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GENERALIZED $\triangle_q^m$ DIFFERENCE OPERATOR AND TOPOLOGICAL PROPERTIES

MASTER DEGREE THESIS

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Mahir Salih Abdulrahman ASSAFI
SİİRT-2017

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LIST OF SYMBOLS

Symbol		Explanation
$\mathbb{N}$	:	The set of Natural numbers
$\mathbb{R}$	:	The set of Real numbers
$\mathbb{R}^{\boldsymbol{n}}$	:	n-dimensional Euclidean space
$\mathbb{C}$	:	The set of complex numbers
$\boldsymbol{\omega}$	:	The spaces of all sequences defined on $\mathbb{C}$
$\boldsymbol{l}_\infty$	:	The spaces of all bounded scalar sequences
$\boldsymbol{l_p}$	:	The spaces of all $\sum_{\boldsymbol{k=1}}^{\infty}\lvert\boldsymbol{x_k}\rvert^{\boldsymbol{p}} < \infty$ is the convergent sequences
$\boldsymbol{c}$	:	The spaces of all convergent scalar sequences
$\boldsymbol{c_0}$	:	The spaces of all null scalar sequences
$\boldsymbol{l}_\infty(\Delta_q^m)$	:	$\boldsymbol{l}_\infty(\Delta_q^m) = \{\boldsymbol{x} = (\boldsymbol{x_k}): \Delta_q^m\boldsymbol{x} \in \boldsymbol{l}_\infty\}$
$\boldsymbol{c}(\Delta_q^m)$	:	$\boldsymbol{c}(\Delta_q^m) = \{\boldsymbol{x} = (\boldsymbol{x_k}): \Delta_q^m\boldsymbol{x} \in \boldsymbol{c}\}$
$\boldsymbol{c}_0(\Delta_q^m)$	:	$\boldsymbol{c_0}(\Delta_q^m) = \{\boldsymbol{x} = (\boldsymbol{x_k}): \Delta_q^m\boldsymbol{x} \in \boldsymbol{c_0}\}$
$\boldsymbol{\mathcal{M}}$	:	$\boldsymbol{\mathcal{M}}$ is a family of Orlicz functions.
$\Delta_{\boldsymbol{q}}^{\boldsymbol{m}}$	:	$\boldsymbol{m}$.th difference operator
BK	:	Banach Coordinatial

ÖZET

YÜKSEK LİSANS

GENELLEŞTİRİLMİŞ Δ_q^m FARK OPERATÖRÜ VE TOPOLOJİK ÖZELLİKLERİ

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Bu tezi dört bölüme ayırdık. Tezin birinci kısmında, konunun tarihsel bir gelişimi ile ilişkili giriş verildi. İkinci ve üçüncü kısımlarda, tanımlarımız ve sonuçlarımızla direkt ilişkili olan çeşitli yazarların yaptığı, farklı çalışmalarla ilgili bilgi verildi. Tez boyunca kullandığımız tanımlar ve notasyonlar gibi kavramların çoğu şuan standarttır. Dördüncü bölümde, Orlicz fonksiyonunun hakkında bilgi verildi. Peralta (2010) nın çalışması, Karakaş, ark. (2016) tarafından tanımlanan Δ_q^m fark operatörü kullanılarak genelleştirildi. $l_p(\Delta_q^m)$ fark dizi uzayı elde ederek bu dizi uzayının özelliklerinin bir kısmı araştırıldı. $l_p(\Delta_q^m)$, $\|\cdot\|_{p,\Delta_q^m}$ normu ile birlikte verilirse bir Banach uzayı olacağı gösterildi. Üstelik $\left(l_p(\Delta_q^m), \|\cdot\|_{p,\Delta_q^m}\right)$ ve $\left(l_p, \|\cdot\|_p\right)$ dizi uzaylarının lineer izometrik olduğu gösterildi. Bu bölümün sonunda, Orlicz fonksiyonlarının bir ailesi olan $l_p(\Delta_q^m) \subset l_p(\mathcal{M}, \Delta_q^m)$ kapsaması gösterildi.

Anahtar Kelimeler: Dizi uzayları, Fark dizi uzayları, İzometrik dizi uzayları.

ABSTRACT

MS. THESIS

GENERALIZED Δ_q^m DIFFERENCE OPERATOR AND TOPOLOGICAL PROPERTIES

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The Graduate School of Natural and Applied Science of Siirt University
The Degree of Master of Science
In Mathmetics

Supervisior : Asst. Prof. Abdulkadir KARAKAŞ

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We divided this thesis into the four chapters. The first chapter of the thesis gives the introduction deals with a historical review. The second and the third chapters give the background of different kinds of work done by various authors, which are related directly to our definitions and results. Most of concepts which we have used throughout the thesis such as notations and definitions are currently standard. We presented the history of the Orlicz function in the the fourth chapter. We used the Peralta' s (2010) studies and extented it by using difference operator Δ_q^m given by Karakaş et al. (2016), we generated the difference sequence space $l_p(\Delta_q^m)$ and investigated some of their properties. We showed that, if $l_p(\Delta_q^m)$ is supplied with an aproper norm $\|\cdot\|_{p,\Delta_q^m}$ then it will be a Banach space. We further more showed that, the sequence spaces $\left(l_p(\Delta_q^m), \|\cdot\|_{p,\Delta_q^m}\right)$ and $\left(l_p, \|\cdot\|_p\right)$ are linearly isometric. At the end of this chapter, it was shown that $l_p(\Delta_q^m) \subset l_p(\mathcal{M}, \Delta_q^m)$.

Keywords: Difference sequence spaces, Isometric sequence spaces, Sequence spaces.

1. INTRODUCTION

Let c, ℓ_∞ and c_0 be the Banach spaces of convergent, bounded and null sequences $x=(x_k)_{k=1}^{\infty}$ respectively with complex terms, normed by

$$\|x\|_\infty = \sup_k |x_k|,$$

where $k \in \mathbb{N}$.

Kizmaz (1981) presented the difference sequence spaces,

$$X(\Delta) = \{x = (x_k) : \Delta x \in X\}$$

for $X = c$, ℓ_∞ and c_0 where

$$\Delta x = (\Delta x_k) = (x_k - x_{k+1}).$$

These are Banach spaces with the norm

$$\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty.$$

He also studied their topological properties. Recently Çolak and Et (1997) extended the spaces $X(\Delta)$ to the spaces $X(\Delta^m)$ for $X = c$, ℓ_∞ and c_0. Let X be any sequence spaces and defined

$$X(\Delta^m) = \{x = (x_k) : \Delta^m x \in X\}$$

where $m \in \mathbb{N}$ and $\Delta^m x = ((\Delta \circ \Delta^{m-1})x_k)$ for all $k \in \mathbb{N}$ and prove that $c(\Delta^m)$, $l_\infty(\Delta^m)$ and $c_0(\Delta^m)$ are Banach spaces with the norm

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}, \|x\|_{\Delta^m} = \sum_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty$$

Karakaş et al. (2015) defined the sequence spaces $l_\infty(\Delta_q)$, $c(\Delta_q)$ and $c_0(\Delta_q)$, where $q \in \mathbb{N}$ and

$$\Delta_q x = (\Delta_q x_k) = (q x_k - x_{k+1}).$$

The following sequence spaces have been given by Karakaş et al (2016),

$$X\left(\Delta_q^m\right) = \left\{x = (x_k) : \Delta_q^m x \in X\right\}$$

for $X = c$, ℓ_∞ and c_0, where $q, m \in \mathbb{N}$. They show that the spaces $X\left(\Delta_q^m\right)$ are Banach spaces by the norm

$$\|x\|_{\Delta_q^m} = \sum_{i=1}^{m} |x_i| + \left\|\Delta_q^m x\right\|_\infty ,$$

where

$$\Delta_q^m = \left(\Delta_q^m x_k\right) = \left(\Delta_q^{m-1} x_k - \Delta_q^{m-1} x_{k+1}\right)$$

and

$$\Delta_q^m x = \left(\Delta_q^m x_k\right) = \sum_{v=0}^{m} (-1)^v \binom{m}{v} q^{m-v} x_{k+v}.$$

Recently, Peralta (2010) studied $\ell_p(\Delta^m)$ and examined the topological properties of this space.

