T.R.
SIIRT UNIVERSITY
INSTITUTE OF SCIENCE

# SOME SPECIAL INTEGER SEQUENCES RELATED TO BIPARTITE GRAPHS

MASTER DEGREE THESIS
DIYAR OMAR MUSTAFA ZANGANA
(153114021)

Department of Mathematics

Supervisor: Asst. Prof. Dr. Abdulkadir KARAKAŞ
Co-Supervisor: Dr. Ahmet ÖTELEŞ

AUGUST - 2017
SİİRT

# THESIS NOTIFICATION

I hereby declare that this paper is my unique authorial work, which I have worked out by my own. Every information bases, references and literature used or excerpted through explanation of this work are correctly cited and listed in complete reference to the owing cause.

*Signature*
*Diyar Omar Mustafa ZANGANA*

# ACKNOWLEDGMENTS

Firstly the name of Allah we started and starting.

And before of all things, I want to thank God for helping me to finish my thesis, this is a great opportunity to acknowledge and to thanks all persons without their support, and help this thesis would have been impossible.

I would like to add a few heartfelt words for the people who were part of this thesis in numerous ways.

I wish to thanks my supervisors Res. Asst. Dr. Ahmet Öteleş and Asst. Prof. Dr. Abdulkadir Karakaş for our indefatigable guidance, valuable suggestion, moral support, constant encouragement and contribution of time for the successful completion of thesis work. I am very grateful to there, for providing all the facilities needed during the thesis development.

My special thanks to my family for indispensable support and encouragement thing my study. Finally, I would like to thank all those who helped me directly or indirectly.

SİİRT-2017
Diyar Omar Mustafa ZANAGANA

# Contents

# LIST OF TABLES

# LIST OF FIGURES

Page

# LIST OF SYMBOLS

| Symbols | Explanation |
|---|---|
| $G$ | graph $G$ |
| $A_n = (a_{ij})$ | matrix $A_n$ of order $n$ with entries $a_{ij}$ |
| $A_{(n,k)} = (a_{ij})$ | matrix $A_{(n,k)}$ of order $n$ with entries $a_{ij} = \begin{cases} 1, & if \quad -1 \leq \lvert j-i \rvert \leq k-1, \\ 0, & otherwise. \end{cases}$ |
| $G(A_n)$ | graph with adjacency matrix $A_n$ |
| $J(n)$ | $n^{th}$ Jacobsthal number |
| $F(n)$ | $n^{th}$ Fibonacci number |
| $L(n)$ | $n^{th}$ Lucas number |
| $A_n^k$ | $k^{th}$ contraction of the matrix $A_n$ |
| per$A_n$ | permanent of the matrix $A_n$ |

# ÖZET

YÜKSEK LİSANS

İKİ PARÇALI GRAFLARLA İLİŞKİLİ BAZI ÖZEL TAMSAYI DİZİLERİ

Diyar Omar Mustafa ZANGANA

Siirt Üniversitesi Fen Bilimleri Enstitüsü
Matematik Anabilim Dalı

Danışman: Yrd. Doç. Dr. Abdulkadir KARAKAŞ
ll. Danışman: Arş. Gör. Dr. Ahmet ÖTELEŞ

2017, 56 Sayfa

Bu tezde, iki parçalı komşuluk matrisleri $n \times n$ $(0,1)-$matrisler olan bazı belli tip iki parçalı grafları ele aldık. Sonra bu iki parçalı grafların mükemmel eşlemeleri sayılarının çok iyi bilinen sayı dizilerine (örneğin; Fibonacci Lucas Jacobsthal) eşit olduğunu gösterdik. Daha sonra, ele aldığımız bu graflar ve onların mükemmel eşlemeleriyle ilgili bazı örnekler verdik. Son olarak, bu iki parçalı grafların mükemmel eşlemeleri sayılarını hesaplamak için Maple 2016 prosedürleri sunduk.

**Anahtar Kelimeler:** Fibonacci dizisi, İki parçalı graf, Jacobsthal dizisi, Lucas dizisi, Mükemmel eşleme, Permanent.

# ABSTRACT

M.Sc. THESIS

SOME SPECIAL INTEGER SEQUENCES RELATED TO BIPARTITE GRAPHS

Diyar Omar Mustafa ZANGANA

The Graduate School of Natural and Applied Science of Siirt University
The Degree of Master of Science
In Mathematics

Supervisor: Asst. Prof. Dr. Abdulkadir KARAKAŞ
Co-Supervisor: Res. Asst. Dr. Ahmet ÖTELEŞ

2017, 56 Pages

In this thesis, we consider some certain types of bipartite graphs whose bipartite adjacency matrices are the $n \times n$ $(0,1)-$matrices. Then we show that the numbers of perfect matchings ($1-$factors) of these bipartite graphs are equal to the famous integer sequences (e.g. Fibonacci, Lucas, Jacobsthal). After that, we give some examples concerned with these graphs and their perfect matchings. Finally, we present Maple procedures in order to calculate the numbers of perfect matchings of the bipartite graphs.

**Keywords:** Bipartite graph, Fibonacci sequence, Jacobsthal sequence, Lucas sequence, Perfect matching, Permanent.

# 1. INTRODUCTION

Bipartite graph is a graph in which the vertices can be divided into two parts such that no two vertices in the same part are joined by an edge. The investigation of the properties of bipartite graphs was begun by König. His work was motivated by an attempt to give a new approach to the investigation of matrices on determinants of matrices. As a practical matter, bipartite graphs form a model of the interaction between two different types of objects. For example; social network analysis, railway optimization problem, marriage problem etc (Asratian et al., 1998).

A perfect matching (1-factors) of a graph is a matching ( i.e., an independent edge set ) in which every vertex of the graph is incident to exactly one edge of the matching. ″The enumeration or actual construction of perfect matching of a bipartite graph has many applications, for example, in maximal flow problems and in assignment and scheduling problems arising in operational research″ (Minc, 1978).
The number of perfect matchings of bipartite graphs also plays a significant role in organic chemistry (Wheland, 1953).

Fibonacci, Lucas and Jacobsthal numbers which are respectively defined by the recurrence relation

$$\begin{aligned} F(n) &= F(n-1)+F(n-2), && F(0)=0 \ \ and\ F(1)=1, \\ L(n) &= L(n-1)+L(n-2), && L(0)=2 \ \ and\ L(1)=1, \\ J(n) &= J(n-1)+2J(n-2), && J(0)=0 \ \ and\ J(1)=1, \end{aligned}$$

for $n \geq 2$, belong to a large family of positive integers. They have many interesting properties and applications to almost every field of science and art. They continue to contribute significant innovations for investigations, and reveal the niceness of mathematics in many fields, particularly number theory (Koshy, 2001; Koshy, 2011; Horadam, 1988).

″The permanent of an $n \times n$ matrix $A = (a_{ij})$ is defined by

$$\text{per}(A) = \sum_{\sigma \epsilon S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations $\sigma$ of the symmetric group $S_n$. The permanent of a matrix is analogous to the determinant, where all of the signs used in the Laplace expansion of minors are positive″ (Minc, 1978). Permanents have many applications in physics, chemistry, electrical engineering, graph theory etc. Some of the most important applications of permanents are via graph theory.

A more difficult problem with many applications is the enumeration of perfect matchings of a graph. Therefore, counting the number of perfect matchings in bipartite graphs has been very popular problem.

## 1.1. Structure of the Thesis

The rest of the thesis is structured as follows.

In Chapter 2, we present a discussion of previously published work that I did in this area in conjunction with other authors.

In Chapter 3, we give the fundamental definitions, structures and theorems which are necessary to better understand the topics contained within this text.

In Chapter 4.1, we consider a bipartite graph. Then we show that the numbers of perfect matchings of this graph generate the Jacobsthal numbers by the contraction method. Finally, a Maple procedure is presented in order to compute the numbers of perfect matchings of the graph.

In Chapter 4.2, we firstly introduce two lemmas related to bipartite graphs associated with Fibonacci numbers. After that, we define a bipartite graph associated with $n \times n$ $(0;1)-$circulant matrix whose the numbers of perfect matchings generate the Lucas numbers. Finally, two Maple procedures are presented to compute the numbers of perfect matchings in these graphs.

# 2. LITERATURE RESEARCH

The purpose of this chapter is to further motivate the rest of the thesis by presenting a discussion of previously published work that I did in this area in conjunction with other authors.

Lee et al. (1997) consider "a bipartite graph $G\left(\mathcal{F}_{(n,2)}=\left(f_{i,j}\right)\right)$ with bipartite adjacency matrix is the $n\times n$ tridiagonal matrix of the form

$$\mathcal{F}_{(n,2)}=\begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 1 & \ddots & & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & & \ddots & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}, \tag{1}$$

with the entries are

$$f_{ij}=\begin{cases} 1, & if \quad |j-i|\leq 1, \\ 0, & \qquad otherwise, \end{cases}$$

". Then they obtain "the number of perfect matchings of $G\left(\mathcal{F}_{(n,2)}\right)$ is the $(n+1)$st Fibonacci number $F\,(n+1)$". In other words,

$$\text{per}\mathcal{F}_{(n,2)}=F\,(n+1)\,. \tag{2}$$

They also consider "a bipartite graph $G\left(\mathcal{F}_{(n,k)}=\left(f_{i,j}\right)\right)$ with bipartite adjacency matrix is the $n\times n$ $(0,1)-$matrix of the form

$$\mathcal{F}_{(n,k)}=\begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 1 & \cdots & \cdots & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \ddots & & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & & & & & & \ddots & 0 \\ \vdots & & & & \ddots & & & & & 1 \\ & & & & & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & \cdots & \cdots & & & 0 & 1 & 1 \end{pmatrix}, \tag{3}$$

with the entries are

$$f_{ij} = \begin{cases} 1, & if \quad -1 \leq |j-i| \leq k-1, \\ 0, & \quad otherwise, \end{cases}$$

″. Then they obtain ″the number of perfect matchings of $G\left(\mathcal{F}_{(n,k)}\right)$ is $g^k\,(n+k-1)$, where $g^k\,(n)$ is the $n^{th}$ $k$-Fibonacci number″ (Lee et al., 1997). This time, they consider ″an another bipartite graph $G\left(U_n = \left(u_{i,j}\right)\right)$ with bipartite adjacency matrix is the $n \times n$ $(0,1)-$matrix of the form

$$U_n = \begin{pmatrix} 1 & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & 0 & 1 & \cdots & \cdots & 1 \\ 0 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix},$$

with the entries are

$$u_{ij} = \begin{cases} 1, & if \quad i=j=1 \ or\ i=j=n, \\ 1, & if \quad i<j \ or\ j-i=-1, \\ 0, & \quad otherwise, \end{cases}$$

″. Then they obtain ″the number of perfect matchings of $G\,(U_n)$ is the $(n+1)$st Fibonacci number $F\,(n+1)$″ (Lee et al., 1997).

Lee (2000) considers ″a bipartite graph $G\left(\mathcal{L}_{(n,2)} = \left(l_{i,j}\right)\right)$ with bipartite adjacency matrix is the $n \times n$ matrix of the form

$$\mathcal{L}_{(n,2)} = \begin{pmatrix} 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix},$$

with the entries are

$$l_{i,j} = \begin{cases} 1, & if \quad i=j=1 \ or\ i=1\ and\ j=3, \\ 1, & if \quad i>2\ and\ |j-i| \leq 1, \\ 0, & \quad otherwise, \end{cases}$$

". Then he shows that "the number of perfect matchings of $G\left(\mathcal{L}_{(n,2)}\right)$ is $(n-1)$st Lucas number $L\left(n-1\right)$ for $n \geq 3$ ". He also considers "a bipartite graph $G\left(\mathcal{L}_{(n,k)}\right)$ with bipartite adjacency matrix $\mathcal{L}_{(n,k)} = \mathcal{F}_{(n,k)} + E_{1,k+1} - \sum_{j=2}^{k} E_{1,j}$ for $n \geq 3$, where $\mathcal{F}_{(n,k)}$ is the matrix in (3) and $E_{i,j}$ denotes the $n \times n$ matrix with 1 at the $(i,j)$-entry and zeros elsewhere". Namely,

$$\mathcal{L}_{(n,k)} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 1 & \cdots & \cdots & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \ddots & & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & & & & & & \ddots & 0 \\ \vdots & & & & \ddots & & & & & 1 \\ & & & & & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & \cdots & \cdots & & & 0 & 1 & 1 \end{pmatrix}.$$

Then he shows that "the number of perfect matchings of $G\left(\mathcal{L}_{(n,k)}\right)$ is $l^{k}\left(n-1\right),$ where $l^{k}\left(n\right)$ is the $n^{th}$ $k$-Lucas number" (Lee, 2000). This time, "he defines the matrix $B_n$ as

$$B_n = \mathcal{F}_{(n,2)} + E_{13} - E_{23} + E_{24} - E_{34}$$

where $\mathcal{F}_{(n,2)}$ is the matrix in (1) and $E_{i,j}$ denotes the $n \times n$ matrix with 1 at the $(i,j)$-entry and zeros elsewhere". Let $G\left(B_n\right)$ be the bipartite graph with bipartite adjacency matrix $B_n$. Then he shows that "the number of perfect matchings of $G\left(B_n\right)$ is $(n-1)$st Lucas number $L\left(n-1\right)$" (Lee, 2000).

Shiu et al. (2003) firstly define the $(k,\alpha)$-sequences $s_{\alpha}^{k}\left(n\right)$. Then they give the following result:

"For a fixed $m \geq 1$, suppose $n,k \geq 2$ and $n \geq m$. Let $G\left(\mathcal{B}_{(n,k)}\right)$ a bipartite graph with bipartite adjacency matrix has the form

$$\mathcal{B}_{(n,k)} = \begin{pmatrix} a_1 & a_2 & \ldots & a_m & 0 & \ldots & 0 \\ 1 & & & & & & \\ 0 & & & \mathcal{F}_{(n-1,2)} & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{pmatrix},$$

for some elements $a_1, a_2, \ldots, a_m$ in a ring $R$.Then the number of perfect matching of $G\left(\mathcal{B}_{(n,k)}\right)$ is $n^{th}$ $(k,\alpha)$-number $s_{\alpha}^{k}\left(n\right)$ with $\alpha = \left(a_1, a_2, \ldots, a_m\right)$."

Kılıç and Taşcı (2008a) consider ″a bipartite graph $G\left(V_n = \left(v_{ij}\right)\right)$ with bipartite adjacency matrix has the form

$$V_n = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix},$$

with the entries are

$$v_{ij} = \begin{cases} 1, & if\ -1 \leq j-i \leq 1\ or\ i=1, \\ 0, & otherwise, \end{cases}$$

″. Then they obtain ″the number of perfect matchings of $G\left(V_n\right)$ is $\sum_{i=0}^{n} F\left(i\right) = F\left(n+2\right) - 1,$ where $F\left(n\right)$ is the $n^{th}$ Fibonacci number″. They also consider ″a bipartite graph $G\left(W_n\right)$ with bipartite adjacency matrix $W_n = V_n + Y_n,$ where $Y_n$ denotes the $n \times n$ matrix with $-1$ at the $(1,2)$-entry, 1 at the $(2,4)$-entry and zeros elsewhere″. Clearly,

$$W_n = \begin{pmatrix} 1 & 0 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}.$$

Then for $n \geq 4,$ they obtain ″that the number of perfect matchings of $G\left(W_n\right)$ is $\sum_{i=0}^{n-2} L\left(i\right) = L\left(n\right) - 1,$ where $L\left(n\right)$ is the $n^{th}$ Lucas number (Kılıç and Taşcı, 2008a)″.

