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SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS BY REPRODUCING KERNEL METHOD AND GROUP PRESERVING SCHEME

MASTER DEGREE THESIS

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THESIS ACCEPTANCE AND APPROVAL

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This thesis is prepared in accordance with the thesis writing rules and it is comply with the scientific code of ethics. In case of utilization others' works or results, it is clearly referred to in accordance with the scientific norms, innovations contained in the thesis. I declare that in any part of the thesis, there is no tampering with the used data and also the data is not presented in another thesis work at this university or another university

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SİİRT-2017

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CONTENTS

Page

LIST OF SYMBOLS

Symbol	**Explanation**
$\mathbb{N}$:	The set of Natural numbers
$\mathbb{R}$:	The set of Real numbers
$\mathbb{C}^{\boldsymbol{n}}$:	n-dimensional complex number
$\mathbb{C}$:	The set of complex numbers
$W_2^m[a,b]$:	Reproducing kernel space
$\lVert u \rVert$:	The norms
$\delta(x,y)$:	Dirac dalta function
$\boldsymbol{L}^*$:	Adjoint operator
G(p,q) :	Green function
H :	Hilbert space

ÖZET

YÜKSEK LİSANS

LİNEER OLMAYAN DİFERANSİYEL DENKLEMLERİN ÇEKİRDEK ÜRETEN METOD VE GRUP KORUMA METODU İLE ÇÖZÜMLERİ

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Siirt Üniversitesi Fen Bilimleri Enstitüsü
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Bu tez 6 bölümden oluşmaktadır. Bu tezin ilk bölümü tarihsel gelişim ile alakalı bilgilerden oluşmaktadır. İkinci bölümü üretilen çekirdekler ve diferansiyel denklemlerin genel kavramlarını vermektedir. Çekirdek fonksiyonlar detaylı bir şekilde üçüncü bölümde ele alındı. Dördüncü bölümde çekirdek üreten uzaylar ve fonksiyonlar verildi. Yeni uygulamalar beşinci bölümde ele alındı. Son bölümde sonuç verildi.

Anahtar Kelimeler: Çekirdek Üreten Metod, Diferansiyel Denklemler, Grup Koruyan Metod, Lineer Olmayan.

ABSTRACT

MS. THESIS

SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS BY REPRODUCING KERNEL METHOD AND GROUP PRESERVING SCHEME

Salar Ameen RAHEEM

The Graduate School of Natural and Applied Science of Siirt University
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We divided this thesis into the six sections. The first section of the thesis presents the introduction deals with a historical review. The second section gives the general concepts of differential equations and reproducing kernels. Kernels are explained clearly in the third section. Section 4 gives the reproducing kernel spaces and functions. New applications are shown in Section 5. Conclusion is given in Section 6.

Keywords: Group Preserving Scheme, Nonlinear Differential Equations, Reproducing Kernel Method,
.

1. INTRODUCTION

The studies of reproducing kernels began with Szegö (1921) and Bergman (1922). After all, definitely, reproducing kernels seemed at the first in the twentieth century in Zaremba (1907). Mean while, the common theory of these kernels was constituted in Aronszajn (1950). Furthemore, L. Schwartz (1964) Enhanced the general theory remarkably in 1964. When we take into consideration linear mappings in the framework of Hilbert space we will see the notion of reproducing kernels. It is a main notion and valuable mathematics (Saitoh and Sawano, 2016).

Definition 1.1.1

An ordinary differential equation (ODE) is an equation that contains an unknown function of a single variable and one or more of its derivatives (Modern, 2009).

Example 1.1.2

Here's a typical simple ODE, with some of its constituents stated:
unknown function, $z \downarrow$

$$3\frac{dz}{dx} = z$$

independent variable, $x \uparrow$.

This equation defines an unknown function of x that is equal to three times its own derivative.The Leibniz formula for a derivative,$\frac{d()}{d()}$, is helpful because the independent variable seems in the denominator, the dependent variable in the numerator. The following equations

$$\begin{cases} \frac{dy}{dx} + 2xy = e^{-x^2} \\ x^{''}(t) - 5x^{'}(t) + 6x(t) = 0 \\ \frac{dm}{dn} = \frac{3n^2+4n+2}{2(m-1)} \end{cases}$$

leave no suspicions with the relationship between independent and dependent variables. However, in an equation such as $(p^{'})^2 + 2x^3p^{'} - 4x^2p = 0$, we must make out that the unknown function p is really $p(x)$, a function of the independent variable x. In many implementations, the independent variable is time, given by x, and we can define the function's derivative using Newton's dot notation, as in the equation $\ddot{x} + 3t\dot{x} + 2x = \sin(wt)$. The following ordinary differential equations

$$(A)\frac{d^2u}{dt^2} - 3\frac{du}{dt} + 7u = 0 and (B)\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 7y = 0$$

are the same that is, they are defining the same mathematical or physical behavior (Szegö, 1921).