In this paper, we chose $p \in [1, \infty)$. By ω, we shall denote the space of all sequences $x = (x_k)$, for $x_k \in \mathbb{C}$ and all $k \in \mathbb{N}$. Taken $x \in \omega$, describe

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}$$

and let

$$\ell_p = \{x = (x_k) : \|x\|_p < \infty\}.$$

The linear operator $\Delta_q^m : \omega \to \omega$ is presented recursively as the composition $\Delta_q^m = \Delta_q \circ \Delta_q^{m-1}$ for $m \geq 2$ and $q \in \mathbb{N}$. It is obvious that for $x \in \omega$ and $m \geq 1$ we have the following Binomial representation

$$\Delta_q^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} q^{m-v} x_{k+v}$$

for all $k \in \mathbb{N}$.

Let $m \in \mathbb{N}$ and define the sequence spaces $\ell_p(\Delta_q^m)$ by

$$\ell_p(\Delta_q^m) = \{x = (x_k) : \Delta_q^m x \in \ell_p\}$$

The sequence spaces are Banach spaces normed by

$$\|x\|_{p,\Delta_q^m} = \left(\sum_{i=1}^{m} |x_i|^p + \left\|\Delta_q^m x\right\|_p^p\right)^{1/p} . \tag{1.1}$$

2. BASIC CONCEPTS

2.1. Basic Definitions and Theorems

Definition 2.1.1

Let X be a nonempty set and K be the field of complex number. If operations

$$+ : X \times X \rightarrow X$$

$$\cdot : K \times X \rightarrow X$$

Satisfy the following statements then X is called a vector (linear) space on K.

1) $x + y = y + x$,
2) $(x + y) + z = x + (y + z)$,
3) $x + \theta = x$ so that there is a zero vector given name $\theta \in X$,
4) For each $x \in X$, $x + (-x) = \theta$ so that there is an $-x \in X$,
5) $1x = x$
6) $\lambda (x + y) = \lambda x + \lambda y$,
7) $(\lambda + \mu) x = \lambda x + \mu x$,
8) $\lambda(\mu x) = (\lambda\mu)x$

for every $\lambda, \mu \in K$ and $x, y, z \in X$ (Maddox, 1988).

Definition 2.1.2

$X \neq \emptyset$, $d : XxX \rightarrow \mathbb{R}$ be a function. If the following statements are satisfied for any $x, y, z \in X$, then d is called a metric on X and (X, d) is called a metric space;

M1) $d(x, y) \geq 0$
M2) $d(x, y) = 0 \Longleftrightarrow x = y$
M3) $d(x, y) = d(y, x)$
M4) $d(x, z) \leq d(x, y) + d(y, z)$ (Maddox, 1988).

Definition 2.1.3

Let (X, d) be a metric space and $x = (x_n)$ be a sequence on X. $x = (x_n)$ converges to a number $x \in \mathbb{R}$ and denoted by $x_n \rightarrow x$, if for every $\varepsilon > 0$ there exists a $N(\varepsilon) \in \mathbb{N}$, such that, for any $n \geq N(\varepsilon)$

$$d (x_n, x) < \varepsilon.$$

In other words, (x_n) converges to x if $\forall\varepsilon > 0$, $|x_n - x| < \varepsilon$ holds except finitely many terms of the sequence x (Maddox, 1988).

Definition 2.1.4

A sequence $x = (x_n)$ is called a Cauchy sequence if $\forall \varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$, with $n, m \geq N(\varepsilon)$,

$$|x_n - x_m| < \varepsilon.$$

(Maddox, 1988).

Theorem 2.1.5

A sequence of real numbers is convergent if and only if it is a Cauchy sequence (Maddox, 1988).

Definition 2.1.6

Let (X, d) be a metric space. If each Cauchy sequences converges in a metric space (X, d), this space is called complete metric space (Maddox, 1988).

Definition 2.1.7

Let $p \geq 1$ fixed a real number. Each element in the space l_p is a sequence $x = (x_n) = (x_1, x_2, x_3, ..)$ of numbers such that $|x_1|^p + |x_2|^p + ...$ converges; thus

$$l_p = \left\{ (x_n) : \sum_{k=1}^{\infty} |x_k|^p < \infty \right\}$$

and the metric is defined by

$$d(x, y) = \left(\sum_{k=1}^{\infty} |x_k - y_k|^p \right)^{\frac{1}{p}}$$

where $y = (y_k)$ and $\sum_{k=1}^{\infty} |y_k|^p < \infty$ (Maddox, 1988).

Definition 2.1.8

Let be a vector space on K. If the following conditions are satisfied then the mapping $\|.\| : X \to \mathbb{R}^+$, $x \to \|x\|$ is called a norm,

N1) $\|x\| \geq 0$

N2) $\|x\| = 0 \Leftrightarrow x = \theta$

N3) $\|\alpha x\| = |\alpha| \, \|x\|$, $(\alpha \in K, x \in X)$

N4) $\|x + y\| \leq \|x\| + \|y\|$ for $\forall x, y \in X$

then the $(X, \|.\|)$ is called as a norm space (Kreyszig, 1978).

Definition 2.1.9

If a Cauchy sequence in a $(X, \|\cdot\|)$ normed space converges then it is called Banach space (Kreyszig, 1978).

Definition 2.1.10

Let us show set of all sequence with complex variables by ω, for $x = (x_k)$, $y = (y_k)$, $(k = 1, 2, 3, ...)$ and a constant α, the set ω is a vector space under the operation defined by

$$x + y = (x_k) + (y_k)$$

$$\alpha x = (\alpha x_k).$$

Every sub-vector space of ω is termed a sequence space (Goes and Goes, 1970).

Let c, ℓ_∞ and c_0 be the linear spaces of convergent,bounded and null sequences $x = (x_k)$ with complex terms, that is

$$c = \left\{ x = (x_k) : \lim_k x_k = L \text{ and for } \exists L \right\}$$

$$l_\infty = \left\{ x = (x_k) : \sup_k |x_k| < \infty \right\}$$

$$c_0 = \left\{ x = (x_k) : \lim_k x_k = 0 \right\}.$$

All the above sequence spaces are Banach spaces normed by

$$\|x\|_\infty = \sup_k |x_k|,$$

for $k \in \mathbb{N}$.

Definition 2.1.11

Let X be a Banach space.

If;

$$\tau_k : X \to \mathbb{C}, \tau_k(x) = x_k, \ (k = 1, 2, 3, ...)$$

transformation is continuous, then X is called BK$-$space (Goes and Goes, 1970).

Definition 2.1.12

Let $(X, \|\cdot\|)$ and $\left(Y, \|\cdot\|^1\right)$ be two normed spaces.

Then;

i) A mapping $T : X \to Y$ is known to be isometric (or an isometry) if preserves T norm, that is, for all $x, y \in X$,;

$$\|T_x - T_y\|^1 = \|x - y\|$$

where T_x and T_y are the images of x and y, respectively.

ii) The space X is said to be isometric with the space Y if there exist a one-to-one and onto isometry X onto Y. Then X and Y are called isometric spaces

(Kreyszig, 1978).

Definition 2.1.13

Let X and Y be two vector spaces. If a function $T : X \to Y$ possesses the properties,

$i)$ $T(x_1 + x_2) = T(x_1) + T(x_2)$ for all $x_1, x_2 \in X$ (additivity),

$ii)$ $T(\alpha x_1) = \alpha T(x_1)$ for all $x_1 \in X$ and $\alpha \in \mathbb{C}$ (homogeneity)

then T is called linear mapping (Şuhubi, 2001).

Definition 2.1.14

Let X be any vector space and $Y \subset X$, for all $y_1, y_2 \in Y$ and $\lambda \in [0, 1]$. If

$$M(y_1, y_2) = \{y \in Y : y = \lambda y_1 + (1 - \lambda)y_2, 0 \leq \lambda \leq 1\} \subset Y$$

Y is called convex space (Kreyszig, 1978).

Theorem 2.1.15

In order for an X subspace of Banach space to be complete, it is necessary and sufficient that condition Y is closed in X (Kreyszig, 1978).