Kılıç and Taşcı (2008b) consider ″a bipartite graph $G\left(M_{(n,k)}\right)$ with bipartite adjacency matrix $M_{(n,k)} = \mathcal{F}_{(n,k)} + U_{(n,k)},$ where $\mathcal{F}_{(n,k)}$ is the matrix given by (3) and $U_{(n,k)} = \left(u_{ij}\right)$ is the $n \times n$ $(0,1)$-matrix with $u_{n-k-1,n-1} = u_{n-k,n} = 1$ and otherwise 0". Clearly,

$$
M_{(n,k)}=\begin{pmatrix}
1 & \cdots & 1 & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0\\
1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0\\
0 & 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 & \cdots & 0\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
\vdots & & 0 & 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0\\
\vdots & & & 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0\\
\vdots & & & & 0 & 1 & 1 & & 1 & 1 & 0\\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 & & 1 & 1\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 & 1\\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1
\end{pmatrix}. \tag{4}
$$

Then for $n \geq 3$, they obtain that "the number of perfect matchings of $G\left(M_{(n,k)}\right)$ is the $n^{th}$ $k$-Lucas number, $l^k(n)$". They also consider "a bipartite graph $G\left(T_{(n,k)}\right)$ with bipartite adjacency matrix $T_{(n,k)} = \mathcal{F}_{(n,k)} + V_{(n,k)}$ for $2 \leq k < n$, where $\mathcal{F}_{(n,k)}$ is the matrix given by (3) and $V_{(n,k)} = (v_{ij})$ is the $n \times n$ $(0,1)$-matrix with $v_{1j} = 1$ if $k+1 \leq j \leq n$ and otherwise 0". Clearly,

$$
T_{(n,k)}=\begin{pmatrix}
1 & \cdots & 1 & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 & 1\\
1 & 1 & \cdots & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0\\
0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 0 & \cdots & \cdots & 0\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots\\
\vdots & & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & \ddots & \vdots\\
\vdots & & & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 0\\
\vdots & & & & 0 & 1 & 1 & \cdots & \cdots & 1 & 1\\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 & \cdots & \cdots & 1\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 & 1\\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1
\end{pmatrix}.
$$

Then they obtain that "the number of perfect matchings of $G\left(T_{(n,k)}\right)$ is the sums of $k$-Fibonacci numbers, $\sum_{j=1}^{n} g^k(j)$ for $n \geq 3$" (Kılıç and Taşcı, 2008b). This time, they consider "a bipartite graph $G\left(E_{(n,k)}\right)$ with bipartite adjacency matrix $E_{(n,k)} = M_{(n,k)} + D_{(n,k)}$ for $2 \leq k < n$, where $M_{(n,k)}$ is the matrix given by (4) and $D_{(n,k)} = (d_{ij})$ is the $n \times n$ $(0,1)$-matrix with $d_{1j} = 1$ if $k+1 \leq j \leq n$ and otherwise 0". Clearly,

$$E_{(n,k)} = \begin{pmatrix} 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ \vdots & & & 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 \\ \vdots & & & & 0 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}.$$

Then they obtain that ″the number of perfect matchings of $G\left(E_{(n,k)}\right)$ is the sums of $k$-Lucas numbers, $\sum_{j=1}^{n-1} l^k(j)$ for $n \geq 3$″ (Kılıç and Taşcı, 2008b).

Kılıç and Stakhov (2009) consider ″a bipartite graph $G\left(M_{(n,p)}\right)$ with bipartite adjacency matrix $M_{(n,p)} = \left(m_{ij}\right)$ with $m_{i+1,i} = m_{i,i} = m_{i,i+p} = 1$ for a fixed integer $p$ and otherwise 0″. Clearly,

$$M_{(n,p)} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 & 1 & \ddots & \vdots \\ 0 & 1 & 1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & 1 \end{pmatrix}. \tag{5}$$

Then they obtain that ″the number of perfect matchings of $G\left(M_{(n,p)}\right)$ is the $(n+1)$st generalized Fibonacci $p$-number $F_p(n+1)$ for $n \geq 3$″. They also consider ″a bipartite graph $G\left(T_{(n,p)}\right)$ with bipartite adjacency matrix has the form

$$T_{(n,p)} = \begin{pmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ 1 & & & & & & \\ 0 & & & M_{(n-1,p)} & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{pmatrix},$$

where $M_{(n-1,p)}$ is the matrix of order $(n-1)$ given by (5)." Then they obtain that "the number of perfect matchings of $G\left(T_{(n,p)}\right)$ is the sums of the consecutive generalized Fibonacci $p$-numbers, $\sum_{i=1}^{n} F_p(i)$ for $n \geq 3$". (Kılıç and Stakhov, 2009).

Akbulak and Öteleş (2013) give the following results for the number of perfect matchings in bipartite graphs with upper Hessenberg adjacency matrix related to Fibonacci, Lucas and Padovan numbers.

"Let $G(U_n = (u_{ij}))$ be a bipartite graph with bipartite adjacency matrix is the $n \times n$ $(0,1)$-upper Hessenberg matrix defined by

$$U_n = \begin{pmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}, \quad (6)$$

where

$$u_{ij} = \begin{cases} 1, & if \quad j-i=-1, \\ 1, & if \quad i \leq j \; and \; j-i \equiv 0 \ (\ mod\, 2), \\ 0, & \quad otherwise. \end{cases}$$

Then the number of perfect matchings of $G(U_n)$ is the $n$th Fibonacci number $F(n)$ for $n \geq 3$" (Akbulak and Öteleş, 2013).

"Let $G(V_n = (v_{ij}))$ be a bipartite graph with bipartite adjacency matrix is the $n \times n$ $(0,1)$-upper Hessenberg matrix defined by

$$V_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 0 & 1 & \cdots & \cdots \\ 0 & 1 & & & & & & \\ 0 & 0 & & & U_{n-2} & & & \\ \vdots & \vdots & & & & & & \\ 0 & 0 & & & & & & \end{pmatrix},$$

where $U_{n-2}$ is the matrix of order $(n-2)$ in (6). Then the number of perfect matchings of $G(V_n)$ is the $(n-1)$st Lucas number $L(n-1)$ for $n \geq 3$" (Akbulak and Öteleş, 2013).

"Let $G(W_n = (w_{ij}))$ be a bipartite graph with bipartite adjacency matrix is the $n \times n$ $(0,1)$-upper Hessenberg matrix defined by

$$W_n = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix},$$

where

$$w_{ij} = \begin{cases} 1, & if \quad i = 1 \; or \; j - i = -1, \\ 1, & if \quad i \leq j \; and \; j - i \equiv 1 \; (\; mod 3), \\ 0, & \qquad otherwise. \end{cases}$$

Then the number of perfect matchings of $G(W_n)$ is the $n$th Padovan number $P(n)$ for $n \geq 3$". (Akbulak and Öteleş, 2013)

Öteleş (2017) gives the following results for the number of perfect matchings in bipartite graphs related to the well-known integer sequences (Fibonacci, Lucas i.e.).

"Let $G\left(H_n = \left(h_{i,j}\right)\right)$ $(n = 2t, \; t \in \mathbb{N})$ be a bipartite graph with bipartite adjacency matrix is the $n \times n$ $(0,1)$-tridiagonal matrix has the form

$$H_n = \begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & 1 & \ddots & & & \vdots \\ 0 & 1 & 1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & 1 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 1 & \ddots & 1 & 0 \\ \vdots & & & \ddots & \ddots & & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & \end{pmatrix}, \tag{7}$$

where

$$h_{i,j} = \begin{cases} 1, & if \quad |j - i| = 1, \\ 1, & if \quad i = j = 2m - 1 \; (m \in \mathbb{N}), \\ 0, & \qquad otherwise. \end{cases}$$

Then the number of perfect matchings of $G(H_n)$ is 1" (Öteleş, 2017).

"Let $G(H_n)$ $(n = 2t + 1, \; t \in \mathbb{N})$ be a bipartite graph with bipartite adjacency matrix $H_n$ given by (7). Then the number of perfect matchings of $G(H_n)$ is $\frac{n+1}{2}$" (Öteleş, 2017).

"Let $G\left(K_n = \left(k_{i,j}\right)\right)$ be a bipartite graph with bipartite adjacency matrix is the $n \times n$ $(0,1)$-tridiagonal matrix has the form

$$K_n = \begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & 1 & \ddots & & & \vdots \\ 0 & 1 & 1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & 1 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 1 & 1 & 1 & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix},$$

where $k_{2,2} = k_{4,4} = 0,$ all other terms on the main diagonal are $1,$ all terms on the sub-diagonal and super-diagonal are 1 and otherwise $k_{i,j} = 0$. Then for $n \geq 3$, the number of perfect matchings of $G(K_n)$ is the $(n-3)$rd Lucas number $L(n-3)$" (Öteleş, 2017).

"Let $G\left(B_n = \left(b_{i,j}\right)\right)$ be a bipartite graph with bipartite adjacency matrix is the $n \times n$ $(0,1)$-anti-tridiagonal matrix has the form

$$B_n = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 & 1 \\ \vdots & & \iddots & 1 & 1 & 1 \\ \vdots & \iddots & \iddots & \iddots & 1 & 0 \\ 0 & \iddots & \iddots & \iddots & \iddots & \vdots \\ 1 & 1 & \iddots & \iddots & & \vdots \\ 1 & 1 & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

where

$$b_{i,j} = \begin{cases} 1, & if \quad |i+j \; mod \; (n+1)| \leq 1, \\ 0, & \quad otherwise. \end{cases}$$

Then for $n \geq 2$, the number of perfect matchings of $G(B_n)$ is the $(n+1)$st Fibonacci number $F(n+1)$" (Öteleş, 2017).

"Let $G\left(D_n = \left(d_{i,j}\right)\right)$ be a bipartite graph with bipartite adjacency matrix has the form

$$D_n = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & 1 \\ \vdots & & 0 & 1 & 1 & 1 \\ \vdots & \iddots & \iddots & \iddots & 1 & 0 \\ 0 & \iddots & \iddots & \iddots & \iddots & \vdots \\ 1 & 1 & 1 & \iddots & & \vdots \\ 1 & 1 & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

where

$$d_{i,j} = \begin{cases} 1, & if \quad i = 1 \ \ and \ \ j = n-2, \ j = n, \\ 1, & if \quad |i + j \, mod \ \ (n+1)|_{2 \le i \le n} \le 1, \\ 0, & \qquad otherwise. \end{cases}$$

Then for $n \geq 3$, the number of perfect matchings of $G(D_n)$ is the $(n-1)$st Lucas number $L(n-1)$" (Öteleş, 2017).

"Let $G(U_n = (u_{i,j}))$ be a bipartite graph with bipartite adjacency matrix has the form

$$U_n = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 \\ \vdots & & \iddots & 1 & 1 & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & 1 & \iddots & \\ 0 & 1 & \iddots & \iddots & \iddots & & \vdots \\ 1 & 1 & 1 & \iddots & & & 0 \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix},$$

where

$$u_{i,j} = \begin{cases} 1, & if \quad i = 1 \ \ and \ 1 \le j \le n, \\ 1, & if \quad |i + j \, mod \ \ (n+1)|_{2 \le i \le n} \le 1, \\ 0, & \qquad otherwise. \end{cases}$$

Then for $n \geq 2$, the number of perfect matchings of $G(U_n)$ is $F(n+2) - 1$, where $F(n)$ is the $n^{th}$ Fibonacci number" (Öteleş, 2017).

"Let $G(V_n = (v_{i,j}))$ be a bipartite graph with bipartite adjacency matrix has the form

$$V_n = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 & 1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 & 1 \\ \vdots & & \iddots & 0 & 1 & 1 & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & 1 & 1 & 0 & 0 \\ 0 & \iddots & \iddots & \iddots & 1 & \iddots & \iddots & \vdots \\ 0 & 1 & 1 & \iddots & \iddots & \iddots & & \vdots \\ 1 & 1 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix},$$

where

$$v_{i,j} = \begin{cases} 1, & if \quad i = 1 \ \ and \ 1 \le j \le n-2, \ j = n, \\ 1, & if \quad i = 2 \ \ and \ \ j = n-3, \\ 1, & if \quad |i + j \, mod \ \ (n+1)|_{2 \le i \le n} \le 1, \\ 0, & \qquad otherwise. \end{cases}$$

Then for $n \geq 3$, the number of perfect matchings of $G\left(V_n\right)$ is $L\left(n\right) - 1$, where $L\left(n\right)$ is the $n^{th}$ Lucas number"(Öteleş, 2017).

"Let $G\left(P_n = \left(p_{ij}\right)\right)$ be a bipartite graph with bipartite adjacency matrix is the $n \times n$ $(0,1)$-pentadiagonal matrix has the form

$$P_n = \begin{pmatrix} 1 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & 1 & \ddots & & \vdots \\ 1 & 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 1 \\ \vdots & & \ddots & 1 & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Then the number of perfect matchings of $G\left(P_n\right)$ is obtained as

$$\operatorname{per}\left(P_n\right) = \begin{cases} F\left(\frac{n+1}{2}\right) F\left(\frac{n+3}{2}\right), & if\ n\ is\ odd, \\ \left(F\left(\frac{n}{2}+1\right)\right)^2, & if\ n\ is\ even, \end{cases}$$

where $F\left(n\right)$ is the $n$th Fibonacci number" (Öteleş, 2017).

# 3. PRELIMINARIES

In this chapter, we give the fundamental definitions, structures and theorems which are necessary to better understand the topics contained within this text.

## 3.1. Special Types of Matrices

### 3.1.1. Diagonal matrices

″A matrix $D = [d_{ij}] \in M_{n,m}(F)$ is diagonal if $d_{ij} = 0$ whenever $j \neq i$″ (Horn and Johnson, 1990).

### 3.1.2. Triangular matrices

″A matrix $T = [t_{ij}] \in M_{n,m}(F)$ is upper triangular if $t_{ij} = 0$ whenever $i > j$. If $t_{ij} = 0$ whenever $i \geq j$, then $T$ is said to be strictly upper triangular. Analogously, $T$ is lower triangular (or strictly lower triangular) if its transpose is upper triangular (or strictly upper triangular). A triangular matrix is either lower or upper triangular; a strictly triangular matrix is either strictly upper triangular or strictly lower triangular″ (Horn and Johnson, 1990).

### 3.1.3. Permutation matrices

″A square matrix $P$ is a permutation matrix if exactly one entry in each row and column is equal to 1 and all other entries are 0. Multiplication by such matrices effects a permutation of the rows or columns of the matrix multiplied.

