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GENELLEŞTİRİLMİŞ ÇOK EKSENSEL SİMETRİK POTANSİYEL KURAMININ HELMHOLTZ DENKLEMLERİ İÇİN DIRICHLET PROBLEMİ VE BİR ELEMANTER ÇÖZÜM.

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DOKTORA TEZİ

174000

Abdulkadir Ertaş

Dicle Üniversitesi Fen Edebiyat Fakültesi

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Ö N S Ö Z

Doktora tezi olarak düşünülen bu çalışma iki bölümden oluşur. Birinci Bölümde genelleştirilmiş çok eksensel simetrik potansiyel kuramının Helmholtz tipinde bir denklemi için bir Dirichlet problemi çözüldü. İkinci Bölümde ise aynı tipden bir başka denklem için bir elemanter çözüm verildi.

Bu çalışmayı yöneten sayın hocam Doç.Dr.İhsan Dağ'a yardımları için teşekkür ederim.

Abdulkadir ERTAŞ

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GENELLEŞTİRİLMİŞ ÇOK EKSENSEL SİMETRİK POTANSİYEL KURAMININ HELMHOLTZ DENKLEMLERİ İÇİN DİRICHLET PROBLEMİ VE BİRELEMANTER ÇÖZÜM

0.1 GİRİŞ

Eksensel simetrik potansiyel kuramı ilk kez Weinstein tarafından ortaya konulmuş ve daha sonra geliştirilmiştir. n boyutlu uzayda

$$\frac{\partial^2 u}{\partial t_1^2} + \cdots + \frac{\partial^2 u}{\partial t_n^2} = 0$$

Laplace denklemi

$$x = t_1 \quad , \quad y = (t_2^2 + \cdots + t_n^2)^{1/2}$$

dönüşümleri altında, 2 boyutlu uzayın

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{k}{y}\frac{\partial u}{\partial y} = 0 \qquad (0.1\text{-}1)$$

denklemine indirgenir. Burada $k = n-2$ dir. k pozitif tamsayı ise (0.1-1) denkleminin çözümlerine k + 2 boyutlu uzayda eksensel potansiyeller denir [1]. Eksensel simetrik, potansiyeller,

$$\frac{\partial \phi}{\partial x} - y^{-k}\frac{\partial \psi}{\partial y} = 0 \quad , \quad \frac{\partial \psi}{\partial x} + y^{k}\frac{\partial \phi}{\partial y} = 0$$

genelleştirilmiş Stoks-Beltrami denklemleri incelenirken ortaya çıkar. Çünkü ϕ ve ψ fonksiyonları sırasıyla k ve -k için (0.1-1) denkleminin birer çözümü olur. Akışkanlar mekaniğinde ϕ fonksiyonuna hız potansiyeli ψ fonksiyonuna da akım fonksiyonu denir. Özel olarak bu fonksiyonlar, $k = o$ için iki boyutlu dönmesiz akıyı ve $k = 1$ için eksensel simetrik dönmesiz akıyı gösterir. Burada simetri ekseni ox-eksenidir [2]. k nın ger-

çel değerleri için (0.1-1) denkleminin çözümlerine genelleştirilmiş eksensel simetrik potansiyeller denir [1,2,3]. Daha sonra bu kuram geliştirilerek genelleştirilmiş iki eksensel simetrik potansiyeller tanımlandı [4]. Buna göre,

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \frac{\alpha}{x_{n-1}} \frac{\partial u}{\partial x_{n-1}} + \frac{\beta}{x_n} \frac{\partial u}{\partial x_n} = 0$$

tipindeki denklemlerin çözümlerine iki eksensel simetrik potansiyeller denir. Gerek yukarıda sözü edilen denklemler ve gerekse onların değişik türlerinin çözümleri birçok araştırmacı tarafından incelenmiştir [5,6, 7,8,9,10]. Biz bu çalışmada

$$(0.1\text{-}2) \qquad \sum_{i=2}^{m+n} \frac{\partial^2 u}{\partial x_i^2} + \frac{\alpha_{m+1}}{x_{m+1}} \frac{\partial u}{\partial x_{m+1}} + \ldots + \frac{\alpha_{m+n}}{x_{m+n}} \frac{\partial u}{\partial x_{m+n}} - \lambda^2 u = 0$$

denklemini gözönüne alarak Dirichlet tipinde bir sınır değer problemi çözmeyi düşünüyoruz. Bu amaçla

$$(0.1\text{-}3) \qquad L_{\alpha_{m+1},\ldots,\alpha_{m+n}}[v] = \sum_{i=1}^{m+n} \frac{\partial^2 v}{\partial x_i^2} + \frac{\alpha_{m+1}}{x_{m+1}} \frac{\partial v}{\partial x_{m+1}} + \ldots + \frac{\alpha_{m+n}}{x_{m+n}} \frac{\partial v}{\partial x_{m+n}} = 0$$

denklemi için bir Poisson formülü oluşturacağız, sonra eksiltme (descent) yöntemini kullanarak, (0.1-3) denkleminin verilen sınır değerlerini sağlayan bir çözümünden (0.1-2) denkleminin çözümü elde edilecektir.

Tezin ikinci bölümünde

$$(0.1\text{-}4) \qquad \sum_{i=1}^{n} \frac{\partial^2 w}{\partial x_i^2} + \frac{\alpha_i}{x_i} \frac{\partial w}{\partial x_i} + \lambda w = 0$$

denkleminin dik koordinatlardaki bir elemanter çözümü araştırılacaktır.

Burada değişkenlerin ayrılması yöntemi ve bazı dönüşüm formülleri kullanılarak çözüm Bessel fonksiyonları cinsinden ifade edilecektir [5]. (0.1-2) ve (0.1-4) denklemlerinin çözümlerine genelleştirilmiş çok eksensel simetrik Helmholtz potansiyelleri diyeceğiz.

BİRİNCİ BÖLÜM

I.I GENELLEŞTİRİLMİŞ POISSON İNTEGRALİ

Bu bölümde $\alpha_{m+1} \geq 1, \ldots, \alpha_{m+n} \geq 1$ hali için m+n boyutlu Euclid uzayında bir çeyrek küre bölgesinde (0.1-3) denklemine ait genelleştirilmiş Poisson integral formülü çıkaracağız. $p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}$ boyutlu uzayda Laplace denklemi için bilinen Poisson integral formülünü alarak eksensel simetriyi uygulayacağız. Burada $\alpha_{m+1}, \ldots, \alpha_{m+n}$ hepsi 1 den büyük pozitif tamsayılardır. Basit bir gösterim değişikliği ile p boyotlu Laplace denklemi için Poisson integral formülü

$$(1.1\text{-}1) \qquad v(x)= \frac{\rho^2-r^2}{\rho\left[2\pi^{p/2}/\Gamma\left(\frac{p}{2}\right)\right]} \int_{S(p)} f(\xi_1,\ldots,\xi_p)|\xi-x|^{-p} dS$$

biçiminde yazılır ([14]265. Sayfa). Burada $\rho=|\xi|, r=|x|$, $S(\rho)$ ρ yarıçaplı ve p boyutlu küre yüzeyi olup, ξ yüzey üzerinde bir noktadır.

Farzedelim ki f fonksiyonu eksensel simetriye sahip olup

$$\xi_1,\ldots,\xi_{p_n}, \sqrt{\xi^2_{p_n+1}+\ldots+\xi^2_{p_n+\alpha_{m+n}+1}}$$

değişkenlerine bağlı olsun. Burada α_{m+n} pozitif tamsayı ve $p=p_n+\alpha_{m+n}+1$ dir.

Şimdi

$$\begin{aligned}
\xi_1 &= \rho \cos\theta_1 \\
\xi_2 &= \rho \sin\theta_1 \cos\theta_2 \\
&\ldots\ldots \qquad 0<\theta_i<\pi \\
&\ldots\ldots \qquad 1<i<p_n+\alpha_{m+n}-1 \\
&\ldots\ldots \qquad 0<p_n+\alpha_{m+n}<2\pi
\end{aligned}$$

$$\xi_{p_n+\alpha_{m+n}}=\rho \sin\theta_1 \ldots \sin\theta_{p_n+\alpha_{m+n}-1} \cos\theta_{p_n+\alpha_{m+n}}$$

$$\xi_{p_n+\alpha_{m+n}+1}=\rho \sin\theta_1 \ldots \sin\theta_{p_n+\alpha_{m+n}-1} \sin\theta_{p_n+\alpha_{m+n}}$$

küresel koordinatlarını kullanalım.

$$\sqrt{\xi^2_{p_n+1}+\ldots+\xi^2_{p_n+\alpha_{m+n}+1}}=\sqrt{\rho^2\sin^2\theta_1 \ldots \sin^2\theta_{p_n}}$$

eşitliğinden f fonksiyonunun $\theta_1,\ldots,\theta_{p_n}$ ye bağlı olduğu görülür.

(1.1-2) $$I_1=\int_{S(p)} f(\xi)|\xi-x|^{-p} dS$$

olsun. Küresel koordinatlar ve

$$\int \ldots \int_R f(\xi_1,\ldots,\xi_{p_n+\alpha_{m+n}+1}) d\xi_1 \ldots d\xi_{p_n+\alpha_{m+n}+1}$$

$$= \int \ldots \int_{R'} f(\rho\cos\theta_1,\ldots,\rho\sin\theta_1 \ldots \sin\theta_{p_n+\alpha_{m+n}}) \cdot \frac{\partial(\xi_1,\ldots,\xi_{p_n+\alpha_{m+n}+1})}{\partial(\rho,\theta_1,\ldots,\theta_{p_n+\alpha_{m+n}+1})} d\theta_1 \ldots d\theta_{p_n+\alpha_{m+n}+1}$$

dönüşümü gözönüne alınarak (1.1-2) integrali

(1.1-3) $$\rho^{p_n+\alpha_{m+n}} \int_0^{\pi} \int_0^{\pi} \int_0^{2\pi} \frac{f(\xi).\sin^{p_n+\alpha_{m+n}-1}\theta_1 \ldots \sin^{p_n+\alpha_{m+n}-1}\theta_{p_n+\alpha_{m+n}-1} \, d\theta_1 \ldots d\theta_{p_n+\alpha_{m+n}}}{\left[\sum_{i=1}^{p_n+\alpha_{m+n}+1} (\xi_i-x_i)^2\right]^{\frac{p_n+\alpha_{m+n}+1}{2}}}$$

biçiminde yazılır. f fonksiyonu $\xi_{p_n+1},\ldots,\xi_{p_n+\alpha_{m+n}+1}$ değişkenlerine bağlı olduğundan, $x_{p_{n+1}}=x_{m+n}$ ve $x_{p_{n+2}}=x_{p_{n+3}}=\ldots=x_{p_n+\alpha_{m+n}+1}=0$ yazılır. Eğer

$$\eta_{m+n}=\left[\sum_{i=1}^{p_n+\alpha_{m+n}+1}\xi_i^2\right]^{1/2} \tag{1.1-4}$$

biçiminde tanımlanırsa $\eta_{m+n}=\rho\sin\theta_1\ldots\sin\theta_{p_n}$ ve $\xi_{p_n+1}=\rho_{m+n}\cdot\cos\theta_{p_n+1}$ olur. Böylece

$$\sum_{i=1}^{p}(\xi_i-x_i)^2=\sum_{i=1}^{p_n}(\xi_i-x_i)^2+(\xi_{p_n+1}-x_{p_n+1})^2+\sum_{i=p_n+2}^{p_n+\alpha_{m+n}+1}(\xi_i-x_i)^2$$

$$=\sum_{i=1}^{p_n}(\xi_i-x_i)^2+\xi_{p_n+1}^2-2\xi_{p_n+1}x_{p_n+1}+x_{p_n+1}^2+\eta_{m+n}^2\sin^2\theta_{p_n+1}$$

$$=\sum_{i=1}^{p_n}(\xi_i-x_i)^2+\eta_{m+n}^2-2\eta_{m+n}x_{m+n}\cos\theta_{p_n+1}+x_{m+n}^2$$

elde edilir. Buna

$$\phi_1^2=\sum_{i=1}^{p_n}(\xi_i-x_i)^2+\eta_{m+n}^2-2x_{m+n}\eta_{m+n}\cos t_{m+n}+x_{m+n}^2 \tag{1.1-5}$$

diyelim.

$$\sin^{p_n+\alpha_{m+n}-1}\theta_1.\sin^{p_n+\alpha_{m+n}-2}\theta_2\ldots\sin\theta_{p_n+\alpha_{m+n}-1}\cdot d\theta_1\ldots d\theta_{p_n+\alpha_{m+n}-1}$$

$$=(\sin^{p_n-1}\theta_1\ldots\sin\theta_{p_n}d\theta_1\ldots d\theta_{p_n})(\sin\theta_1\ldots\sin\theta_{p_n})^{\alpha_{m+n}}$$

$$.(\sin^{\alpha_{m+n}-1}\theta_{p_n+1}d\theta_{p_n+1})(\sin^{\alpha_{m+n}-2}\theta_{p_n+2}\ldots\sin\theta_{p_n+\alpha_{m+n}-1}d\theta_{p_n+2}\ldots d\theta_{p_n+\alpha_{m+n}-1})$$

$$(1.1\text{-}6) \quad = \frac{\eta^{m+n}}{\rho^{m+n}} \left[\sin^{\alpha_{m+n}-1}\theta_{p_n+1}\, d\theta_{p_n+1}\right] dH(p_n+1).\sin^{\alpha_{m+n}-2}\theta_{p_n+2}$$

$$\ldots \sin\theta_{p_n+\alpha_{m+n}-1}\, d\theta_{p_n+2} \ldots d\theta_{p_n+\alpha_{m+n}-1}$$

biçiminde yazılır. Burada $dH(p_n+1)$ p_n+1 boyutlu küre yüzeyinin elamanıdır. (1.1-4),(1.1-5) ve (1.1-6) bağıntılarından yararlanarak

$$I_1=\rho^{p_n+\alpha_{m+n}} \int_0^{\pi}\cdots\int_0^{\pi} f(\xi_1,\ldots,\xi_{p_n+1},\eta_{m+n})$$

$$\cdot \frac{\eta_{m+n}^{\alpha_{m+n}}}{\rho^{\alpha_{m+n}}} \cdot \left[\int_0^{\pi} \frac{\sin^{\alpha_{m+n}-1}\theta_{p_n+1}\, d\theta_{p_n+1}}{\phi_1^{p_n+\alpha_{m+n}+1}}\right] \left(\int_0^{2\pi} d\theta_{p_n+\alpha_{m+n}}\right)$$

$$\left(\int_0^{\pi} \sin^{\alpha_{m+n}-2}\theta_{p_n+2}\right)\ldots\left(\int_0^{\pi} \sin\theta_{p_n+\alpha_{m+n-1}}\, d\theta_{p_n+\alpha_{m+n-1}}\right)$$

bu kısaca

$$(1.1\text{-}7) \quad \rho^{p_n} \int_{S(p_n+1)} f.\eta_{n+m}^{\alpha_{m+n}} \left[\int_0^{\pi} \frac{\sin^{\alpha_{m+n}-1}\theta_{p_n+1}\, d\theta_{p_n+1}}{\phi_1^{p_n+\alpha_{m+n}+1}}\right] dH(p_n+1)$$

$$2\pi. \prod_{j=2}^{\alpha_{m+n}-1} \int_0^{\pi} \sin^{\alpha_{m+n}-j}\theta\, d\theta$$

biçiminde yazılır. Burada $S(p_n+1)$,P_n+1 boyutlu ve ρ yarıçaplı yarı kürenin yüzeyidir.

$$\int_0^{\pi} \sin^{t-1}w\, dw = \frac{\sqrt{\pi}\,\Gamma\left(\frac{t}{2}\right)}{\Gamma\left(\frac{t+1}{2}\right)}, \quad t\geq 1$$

bağıntısından yararlanarak

$$(1.1\text{-}8)\qquad \prod_{j=2}^{\alpha_{m+n}-1}\int_0^{\pi}\sin^{\alpha_{m+n}-j}\theta\,d\theta=\prod_{j=2}^{\alpha_{m+n}-1}\left[\frac{\sqrt{\pi}\,\Gamma\left(\frac{\alpha_{m+n}-j+1}{2}\right)}{\Gamma\left(\frac{\alpha_{m+n}-j+2}{2}\right)}\right]=\frac{\pi^{\frac{\alpha_{m+n}-2}{2}}}{\Gamma\left(\frac{\alpha_{m+n}}{2}\right)}$$

bulunur. Buna göre

$$I_1=2\pi\rho^{p_n}\frac{\pi^{\frac{\alpha_{m+n}-2}{2}}}{\Gamma\left(\frac{\alpha_{m+n}}{2}\right)}\int_{S(p_n+1)}f.\,\eta_{m+n}^{\alpha_{m+n}}\left[\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1}\theta_{p_n+1}d\theta_{p_n+1}}{\phi_1^{p_n+\alpha_{m+n}+1}}\right]dH(p_n+1)$$

$$(1.1\text{-}9)\qquad =\frac{2\rho^{p_n}\pi^{\frac{\alpha_{m+n}}{2}}}{\Gamma\left(\frac{\alpha_{m+n}}{2}\right)}\int_{S(p_n+1)}f.\eta_{m+n}^{\alpha_{m+n}}\left[\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1}\theta_{p_n+1}\,d\theta_{p_n+1}}{\phi_1^{p_n+\alpha_{m+n}+1}}\right]dH(p_n+1)$$

olur. (1.1-1),(1.1-2) ve (1.1-9) ifadelerinden

$$v(x)=\frac{\rho^{p_n-1}(\rho^2-r^2)\Gamma\left(\frac{p_n+\alpha_{m+n}+1}{2}\right)}{\pi^{\frac{p_n+1}{2}}\Gamma\left(\frac{\alpha_{m+n}}{2}\right)}\int_{S(p_n+1)}f.\,\eta_{m+n}^{\alpha_{m+n}}\left[\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1}\theta_{p_n+1}d\theta_{p_n+1}}{\phi_1^{p_n+\alpha_{m+n}+1}}\right]dH(p_n+1)$$

bulunur. Bu ifade

$$(1.1\text{-}10)\qquad v(x)=\frac{(\rho^2-r^2)\Gamma\left(\frac{p_n+\alpha_{m+n}+1}{2}\right)}{\rho\pi^{\frac{p_n+1}{2}}\Gamma\left(\frac{\alpha_{m+n}}{2}\right)}\int_{S(p_n+1)}f.\,\eta_{m+n}^{\alpha_{m+n}}$$

$$\left[\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1}t_{m+n}\,dt_{m+n}}{\left[\sum_{i=1}^{p_n}(\xi_i-x_i)^2+\eta_{m+n}^2-2\eta_{m+n}x_{m+n}\cos t_{m+n}+x_{m+n}^2\right]^{\frac{p_n+\alpha_{m+n}+1}{2}}}\right]dS(p_n+1)$$

biçiminde yazılabilir. Bu ifade Huber tarafından çıkarılmış olup, p_n+1 boyutlu yarı kürede

$$(1.1\text{-}11) \qquad \sum_{i=1}^{p_n} \frac{\partial^2 v}{\partial x_i^2} + \frac{\partial^2 v}{\partial x_{m+n}^2} + \frac{\alpha_{m+n}}{x_{m+n}} \frac{\partial v}{\partial x_{m+n}} = 0$$

denkleminin çözümüdür ([3] sayfa 352). Burada $v(x_1,\ldots x_{p_n}, x_{m+n})$ ve $\alpha_{m+n} \geq 1$, $p_n = m+n-1$ dir.

