathbf{r}_{t+1} + \frac{1}{2}\mathbf{w}_{t+1}^{\mathrm{T}}(\mathrm{diag}(\Omega_{t+1}) - \Omega_{t+1}\mathbf{w}_{t+1}) \quad \textbf{(3.36)}$$

Such combinations are used to obtain profit or loss estimates for the overall portfolio at time t. When the return estimates are sorted, the ES of the portfolio at the $100(1-\alpha)$ percent confidence level is the average value of the worst 100α percent cases of the combinations.

4. PRACTICE IN XU100

An identical practice over XU100 index is tackled to illustrate the impact of the approximations. XU100 is designed by İstanbul Stock Exchange (ISE) as the basic index for ISE stock market. It is the successor of the Composite Index which was introduced in 1986 including the stocks of 40 companies and was in time limited to the stocks of 100 companies. It consists of one hundred stocks which are selected among the stocks of companies listed on National Market and the stocks of real estate investment trusts and venture capital investment trusts listed on Corporate Products Market. The combination of such one hundred stocks is thereby changeable as the sorted list is changeable.

ISE stock indices are calculated both in terms of price and return. The only difference between the price index and return index is related to the cash dividend payments. In cash dividend payments, the divisor of the return index is adjusted assuming that the dividend paid is invested in the stocks included in the index in proportion to the weight of the stocks, whilst the divisor of the price index is not adjusted assuming that the dividend paid is excluded from the portfolio (Stock indices, 2012). Since it is desired to reflect only the changes in price, XU100 price index is taken into account.

In order to highlight the impact of the financial crisis of 2007 – 2009, the time interval of 01.07.2008 and 03.07.2009 is considered for the identical practice. It consists of 251 daily observations starting from 01.07.2008 and ending with 03.07.2009.

Because the stock price values of 20 companies from the current composition of XU100 index is not available for the desired time period, a new index over 80 stocks which are already in XU100 index is defined: AEFES, AFYON, AKBNK, AKENR, AKGRT, AKSA, ALARK, ANSGR, ARCLK, ASELS, AYGAZ, BAGFS, BANVT, BIMAS, BJKAS, BOYNR, BRISA, BRSAN, DEVA, DOAS, DOHOL, DYHOL, ECILC, ECZYT, EGGUB, ENKAI, EREGL, FENER, FROTO, GARAN, GLYHO, GOLDS, GOODY, GSDHO, GSRAY, GUBRF, HURGZ, IHEVA, IHLAS,

IPEKE, ISCTR, ISFIN, ISGYO, ISYHO, IZMDC, KARSN, KARTN, KCHOL, KONYA, KOZAA, KRDMD, METRO, MGROS, MNDRS, MUTLU, NETAS, NTHOL, NTTUR, OTKAR, PETKM, PRKME, RHEAG, SAHOL, SASA, SISE, SKBNK, TCELL, TEKST, TEKTU, THYAO, TIRE, TOASO, TRCAS, TRKCM, TSKB, TSPOR, TTRAK, TUPRS, VESTL, YKBNK.

To illustrate how converges the first-order and second-order approximations, a portfolio of the predefined stocks each of which has only 1 share is provided at 01.07.2008. The portfolio is kept until 03.07.2009.

The relation discussed in Section 2.2 is tackled in the Figure 4.1 by showing the weighted combination of the single equities geometric returns is not exactly equal to the portfolio geometric return.

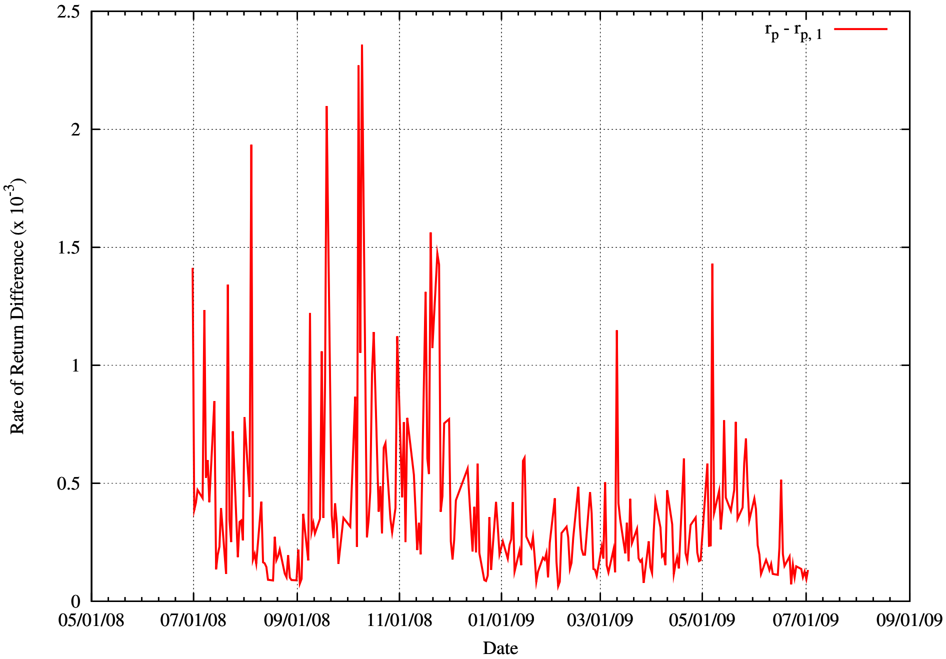

Figure 4.1: Portfolio rate of return differences.