For example,

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

illustrates how a permutation matrix produces a permutation of the rows (entries) of a vector: it sends the first entry to the second position, sends the second entry to the first position, and leaves the third entry in the third position″ (Horn and Johnson, 1990).

### 3.1.4. Circulant matrices

″A matrix $A \in M_n(F)$ of the form

$$A = \begin{pmatrix} a_1 & a_2 & & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_n & a_1 \end{pmatrix}$$

is a circulant matrix. Each row is the previous row cycled forward one step; the entries in each row are a cyclic permutation of those in the first. The $n \times n$ permutation matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & & & & 1 \\ 1 & 0 & & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_{n-1} \\ 1 & 0_{1,n-1} \end{pmatrix}$$

is the basic circulant permutation matrix. A matrix $A \in M_n(F)$ can be written in the form

$$A = \sum_{k=0}^{n-1} a_{k+1} C_n^k$$

(a polynomial in the matrix $C_n$) if and only if it is a circulant. We have $C_n^0 = I = C_n^n$, and the coefficients $a_1, \ldots, a_n$ are the entries of the first row of $A$″ (Horn and Johnson, 1990).

### 3.1.5. Toeplitz matrices

″A matrix $A = a_{ij} \in M_{n+1}(F)$ of the form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_n \\ a_{-1} & a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{-2} & a_{-1} & a_0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & a_1 \\ a_{-n} & a_{-n+1} & \cdots & \cdots & a_{-1} & a_0 \end{pmatrix}$$

is a Toeplitz matrix. The entry $a_{ij}$ is equal to $a_{j-i}$ for some given sequence

$a_{-n}, a_{-n+1}, \ldots, a_{-1}, a_0, a_1, a_2, \ldots, a_{n-1}, a_n \in C$. The entries of $A$ are constant down the diagonals parallel to the main diagonal. The Toeplitz matrices

$$B = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

and

$$F = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}$$

are called the backward shift and forward shift because of their effect on the elements of the standard basis $\{e_1, \ldots, e_{n+1}\}$. Moreover, $F = B^T$ and $B = F^T$. A matrix $A \in M_{n+1}$ can be written in the form

$$A = \sum_{k=1}^{n} a_{-k} F^k + \sum_{k=0}^{n} a_k B^k$$

if and only if it is a Toeplitz matrix. Toeplitz matrices arise naturally in problems involving trigonometric moments″ (Horn and Johnson, 1990).

### 3.1.6. Hankel matrices

″A matrix $A \in M_{n+1}(F)$ of the form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & \iddots & a_{n+1} \\ a_2 & & \iddots & & \vdots \\ \vdots & a_n & & & a_{2n-1} \\ a_n & a_{n+1} & \cdots & a_{2n-1} & a_{2n} \end{pmatrix}$$

is a Hankel matrix. Each entry $a_{ij}$ is equal to $a_{i+j-2}$ for some given sequence $a_0, a_1, a_2, \ldots, a_{2n-1}, a_{2n}$. The entries of $A$ are constant along the diagonals perpendicular to the main diagonal. Hankel matrices arise naturally in problems involving power moments″ (Horn and Johnson, 1990).

### 3.1.7. Hessenberg matrices

"A matrix $A = [a_{ij}] \in M_n(F)$ is said to be in upper Hessenberg form or to be an upper Hessenberg matrix if $a_{ij} = 0$ for all $i > j+1$ :

$$A = \begin{pmatrix} a_{11} & & & & \bigstar \\ a_{21} & a_{22} & & & \\ & a_{32} & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & a_{n,n-1} & a_{nn} \end{pmatrix}.$$

An upper Hessenberg matrix $A$ is said to be unreduced if all its sub-diagonal entries are nonzero, that is, if $a_{i+1}, i = 0$ for all $i = 1, \ldots, n-1$; the rank of such a matrix is at least $n-1$ since its first $n-1$ columns are independent" (Horn and Johnson, 1990).

### 3.1.8. Tridiagonal matrices

"A matrix $A = [a_{ij}] \in M_n(F)$ that is both upper and lower Hessenberg is called tridiagonal, that is, $A$ is tridiagonal if $a_{ij} = 0$ whenever $|i-j| > 1$:

$$A = \begin{pmatrix} a_1 & b_1 & & 0 \\ c_1 & a_2 & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ 0 & & c_{n-1} & a_n \end{pmatrix}$$

"(Horn and Johnson, 1990).

## 3.2. Graph Theory

**Definition 1** *"A graph G consists of a set of objects $V = \{v_1, v_2, v_3, \ldots\}$ called vertices (also called points or nodes) and other set $E = \{e_1, e_2, e_3, \ldots\}$ whose elements are called edges (also called lines or arcs)" (Vasudev, 2006).*

**Definition 2** *"The set $V(G)$ is called the vertex set of G and $E(G)$ is the edge set. Usually the graph is denoted as $G = (V, E)$" (Vasudev, 2006).*

Let $G$ be a graph and $\{u, v\}$ an edge of $G$. Since $\{u, v\}$ is 2-element set, we may write $\{v, u\}$ instead of $\{u, v\}$. It is often more convenient to represent this edge by $uv$ or $vu$. If $e = uv$ is an edge of a graph $G$, then we say that $u$ and $v$ are adjacent in $G$ and that $e$ joins $u$ and $v$. (We may also say that each that of $u$ and $v$ is adjacent to or with the other).

**Example 3** *A graph G is defined by the sets*

$$V(G) = \{u, v, w, x, y, z\}$$

*and*

$$E(G) = \{uv, uw, wx, xy, xz\}.$$

*Now we have the graph in the Figure 3.1 by considering these sets.*


Figure 3.1. A graph with 6−vertices and 5−edges

Every graph has a diagram associated with it. The vertex $u$ and an edge $e$ are incident with each other as are $v$ and $e$. If two distinct edges say $e$ and $f$ are incident with a common vertex, then they are adjacent edges.

**Example 4** *In Figure 3.2 the vertices a and b are adjacent but a and c are not. The edges x and y are adjacent but x and z are not. Although the edges x and z intersect in the diagram, their intersection is not a vertex of the graph.*

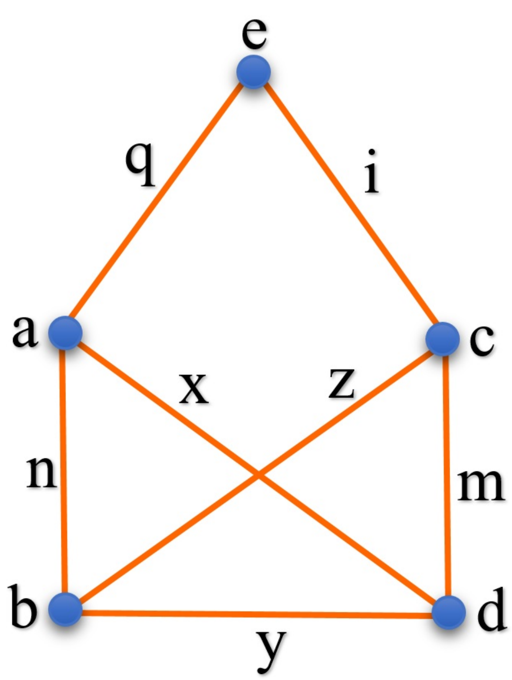



Figure 3.2. A graph with 5−vertices and 7−edges

**Definition 5** *"A graph with $p$-vertices and $q$-edges is called a $(p,q)$ graph. The $(1,0)$ graph is called trivial graph" (Vasudev, 2006).*

**Example 6** *Let $V = \{1,2,3,4\}$ and $E = \{\{1,2\},\{1,3\},\{3,2\},\{4,4\}\}$.Then $G(V,E)$ is a graph.*

**Example 7** *Let $V = \{1,2,3,4\}$ and $E = \{\{1,5\},\{2,3\}\}$.Then $G(V,E)$ is not a graph, as 5 is not in $V$.*

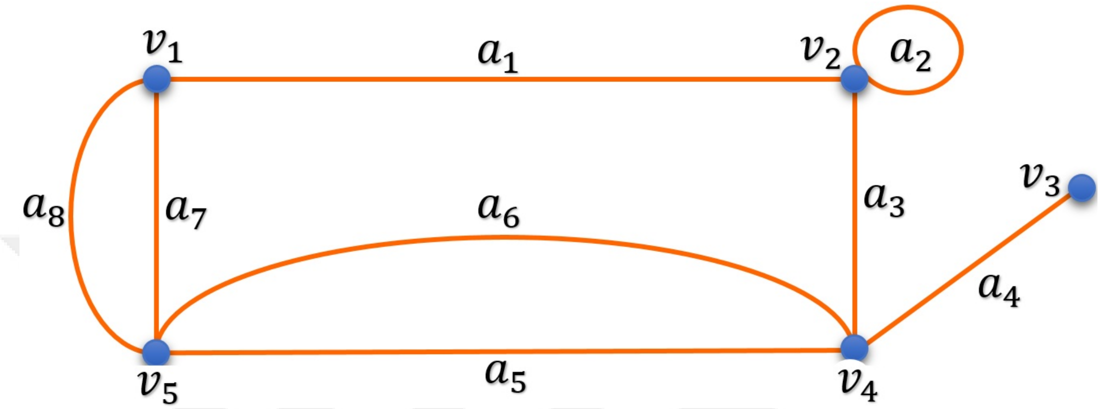



Figure 3.3. A graph with $5-$vertices and $8-$edges is called a $(5,8)$ graph

**Definition 8** *"A directed graph or digraph $G$ consists of a set $V$ of vertices and a set $E$ of edges such that $e \in E$ is associated with an ordered pair of vertices. In other words, if each edge of the graph $G$ has a direction then the graph is called directed graph" (Vasudev, 2006).*

In the diagram of directed graph, each edge $e = (u,v)$ is represented by an arrow or directed curve from initial point $u$ of $e$ to the terminal point $v$. Figure 3.4 is an example of a directed graph.

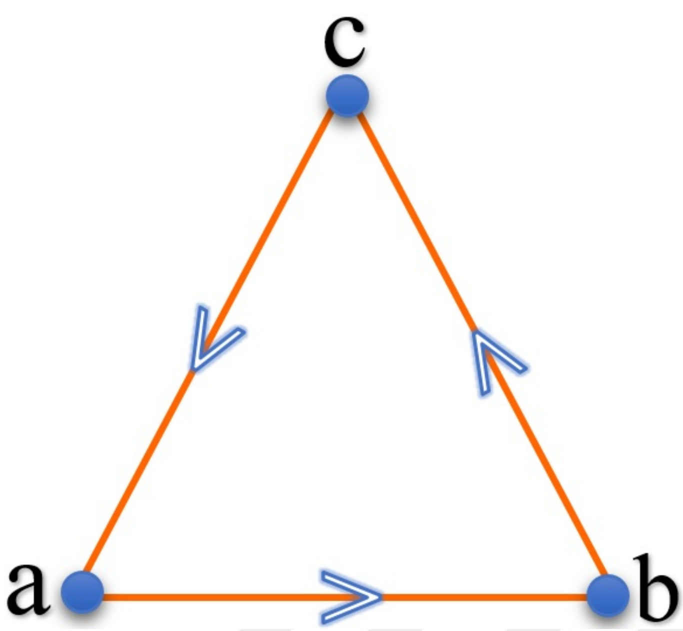


Figure 3.4. Directed graph

Suppose $e = (u, v)$ is a directed edge in a digraph, then

(i) $u$ is called the initial vertex of $e$ and $v$ is the terminal vertex of $e$

(ii) $e$ is said to be incident from $u$ and to be incident to $v$.

(iii) $u$ is adjacent to $v$, and $v$ is adjacent from $u$.

**Definition 9** *"An un-directed graph G consists of set V of vertices and a set E of edges such that each edge $e \in E$ is associated with an unordered pair of vertices. In other words, if each edge of the graph G has no direction then the graph is called un-directed graph" (Vasudev, 2006)*.

We can refer to an edge joining the vertex pair $i$ and $j$ as either $(i, j)$ or $(j, i)$. Figure 3.5 is an example of an undirected graph.

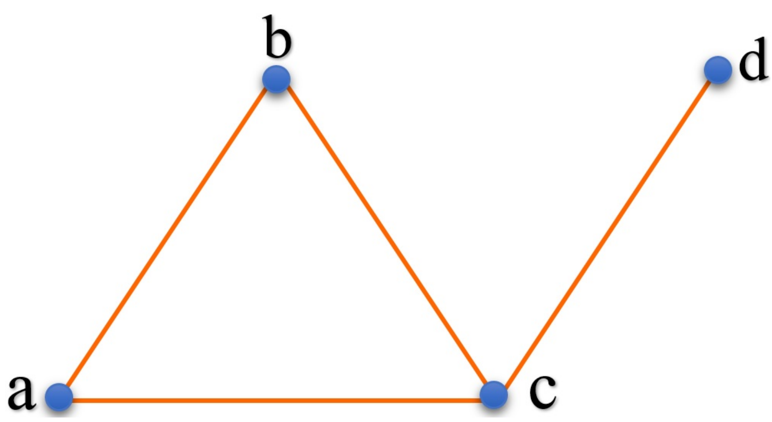


Figure 3.5. Un-Directed graph

**Definition 10** *"An edge of a graph that joins a node to itself is called loop or self loop" (Vasudev, 2006)***.**

i.e., a loop is an edge $(v_i, v_j)$ where $v_i = v_j$.

**Definition 11** *"A multi-graph is a graph which is permitted to have multiple edges (also called parallel edges) that is, edges that have the same end nodes" (Vasudev, 2006).*

Two edges $(v_i, v_j)$ and $(v_f, v_r)$ are parallel edges if $v_i = v_f$ and $v_j = v_r$.

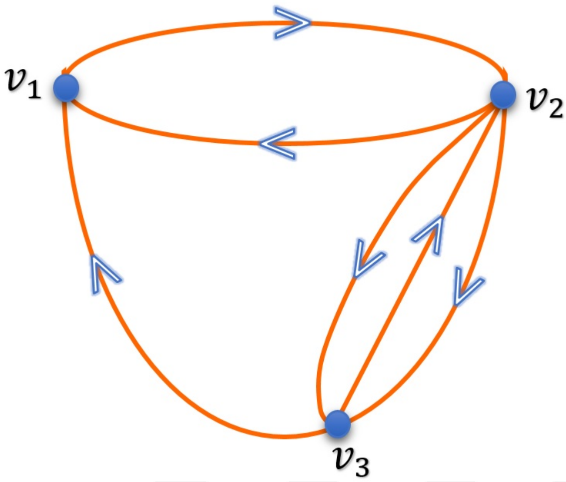



Figure 3.6.
Directed multiple

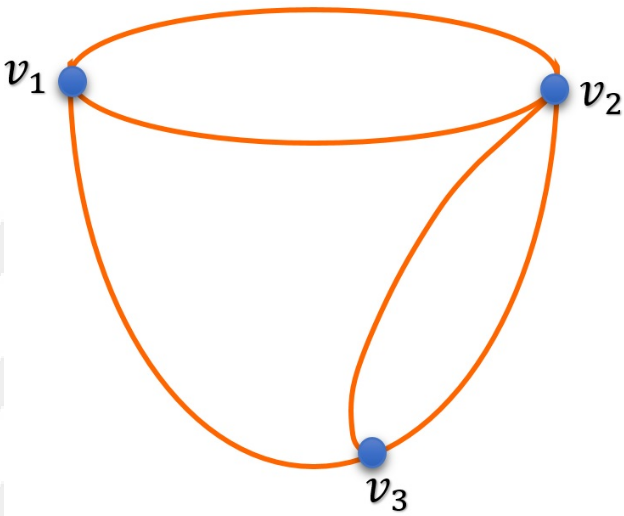



Figure 3.7.
Un-Directed multiple

In Figure 3.6, there are two parallel edges associated with $v_2$ and $v_3$. In Figure 3.7, there are two parallel edges joining nodes $v_1$ and $v_2$ and $v_2$ and $v_3$ (Vasudev, 2006).