2.GENERAL CONCEPTS

2.1. The Order of an Ordinary Differential Equation

Definition 2.1.1

An ordinary differential equation is of order n if the highest derivative of the unknown function in the equation is the n th derivative.

$$\frac{dz}{dy} + 2yz = e^{-y^2} \tag{4}$$

$$(p^{'})^2 - 2x^3p^{'} - 4x^2p = 0$$

$$\frac{dg}{dm} = \frac{3m^2 + 4m + 2}{2(g-1)}$$

are all first-order differential equations.

$$w^{''}(x) - 5w^{'}(x) + 6wx = 0 \tag{5}$$

and

$$\ddot{w}(x) - 5\dot{w}(x) + 2w = sin(d\ x) \tag{6}$$

are second-order equations, and $e^{-p}w^{(5)} + (\sin p)w''' = 3e^p$ is of order 5 (Szegö, 1921).

2.2. A Common Manner for an Ordinary Differential Equation

If z is the unknown function with a single independent variable t, and $z^{(k)}$ describes the kth derivative of z, we state an n th-order differential equation as:

$$M(t, z, z^{'}, z^{''}, z^{'''}, ..., z^{(n-1)}, z^{(n)}) = 0 \tag{7}$$

or often as

$$z^{(n)} = N(t, z, z^{'}, z^{''}, z^{'''}, ..., z^{(n-1)}). \tag{8}$$

2.3. Common Manner for a Second-Order ODE

If z is an unknown function of p, then the second-order ordinary differential equation $2\frac{d^2z}{dp^2} + e^p\frac{dz}{dp} = z + \sin(p)$ can be expressed as $2\frac{d^2z}{dp^2} + e^p\frac{dz}{dp} - z - \sin(p) = 0$ or as

$$\underbrace{2z^{''} + e^pz^{'} - z - \sin p}_{F\ (\ p,\ z,\ z',z''\ )} = 0. \tag{9}$$

F defines a mathematical expression containing the independent variable p, the unknown function z, and the first and second derivatives of z. We can utilize ordinary algebra to solve the original differential equation for its highest derivative and obtaion the equation as

$$z'' = \underbrace{\frac{1}{2}\sin p + \frac{1}{2}z - \frac{1}{2}e^p z'}_{G(\ p,\ z,\ z')}.$$

2.4. Main Notions Of Reproducing Kernels

We give main notions of reproducing kernels. These notions are main, extensive and have involved backgrounds. Take into consideration $\delta(s,p)$ at $p \in D$ with any continuous function f on D we get

$$h(p) = \int_D h(s)\delta(s,p)ds. \tag{10}$$

This main notion was obtained by a physician Paul Adrien Maurice Dirac and by L. Schwartz. The performance will demonstrate that $\delta(s,p)$ is not the normal function, but a distribution. Mean while, assume that a solution $G(s,p)$ for some linear (differential) operator L on some function space on D is presented by the equation, symbolically, for any fixed $p \in D$

$$LG(s,p) = \delta(s,p) \tag{11}$$

whose similarity is applicable on D apart from for the point $p \in D$ when G turns only on the range $|s-p|$, then the function $G\ |s-p|$ will be called a basic solution for the operator L and further when some boundary conditions are satisfied, then the function $G(s-p)$ will be called a Green's function for the operator L fulfilling the enjoined boundary conditions.

For the adjoint operator $L^*$of L, we take into consideration the self-adjoint operator L^*L and its Green's function $G(p,q)$ fulfilling

$$L^*LG(s,p) = \delta(s,p) \tag{12}$$

at the formal level, whose identity is applicable on D apart from for the point $q \in D$. Then, from (10) we get:

$$h(p) = \int_D h(s)L^*LG(s,p)ds.$$

Therefore, we get the presentment symbolically, utilizing the Green Stokes formula,

$$h(p) = \int_D Lh(s)LG(s,p)ds + \text{some boundary integrals.} \tag{14}$$

If the boundary integrals are zero, then we obtain

$$h(p) = \int_D Lh(s)LG(s,p)ds.$$

If the function space is made up of all fulfilling

$$h(p) = \int_D |Lh(s)|^2 \, ds < \infty, \tag{15}$$

then the space constitutes a Hilbert space, and if the function $G(s,p)$ is the normal function on D refering to this Hilbert space, then the function $G(s,p)$ will show the reproducing property for the Hilbert space. We obtain many cases fulfilling these features.

3.KERNELS

3.1.The Bergman Kernel

It is difficult to obtain an explicit integral formula, with holomrphic reproducing kernel, for holomorphic functions on an arbitrary domain in $\mathbb{C}^n$. Standart works which carry out such constraction tend to center on domains getting a perfect agreement of symmetry Hua (1963) Bungart Bungart (1964). and Gleson Gleason (1962) have presented that any bounded domain in $\mathbb{C}^n$ has a reproducing kernel for holomorphic function as:

$$h(x) = \int_\Omega h(p)K(x,p)dV(p),$$

and K is holomorphic in the x varible. Bungarts and Gleasons' proofs are highly nonconstractive. The venerable Bochner-Martinelli kernel is well constracted on any bounded domain with reasonable boundary and the kernel is explicit just like the Caushy kernel in one complex varible. Henkin Henkin (1969), Kerzman Kerzman et al. (1971), E. Ramirez Ramirez (1970) and Grauret-Lieb Grauert and Lieb. (1970) have given very explicit constractions of reproducing kernels on strictly pseudoconvex domains.