Şimdi bundan başka bir eksensel simetrinin olduğunu kabul edelim. Farzedelim ki f fonksiyonu

$$\xi_1,\ldots,\xi_{p_{n-1}}, \sqrt{\xi^2_{p_{n-1}+1}+\ldots+\xi^2_{p_{n-1}+\alpha_{m+n-1}+1}}, \eta_{m+n}$$

bağlıdır. Burada $p_n = p_{n-1}+\alpha_{m+n-1}+1$ ve α_{m+n-1} pozitif tamsayıdır.

Şimdi

$$\eta_{m+n} = \rho\cos\theta_1$$

$$\xi_1 = \rho\sin\theta_1\cos\theta_2$$

$$\ldots\ldots\ldots \quad 0<\theta_i<\pi$$

$$\ldots\ldots\ldots \quad 1<i<p_{n-1}+\alpha_{m+n-1}-1$$

$$\ldots\ldots\ldots \quad 0<\theta p_{n-1}+\alpha_{m+n-1}<2\pi$$

$$\xi_{p_{n-1}+\alpha_{m+n-1}+1} = \rho\sin\theta_1\ldots\sin\theta_{p_{n-1}+\alpha_{m+n-1}}$$

küresel koordinatlarını kullanalım.

$$\sqrt{\xi^2_{p_{n-1}+1}+\ldots+\xi^2_{p_{n-1}+\alpha_{m+n-1}+1}} = \sqrt{\rho^2\sin^2\theta_1\ldots\sin^2\theta_{p_{n-1}}}$$

eşitliğinden f fonksiyonunun $\theta_1,\ldots,\theta_{p_{n-1}}$ değişkenlerine bağlı olduğu görülür. $x_{p_{n-1}+1} = x_{m+n-1}$ ve $x_{p_{n-1}+2} = x_{p_{n-1}+3} = \ldots = x_{p_{n-1}+\alpha_{m+n-1}+1} = 0$ ya-

zılır. Eğer

$$(1.1\text{-}12) \qquad \eta_{m+n-1}=\left[\sum_{i=1}^{p_{n-1}+\alpha_{m+n-1}+1} \xi_i^2\right]^{1/2}$$

biçiminde tanımlanırsa $\eta_{m+n-1}=\rho\sin\theta_1\ldots\sin\theta_{p_{n-1}+1}$ ve $\xi_{p_{n-1}+1}=\eta_{m+n-1}$ $\cos\theta_{p_{n-1}+2}$ olur. Böylece

$$\sum_{i=1}^{p_{n-1}+\alpha_{m+n-1}+1}(\xi_i-x_i)^2=\sum_{i=1}^{p_{n-1}}(\xi_i-x_i)^2+(\xi_{p_{n-1}+1}-x_{p_{n-1}+1})^2+\sum_{i=p_{n-1}+2}^{p_{n-1}+\alpha_{m+n-1}+1}(\xi_i-x_i)^2$$

$$=\sum_{i=1}^{p_{n-1}}(\xi_i-x_i)^2+\eta_{m+n-1}^2-2x_{m+n-1}\eta_{m+n-1}\cos\theta_{p_{n-1}+2}+x_{m+n-1}^2$$

$$+\eta_{m+n}^2-2\eta_{m+n}x_{m+n}\cos\theta_{p_{n-1}+2}$$

elde edilir. Uygunluk olması bakımından bu ifadeye

$$\phi_2^2=\sum_{i=1}^{p_{n-1}}(\xi_i-x_i)^2+\eta_{m+n-1}^2-2\eta_{m+n-1}x_{m+n-1}\cos t_{m+n-1}$$

$$+\eta_{m+n}^2-2\eta_{m+n}x_{m+n}\cos t_{m+n}+x_{m+n}^2$$

diyelim. Buna göre (1.1-10) integralini

$$(1.1\text{-}13) \qquad I_2=\int_{S(p_n+1)} f.\eta_{m+n}^{\alpha_{m+n}}\left[\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1}t_{m+n}dt_{m+n}}{\phi_1^{p_n+\alpha_{m+n}+1}}\right]dS(p_n+1)$$

alarak yeni bağıntılara göre dönüşüm uygulanırsa

$$(1.1\text{-}14) \qquad \rho^{p_{n-1}+\alpha_{m+n-1}+1}\int_0^{\pi/2}\int_0^{\pi}\cdots\int_0^{\pi} f.\eta_{m+n}^{\alpha_{m+n}}\left[\frac{\sin^{\alpha_{m+n}-1}t_{m+n}dt_{m+n}}{\phi_2^{p_{n-1}+\alpha_{m+n-1}+\alpha_{m+n}+2}}\right]$$

$$\sin^{p_{n-1}+\alpha_{m+n-1}}\theta_1.\sin^{p_{n-1}+\alpha_{m+n-1}-1}\theta_2\ldots\sin\theta_{p_{n-1}+\alpha_{m+n-1}+1}$$

$$d\theta_1\ldots d\theta_{p_{n-1}+\alpha_{m+n-1}+1}$$

yazılır. (1.1-6) da yapılan işlemleri bunun için yaparak

$$(1.1\text{-}15)\quad \frac{\eta^{\alpha_{m+n-1}}}{\rho^{\alpha_{m+n-1}}}\left[\sin^{\alpha_{m+n-1}-1}\theta_{p_{n-1}+2}d\theta_{p_{n-1}+2}\right]dH(p_{n-1}+2).\sin^{\alpha_{m+n-1}-2}\theta_{p_{n-1}+3}$$

$$\ldots\sin\theta_{p_{n-1}+\alpha_{m+n-1}}.d\theta_{p_{n-1}+3}\ldots d\theta_{p_{n-1}+\alpha_{m+n-1}+1}$$

elde ederiz, burada $dH(p_{n-1}+2)$ $p_{n-1}+2$ boyutlu yüzey elemanıdır. Buna göre (1.1-14) integrali

$$\rho^{p_{n-1}+\alpha_{m+n-1}+1}\int_0^{\pi/2}\int_0^{\pi}\ldots\int_0^{\pi} f.\eta_{m+n}^{\alpha_{m+n}}.\frac{\eta_{m+n-1}^{\alpha_{m+n-1}}}{\rho^{\alpha_{m+n-1}}}.\left[\int_0^{\pi}\int_0^{\pi}\right.$$

$$\left.\frac{\sin^{\alpha_{m+n}-1}t_{m+n}\sin^{\alpha_{m+n-1}-1}\theta_{p_{n-1}+2}d\theta_{p_{n-1}+2}dt_{m+n}}{\phi_2^{p_{n-1}+\alpha_{m+n}+\alpha_{m+n-1}+2}}\right]dH(p_{n-1}+2)$$

$$\left(\int_0^{\pi}\sin^{\alpha_{m+n-1}-1}\theta_{p_{n-1}+3}d\theta_{p_{n-1}+3}\right)\ldots\left(\int_0^{\pi}\sin\theta_{p_{n-1}+\alpha_{m+n-1}}d\theta_{p_{n-1}+\alpha_{m+n-1}}\right)$$

$$.\left(\int_0^{2\pi}d\theta_{p_{n-1}+\alpha_{m+n-1}+1}\right)=\rho^{p_{n-1}+1}\int_{S(p_{n-1}+2}f.\eta_{m+n}^{\alpha_{m+n}}\eta_{m+n-1}^{\alpha_{m+n-1}}$$

$$\left[\int_0^{\pi}\int_0^{\pi}\frac{\sin^{\alpha_{m+n-1}-1}\theta_{p_{n-1}+2}\sin^{\alpha_{m+n-1}}t_{m+n}dt_{m+n}d\theta_{p_{n-1}+2}}{\phi_2^{p_{n-1}+\alpha_{m+n-1}+\alpha_{m+n}+2}}\right].$$

$$dH(p_{n-1}+2).2\pi. \prod_{j=2}^{\alpha_{m+n-1}-1} \int_0^{\pi} \sin^{\alpha_{m+n-1}-j}\theta \, d\theta$$

olur. Burada $S(p_{n-1}+2)$ $p_{n-1}+2$ boyutlu bir çeyrek küre yüzey elamanıdır. (1.1-8) eşitliğinden

$$\prod_{j=2}^{\alpha_{m+n-1}-1} \int_0^{\pi} \sin^{\alpha_{m+n-1}-j}\theta \, d\theta = \pi \frac{\frac{\alpha_{m+n-1}-2}{2}}{\Gamma(\frac{\alpha_{m+n-1}}{2})}$$

yazarız. Basit bir gösterim değişikliği ile I_2 integrali

$$(1.1\text{-}16)\quad \frac{2\pi^{\frac{\alpha_{m+n-1}}{2}} \rho^{p_{n-1}+1}}{\Gamma(\frac{\alpha_{m+n-1}}{2})} \int_{S(p_{n-1}+2)} f.\eta_{m+n}^{\alpha_{m+n}}.\eta_{m+n-1}^{\alpha_{m+n-1}} \left[\int_0^{\pi}\int_0^{\pi} \frac{\sin^{\alpha_{m+n}-1}t_{m+n}.\sin^{\alpha_{m+n-1}-1}t_{m+n-1}dt_{m+n-1}dt_{m+n}}{\phi_2^{p_{n-1}+\alpha_{m+n-1}+\alpha_{m+n}+2}}\right] dH(p_{n-1}+2)$$

yazılır. (10),(13) ve (16) ifadelerinden

$$(1.1\text{-}17)\quad v(x)=\frac{2\rho^{p_{n-1}}(\rho^2-r^2)\Gamma(\frac{p_{n-1}+\alpha_{m+n-1}+\alpha_{m+n}+2}{2})}{\pi^{\frac{p_{n-1}+2}{2}}\Gamma(\frac{\alpha_{m+n-1}}{2})\Gamma(\frac{\alpha_{m+n}}{2})} .\int_{S(p_{n-1}+2)} f.\eta_{m+n-1}^{\alpha_{m+n-1}}$$

$$\eta_{m+n}^{\alpha_{m+n}}\left[\int_0^{\pi}\int_0^{\pi} \frac{\sin^{\alpha_{m+n-1}-1}t_{m+n-1}\sin^{\alpha_{m+n}-1}t_{m+n}dt_{m+n}dt_{m+n-1}}{\phi_2^{p_{n-1}+\alpha_{m+n-1}+\alpha_{m+n}+2}}\right] dH(p_{n-1}+2)$$

elde edilir. Bu ifade $n=2$, $\lambda=0$ için (0.2) denkleminin bir çözümü olduğu Hall, Quinn ve Weinacht tarafından gösterilmiştir. (1.1-17) denklemi

$$(1.1\text{-}18)\quad v(x)=\frac{2(\rho^2-r^2)\Gamma\left(\frac{p_{n-1}+\alpha_{m+n-1}+\alpha_{m+n}+2}{2}\right)}{\rho\,\pi\,\frac{p_{n-1}+2}{2}\,\Gamma\left(\frac{\alpha_{m+n-1}}{2}\right)\Gamma\left(\frac{\alpha_{m+n}}{2}\right)}\int_{S(p_{n-1}+2)} f.\eta_{m+n-1}^{\alpha_{m+n-1}}.\eta_{m+n}^{\alpha_{m+n}}$$

$$\left[\int_0^{\pi}\int_0^{\pi}\frac{\sin^{\alpha_{m+n-1}-1}t_{m+n-1}\sin^{\alpha_{m+n}-1}t_{m+n}\,dt_{m+n}\,dt_{m+n-1}}{\phi_2^{\,p_{n-1}+\alpha_{m+n-1}+\alpha_{m+n}+2}}\right]dS(p_{n-1}+2)$$

biçiminde yazılır. Aynı şekilde işlemlere devam edilerek farzedelim ki f fonksiyonu eksensel simetriye sahip olup

$$\xi_1,\ldots,\xi_{p_{n-2}},\sqrt{\xi^2_{p_{n-2}+1}+\ldots+\xi^2_{p_{n-2}+\alpha_{m+n-2}+1}},\ \eta_{m+n-1}^{\alpha_{m+n-1}},\eta_{m+n}^{\alpha_{m+n}}$$

değişikenlerine bağlı olsun. Burada $p_{n-1}=p_{n-2}+\alpha_{m+n-2}+1$ ve α_{m+n-2} pozitif tamsayıdır.

Şimdi

$$\eta_{m+n}=\rho\cos\theta_1$$

$$\eta_{m+n-1}=\rho\sin\theta_1\cos\theta_2$$

$$\xi_1=\rho\sin\theta_1\sin\theta_2\cos\theta_3$$

$$\ldots\ldots\quad 0<\theta_i<\pi$$

$$\ldots\ldots\quad 1<i<p_{n-2}+\alpha_{m+n-2}-1$$

$$\ldots\ldots\quad 0<p_{n-2}+\alpha_{m+n-2}<2\pi$$

$$\xi_{p_{n-2}+\alpha_{m+n-2}+1}=\rho\sin\theta_1\ldots\sin\theta_{p_{n-2}+\alpha_{m+n-2}}$$

küresel koordinatlarını kullanalım.

$$\sqrt{\xi^2_{p_{n-2}+1}+\ldots+\xi^2_{p_{n-2}+\alpha_{m+n-2}+1}} = \sqrt{\rho^2\sin^2\theta_1\ldots\sin^2\theta_{p_{n-2}+1}}$$

eşitliğinden f fonksiyonunun $\theta_1,\ldots,\theta_{p_{n-2}+1}$ değişkenlerine bağlı olduğu görülür. $x_{p_{n-2}+1}=x_{m+n-3}$ ve $x_{p_{n-2}+2}=x_{p_{n-2}+3}=\ldots=x_{p_{n-2}+\alpha_{m+n-2}+1}=0$ yazılır. Eğer

$$\eta_{m+n-2}=\left[\sum_{i=1}^{p_{n-2}+\alpha_{m+n-2}+1}\xi_i^2\right]^{1/2}$$

biçiminde tanımlanırsa $\eta_{m+n-2}=\rho\sin\theta_1\ldots\sin\theta_{p_{n-2}+1}$ ve $\xi_{p_{n-2}+1}=\eta_{m+n-2}\cos\theta_{p_{n-2}+2}$ olur.