**Definition 12** *"A graph in which loops and multiple edges are allowed, is called a pseudo graph."*

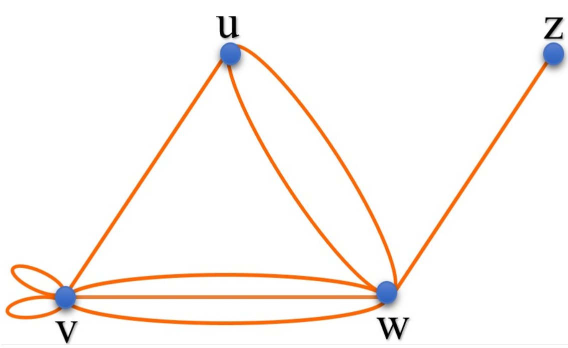



Figure 3.8.
Un-Directed Pseudo graph

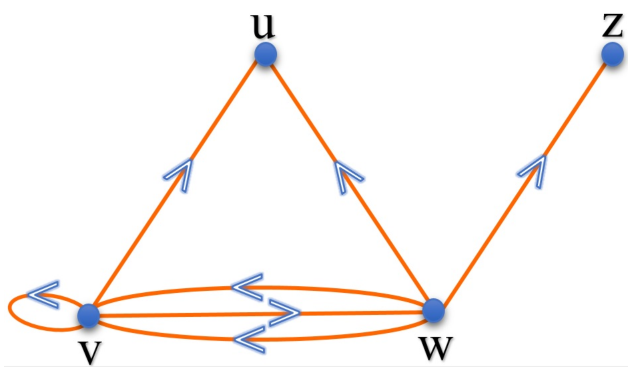



Figure 3.9.
Directed Pseudo graph

(Vasudev, 2006).

**Definition 13** *"A graph which has neither loops nor multiple edges. i.e., where each edge connects two distinct vertices and no two edges connect the same pair of vertices is called a simple graph" (Vasudev, 2006).*

Figure 3.4 and 3.5 represents simple un-directed and directed graph because the graphs do not contain loops and the edges are all distinct.

**Definition 14** *"A graph with finite number of vertices as well as a finite number of edges is called a finite graph. Otherwise, it is an infinite graph" (Vasudev, 2006).*

**Definition 15** *"The number of edges incident on a vertex $v_i$ with self-loops counted twice (is called the degree of a vertex $v_i$ and is denoted by $\deg_G(v_i)$ or $\deg v_i$ or $d(v_i)$" (Vasudev, 2006).*

The degrees of vertices in the graph $G$ and $H$ are shown in Figure 3.10 and 3.11.

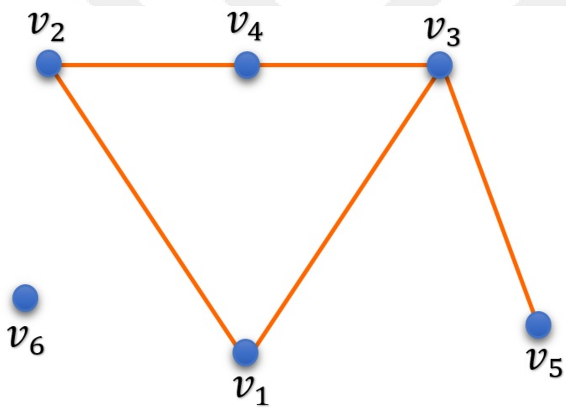



Figure 3.10.
A graph with 6−vertices and 5−edges

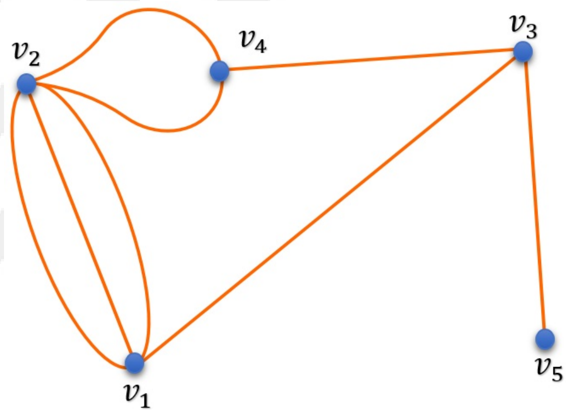



Figure 3.11.
A graph with 5−vertices and 8−edges

In $G$ as shown in Figure 3.10, $\deg_G(v_2) = 2 = \deg_G(v_4) = \deg_G(v_1)$, $\deg_G(v_3) = 3$ and $\deg_G(v_5) = 1$

In $H$ as shown in Figure 3.11 $\deg_H(v_2) = 5$, $\deg_H(v_4) = 3$, $\deg_H(v_3) = 5$, $\deg_H(v_1) = 4$ and $deg_H(v_5) = 1$ (Vasudev, 2006).

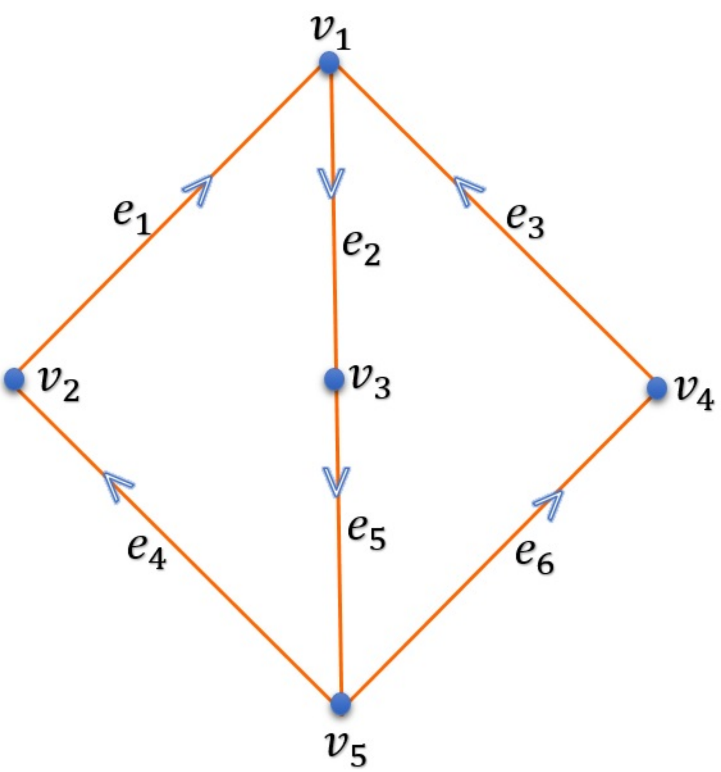
The degree of a vertex is some times also referred to as its valency (Vasudev, 2006).

## 3.3. Graphs and Matrices

### 3.3.1. Incidence matrix

"Let $G$ be a graph with $V(G) = \{1, \ldots, n\}$ and $E(G) = \{e_1, \ldots, e_m\}$ . Suppose each edge of $G$ is assigned an orientation, which is arbitrary but fixed. The (vertex-edge) incidence matrix of $G$, denoted by $Q(G)$, is the $n \times m$ matrix defined as follows. The rows and the columns of $Q(G)$ are indexed by $V(G)$ and $E(G)$, respectively. The $(i,j)$-entry of $Q(G)$ is 0 if vertex $i$ and edge $e_j$ are not incident, and otherwise it is 1 or $-1$ according as $e_j$ originates or terminates at $i$, respectively. We often denote $Q(G)$ simply by $Q$. Whenever we mention $Q(G)$ it is assumed that the edges of $G$ are oriented" (Bapat, 2010).

**Example 16** *Consider a graph as the following.*




Then Its incidence matrix is given by $Q$ as

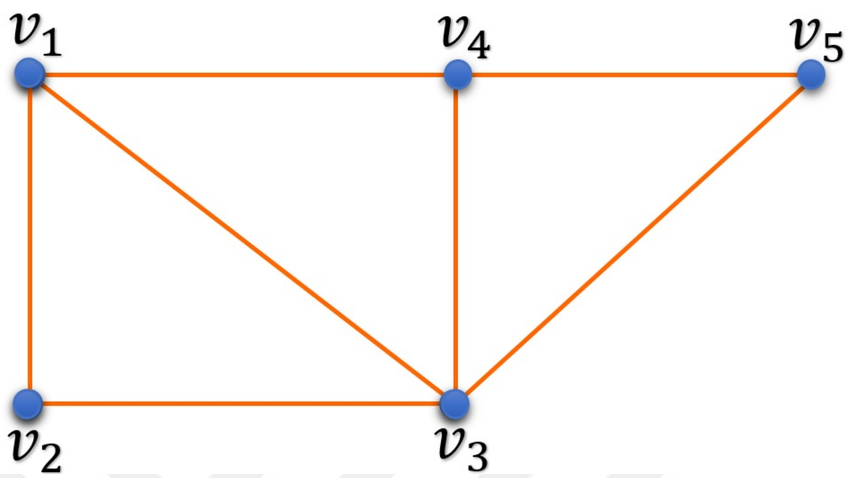
$$Q = \begin{pmatrix} -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}.$$

#### 3.3.2. Adjacency matrix

"Let $G$ be a graph with $V(G) = \{1, \ldots, n\}$ and $E(G) = \{e_1, \ldots, e_m\}$ . The adjacency matrix of $G$, denoted by $A(G)$, is the $n \times n$ matrix defined as follows. The rows and the columns of $A(G)$ are indexed by $V(G)$. If $i \neq j$ then the $(i,j)$-entry of $A(G)$

is 0 for vertices $i$ and $j$ non-adjacent, and the $(i,j)$-entry is 1 for $i$ and $j$ adjacent. The $(i,i)$-entry of $A(G)$ is 0 for $i = 1,\dots,n$. We often denote $A(G)$ simply by $A$" (Bapat, 2010).

**Example 17** *Consider a graph G as the following*




Then its adjacency matrix is given $A(G)$ as

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

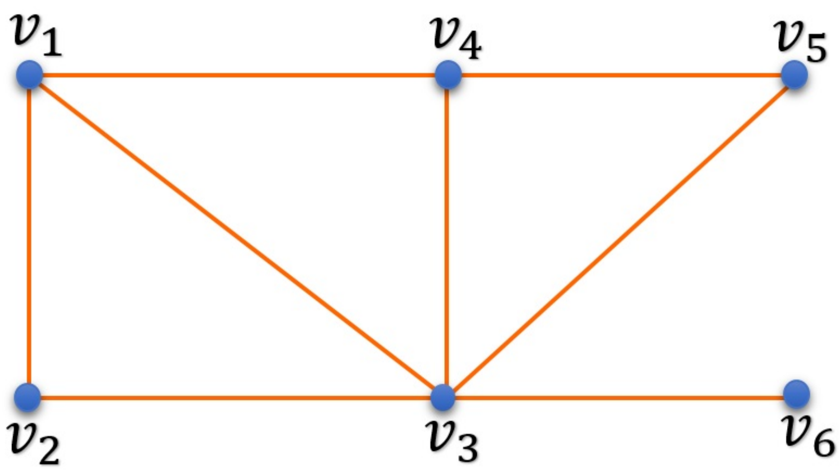
Clearly, $A$ is a symmetric matrix with zeros on the diagonal. For $i \neq j$, the principal sub-matrix of $A$ formed by the rows and the columns $i,j$ is the the zero matrix if $i \nsim j$ and otherwise it equals $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

#### 3.3.3. Laplacian matrix

"Let $G$ be a graph with $V(G) = \{1,\dots,n\}$ and $E(G) = \{e_1,\dots,e_m\}$ . The Laplacian matrix of $G$, denoted by $L(G)$, is the $n \times n$ matrix defined as follows. The rows and columns of $L(G)$ are indexed by $V(G)$. If $i \neq j$ then the $(i,j)$-entry of $L(G)$ is 0 if vertex $i$ and $j$ are not adjacent, and it is $-1$ if $i$ and $j$ are adjacent. The $(i,i)$-entry of $L(G)$ is $d_i$, the degree of the vertex $i, i = 1,2,\dots,n$ "(Bapat, 2010).

"Let $D(G)$ be the diagonal matrix of vertex degrees. If $A(G)$ is the adjacency matrix of $G$, then note that $L(G) = D(G) - A(G)$" (Bapat, 2010). "Suppose each edge of $G$ is assigned an orientation, which is arbitrary but fixed. Let $Q(G)$ be the incidence matrix of $G$. Then observe that $L(G) = Q(G)Q(G)'$. This can be seen as follows. The rows of $Q(G)$ are indexed by $V(G)$. The $(i,j)$-entry of $Q(G)Q(G)'$ is the inner product of the rows $i$ and $j$ of $Q(G)$. If $i = j$ then the inner product is clearly $d_i$, the degree of the vertex $i$. If $i \neq j$, then the inner product is $-1$ if the vertices $i$ and $j$ are adjacent, and zero otherwise" (Bapat, 2010).