Figure 4.2 plots the portfolio geometric return and approximated geometric returns in the range 01.07.2008 and 03.07.2009. It is noticed that the second-order approximation method converges more than the first-order one. This is because of the sum of terms other than aggregating single equities returns in Equation (2.11) always has a positive value.

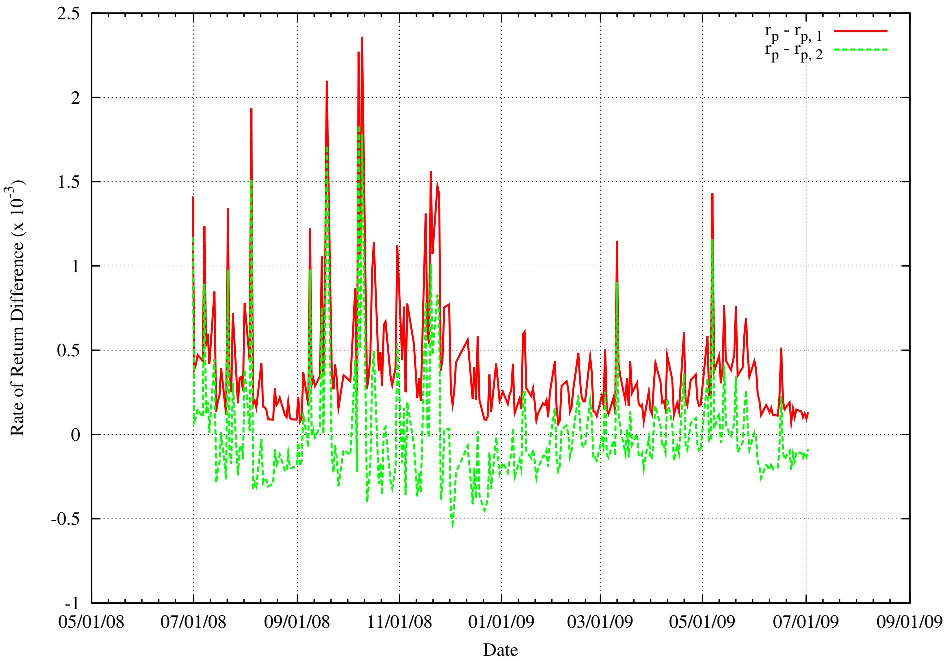

Figure 4.2: Portfolio rate of return differences (including proposed correction).

Geometric return evaluation on a portfolio is implemented using C programming language. The pseudocode is given as follows.

```
READ prices of equities
FOR each day in time period
    FOR each equity in portfolio
        CALCULATE geometric returns
CALCULATE portfolio value
DETERMINE covariance matrix at the beginning of time period
FOR each day in time period
    DETERMINE covariance matrix via EWMA model
FOR each day in time period
    CALCULATE portfolio logaritmic return
    CALCULATE first-order approximated
              portfolio geometric return
    CALCULATE second-order approximated
              portfolio geometric return
```

4.1 ES Estimation

A new index over fixed 80 stocks is built utilizing the market values of underlying equities. Market value is calculated by multiplying total number of assets that represent the capital by the asset price.

$$MV_{i,t} = S_{i,t} s_{i,t} \tag{4.1}$$

where $S_{i,t}$ is the asset price, and $s_{i,t}$ is the total number of assets of asset i at time t.

The index value is simply constructed as

$$I_t = I_{t-1} \frac{\sum_{i=1}^{n} MV_{i,t}}{\sum_{i=1}^{n} MV_{i,t-1}} \quad \textbf{(4.2)}$$

where I_t is the index value at time t.

Table 4.1 presents Monte Carlo based daily ES estimates at 03.07.2009 by means of the first-order and second-order approximations for confidence levels 95% and 99%. For the same confidence level, the results obtained via second-order approximation are less than the ones via first-order approximation as predicted. Accordigly the MES estimates at the confidence level 95% are listed in Table 4.2. In general, MES estimates obtained via second-order approximation are lower loss values.

Table 4.1: Index ES estimates by means of approximations.