3. DIFFERENCE SEQUENCES

3.1. Δ Difference Sequences and Some Properties of Δ

The notion of difference sequence spaces was firstly introduced by Kizmaz (1981) as follows:

$$X(\Delta) = \{x = (x_k) : (\Delta x_k) \in X\}$$

for $X = \ell_\infty$, c and c_0. In 1981, He defined the following sequence spaces

$$\begin{aligned} l_\infty(\Delta) &= \{x = (x_k) : \Delta x \in l_\infty\} \\ c(\Delta) &= \{x = (x_k) : \Delta x \in c\} \\ c_0(\Delta) &= \{x = (x_k) : \Delta x \in c_0\}. \end{aligned}$$

where

$$\Delta x = (\Delta x_k) = (x_k - x_{k+1}),$$

and showed that these spaces are Banach spaces with norm

$$||x||_\Delta = |x_1| + \|\Delta x\|_\infty .$$

Here, he showed that $(l_\infty(\Delta), ||\cdot||_\Delta)$ is a Banach space.

Let (x^n) be a Cauchy sequence in $l_\infty(\Delta)$, where $x^n = (x_i^n) = (x_1^n, x_2^n, ...) \in l_\infty(\Delta)$, for each $n \in \mathbb{N}$. Then

$$\|x^n - x^m\|_\Delta = |x_1^n - x_1^m| + \|\Delta x^n - \Delta x^m\|_\infty \to 0 \ (n, m \to \infty).$$

Therefore, we obtain $|x_k^n - x_k^m| \to 0$, for $n, m \to \infty$ and each $k \in \mathbb{N}$.

Hence, $(x_k^n) = (x_k^1, x_k^2, ...)$ is a Cauchy sequence in $\mathbb{C}$ (complex numbers) whence by the completeness of $\mathbb{C}$, it converges to x_k say, i.e., there exists

$$\lim_n x_k^n = x_k, \text{ for each } k \in \mathbb{N}.$$

Further, for each $\varepsilon > 0$, there exists $N = N(\varepsilon)$, such that for all $n, m \geq N$ and, for all $k \in \mathbb{N}$,

$$\begin{aligned} |x_1^n - x_1^m| &< \varepsilon, \\ \left|x_{k+1}^n - x_{k+1}^m - (x_k^n - x_k^m)\right| &< \varepsilon \end{aligned}$$

and

$$\lim_m |x_1^n - x_1^m| = |x_1^n - x_1| \leq \varepsilon,$$
$$\lim_m \left|x_{k+1}^n - x_{k+1}^m - (x_k^n - x_k^m)\right| = \left|x_{k+1}^n - x_{k+1} - (x_k^n - x_{k)}\right| \leq \varepsilon,$$

for all $n \geq N$. Since ε is not dependent on k,

$$\sup_k |x_{k+1}^n - x_{k+1} - (x_k^n - x_k)| \leq \varepsilon.$$

Consequently we have $||x^n - x||_\Delta \leq 2\varepsilon$, for $n \geq N$. Hence we obtain $x^n \to x$ $(n \to \infty)$ in $l_\infty(\Delta)$, where $x = (x_k)$.

Now we must show that $x \in l_\infty(\Delta)$. We have

$$|x_k - x_{k+1}| = \left|x_k - x_k^N + x_k^N - x_{k+1}^N + x_{k+1}^N - x_{k+1}\right| \leq \left|x_k^N - x_{k+1}^N\right| + \left\|x^N - x\right\|_\Delta = O(1).$$

This implies $x = (x_k)$, where $x \in l_\infty(\Delta)$ (Kizmaz, 1981).

3.2. Δ^m Difference Sequences and Some Properties of Δ^m

In 1993, Et defined the sequence spaces $l_\infty(\Delta^2)$, $c(\Delta^2)$ and $c_0(\Delta^2)$ as:

$$l_\infty(\Delta^2) = \{x = (x_k) : \Delta^2 x \in l_\infty\},$$
$$c(\Delta^2) = \{x = (x_k) : \Delta^2 x \in c\},$$
$$c_0(\Delta^2) = \{x = (x_k) : \Delta^2 x \in c_0\}$$

where

$$\Delta^2 x = (\Delta^2 x_k) = (\Delta x_k - \Delta x_{k+1}),$$

and showed that these are Banach spaces with norm

$$\|x\|_\Delta = |x_1| + |x_2| + \left\|\Delta^2 x\right\|_\infty.$$

After then Et and Çolak (1995) defined the sequence spaces

$$
\begin{aligned}
l_\infty(\Delta^m) &= \{x = (x_k) : \Delta^m x \in l_\infty\}, \\
c(\Delta^m) &= \{x = (x_k) : \Delta^m x \in c\}, \\
c_0(\Delta^m) &= \{x = (x_k) : \Delta^m x \in c_0\}
\end{aligned}
$$

for $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$,

$$\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$$

and hence

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}.$$

It is obvious that $l_\infty(\Delta^m)$, $c_0(\Delta^m)$ and $c(\Delta^m)$ are linear spaces. It is dearly observed that these sequence spaces are normed spaces with norm

$$\|x\|_{\Delta^m} = \sum_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty . \tag{3.1}$$

We write $\lim_n$ for $\lim_{n\to\infty}$and $\sum_k$ for $\sum_{k=1}^{\infty}$.

Theorem 3.2.1

The sequence spaces $l_\infty(\Delta^m)$, $c_0(\Delta^m)$ and $c(\Delta^m)$ are Banach spaces with norm (3.1) (Et and Çolak, 1997).

Proof:

Let (x^s) be a Cauchy sequence in $l_\infty(\Delta^m)$, where $x^s = (x_i^s) = (x_1^s, x_2^s, ...) \in l_\infty(\Delta^m)$ for each $s \in \mathbb{N}$.

Then

$$\left\|x^s - x^l\right\|_{\Delta^m} = \sum_{i=1}^{m} \left|x_i^s - x_i^l\right| + \sup_k \left|\Delta^m (x_k^s - x_k^l)\right| \to 0$$

as $s, t \to \infty$.

Hence we obtain

$$\left|x_k^s - x_k^l\right| \to 0$$

as $s, t \to \infty$, for each $k \in \mathbb{N}$. Thus $(x_k^s) = (x_k^1, x_k^2, ...)$ is a Cauchy sequence in $\mathbb{C}$,

since $\mathbb{C}$ is complete, it is convergent.

$$\lim_s x_k^s = x_k$$

for each $k \in \mathbb{N}$. Since (x^s) is a Cauchy sequence for each $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that $\left\|x^s - x^l\right\|_{\Delta^m} < \varepsilon$ for all $s, t \geq N$.

Hence;

$$\sum_{i=1}^{m} \left|x_i^s - x_i^l\right| \leq \varepsilon \text{ and } \left|\sum_{v=0}^{m} (-1)^v \binom{m}{v} (x_{k+v}^s - x_{k+v}^l)\right| \leq \varepsilon$$

for all $k \in \mathbb{N}$, and all $s, t \geq N$.

We have

$$\lim_l \sum_{i=1}^{m} \left|x_i^s - x_i^l\right| = \sum_{i=1}^{m} \left|x_i^s - x_i\right| \leq \varepsilon$$

and

$$\lim_l \left|\Delta^m (x_k^s - x_k^l)\right| = \left|\Delta^m (x_k^s - x_k)\right| \leq \varepsilon$$

for all $s \geq N$. This implies that $\|x^s - x\|_{\Delta^m} < 2\varepsilon$ for all $s \geq N$, that is, $x^s \to x$ as $s \to \infty$ where $x = (x_k)$.

Since

$$\begin{aligned}
|\Delta^m x_k| &= \left|\sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}\right| \\
&= \left|\sum_{v=0}^{m} (-1)^v \binom{m}{v} (x_{k+v} - x_{k+v}^N + x_{k+v}^N)\right| \\
&\leq \left|\sum_{v=0}^{m} (-1)^v \binom{m}{v} (x_{k+v}^N - x_{k+v})\right| + \left|\sum_{v=0}^{m} (-1)^v \binom{m}{v} (x_{k+v}^N)\right| \\
&\leq \left\|x^N - x\right\|_{\Delta^m} + \left|\Delta^m x_k^N\right| = O(1)
\end{aligned}$$

we obtain $x \in l_\infty(\Delta^m)$. Therefore $l_\infty(\Delta^m)$ is a Banach space.

It can be shown that $c_0(\Delta^m)$ and $c(\Delta^m)$ are closed subspaces of $l_\infty(\Delta^m)$. Therefore, these sequence spaces are Banach spaces with norm (3.1). Some of inclusion relations between these sequence spaces are given below:

Lemma 3.2.2

i) $c(\Delta^m) \subset c(\Delta^{m+1})$,

ii) $c_0(\Delta^m) \subset c_0(\Delta^{m+1})$,

iii) $l_\infty(\Delta^m) \subset l_\infty(\Delta^{m+1})$

and the inclusions are strictly (Et and Çolak, 1995).