**Example 18** *Consider a graph G as the following*




Then its Laplacian matrix is given by $L(G)$ as

$$L(G) = \begin{pmatrix} 3 & -1 & 0 & -1 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 & 0 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ -1 & -1 & -1 & -1 & 5 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

## 3.4. Bipartite Graph and Perfect Matching

**Definition 19** *"A bipartite graph G is a graph whose vertex set V can be partitioned into two subsets $V_1$ and $V_2$ such that every edge of G joins a vertex in $V_1$ and a vertex in $V_2$. In other words, there are no edges which connect two vertices in $V_1$ or in $V_2$" (Diestel, 2005).*

**Example 20** *Figure 3.12 can be given as an example for a bipartite graph.*

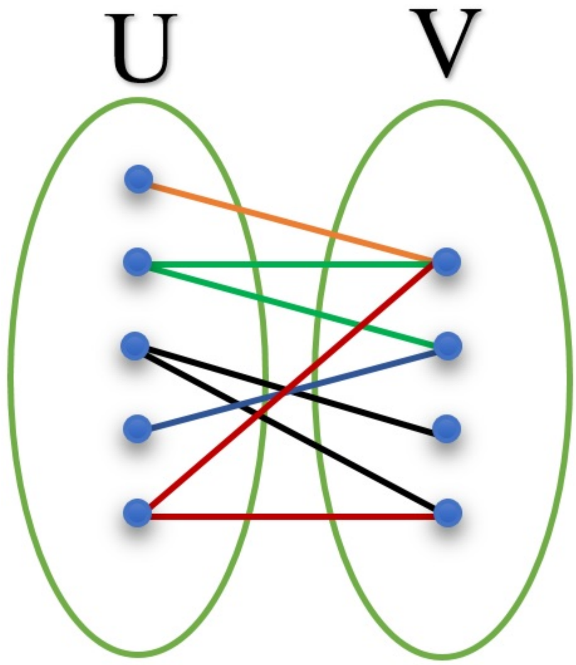



Figure 3.12. Bipartite Graph

Bipartite graphs have a huge of applications in modern science. For example;

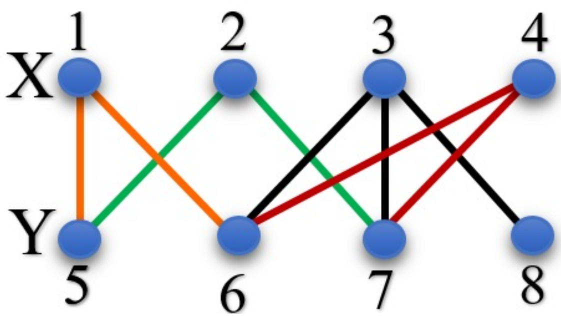
- When modeling relations between two different classes of objects, bipartite graphs very often arise naturally. For instance, ″a graph of football players and clubs, with an edge between a player and a club if the player has played for that club, is a natural example of an affiliation network, a type of bipartite graph used in social network analysis″ (Wasserman and Faust, 1994).

- Another example where bipartite graphs appear naturally is in the railway optimization problem, in which ″the input is a schedule of trains and their stops, and the goal is to find a set of train stations as small as possible such that every train visits at least one of the chosen stations. This problem can be modeled as a dominating set problem in a bipartite graph that has a vertex for each train and each station and an edge for each pair of a station and a train that stops at that station″ (Niedermeier, 2002).

- A third example is in the academic field of numismatics. ″Ancient coins are made using two positive impressions of the design (the obverse and reverse). The charts numismatists produce to represent the production of coins are bipartite graphs″ (Bracey, 2012).

**Definition 21** *Bipartite adjacency matrix ″Let G be a bipartite graph whose vertex set V is partitioned into two subsets* $V_1$ *and* $V_2$ *such that* $|V_1| = |V_2| = n$*. We construct the bipartite adjacency matrix* $B(G) = (b_{ij})$ *of G as following:* $b_{ij} = 1$ *if and only if G contains an edge from* $v_i \in V_1$ *to* $v_j \in V_2$*, and otherwise* $b_{ij} = 0$*″ (Minc, 1978).*

**Example 22** *Let G be a bipartite graph as the following*

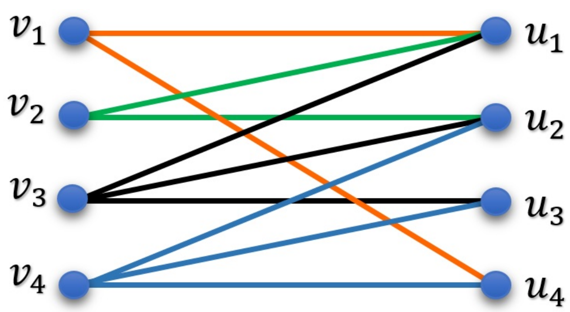



Then its bipartite adjacency matrix $B(G)$ is

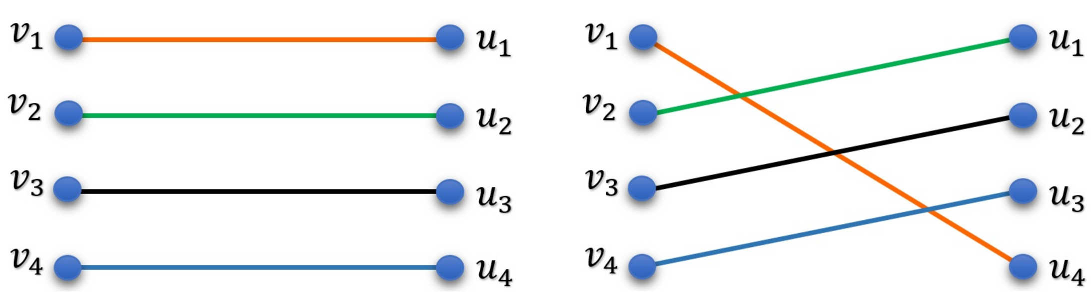
$$B(G) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

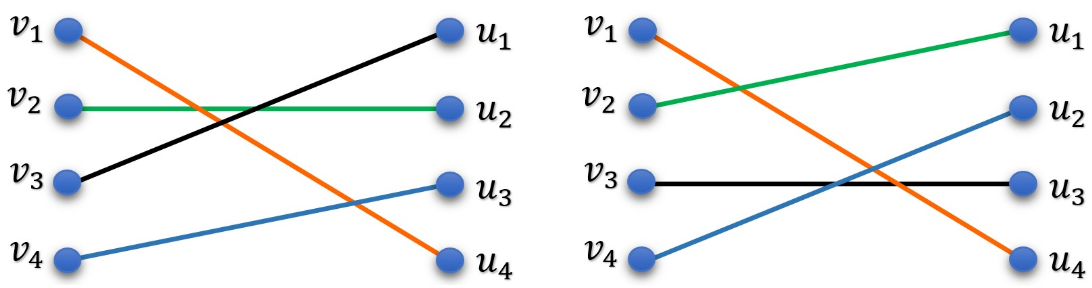
**Definition 23 (Perfect matching)** *"A perfect matching (or $1$-factor) of a graph is a matching in which each vertex has exactly one edge incident on it. Namely, every vertex in the graph has degree $1$" (Minc, 1978).*

**Example 24** *Let G be a bipartite graph as the following*




*Then its perfect matchings can be given as*






**Lemma 25** *"The number of perfect matchings of a bipartite graph is equal to the permanent of its bipartite adjacency matrix" (Minc, 1978).*

## 3.5. Permanents

**Definition 26** *"The permanent of an $n \times n$ matrix $A = (a_{ij})$ is defined by*

$$\text{per}A = \sum_{\sigma \epsilon S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

*where the summation extends over all permutations $\sigma$ of the symmetric group $S_n$" (Minc, 1978).*

"The permanent of a matrix is analogous to the determinant, where all of the signs used in the Laplace expansion of minors are positive" (Minc, 1978).

**Lemma 27** *"Let $A$ be an $n \times n$ matrix, then*

$$\text{per}\,(PAQ) = \text{per}A \tag{8}$$

*for all permutation matrices $P$ and $Q$ of order $n$" (Brualdi and Cvetkovic, 2009).*

**Lemma 28** *"Let $B$ and $C$ are square matrices. If*

$$A = \begin{pmatrix} B & 0 \\ X & C \end{pmatrix},$$

*then*

$$\text{per}A = \text{per}B\text{per}C \tag{9}$$

*" (Brualdi and Cvetkovic, 2009).*

**Lemma 29** *"Let $\{T_n,\ n = 1, 2, \dots\}$ be sequence of tridiagonal matrices of type $n \times n$ in the following form*

$$T_n = \begin{pmatrix} t_{1,1} & t_{1,2} & 0 & \cdots & \cdots & 0 \\ t_{2,1} & t_{2,2} & t_{2,3} & \ddots & & \vdots \\ 0 & t_{3,2} & t_{3,3} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & t_{n-1,n} \\ 0 & \cdots & \cdots & 0 & t_{n,n-1} & t_{n,n} \end{pmatrix}.$$

*Then the successive permanents of $T_n$ are given by the recursive formula:*

$$\begin{aligned} \mathrm{per}T_1 &= t_{11}, \\ \mathrm{per}T_2 &= t_{11}t_{22} + t_{12}t_{21}, \\ \mathrm{per}T_n &= t_{n,n}\mathrm{per}T_{n-1} + t_{n-1,n}t_{n,n-1}\mathrm{per}T_{n-2} \end{aligned}$$

*" (Kılıç and Taşcı, 2007)*

**Lemma 30** *"Let $H_n = (h_{i,j})$ be the $n \times n$ Hessenberg matrices in the following form*

$$H_n = \begin{pmatrix} h_{1,1} & h_{1,2} & 0 & \cdots & \cdots & 0 \\ h_{2,1} & h_{2,2} & h_{2,3} & \ddots & & \vdots \\ h_{3,1} & h_{3,2} & h_{3,3} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & h_{n-1,n} \\ h_{n,1} & \cdots & \cdots & h_{n,n-2} & h_{n,n-1} & h_{n,n} \end{pmatrix}$$

*where $h_{i,j} = 0$ if $j > i+1$ and $h_{i,i+1} \neq 0$ for some $i$. Then for $n \geq 2$*

$$\mathrm{per}H_n = h_{n,n}\mathrm{per}H_{n-1} + \sum_{r=1}^{n-1}\left(h_{n,r}\prod_{j=r}^{n-1} h_{j,j+1}\mathrm{per}H_{r-1}\right)$$

*with $\mathrm{per}H_0 = 1$ and $\mathrm{per}H_1 = h_{1,1}$" (Kaygısız and Şahin, 2013).*

**Definition 31** *"Let $A = [a_{ij}]$ be an $m \times n$ real matrix with row vectors $\alpha_1, \alpha_2, ..., \alpha_m$. We say $A$ is contractible on column (resp. row) $k$ if column (resp. row) $k$ contains exactly two nonzero entries. Suppose $A$ is contractible on column $k$ with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from $A$ by replacing row $i$ with $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row $j$ and column $k$ is called the contraction of $A$ on column $k$ relative to rows $i$ and $j$. If $A$ is contractible on row $k$ with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = \left[A_{ij:k}^T\right]^T$ is called the contraction of $A$ on row $k$ relative to columns $i$ and $j$. We say that $A$ can be contracted to a matrix $B$ if either $B = A$ or there exist* *matrices $A_0, A_1, ..., A_t$ $(t \geq 1)$ such that $A_0 = A,\ A_t = B$, and $A_r$ is a contraction of $A_{r-1}$ for $r = 1, ..., t$" (Brualdi and Gibson, 1977).*

**Lemma 32** *"Let $A$ be a nonnegative integral matrix of order $n$ for $n > 1$ and let $B$ be a contraction of $A$. Then*

$$perA = perB \tag{10}$$

*" (Brualdi and Gibson, 1977).*

## 3.6. Number Sequences

**Definition 33** *"The well-known Fibonacci sequence $\{F(n)\}$ is defined by the recurrence relation*

$$F(n) = F(n-1) + F(n-2), \tag{11}$$

*with $F(0) = 0$ and $F(1) = 1$ for $n \geq 2$" (Koshy, 2001).*

The number $F(n)$ is called $n$th Fibonacci numbers. The Fibonacci numbers are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...$$

for $n = 0, 1, 2, ....$ The Fibonacci sequence is named as $A000045$ (The OEIS, 2013).

**Definition 34** *"The k-generalized Fibonacci sequence $\{g^k(n)\}$ is defined as*

$$g^k(1) = \ldots = g^k(k-2) = 0,\ g^k(k-1) = g^k(k) = 1$$

*and for $n > k \geq 2$,*

$$g^k(n) = g^k(n-1) + g^k(n-2) + ... + g^k(n-k)$$

*" (Lee et al., 1997).*

"$\{g^k(n)\}$ is also called $k$- Fibonacci sequence. We call $g^k(n)$ the $n^{th}$ $k$-Fibonacci number. By the definition of the $k$-Fibonacci sequence, we know that

$$\begin{aligned}
g^k(k+1) &= g^k(k) + g^k(k-1) \\
&= 1+1 = 2, \\
g^k(k+2) &= g^k(k+1) + g^k(k) + g^k(k-1) \\
&= 2+1+1 = 2^2, \\
g^k(k+3) &= g^k(k+2) + g^k(k+1) + g^k(k) + g^k(k-1) \\
&= 2^2+2+1+1 = 2^3, \\
&\vdots \\
g^k(2k-2) &= g^k(2k-3) + \ldots + g^k(k) + g^k(k-1) \\
&= 2^{k-3} + \ldots + 2 + 1 + 1 = 2^{k-2}, \\
g^k(2k-1) &= g^k(2k-2) + \ldots + g^k(k) + g^k(k-1) \\
&= 2^{k-2} + 2^{k-3} + \ldots + 2 + 1 + 1 = 2^{k-1}.
\end{aligned}$$

Thus, we have that $g^k(j) = 2^{j-k}$ for $j = k, k+1, \ldots, 2k-1$. For example, if $k = 2$, then $\{g^2(n)\}$ is the Fibonacci sequence. If $k = 5$, then the 5-Fibonacci sequence is

$$0, 0, 0, 1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, 3535, 6930, \ldots$$

" (Lee et al., 1997).

"Let $E$ be the $1 \times (k-1)$- matrix all of whose entries are ones and let in be the identity matrix of order $n$. For any $k \geq 2$, the fundamental recurrence relation, $n > k$,

$$g^k(n) = g^k(n-1) + g^k(n-2) + ... + g^k(n-k)$$

can be defined by the vector recurrence relation

$$\begin{pmatrix} g^k(n+1) \\ g^k(n+2) \\ \vdots \\ g^k(n+k) \end{pmatrix} = Q_k \begin{pmatrix} g^k(n) \\ g^k(n+1) \\ \vdots \\ g^k(n+k-1) \end{pmatrix} \tag{12}$$

where

$$Q_k = \begin{pmatrix} 0 & I_{k-1} \\ 1 & E \end{pmatrix}.$$

By applying (12), we have

$$\begin{pmatrix} g^k(n+1) \\ g^k(n+2) \\ \vdots \\ g^k(n+k) \end{pmatrix} = Q_k^n \begin{pmatrix} g^k(1) \\ g^k(2) \\ \vdots \\ g^k(k) \end{pmatrix}$$

" (Lee et al, 1997).

One can find the relationships between the $k$-Fibonacci numbers and their associated matrices (Lee and Lee, 1995; Lee et al, 1997; Miles, 1960).

**Definition 35** *"The well-known Lucas sequence $\{L(n)\}$ is defined by the recurrence relation, for $n \geq 2$*

$$L(n) = L(n-1) + L(n-2),$$

*with $L(0) = 2$ and $L(1) = 1$" (Koshy, 2001).*

The number $L(n)$ is called $n^{th}$ Lucas numbers. The Lucas numbers are

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, ...$$

for $n = 0, 1, 2, ....$ The Lucas sequence is named as $A000032$ (The OEIS, 2013).

The Lucas numbers are related to the Fibonacci numbers by the identities

$$L(n) = F(n-1) + F(n+1) = F(n) + 2F(n-1).$$

The $k$-generalized Lucas sequence $\{l^k(n)\}$is defined by

$$l^k(n) = g^k(n-1) + g^k(n+k-1).$$

"$\{l^k(n)\}$ is also called $k$-Lucas sequence. We call $l^k(n)$ the $n^{th}$ $k$-Lucas number. Then we have $l^k(j) = 2^{j-1}$, $j = 1, 2, \ldots, k-1$, and $l^k(k) = 1 + 2^{k-1}$. If $k = 2$, then $l^2(n) = L(n)$. For example, if $k = 5$, then the 5-Lucas sequence is

$$1, 2, 4, 8, 17, 32, 63, 124, 244, 480, 943, 1854, 3645, 7166, \ldots$$

" (Lee, 2000).