Let $\Omega \subseteq \mathbb{C}^n$ be bounded domains. If the domain is smoothly bounded, then we can consider it as specified by a defining function:

$$\Omega = \{x \in \mathbb{C}^n : \rho(x) < 0\}.$$

It is customary to get that $\nabla\rho \neq 0$ on $\partial\Omega$. We can show the existence of a definiting function by utilizing the implicite function theorem Kratz and Parks. (2002) for the latter and Krantz and Parks. (1996) for a detailed opinion of defining functions.

Give a domain Ω as defined in the last paragraph and a point $P \in \partial\Omega$, w is a complex tangent vector at p and $w \in \tau_p(\partial\Omega)$ if

$$\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(P)w_j = 0.$$

The point P is said to be strongly pseudoconvex if

$$\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j\ \partial\bar{z}_k}(P)w_j\bar{w}_k \geq 0$$

for $0 \neq w \in \tau_p(\partial\Omega)$. In fact a little simple analysis presents that we may give the defining feature of powerful pseudoconvexity by:

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(P) w_j \overline{w}_k \geq C |w|^2$$

and perform the estimate uniform when p aligns over a compact, hard pesudoconvex bounday neighborhood of Ω.

Let dV shows the Lebesgue volume measure on Ω. Define the Bergman space

$$A^2(\Omega) = \{f \text{ holomorphic on } \Omega : \int_{\Omega} |f(z)|^2 \, dV(z)^{\frac{1}{2}} \equiv \|f\|_{A^2(\Omega)} < \infty\}.$$

We provide the Bergeman space with the inner product .

$$\langle u, v \rangle = \int_{\Omega} u(x)\bar{v}(x) dV(x).$$

3.2. Official Ideas of Aronszajn

One of the first canonical integral notations ever found was that of Begman Bergman and Schiffe (1953) . and Krantz and Parks (1996). We give the Bergman idea in the context of a more general structure by the reason of Nechman Aronszajn Aronszajn (1950). This is the thought of a Hilbert space with reproducing kernel and several other important reproducing kernels.

Definition 3.2.1

Let X be any set. Let H be a Hilbert space of complex- valued functions on X. Then, H is a Hilbert space with reproducing kernel if, for each $x \in X$, the linear map of the form

$$\begin{aligned} L_x \quad &: \quad H \longrightarrow \mathbb{C} \\ f \quad &\mapsto \quad f(x), \end{aligned}$$

is continuous. We have

$$|f(x)| \leq C \|f\|_H . \tag{16}$$

The classical Riesz representation theorem gives us that, for each $x \in X$, there is a unique element $K_x \in H$ such that

$$f(x) = \langle f, k_x \rangle, \forall f \in H. \tag{17}$$

we then define a function

$$K : X \times X \longrightarrow \mathbb{C}$$

by the formula

$$R(z,t) \equiv \overline{R_z(t)}.$$

The function R is the reproducing kernel for the H. We know that R is uniquely defined by H. If $\{\varphi_j(z)\}$ is a complete orthonormal basis for the Bergman space, then, we obtain

$$R(z,t) = \sum \varphi_j(z)\overline{\varphi_j(t)},$$

where the convergence is in the Hilbert space topology in each variable (Berlinet, 2004).

Theorem 3.2.2

Let K be a positive function on $E \times E$. We have only one Hilbert space H of functions on E with K as reproducing kernel. The subspace H_0 of H spanned by the functions $(K(.,x)_{x\in E})$ is dense in H. H is the set of functions on E which are pointwise limits of Cauchy sequences in H_0 with the inner product

$$\langle f, g\rangle_{H_0} = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i \overline{\beta_j} K(y_i, x_i) \tag{18}$$

there $f = \sum_{i=1}^{n} \alpha_i K(., x_i)$ and $g = \sum_{j=1}^{m} \beta_j K(., y_i)$ (Berlinet, 2004).

3.3.Basic Properties Of Reproducing Kernels

Theorem 3.3.1

Let K_1 and K_2 be reproducing kernels of spaces H_1 and H_2 of functions on E with the norms $\|.\|_{H_1}$ and $\|.\|_{H_2}$. Then $K = K_1 + K_2$ is the reproducing kernel of the space $H = H_1 \oplus H_2 = \{f \mid f = f_1 + f_2, f_1 \in H_1, f_2 \in H_2\}$ with the norm $\|.\|_H$ defined by (Berlinet, 2004)

$$\forall f \in H \quad \|f\|_H^2 = \min_{\substack{f=f_1+f_2 \\ f_1\in H_1, f_2\in H_2}} (\|f_1\|_{H_1}^2 + \|f_1\|_{H_2}^2).$$

3.4. Support Of A Reproducing Kernel

Definition 3.4.1

Let K be a non null complex function defined on $E \times E$. A subset A of E is said to be association for K if and only if we have elements $x_1, ..., x_n$ in A such that the functions $K(.,x_1),...,K(.,x_n)$ are linearly depedent in the vector space $\mathbb{C}^E$ (Berlinet, 2001).

Theorem 3.4.2

The set Λ_k of non-overarching sets for K partially ordered by inclusion is inductive. Thus, it agrees at least a maximal element (Berlinet, 2001).

Definition 3.4.3

Let K be a non null complex function defined on $E \times E$. A subset S of E is called a support of K if and only if S is a maximal element of the set Λ_k of non-overarching sets for K (Berlinet, 2001).

Theorem 3.4.4.