Proof:

ii) Let $x \in c_0(\Delta^m)$. Since

$$\begin{aligned} \left|\Delta^{m+1} x_k\right| &= \left|\Delta^m x_k - \Delta^m x_{k+1}\right| \\ &\leq \left|\Delta^m x_k\right| + \left|\Delta^m x_{k+1}\right| \to 0, (k \to \infty) \end{aligned}$$

we obtain $x \in \Delta^{m+1}(c_0)$. Thus $c_0(\Delta^m) \subset c_0(\Delta^{m+1})$. This inclusion is strict since the sequence $x = (k^m)$. For example, it belongs to $c_o(\Delta^{m+1})$, but it is not belong to $c_0(\Delta^m)$.

The proofs of (i) and (iii) are similar to the proof of (ii).

Lemma 3.2.3

$i)$ $c(\Delta^m) \subset l_\infty(\Delta^m)$

$ii)$ $c_0(\Delta^m) \subset c(\Delta^m)$

and the inclusions are strictly.

The proof is identical with the proof of Lemma 3.2.2.

Additionally, since $l_\infty(\Delta^m)$, $c_0(\Delta^m)$ and $c(\Delta^m)$ are Banach spaces with continuous coordinates, that is, $\|x^s - x\|_{\Delta^m} \to 0$ implies $|x_k^s - x_k| \to 0$ for each $k \in \mathbb{N}$ as $s \to \infty$ they are also BK$-$spaces (Et and Çolak, 1995).

3.3. Δ_q Difference Sequences and Some Properties of Δ_q

All throughout this paper we let $1 \leq q < \infty$, $q \in \mathbb{N}$. After Karakaş et al (2015) defined the sequence spaces $l_\infty(\Delta_q)$, $c(\Delta_q)$ and $c_0(\Delta_q)$ by

$$\begin{aligned} l_\infty(\Delta_q) &= \{x = (x_k) : \Delta_q x \in l_\infty\}, \\ c(\Delta_q) &= \{x = (x_k) : \Delta_q x \in c\}, \\ c_0(\Delta_q) &= \{x = (x_k) : \Delta_q x \in c_0\} \end{aligned}$$

where $q \in \mathbb{N}$ and

$$\Delta_q x = (\Delta_q x_k) = (q x_k - x_{k+1}).$$

Subsequently difference sequence spaces have been investigated by Altin (2009), Et (1993, 2003, 2004, 2013), (Bektas et al, 2004). Recently, Başar and Altay (2003) and Altay and Başar (2007) introduced the difference space bvp consisting of all sequence whose backwared differences are in the space l_p of absolutely $p-$summable sequences in the cases $1 \leq p \leq \infty$ and $0 < p < 1$, respecitively.

Theorem 3.3.1

The sequence spaces $l_\infty(\Delta_q)$, $c(\Delta_q)$ and $c_0(\Delta_q)$ are Banach spaces with norm

$$\|x\|_{\Delta_q} = |x_1| + \|\Delta_q x\|_\infty . \tag{3.2}$$

(Karakaş et al., 2016).

Proof:

Let (x^n) be a Cauchy sequence in $l_\infty(\Delta_q)$, where $x^n = (x_i^n) = (x_1^n, x_2^n, ...) \in l_\infty(\Delta_q)$ for each $n \in \mathbb{N}$.

Then

$$\|x^n - x^m\|_{\Delta_q} = |x_1^n - x_1^m| + \|\Delta_q x^n - \Delta_q x^m\|_\infty \to 0 \tag{3.3}$$

as $n, m \to \infty$. Hence, we obtain

$$|x_k^n - x_k^m| \to 0$$

for $n, m \to \infty$ and each $k \in \mathbb{N}$.

Therefore, $(x_k^n) = (x_k^1, x_k^2, ...)$ is a Cauchy sequence in $\mathbb{C}$ whence by the completeness of $\mathbb{C}$, it is convergent, say $x_k \in \mathbb{C}$, that is

$$\lim_n x_k^n = x_k$$

for each $k \in \mathbb{N}$. Since (x^n) is a cauchy sequence, for each $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that $\|x^n - x^m\|_{\Delta_q} < \varepsilon$ for all $n, m \geq N$.

Hence

$$\begin{aligned} |x_1^n - x_1^m| &< \varepsilon, \\ \left|x_{k+1}^n - x_{k+1}^m - (qx_k^n - qx_k^m)\right| &< \varepsilon, \end{aligned}$$

and

$$\begin{aligned} \lim_m |x_1^n - x_1^m| &= |x_1^n - x_1| \leq \varepsilon, \\ \lim_m \left|x_{k+1}^n - x_{k+1}^m - (qx_k^n - qx_k^m)\right| &= \left|x_{k+1}^n - x_{k+1} - (qx_k^n - qx_{k)}\right| \leq \varepsilon \end{aligned}$$

for all $n \geq N$.

Then,

$$\sup_k |x_{k+1}^n - x_{k+1} - (qx_k^n - qx_k)| < \varepsilon.$$

This implies that $\|x^n - x\|_{\Delta_q} \leq 2\varepsilon$ for $n \geq N$, that is, $x^n \to x$ as $n \to \infty$ where $x = (x_k)$.

Since

$$\begin{aligned} |qx_k - x_{k+1}| &= \left|qx_k - qx_k^N + qx_k^N - x_{k+1}^N + x_{k+1}^N - x_{k+1}\right| \\ &\leq \left|qx_k - x_{k+1}^N\right| + \left\|x^N - x\right\|_{\Delta_q} = O(1), \end{aligned}$$

We obtain $x \in l_\infty(\Delta_q)$. Thus, $l_\infty(\Delta_q)$ is a Banach space.

In the same way, it can be shown that $c(\Delta_q)$ and $c_0(\Delta_q)$ are the Banach spaces with norm (3.2).

Moreover, since $\ell_\infty(\Delta_q)$, $c(\Delta_q)$ and $c_0(\Delta_q)$ are Banach spaces with continuous coordinates, that is, $\|x^n - x\|_{\Delta_q} \to 0$ implies $|qx_k^n - x| \to 0$ for each $k \in \mathbb{N}$, as $n \to \infty$, they are also BK-spaces.

Let us define that operator

$$D : \ell_\infty(\Delta_q) \to \ell_\infty$$

as $Dx = (0, x_2, x_3, x_4, ...)$, where $x = (x_1, x_2, x_3, x_4, ...)$. It is easy to show that D is a bounded linear operator on $\ell_\infty(\Delta_q)$.

Furthermore the set

$$D\left[\ell_\infty(\Delta_q)\right] = D\ell_\infty(\Delta_q) = \{x = (x_k) : x \in \ell_\infty(\Delta_q), x_1 = 0\}$$

is a subspace of $\ell_\infty(\Delta_q)$, and

$$\|x\|_{\Delta_q} = \|\Delta_q x\|_\infty$$

in $D\ell_\infty(\Delta_q)$.

Now let us define

$$\Delta_q : D\ell_\infty(\Delta_q) \to \ell_\infty, \Delta_q x = y = (qx_k - x_{k+1}).$$

It can be shown that Δ_q is a linear homeomorphism. Hence, $D\ell_\infty(\Delta_q)$ and ℓ_∞ are equivalent as topological spaces. Δ_q and $(\Delta_q)^{-1}$ are norm preseriving and

$$\|\Delta_q\| = \left\|(\Delta_q)^{-1}\right\| = 1.$$

Following Karakaş et al. (2015) give some properties of $\Delta_q(x)$.

Theorem 3.3.2

Let X be a vector space and A subset of X. If A is a convex set, then $\Delta_q(A)$ is a convex in $\Delta_q(X)$ (Karakaş et al., 2015).

Proof:

Let $x, y \in \Delta_q(A)$. Then $\Delta_q x$, $\Delta_q y \in A$. Since Δ_q is linear, we have

$$\lambda\Delta_q x + (1-\lambda)\Delta_q y = \Delta_q(\lambda x + (1-\lambda)y), (0 \leq \lambda \leq 1).$$

Since A is convex, so $\lambda\Delta_q x + (1-\lambda)\Delta_q y \in A$ and $\lambda x + (1-\lambda)y \in \Delta_q(A)$, where $0 \leq \lambda \leq 1$.