	First-order approximation	Second-order approximation
$ES_{95\%}$	3.2774%	3.2471%
$ES_{99\%}$	3.7574%	3.7271%

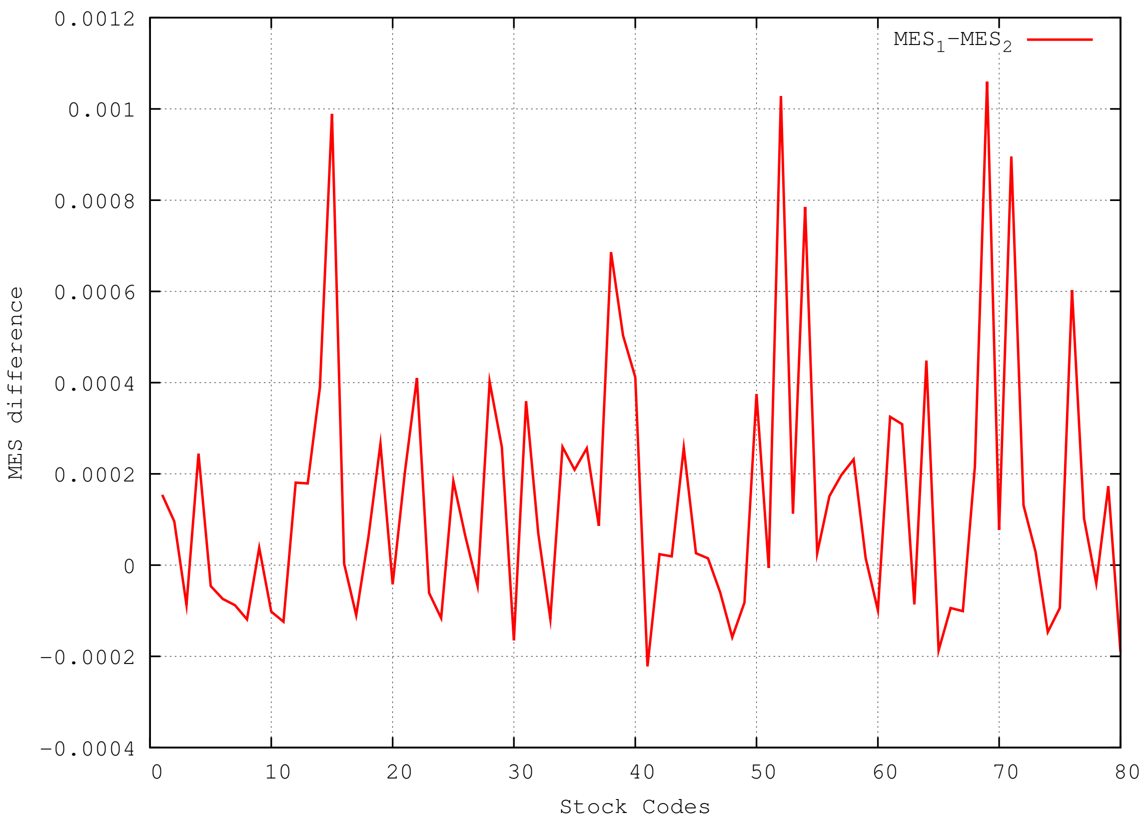

Figure 4.3: MES differences.

Table 4.2: MES estimates at the confidence level 95% by means of approximations.

	First-order approximation (%)	Second-order approximation (%)		First-order approximation (%)	Second-order approximation (%)
AEFES	1.9075	1.8921	ISCTR	3.2666	3.2888
AFYON	1.9256	1.9160	ISFIN	2.8050	2.8026
AKBNK	4.4488	4.4575	ISGYO	2.4644	2.4625
AKENR	3.1248	3.1004	ISYHO	3.7357	3.7100
AKGRT	3.9729	3.9775	IZMDC	2.6263	2.6237
AKSA	1.3823	1.3897	KARSN	3.0625	3.0610
ALARK	1.1947	1.2035	KARTN	0.7628	0.7688
ANSGR	2.2134	2.2253	KCHOL	3.3292	3.3450
ARCLK	3.2912	3.2874	KONYA	0.9936	1.0018
ASELS	1.8589	1.8691	KOZAA	4.9772	4.9397
AYGAZ	1.5910	1.6034	KRDMD	3.4578	3.4584
BAGFS	4.8888	4.8707	METRO	5.0978	4.9950
BANVT	2.9593	2.9414	MGROS	1.3476	1.3363
BIMAS	4.0808	4.0418	MNDRS	6.6329	6.5544
BJKAS	5.6613	5.5624	MUTLU	3.0679	3.0653
BOYNR	3.1196	3.1193	NETAS	2.4605	2.4454
BRISA	1.1353	1.1463	NTHOL	2.8532	2.8334
BRSAN	3.1161	3.1100	NTTUR	2.6954	2.6722
DEVA	3.4603	3.4337	OTKAR	2.4950	2.4935
DOAS	2.9562	2.9604	PETKM	2.0292	2.0391
DOHOL	3.6644	3.6441	PRKME	3.9559	3.9234
DYHOL	4.8580	4.8170	RHEAG	2.7774	2.7465
ECILC	1.5878	1.5939	SAHOL	3.6676	3.6762
ECZYT	1.0290	1.0406	SASA	3.4092	3.3644
EGGUB	4.5019	4.4835	SISE	2.3649	2.3836
ENKAI	3.8964	3.8901	SKBNK	3.9188	3.9282
EREGL	3.7826	3.7872	TCELL	1.8739	1.8840
FENER	2.5168	2.4766	TEKST	5.6271	5.6055
FROTO	3.6611	3.6352	TEKTU	3.2986	3.1926
GARAN	4.0641	4.0806	THYAO	2.4710	2.4633
GLYHO	4.5944	4.5585	TIRE	4.1361	4.0465
GOLDS	3.8230	3.8161	TOASO	4.3955	4.3824
GOODY	1.5836	1.5954	TRCAS	2.8791	2.8762
GSDHO	5.9319	5.9060	TRKCM	1.9594	1.9741
GSRAY	1.4259	1.4050	TSKB	2.3611	2.3705
GUBRF	5.0797	5.0541	TSPOR	3.1301	3.0698
HURGZ	3.9387	3.9301	TTRAK	2.7988	2.7886
IHEVA	6.4144	6.3458	TUPRS	2.7194	2.7234
IHLAS	5.1726	5.1223	VESTL	3.2549	3.2376
IPEKE	4.5460	4.5048	YKBNK	3.5739	3.5930