$$\sum_{i=1}^{p_{n-2}+\alpha_{m+n-2}+1}(\xi_i-x_i)^2=\sum_{i=1}^{p_{n-2}}(\xi_i-x_i)^2+(\xi_{p_{n-2}+1}-x_{p_{n-2}+1})^2+$$

$$\sum_{i=p_{n-2}+2}^{p_{n-2}+\alpha_{m+n-2}+1}(\xi_i-x_i)^2=\sum_{i=1}^{p_{n-2}}(\xi_i-x_i)^2+\eta^2_{m+n-2}-2\eta_{m+n}x_{m+n}\cos\theta_{p_{n-2}+2}+x^2_{m+n-2}$$

elde edilir. Bunu

$$\phi_3^2=\sum_{i=1}^{p_{n-2}}(\xi_i-x_i)^2+\eta^2_{m+n-2}-2\eta_{m+n-2}x_{m+n-2}\cos t_{m+n-2}+x^2_{m+n-2}$$

$$+\eta^2_{m+n-1}-2\eta_{m+n-1}x_{m+n-1}\cos t_{m+n-1}+x^2_{m+n-1}+\eta^2_{m+n}-2\eta_{m+n}x_{m+n}.\cos t_{m+n}+x^2_{m+n}$$

yazarız. Buna göre (18) integralini

$$(1.1\text{-}19)\quad I_3=\int_{S(p_{n-2}+2)} f.\eta_{m+n-1}^{\alpha_{m+n-1}}.\eta_{m+n}^{\alpha_{m+n}}.\left[\int_0^{\pi}\int_0^{\pi}\frac{\sin^{\alpha_{m+n-1}-1}t_{m+n-1}}{}\right.$$

$$\left.\frac{\cdot\sin^{\alpha_{m+n}-1}t_{m+n}dt_{m+n}.dt_{m+n-1}}{\phi_2^{p_{n-1}+\alpha_{m+n-1}+\alpha_{m+n-2}+2}}\right] dS(p_{n-2}+2)$$

olarak yeni bağıntılara göre dönüşüm uygulanarak

$$(1.1\text{-}20)\quad \rho^{p_{n-2}+\alpha_{m+n-2}+1}\int_0^{\pi/2}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi} f.\eta_{m+n-1}^{\alpha_{m+n-1}}.\eta_{m+n}^{\alpha_{m+n}}.$$

$$\left[\int_0^{\pi}\int_0^{\pi}\frac{\sin^{\alpha_{m+n-1}-1}t_{m+n-1}\sin^{\alpha_{m+n}-1}t_{m+n}dt_{m+n}dt_{m+n-1}}{\phi_3^{p_{n-2}+\alpha_{m+n-2}+\alpha_{m+n-1}+\alpha_{m+n}+3}}\right]$$

$$\sin^{p_{n-2}+\alpha_{m+n-2}}\theta_1.\sin^{p_{n-2}+\alpha_{m+n-2}-1}\theta_2\ldots\sin\theta_{p_{n-2}+\alpha_{m+n-2}}$$

$$d\theta_1\ldots d\theta_{p_{n-2}+\alpha_{m+n-2}+1}$$

bulunur. $\sin^{p_{n-2}+\alpha_{m+n-2}}\theta_1.\sin^{p_{n-2}+\alpha_{m+n-2}-1}\theta_2\ldots\sin\theta_{p_{n-2}+\alpha_{m+n-2}}.d\theta_1$ $\ldots d\theta_{p_{n-2}+\alpha_{m+n-2}+1}$ ifadesine (1.1-6) da yapılan işlemler uygulanırsa

$$(1.1\text{-}21)\quad \frac{\eta_{m+n-2}^{\alpha_{m+n-2}}}{\rho^{\alpha_{m+n-2}}}\left[\sin^{\alpha_{m+n-2}-1}\theta_{p_{n-2}+2}d\theta_{p_{n-2}+2}\right] dH(p_{n-2}+2).$$

$$\sin^{\alpha_{m+n-2}-2}\theta_{p_{n-2}+3}\ldots\sin\theta_{p_{n-2}+\alpha_{m+n-2}}d\theta_{p_{n-2}+3}\ldots d\theta_{p_{n-2}+\alpha_{m+n-2}+1}$$

elde edilir. Burada $dH(p_{n-2}+2)$ $p_{n-2}+2$ boyutlu yüzey elemanıdır.(1.1-21) bağıntısından yararlanarak (1.1-19) integralini

$$\rho^{p_{n-2}+\alpha_{m+n-2}+1}\int_0^{\pi/2}\int_0^{\pi}\cdots\int_0^{\pi} f.\eta_{m+n-1}^{\alpha_{m+n-1}}.\eta_{m+n}^{\alpha_{m+n}}.\frac{\eta_{m+n-2}^{\alpha_{m+n-2}}}{\rho^{m+n-2}}.$$

$$\left[\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1}t_{m+n}.\sin^{\alpha_{m+n-1}-1}t_{m+n-1}.\sin^{\alpha_{m+n-2}-1}\theta_{p_{n-2}+2}}{\phi_3^{p_{n-1}+\alpha_{m+n-2}+\alpha_{m+n-1}+\alpha_{m+n}+3}}\right.$$

$$\left..dt_{m+n}.dt_{m+n-1}.d\theta_{p_{n-2}+2}\right]dH(p_{n-2}+2).\left(\int_0^{\pi}\sin^{\alpha_{m+n-2}}\theta_{p_{n-2}+3}\right.$$

$$d\theta_{p_{n-2}+3})\ldots(\int_0^{\pi}\sin\theta_{p_{n-2}+\alpha_{m+n-2}}d\theta_{p_{n-2}+\alpha_{m+n-2}})(\int_0^{\pi}d\theta_{p_{n-2}+\alpha_{m+n-2}+1})$$

$$=\rho^{p_{n-2}+1}\int_{S(p_{n-2}+3)} f.\eta_{m+n-2}^{\alpha_{m+n-2}}.\eta_{m+n-1}^{\alpha_{m+n-1}}.\eta_{m+n}^{\alpha_{m+n}}\left[\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\right.$$

$$\frac{\sin^{\alpha_{m+n}-1}t_{m+n}\sin^{\alpha_{m+n-1}-1}t_{m+n-1}\sin^{\alpha_{m+n-2}-1}\theta_{p_{n-2}+2}dt_{m+n}}{\phi_3^{p_{n-1}+\alpha_{m+n-2}+\alpha_{m+n-1}+\alpha_{m+n}+3}}$$

$$\left.dt_{m+n-1}d\theta_{p_{n-2}+2}\right]dH(p_{n-2}+3).2\pi\prod_{j=2}^{\alpha_{m+n-2}-1}\int_0^{\pi}\sin^{\alpha_{m+n-2}-j}\theta d\theta$$

elde ederiz. (1.1-8) eşitliğini gözönüne alarak

$$\sum_{j=2}^{\alpha_{m+n-2}-1}\int_0^{\pi}\sin^{\alpha_{m+n-2}-j}\theta d\theta=\frac{\pi\,\frac{\alpha_{m+n-2}-2}{2}}{\Gamma(\frac{\alpha_{m+n-2}}{2})}$$

buluruz. Basit bir gösterim değişikliği ile I_3 integrali

$$(1.1\text{-}22)\quad \frac{2\pi^{\frac{\alpha_{m+n-2}}{2}}\rho^{p_{n-2}+1}}{\Gamma(\frac{\alpha_{m+n-2}}{2})}\int_{S(p_{n-2}+3)} f.\eta_{m+n-2}^{\alpha_{m+n-2}}.\eta_{m+n-1}^{\alpha_{m+n-1}}.\eta_{m+n}^{\alpha_{m+n}}$$

$$\left[\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \frac{\sin^{\alpha_{m+n-1}}t_{m+n}\sin^{\alpha_{m+n-1}-1}t_{m+n-1}\sin^{\alpha_{m+n-2}-1}\theta_{p_{n-2}+3}}{\phi_3^{p_{n-2}+\alpha_{m+n-2}+\alpha_{m+n-1}+\alpha_{m+n}+3}}\right.$$

$$\left.\cdot dt_{m+n}\cdot dt_{m+n-1}\cdot d\theta_{p_{n-2}+3}\right] dH(p_{n-2}+3)$$

olur. (1.18), (1.19) ve (1.1-22) integrallerinden

$$(1.1\text{-}23)\quad v(x)=\frac{2^2\rho^{p_{n-2}+1}(\rho^2-r^2)\Gamma(\frac{p_{n-2}+\alpha_{m+n-2}+\alpha_{m+n-1}+\alpha_{m+n}+3}{2})}{\pi^{\frac{p_{n-2}+3}{2}}\Gamma(\frac{\alpha_{m+n-2}}{2})\Gamma(\frac{\alpha_{m+n-1}}{2})\Gamma(\frac{\alpha_{m+n}}{2})}\cdot$$

$$\int_{S(p_{n-2}+3)} f.\,\eta_{m+n-2}^{\alpha_{m+n-2}}.\eta_{m+n-1}^{\alpha_{m+n-1}}.\eta_{m+n}^{\alpha_{m+n}}\left[\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\right.$$

$$\left.\frac{\sin^{\alpha_{m+n}-1}t_{m+n}\sin^{\alpha_{m+n-1}-1}t_{m+n-1}\sin^{\alpha_{m+n-2}-1}\theta_{p_{n-2}+3}}{\phi_3^{p_{n-2}+\alpha_{m+n-2}+\alpha_{m+n-1}+\alpha_{m+n}+3}}\right.$$

$$\left.dt_{m+1}\cdot dt_{m+n-1}\cdot d\theta_{p_{n-2}+3}\right] dH(p_{n-2}+3)$$

elde edilir. (1.1-23) integrali

$$v(x)=\frac{2^2(\rho^2-r^2)\Gamma(\frac{p_{n-2}+\alpha_{m+n-2}+\alpha_{m+n-1}+\alpha_{m+n}+3}{2})}{\rho.\pi^{\frac{p_{n-2}+3}{2}}\Gamma(\frac{\alpha_{m+n-2}}{2})\Gamma(\frac{\alpha_{m+n-1}}{2})\Gamma(\frac{\alpha_{m+n}}{2})}$$

$$\int_{S(p_{n-2}+3)} f.\eta_{m+n-2}^{\alpha_{m+n-2}} \cdot \eta_{m+n-1}^{\alpha_{m+n-1}} \cdot \eta_{m+n}^{\alpha_{m+n}} \left[\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \frac{\sin^{\alpha_{m+n}-1} t_{m+n}}{\sin^{\alpha_{m+n-1}-1} t_{m+n-1} \sin^{\alpha_{m+n-2}-1} t_{m+n-2} dt_{m+n} \cdot dt_{m+n-1} \cdot dt_{m+n-2}} \right] dS(p_{n-2}+3)$$

$$\phi_3^{p_{n-2}+\alpha_{m+n-2}+\alpha_{m+n-1}+\alpha_{m+n}+3}$$

yazılır. Burada $S(p_{n-2}+3)$ $p_{n-2}+3$ boyutlu yüzey elamanıdır.

İşlemler aynı şekilde devam edilerek (n-1) inci bağıntıda farzedelim ki f fonksiyonu

$$\xi_1,\ldots,\xi_{p_1}, \sqrt{\xi_{p_1+1}^2+\ldots+\xi_{p_1+\alpha_{m+1}+1}^2}, \eta_{m+n},\ldots,\eta_{m+1}$$

bağlı olsun. Burada $p_1=p_2+\alpha_{m+2}+1$ ve α_{m+2} pozitif bir tamsayıdır.

Şimdi

$$\eta_{m+n} = \rho\cos\theta_1 \qquad 0<\theta_i<\pi$$

$$\ldots \qquad 1<i<p_1+\alpha_{m+1}-1$$

$$\ldots \qquad 0<p_1+\alpha_{m+1}<2\pi$$

$$\ldots$$

$$\eta_{m+1} = \rho\sin\theta_1\ldots\sin\theta_m\cos\theta_{m+1}$$

$$\xi_1=\rho\sin\theta_1\ldots\sin\theta_{m+1}\cos\theta_{m+2}$$

$$\ldots$$

$$\ldots$$

$$\ldots$$

$$\xi_{p_1+\alpha_{m+1}+1}=\rho\sin\theta_1\ldots\sin\theta_{p_1+\alpha_{m+1}}$$

küresel koordinatlarını kullanalım.

$$\sqrt{\xi^2_{p_1+1}+\ldots+\xi^2_{p_1+\alpha_{m+1}+1}} = \sqrt{\rho^2\sin^2\theta\ldots\sin^2\theta_{p_1+1}}$$

eşitliğinden f fonksiyonunun $\theta_1,\ldots,\theta_{p_1+1}$ değişkenlerine bağlı olduğu görülür. $x_{p_1+1}=x_{m+1}$ ve $x_{p_1+2}=x_{p_1+3}=x_{p_1+\alpha_{m+1}+1}=0$ yazılır. Eğer

$$\eta_{m+1}=\left[\sum_{i=1}^{p_1}\xi_i^2\right]^{1/2}$$

biçiminde tanımlanırsa $\eta_{m+1}=\rho\sin\theta_1\ldots\sin\theta_{p_1+1}$ ve $\xi_{p_1+1}=\eta_{m+1}\cos\theta_{p_1+2}$ olur. Böylece

$$\sum_{i=1}^{p_1}(\xi_i-x_i)^2+(\xi_{p_1+1}-x_{p_1+1})^2+\sum_{i=p_1+2}^{p_1+\alpha_{m+1}+1}(\xi_i-x_i)^2$$

$$=\sum_{i=1}^{p_1}(\xi_i-x_i)^2+\eta^2_{m+1}-2\eta_{m+1}x_{m+1}\cos\theta_{p_1+2}+x^2_{m+1}$$

bulunur. Benzerlik olması bakımından bu eşitliğe

$$\phi^2_n=\sum_{i=1}^{p_1}(\xi_i-x_i)^2+\eta^2_{m+1}-2\eta_{m+1}x_{m+1}\cos t_{m+1}+x^2_{m+1}$$

$$+\ldots+\eta^2_{m+n}-2\eta_{m+n}x_{m+n}\cos t_{m+n}+x^2_{m+n}$$

diyelim. Buna göre enson integral

$$(1.1\text{-}24)\quad I_n=\int_{S(p_2+n-1)} f.\eta_{m+2}^{\alpha_{m+2}}\ldots\eta_{m+n}^{\alpha_{m+n}}\left[\int_0^{\pi}\ldots\int_0^{\pi}\sin^{\alpha_{m+n}-1}t_{m+n}\right.$$

$$\frac{\ldots \sin^{\alpha_{m+2}-1} t_{m+2} dt_{m+n} \ldots dt_{m+2}}{\phi_1^{p_2+\alpha_{m+2}+\ldots+\alpha_{m+n}+(n-1)}} \Bigg] dS(p_2+n-2)$$

olur. Bu integralde yeni bağıntılara göre dönüşüm yapılırsa

$$\rho^{p_1+\alpha_{m+1}+1} \int_0^{\pi/2} \int_0^{\pi} \int_0^{2\pi} f.\eta_{m+2}^{\alpha_{m+2}} \ldots \eta_{m+n}^{\alpha_{m+n}} \Bigg[\int_0^{\pi} \ldots \int_0^{\pi} \sin^{p_1+\alpha_{m+1}-1} \theta_1 \sin^{\alpha_{m+n}-1} t_{m+n} \ldots \sin^{\alpha_{m+2}-1} t_{m+2} dt_{m+n} \ldots dt_{m+2} \Bigg]$$

$$\phi_n^{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}$$

$$\sin^{p_1+\alpha_{m+1}-1} \theta_1 \sin^{p_1+\alpha_{m+1}-2} \theta_2 \ldots \sin\theta_{p_1+\alpha_{m+1}} d\theta_1 \ldots d\theta_{p_n+\alpha_{m+1}+1}$$

elde edilir. Buradaki $\sin^{p_1+\alpha_{m+1}} \theta_1 \sin^{p_1+\alpha_{m+1}-2} \theta_2 \ldots \sin\theta_{p_1+\alpha_{m+1}} . d\theta_1 \ldots d\theta_{p_n+\alpha_{m+1}+1}$ elde edilir. Buradaki $\sin^{p_1+\alpha_{m+1}} \theta_1 . \sin^{p_1+\alpha_{m+1}-1} \theta_2 \ldots \sin\theta_{p_1+\alpha_{m+1}} d\theta_1 \ldots d\theta_{p_1+\alpha_{m+1}+1}$. ifadesine (1.1-6) da yapılan işlemleri iygulayarak