Theorem 3.3.3

The following statements hold:

$i)$ $\ell_\infty \subset \ell_\infty(\Delta_q)$ and the inclusion is strict.

$ii)$ $c(\Delta_q) \subset \ell_\infty(\Delta_q)$ and the inclusion is strict .

$iii)$ $c(\Delta) \subset c(\Delta_q)$ and the inclusion is strict .

$iv)$ The sequence space $\ell_\infty(\Delta)$ is different from the sequence space $\ell_\infty(\Delta_q)$ and $\ell_\infty(\Delta) \cap \ell_\infty(\Delta_q) \neq \theta$ (Karakaş et al., 2015).

Proof:

$i)$ Let $x \in \ell_\infty$. The incluction follows from the inequality

$$|qx_k - x_{k+1}| < q\,|x_k| + |x_{k+1}| < K$$

for some $K > 0$.

To show that the inclusion is strict, let us take

$$x_k = q^k - \sum_{i=1}^{k-1} q^i$$

such that $\Delta_q x = (q, q, q, ...)$ then, we obtain $(\Delta_q x_k) \in \ell_\infty$ but $(x_k) \notin \ell_\infty$.

$ii)$ Let $x \in c(\Delta_q)$.Then, we have $\Delta_q x \in c \subset \ell_\infty$ that is, $x \in \ell_\infty(\Delta_q)$.

Hence, $c(\Delta_q) \subset \ell_\infty(\Delta_q)$. To display that the inclusion is strict, define $x = (x_k)$ such that

$$x_k = (0, q, 0, q, 0, ...).$$

Then $x \in \ell_\infty(\Delta_q)\backslash c(\Delta_q)$.

$iii)$ If we choose $x = (x_k)$ such that

$$x_k = (q, 2q, 3q, 4q, ...).$$

Then we obtain $x \in c(\Delta)$ but $x \notin c(\Delta_q)$.

$iv)$ If we choose $x_k = (1, 2, 3, ...)$, then $x \in \ell_\infty(\Delta)$, but $x \notin \ell_\infty(\Delta_q)$. Let us take $x = (x_k)$ such that

$$x_k = q^k - \sum_{i=1}^{k-1} q^i,$$

then we obtain $x \notin \ell_\infty(\Delta)$ but $x \in \ell_\infty(\Delta_q)$. Hence, the spaces $\ell_\infty(\Delta)$ and $\ell_\infty(\Delta_q)$ are overlap.

3.4. Δ_q^m Difference Sequences and Some Properties of Δ_q^m

Later Karakaş et al. (2016), defined equence spaces $\ell_\infty\left(\Delta_q^m\right)$, $c\left(\Delta_q^m\right)$ and $c_0\left(\Delta_q^m\right)$ by

$$\begin{aligned} \ell_\infty\left(\Delta_q^m\right) &= \left\{x = (x_k) : \Delta_q^m x \in \ell_\infty\right\}, \\ c\left(\Delta_q^m\right) &= \left\{x = (x_k) : \Delta_q^m x \in c\right\}, \\ c_0\left(\Delta_q^m\right) &= \left\{x = (x_k) : \Delta_q^m x \in c_0\right\}, \end{aligned}$$

where

$$\Delta_q^m x = \left(\Delta_q^m x_k\right) = \sum_{v=0}^{m} (-1)^v \binom{m}{v} q^{m-v} x_{k+v} = \Delta(\Delta_q^{m-1} x_k - \Delta_q^{m-1} x_{k+1}).$$

Theorem 3.4.1

The sequence spaces $c\left(\Delta_q^m\right)$, $\ell_\infty\left(\Delta_q^m\right)$ and $c_0\left(\Delta_q^m\right)$ are Banach spaces with norm

$$\|x\|_{\Delta_q^m} = \sum_{i=1}^{m} |x_i| + \left\|\Delta_q^m x\right\|_\infty \tag{3.4}$$

(Karakaş et al., 2016).

Proof:

Let (x^n) be a Cauchy sequence in $\ell_\infty\left(\Delta_q^m\right)$, where $x^n = (x_i^n) = (x_1^n, x_2^n, x_3^n, ...) \in \ell_\infty\left(\Delta_q^m\right)$ for each $n \in \mathbb{N}$. Then

$$\left\|x^n - x^t\right\|_{\Delta_q^m} = \sum_{i=1}^{m} \left|x_i^n - x_i^t\right| + \sup_k \left|\Delta_q^m (x_k^n - x_k^t)\right| \to 0$$

as $n, t \to \infty$.

Hence, we obtain

$$\left|x_k^n - x_k^t\right| \to 0$$

as $n, t \to \infty$ and each $k \in \mathbb{N}$.

Therefore, $(x_k^n) = (x_k^1, x_k^2, x_k^3, ...)$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, it is convergent, say $x_k \in \mathbb{C}$, that is,

$$\lim_n x_{_k}^n = x_{_k}, \ (k = 1, 2, 3...)$$

for each $k \in \mathbb{N}$. Since (x^n) is a Cauchy sequence, for each $\varepsilon > 0$, there exits $N = N(\varepsilon)$ such that $\|x^n - x^t\|_{\Delta_q^m} < \varepsilon$ for all $n, t \geq N$.

Hence

$$\sum_{i=1}^{m} \left| x_{_i}^n - x_{_i}^t \right| \leq \varepsilon, \left| \sum_{v=0}^{m} (-1)^v \binom{m}{v} q^{m-v} (x_{k+v}^n - x_{k+v}^t) \right| \leq \varepsilon$$

for all $k \in \mathbb{N}$, and $n, t \geq N$. So we have

$$\lim_t \sum_{i=1}^{m} \left| x_{_i}^n - x_{_i}^t \right| = \sum_{i=1}^{m} \left| x_{_i}^n - x_{_i} \right| \leq \varepsilon,$$

and

$$\lim_t \left| \Delta_q^m (x_k^n - x_k^t) \right| = \left| \Delta_q^m (x_k^n - x_k) \right| \leq \varepsilon,$$

for all $n \geq N$.

$$\sup_k \left| \Delta_q^m (x_k^n - x_k) \right| < \varepsilon.$$

This implies that $\|x^n - x\|_{\Delta_q^m} \leq 2\varepsilon$ for $n \geq N$, that is, $x^n \to x$ as $n \to \infty$ where $x = (x_k)$. Since

$$\begin{aligned}
\left| \Delta_q^m x_k \right| &= \left| \sum_{v=0}^{m} (-1)^v \binom{m}{v} q^{m-v} x_{k+v} \right| \\
&= \left| \sum_{v=0}^{m} (-1)^v \binom{m}{v} q^{m-v} (x_{k+v} - x_{k+v}^N + x_{k+v}^N) \right| \\
&\leq \left| \sum_{v=0}^{m} (-1)^v \binom{m}{v} q^{m-v} (x_{k+v}^N - x_{k+v}) \right| + \left| \sum_{v=0}^{m} (-1)^v \binom{m}{v} q^{m-v} (x_{k+v}^N) \right| \\
&\leq \left\| x^N - x \right\|_{\Delta_q^m} + \left| \Delta_q^m x_k^N \right| = O(1)
\end{aligned}$$

we obtain $x \in \ell_\infty \left(\Delta_q^m\right)$. Thus, $\ell_\infty \left(\Delta_q^m\right)$ is a Banach space.

In the same way, it can be shown that $c\left(\Delta_q^m\right)$ and $c_0\left(\Delta_q^m\right)$ are Banach spaces with the norm (3.4).

Futhermore, since $\ell_\infty \left(\Delta_q^m\right)$, $c\left(\Delta_q^m\right)$ and $c_0\left(\Delta_q^m\right)$ are Banach spaces with continuous coordinates, that is, $\|x^n - x\|_{\Delta_q^m} \to 0$ implies $|qx_k^n - x| \to 0$ for each $k \in \mathbb{N}$, as $n \to \infty$, they are also BK-spaces. Let us define the operator

$$D : \ell_\infty \left(\Delta_q^m\right) \to \ell_\infty \left(\Delta_q^m\right)$$

as $Dx = (0, 0, ...0, x_{m+1}, x_{m+2}, ...)$, where $x = (x_1, x_2, x_3, x_4, ...)$. Its easy to show that D is a bounded linear operator on $\ell_\infty \left(\Delta_q^m\right)$. Furthermore the set

$$D[\ell_\infty \left(\Delta_q^m\right)] = D\ell_\infty \left(\Delta_q^m\right) = \left\{x = (x_k) : x \in \ell_\infty \left(\Delta_q^m\right),\ x_1 = x_1 = x_2 = ... = x_m = 0\right\}$$

is a subspace of $\ell_\infty \left(\Delta_q^m\right)$, and

$$\|x\|_{\Delta_q^m} = \left\|\Delta_q^m x\right\|_\infty$$

in $D\ell_\infty \left(\Delta_q^m\right)$.