4.1.1 Serial computing

The serial pseudocode for the implementation of both ES and MES estimation procedure simultaneously:

```
READ prices of equities
READ number of equities
CALCULATE market value
FOR each equity in index
    CALCULATE index weight
DETERMINE covariance matrix at the end of time period
SET Cholesky decomposition to covariance matrix
DETERMINE covariance matrix via EWMA model at the end of
          specified time period
SET random number generation
SET correlated random number generation
CALCULATE geometric return via geometric Brownian motion
CALCULATE simulated market ES
SET simulated market ES sorting
CALCULATE market ES by means of a first-order approximation
FOR each equity in index
    CALCULATE relative MES
CALCULATE market ES by means of a second-order approximation
FOR each equity in index
    CALCULATE relative MES
```

Particularly, C language is applied for programming. Getting price and number values of equities into the computation is done via `fread` command due to read at once. The original files therefore converted to binary versions before serial computing starts. Secondly, random number generation is achieved utilizing Intel® Math Kernel Library (MKL). Mersenne Twister pseudorandom number generator (MT19937) has the period length of $2^{19937} - 1$ and is 623-dimensionally equidistributed with up to 32-bit accuracy that attract attention to the generator for simulations in various fields of science and engineering (Statistical functions, 2012). Last, in-place version of quicksort algorithm which is more complex, but more efficient in terms of space reqirement than the standard version is served to the purpose of sorting. Intel compiler is the one that is applied to compile the serial program.

4.1.2 Parallel computing

Emerging scientific and engineering applications steadily require greater computational speed from a computer system than is available. Furthermore, such applications are often repeated on large amounts of data to achieve valid results. A natural way to increase the related computational performance is to use multiple processors to solve a single problem. The overall problem is broken into a number of subproblems, each of which is solved simultaneously on a different processor (Kumar et al, 1994; Wilkinson and Allen, 1999).

Parallel programming is described as writing programs for this way of computation. It is ideally expected that the problem would be completed in $(1/\text{number of processors})$th of the time spent by a single processor. However, this is rarely performed in practice because of non-perfect division of the problem into independent parts and interconnection requirement of the parts for data transfer and synchronization (Wilkinson and Allen, 1999).

A parallel computer is a specially designed computing platform containing multiple processors or several independent computers interconnected in some way. The three types of parallel computers are:

Shared memory: Multiple processors are connected to multiple memory modules where each processor can access any of. The connection between the processors and memory is provided via an network interconnect. Each location in the whole main memory has a unique address which is known as a single address space. Each processor employs such a space to access the location

Distributed memory: Each processor has a local memory that is not accessable by other processors. A processor only has access to a location in its own memory. An network interconnect is provided for communication between processors.

Distributed shared memory: Each processor can access the whole memory using a single memory address space. A processor must communicate in order to access a location which doesn't exist in its local memory (Wilkinson and Allen, 1999).

4.1.2.1 Embarrassingly parallel computing

An embarrassingly parallel computation is considered ideal from a parallel computing standpoint. The computation is divided into a number of completely independent parts which can be executed simultaneously. In the case of truly embarrassingly parallelism, there will be no communication between separate processors. Each processor demand data and supply results without any need from other processors. According to Wilkinson and Allen (1999), a nearly embarrassingly parallel computation is the one that require data to be scattered, and results to be gathered in some way.

Monte Carlo methods are based on utilization of random selections in calculations which lead to the solution to numerical and physical problems. Due to the fact that each calculation is independent of the others, Monte Carlo methods are represented as a clean example of an embarrassingly parallel computation.