Let H be $RKHS$ with kernel K on $E \times E$. Let H_0 be the subspace of H_0 be the subspace of H spanned by $\{K(.,x) : x \in E\}$. If a subset S of E is a support of K then $\{K(.,x) : x \in S\}$ is basis of H. On the contrary if $K(.,x),...,K(\in .,x_n)$ are linearly independent, we have a support S of K involving $\{x_1,...,x_n\}$ (Berlinet, 2001).

3.5. Kernel Of An Operator

Definition 3.5.1

Let E be pre-Hilbert space of functions defined on E. Let u be an operator in E. A function $U : E \times E \longrightarrow \mathbb{C}; (x,y) \mapsto U(x,y)$ is said to be a kernel of U if and only if

$$\forall y \in E, U(.,y) \in \varepsilon$$

$$\forall y \in E, \forall f \in \varepsilon, u(f)(y) = \langle f, U(.,y)\rangle_{\varepsilon}.$$

If u has two kernels U_1 and U_2, we get

$$\forall y \in E, \forall f \in \varepsilon, \langle f, U_1(.,y) - U_2(y)\rangle_{\varepsilon} = u(f)(y) - u(f)(y) = 0.$$

Therefore, $\forall y \in E$, $U_1(.,y) = U_2(y)$ and $U_1 = U_2$.

Thus, for any operator there is at most one kernel. It is also obvious from Definition 3.4.3 that a Hilbert space of functions H has a reproducing kernel K if and only if K is the kernel of the identity operator in H (Berlinet, 2001).

Theorem 3.5.2

In a Hilbert space H of functions with reproducing kernel K any continous operator u has a kernel U presented by

$$U(x, y) = [u^*(K(., y))](x), \tag{20}$$

where u^*denotes the adjoint operator of u (Berlinet, 2001).

Proof:

By Rieszs theorem, in the Hilbert space H any continous operator u has an adjoint defined as:

$$\forall (f, g) \in H \times H; \langle u(f), g\rangle_H = \langle f, u^*(g)\rangle_H.$$

Thus we have

$$\forall y \in E, \forall f \in H, \langle f, u^*(K(., y))\rangle_H = \langle u(f), K(., y)\rangle_H = u(f)(y).$$

Theorem 3.5.3

Let V be a closed subspace of a Hilbert space H with reproducing kernel K. Then V is a RKHS. Reproducing kernel K_v of this space is presented as:

$$K_v(x, y) = [\Pi_v(K(., y)](x),$$

where Π_v defines the orthogonal projection on to the space V.

Lemma 3.5.4

In a Hilbert space H of functions with reproducing kernel K any continous linear form $u : H \longrightarrow \mathbb{C}$ has a Riesz representer $\widetilde{u}$ presented as (Berlinet, 2001):

$$\widetilde{u}(x) = u(k(., x))$$

3.6. Condition For $H_K \subset H_R$.

Theorem 3.6.1

Let K be a continous nonegative kernel on $T \times T$. The following statements are equivalent (Alin Berlinet, 2001) :

i) $H_K \subset H_R$

ii) We have a constant B such that $B^2R - K$ is a nonnegative kernel. Yivisaker (1962) presented an alternative condition which is that if $\sum_{j=1}^{N(n)} c_{jn}R(., t_{jn})$ is a Cauchy sequence in H_R, then $\sum_{jn}^{(n)} K(., t_{jn})K(., t_{jn})$ must be a Cauchy sequence in H_K.

iii) We have an operator $L : H_R \rightarrow H_K$ such that $\|L\| \leq B$ and $LR(t, .) = K(t, .)$, $\forall t \in T_0$ where T_0 is a countable dense of T.

Furthermore (i) indicates that there exists a contant B such that

$$\forall g \in H_{K,} \|g\|_R \leq B \|g\|_K ,$$

and either of these condition indicates that there exists adjoint operator $L : H_R \longrightarrow H_K$ such that $\|L\| \leq B$ and

$$\forall t \in T, LR(t, .) = K(t, .).$$

3.7. Tensor Products Of RKHS

Product of function and kernel perform a valuable status in multidimensional setting. Direct product RKHS are invetigated in (Arzen, 1963).

Let H_1 nd H_2 be two vector spaces of comlex functions defined on E_1 and E_2. The tensor product $H \tilde{\otimes} H_2$ is defined as the vector spaces produced by the functions

$$f_1 \otimes f_2 : E_1 \times E_2 \longrightarrow \mathbb{C}$$

$$(x_1, x_2) \longmapsto f_1(x) f_2(x_2),$$

where f_1 diversifies in H_1 and f_2 diversifies in H_2. If $\langle ., . \rangle_1$ and $\langle ., . \rangle_2$ are inner products repectively on H_1 and H_2, it can be presented that the mapping

$$\begin{array}{ccc} H_1 \tilde{\otimes} H_2 & \longrightarrow & \mathbb{C} \\ (f_1 \times f_2, f_1^{'} \times f_2^{'}) & \longmapsto & < f_1, f >_1 < f_1^{'}, f_2^{'} >_2 \end{array}$$

is an inner product on $H_1 \tilde{\otimes} H_2$ which is therefore a pre-Hilbert space. Its completion is called the tensor product of the Hilbert space H_1 and H_2 and is defined by $H_1 \tilde{\otimes} H_2$.