Now let us define

$$\Delta_q^m : D\ell_\infty \left(\Delta_q^m\right) \to \ell_\infty,\ \Delta_q^m x = y = \sum_{v=0}^{m} (-1)^v \binom{m}{v} q^{m-v} x_{k+v} \tag{3.5}$$

$$(\Delta_q^m)^{-1} : \ell_\infty \to D\ell_\infty \left(\Delta_q^m\right),\ ((\Delta_q^m)^{-1} x_k) = (y_k) = \sum_{v=1}^{k-m} (-1)^v \binom{k-v-1}{m-1} q^{k-m-v} x_v$$

It can be shown that Δ_q^m is a linear homeomorphism. Hence $D\ell_\infty \left(\Delta_q^m\right)$ and ℓ_∞ are equivalent as topological spaces. Δ_q^m and $\left(\Delta_q^m\right)^{-1}$ are norm preserving and

$$\left\|\Delta_q^m\right\| = \sup_{x\neq 0} \frac{\left\|\Delta_q^m x\right\|_\infty}{\|x\|_{\Delta_q^m}} = \sup_{x\neq 0} \frac{\left\|\Delta_q^m x\right\|_\infty}{\left\|\Delta_q^m x\right\|_\infty} = 1$$

$$\left\|\left(\Delta_q^m\right)^{-1}\right\| = \sup_{x\neq 0} \frac{\|y\|_{\Delta_q^m}}{\|x\|_\infty} = \sup_{x\neq 0} \frac{\sup_k \left\|\Delta_q^m (\Delta_q^m)^{-1} x_k\right\|_\infty}{\sup_k |x_k|} = 1$$

hence

$$\left\|\Delta_q^m\right\| = \left\|\left(\Delta_q^m\right)^{-1}\right\| = 1.$$

Following Karakaş et al. (2016) give some properties of some topological properties of $\Delta_q^m (X)$ and inclusion relations.

Theorem 3.4.2

Let X be a vector space and A subset of X. If A is a convex set, then $\Delta_q^m(A)$ is a convex set in $\Delta_q^m(X)$ (Karakaş et al., 2016).

Proof:

Let $x, y \in \Delta_q^m(A)$. Then $\Delta_q^m x,\ \Delta_q^m y \in A$. Since Δ_q^m is linear, then

$$\lambda \Delta_q^m x + (1-\lambda)\Delta_q^m y = \Delta_q^m(\lambda x + (1-\lambda) y), (0 \le \lambda \le 1).$$

Since A is convex , so $\lambda \Delta_q^m x + (1-\lambda)\Delta_q^m y \in A$ and so $\lambda x + (1-\lambda) y \in \Delta_q^m(A)$, where $0 \le \lambda \le 1$.

Theorem 3.4.3

The following statements hold:

i) $\ell_\infty \subset \ell_\infty(\Delta_q^m)$ and the inclusion is strict.

ii) $c(\Delta_q^m) \subset \ell_\infty(\Delta_q^m)$ and the inclusion is strict.

iii) $c(\Delta) \subset c(\Delta_q^m)$ and the inclusion is strict.

iv) The sequence space $\ell_\infty(\Delta)$ is different from the sequence space $\ell_\infty(\Delta_q^m)$ and $\ell_\infty(\Delta) \cap \ell_\infty(\Delta_q^m) \neq \emptyset$ (Karakaş et al., 2016).

Proof:

i) Let $x \in \ell_\infty$. The inclusion follows from the inequality

$$\begin{aligned} \left|\Delta_q^m x\right| &= \left|\sum_{v=0}^{m}(-1)^v \binom{m}{v} q^{m-v} x_{k+v}\right| \\ &\le \binom{m}{0} q^m |x_k| + \binom{m}{1} q^{m-1}|x_{k+1}| + \binom{m}{2} q^{m-2}|x_{k+2}| + \ldots \binom{m}{m-1} q\,|x_{k+v}| < K \end{aligned}$$

for some $K > 0$. Hence $(\Delta_q^m x_k) \in \ell_\infty \Rightarrow x \in \ell_\infty(\Delta_q^m)$.

To show that the inclusion is strict, let us take

$$x_k = q^k - \sum_{i=1}^{k-1} q^i$$

such that

$$\Delta_q^m = \left(q(q-1)^{m-1}, q(q-1)^{m-1}, q(q-1)^{m-1}, \ldots\right)$$

then, we obtain $(\Delta_q^m x_k) \in \ell_\infty$ but $(x_k) \notin \ell_\infty$.

ii) Let $x \in c(\Delta_q^m)$. Then, we have $\Delta_q^m x \in c \subset \ell_\infty$, that is, $x \in \ell_\infty(\Delta_q^m)$. Hence, $c(\Delta_q^m) \subset \ell_\infty(\Delta_q^m)$. To prove that the inclusion is strict, define $x = (x_k)$ such that

$$x_k = (0, q, 0, q, 0, \ldots)$$

Then $x \in \ell_\infty\left(\Delta_q^m\right) \backslash c\left(\Delta_q^m\right)$.

iii) If we choose $x = (x_k)$ such that

$$x_k = (q, 2q, 3q, 4q, ...).$$

Then, we obtain $x \in c(\Delta)$ but $x \notin c\left(\Delta_q^m\right)$.

iv) If we choose $x_k = (1, 2, 3, ...)$, then $x \in \ell_\infty(\Delta)$, but $x \notin \ell_\infty\left(\Delta_q^m\right)$. Let us take $x = (x_k)$

such that

$$x_k = q^k - \sum_{i=1}^{k-1} q^i$$

then we obtain $x \notin \ell_\infty(\Delta)$ but $x \in \ell_\infty\left(\Delta_q^m\right)$. Hence, the spaces $\ell_\infty(\Delta)$ and $\ell_\infty\left(\Delta_q^m\right)$ are overlap.

4. ISOMETRY of $\ell_p(\Delta_q^m)$ SEQUENCE SPACE GENERATED BY Δ_q^m DIFFERENCE OPERATOR

In this section we give the original part of this thesis.

Peralta (2010), studied $\ell_p(\Delta^m)$ and examined some topological properties of this space.

Let $p \in [1, \infty)$. We define the space of all sequences $x = (x_k)$, where $x_k \in \mathbb{C}$ for all $k \in \mathbb{N}$ by ω. Taken $x \in \omega$, describe

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}$$

and let

$$\ell_p = \{x = (x_k) : \|x\|_p < \infty\}.$$

We obtain the linear difference operator $\Delta : \omega \to \omega$ which maps a sequence $x \in \omega$ into $\Delta x = (\Delta x_k) \in \omega$ having components

$$\Delta x_k = x_k - x_{k+1}.$$

The linear operator $\Delta_q^m : \omega \to \omega$ is presented recursively as the composition $\Delta_q^m = \Delta_q \circ \Delta_q^{m-1}$ for $m \geq 2$ and $q \in \mathbb{N}$. It is obvious that for $m \geq 1$ and $x \in \omega$ we present the following Binomial representation

$$\Delta_q^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} q^{m-v} x_{k+v},$$

for each $k \in \mathbb{N}$.

Taken $m \in \mathbb{N}$, we find the sequence space as:

$$\ell_p(\Delta_q^m) = \{x = (x_k) : \Delta_q^m x \in \ell_p\}$$

and for $x \in \ell_p(\Delta_q^m)$ we get

$$\|x\|_{p,\Delta_q^m} = \left(\sum_{i=1}^{m} |x_i|^p + \left\|\Delta_q^m x\right\|_p^p \right)^{1/p} \tag{4.1}$$

It can be dearly observed that the pair $(\ell_p(\Delta_q^m), \|.\|_{p,\Delta_q^m})$ is a normed space.