The embarrassingly parallel computations apply partitioning even though the results of the parts need to be combined to obtain the desired result in most partitioning formulations. Partitioning can be performed into the program by data or functions. Data partitioning or domain decomposition is based on dividing the data and performing upon the divided data concurrently. On the other hand, functional decomposition achieves dividing the program into independent functions and executing them simultaneously (Wilkinson and Allen, 1999).

Distributed memory system is the one that is utilized to fit the nearly embarrassingly parallel computation. C programming language based on Message Passing Interface (MPI) is applied to parallelize the serial program, also compiled with Intel MPI compiler. The pseudocode of Monte Carlo simulation procedure for parallel computing is:

```
PARALLEL
    READ prices of equities
    READ number of equities
    CALCULATE market value
    FOR each equity in index
        CALCULATE index weight
MASTER
    DETERMINE covariance matrix at the end of time period
    SET Cholesky decomposition to covariance matrix
    DETERMINE covariance matrix via EWMA model at the end of
              specified time period
PARALLEL
    SET random number generation
    SET correlated random number generation
    CALCULATE geometric return via geometric Brownian motion
    CALCULATE simulated market ES
MASTER
    SET simulated market ES sorting
```

```
    CALCULATE market ES by means of a first-order approximation
    CALCULATE market ES by means of a second-order approximation
PARALLEL
    FOR each equity in index
        CALCULATE MES in sorting
MASTER
    FOR each equity in index
        CALCULATE MES by means of a first-order approximation
        CALCULATE MES by means of a second-order approximation
```

Data partitioning is hereby implemented considering block decomposition model. In particular, that gives fall to the relative computational complexity which is classified as time and space. Time complexity of the serial program that implies time requirement in big O notation as a function of program input is $O(\text{number of equities}x\text{number of simulations})$. Space complexity that refers to memory requirement is also $O(\text{number of equities}x\text{number of simulations})$. Random number generation on block partitions is locally set up at each processor in the light of such complexities. Therefore, both complexities become $O(\frac{\text{number of equities}x\text{number of simulations}}{\text{number of processors}})$. Since sending and receiving tasks are done with the entire group of processors, collective communication routines are used in order to reduce time complexity. In addition, parallel file read operation is performed to a single file that consists of binary versions of equities price and number values. The advantage of doing parallel input/output is that it is straightforward to read the file in parallel with a different number of processors. Because of embarrassingly parallelizing property, MT19937 is based on different *seed*s on separate processors.

4.1.3 Performance and scalability

Generally, a serial program is evaluated in terms of execution time which is expressed as a function of input size. The execution time of a parallel program also depends on the architecture of the parallel computer and the number of processors. Therefore, a parallel program cannot be evaluated isolating from a parallel achitecture (Kumar et al, 2003). The architecture details where the applications run on are in Table 4.3 (Resources, 2012).

The computing resources are provided by National Center for High Performance Computing of Turkey (UHeM). In order to test the parallel program, wall clock time

Table 4.3: Computing server system technical specifications.

System Name	ANADOLU (HP ProLiant DL360 G5)
Processor	Intel Xeon 2.33 GHz (5140 dual-core, E5345 quad-core)
Number of Compute Nodes	192
Number of Compute Cores	1004
Memory Architecture	Distributed
Compute Node Memory Amount	8 GB (dual-core servers), 16 GB (quad-core servers)
Compute Node Disk Amount	2 x 60 GB RAID1 + 60 GB
High Performance Network	InfiniBand 20 Gbps
Operating System	RHEL 5.1 x86_64

in seconds is used as the performance index. Speedup factor and efficiency are also analysed related to the test results.

The time elapsed from the beginning to the end of execution of a program on a sequential computer gives the serial wall clock time of that program. On the other side, the parallel wall clock time is the time elapsed between the moment that a parallel computation starts and the moment the last processor finishes its execution (Kumar et al, 2003).