If H_1 is a $RKHS$ with kernel K_1 and H_2 is a $RKHS$ with kernel K_2. Therefore, the mapping

$$K_1 \otimes K_2 : (E_1 \times E_2) \longrightarrow \mathbb{C}$$

$$((x_1, x_2), (y_1, y_2)) \longmapsto K_1(x_{1,} y_1) K_2(x_2, y_2)$$

is a positive type function on $(E_1 \times E_2)^2$. More precisely we have the following theorem (Neveu, 1968).

Theorem 3.7.1

Let H_1 and H_2 be two $RKHS$ with reproducing kernels K_1 and K_2. The tensor product $H_1 \overset{\sim}{\otimes} H_2$ of the vector spaces H_1 and H_2 admits a functional completion $H_1 \otimes H_2$ which is a RKHS with reproducing kernel $K = K_1 \otimes K_2$.

It follows from this theorem that the product of a family of reproducing kernels on the same set E^2 is a reroducing kernel on E^2.

3.8. Schwartz Kernels

Definition 3.8.1

A subspace H of ε is called a Hilbertian subspace of ε if and only if H is a Hilbert space and the natural emdedding

$$I : H \longrightarrow \varepsilon$$

$$h \longmapsto h$$

is continous.

This means that any sequence (f_n) is H converging to some f in the sense of the norm of H also converges to f in the sense of the topology defined on ε. Namely, the Hilbert sace topology on H is finer than the topology induced on H by the one on ε. Let Hilb(ε) be the set of Hilbertion subsaces of ε. Hilb(ε) has a significant structure.

1) For any $\lambda \geq 0$ and any H in Hilb(ε), λH is $\{0\}$ if $\lambda = 0$, otherwise λH is the space H endowed with the inner product

$$\langle h, k \rangle_{\lambda H} = \frac{1}{\lambda} \langle h, k \rangle_H.$$

2) $H_1 + H_2$ is the sum of the vector space H_1 and H_2 given with the norm defined by

$$\|h\|^2_{H_1+H_2} = \inf_{\substack{h_1+h_2=h \\ h_1 \in H_1, h_2 \in H_2}} (\|h_1\|^2_{H_1} + \|h_1\|^2_{H2}).$$

$H_1 + H_2 \in$ Hilb (ε). The internal addition in Hilb (ε) is associative and commutative, $\{0\}$ is a natural element and we obtain

$$\begin{aligned} (\lambda + \mu)H &= \lambda H + \mu H \\ \lambda(H_1 + H_2) &= \lambda H_1 + \lambda H_2. \end{aligned}$$

4. REPRODUCING KERNEL SPACES AND FUNCTIONS

Operator generalizations of the Schur class contains functions $S(z)$ defined on a subregion $\Omega(S)$ of the unit disk involving the origin whose values are operators in $\pounds(\Im, \eth)$ for some Hilbert, Pontryagin, or Krein space $\Im$ and $\eth$. We associate with such functions $S(z)$ the three kernels

$$Ks(w,z) = \frac{1 - S(z)S(w)^*}{1 - z\widetilde{w}}.$$

$$K_{\widetilde{S}}(w,z) = \frac{1 - \widetilde{S}(z)\widetilde{S}(w)^*}{1 - z\widetilde{w}}$$

$$D_s(w,z) = \begin{pmatrix} K_s(w,z) & \frac{S(z)-S(\widetilde{w})}{z-\widetilde{w}} \\ \frac{\widetilde{S}(z)-\widetilde{S}(\widetilde{w})}{z-\widetilde{w}} & K_{\widetilde{S}}(w,z) \end{pmatrix}, \tag{21}$$

where $\widetilde{S}(z) = S(\widetilde{z})^*$ and 1 denotes either the scalar unit or an identity operator, depending on context. When these kernels are nonnegative, they are reroducing kernels for Hilbert space $\Re(S)$, $\Re(\widetilde{S})$, $\check{D}(S)$ of vector-valued functions. (Alpay, 1997).

4.1. Definition of Reproducing Kernel Space

$H = \{f(x)$ is a real value funtion or complex function, $x \in \chi, \chi$ is an abstract set$\}$ is Hilbert space, with inner product

$$\langle f(x), g(x)\rangle_H = (f(x), g(x) \in H). \tag{22}$$

if there exists a function $R_y(x)$ for each fixed $y \in X$ then, $R_y(x) \in H$, and any $f(x) \in H$, which satisfies

$$\langle f(x), R_y(x)\rangle_H = f(y) \tag{23}$$

then $R_y(x)$ is called the reproducing kernel of H and Hilbert space H is called the reproducing Kernel Space (Cui et al., 2009).

4.2. Absolutely continous function and some properties

Definition 4.2.1

Given a function $f(x)$ on interval $[a, b]$, let $\{(a_k, b_k)\}_{k=1}^n$ is a set of mutually disjoint open intervals $(a_k, b_k) \in [a, b]$, if for $\forall\varepsilon$, $\exists\delta$ which has no relation with n, such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon \text{ for } \sum_{i=1}^{n} |(b_i) - (a_i)| < \delta, \tag{24}$$

then, $f(x)$ is said to absolutely continous on interval $[a, b]$.