We have the following inclusions $\ell_p(\Delta_q^m) \subset \ell_\infty(\Delta_q^m)$ and $\ell_p(\Delta_q^m) \subset c_0(\Delta_q^m) \subset c(\Delta_q^m)$.

The researchers may consult with Altay and Polat (2006) and Qamaruddin and Saifi (2005).

In this section, we will prove $\ell_p(\Delta_q^m)$ with norm $\|.\|_{p,\Delta_q^m}$ is a Banach space and linearly isometric to the ordinary sequence space ℓ_p. Moreover, a sufficient condition for the inclusion $\ell_p(\Delta_q^m) \subset \ell_p(\mathcal{M}, \Delta_q^m)$, where $\mathcal{M}$ is a family of Orlicz functions satisfying the Δ_2-condition, will be given.

Theorem 4.1.1

The sequence space $\ell_p(\Delta_q^m)$ is a Banach space with the norm $\|.\|_{p,\Delta_q^m}$.

Proof:

Let $(x^{(n)}) = ((x_k^{(n)}))$ is a Cauchy sequence in $\ell_p(\Delta_q^m)$. Thus, for $\varepsilon > 0$ we can find a positive integer N such that

$$\left\|x^{(n)} - x^{(r)}\right\|_{p,\Delta_q^m} < \varepsilon$$

whenever $n, r \geq N$, that is,

$$\left(\sum_{i=1}^{m} \left|x_i^{(n)} - x_i^{(r)}\right|^p + \left\|\Delta_q^m x^{(n)} - \Delta_q^m x^{(r)}\right\|_p^p\right)^{\frac{1}{p}} < \varepsilon,$$

for $n, r \geq N$.

Since

$$\left|x_i^{(n)} - x_i^{(r)}\right| \leq \left\|x^{(n)} - x^{(r)}\right\|_{p,\Delta_q^m}$$

for $i = 1, 2, 3, ..., m$ and

$$\left\|\Delta_q^m x^{(n)} - \Delta_q^m x^{(r)}\right\|_p \leq \left\|x^{(n)} - x^{(r)}\right\|_{p,\Delta_q^m}.$$

Therefore, $(x_i^{(n)})$ and $(\Delta_q^m x^{(n)})$ are Cauchy sequences in $\mathbb{C}$ and ℓ_p, respectively. The completeness of the spaces $\mathbb{C}$ and ℓ_p show the existence of elements $y_i \in \mathbb{C}$, $i = 1, 2, ..., m$, and $z = (z_k) \in \ell_p$ such that

$$\lim_{n\to\infty} |x_i^{(n)} - y_i| = 0 \tag{4.2}$$

for $i = 1, 2, ..., m$ and

$$\lim_{n\to\infty} \left\|\Delta_q^m x^{(n)} - z\right\|_p = 0. \tag{4.3}$$

Since

$$|\Delta_q^m x_k^{(n)} - z_k| \leq \left\|\Delta_q^m x^{(n)} - z\right\|_p$$

we get

$$|\Delta_q^m x_k^{(n)} - z_k| \to 0$$

as $n \to \infty$ for all $k \in \mathbb{N}$ by equation (4.3).

We obtain a recursive formula for $\lim\limits_{n\to\infty} x^{(n)}_{m+i}, i \geq 1$, as $n \to \infty$.

We have

$$(-1)^m x^{(n)}_{m+1} = \Delta^m_q x^{(n)}_1 - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} q^{m-v} x^{(n)}_{v+1}$$

and so

$$w_{m+1} := \lim_{n\to\infty} x^{(n)}_{m+1} = (-1)^m \left[z_1 - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} q^{m-v} y_{v+1} \right].$$

Assume that $w_{m+1}, ..., w_{m+k-1}$, $1 < k \leq m$, have been established. Where

$$w_{m+i} := \lim_{n\to\infty} x^{(n)}_{m+i}, i = 1, 2, ..., k-1.$$

Using these, we acquire, for $1 < k \leq m$

$$w_{m+k} := \lim_{n\to\infty} x^{(n)}_{m+k} = (-1)^m \begin{bmatrix} z_k - \sum\limits_{v=0}^{m-k} (-1)^v \binom{m}{v} q^{m-v} y_{v+k} \\ - \sum\limits_{v=1}^{k-1} (-1)^{m-k+v} \binom{m}{m-k+v} q^{k-v} w_{m+v} \end{bmatrix}.$$

On the other side, for $k > m$ we get

$$(-1)^m x^{(n)}_{m+k} = \Delta^m_q x^{(n)}_k - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} q^{m-v} x^{(n)}_{v+k}.$$

So that

$$w_{m+k} = \lim_{n\to\infty} x^{(n)}_{m+k} = (-1)^m \left[z_k - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} q^{m-v} w_{k+v} \right].$$

Let $w = (y_1, ..., y_m, w_{m+1}, w_{m+2}, ...)$. We assert that $w \in \ell_p(\Delta^m_q)$, that is, $\Delta^m_q w \in \ell_p$. First, show that

$$\begin{aligned}
(\Delta^m_q w)_1 &= \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} q^{m-v} y_{v+1} + (-1)^m w_{m+1} \\
&= \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} q^{m-v} y_{v+1} + \left[z_1 - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} q^{m-v} y_{v+1} \right] \\
&= z_{1.}
\end{aligned}$$

Also, for $k = 2, ..., m$. We get

$$
\begin{aligned}
(\Delta_q^m w)_k &= \sum_{v=0}^{m-k} (-1)^v \binom{m}{v} q^{m-v} y_{v+k} + \sum_{v=m-k+1}^{m-1} (-1)^v \binom{m}{v} q^{m-v} w_{v+k} + (-1)^m w_{m+k} \\
&= z_k.
\end{aligned}
$$

Similarly, for $k > m$ we acquire

$$
\begin{aligned}
(\Delta_q^m w)_k &= \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} q^{m-v} w_{v+k} + (-1)^m w_{m+k} \\
&= z_k.
\end{aligned}
$$

Thus we have presented that $\Delta_q^m w = z \in \ell_p$. It remains to prove that $\left\|x^{(n)} - w\right\|_{p,\Delta_q^m} \to 0$ as $n \to \infty$. This follows directly from Equations (4.2) and (4.3) and

$$
\begin{aligned}
\lim_{n\to\infty} \left\|x^{(n)} - w\right\|_{p,\Delta_q^m}^p &= \lim_{n\to\infty} \left(\sum_{k=1}^{m} \left| x_k^{(n)} - y_k \right|^p + \left\| \Delta_q^m x^{(n)} - \Delta_q^m w \right\|_p^p \right) \\
&= \sum_{k=1}^{m} \lim_{n\to\infty} \left| x_k^{(n)} - y_k \right|^p + \lim_{n\to\infty} \left\| \Delta_q^m x^{(n)} - z \right\|_p^p \\
&= 0.
\end{aligned}
$$

This completes the proof of the theorem.

Theorem 4.1.2

The sequence spaces $(\ell_p(\Delta_q^m), \|.\|_{p,\Delta_q^m})$ and $(\ell_p, \|.\|_p)$ are linearly isometric.

Proof:

Take in to consideration the map $T : \ell_p(\Delta_q^m) \to \ell_p$ given by $Ty = x$, where $y = (y_k) \in \ell_p(\Delta_q^m)$ and $x = (x_k)$ with

$$
x_k = \begin{cases} y_k, & \text{if } 1 \le k \le m; \\ \Delta_q^m y_{k-m.}, & \text{if } k > m. \end{cases}
$$

The linearity of the difference operator Δ implies the linearity of T. If $y \in \ell_p(\Delta_q^m)$ and $Ty = x$, then

$$
\begin{aligned}
\|Ty\|_p^p = \|x\|_p^p &= \sum_{k=1}^{m} |y_k|^p + \sum_{k=m+1}^{\infty} \left|\Delta_q^m y_{k-m}\right|^p \\
&= \sum_{k=1}^{m} |y_k|^p + \sum_{k=1}^{\infty} \left|\Delta_q^m y_k\right|^p \\
&= \|y\|_{p,\Delta_q^m}^p < \infty.
\end{aligned}
$$

This shows that T is well-defined and it is also norm preserving. We presented that T is one-to-one and onto. Assume that $Ty = 0$.