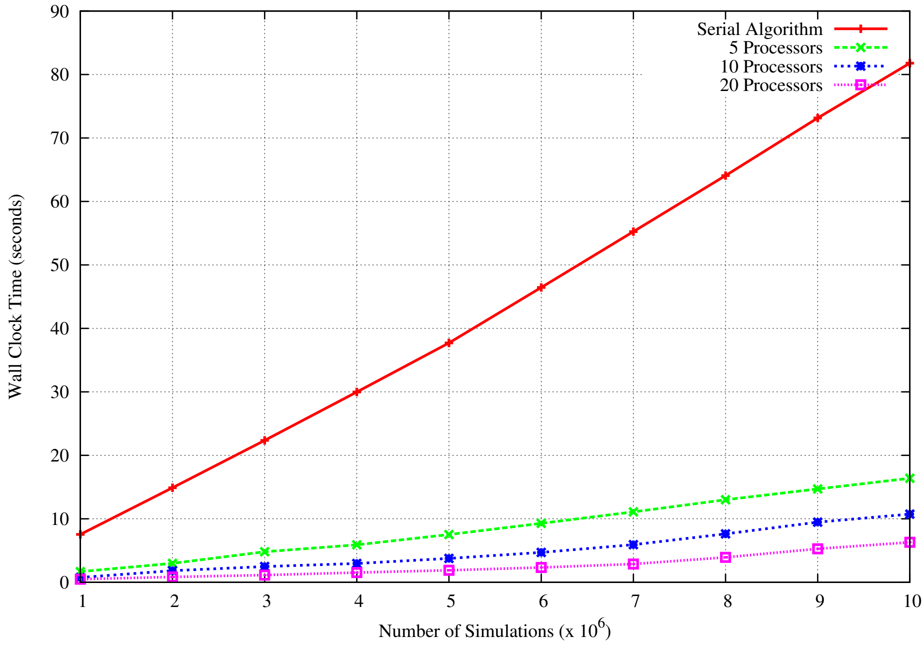

Figure 4.4: Benchmark run time results based on the number of processors.

As seen in Figure 4.4, the parallel algorithms reduce the wall clock time approximately to the execution time of the serial algorithm proportioned by the number of processors. The foremost reason is that multiple processors are included by dividing the whole process into separate tasks rather than implementing on a single processor. Furthermore, the computation time is declined due to the decomposed task implementation concurrently. By increasing the random number size of each risk factor for the fixed number of pocessors, wall clock time index is linearly rising. In addition, doubling number of processors almost takes halfway down the parallel computing time.

For the highest number of simulations, it can be mentioned that the wall clock time is decreased in descending order due to smaller sized tasks of processors at each step of incrementing number of processors, and on the other hand greater communication cost. Speedup factor is defined as the ratio of the serial run time of the best sequential

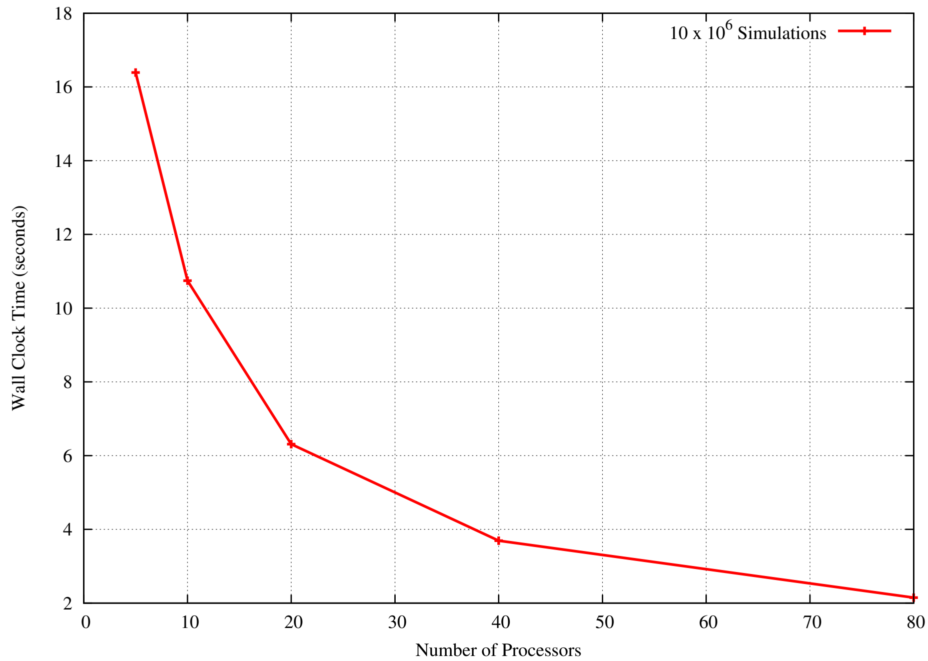

Figure 4.5: Run time results for $10x10^6$ simulations.

algorithm for solving a problem to the time taken by the parallel algorithm to solve the same problem on p identical processors (Kumar et al, 2003).

$$Speedup(p) = \frac{\text{Run time using one processor (best sequential algorithm)}}{\text{Execution time using a multiprocessor with p processors}} = \frac{t_s}{t_p} \quad \textbf{(4.3)}$$

Speedup factor should take place between 0 and p. The lower bound 0 occurs when a parallel program never terminates. However, a speedup factor greater than p can be obtained due to a specific reason such as a parallel program does less work than the corresponding serial version (Wilkinson and Allen, 1999).