4.3. $W_2^m[a, b]$ is a Hilbert Space

$W_2^m[a, b]$ is defined as:

$$W_2^m[a, b] = \left\{ f(x) \mid f^{(m-1)}(x) \text{ is absolutely continous; } f^{(m)}(x) \in L^2[a, b], \ x \in [a, b] \right\} \quad (25)$$

For any functions $f(x)$, $g(x) \in W_2^m[a, b]$ the inner product and the norm in the function space $W_2^m[a, b]$ are defined as:

$$\begin{aligned} \langle f, g \rangle_{W_2^m} &= \sum_{i=0}^{m-1} f^{(i)}(a) g^{(i)}(a) + \int_a^b f^{(m)}(x) g^{(m)}(x) dx, \quad (26) \\ \|f\|_{W_2^m} &= \sqrt{\langle f, f \rangle_{W_2^m}}. \end{aligned}$$

4.4. Reproducing kernel function for differential equations :

Definition 4.4.1

Let $E \neq \phi$. A function $K : E \times E \to \mathbb{C}$ is called reproducing kernel function of the Hilbert space H if only if

$a)$ $K(., t) \in H$ for all $t \in E$

$b)$ $\langle \varphi, K(., t) \rangle = \varphi(t)$ for all $t \in E$ and all $\varphi \in H$.

Definition 4.4.2 (REPRODUCING KERNEL HILBERT SPACE)

A Hilbert space H which is defined on a nonempty set E is called reproducing Kernel Hilbert Space if there exists a reproducing kernel function

$$K : E \times E \to \mathbb{C}.$$

Definition 4.4.3

We define the space $G_2^1[a, b]$ by of absolutely continous functions. The inner product and the norm in $G_2^1[a, b]$ are defined by

$$\begin{aligned} \langle u, v \rangle_{G_2^1} &= u(a)v(a) + \int_a^b u'(x) v'(x) dx, \ u, v \in G_2^1[a, b], \quad (27) \\ \|u\|_{G_2^1} &= \sqrt{\langle u, u \rangle_{G_2^1}}. \end{aligned}$$

Theorem 4.4.4

The space $G_2^1[a,b]$ is a reproducing kernel space and its reproducing kernel function $\tilde{O}_y$ is obtained as:

$$\tilde{O}_y(x) = \begin{cases} \sum_{i=1}^{2} c_i(y)x^{i-1}, \ a \le x \le y \le b \\ \sum_{i=1}^{2} d_i(y)x^{i-1}, \ a \le y < x \le b \end{cases} \tag{28}$$

Proof:

By Definition 4.4.3, we have

$$\left\langle u, \tilde{O}_y \right\rangle_{G_2^1} = u(a)\tilde{O}_y(a) + \int_a^b u^{'}(x)\tilde{O}_y^{'}(x)dx, \tag{29}$$

integrating this equation by parts one time, we get

$$\left\langle u, \tilde{O}_y \right\rangle_{G_2^1} = u(a)\tilde{O}_y(a) + u(b)\tilde{O}_y^{'}(b) - u(a)\tilde{O}_y^{'}(a) - \int_a^b u(x)\tilde{O}_y^{''}(x)dx. \tag{30}$$

We have

$$\left\langle u(x), \tilde{O}_y(x) \right\rangle_{G_2^1} = u(y), \tag{31}$$

by reproducing property. If

$$\begin{aligned} \tilde{O}_y(a) - \tilde{O}_y^{'}(a) &= 0, \\ \tilde{O}_y^{'}(b) &= 0, \end{aligned} \tag{32}$$

then (30) gives

$$-\tilde{O}_y^{''}(x) = \delta(x-y). \tag{33}$$

when $x \neq y$, we have

$$\tilde{O}_y^{''}(x) = 0. \tag{34}$$

Therefore, we get

$$\tilde{O}_y(x) = \begin{cases} c_1(y) + c_2(y)x, a \le x \le y \le b, \\ \\ d_1(y) + d_2(y)x, a \le y < x \le b. \end{cases} \tag{35}$$

Since

$$-\tilde{O}_y^{''}(x) = \delta(x-y), \tag{36}$$

we obtained

$$\begin{aligned}\tilde{O}_{y+}(y) &= \tilde{O}_{y-}(y) \\ \tilde{O}'_{y+}(y) - \tilde{O}'_{y-}(y) &= -1.\end{aligned} \tag{37}$$

The unknown coefficients $c_i(y)$ and $d_i(y)(i = 1, 2)$ can be obtained by $(32) - (37)$ Thus, $\tilde{O}_y$ is acquired as:

$$\tilde{O}_y(x) = \left\{ \begin{array}{l} 1 + x - a, a \leq x \leq y \leq b, \\ \\ 1 + y - a, a \leq y < x \leq b. \end{array} \right\} \tag{38}$$

Definition 4.4.5

We define the space $H_2^2\,[a, b]$ by

$$H_2^2\,[a, b] = \left\{ u \in AC\,[a, b] : u^{'} \in AC\,[a, b]\,, u^{''} \in L^2\,[a, b] \right\}. \tag{39}$$

The inner product and the norm in $H_2^2\,[a, b]$ are defined by

$$\begin{aligned}\langle u, v\rangle_{H_2^2} &= u(a)v(a) + u^{'}(a)v^{'}(a) + \int_a^b u^{''}(x)v^{''}(x)dx, u, v \in [a, b]\,, \\ \|u\|_{H_2^2} &= \sqrt{\langle u, u\rangle_{H_2^2}}, u \in H_2^2\,[a, b]\,.\end{aligned} \tag{40}$$