**Lemma 36** *"For $n > k$,*

$$l^k(n) = l^k(n-1) + l^k(n-2) + ... + l^k(n-k)$$

*" (Lee, 2000).*

**Proof.** See (Lee, 2000). ■

**Remark 37** *The following well-known identity gives the relationship between Lucas numbers and Fibonacci numbers. For $n \geq 1$,*

$$L(n) = F(n-1) + F(n+1) = F(n) + 2F(n-1). \tag{13}$$

**Definition 38** *"The well-known Pell sequence $\{P(n)\}$ is defined by the recurrence relation, for $n \geq 2$*

$$P(n) = 2P(n-1) + P(n-2)$$

*with $P(0) = 0$ and $P(1) = 1$" (Horadam, 1988).*

The number $P(n)$ is called $n$th Pell number. The Pell numbers are

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, \ldots .$$

for $n = 0, 1, 2, \ldots$. The Pell sequence is named as $A000129$ (The OEIS, 2013).

**Definition 39** *"The generalized k-Pell sequence $\{P^k(n)\}$ is defined as*

$$P^k(1-k) = 1,\ P^k(2-k) = \ldots = P^k(-1) = P^k(0) = 0$$

*and for $n > k \geq 2$*

$$P^k(n) = 2P^k(n-1) + P^k(n-2) + \ldots + P^k(n-k)$$

*" (Kılıc, 2008).*

"First few terms of the generalized $k$-Pell numbers are as the following:

$$\begin{aligned}
P^k(1) &= 2P^k(0) + P^k(-1) + \ldots + P^k(1-k) = 1, \\
P^k(2) &= 2P^k(1) + P^k(0) + \ldots + P^k(2-k) = 2(1) = 2, \\
P^k(3) &= 2P^k(2) + P^k(1) + \ldots + P^k(3-k) = 2(2) + 1 = 5, \\
P^k(4) &= 2P^k(3) + P^k(2) + \ldots + P^k(4-k) = 2(5) + 2 + 1 = 13, \\
P^k(5) &= 2P^k(4) + P^k(3) + \ldots + P^k(5-k) = 2(13) + 5 + 2 + 1 = 34, \ldots .
\end{aligned}$$

It is clearly seen that $\{P^k(n)\}$ is the usual Pell sequence $\{P(n)\}$ for $k = 2$" (Kılıc, 2008).

**Definition 40** *"The well-known Jacobsthal sequence* $\{J(n)\}$ *is defined by the recurrence relation, for* $n \geq 2$

$$J(n) = J(n-1) + 2J(n-2) \tag{14}$$

*with* $J(0) = 0$ *and* $J(1) = 1$*" (Horadam, 1988).*

The number $J(n)$ is called $n^{th}$ Jacobsthal number. The Jacobsthal numbers are

$$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, ... .$$

for $n = 0, 1, 2, ....$ The Jacobsthal sequence is named as $A001045$ (The OEIS, 2013).

**Definition 41** *"The Padovan sequence* $\{\mathcal{P}(n)\}$ *is defined by the recurrence relation, for* $n > 2$

$$\mathcal{P}(n) = \mathcal{P}(n-2) + \mathcal{P}(n-3)$$

*with* $\mathcal{P}(0) = \mathcal{P}(1) = \mathcal{P}(2) = 1$*" (Shannon et al, 2006).*

The number $\mathcal{P}(n)$ is called $n^{th}$ Padovan number. The Padovan numbers are

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, ... .$$

for $n = 0, 1, 2, ....$ The Padovan sequence is named as $A000931$ (The OEIS, 2013).

**Definition 42** *"The Perrin sequence* $\{R(n)\}$ *is defined by the recurrence relation, for* $n > 2$

$$R(n) = R(n-2) + R(n-3)$$

*with* $R(0) = 3$*,* $R(1) = 0$ $R(2) = 2$*" (Adams and Shanks, 1982).*

The number $R(n)$ is called $n$th Perrin number. The Perrin numbers are

$$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, ... .$$

for $n = 0, 1, 2, ....$ The Perrin sequence is named as $A001608$ (The OEIS, 2013).

**Definition 43** *"The* $(k, \alpha)$*-sequence* $\{s_\alpha^k(n)\}$ *is defined as*

$$\begin{aligned} s_\alpha^k(n) &= a_1 f^k(n+k-2) + a_2 f^k(n+k-3) + \ldots + a_m f^k(n+k-m-1) \\ &= \sum_{i=1}^{m} a_i f^k(n-1+k-i), \end{aligned}$$

*for a fixed* $k \geq 2$*,* $n \geq 1$ *and* $\alpha = (a_1, a_2, \ldots, a_m) \in R^m$*, where* $R$ *is a ring" (Shiu et al, 2003).*

The number $s_\alpha^k(n)$ is called $n^{th}$ $(k,\alpha)$-number. Note that, if $\alpha = (1,1,\ldots,1) \in \mathbb{Z}^k$, then $s_\alpha^k(n)$ is the $(n-1+k)^{th}$ $k$-Fibonacci number $g^k(n-1+k)$; if $\alpha = (1,0,\ldots,0,1) \in \mathbb{Z}^{k+1}$, then $s_\alpha^k(n)$ is the $(n-1)$st $k$-Lucas number $l^k(n-1)$.

**Definition 44** *"The Mersenne sequence $\{M(n)\}$ is defined by the recurrence relation, for $n > 2$*

$$M(n) = 2M(n-1) + 1 \tag{15}$$

*with initial conditions $M(0) = 0$ and $M(1) = 1$" (Catarino et al, 2016).*

Since this recurrence is inhomogeneous, substituting $n$ by $n+1$, we obtain the new form

$$M(n+1) = 2M(n) + 1. \tag{16}$$

Subtracting (15) to (16), we have that $M(n+1) - M(n) = 2M(n) + 1 - 2M(n-1) - 1$ and then

$$M(n+1) = 3M(n) - 2M(n-1),$$

other form for the recurrence relation of Mersenne sequence, with initial conditions $M(0) = 0$ and $M(1) = 1$.

The number $M(n)$ is called $n^{th}$ Mersenne number. The Mersenne numbers are

$$0, 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047... .$$

for $n = 0, 1, 2, ....$ The Mersenne sequence is named as $A000225$ (The OEIS, 2013).

The first few values of these famous integer sequences can be seen at the following table:

Table 3.1. Some famous integer sequences and their several values

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $F(n)$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | $\cdots$ |
| $L(n)$ | 2 | 1 | 3 | 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 | 199 | 322 | $\cdots$ |
| $P(n)$ | 0 | 1 | 2 | 5 | 12 | 29 | 70 | 169 | 408 | 985 | 2378 | 5741 | 13860 | $\cdots$ |
| $J(n)$ | 0 | 1 | 1 | 3 | 5 | 11 | 21 | 43 | 85 | 171 | 341 | 683 | 1365 | $\cdots$ |
| $\mathcal{P}(n)$ | 1 | 1 | 1 | 2 | 2 | 3 | 4 | 5 | 7 | 9 | 12 | 16 | 21 | $\cdots$ |
| $R(n)$ | 3 | 0 | 2 | 3 | 2 | 5 | 5 | 7 | 10 | 12 | 17 | 22 | 29 | $\cdots$ |
| $M(n)$ | 0 | 1 | 3 | 7 | 15 | 31 | 63 | 127 | 255 | 511 | 1023 | 2047 | 4095 | $\cdots$ |

# 4. SOME SPECIAL INTEGER SEQUENCES RELATED TO BIPARTITE GRAPHS

## 4.1. Jacobsthal Numbers and Associated Bipartite Graphs

In this section, we consider a bipartite graph. Then we show that the numbers of perfect matchings of this graph generate the Jacobsthal numbers by the contraction method. Finally, we give a Maple procedure in order to calculate the numbers of perfect matchings of above-mentioned bipartite graph.

**Theorem 45** *Let $G(A_n)$ be the bipartite graph with bipartite adjacency matrix $A_n$ has the form*

$$A_n = \begin{pmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 & \cdots \\ 1 & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & \cdots & \cdots & 1 \\ \vdots & 0 & 1 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 1 & 1 & 1 \\ \vdots & & & & 0 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}. \tag{17}$$

*Then, the number of perfect matchings of $G(A_n)$ is nth Jacobsthal number equal to $J(n)$.*

**Proof.** If $n = 3$; then we have

$$\text{per}A_3 = \text{per}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 3 = J(3).$$

Let $A_n^k$ be the $k$th contraction of $A_n$, $1 \leq k \leq n-2$. Since the definition of the matrix $A_n$; the matrix $A_n$ can be contracted on column 1 so that

$$A_n^1 = \begin{pmatrix} 1 & 2 & 1 & 2 & \cdots & \cdots & 1 & 2 \\ 1 & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ & 1 & 1 & 1 & 1 & \cdots & \cdots & 1 \\ & & 1 & \ddots & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & & \ddots & 1 & 1 & 1 \\ & & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{pmatrix}_{(n-1)\times(n-1)}$$

Since the matrix $A_n^1$ can be contracted on column 1 and $J(3) = 3$ and $J(2) = 1$

$$A_n^2 = \begin{pmatrix} 3 & 2 & 3 & 2 & \cdots & \cdots & 3 & 2 \\ 1 & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ & 1 & 1 & 1 & 1 & \cdots & \cdots & 1 \\ & & 1 & \ddots & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & & \ddots & 1 & 1 & 1 \\ & & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{pmatrix}_{(n-2)\times(n-2)}$$

$$= \begin{pmatrix} J(3) & 2J(2) & 3 & 2 & \cdots & \cdots & 3 & 2 \\ 1 & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ & 1 & 1 & 1 & 1 & \cdots & \cdots & 1 \\ & & 1 & \ddots & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & & \ddots & 1 & 1 & 1 \\ & & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{pmatrix}_{(n-2)\times(n-2)}$$

Furthermore, the matrix $A_n^2$ can be contracted on column 1 and $J(4) = 5$ so that

$$A_n^3 = \begin{pmatrix} 5 & 6 & 5 & 6 & \cdots & \cdots & 5 & 6 \\ 1 & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ & 1 & 1 & 1 & 1 & \cdots & \cdots & 1 \\ & & 1 & \ddots & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & & \ddots & 1 & 1 & 1 \\ & & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{pmatrix}_{(n-3)\times(n-3)}$$

$$= \begin{pmatrix} J(4) & 2J(3) & 5 & 6 & \cdots & \cdots & 5 & 6 \\ 1 & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ & 1 & 1 & 1 & 1 & \cdots & \cdots & 1 \\ & & 1 & \ddots & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & & \ddots & 1 & 1 & 1 \\ & & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{pmatrix}_{(n-3)\times(n-3)}$$

Continuing this process, we have

$$A_n^k = \begin{pmatrix} J(k+1) & 2J(k) & \cdots & J(k+1) & 2J(k) & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ & 1 & 1 & 1 & 1 & \cdots & \cdots & 1 \\ & & 1 & \ddots & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & & \ddots & 1 & 1 & 1 \\ & & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{pmatrix}_{(n-k)\times(n-k)}$$

for $3 \leq k \leq n-4$. Hence,

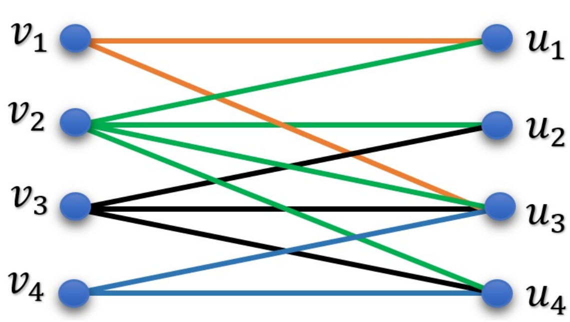
$$A_n^{n-3} = \begin{pmatrix} J(n-2) & 2J(n-3) & J(n-2) \\ 1 & 1 & 1 \\ & 1 & 1 \end{pmatrix}_{3\times 3}$$

which, by contraction of $A_n^{n-3}$ on column 1, gives

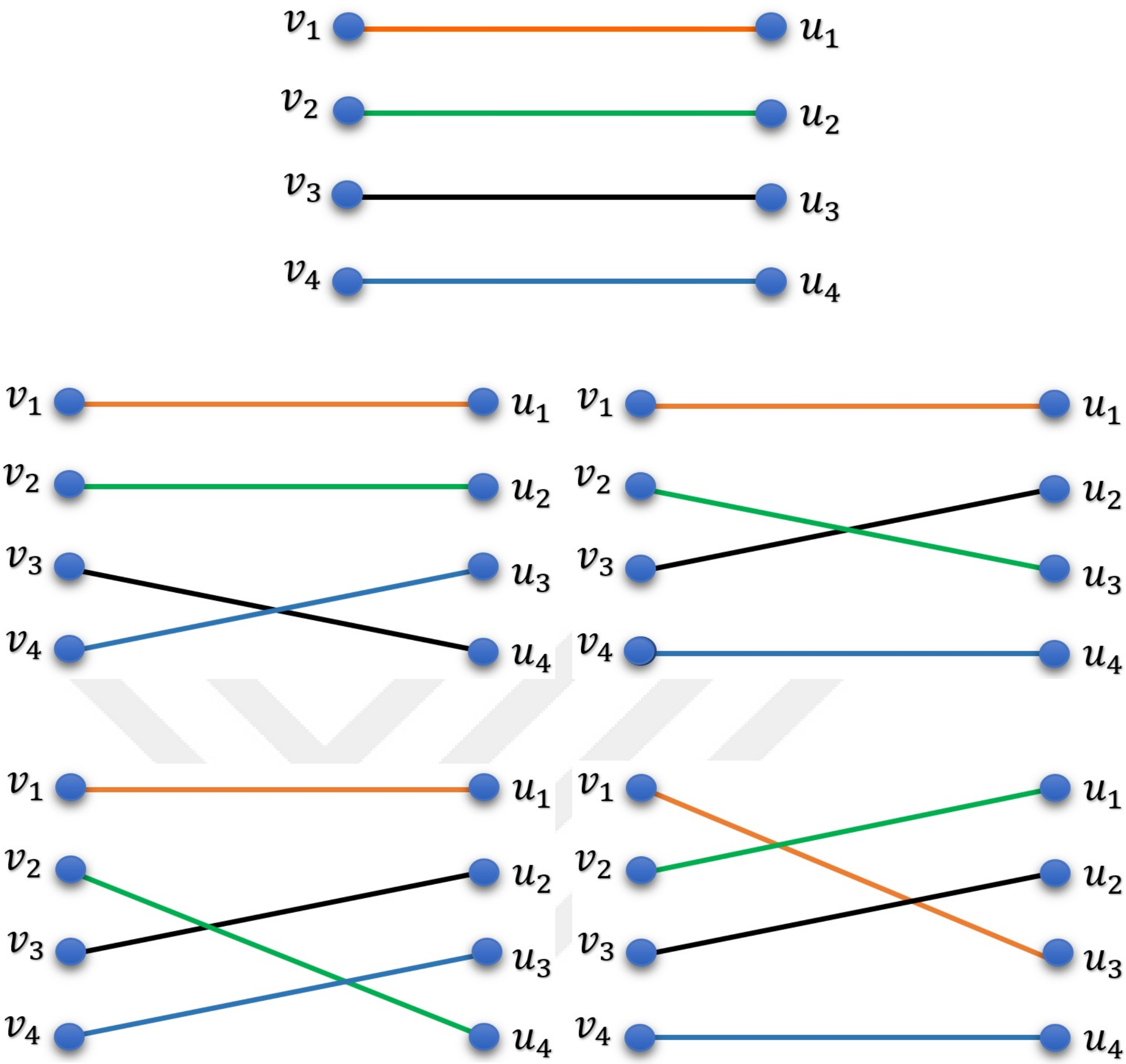
$$A_n^{n-2} = \begin{pmatrix} J(n-1) & 2J(n-2) \\ 1 & 1 \end{pmatrix}_{2\times 2}$$