Then, we obtain

$$\Delta_q^m y_k = 0 \text{ for all } k \geq 1, \tag{4.4}$$

$$y_1 = y_2 = ... = y_m = 0. \tag{4.5}$$

We show that the difference equation (4.4) with initial conditions (4.5) implies that $y_k = 0$ for all $k \geq 1$, that is, $y = (0, 0, ...)$. Therefore, T is one-to-one.

Suppose that $x = (x_k) \in \ell_p$. Describe the sequence $y = (y_k)$ as follows. Let $y_k = x_k$ for $x_{m+k} = \Delta_q^m x_k, k = 1, 2, ..., m$. The succeeding terms of the sequence y is then showed recursively by

$$y_{m+1} = (-1)^m \left[x_{m+1} - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} q^{m-v} x_{v+1} \right]$$

$$y_{m+k} = (-1)^m \left[\begin{array}{c} x_{m+k} - \sum\limits_{v=0}^{m-k} (-1)^v \binom{m}{v} q^{m-v} x_{v+k} \\ - \sum\limits_{v=1}^{k-1} (-1)^{m-k+v} \binom{m}{m-k+v} q^{k-v} y_{m+v} \end{array} \right], 1 < k \leq m$$

and

$$y_{m+k} = (-1)^m \left[x_{m+k} - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} q^{m-v} y_{v+k} \right], k > m.$$

Utilizing a similar argument as in the proof of the previous theorem, we can prove that

$$\Delta_q^m y_k = x_{k+m}$$

for $k \in \mathbb{N}$. Therefore it follows that $Ty = x$.

Thus, we obtain

$$
\begin{aligned}
\left\|\Delta_q^m y\right\|_p^p &= \sum_{k=1}^{\infty} \left|\Delta_q^m y_k\right|^p \\
&= \sum_{k=1}^{\infty} |x_{k+m}|^p \\
&\leq \|x\|_p^p < \infty.
\end{aligned}
$$

So that $y \in \ell_p(\Delta_q^m)$. Since. T is onto, $\ell_p(\Delta_q^m)$ and ℓ_p are linearly isometric.

Definition 4.1.3

An Orlicz function is a continuous, non decreasing and convex function $M : [0, \infty) \to [0, \infty)$ such that $M(u) = 0$ if and only if $u = 0$, $M(x) > 0$, and $M(x) \to \infty$ as $x \to \infty$. M is said to fulfil $\Delta_2 - condition$ if there exists a positive constant K such that $M(2u) \leq KM(u)$ for all $u \geq 0$. Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions meeting the $\Delta_2 - condition$ (Kamthan and Gupta, 1981).

The study Orlicz sequence spaces was initiated with a certain specific goal in Banach spaces theory. Actually, Lindberg (1970) and Lindberg (1973) concerned on Orlicz spaces in connection with finding Banach spaces with symmetric Schauder bases that have complementary subspaces isomorphic to c_0 or ℓ_p $(1 \leq p < \infty)$. There after, these Orlicz sequence spaces were further investigated in details by Lindenstrauss and Tzafriri that lead to solve main structural problems in Banach spaces (Lındenstrauss and Tzafriri, 1971), (Lındenstrauss and Tzafriri, 1972), (Lındenstrauss and Tzafriri, 1973), (Lındenstrauss and Tzafriri, 1973), (Lındenstrauss and Tzafriri, 1977). In the meantime, Woo (1973), generalized the concept of Orlicz sequence spaces to modular sequence spaces and this led him to sharpen some of the results of Lindberg and of Lindenstrauss and Tzafriri; he carried thise study further in (Woo, 1975). The Orlicz sequence spaces are the special cases of Orlicz spaces introduced in Orlicz (1932) and extensively studied in Krasnoselskii and Rutitsky (1961). Orlicz spaces find several beneficial implementations in the theory of nonlinear integral equations. Where as the Orlicz sequence spaces are the generalizations of ℓ_p-spaces, the L^p-spaces find themselves enveloped in Orlicz spaces.

An Orlicz function M can always be represented see (Krasnoselskii and Rutitsky, 1961) also see (Kamthan, 1963) for a more general representation in thise direction) in the following integral from:

$$
M(x) = \int_0^x p(t)dt
$$

where p, know as the kernel of M, is right-differentable for $t \geq 0$, $p(0) = 0$, $p(t) > 0$

for $t > 0$, p is nondecreasing, and $p(t) \to \infty$ as $t \to \infty$.

Note: An Orlicz function is sometimes referred to as an 0-function as well.

Consider the kernel $p(t)$ associated with an Orlicz function $M(t)$, and let

$$q(s) = \sup \{t : p(t) \leq s\}$$

Then q possesses the same properties as the function P. Suppose now

$$N(x) = \int_0^x q(s)ds$$

Then N is an Orlicz function. The functions M and N are called mutually complementary 0-functions (or mutually complementary Orlicz function).

The following result on complementary 0-functions is quoted from Krasnoselskii and Rutitsky (1961).

Proposition 4.1.4

Let M and N be mutually complementary functions. Then we have (young' s inequality)

i) For $x, y \geq 0$, $xy \leq M(x) + N(y)$.

We also have

ii) For $x \geq 0$, $xp(x) = M(x) + N(p(x))$.

Peralta (2010) describe the sequence spaces as:

$$\ell_p(M) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} M_k(|x_k|/\rho)|^p < \infty, \text{ for some } \rho > 0 \right\}$$

and

$$\ell_p(M, \Delta_q^m) = \left\{ x = (x_k) : \Delta_q^m x \in \ell_p(M) \right\}.$$

Theorem 4.1.5

Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions fulfil the $\Delta_2 - \textit{condition}$. If

$$\sum_{k=1}^{\infty} |M_k(t/\rho)|^p < \infty \tag{4.6}$$

for all $t, \rho > 0$ then $\ell_p(\Delta_q^m) \subset \ell_p(\mathcal{M}, \Delta_q^m)$.

Proof:

Suppose condition (4.6) holds and let $x = (x_k) \in \ell_p(\Delta_q^m)$. Then, we get

$$\sum_{k=1}^{\infty} |\Delta_q^m x_k|^p < \infty. \tag{4.7}$$

The convergence of

$$\sum_{k=1}^{\infty} |\Delta_q^m x_k|^p < \infty$$

implies that

$$\lim_{k\to\infty} |\Delta_q^m x_k| = 0.$$

Thus, we can find $n \in \mathbb{N}$ such that $|\Delta_q^m x_k| \leq 1$ for all $k \geq N$.

Let

$$K = max(\left|\Delta_q^m x_1\right|, ..., \left|\Delta_q^m x_{N-1}\right|, 1).$$

Then $\left|\Delta_q^m x_k\right| \leq K$ for all $k \in \mathbb{N}$. For $\rho > 0$, utilizing the monotonicity of M_k, we get $M_k(|\Delta_q^m x_k|/\rho) \leq M_k(K/\rho)$ for all $k \in \mathbb{N}$.

This inequality shows that

$$\sum_{k=1}^{\infty} \left|M_k(\left|\Delta_q^m x_k\right|/\rho)\right|^p \leq \sum_{k=1}^{\infty} |M_k(K/\rho)|^p .$$

This estimate proves that $\Delta_q^m x \in \ell_p(\mathcal{M})$, that is, $x \in \ell_p(\mathcal{M}, \Delta_q^m)$. By equation (4.6). Therefore, the inclusion $\ell_p(\Delta_q^m) \subset \ell_p(\mathcal{M}, \Delta_q^m)$ holds.

5. CONCULUSION

Peralta (2010) studied $\ell_p(\Delta^m)$ and examined the topological properties of this space. Later Karakaş et al. (2016) defined difference operator Δ_q^m. We used Peralta' s (2010) studies and extented it by used the generalized difference operator Δ_q^m. We generated the difference sequence space $\ell_p(\Delta_q^m)$ and $\|x\|_{p,\Delta_q^m}$, and investigated some of their properties. We showed that, if $\ell_p(\Delta_q^m)$ is equipped with an appropriate norm $\|.\|_{p,\Delta_q^m}$ is a Banach space. We further more showed that, the sequence spaces $(\ell_p(\Delta_q^m), \|.\|_{p,\Delta_q^m})$ and $(\ell_p, \|.\|_p)$ are linearly isometric. It is shown that $\ell_p(\Delta_q^m) \subset \ell_p(\mathcal{M}, \Delta_q^m)$. Where $\mathcal{M}$, a family of Orlicz functions, is meeting the Δ_2-condition.

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