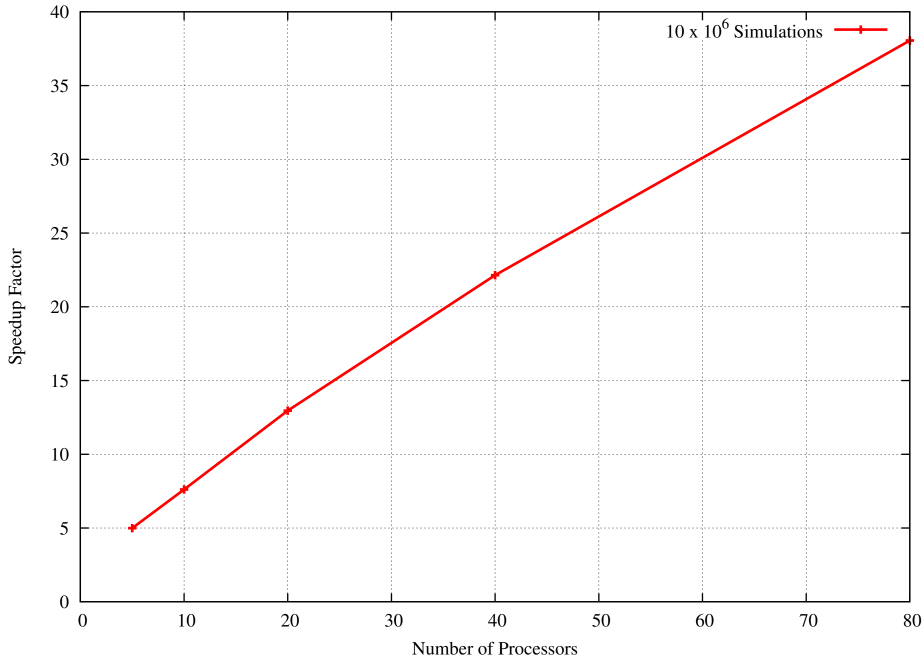

Figure 4.6: Speedup factor results.

Figure 4.6 almost presents linear speedup that is reducing to p number of processors proportioned by 2 for random number size $10x10^6$ at each stock in the market index. Equally divided tasks for parallelism primarily cause to get a maximum speedup. In addition, cost of communication and cost of contention for resources are reasons of not observing perfect speedup which leads to scaled one.

Efficiency is a measure of the fraction of time for which a processor is usefully employed. It is the ratio of speedup to the number of processors.

$$\text{Efficiency}(p) = \frac{\text{Speedup}(p)}{p} \qquad \textbf{(4.4)}$$

For perfect speedup, efficiency is equal to 1. Practically, due to the fact that speedup factor p is rarely obtained, efficiency is between 0 and 1 (Kumar et al, 2003).

The speedup benchmark results lead that efficiency drops by increasing the number of processors for the fixed size problem as in Figure 4.5.

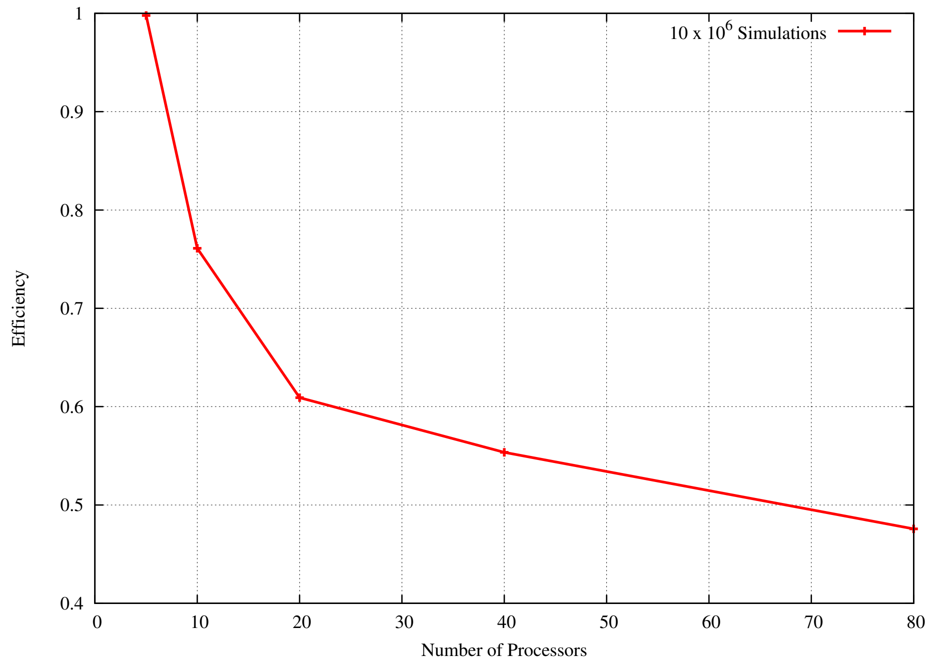

Figure 4.7: Efficiency results.