By applying equation(10), we obtain per$A_n$ = per$A_n^{n-2} = J(n-1) + 2J(n-2)$ and by equation (14), we have $J(n) = J(n-1) + 2J(n-2)$. So that per$A_n = J(n)$, which is desired. ■

**Example 46** *Let $G(A_4)$ be a bipartite graph whose bipartite adjacency matrix is $A_4$ given by (17) for $n = 4$. Then the bipartite graph $G(A_4)$ can be seen as:*




*and its perfect matchings can be given as:*


$v_1$ $u_1$
$v_2$ $u_2$
$v_3$ $u_3$
$v_4$ $u_4$
$v_1$ $u_1$
$v_2$ $u_2$
$v_3$ $u_3$
$v_4$ $u_4$
$v_1$ $u_1$
$v_2$ $u_2$
$v_3$ $u_3$
$v_4$ $u_4$
$v_1$ $u_1$
$v_2$ $u_2$
$v_3$ $u_3$
$v_4$ $u_4$
$v_1$ $u_1$
$v_2$ $u_2$
$v_3$ $u_3$
$v_4$ $u_4$

### 4.1.1. Maple procedure

The following Maple procedure calculates the numbers of perfect matchings of bipartite graph $G(A_n)$ given in Theorem 45.

```
> restart:
with(LinearAlgebra):
permanent:=proc(n)
local i, j, r, f, A;
f := (i, j) - >piecewise(i = 1 and jmod2 = 1, 1, i>1 and j - i> - 2, 1, 0);
A:=Matrix(n, n, f) :
for r from 0 to n - 2 do
print(r,A):
for j from 2 to n - r do
A[1, j] := A[2, 1] * A[1, j] + A[1, 1] * A[2, j] :
od:
A:=DeleteRow(DeleteColumn(Matrix(n - r, n - r, A), 1), 2):
od:
print(r,eval(A)):
end proc:with(LinearAlgebra):
permanent(n);
```

## 4.2. Bipartite Graphs Associated with Circulant Matrices

In this section, we firstly introduce two lemmas related to bipartite graphs associated with Fibonacci numbers. After that, we define a bipartite graph associated with $n \times n$ $(0,1)$-circulant matrix whose the numbers of perfect matchings generate the Lucas numbers. Finally, we give some Maple procedures in order to calculate the numbers of perfect matchings of above-mentioned bipartite graph.

**Lemma 47** *Let $G(U_n)$ be the bipartite graph with bipartite adjacency matrix $U_n$ has the form*

$$U_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & & \ddots & \ddots & 0 \\ 1 & 1 & 1 & \ddots & & \ddots & 0 \\ 0 & 1 & 1 & \ddots & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 \end{pmatrix}_{n\times n}. \tag{18}$$

*Then, the number of perfect matchings of $G(U_n)$ is $F(n)+1$, where $F(n)$ is nth Fibonacci number.*

**Proof.** Let $U_n^k$ be the $k$th contraction of $U_n$, $1 \leq k \leq n-3$. Since the definition of the matrix $U_n$; the matrix $U_n$ can be contracted according to last column so that

$$U_n^1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & & \ddots & \ddots & 0 \\ 1 & 1 & 1 & \ddots & & \ddots & 0 \\ 0 & 1 & 1 & \ddots & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 \end{pmatrix}_{(n-1)\times(n-1)}.$$

Since the matrix $U_n^1$ can be contracted according to last column and $F(3) = 2$ and $F(2) = 1$

$$U_n^2 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & & \ddots & \ddots & 0 \\ 1 & 1 & 1 & \ddots & & \ddots & 0 \\ 0 & 1 & 1 & \ddots & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 \end{pmatrix}_{(n-2)\times(n-2)}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & F(2) & F(3) \\ 1 & 1 & 0 & & \ddots & \ddots & 0 \\ 1 & 1 & 1 & \ddots & & \ddots & 0 \\ 0 & 1 & 1 & \ddots & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 \end{pmatrix}_{(n-2)\times(n-2)}.$$

Furthermore, the matrix $U_n^2$ can be contracted according to last column so that

$$U_n^3 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & F(3) & F(4) \\ 1 & 1 & 0 & & \ddots & \ddots & 0 \\ 1 & 1 & 1 & \ddots & & \ddots & 0 \\ 0 & 1 & 1 & \ddots & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 \end{pmatrix}_{(n-3)\times(n-3)}.$$

Continuing this process, we have

$$U_n^k = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & F(k) & F(k+1) \\ 1 & 1 & 0 & & \ddots & \ddots & 0 \\ 1 & 1 & 1 & \ddots & & \ddots & 0 \\ 0 & 1 & 1 & \ddots & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 \end{pmatrix}_{(n-k)\times(n-k)}$$

for $1 \leq k \leq n-3$. Hence,

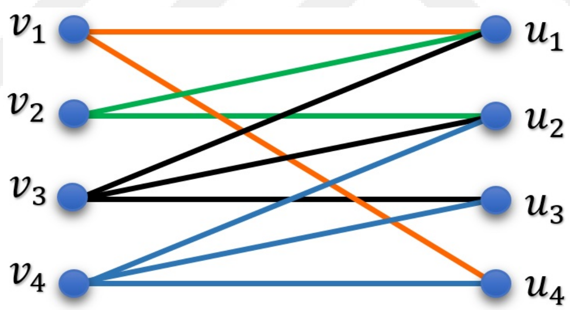
$$U_n^{n-3} = \begin{pmatrix} 1 & F(n-3) & F(n-2) \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}_{3\times 3}$$

which, by contraction of $U_n^{n-3}$ according to last column, gives

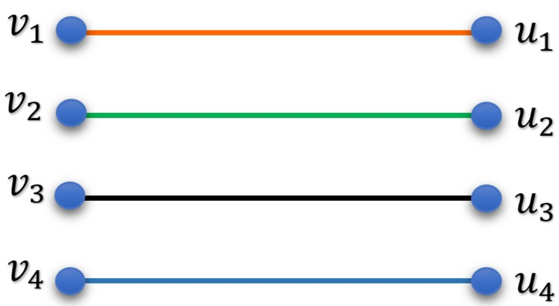
$$\begin{aligned} U_n^{n-2} &= \begin{pmatrix} F(n-2)+1 & F(n-2)+F(n-3) \\ 1 & 1 \end{pmatrix}_{2\times 2} \\ &= \begin{pmatrix} F(n-2)+1 & F(n-1) \\ 1 & 1 \end{pmatrix}_{2\times 2}. \end{aligned}$$

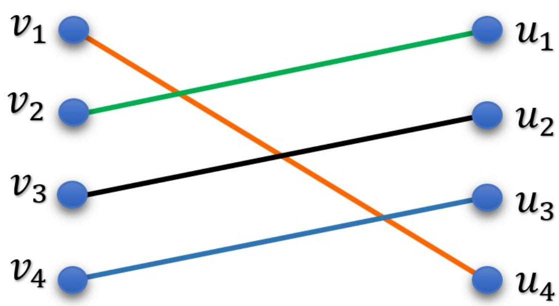
By applying equation (10), we obtain $\mathrm{per}U_n = \mathrm{per}U_n^{n-2} = F(n-1) + F(n-2) + 1$ and by equation (11), we have $F(n) = F(n-1) + F(n-2)$. So that $\mathrm{per}U_n = F(n) + 1,$ which is desired. ■

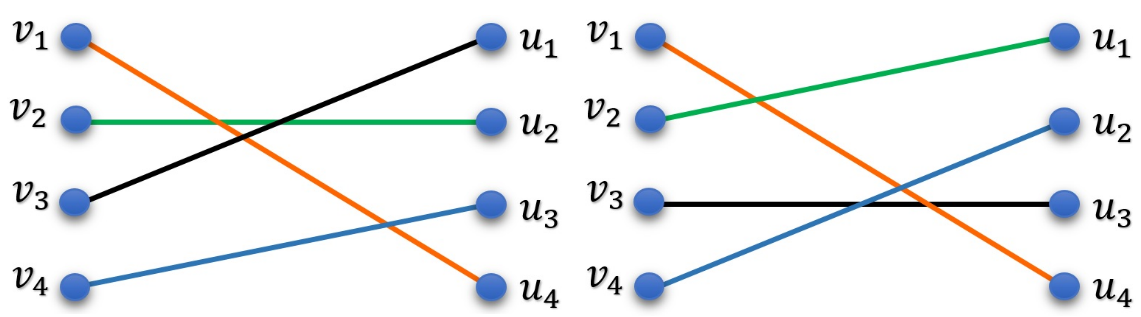
**Example 48** *Let $G(U_4)$ be a bipartite graph whose bipartite adjacency matrix is $U_4$ given by (18) for $n = 4$. Then the bipartite graph $G(U_4)$ can be seen as:*




*and its perfect matchings can be given as:*






**Lemma 49** *Let $G(V_n)$ be the bipartite graph with bipartite adjacency matrix $V_n$ has the form*

$$V_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 & 1 & 0 \\ \vdots & & & 0 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}_{n\times n} . \tag{19}$$

*Then, the number of perfect matchings of $G(V_n)$ is $F(n)+1$, where $F(n)$ is nth Fibonacci number.*

**Proof.** Let $V_n^k$ be the $k$th contraction of $V_n$, $1 \leq k \leq n-3$. Since the definition of the matrix $V_n$; the matrix $V_n$ can be contracted on column 1 so that

$$V_n^1 = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 & 1 & 0 \\ \vdots & & & 0 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}_{(n-1)\times(n-1)} .$$

Since the matrix $V_n^1$ can be contracted on column 1 and $F(3)=2$ and $F(2)=1$

$$V_n^2 = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 & 1 & 0 \\ \vdots & & & 0 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}_{(n-2)\times(n-2)}$$

$$= \begin{pmatrix} F(3) & F(2) & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 & 1 & 0 \\ \vdots & & & 0 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}_{(n-2)\times(n-2)}.$$

Furthermore, the matrix $V_n^2$ can be contracted on column 1 so that

$$V_n^3 = \begin{pmatrix} F(4) & F(3) & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 & 1 & 0 \\ \vdots & & & 0 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}_{(n-3)\times(n-3)}.$$

Continuing this process, we have

$$V_n^k = \begin{pmatrix} F(k+1) & F(k) & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 & 1 & 0 \\ \vdots & & & 0 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}_{(n-k)\times(n-k)}$$

for $1 \leq k \leq n-3$. Hence,

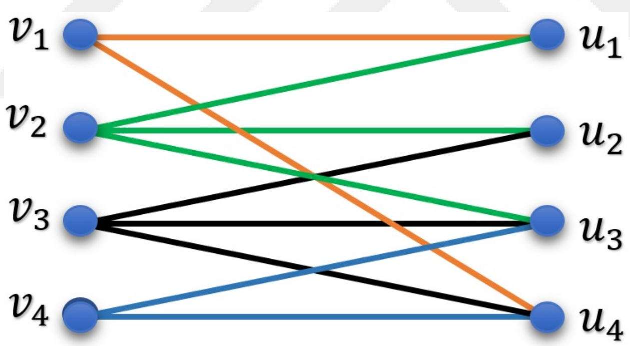
$$V_n^{n-3} = \begin{pmatrix} F(n-2) & F(n-3) & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}_{3\times 3}$$

which, by contraction of $V_n^{n-3}$ on last column, gives

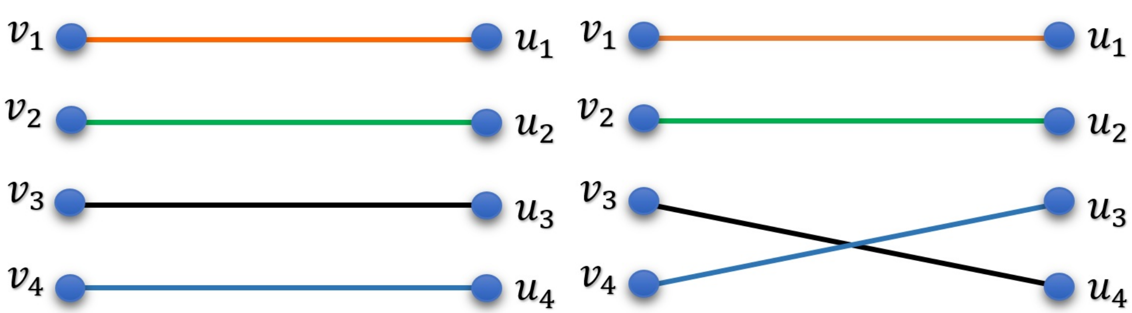
$$\begin{aligned} V_n^{n-2} &= \begin{pmatrix} F(n-2)+F(n-3) & F(n-2)+1 \\ 1 & 1 \end{pmatrix}_{2\times 2} \\ &= \begin{pmatrix} F(n-1) & F(n-2)+1 \\ 1 & 1 \end{pmatrix}_{2\times 2}. \end{aligned}$$

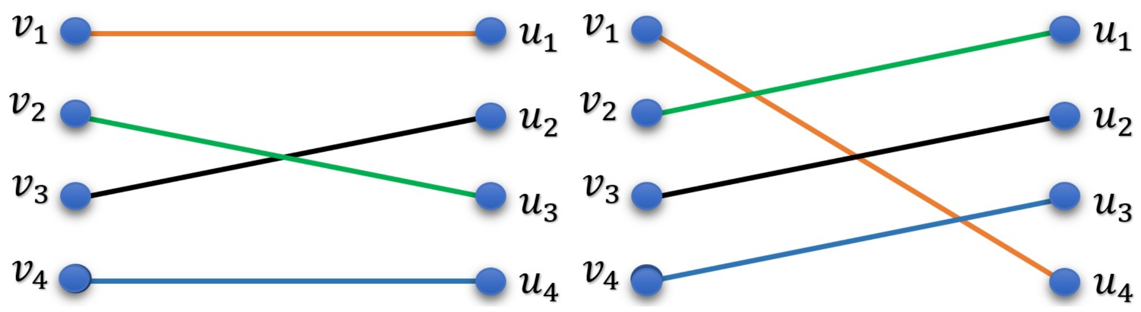
By applying equation (10), we obtain $\operatorname{per}V_n = \operatorname{per}V_n^{n-2} = F(n-1)+F(n-2)+1$ and by equation (11), we have $F(n) = F(n-1)+F(n-2)$. So that $\operatorname{per}V_n = F(n) + 1,$ which is desired. ■