Theorem 4.4.6

The Space $H_2^2\,[a, b]$ is reproducing Kernel space, and its reproducing kernel function $\check{T}_y$ is given by

$$\check{T}_y(x) = \left\{ \begin{array}{l} \sum\limits_{i=1}^{4} c_i(y)x^{i-1}, a \leq x \leq y < b, \\ \\ \sum\limits_{i=1}^{4} d_i(y)x^{i-1}, a \leq y < x \leq b. \end{array} \right. \tag{41}$$

Proof:

By Definition 4.4.5., we have:

$$\left\langle u, \check{T}_y \right\rangle_{H_2^2} = u(a)\check{T}_y(a) + u^{'}(a)\check{T}_y^{'}(a) + \int_a^b u^{''}(x)\check{T}_y^{''}(x)dx. \tag{42}$$

Integrating (42) by parts two times, we get

$$\left\langle u, \check{T}_y \right\rangle_{H_2^2} = u(a)\check{T}_y(a) + u^{'}(a)\check{T}_y^{'}(a) + u^{'}(b)\check{T}_y^{''}(b) - u^{'}(a)\check{T}_y^{''}(a) - u(b)\check{T}_y^{'''}(b) + u(a)\check{T}_y^{'''}(a) + \int_a^b u(x)\check{T}_y^{(4)}(x)dx. \tag{43}$$

Note that property of the reproducing kernel is

$$\left\langle u(x), \check{T}_y(x) \right\rangle_{H_2^2} = u(y). \tag{44}$$

We have

$$\begin{aligned} \check{T}_y(a) + \check{T}_y^{'''}(a) &= 0, \\ \check{T}_y^{'}(a) - \check{T}_y^{''}(a) &= 0, \\ \check{T}_y^{''}(a) &= 0, \\ \check{T}_y^{'''}(b) &= 0. \end{aligned} \tag{45}$$

Then (43) gives

$$\check{T}_y^{4}(x) = \delta(x-y) \tag{46}$$

when $x \neq y$, we get

$$\check{T}_y^{4}(x) = 0. \tag{47}$$

Thus, we obtain

$$\tilde{O}_y(x) = \left\{ \begin{array}{l} c_1(y) + c_2(y)x + c_3(y)x^2 + c_4(y)x^3, \ a \leq x \leq y \leq b, \\ \\ d_1(y) + d_2(y)x + d_3(y)x^2 + d_4(y)x^3, \ a \leq y < x \leq b. \end{array} \right\} \tag{48}$$

since

$$\check{T}_y^{4}(x) = \delta(x-y), \tag{49}$$

we get

$$\begin{aligned} \check{T}_{y+}^{(k)}(y) &= \check{T}_{y-}^{(k)}(y), \ k = 0,1,2 \\ \check{T}_{y+}^{'''}(y) - \check{T}_{y-}^{'''}(y) &= 1. \end{aligned} \tag{50}$$

The unknown coefficients $c_i(y)$ and $d_i(y)(i = 1,2,3,4)$ can be obtained by (39)-

(44). Therefore, $\check{T}_y$ is achieved as:

$$\check{T}_y(x) = \left\{ \begin{array}{c} 1 + a^2 + xy - xa - ay + \frac{(x-a)(y-a)^2}{2} \\ \\ +\frac{(y-x)^3}{6} - \frac{(y-a)^3}{6}, a \le x \le y \le b \\ \\ \\ 1 + a^2 + xy - ya - ax + \frac{(y-a)(x-a)^2}{2} \\ \\ +\frac{(x-y)^3}{6} - \frac{(x-a)^3}{6}, a \le y < x \le b \end{array} \right\} \tag{51}$$

Definition.4.4.7

We define the space $W_2^3\,[a,b]$ by

$$W_2^3\,[a,b] = \left\{ u \in AC\,[a,b] : \; u^{'}, u^{''} \in AC\,[a,b]\,, u^{(3)} \in L^2\,[a,b] \right\}. \tag{52}$$

The inner product and the norm in $W_2^3\,[a,b]$ are defined by :

$$\langle u, v \rangle_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(a) v^{(i)}(a) + \int_a^b u^{(3)}(x) v^{(3)}(x) dx, u, v \in W_2^3\,[a,b] \tag{53}$$

$$\|u\|_{W_2^3} = \sqrt{\langle u, u \rangle}, u \in W_2^3\,[a,b]\,.$$

Theorem 4.4.8. The space $W_2^3\,[a,b]$ is a reproducing kernel space. Reproducing kernel function $\overline{R}_y$ of this space is obtained as:

$$\overline{R}_y(x) = \left\{ \begin{array}{l} \sum\limits_{i=1}^{6} c_i(y) x^{i-1}, a \le x \le y \le b, \\ \\ \sum\limits_{i=1}^{6} d_i(y) x^{i-1}, a \le y < x \le b. \end{array} \right\} \tag{54}$$

Proof:

By Definition 4.4.7, we get

$$\langle u, \overline{R}_y \rangle_{W_2^3} = \sum_{i=1}^{2} u^{(i)}(a) \overline{R}_y^{(i)}(a) + \int_a^b u^{(3)}(x) \overline{R}_y^{(3)}(x) dx. \tag{55}$$