(1.1-26) $$\frac{\eta_{m+1}^{\alpha_{m+1}}}{\rho^{m+1}} \left[\sin^{\alpha_{m+1}-1} \theta_{p_1+2} d\theta_{p_1+2} \right] dH(p_1+n) . \sin^{\alpha_{m+1}-2} \theta_{p_1+3} \ldots \sin\theta_{p_1+\alpha_{m+1}} d\theta_{p_1+3} \ldots d\theta_{p_1+\alpha_{m+1}+1}$$

elde edilir. Buna göre (1.24) integrali

$$\rho^{p_1\alpha_{m+1}+1} \int_0^{\pi/2} \int_0^{\pi} \ldots \int_0^{\pi} f.\eta_{m+2}^{\alpha_{m+2}} \ldots \eta_{m+n}^{\alpha_{m+n}} \frac{\eta_{m+1}^{\alpha_{m+1}}}{\rho^{\alpha_{m+1}}}$$

$$\left[\int_0^{\pi}\cdots\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1}t_{m+n}\cdots\sin^{\alpha_{m+1}-1}\theta_{p_1+2}\,dt_{m+n}\cdots d\theta_{p_1+2}}{\phi_n^{p_1+\alpha_{m+1}+\cdots+\alpha_{m+n}+n}}\right]$$

$$.dH(p_1+n).\left(\int_0^{\pi}\sin^{\alpha_{m+1}-2}\theta_{p_1+3}d\theta_{p_1+3}\right)\cdots\left(\int_0^{\pi}\sin\theta_{p_1+\alpha_{m+1}}\right.$$

$$\left.d\theta_{p_1+\alpha_{m+1}}\right)\left(\int_0^{2\pi}d\theta_{p_1+\alpha_{m+1}+1}\right)=\rho^{p_1+1}\cdot\int_{S(p_1+n)}f.$$

$$\eta_{m+1}^{\alpha_{m+1}}\cdots\eta_{m+n}^{\alpha_{m+n}}\left[\int_0^{\pi}\cdots\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1}t_{m+n}\cdots\sin^{\alpha_{m+1}-1}\theta_{p_1+2}}{\phi_n^{p_1+\alpha_{m+1}+\cdots+\alpha_{m+n}+n}}\right.$$

$$\left.d\theta_{p_1+2}.dt_{m+2}\cdots dt_{m+n}\right]dH(p_1+n).2\pi\prod_{j=2}^{\alpha_{m+1}-1}\int_0^{\pi}\sin^{\alpha_{m+1}-j}\theta d\theta$$

olur. Burada $S(p_1+n)$ p_1+n boyutlu çeyrek küre yüzeyini belirtir.(1.1-8) den yararlanarak

$$\prod_{j=2}^{\alpha_{m+1}-1}\int_0^{\pi}\sin^{\alpha_{m+1}-j}\theta d\theta=\frac{\pi\,\frac{\alpha_{m+1}-2}{2}}{\Gamma\left(\frac{\alpha_{m+1}}{2}\right)}$$

buluruz. Bu değeri basit bir gösterim değişikliği ile (1.1-24) integralinde yerine yazarak

$$(1.1\text{-}27)\quad\frac{2^{n-1}\pi^{\frac{\alpha_{m+1}+1}{2}}\rho^{p_1+1}}{\Gamma\left(\frac{\alpha_{m+1}}{2}\right)}\cdot\int_{S(p_1+n)}f.\eta_{m+1}^{\alpha_{m+1}}\cdots\eta_{m+n}^{\alpha_{m+n}}\left[\int_0^{\pi}\cdots\int_0^{\pi}\right.$$

$$\left.\frac{\sin^{\alpha_{m+n}-1}t_{m+n}\cdots\sin^{\alpha_{m+1}-1}\theta_{p_1+2}d\theta_{p_1+2}\cdots dt_{m+n}}{\phi_n^{p_1+\alpha_{m+1}+\cdots+\alpha_{m+n}+n}}\right]dH(p_1+n)$$

elde ederiz. (I_{n-1}),(1.1-24) ve (1.1-27) integrallerinde

$$(1.1\text{-}28)\qquad v(x)=\frac{2^{n-1}\rho^{p_1}(\rho^2-r^2)\Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)}{\pi^{\frac{p_1+n}{2}}\Gamma\left(\frac{\alpha_{m+1}}{2}\right)\ldots\Gamma\left(\frac{\alpha_{m+n}}{2}\right)}\int_{S(p_1+n)} f.$$

$$\eta_{m+1}^{\alpha_{m+1}}\ldots\eta_{m+n}^{\alpha_{m+n}}\left[\int_0^{\pi}\ldots\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1}t_{m+n}\ldots\sin^{\alpha_{m+1}-1}t_{m+1}}{\phi_n^{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}}\right.$$

$$\left.dt_{m+1}\ldots dt_{m+n}\right]dH(p_1+n)$$

bulunur. Bu genelleştirilmiş Poisson integralidir. $\lambda=0$ için (0.1-2) denkleminin p_1+n boyutlu uzayda çeyrek küre bölgesinde Dirichlet probleminin bir çözümü olduğunu göstereceğiz. (1.1-28) denklemi

$$(1.1\text{-}29)\qquad v(x)=\frac{2^{n-1}(\rho^2-r^2)\Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)}{\rho\pi^{\frac{p_1+n}{2}}\Gamma\left(\frac{\alpha_{m+1}}{2}\right)\ldots\Gamma\left(\frac{\alpha_{m+n}}{2}\right)}\int_{S(p_1+n)} f.\eta_{m+1}^{\alpha_{m+1}}$$

$$\ldots\eta_{m+n}^{\alpha_{m+n}}\left[\int_0^{\pi}\ldots\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1}t_{m+n}\ldots\sin^{\alpha_{m+1}-1}t_{m+1}}{\phi_n^{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}}\right.$$

$$\left.dt_{m+1}\ldots dt_{m+n}\right]d\mathfrak{S}(p_1+n)$$

biçiminde yazılır.

1.2 DIRICHLET PROBLEMİ

Bu kesimde (1.1-29) formülü ile verilen genelleştirilmiş Poisson integral formülü m+n boyutlu bir çeyrek küre bölgesinde (0.1-3) denkleminin Dirichlet tipinde bir çözümü olduğunu aşağıda vereceğimiz bir teoremle göstereceğiz. Bunun için kullanacağımız uygun gösterimleri yazalım.

p boyutlu uzayda bir nokta $x=(x_1,\ldots,x_p)$ biçiminde gösterilir. Burada p=m+n olmak üzere,

$$Q=\{x|x_{m+1}>0,\ldots,x_{m+n}>0\}$$

$$S(\rho)=Q\cap\{x|\ |x|\ =\rho\}$$

$$B(\rho)=\{x|\ |x|\ <\rho\}$$

$$B(\rho)^{+}=B(\rho)\cap Q$$

$$D(x_{m+1})=\{x|x_{m+1}=0,x_{m+2}\geq 0,\ldots,x_{m+n}\geq 0\}$$

$$D(x_{m+2})=\{x|x_{m+1}\geq 0,x_{m+2}=0,\ldots,x_{m+n}\geq 0\}$$

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$$D(x_{m+n})=\{x|x_{m+1}\geq 0,x_{m+n2}\geq 0,\ldots,x_{m+n}=0\}$$

$$D=D(x_{m+1})\cup D(x_{m+2})\cup\ldots\cup D(x_{m+n})$$

$$D(\rho,x_{m+1})=D(x_{m+1})\cap B(\rho)$$

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$$D(\rho,x_{m+n})=D(x_{m+n})\cap B(\rho)$$

$D(\rho)=D\cap B(\rho)$

dır. $S(\rho)$ bölgesinin kapanışına $\overline{S(\rho)}$ ve $S(\rho)$ üzerindeki bir noktaya $\xi=(\xi_1,\ldots,\xi_m,\ldots,\xi_{m+n})$ diyelim.

TEOREM: Farzedelim ki $\alpha_{m+1}\geq 1,\ldots,\alpha_{m+n}\geq 1$ ve $f(\xi)$, $S(\rho)$ üzerinde tanımlı ve sürekli olsun. Bundan başka, $S(\rho)$ bölgesinin eksensel sınırlarında, f fonksiyonunun aşağıdaki özeliklere sahip olduğunu kabul edelim. $\xi_0\in \overline{S(\rho)}\cap D$ alalım. O zaman,

(2.1-1) Eğer $\alpha_{m+1}=1,\alpha_{m+2}=1,\ldots,\alpha_{m+n}=1$ ise

$$\lim_{\xi\to\xi_0} f(\xi).(\log\eta_{m+1})^{-1}.(\log\eta_{m+2})^{-1}\ldots(\log\eta_{m+n})^{-1}=0$$

(2.1-2) Eğer $\alpha_{m+1}>1,\alpha_{m+2}=1,\ldots,\alpha_{m+n}=1$ ise

$$\lim_{\xi\to\xi_0} f(\xi)\eta_{m+1}^{\alpha_{m+1}-1}.(\log\eta_{m+2})^{-1}\ldots(\log\eta_{m+n})^{-1}=0$$

(2.1-3) Eğer $\alpha_{m+1}=1,\ldots,\alpha_{m+n-1}=1,\alpha_{m+n}>1$ ise

$$\lim_{\xi\to\xi_0} f(\xi).(\log\eta_{m+1})^{-1}\ldots(\log\eta_{m+n-1})^{-1}.\eta_{m+n}^{\alpha_{m+n}-1}=0$$

(2.1-4) Eğer $\alpha_{m+1}>1,\alpha_{m+2}>1,\ldots,\alpha_{m+n}>1$ ise

$$\lim_{\xi\to\xi_0} f(\xi)\,\eta_{m+1}^{\alpha_{m+1}-1}.\eta_{m+2}^{\alpha_{m+2}-1}\ldots\eta_{m+n}^{\alpha_{m+n}-1}=0$$

olur. $x\in B(\rho)$ için tanımlanan

$$(2.1\text{-}5)\quad v(x)=\frac{2^{n-1}(\rho^2-r^2)\Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)}{\rho^n\,\frac{p_1+n}{2}\,\Gamma\left(\frac{\alpha_{m+1}}{2}\right)\ldots\Gamma\left(\frac{\alpha_{m+n}}{2}\right)}\int_{S(\rho)} f.\,\eta_{m+1}^{\alpha_{m+1}}$$

$$\cdots \eta_{m+n}^{\alpha_{m+n}} \left[\int_0^{\pi} \cdots \int_0^{\pi} \frac{\sin^{\alpha_{m+n}-1} t_{m+n} \cdots \sin^{\alpha_{m+1}-1} t_{m+1}}{\phi^{p_1+\alpha_{m+1}+\cdots+\alpha_{m+n}+n}} \right.$$

$$\left. dt_{m+1} \cdots dt_{m+n} \right] dS$$

burada

$$\phi = \left[\sum_{i=1}^{p_1} (\xi_1 - x_i)^2 + \eta_{m+1}^2 - 2\eta_{m+1} x_{m+1} \cos t_{m+1} + x_{m+1}^2 \right.$$

$$\left. + \cdots + \eta_{m+n}^2 - 2\eta_{m+n} x_{m+n} \cos t_{m+n} + x_{m+n}^2 \right]^{1/2}$$

$$|\rho| = |\xi| \text{ ve } r = |x|$$

fonksiyonu aşağıdaki özeliklere sahiptir.

(a) $v, x_{m+1}, \ldots, x_{m+n}$ değişkenlerine göre çifttir.

(b) $v, B(\rho)$ bölgesinde analitik ve (0.1-3) denklemini sağlar.

(c) $v, S(\rho)$ üzerinde $f(\xi)$ değerini alır.

(d) $x \varepsilon B(\rho)$ ve $z \varepsilon \overline{D(\rho)}$ olmak üzere,

$$\alpha_{m+1} = 1, \alpha_{m+2} = 1, \ldots, \alpha_{m+n} = 1 \text{ ise}$$

$$\lim_{x \to z} v(x) (\log x_{m+1})^{-1} . (\log x_{m+2})^{-1} \cdots (\log x_{m+n})^{-1} = 0$$

$$\alpha_{m+1} > 1, \alpha_{m+2} = 1, \ldots, \alpha_{m+n} = 1 \text{ ise}$$

$$\lim_{x \to z} v(x) \, \eta_{m+1}^{\alpha_{m+1}-1} . (\log \eta_{m+2})^{-1} \cdots (\log \eta_{m+n})^{-1} = 0$$

$$\alpha_{m+1} = 1, \ldots, \alpha_{m+n-1} = 1, \alpha_{m+n} > 1 \text{ ise}$$

$$\lim_{x \to z} v(x) (\log \eta_{m+1})^{-1} \cdots (\log \eta_{m+n-1})^{-1} . \eta_{m+n}^{\alpha_{m+n}-1} = 0$$

$$\alpha_{m+1}>1,\alpha_{m+2}>1,\ldots,\alpha_{m+n}>1$$

$$\lim_{x\to z} v(x).\ \eta_{m+1}^{\alpha_{m+1}-1}.\eta_{m+2}^{\alpha_{m+2}-1}\ \ldots\ \eta_{m+n}^{\alpha_{m+n}-1}=0$$

dır.

KANIT: (a) v nin x_{m+1} değişkenine göre çift olduğunu göstereceğiz.

$$\frac{2^{n-1}(\rho^2-r^2)\ \Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)}{\rho\ \pi^{\frac{p_1+n}{2}}\ \Gamma\left(\frac{\alpha_{m+1}}{2}\right)\ldots\Gamma\left(\frac{\alpha_{m+n}}{2}\right)}$$

çarpanının x_{m+1} değişkenine göre çift olduğu açıktır.

$$\int_0^{\pi}\ldots\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1}t_{m+n}\ldots\sin^{\alpha_{m+1}-1}t_{m+1}\,dt_{m+1}\ldots dt_{m+n}}{\phi^{\,p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}}$$

ifadesindeki ϕ yi yeniden $\left[G+x_{m+1}^2-2\eta_{m+1}x_{m+1}\cos t_{m+1}\right]^{1/2}$ biçiminde yazalım. Burada G,ϕ nin x_{m+1} e bağlı olmayan bütün terimlerini içine alır.

$W=\pi-t_{m+1}$ denilirse $dw=-dt_{m+1}$ olur. Buna göre yukarıdaki integral

$$\int_{\pi}^{0}\int_{0}^{\pi}\ldots\int_{0}^{\pi}\frac{\sin^{\alpha_{m+n}-1}t_{m+n}\ldots\sin^{\alpha_{m+1}-1}(\pi-w)(-dw)}{\left[G+x_{m+1}^2+2\eta_{m+1}x_{m+1}\cos(\pi-w)\right]^{\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}}}$$

$n-1$

biçiminde yazılır. Basit bir sadeleştirme yaparak

$$\int_0^{\pi}\ldots\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1}t_{m+n}\ldots\sin^{\alpha_{m+1}-1}w\ dw\ldots dt_{m+n}}{\left[G+x_{m+1}^2-2\eta_{m+1}x_{m+1}\cos w\right]^{\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}}}$$

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elde edilir. Buradan, x_{m+1} e göre v nin çift olduğu görülür. Aynı yolla

$x_{m+2},\ldots,x_{m+n}$ içinde v nin çift olduğu gösterilir.

(b) v nin $B(\rho)$ bölgesinde $\alpha_{m+1}>1,\ldots,\alpha_{m+n}>1$ için (0.1-2) denklemini sağladığını göstereceğiz. v nin türevlerini bulmak için integral altında türev alınır. Çünkü integral ve türevi alınan integral $S(\rho)$ üzerinde ξ ve $B(\rho)$ içinde verilen herhangi bir noktanın bazı komşuluğundaki x için ξ ve x e göre süreklidirler. Bundan dolayı $v,B(\rho)$ açık bölgesinde C^2(ikinci mertebeden sürekli türevlere sahip) dir. O halde

$$(1.2.6)\qquad L_{\alpha_{m+1},\ldots,\alpha_{m+n}}[v]=\frac{2\Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)}{\rho\,\pi^{\frac{p_1+n}{2}}\,\Gamma\left(\frac{\alpha_{m+1}}{2}\right)\ldots\Gamma\left(\frac{\alpha_{m+n}}{2}\right)}$$

$$\int_{S(\rho)} f(\xi).\eta_{m+1}^{\alpha_{m+1}}\ldots\eta_{m+n}^{\alpha_{m+n}}\int_0^{\pi}\ldots\int_0^{\pi}\sin^{\alpha_{m+n}-1}t_{m+n}$$

$$\ldots\sin^{\alpha_{m+1}-1}t_{m+1}\,L_{\alpha_{m+1},\ldots,\alpha_{m+n}}\Big[(\rho^2-r^2)\phi^{-(p_1\alpha_{m+1}+\ldots+}$$

$$\alpha_{m+n}+n)\Big]dt_{m+1}\ldots dt_{m+n}.dS$$

yazılır. Burada, kolaylık için $\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)}=w$ diyelim. Buna göre

$$L_{\alpha_{m+1},\ldots,\alpha_{m+n}}[\rho^2w-r^2w]=\rho^2\,L_{\alpha_{m+1},\ldots,\alpha_{m+n}}[w]-L_{\alpha_{m+1},\ldots,\alpha_{m+n}}[r^2w]$$

yi hesaplayalım.

$$\frac{\partial w}{\partial x_i}=(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n).\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}\cdot\sum_{i=1}^{p_1}(\xi_i-x_i)$$

$$\frac{\partial w}{\partial x_{m+1}}=(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)\,\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}.(x_{m+1}-\eta_{m+1}\cos t_{m+1})$$