**Example 50** *Let $G(V_4)$ be a bipartite graph whose bipartite adjacency matrix is $V_4$ given by (19) for $n=4$. Then the bipartite graph $G(V_4)$ can be seen as:*




*and its perfect matchings can be given as:*






**Theorem 51** *Let $G(W_n)$ be the bipartite graph whose bipartite adjacency matrix is the $(0,1)$-circulant matrix $W_n$ as the following*

$$W_n = \begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 1 & 1 & 0 & & & 0 \\ 0 & 1 & 1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 & 1 & 0 \\ 0 & & & 0 & 1 & 1 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix}_{n\times n}. \tag{20}$$

*Then, the number of perfect matchings of $G(W_n)$ is $L(n)+2$, where $L(n)$ is $n^{th}$ Lucas number.*

**Proof.** If $n=4$; then we have

$$\begin{aligned} \mathrm{per}W_4 &= \mathrm{per}\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \\ &= \mathrm{per}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \mathrm{per}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \mathrm{per}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \\ &= 3+3+3 = 9 = L(4)+2. \end{aligned}$$

By applying the Laplace expansion for permanent according to first column of $W_4$, we get

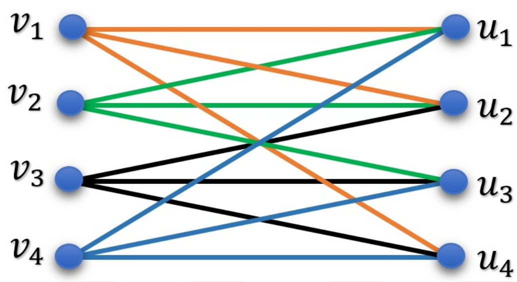
$$\mathrm{per}W_n = \mathrm{per}\mathcal{F}_{(n,2)} + \mathrm{per}U_{n-1} + \mathrm{per}V_{n-1},$$

where $\mathcal{F}_{(n,2)}$, $U_n$ and $V_n$ are respectively the matrices given by (1), (18) and (19). Taking into account (2), Lemma (47) and Lemma (49), we get the last equation as

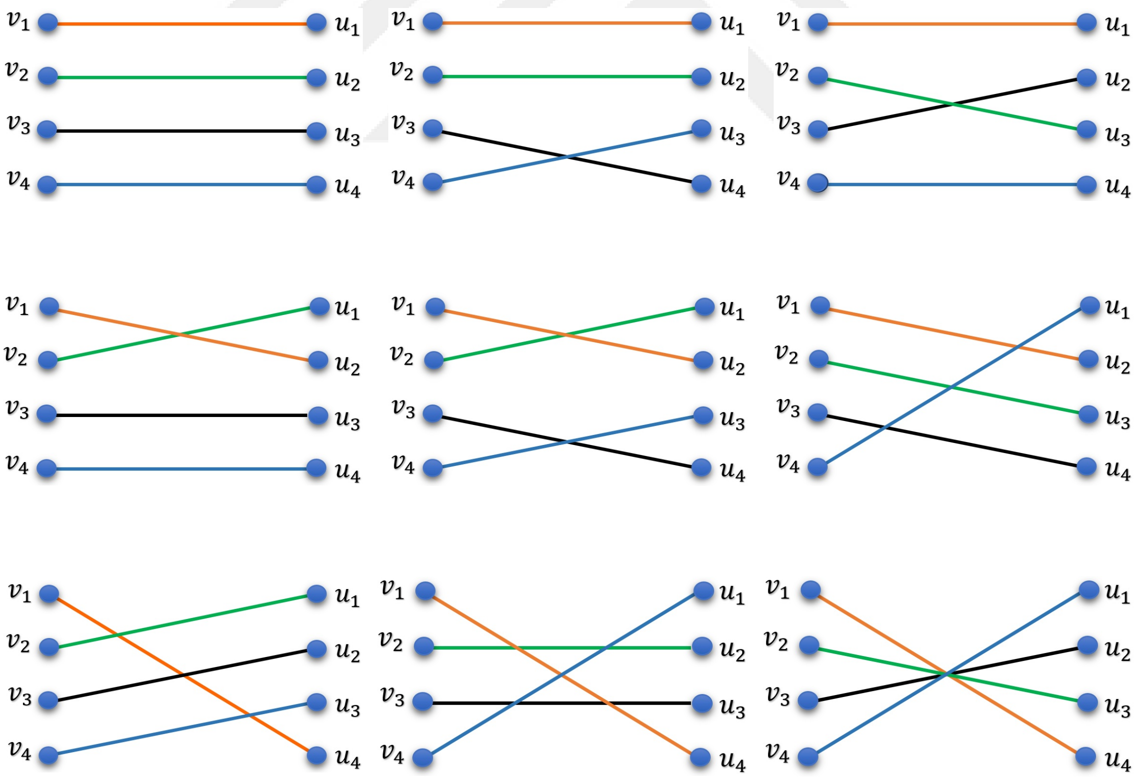
$$\begin{aligned}\operatorname{per}W_n &= F(n)+F(n-1)+1+F(n-1)+1\\ & F(n+1)+F(n-1)+2.\end{aligned}$$

The result follows by using (13). ■

**Example 52** *Let $G(W_4)$ be a bipartite graph whose bipartite adjacency matrix is $W_4$ given by (20) for $n=4$. Then the bipartite graph $G(W_4)$ can be seen as:*




*and its perfect matchings can be given as:*



#### 4.2.1. Maple procedure

**Procedure A.**

The following Maple procedure calculates the numbers of perfect matchings of bipartite graph $G(U_n)$ given in Lemma 47.

```
> restart:
with(LinearAlgebra):
permanent:=proc(n)
local i,j,r,f,U;
f := (i,j) - >piecewise(j - i = 0,1,j - i = -1,1,
j - i = -2,1,j - i = n - 1,1,0);
U:=Matrix(n,n,f) :
for r from 0 to n - 2 do
print(r,U) :
for j from 1 to n - r do
U[1,j] := U[n - r,n - r] * U[1,j] + U[1,n - r] * U[n - r,j] :
od:
U:=DeleteRow(DeleteColumn(Matrix(n - r,n - r,U),n - r),n - r) :
od:
print(r,eval(U)):
end proc:with(LinearAlgebra):
permanent(n);
```

**Procedure B.**

The following Maple procedure calculates the numbers of perfect matchings of bipartite graph $G\left(V_n\right)$ given in Lemma 49.

```
>restart:
with(LinearAlgebra):
permanent:=proc(n)
local i, j, r, f, V;
f := (i, j) - >piecewise(j - i = 0, 1, i>1 and j - i = 1, 1,
j - i = -1, 1, j - i = n - 1, 1, 0);
V:=Matrix(n, n, f) :
for r from 0 to n - 2 do
print(r, V) :
for j from 2 to n - r do
V[1, j] := V[2, 1] * V[1, j] + V[1, 1] * V[2, j] :
od:
V:=DeleteRow(DeleteColumn(Matrix(n - r, n - r, V), 1), 2) :
od:
print(r,eval(V)):
end proc:with(LinearAlgebra):
permanent(n);
```

# 5. CONCLUSION

Permanents have many applications in physics, chemistry, electrical engineering, graph theory etc. Some of the most important applications of permanents are via graph theory. A more difficult problem with many applications is the enumeration of perfect matchings of a graph. Besides, ”the enumeration or actual construction of perfect matching of a bipartite graph has many applications, for example, in maximal flow problems and in assignment and scheduling problems arising in operational research” (Minc, 1978). The numbers of perfect matchings of bipartite graphs also play a significant role in organic chemistry (Wheland, 1953). Fibonacci, Lucas and Jacobsthal numbers belong to a large family of positive integers. They have many interesting properties and applications to almost every field of science and art. They continue to provide invaluable opportunities for exploration, and contribute handsomely to the beauty of mathematics, especially number theory (Koshy, 2001; Koshy, 2011). Therefore, Many authors have investigated the relationship between the well-known integer sequences and the number of perfect matchings in bipartite graphs. In relation to that they found many considerable results. We speak of them in Chapter 2. Consequently, we have shown that the numbers of perfect matchings in some bipartite graphs are equal to Fibonacci, Lucas and Jacobsthal numbers. This results are also very significant because linear algebra, graph theory and number theory are used together.

## 6. REFERENCES


Adams, W. and Shanks, D., 1982. Strong primality tests that are not sufficient, *Mathematics of Computation,* 39 (159), 255-300.

Akbulak, M. and Öteles, A., 2013. On the number of 1-factors of bipartite graphs, *Mathematical Science Letters,* 2 (3), 1-7.

Asratian, A.S., Denley, T.M.J. and Häggkvist, R., 1998. Bipartite Graphs and their Applications, Cambridge Tracts in Mathematics, 131, Cambridge University Press.

Bapat, R. B., 2010. Graphs and matrices, Universitext, Springer/Hindustan Book Agency, London, New Delhi.

Bracey, R., 2012. On the Graphical Interpreation of Herod.s Coinage in Judaea and Rome in Coins, Spink & Son of London, 65-84.

Brualdi, R.A. and Gibson, P.M., 1977. Convex polyhedra of doubly stochastic matrices I: applications of the permanent function, *J. Combin. Theory A,* 22, 194-230.

Brualdi, R.A. and Cvetkovic, D., 2009. A Combinatorial Approach to Matrix Theory and Its Applications, CRC Press.

Catarino, P., Campos, H., and Vasco, P., 2016. On the Mersenne sequence, *Annales Mathematicae et Informaticae,* 46, 37-53.

Diestel, R., 2005. Graph Theory, Grad. Texts in Math, Springer.

Fonseca, C.M.da, Sogabe, T. and Yilmaz, F., 2015. Lower k-Hessenberg Matrices and *k*-Fibonacci, Fibonacci-p and Pell (p,i) Number, *General Mathematics Notes,* 31(1), 10-17.

Harary, F., 1969. Determinants, permanents and bipartite graphs, *Mathematics Magazine,* 42, 146-148.

Horadam, A.F., 1988. Jacobstal and Pell Curves, *The Fibonacci Quarterly*, 26, 79-83.

Horn, R.A. and Johnson, C.R., 1990. Matrix Analysis, Cambridge University Press.

Kaygisiz, K. and Sahin, A., 2013. Determinants and permanents of Hessenberg matrices and generalized Lucas polynomials, *Bull. Iranian Math. Soc.*, 39 (6), 1065-1078.

Kılıç, E. and Tasçı, D., 2007. On the permanents of some tridiagonal matrices with applications to the Fibonacci and Lucas numbers, *Rocky Mountain Journal of Mathematics*, 37 (6), 1953-1969.

Kılıç, E., 2008. On the usual Fibonacci and generalized order-*k* Pell numbers, *Ars Combinatoria*, 88, 33-45.

Kılıç, E. and Tasçı, D., 2008a. On families of bipartite graphs associated with sums of Fibonacci and Lucas numbers, *Ars Combinatoria,* 89, 31-40.

Kılıç, E. and Tasçı, D., 2008b. On families of bipartite graphs associated with sums of generalized order-*k* Fibonacci and Lucas numbers, *Ars Combinatoria,* 94, 13-23.

Kılıç, E. and Stakhov, A. P., 2009. On the Fibonacci and Lucas p-numbers, their sums, families of bipartite graphs and permanents of certain matrices, *Chaos, Solitons&Fractals,* 40 (22), 10-21.

Koshy, T., 2001. Fibonacci and Lucas Numbers with Applications, Wiley-Interscience, New York.

Koshy, T., 2011. Fibonacci, Lucas, and Pell numbers, and Pascal's triangle, *Mathematical Spectrum,* 43 (3), 125-132.

Lee, G.Y. and Lee, S.G., 1995. A note on generalized Fibonacci numbers, *The Fibonacci Quarterly,* 33, 273-278.

Lee, G.Y., Lee, S.G. and Shin, H.G., 1997. On the *k*-generalized Fibonacci matrix Q_k, *Linear Algebra and its Applications,* 251, 73-88.

Lee, G. Y. 2000. *k*-Lucas numbers and associated bipartite graphs, *Linear Algebra and its Applications,* 320, 51-61.

Marcus, M. and Minc, H., 1965. Permanents, *American Mathematical Monthly,* 72, 577-591.

Miles Jr., E. P., 1960. Generalized Fibonacci numbers and associated matrices, *American Mathematical Monthly,* 67 (8), 745-752.

Minc, H., 1978. Permanents, Encyclopedia of mathematics and its applications, Addison-Wesley, New York.

Niedermeier, R., 2002. Invitation to Fixed Parameter Algorithms, Habilitationschrift, University of Tubingen, September, 2002.

Öteleş, A. 2017. On the Number of Perfect Matchings for Some Certain Types of Bipartite Graphs, *Filomat,* (In Press).

Öteleş, A. and Akbulak, M., 2016. On the number of 1-factors in Bipartite Graphs, *20th Conference of the International Linear Algebra Society-ILAS 2016,* 11-15 July, Leuven, Belgium.

Shannon, A.G., Anderson, P.G. and Horadam, A.F., 2006. Properties of Cordonnier, Perrin and Van der Laan numbers, *International Journal of Mathematical Education in Science and Technology,* 37, 825-831.

Shiu, W.C. and Lam, P.C.B., 2003. More on the generalized Fibonacci numbers and associated bipartite graphs, *Int. Math. J.*, 3, 5-9.

The OEIS Foundation Inc., 2013. The On-Line Encyclopedia of Integer Sequences, http://oeis.org [last visited: 16 April, 2017].

Vasudev, C., 2006. Graph Theory with Applications, New Age Publishers, Bangolare.

Wasserman, S. and Faust, K., 1994. Social Network Analysis: Methods and Applications (Structural Analysis in the Social Sciences), Cambridge University, Press, 29-302.

Wheland, G.W., 1953. The Theory of Resonant and its Application to Organic Chemistry, Wiley, New York.

## CURRICULUM VITAE

## PERSONEL INFORMATION


| | |
|---|---|
| **Name and Surname** | **:** Diyar Omar Mustafa ZANGANA |
| **Nationality** | **:** Iraq |
| **Date and Place of Birth** | **:** 10.05.1993 and Sulaymaniyah / Kalar |
| **Phone** | **:** +964 750 195 8562 |
| **E-mail** | **:** diyar.math2@gmail.com |


## EDUCATION

| Degree | Institution | Year of Graduation |
|---|---|---|
| High School | Hamreen | 2010 |
| B.S. | University of Sulaymaniah | 2014 |
| M.Sc. | University of Siirt | |

## RESEARCH INTERESTS

Algebra and Number Theory, Linear Algebra, Graph Theory, Matrix Theory

## FOREIGN LANGUAGE

English, Turkish, Arabic