5. CONCLUSION

The financial crisis of 2007-2009 has highlighted two broad risk management strategies open to any financial institution. One approach is to identify risk factors and tackle each one separately, which sometimes refers to decomposition. The other is to reduce risks via diversification. This study clearly attract attention to define individual risk factors by proposing a correction. It is firstly shown over an XU100 portfolio that the second-order approximation converges to the portfolio geometric return more than the weighted combination of single risk factors geometric returns. In such a case, applying the proposed approximation in market ES assessment produces a lower loss value which relates to a less conservative result. Sensitivity analysis are then implemented via first derivatives of market ES with respect to market allocation. The sensitivities to individual risk factors generally present lower loss values than the ones estimated by means of a first-order approximation.

In addition, Monte Carlo simulation procedure is the one that is utilized for market ES estimation. Since Monte Carlo methods consider the behavior of possible future events, it is intended to minimize doubt on the validity of the results which is caused by simplifications and approximations of other estimation methods. Space and time complexity of the procedure is reduced by applying parallel computing techniques. It is demonsrated via several performance criteria that acceleration is provided with processors up to the number of risk factors.

In order to test how the second-order approximation converges to the rate of return on a portfolio, it is recommended to consider richer covariance matrix estimation methods, e.g. GARCH(1, 1), for further research. Because it is highly accurate in short time intervals, the approximation can be illustrated making use of realized variance and covariance estimates.

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APPENDICES

APPENDIX A: Stock code list

APPENDIX A

Table A.1: Stock code list

Stock code	Stock name	Stock code	Stock name
AEFES	ANADOLU EFES	ISCTR	İŞ BANKASI (C)
AFYON	AFYON ÇİMENTO	ISFIN	İŞ FİN. KİR.
AKBNK	AKBANK	ISGYO	İŞ GMYO
AKENR	AK ENERJİ	ISYHO	IŞIKLAR YAT. HOLDİNG
AKGRT	AKSİGORTA	IZMDC	İZMİR DEMİR ÇELİK
AKSA	AKSA	KARSN	KARSAN OTOMOTİV
ALARK	ALARKO HOLDİNG	KARTN	KARTONSAN
ANSGR	ANADOLU SİGORTA	KCHOL	KOÇ HOLDİNG
ARCLK	ARÇELİK	KONYA	KONYA ÇİMENTO
ASELS	ASELSAN	KOZAA	KOZA MADENCİLİK
AYGAZ	AYGAZ	KRDMD	KARDEMİR (D)
BAGFS	BAGFAŞ	METRO	METRO HOLDİNG
BANVT	BANVİT	MGROS	MİGROS TİCARET
BIMAS	BİM MAĞAZALAR	MNDRS	MENDERES TEKSTİL
BJKAS	BEŞİKTAŞ FUTBOL YAT.	MUTLU	MUTLU AKÜ
BOYNR	BOYNER MAĞAZACILIK	NETAS	NETAŞ TELEKOM.
BRISA	BRİSA	NTHOL	NET HOLDİNG
BRSAN	BORUSAN MANNESMANN	NTTUR	NET TURİZM
DEVA	DEVA HOLDİNG	OTKAR	OTOKAR
DOAS	DOĞUŞ OTOMOTİV	PETKM	PETKİM
DOHOL	DOĞAN HOLDİNG	PRKME	PARK ELEK. MADENCİLİK
DYHOL	DOĞAN YAYIN HOL.	RHEAG	RHEA GİRİŞİM
ECILC	ECZACIBAŞI İLAÇ	SAHOL	SABANCI HOLDİNG
ECZYT	ECZACIBAŞI YATIRIM	SASA	SASA POLYESTER
EGGUB	EGE GÜBRE	SISE	ŞİŞE CAM
ENKAI	ENKA İNŞAAT	SKBNK	ŞEKERBANK
EREGL	EREĞLİ DEMİR CELİK	TCELL	TURKCELL
FENER	FENERBAHÇE SPORTİF	TEKST	TEKSTİLBANK
FROTO	FORD OTOSAN	TEKTU	TEK-ART TURİZM
GARAN	GARANTİ BANKASI	THYAO	TÜRK HAVA YOLLARI
GLYHO	GLOBAL YAT. HOLDİNG	TIRE	MONDİ TİRE KUTSAN
GOLDS	GOLDAS KUYUMCULUK	TOASO	TOFAŞ OTO. FAB.
GOODY	GOOD-YEAR	TRCAS	TURCAS PETROL
GSDHO	GSD HOLDİNG	TRKCM	TRAKYA CAM
GSRAY	GALATASARAY SPORTİF	TSKB	T.S.K.B.
GUBRF	GÜBRE FABRİK.	TSPOR	TRABZONSPOR SPORTİF
HURGZ	HÜRRİYET GZT.	TTRAK	TÜRK TRAKTÖR
IHEVA	İHLAS EV ALETLERİ	TUPRS	TÜPRAŞ
IHLAS	İHLAS HOLDİNG	VESTL	VESTEL
IPEKE	İPEK DOĞAL ENERJİ	YKBNK	YAPI VE KREDİ BANK.

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