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$$\frac{\partial w}{\partial x_{m+n}} = (p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)} \cdot (x_{m+n}-\eta_{m+n}\ \cos t_{m+n})$$

$$\frac{\partial^2 w}{\partial x_i^2} = (p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+2)$$

$$\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+4)} \cdot \sum_{i=1}^{p_1} (\xi_i-x_i)^2 - \sum_{i=1}^{p_1} (p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)$$

$$\cdot\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}$$

$$\frac{\partial^2 w}{\partial x_{m+1}^2} = (p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)\ (p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)$$

$$\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+4)} (x_{m+1}-\eta_{m+1}\ \cos t_{m+1})^2 - (p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+n)$$

$$\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}$$

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$$\frac{\partial^2 w}{\partial x_{m+n}^2} = (p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)$$

$$\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+4)} \cdot (x_{m+n}-\cos t_{m+n})^2 - (p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n) \cdot$$

$$\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}$$

bu değerleri (0.1-3) denkleminde yerlerine yazılırsa

$$(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)\ (p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2).$$

$$\left[-p_1-1-\ldots-1-\frac{\alpha_{m+1}}{x_{m+1}}\ (x_{m+1}-\eta_{m+1}\ \cos t_{m+1})-\ldots-\frac{\alpha_{m+n}}{x_{m+n}}(x_{m+n}-\eta_{m+n}\cos t_{m+n})\right]$$

$$=(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)\left[\sum_{i=1}^{p_1}(\xi_i-x_i)^2+\eta_{m+1}^2-2\eta_{m+1}.\right.$$

$$\left.x_{m+1}\ \cos t_{m+1}+x_{m+1}^2+\ldots+\ \eta_{m+n}^2-2\eta_{m+n}\ x_{m+n}\ \cos t_{m+n}\ +\ x_{m+n}^2\right]$$

$$.\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+4)}+(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}$$

$$\left[-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)+\frac{\alpha_{m+1}}{x_{m+1}}\ \eta_{m+1}\cos t_{m+1}+\ldots+\frac{\alpha_{m+n}}{x_{m+n}}\ \eta_{m+n}\cos t_{m+n})\right]=$$

$$(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)\left[\sum_{i=1}^{p_1}(\xi_i-x_i)^2+\eta_{m+1}^2-2\eta_{m+1}x_{m+1}\right.$$

$$\left.\cos t_{m+1}+x_{m+1}^2+\ldots+\eta_{m+n}^2-2\eta_{m+n}\cos t_{m+n}+\eta_{m+1}^2\sin^2 t_{m+1}+\ldots+\eta_{m+n}^2\ \sin^2 t_{m+n}\ \right]$$

$$\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+4)}+(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2\ )}$$

$$\left[-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)+(\ \frac{\alpha_{m+1}}{x_{m+1}}\ \eta_{m+1}\cos t_{m+1}+\ldots+\frac{\alpha_{m+n}}{x_{m+n}}\ \eta_{m+n}\cos t_{m+n})\right]$$

$$=(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}\{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)$$

$$+(\ \frac{\alpha_{m+1}}{x_{m+1}}\ \eta_{m+1}\cos t_{m+1}+\ldots+\frac{\alpha_{m+n}}{x_{m+n}}\ \eta_{m+n}\cos t_{m+n})+(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)$$

$$.\phi^{-2}\left[\phi^2-(\eta_{m+1}^2\sin^2 t_{m+1}+\ldots+\eta_{m+n}^2\sin^2 t_{m+n})\right]\ \}=(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)$$

$$\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}\Big[-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)+\Big(\frac{\alpha_{m+1}}{x_{m+1}}\eta_{m+1}\cos t_{m+1}+\ldots+\frac{\alpha_{m+n}}{x_{m+n}}\eta_{m+n}\cos t_{m+n}\Big)+(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n})+2-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)$$

$$\phi^{-2}(\eta_{m+1}^2\sin^2 t_{m+1}+\ldots+\eta_{m+n}^2\sin^2 t_{m+n})\Big]=(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)\ \phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}\Big[\Big(2+\frac{\alpha_{m+1}}{x_{m+1}}\eta_{m+1}\cos t_{m+1}+\ldots+\frac{\alpha_{m+n}}{x_{m+n}}\eta_{m+n}\cos t_{m+n}\Big)-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)\ \phi^{-2}.(\eta_{m+1}^2\sin^2 t_{m+1}+\ldots+\eta_{m+n}^2\sin^2 t_{m+n})\Big]$$

bulunur. Ayrıca,

$$\sum_{i=1}^{p_1}\frac{\partial}{\partial x_i}\left[r^2w\right]=\sum_{i=1}^{p_1}2rw\frac{\partial r}{\partial x_i}+\sum_{i=1}^{p_1}r^2\frac{\partial w}{\partial x_i}$$

$$\frac{\partial}{\partial x_{m+1}}\left[r^2w\right]=2rw\frac{\partial r}{\partial x_{m+1}}+r^2\frac{\partial w}{\partial x_{m+1}}$$

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$$\frac{\partial}{\partial x_{m+n}}\left[r^2w\right]=2rw\frac{\partial r}{\partial x_{m+n}}+r^2\frac{\partial w}{\partial x_{m+n}}$$

$$\sum_{i=1}^{p_1}\frac{\partial^2}{\partial x_i^2}\left[r^2w\right]=\sum_{i=1}^{p_1}2w\Big(\frac{\partial r}{\partial x_i^2}\Big)+\sum_{i=1}^{p_1}4r\frac{\partial w}{\partial r}\frac{\partial r}{\partial x_i}+\sum_{i=1}^{p_1}2rw\frac{\partial^2 r}{\partial x_i^2}+\sum_{i=1}^{p_1}r^2\frac{\partial^2 w}{\partial x_i^2}$$

$$\frac{\partial^2}{\partial x_{m+1}^2}\left[r^2w\right]=2w\Big(\frac{\partial r}{\partial x_{m+1}}\Big)^2+4r\frac{\partial w}{\partial x_{m+1}}\frac{\partial r}{\partial x_{m+1}}+2rw\frac{\partial^2 r}{\partial x_{m+1}^2}+r^2\frac{\partial^2 w}{\partial x_{m+1}^2}$$

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$$\frac{\partial^2}{\partial x_{m+n}^2}\left[r^2 w\right]=2w\left(\frac{\partial r}{\partial x_{m+n}^2}\right)+4r\frac{\partial w}{\partial x_{m+n}}\frac{\partial r}{\partial x_{m+n}}+2rw\frac{\partial^2 r}{\partial x_{m+n}^2}+r^2\frac{\partial^2 w}{\partial x_{m+n}^2}$$

elde edilir. $r=|x|$ olduğundan

$$r=\sqrt{\sum_{i=1}^{p_1} x_i^2+x_{m+1}^2+\ldots+x_{m+n}^2}$$

yazılır. Bu değerleri (0.1-3) denkleminde yerlerine yazarsak

$$\sum_{i=1}^{p_1}\left[2w\left(\frac{x_i}{r}\right)^2+4r\frac{x_i}{r}\frac{\partial w}{\partial x_i}+2rw\frac{r^2-x_i^2}{r^3}+r^2\frac{\partial^2 w}{\partial x_i^2}\right]+2w\left(\frac{x_{m+1}}{r}\right)^2+4r\frac{x_{m+1}}{r}\cdot$$

$$\frac{\partial w}{\partial x_{m+1}}+2rw\frac{r^2-x_{m+1}^2}{r^3}+r^2\frac{\partial^2 w}{\partial x_{m+1}^2}+\ldots+2w\left(\frac{x_{m+n}}{r}\right)^2+4r\frac{x_{m+n}}{r}\frac{\partial w}{\partial x_{m+n}}+$$

$$2rw\frac{r^2-x_{m+n}^2}{r^3}+r^2\frac{\partial^2 w}{\partial x_{m+n}^2}+\frac{\alpha_{m+1}}{x_{m+1}}\left(2rw\frac{x_{m+1}}{r}+r^2\frac{\partial^2 w}{\partial x_{m+1}^2}\right)+\ldots+\frac{\alpha_{m+n}}{x_{m+n}}\left(2rw\frac{x_{m+n}}{r}\right.$$

$$\left.+r^2\frac{\partial^2 w}{\partial x_{m+n}^2}\right)=r^2\left[\sum_{i=1}^{p_1}\frac{\partial^2 w}{\partial x_i^2}+\frac{\partial^2 w}{\partial x_{m+1}^2}+\frac{\alpha_{m+1}}{x_{m+1}}\frac{\partial w}{\partial x_{m+1}}+\ldots+\frac{\partial^2 w}{\partial x_{m+n}^2}+\frac{\alpha_{m+n}}{x_{m+n}}\cdot\right.$$

$$\left.\frac{\partial w}{\partial x_{m+n}}\right]+4\left[\sum_{i=1}^{p_1} x_i\frac{\partial w}{\partial x_i}+x_{m+1}\frac{\partial w}{\partial x_{m+1}}+\ldots+x_{m+n}\frac{\partial w}{\partial x_{m+n}}\right]+p_1\cdot 2w+2w+\ldots+2w+$$

$$2w\alpha_{m+1}+\ldots+2w\alpha_{m+n}=r^2 L_{\alpha_{m+1},\ldots,\alpha_{m+n}}[w]+4\left[\sum_{i=1}^{p_1} x_i\frac{\partial w}{\partial x_i}+x_{m+1}\frac{\partial w}{\partial x_{m+1}}+\right.$$

$$\left.\ldots+x_{m+n}\frac{\partial w}{\partial x_{m+n}}\right]+2w(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)$$

buluruz. Buradan

$$L_{\alpha_{m+1},\ldots,\alpha_{m+n}}(\rho^2 w-r^2 w)=\rho^2\cdot L_{\alpha_{m+1},\ldots,\alpha_{m+n}}[w]-L_{\alpha_{m+1},\ldots,\alpha_{m+n}}[r^2 w](\rho^2-r^2)$$

$$L_{\alpha_{m+1},\ldots,\alpha_{m+n}}[w]-2w(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)-4\Big(\sum_{i=1}^{p_1} x_i\frac{\partial w}{\partial x_i}+x_{m+1}\frac{\partial w}{\partial x_{m+1}}$$

$$+\ldots+x_{m+n}\frac{\partial w}{\partial x_{m+n}}\Big)=(\rho^2-r^2)L_{\alpha_{m+1},\ldots,\alpha_{m+n}}[w]-2(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)$$

$$\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)}\Big[-4\Big(\sum_{i=1}^{p_1} x_i(\xi_i-x_i)-x_{m+1}(x_{m+1}-\eta_{m+1}\cos t_{m+1})-\ldots-$$

$$x_{m+n}(x_{m+n}-\eta_{m+n}\cos t_{m+n})\Big]\,(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n).\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}$$

$$=(\rho^2-r^2)L_{\alpha_{m+1},\ldots,\alpha_{m+n}}[w]-2(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)\,\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}$$

$$\Big[\phi^2+2\Big(\sum_{i=1}^{p_1}\xi_i x_i-x_i^2-x_{m+1}^2+x_{m+1}\eta_{m+1}\cos t_{m+1}-\ldots-x_{m+n}^2+x_{m+n}\eta_{m+n}\cos t_{m+n}\Big)\Big]$$

$$=(\rho^2-r^2)L_{\alpha_{m+1},\ldots,\alpha_{m+n}}[w]-2(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)\,\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}$$

$$.\Big(\sum_{i=1}^{p_1}\xi_i^2-2x_i\xi_i+x_i^2+\eta_{m+1}^2-2\eta_{m+1}x_{m+1}\cos t_{m+1}+x_{m+1}^2+\ldots+\eta_{m+n}^2-2\eta_{m+n}x_{m+n}\cos t_{m+n}$$

$$+x_{m+n}^2+\sum_{i=1}^{p_1}2\xi_i x_i-2x_i^2-2x_{m+1}^2+2\eta_{m+n}\cos t_{m+n}-\ldots-2x_{m+n}^2x_{m+n}\cos t_{m+n}\Big)=(\rho^2-r^2)$$

$$L_{\alpha_{m+1},\ldots,\alpha_{m+n}}[w]-2(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)\,\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}$$

$$\Big[\sum_{i=1}^{p_1}\underbrace{\xi_i^2+\eta_{m+1}^2+\ldots+\eta_{m+n}^2}_{r^2}-\Big(\sum_{i=1}^{p_1}\underbrace{x_i^2+x_{m+1}^2+\ldots+x_{m+n}^2}_{\rho^2}\Big)\Big]=(\rho^2-r^2)L_{\alpha_{m+1},\ldots,}$$

$$\alpha_{m+n}[w]-2(\rho^2-r^2)(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)\,\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}=(\rho^2-r^2)$$

$$\Big[L_{\alpha_{m+1},\ldots,\alpha_{m+n}}[w]-2(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)\,\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}\Big]$$

$$=(\rho^2-r^2)(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)\Big[\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}\Big(\frac{\alpha_{m+1}}{x_{m+1}}\eta_{m+1}\cos t_{m+1}+$$

$$\ldots+\frac{\alpha_{m+n}}{x_{m+n}}\eta_{m+n}\cos t_{m+n}\Big)-\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)$$

(1.2-7) $$(\eta_{m+1}^2\sin^2 t_{m+1}+\ldots+\eta_{m+n}^2\sin^2 t_{m+n})\Big]$$

bulunur. (1.2-7) değerini (1.2-6) integralinde yerine yazalım.

$$\int_0^{\pi}\ldots\int_0^{\pi}\sin^{\alpha_{m+n}-1}t_{m+n}\ldots\sin^{\alpha_{m+1}-1}t_{m+1}(\rho^2-r^2)$$

(1.2-8) $$(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)\Big[\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}$$

$$\Big(\frac{\alpha_{m+1}}{x_{m+1}}\eta_{m+1}\cos t_{m+1}+\ldots+\frac{\alpha_{m+n}}{x_{m+n}}\eta_{m+n}\cos t_{m+n}\Big)-$$

$$\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+4)}(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)$$

$$(\eta_{m+1}^2\sin^2 t_{m+1}+\ldots+\eta_{m+n}^2\sin^2 t_{m+n})\Big]dt_{m+1}\ldots dt_{m+n}$$

elde edilir. Burada (ρ^2-r^2) ve $p_1,\alpha_{m+1},\ldots,\alpha_{m+n}$ sabitirler. Dolayısıyla bunlar integral dışına çıkarılabilir. Buna göre sabitleri kaldırarak integrali yeniden yazalım. Butaktirde

$$\int_0^{\pi}\ldots\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1}t_{m+n}\ldots\sin^{\alpha_{m+1}-1}t_{m+1}\;\frac{\alpha_{m+n}}{x_{m+n}}\eta_{m+n}\cos t_{m+n}}{\phi^{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2}}$$

$$dt_{m+1}\ldots dt_{m+n}+\ldots+\int_0^{\pi}\ldots\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1}t_{m+n}}{}$$

$$\frac{\ldots \sin^{\alpha_{m+1}-1} t_{m+1}\left(\frac{\alpha_{m+1}}{x_{m+1}}\eta_{m+1}\cos t_{m+1}\right)dt_{m+1}\ldots dt_{m+n}}{\phi^{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2}}$$

$$(1.2\text{-}9)\quad -\int_0^{\pi}\ldots\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1} t_{m+n}\ldots\sin^{\alpha_{m+1}-1} t_{m+1}(\eta_{m+n}^2\sin^2 t_{m+n})}{\phi^{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+4}}$$

$$(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)\,dt_{m+1}\ldots dt_{m+n}-\ldots$$

$$-\int_0^{\pi}\ldots\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1} t_{m+n}\ldots\sin^{\alpha_{m+1}-1} t_{m+1}(\eta_{m+1}^2\sin^2 t_{m+1})}{\phi^{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+4}}$$

$$(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)dt_{m+1}\ldots dt_{m+n}$$

olur. Bu integrallere sırasıyla $A_1,\ldots,A_n$ ve $B_1,\ldots,B_n$ diyelim. Göstereceğiz ki $A_1-B_1+\ldots+A_n-B_n=0$ dır. Bunun için ilkin t_{m+1} e bağlı olan A_1 integralini gözönüne alalım.

$$\int_0^{\pi}\ldots\int_0^{\pi}\sin^{\alpha_{m+n}-1} t_{m+n}\ldots\sin^{\alpha_{m+2}-1} t_{m+2}$$

$$(1.2\text{-}10)\quad\left[\int_0^{\pi}\frac{\frac{\alpha_{m+1}}{x_{m+1}}\eta_{m+1}\cos t_{m+1}\cdot\sin^{\alpha_{m+1}-1} t_{m+1}\,dt_{m+1}}{\phi^{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2}}\right]$$

integraline kısmi integrasyon uygulanırsa

$$u=\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)}$$

$$du=-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)\,\phi^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+4)}$$

$$\cdot(\eta_{m+1}x_{m+1}\sin t_{m+1})\,dt_{m+1}$$

$$dv=\frac{\alpha_{m+1}}{x_{m+1}}\eta_{m+1}\cos t_{m+1}.\sin^{\alpha_{m+n}-1}t_{m+1}.dt_{m+1}$$

$$v=\frac{\eta_{m+1}}{x_{m+1}}\sin^{\alpha_{m+1}}t_{m+1}$$

olur. Bu değerleri integralde yerlerine yazarak

$$\int_0^{\pi}\cdots\int_0^{\pi}\sin^{\alpha_{m+n}-1}t_{m+n}\ldots\sin^{\alpha_{m+2}-1}t_{m+2}$$

$$\left[\frac{\frac{\eta_{m+1}}{x_{m+1}}\sin^{\alpha_{m+1}}t_{m+1}}{\phi_{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2}}\right]_0^{\pi}-\int_0^{\pi}\left(\frac{\frac{\eta_{m+1}}{x_{m+1}}\sin^{\alpha_{m+1}-1}t_{m+1}}{\phi_{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+4}}\right.$$

$$\left.\left[-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)\right]\eta_{m+1}x_{m+1}\sin t_{m+1}dt_{m+1}\right)$$

$$dt_{m+2}\ldots dt_{m+n}.=\int_0^{\pi}\cdots\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1}t_{m+n}\cdot\cdot}{\phi_{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+4}}$$

$$.\sin^{\alpha_{m+1}-1}t_{m+1}(\eta_{m+1}^2\sin^2 t_{m+1})(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n+2)dt_{m+1}\ldots dt_{m+n}$$

bulunur. Buradan açık olarak $A_1-B_1=0$ olduğu görülür. Çünkü birinci integrasyonun değeri sıfır, ikinci integral de B_1 e eşit çıkar. Yukarıda yapılan işlemler aynı şekilde $t_{m+2},\ldots,t_{m+n}$ değişkenlerine bağlı olarak tekrarlanırsa, $A_2-B_2=0,\ldots,A_n-B_n=0$ bulunur. Buradan (1.2-7) integrali

$$\frac{2^{n-1}\Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)}{\rho\pi^{\frac{p_1+n}{2}}\Gamma\left(\frac{\alpha_{m+1}}{2}\right)\ldots\Gamma\left(\frac{\alpha_{m+n}}{2}\right)}$$

$$\int_{S(\rho)} \eta_{m+1}^{\alpha_{m+1}} \dots \eta_{m+n}^{\alpha_{m+n}} \cdot f(\xi).0.dS=0$$

olur. Dolayısıyla,

$$L_{\alpha_{m+1},\dots,\alpha_{m+n}}[v] = 0$$

olduğu görülür.

(c) v integralinin $S(\rho)$ üzerinde $f(\xi)$ sınır değerini aldığını göstereceğiz. Bunun için kanıtı ([13] sayfa 47) de verilen yardıncı teoremi kullanacağız.

Yardımcı teorem: Farzedelim ki $\alpha_{m+1}>1,\dots,\alpha_{m+n}>1$ olsun. Bu taktirde, $B(\rho)$ daki her x için

$$\int_{S(\rho)} \eta_{m+1}^{\alpha_{m+1}} \dots \eta_{m+n}^{\alpha_{m+n}} .K(x,\xi)dS\equiv 1$$

dir.

Şimdi bir $\xi_0 \varepsilon S(\rho)$ sabit noktasını gözönüne alarak $\epsilon > 0$ seçelim. $S(\rho)$ da $f(\xi)$ fonksiyonunun sürekli olması nedeniyle, $S(\rho)$ daki her ξ ve $|\xi-\xi_0|<\delta$ için $|f(\xi)-f(\xi_0)|< \frac{\varepsilon}{2}$ olacak şekilde bir $\delta > 0$ sayısı vardır. $N(\delta,\xi_0)$, ξ_0 merkezli ve δ yarıçaplı bir komşuluk göstersin. Bu taktirde,

$$S(\rho,\delta)=N(\delta,\xi_0)\cap S(\rho) \text{ ve } B(\rho,\delta)=N(\delta,\xi_0)\cap B(\rho)^+$$

olur. $\xi_0 \varepsilon [\, S(\rho) - S(\rho,\delta)\,]$ olmak üzere $x \to \xi_0$ için $K(x,\xi) \to 0$ (ξ ye göre düzgün yakınsak) dır. Bu aşağıdaki şekilde gösterilebilir.

$$(1.2\text{-}12)\quad K(x,\xi)= \frac{2^{n-1}(\rho^2-r^2)\Gamma\left(\frac{p_1+\alpha_{m+1}+\dots+\alpha_{m+n}+n}{2}\right)}{\rho\,\pi^{\frac{p_1+n}{2}}\,\Gamma\left(\frac{\alpha_{m+1}}{2}\right)\dots\Gamma\left(\frac{\alpha_{m+n}}{2}\right)} \int_0^{\pi}\dots\int_0^{\pi}$$

$$\frac{\sin^{\alpha_{m+n}-1} t_{m+n} \ldots \sin^{\alpha_{m+1}-1} t_{m+1}\, dt_{m+1} \ldots dt_{m+n}}{\left[\left|\xi-x\right|^2 + 2\eta_{m+1}x_{m+1}(1-\cos t_{m+1}) + \ldots + 2\eta_{m+n}x_{m+n}(1-\cos t_{m+n})\right]^{\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}}}$$

diyelim. $|\xi_0 - x| < \delta/2$ olmak üzere $x \in B(\rho, \frac{\delta}{2})$ olsun. Bu taktirde, her

$\xi \in [S(\rho) - S(\rho,\delta)]$ için $|\xi - x| > \frac{\delta}{2}$ ve

$$K(x,\xi) < \frac{2^{n-1}(\rho^2-r^2)\Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)}{\rho^{\frac{p_1+n}{2}}\pi\,\Gamma\left(\frac{\alpha_{m+1}}{2}\right)\ldots\Gamma\left(\frac{\alpha_{m+n}}{2}\right)} \int_0^{\pi}\ldots\int_0^{\pi}$$

$$\frac{\sin^{\alpha_{m+n}-1} t_{m+n} \ldots \sin^{\alpha_{m+1}-1} t_{m+1} dt_{m+1} \ldots dt_{m+n}}{\left[\left|\left(\frac{\delta}{2}\right)^2\right| + 2\eta_{m+1}x_{m+1}(1-\cos t_{m+1}) + \ldots + 2\eta_{m+n}x_{m+n}(1-\cos t_{m+n})\right]^{\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}}}$$

olur. Yukardaki koşullara göre bu ifade,

$$\frac{2^{n-1}(\rho^2-r^2)\Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)}{\rho.\pi^{\frac{p_1+n}{2}}\Gamma\left(\frac{\alpha_{m+1}}{2}\right)\ldots\Gamma\left(\frac{\alpha_{m+n}}{2}\right)} \int_0^{\pi}\ldots\int_0^{\pi}$$

$$\frac{\sin^{\alpha_{m+n}-1} t_{m+n} \ldots \sin^{\alpha_{m+1}-1} t_{m+1} dt_{m+1} \ldots dt_{m+n}}{\left[\left(\frac{\delta}{2}\right)^2\right]^{\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}}}$$

ifadesinden daha küçüktür. Bu da

$$\frac{2^{n-1}(\rho^2-r^2)\Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)}{\rho\,\pi^{\frac{p_1+n}{2}}\,\Gamma\left(\frac{\alpha_{m+1}}{2}\right)\ldots\Gamma\left(\frac{\alpha_{m+n}}{2}\right)\left[\frac{\delta}{2}\right]^{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}}$$

$$\left[\frac{\sqrt{\pi}\,\Gamma\left(\frac{\alpha_{m+1}}{2}\right)}{\Gamma\left(\frac{\alpha_{m+1}+1}{2}\right)}\right]\cdots\left[\frac{\sqrt{\pi}\,\Gamma\left(\frac{\alpha_{m+n}}{2}\right)}{\Gamma\left(\frac{\alpha_{m+n}+1}{2}\right)}\right]$$

ifadesine eşittir. Bundan dolayı

$$K(x,\xi)<\frac{2^{n-1}(\rho^2-r^2)\Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)}{\rho\,\pi^{\frac{p_1}{2}}\,\Gamma\left(\frac{\alpha_{m+1}+1}{2}\right)\ldots\Gamma\left(\frac{\alpha_{m+n}+1}{2}\right)\left[\frac{\delta}{2}\right]^{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}}$$

olup, bu ifade, $x \to \xi_0$ ve $r \to \rho$ için sıfıra gider.

Böylece, her $x\in B(\rho,\delta_1)$ için $\delta > \delta_1 > 0$ olacak şekilde bir δ_1 seçilebilir. Buradan,

$$\int_{S(\rho)-S(\rho,\delta_1)} K(x,\xi)dS < \frac{\varepsilon}{4M}$$

olduğu gerçeklenir. Burada $M=\underset{\xi\in S(\rho)}{\text{Max}}\left[\eta_{m+1}^{\alpha_{m+1}}\ldots\eta_{m+n}^{\alpha_{m+n}}|f(\xi)|\right]$ alınmıştır.

Eğer $x\in B(\rho,\delta_1)$ ise

$$\left|v(x)-(\xi_0)\right|=\left|\int_{S(\rho)}\eta_{m+1}^{\alpha_{m+1}}\ldots\eta_{m+n}^{\alpha_{m+n}}K(x,\xi)f(\xi)dS-f(\xi_0).1\right|$$

$$=\left|\int_{S(\rho)}\eta_{m+1}^{\alpha_{m+1}}\ldots\eta_{m+n}^{\alpha_{m+n}}K(x,\xi)|f(\xi)-f(\xi_0)|dS\right| \leqslant$$

$$\left| \int_{S(\rho,\delta)} \eta_{m+1}^{\alpha_{m+1}} \ldots \eta_{m+n}^{\alpha_{m+n}} K(x,\xi)[f(\xi)-f(\xi_0)]dS \right| + \left| \int_{S(\rho)-S(\rho,\delta)} \eta_{m+1}^{\alpha_{m+1}} \right.$$

$$\left. \ldots \eta_{m+n}^{\alpha_{m+n}} [f(\xi)-f(\xi_0)] K(x,\xi) dS \right| \leq \int_{S(\rho,\delta)} \eta_{m+1}^{\alpha_{m+1}} \ldots \eta_{m+n}^{\alpha_{m+n}}$$

$$K(x,\xi)|f(\xi)-f(\xi_0)|dS + \int_{S(\rho)-S(\rho,\delta)} |\eta_{m+1}^{\alpha_{m+1}} \ldots \eta_{m+n}^{\alpha_{m+n}} f(\xi) - \eta_{m+1}^{\alpha_{m+1}}$$

$$\ldots \eta_{m+n}^{\alpha_{m+n}} f(\xi_0)| K(x,\xi) dS < \frac{\varepsilon}{2} \int_{S(\rho,\delta)} \eta_{m+1}^{\alpha_{m+1}} \ldots \eta_{m+n}^{\alpha_{m+n}} K(x,\xi)\, dS$$

$$+ \int_{S(\rho)-S(\rho,\delta)} \left[\eta_{m+1}^{\alpha_{m+1}} \ldots \eta_{m+n}^{\alpha_{m+n}} |f(\xi)| + \eta_{m+1}^{\alpha_{m+1}} \ldots \eta_{m+n}^{\alpha_{m+n}} |f(\xi_0)| \right] K(x,\xi)\, dS$$

$$< \frac{\varepsilon}{2} \cdot 1 + 2M \int_{S(\rho)-S(\rho,\delta)} K(x,\xi)\, dS < \frac{\varepsilon}{2} + 2M \cdot \frac{\varepsilon}{4M} = \varepsilon$$

bulunur.

(d) Eğer $\alpha_{m+1}>1,\ldots,\alpha_{m+n}>1$ $x\varepsilon B(\rho)^+, z\varepsilon\overline{D(\rho)}$ ise göstereceyiz ki

$$\lim_{x\to z} x_{m+1}^{\alpha_{m+1}-1} \ldots x_{m+n}^{\alpha_{m+n}-1} v(x)=0$$

olur. $\alpha_{m+1},\ldots,\alpha_{m+n}$ lerin diğer durumları içinde benzer işlemler yapılacağından yalnız bu hali göstereceğiz. (1.2-5) den

$$\lim_{\xi\to\xi_0} f(\xi).\, \eta_{m+1}^{\alpha_{m+1}-1} \ldots \eta_{m+n}^{\alpha_{m+n}-1} = 0$$

verilmiştir. $\varepsilon>0$ ve ξ_0, $S(\rho) \cap D$ kümesinin herhangi bir noktası olmak üzere $|\xi-\xi_0|<\delta_1$ olacak şekilde ε ve ξ_0 a bağlı bir $\delta_1>0$ sayısı vardır. Bu taktirde

$$\left| f(\xi)\ \eta_{m+1}^{\alpha_{m+1}-1} \ldots \eta_{m+1}^{\alpha_{m+n}-1} \right| < \varepsilon$$

olur.

$|\xi-\xi_0|<\delta$ olacak şekilde $S(\rho)$ üzerinde ξ lerin oluşturduğu $S(\rho,b)$ cümlesini gözönüne alalım. ξ bir δ üzaklığından ya da D den daha küçüktür. Eğer $\xi \notin S(\rho,b)$ ise buna göre $\eta_{m+1},\ldots,\eta_{m+n}>\delta$ olur. Buradan

$$\left| x_{m+1}^{\alpha_{m+1}-1} \ldots x_{m+n}^{\alpha_{m+n}-1} v(x) \right| = \left| x_{m+1}^{\alpha_{m+1}-1} \ldots x_{m+n}^{\alpha_{m+n}-1} \right|$$

$$(1.2\text{-}13) \qquad \left| \int_{S(\rho)} \eta_{m+1}^{\alpha_{m+1}} \ldots \eta_{m+n}^{\alpha_{m+n}} . f(\xi) . K(x,\xi)\ dS \right| <$$

$$\left| x_{m+1}^{\alpha_{m+1}-1} \ldots x_{m+n}^{\alpha_{m+n}-1} \right| \int_{S(\rho,b)} \eta_{m+1}^{\alpha_{m+1}} \ldots \eta_{m+n}^{\alpha_{m+n}} . f(\xi) .$$

$$K(x,\xi)\ dS| + |x_{m+1}^{\alpha_{m+1}-1} \ldots x_{m+n}^{\alpha_{m+n}-1} \int_{S(\rho)-S(\rho,b)} \eta_{m+1}^{\alpha_{m+1}} .$$

$$.. \eta_{m+n}^{\alpha_{m+n}} . f(\xi) K(x,\xi)\ dS \Big|$$

integrasyon bölgesi $S(\rho)-S(\rho,b)$ olursa

$$0< \eta_{m+1}^{\alpha_{m+1}} \ldots \eta_{m+n}^{\alpha_{m+n}} |f(\xi)| < M$$

olur. Eğer $x_{m+1} \varepsilon \frac{\delta}{2},\ldots,x_{m+n} \varepsilon \frac{\delta}{2}$ olarak belitirsek buna göre $x,D(\rho)$ da $\frac{\delta}{2}$ nin komşuluğunda olur. Bu taktirde,

$$(1.2\text{-}14) \qquad K(x,\xi)= \frac{2^{n-1}(\rho^2-r^2)\Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)}{\rho.\pi^{\frac{p_1+n}{2}}\ \Gamma\left(\frac{\alpha_{m+1}}{2}\right)\ldots\Gamma\left(\frac{\alpha_{m+n}}{2}\right)}$$

$$\int_0^{\pi}\cdots\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1}t_{m+n}\cdots\sin^{\alpha_{m+1}-1}t_{m+1}\,dt_{m+1}\cdots}{\left[|x-\xi|^2+2\eta_{m+1}x_{m+1}(1-\cos t_{m+1})+\ldots+\right.}$$

$$\frac{.\,dt_{m+n}}{\left.2\eta_{m+n}x_{m+n}(1-\cos t_{m+n})\right]^{\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}}{2}}}$$

$$<\frac{2^{n-1}(\rho^2-r^2)\Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)}{\rho\,\pi^{\frac{p_1+n}{2}}\Gamma\left(\frac{\alpha_{m+1}}{2}\right)\cdots\Gamma\left(\frac{\alpha_{m+n}}{2}\right)}\int_0^{\pi}\cdots\int_0^{\pi}$$

$$\frac{\sin^{\alpha_{m+n}-1}t_{m+n}\cdots\sin^{\alpha_{m+1}-1}t_{m+1}\,dt_{m+1}\cdots dt_{m+n}}{\left[\left(\frac{\delta}{2}\right)^2\right]^{\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}}{2}}}$$

$$=\frac{2^{n-1}(\rho^2-r^2)\Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)}{\rho.\pi^{\frac{p_1}{2}}\Gamma\left(\frac{\alpha_{m+1}}{2}\right)\cdots\Gamma\left(\frac{\alpha_{m+n}}{2}\right)\left[\left(\frac{\delta}{2}\right)^2\right]^{\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}}{2}}}$$

$$\cdot\left[\frac{\sqrt{\pi}\,\Gamma\left(\frac{\alpha_{m+1}}{2}\right)}{\Gamma\left(\frac{\alpha_{m+1}+1}{2}\right)}\right]\cdots\left[\frac{\sqrt{\pi}\,\Gamma\left(\frac{\alpha_{m+n}}{2}\right)}{\Gamma\left(\frac{\alpha_{m+n}+1}{2}\right)}\right]$$

$$=\frac{2^{n-1}(\rho^2-r^2)\,\Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)}{\rho\,\pi^{\frac{p_1}{2}}\Gamma\left(\frac{\alpha_{m+1}}{2}\right)\cdots\Gamma\left(\frac{\alpha_{m+n}}{2}\right)\left[\left(\frac{\delta}{2}\right)^2\right]^{\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}}}$$

olur. $r<\rho$ olduğundan

$$\frac{\rho^2-r^2}{\rho} < \frac{\rho^2}{\rho} = \rho$$

olur ve böylece

$$(1.2\text{-}15)\quad K(x,\xi) < \frac{2^{n-1}\rho\ \Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)}{\pi^{\frac{p_1}{2}}\ \Gamma\left(\frac{\alpha_{m+1}}{2}\right)\ldots\Gamma\left(\frac{\alpha_{m+n}}{2}\right)\left[\frac{\delta}{2}\right]^{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}} = H$$

yazılır. Burada H bir δ sabitine bağlıdır. Bundan dolayı ε a da bağlı olur. Bu taktirde,

$$\left| x_{m+1}^{\alpha_{m+1}-1}\ldots x_{m+n}^{\alpha_{m+n}-1} \int_{S(\rho)-S(\rho,b)} \eta_{m+1}^{\alpha_{m+1}}\ldots\eta_{m+n}^{\alpha_{m+n}} f(\xi)K(x,\xi)dS \right|$$

$$< x_{m+1}^{\alpha_{m+1}-1}\ldots x_{m+n}^{\alpha_{m+n}-1} \int_{S(\rho)-S(\rho,b)} \eta_{m+1}^{\alpha_{m+1}}\ldots\eta_{m+n}^{\alpha_{m+n}} \left|f(\xi)\right| K(x,\xi)dS$$

$$< x_{m+1}^{\alpha_{m+1}-1}\ldots x_{m+n}^{\alpha_{m+n}-1}\ M.H \int_{S(\rho)-S(\rho,b)} dS, x_{m+1} < \frac{\delta}{2}, \ldots, x_{m+n} < \frac{\delta}{2}$$

yazılır. Buradan integralin değerinin $S(\rho)$ nun ölçüm yüzeyinde ω dan daha küçük olduğu görülür. Böylece

$$\left| x_{m+1}^{\alpha_{m+1}-1}\ldots x_{m+n}^{\alpha_{m+n}-1} \int_{S(\rho)-S(\rho,b)} \eta_{m+1}^{\alpha_{m+1}}\ldots\eta_{m+n}^{\alpha_{m+n}}.f(\xi)K(x,\xi)dS \right|$$

$$(1.2\text{-}16)\qquad < x_{m+1}^{\alpha_{m+1}-1}\ldots x_{m+n}^{\alpha_{m+n}-1}\ M.H\ W$$

bulunur. x, $D(\rho)$ nın herhangi bir noktasına yaklaştığında (1.2-16) denklemi sıfıra gider.

$S(\rho,b)$ üzerinden integrasyonun değeri ile (1.2-13) denkleminin terimlerini gözönüne alarak, $z_0 \varepsilon\ D(\rho)$ seçilsin. $z_0 \notin S(\rho)$ olduğundan

her $\xi\varepsilon\ S(\rho)$ için $|\xi-z_0|>0$ olur. Gözönüne alınan noktalarda $|\xi-z_0| > \rho-|z_0|$ olur. $S=\rho-|z_0|$ tanımlıyalım. Burada J nin bir z_0 sabitine bağlı olduğu açıktır. O zaman J/2 nin bir z_0 komşuluğundaki her x için $|x-\xi| > \frac{j}{2}$ olur. Aynı düşünceyle (1.2-14) bağıntısından (1.2-15) elde edilir. Buradan

$$K(x,\xi)< \frac{2\rho\ \Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)}{\pi^{\frac{p_1}{2}}\ \Gamma\left(\frac{\alpha_{m+1}+1}{2}\right)\ldots\Gamma\left(\frac{\alpha_{m+n}+1}{2}\right)\left[\frac{j}{2}\right]^{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}} =g$$

olur. Böylece $|x-z_0|< \frac{j}{2}$ eşitsizliğini sağlayan x ler için $K(x,\xi)$ ye bağlı bir g sabiti ile sınırlıdır.

Mademki $S(\rho)$ üzerinde $\alpha_{m+1}>0,\ldots,\alpha_{m+n}>0$ buna göre

$$\left|\eta_{m+1}^{\alpha_{m+1}-1}\ldots\eta_{m+n}^{\alpha_{m+n}-1}\ .f(\xi)\right|=\eta_{m+1}^{\alpha_{m+1}-1}\ldots\eta_{m+n}^{\alpha_{m+n}-1}\left|f(\xi)\right|$$

olacağı açıktır, O zaman eğer $\xi\varepsilon S(\rho,b)$ ise

$$\left|f(\xi)\right| < \varepsilon\ .\ \frac{1}{\eta_{m+1}^{\alpha_{m+1}-1}\ldots\eta_{m+n}^{\alpha_{m+n}-1}}$$

olur. Ayrıca $x\varepsilon B(\rho)^+$ olduğundan $x_{m+1}>0,\ldots,x_{m+n}>0$ buna göre

$$x_{m+1}^{\alpha_{m+1}-1}\ldots x_{m+n}^{\alpha_{m+n}-1}\left|f(\xi)\right| <\varepsilon\ \left(\frac{x_{m+1}}{\eta_{m+1}}\right)^{\alpha_{m+1}-1}\ldots\left(\frac{x_{m+n}}{\eta_{m+n}}\right)^{\alpha_{m+n}-1}$$

eğer $\xi\varepsilon\ S(\rho,b)$ ise bundan dolayı

$$\left|x_{m+1}^{\alpha_{m+1}-1}\ldots x_{m+n}^{\alpha_{m+n}-1}\int_{S(\rho,b)}\eta_{m+1}^{\alpha_{m+1}}\ldots\eta_{m+n}^{\alpha_{m+n}}\ f(\xi)K(x,\xi)dS\right|$$

$$\leq \int_{S(\rho,b)} x_{m+1}^{\alpha_{m+1}-1} \ldots x_{m+1}^{\alpha_{m+n}-1} \left| f(\xi) \right| \eta_{m+1}^{\alpha_{m+1}} \ldots \eta_{m+n}^{\alpha_{m+n}} K(x,\xi) dS$$

$$< \int_{S(\rho,b)} \varepsilon . \left(\frac{x_{m+1}}{\eta_{m+1}} \right)^{\alpha_{m+1}-1} \ldots \left(\frac{x_{m+n}}{\eta_{m+n}} \right)^{\alpha_{m+n}-1} . \eta_{m+1}^{\alpha_{m+1}} \ldots \eta_{m+n}^{\alpha_{m+n}} . g . dS$$

burada $\eta_{m+1}^{\alpha_{m+1}}, \ldots, \eta_{m+n}^{\alpha_{m+n}} < \rho^{\alpha_{m+1}}, \ldots, \rho^{\alpha_{m+n}}$ ve bu

$$(1.2\text{-}17) \qquad \varepsilon . x_{m+1}^{\alpha_{m+1}-1} \ldots x_{m+n}^{\alpha_{m+n}-1} . \rho^{\alpha_{m+1}+\ldots+\alpha_{m+n}} . g \int_{S(\rho,b)} dS$$

ifadesinden daha küçüktür. (1.2-17) ifadesi x, z_0 a gittiğinde $z_0 \in B(\rho)$ olduğundan sıfıra gider. $w \in S(\rho) \cap D$ için gösterilirki

$$\lim_{x \to w} v(x) \, x_{m+1}^{\alpha_{m+1}-1} \ldots x_{m+n}^{\alpha_{m+n}-1} = 0$$

olur. W_0 limiti sıfırdan farklı olmak üzere $x \to w_0$ olacak şekilde bir $w_0 \in S(\rho) \cap D$ noktasının varlığını kabul edelim. Bu taktirde (b) şıkkındaki v nin sürekliliğinden

$$\left| v(x) . x_{m+1}^{\alpha_{m+1}-1} \ldots x_{m+n}^{\alpha_{m+n}-1} \right| > \varepsilon$$

olacak şekilde eğer $x \in N$ ise, v_0 ın bir N komşuluğu ve bir $\varepsilon > 0$ sayısı vardır. $N \cap S(\rho)$ arakesitinin boş olmadığı açıktır. Çünkü $w_0 \in S(\rho)$ dir. (c) şıkkı gereğince $v(x)$ fonksiyonu $S(\rho)$ üzerinde $f(\xi)$ değerini alır. O zaman $\xi \in N \cap S(\rho)$ için

$$\lim_{\xi \to w_0} f(\xi) \, \eta_{m+1}^{\alpha_{m+1}-1} \ldots \eta_{m+n}^{\alpha_{m+n}-1} \neq 0$$

olur. Bu ise (5) denklemi ile bir çelişkidir. Bundan dolayı

$w \in S(\rho) \cap D$ için,

$$\lim_{x \to w} v(x) . x_{m+1}^{\alpha_{m+1}-1} \ldots x_{m+n}^{\alpha_{m+n}-1} = 0$$

olmalıdır. (1.2-15), (1.2-16) ve (1.2-17) denklemlerinden

$$\lim_{x \to z \in D(\rho)} v(x) . x_{m+1}^{\alpha_{m+1}-1} \ldots x_{m+n}^{\alpha_{m+n}-1} = 0$$

bulunur.

1.3 EKSİLTME (DESCENT) YÖNTEMİ

Bu kesimde (0.1-3) denklemi için çözülen Dirichlet problemin-den yararlanarak aynı sınır değerlerinde eksiltme yöntemiyle (0.1-2) denklemi için bir çözüm elde edeceğiz. Bunun için

(1.3-1) $v(x_1, x_2, x_3, \ldots, x_{m+n}) = u(x_2, x_3, \ldots, x_{m+n}) \cos \lambda x_1$

fonksiyonunu tanımlayalım. Burada λ sabit, $0 < x_1 < \pi$ ve $x_2 > 0, \ldots, x_{m+n} > 0$ dır. v verilen bölgede tanımlı birinci ve ikinci mertebeden türev-lere sahip sürekli bir fonksiyondur. (1.3-1) fonksiyonunun türevle-ri alınarak (0.1-3) denkleminde yerlerine yazılırsa

$$\left(\frac{\partial^2 u}{\partial x_2^2}\right) + \frac{\partial^2 u}{\partial x_3^2} + \ldots + \frac{\partial^2 u}{\partial x_{m+1}^2} + \ldots + \frac{\partial^2 u}{\partial x_{m+n}^2} +$$

(1.3-2) $$\frac{\alpha_{m+1}}{x_{m+1}} \frac{\partial u}{\partial x_{m+1}} + \ldots + \frac{\alpha_{m+n}}{x_{m+n}} \frac{\partial u}{\partial x_{m+n}} - \lambda^2 u) \cos \lambda x_1 = 0$$

denklemi elde edilir. $x_1 = 0$ için (2) denklemi (0.1-2) denklemini verir. Burada

$$\frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_{m+1}^2} + \ldots + \frac{\partial^2 u}{\partial x_{m+n}^2}$$

(1.3-3) $$+\frac{\alpha_{m+1}}{x_{m+1}}\frac{\partial u}{\partial x_{m+1}}+\ldots+\frac{\alpha_{m+n}}{x_{m+n}}\frac{\partial u}{\partial x_{m+n}}$$

denklemi gözönüne alınarak ve 1.1 Kesimde yapılan eksenel simetriden yararlanılarak

(1.3-4) $$U=\frac{2^{n-1}(\rho^2-r^2)\Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)}{\rho\,\pi^{\frac{p_1+n}{2}}\Gamma\left(\frac{\alpha_{m+1}}{2}\right)\ldots\Gamma\left(\frac{\alpha_{m+n}}{2}\right)}\int_{S(\rho)}f(\xi).$$

$$\eta_{m+1}^{\alpha_{m+1}}\ldots\eta_{m+n}^{\alpha_{m+n}}\left[\int_0^{\pi}\ldots\int_0^{\pi}\frac{\sin^{\alpha_{m+n}-1}t_{m+n}\ldots}{\sigma^{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}}\right.$$

$$\left.\sin^{\alpha_{m+1}-1}t_{m+1}.dt_{m+1}\ldots dt_{m+n}\right]dS$$

Poisson integrali elde edilir. Burada dS, p_1+n boyutlu çeyrek küre yüzey elemanı ve σ

$$\sigma=\left[\sum_{i=2}^{p_1}(\xi_i-x_i)^2+\eta_{m+1}^2-2\eta_{m+1}x_{m+1}\cos t_{m+1}+x_{m+1}^2\right.$$

$$\left.+\ldots+\eta_{m+n}^2-2\eta_{m+n}x_{m+n}\cos t_{m+n}+x_{m+n}^2\right]^{1/2}$$

biçimindedir. (1.3-1) ve (0.1-3) denklemlerinden (1.3-5) $L[v]=L[u]\cos\lambda x_1-\lambda^2 u\cos\lambda x_1$ yazılır. (4) değeri (5) de yerine yazılarak

$$\int_0^{\pi}\left[\frac{2^{n-1}\Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)}{\rho.\pi^{\frac{p_1+n}{2}}\Gamma\left(\frac{\alpha_{m+1}}{2}\right)\ldots\Gamma\left(\frac{\alpha_{m+n}}{2}\right)}\int_{S(\rho)}f(\xi).\right.$$

$$\eta_{m+1}^{\alpha_{m+1}}\ldots\eta_{m+n}^{\alpha_{m+n}}.\int_0^{\pi}\ldots\int_0^{\pi}\sin^{\alpha_{m+n}-1}t_{m+n}\ldots\sin^{\alpha_{m+1}-1}t_{m+1}.$$

$$L_{\alpha_{m+1},\ldots,\alpha_{m+n}}(\rho^2-r^2)\sigma^{-(p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n)}.dt_{m+1}$$

$$\ldots dt_{m+n}\Big]\cos x_1 dx_1 . dS - \int_0^{\pi} \lambda^2 u \cos\lambda x_1 dx_1 = 0$$

bulunur. Çünkü köşeli parantezin içindeki kısım 1.2 Kesimdeki Teorem (b) de yapılan benzer işlemlerden dolayı sıfır olur. İkinci integralin değeri de sıfır çıkar. Buna göre bölge içinde (1.3-1) fonksiyonu (0.1-2) denkleminin bir çözümü olur. Ayrıca

$$K(x,\xi) = \int_0^{\pi} \cos\lambda x_1 \left[\frac{2^{n-1}(\rho^2 - r^2)\Gamma\left(\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}\right)}{\rho\,\pi^{\frac{p_1+n}{2}}\,\Gamma\left(\frac{\alpha_{m+1}}{2}\right)\ldots\Gamma\left(\frac{\alpha_{m+n}}{2}\right)} \right.$$

$$\int_0^{\pi} \ldots \int_0^{\pi} \frac{\sin^{\alpha_{m+n}-1} t_{m+n} \ldots \sin^{\alpha_{m+1}-1} t_{m+1}\, dt_{m+1} \ldots}{\left[|\xi - x|^2 + 2\eta_{m+1}(1-\cos t_{m+1}) + \ldots + \right.}$$

$$\left. \frac{dt_{m+n}}{2\eta_{m+n}(1-\cos t_{m+n})\Big]^{\frac{p_1+\alpha_{m+1}+\ldots+\alpha_{m+n}+n}{2}}} \right] dx_1$$

biçiminde alınır. Buradan 1.2 Kesimdeki Teorem (c) ye göre u fonksiyonunun sınırdaki değeri f olur. Buna göre (a) fonksiyonu (0.1-2) denkleminin bir çözümü olur.

İKİNCİ BÖLÜM

BİR ELEMANTER ÇÖZÜM

2.1 DEĞİŞKENLERİN AYRILMASI

Bu bölümde (0.1-4) denklemini değişkenlerin ayrılması yöntemiyle çözeceyiz. Bunun için

$$(2.1\text{-}1)\quad w(x_1,x_2,\ldots,x_n)=w_1(x_1).w_2(x_2)\ldots w_n(x_n)=\prod_{i=1}^{n} w_i(x_i)$$

biçiminde bir çözüm arayalım. (1) bağıntısı (0.1-4) denklemini sağlamalıdır. Buradan,

$$\frac{\partial w}{\partial x_1}=\frac{dw_1(x_1)}{dx_1}\,w_2(x_2).\,w_3(x_3)\ldots w_n(x_n)$$

$$\frac{\partial w}{\partial x_2}\quad\frac{dw_2(x_2)}{dx_2}\,w_1(x_1)\,.\,w_3(x_3)\ldots w_n(x_n)$$

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$$\frac{\partial w}{\partial x_n}=\frac{dw_n(x_n)}{dx_n}\,w_1(x_1)\,w_2(x_2)\ldots w_{n-1}(x_{n-1})$$

$$\frac{\partial^2 w}{\partial x_1^2}=\frac{d^2w_1(x_1)}{dx_1^2}\,.w_2(x_2).w_3(x_3)\ldots w_n(x_n)$$

$$\frac{\partial^2 w}{\partial x_2^2}=\frac{d^2w_2(x_2)}{dx_2^2}\,.w_1(x_1).w_3(x_3)\ldots w_n(x_n)$$

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$$\frac{\partial^2 w}{\partial x_n^2}=\frac{d^2w_n(x_n)}{dx_n^2}\,.w_1(x_1)\,w_2(x_2)\ldots w_{n-1}(x_{n-1})$$

değerleri (0.1-4) denkleminde yerlerine yazılırsa,

$$\frac{d^2w_1(x_1)}{dx_1^2}\,w_2(x_2)\ldots w_n(x_n)+\frac{d^2w_2(x_2)}{dx_2^2}\,w_1(x_1)w_3(x_3)$$

$$\ldots w_n(x_n)+\ldots+\frac{dw_n(x_n)}{dx_n^2}\,w_1(x_1)w_2(x_2)\ldots w_{n-1}(x_{n-1})$$

$$+\frac{k_1}{x_1}\,\frac{dw_1(x_1)}{dx_1}\,w_2(x_2).w_3(x_3)\ldots w_n(x_n)+\frac{k_2}{x_2}\,\frac{dw_2(x_2)}{dx_2}$$

(2.1-2) $$.\ w_1(x_1)w_3(x_3)\ldots w_n(x_n)+\frac{k_n}{x_n}\,\frac{dw_n(x_n)}{dx_n}\,w_1(x_1).w_2(x_2)$$

$$\ldots w_{n-1}(x_{n-1})+\lambda w_1(x_1)w_2(x_2)\ldots w_n(x_n)=0$$

olur. (2) denkleminin her iki yanını $w_1(x_1)\cdot w_2(x_2)\ldots w_n(x_n)$ ile bölelim. Böylece (2) denklemi,

$$\frac{1}{w_1(x_1)}\,\frac{d^2w_1(x_1)}{dx_1^2}+\frac{1}{w_2(x_2)}\,\frac{d^2w_2(x_2)}{dx_2^2}+\ldots+\frac{1}{w_n(x_n)}\cdot$$

(2.1-3) $$\frac{d^2w_n(x_n)}{dx_n^2}+\frac{k_1}{x_1w_1(x_1)}\,\frac{dw_1(x_1)}{dx_1}+\frac{k_2}{x_2w_2(x_2)}\,\frac{dw_2(x_2)}{dx_2}$$

$$+\ldots+\frac{k_n}{x_nw_n(x_n)}\,\frac{dw_n(x_n)}{dx_n}+\lambda=0$$

biçiminde yazılır. Bunu aşağıdaki gibi değişkenlerine ayıralım.

$$\sum_{i=1}^{n-1}\frac{1}{w_i(x_i)}\,\frac{d^2w_i(x_i)}{dx_i^2}+\frac{\alpha_i}{x_iw_i(x_i)}\,\frac{dw_i(x_i)}{dx_i}+\lambda=$$

(2.1-4) $$-\left(\frac{1}{w_n(x_n)}\,\frac{d^2w_n(x_n)}{dx_n^2}+\frac{\alpha_n}{x_nw_n(x_n)}\,\frac{dw_n(x_n)}{dx_n}\right)$$

(2.1-4) denkleminin birinci yanı $(x_1,x_2,\ldots,x_{n-1})$ in fonksiyonu ikinci yanı ise yalnız x_n in fonksiyonunudur. Bu iki yanın birbirine eşit olması ancak bir sabite eşit olmaları ile mümkündür. Bu sabite c_n^2 diyelim. Buradan,

$$\sum_{i=1}^{n-1} \frac{1}{w_i(x_i)} \frac{d^2w_i(x_i)}{dx_i^2} + \frac{\alpha_i}{x_i w_i(x_i)} \frac{dw_i(x_i)}{dx_i} = c_n^2$$

$$(2.1\text{-}4)_1 \qquad \frac{1}{w_n(x_n)} \frac{d^2w_n}{dx_n^2} + \frac{\alpha_n}{x_n w_n(x_n)} \frac{dw_n(x_n)}{dx_n} = -c_n^2$$

denklemleri elde edilir. İşleme aynı şekilde devam ederek,

$$\sum_{i=1}^{n-2} \frac{1}{w_i(x_i)} \frac{d^2w_i(x_i)}{dx_i^2} + \frac{\alpha_i}{x_i w_i(x_i)} \frac{dw_i(x_i)}{dx_i} + \lambda - c_n^2$$

$$= -\left(\frac{1}{w_{n-1}(x_{n-1})} \frac{d^2w_{n-1}(x_{n-1})}{dx_{n-1}^2} + \frac{\alpha_{n-1}}{x_{n-1}w_{n-1}(x_{n-1})} \frac{dw_{n-1}(x_{n-1})}{dx_{n-1}} \right.$$

$$\sum_{i=1}^{n-2} \frac{1}{w_i(x_i)} \frac{d^2w_i}{dx_i^2} + \frac{\alpha_i}{x_i w_i(x_i)} \frac{dw_i(x_i)}{dx_i} + \lambda - c_n^2 = c_{n-1}^2$$

$$(2.1\text{-}4)_2 \qquad \frac{1}{w_{n-1}(x_{n-1})} \frac{d^2w_{n-1}(x_{n-1})}{dx_{n-1}^2} + \frac{\alpha_{n-1}}{x_{n-1}w_{n-1}(x_{n-1})} \frac{dw_{n-1}(x_{n-1})}{dx_{n-1}} = -c_{n-1}^2$$

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$$\frac{1}{w_1(x_1)} \frac{d^2w_1(x_1)}{dx_1^2} + \frac{\alpha_1}{x_1 w_1(x_1)} \frac{dw_1(x_1)}{dx_1} + \lambda - (c_n^2 + c_{n-1}^2 + \ldots + c_3^2)$$

$$= -\left(\frac{1}{w_2(x_2)} \frac{d^2w_2(x_2)}{dx_2^2} + \frac{\alpha_2}{x_2 w_2(x_2)} \cdot \frac{dw_2(x_2)}{dx_2} \right)$$

$$(2.1\text{-}4)_{n-1} \qquad \frac{1}{w_2(x_2)}\frac{d^2w_2(x_2)}{dx_2^2}+\frac{\alpha_2}{x_2w_2(x_2)}\frac{dw_2(x_2)}{dx_2}=-c_2^2$$

$$\frac{1}{w_1(x_1)}\frac{d^2w_1(x_1)}{dx_1^2}+\frac{\alpha_2}{x_1w_1(x_1)}\frac{dw_2(x_2)}{dx_2}+\lambda=c_n^2+c_{n-1}^2+\ldots+c_2^2$$

yazılır. Son denkleminde diğerlerine benzemesi bakımından $\lambda=c_1^2+\ldots+c_n^2$ seçeriz ve

$$(2.1\text{-}4)_n \qquad \frac{1}{w_1(x_1)}\frac{d^2w_1(x_1)}{dx_1^2}+\frac{\alpha_1}{x_1w_1(x_1)}\frac{dw_1(x_1)}{dx_1}=-c_1^2$$

$$\lambda = c_n^2 + c_{n-1}^2 + \ldots + c_1^2$$

olur. Burada $c_1,c_2,\ldots,c_n \in C$ (karmaşık sayılardır). Elde edilen $(2.1\text{-}4)_1,(2.1\text{-}4)_2\ldots,(2.1\text{-}4)_{n-1}$ ve $(2.1\text{-}4)_n$ denklemlerinin hepsi aynı tiptendir.

w_i bağlı değişkenini y, x_i bağımsız değişkenini x ile gösterelim. Buna göre çözülecek denklem

$$\frac{1}{y}y''+\frac{k}{xy}y'=-c^2\ ,\ (')=\left(\frac{d}{dx}\right),\ \alpha=k,c_i=c$$

biçiminde yazılır. Eşitliğin heriki yanını xy ile çarparak

$$(2.1\text{-}5) \qquad xy''+ky'+c^2xy=0$$

denklemini elde ederiz.

2.2 DÖNÜŞÜMLER VE ÇÖZÜM

$$(2.2\text{-}1) \qquad cx=u$$

dönüşümünü yapalım [12].

$$cdx=du\ ,\ \frac{1}{c}\frac{d}{dx}=\frac{d}{du}\ ,\ \frac{d}{dx}=c\frac{d}{du}$$

$$\frac{d}{dx}\left(\frac{d}{dx}\right) = \left(c\frac{d}{du}\right)\left(c\frac{d}{du}\right), \quad \frac{d^2}{dx^2} = c^2\frac{d^2}{du^2}$$

bu değerler (2.2-1) denkleminde yerlerine yazılırsa

$$\frac{u}{c} \cdot c^2\frac{d^2y}{du^2} + kc\frac{dy}{du} + c^2\frac{u}{c}y = 0$$

olur. Eşitliğin her iki yanını c ile bölersek

$$uy'' + ky' + uy = 0 \quad , \quad (') = \left(\frac{d}{du}\right)$$

denklemini elde ederiz. Bu denkleme aşağıdaki dönüşüm uygulanarak,

$$(2.2\text{-}3) \qquad y = v(u).e^{-\frac{1}{2}\int\frac{k}{u}du} = v(u).e^{-\frac{k}{2}\ln u} = v(u).u^{-\frac{k}{2}}$$

$$y' = v'u^{-\frac{k}{2}} - \frac{k}{2}u^{-\frac{k+2}{2}}.v$$

$$y'' = v''u^{-\frac{k}{2}} - kv'u^{-\frac{k+2}{2}} + \frac{k(k+2)}{4}u^{-\frac{k+4}{2}}.v$$

(2.2-3) değerlerini (2.2-2) denkleminde yerlerine konularak

$$u^{-\frac{k-2}{2}}v'' - kv'u^{-\frac{k}{2}} + \frac{k(k+2)}{4}vu^{-\frac{k+2}{2}} + kv'u^{-\frac{k}{2}} - \frac{k}{2}vu^{-\frac{k+2}{2}} + vu^{\frac{k}{2}} = 0$$

$$u^{-\frac{k-2}{2}}v'' + \left[\frac{k(k+2)}{4}u^{-\frac{k+2}{2}} - \frac{k}{2}u^{-\frac{k+2}{2}} + u^{\frac{k}{2}}\right]v = 0$$

$$u^{-\frac{k-2}{2}}v'' + \left[\frac{k(2-k)}{4}u^{-\frac{k+2}{2}} + u^{-\frac{k+2}{2}}\right]v = 0$$

elde edilir. Son eşitliğin her iki yanını $u^{-\frac{k}{2}}$ ile bölerek

$$uv'' + \left[\frac{k(2-k)}{4}u^{-1} + u\right]v = 0$$

ve her iki yanı u ile çarparak

$$(2.2\text{-}4) \qquad u^2v''+\left[\frac{k(2-k)}{4}+u^2\right]v=0$$

bulunur. Burada

$$(2.2\text{-}5) \qquad \frac{k(2-k)}{4}=a^2$$

diyelim. Buna göre (2.2-4) denklemi

$$(2.2\text{-}6) \qquad u^2v''+(u^2-a^2)v=0$$

biçiminde yazılır. (11) denklemine de aşağıdaki dönüşüm uygulanarak

$$(2.2\text{-}7) \qquad v=\sqrt{u}\,.f(u)=u^{1/2}f(u)$$

$$v'=\frac{1}{2}u^{-1/2}f+u^{1/2}f'$$

$$v''=u^{1/2}f''+{}^{-1/2}f'-\frac{1}{4}u^{-3/2}f$$

değerleri (11) denkleminde yerlerine yazılırsa,

$$u^2.u^{1/2}.f''+u^{3/2}f'-\frac{1}{4}u^2.u^{-3/2}f+(u^2-a^2)u^{1/2}.f=0$$

$$u^{3/2}f''+u^{3/2}f'+(u^2-a^2-\frac{1}{4})u^{1/2}f=0$$

elde edilir. Bu eşitliğin her iki yanı $u^{1/2}$ ile bölünürse

$$u^2f''+uf'+(u^2-a^2-\frac{1}{4})f=0$$

bulunur. $-a^2$ nin değeri yerine yazılırsa

$$u^2f''+uf'+\left[u^2+\frac{k(2-k)}{4}-\frac{1}{4}\right]f=0$$

$$(2.2\text{-}8) \qquad u^2f''+uf'+\left[u^2-\left(\frac{k-1}{2}\right)^2\right]f=0$$

denklemi bulunur. Bu ise Bessel denklemi tipindedir (ikinci çeşit Bessel denklemidir). Buradan (2.2-8) denkleminin çözümü olarak

(2.2-9) $$f = A.\ J_{\frac{k-1}{2}}(u) + B.\ Y_{\frac{k-1}{2}}(u)$$

bulunur. (2.2-1) den u=cx yerine yazılırsa

(2.2-10) $$f = A.\ J_{\frac{k-1}{2}}(cx) + B.\ Y_{\frac{k-1}{2}}(cx)$$

elde edilir. (2.2-7) dönüşümünden $f=\frac{v}{\sqrt{u}}$ bulunarak yerine yazılırsa

$$\frac{v}{\sqrt{u}} = A.\ J_{\frac{k-1}{2}}(cx) + B.\ Y_{\frac{k-1}{2}}(cx)$$

$$\frac{v}{\sqrt{cx}} = A.\ J_{\frac{k-1}{2}}(cx) + B.\ Y_{\frac{k-1}{2}}(cx)$$

(2.2-11) $$v = \sqrt{cx}\left[A.\ J_{\frac{k-1}{2}}(cx) + B.Y_{\frac{k-1}{2}}(cx)\right]$$

olur. (8) dönüşümünden $y=v.u^{\frac{k}{2}}$ değeri ve (2.2-1) dönüşümü gözönüne alınarak

(2.2-12) $$y = (cx)^{\frac{1-k}{2}}\left[A.\ J_{\frac{k-1}{2}}(cx) + B.\ Y_{\frac{k-1}{2}}(cx)\right]$$

yazılır. y yerine w_i , x yerine x_i,c yerine c_i ve $k=\alpha_i$ değerleri yerlerine yazılarak

(2.2-13) $$w_i(x_i)=(c_ix_i)^{\frac{1-\alpha_i}{2}}.\left[A\ J_{\frac{\alpha_{i-1}}{2}}(c_ix_i)+B\ Y_{\frac{\alpha_{i-1}}{2}}(c_ix_i)\right]$$

elde edilir. (2.1-1) denkleminden,

(2.2-14) $$w(x_1,x_2,\ldots,x_n)= \prod_{i=1}^{n}(c_ix_i)^{\frac{1-\alpha_i}{2}}$$

$$\left[A\ .\ J_{\frac{\alpha_{i-1}}{2}}(c_ix_i) + B\ .\ Y_{\frac{\alpha_{i-1}}{2}}(c_ix_i)\right]$$

bulunur. Böylece (2.2-14) fonksiyonu (0.1-4) denklemi için bir elemanter çözüm olur.

Ö Z E T

Bu çalışmanın amacı Genelleştirilmiş çok eksensel simetrik potansiyel kuramının Helmholtz denklemleri için bir Dirichlet problemi çözmek ve bir elemanter çözüm bulmaktır.

Çalışma iki bölümden oluşur. Birinci bölümde

$$(1)\quad L_{\alpha_{m+1},\ldots,\alpha_{m+n}}[v]=\sum_{i=1}^{m+n}\frac{\partial^2 v}{\partial x_i^2}+\frac{\alpha_{m+1}}{x_{m+1}}\frac{\partial v}{\partial x_{m+1}}+\ldots+\frac{\alpha_{m+n}}{x_{m+n}}\frac{\partial v}{\partial x_{m+n}}=0$$

denklemi için m+n boyutlu Euclidean uzayında çeyrek küre bölgesinde bir Poisson formülü oluşturuldu. Elde edilen Poisson integralinin aynı bölgede Dirichlet probleminin çözümü olduğu bir teoremle ispatlandı.

(1) denklemine eksiltme (descent) yöntemi uygulanarak

$$(2)\quad \sum_{i=2}^{m+n}\frac{\partial^2 u}{\partial x_i^2}+\frac{\alpha_{m+1}}{x_{m+1}}\frac{\partial u}{\partial x_{m+1}}+\ldots+\frac{\alpha_{m+n}}{x_{m+n}}\frac{\partial u}{\partial x_{m+n}}-\lambda^2 u=0$$

denklemi için de aynı bölgede bir Dirichlet problemi çözüldü.

İkinci bölümde

$$(3)\quad \sum_{i=1}^{n}\frac{\partial^2 w}{\partial x_i^2}+\frac{\alpha_i}{x_i}\frac{\partial w}{\partial x_i}+\lambda w=0$$

denkleminin dik koordinatlarda bir elemanter çözümü elde edildi. Bunun için değişkenlerin ayrılması yöntemi kullanılarak (3) denklemi adi diferensiyel denklemlere indirgendi. Bu denklemler uygun dönüşümler altında Bessel denklemlerine dönüştürüldü. Böylece Bessel fonksiyonları türünden bir elemanter çözüm elde edildi.

S U M M A R Y

The purpose of this paper is to sôlve a Dirichlet's problem and find an elementary solution for Helmholtz's equations of generalized poly-axially symmetric potential theory.

The paper contains two chapters. In the first chapter a Poisson's formula is constructed for the equation

$$(1) \qquad \sum_{i=1}^{m+n} \frac{\partial^2 v}{\partial x_i^2} + \frac{\alpha_{m+1}}{x_{m+1}} \frac{\partial v}{\partial x_{m+1}} + \ldots + \frac{\alpha_{m+n}}{x_{m+n}} \frac{\partial v}{\partial x_{m+n}} = 0$$

on the region of a quarter sphere in Euclidean space with dimension m+n. Then by a theorem it is proven that the Poisson's integral is a solution of Dirichlet's problem on the same region. Using method of descent for the equation (1) a Dirichlet's problem is also solved on the same region for the equation

$$(2) \qquad \sum_{i=2}^{m+n} \frac{\partial^2 u}{\partial x_i^2} + \frac{\alpha_{m+1}}{x_{m+1}} \frac{\partial u}{\partial x_{m+1}} + \ldots + \frac{\alpha_{m+n}}{x_{m+n}} \frac{\partial u}{\partial x_{m+n}} - \lambda^2 = 0$$

In the second chapter an elementary solution is obtained in the cartesian coordinates for the equation

$$(3) \qquad \sum_{i=1}^{n} \left(\frac{\partial^2 w}{\partial x_i^2} + \frac{\alpha_i}{x_i} \frac{\partial w}{\partial x_i} \right) + \lambda w = 0$$

To do this, the equation (3) is reduced to some ordinary differential equations by the method of separation of variables. Then the ordinary differential equations are changed to the Bessel's equations using some transformation formulas. Thus the solution is found in terms of Bessel functions.

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