Integration (55) by parts, we obtain

$$\left\langle u,\overline{R}_y\right\rangle_{W_2^3} = u(a)\overline{R}_y(a)+u^{'}(a)\overline{R}_y^{'}(a)+u^{''}(a)\overline{R}_y^{''}(a)+u^{''}(b)\overline{R}_y^{(3)}(b) \quad (56)$$
$$-u^{''}(a)\overline{R}_y^{(3)}(a)-u^{'}(a)\overline{R}_y^{(4)}(a)+u^{'}(a)\overline{R}_y^{(4)}(a)+u(b)\overline{R}_y^{(5)}(b)$$
$$-u(a)\overline{R}_y^{(5)}(a)-\int_a^b u(x)\overline{R}_y^{(6)}(x)dx.$$

We have

$$\left\langle u(x),\overline{R}_y(x)\right\rangle_{W_2^3} = u(y) \quad (57)$$

by reproducing property. If

$$\begin{aligned} \overline{R}_y(a)-\overline{R}_y^{(5)}(a) &= 0,\\ \overline{R}_y^{'}(a)-\overline{R}_y^{(4)}(a) &= 0,\\ \overline{R}_y^{''}(a)-\overline{R}_y^{(3)}(a) &= 0,\\ \overline{R}_y^{(3)}(b) &= 0,\\ \overline{R}_y^{(4)}(b) &= 0,\\ \overline{R}_y^{(5)}(b) &= 0, \end{aligned} \quad (58)$$

then,(50) gives

$$-\overline{R}_y^{(6)}(x) = \delta(x-y). \quad (59)$$

When $x \neq y$ we know

$$\overline{R}_y^{(6)}(x) = 0. \quad (60)$$

Consequently,we attain

$$\overline{R}_y(x) = \begin{cases} c_1(y)+c_2(y)x+c_3(y)x^2+c_4(y)x^3+ \\ c_5(y)x^4+c_6(y)x^5, a \leq x \leq y \leq b, \\ \\ d_1(y)+d_2(y)x+d_3(y)x^2+d_4(y)x^3+ \\ d_5(y)x^4+d_6(y)x^5, a \leq y < x \leq b. \end{cases} \quad (61)$$

Since

$$\overline{R}_y^{(6)}(x) = -\delta(x-y), \quad (62)$$

we have

$$\begin{aligned} \overline{R}_{y^+}^{(k)}(y) &= \overline{R}_{y^-}^{(k)}(y), \;\; k=0,1,2,3,4, \\ \overline{R}_{y+}^{(5)}(y) - \overline{R}_{y-}^{(5)}(y) &= -1. \end{aligned} \tag{63}$$

The unknown coefficients $c_i(y)$ and $d_i(y)$, $(i = 1,2,3,4,5,6)$ can be obtained by $(58)-(63)$.Thus $\overline{R}_y$ is gained as

$$\overline{R}_y(x) = \begin{cases} 1+xy-ax-ay+a^2+\frac{(x-y)^5}{120}-\frac{(a-y)^5}{120}-x\frac{(a-y)^4}{24} \\ +a\frac{(a-y)^4}{24}+\frac{(x-a)^2(y-a)^2}{4}-\frac{(x-a)^2(a-y)^3}{12}, a \le x \le y \le b, \\ \\ 1+xy-ay-ax+a^2+\frac{(y-x)^5}{120}-\frac{(a-x)^5}{120}-y\frac{(a-x)^4}{24} \\ +a\frac{(a-x)^4}{24}+\frac{(y-a)^2(x-a)^2}{4}-\frac{(y-a)^2(x-a)^3}{12}, a \le y < x \le b. \end{cases} \tag{64}$$

4.5.Closed Subspace of the Reproducing Kernel space $W_2^m[a,b]$

We may constract the closed subspace ${}^oW_2^m[a,b]$ of the reproducing kernel space $W_2^m[a,b]$ by imposing several homogenous boundary conditions on ${}^oW_2^m[a,b]$.

Definition 4.5.1

Let function space

$${}^oW_2^m[a,b] = \{f(x)/f(x) \in W_2^m[a,b] \text{ by } (19), f^{'}(a)=0, f(b)=0\}. \tag{65}$$

We can prove it is a Hilbert Reproducing Kernel Space.

Let us try to find the reproducing kernel function $Q_y(x)$ of ${}^oW_2^m[a,b]$. $Q_y(x)$ should satisfy

$$\begin{aligned} \langle f(x), Q_y(x)\rangle_{{}^oW_2^m} &= \sum_{i=0}^{m-1} f^{(i)}(a)\frac{\partial^i Q_y(a)}{\partial x^i} \\ &\quad -\sum_{i=0}^{m-1} f^{(i)}(a)(-1)^{m-i-1}\frac{\partial^{2m-i-1}Q_y(a)}{\partial x^{2m-i-1}} \\ &\quad +\sum_{i=0}^{m-1}(-1)^{m-i-1}f^{(i)}(b)\frac{\partial^{2m-i-1}Q_y(b)}{\partial x^{2m-i-1}} \\ &\quad +(-1)^m\int_a^b f(x)\frac{\partial^{2m}Q_y(x)}{\partial x^{2m}}dx. \end{aligned} \tag{66}$$

Therefore $Q_y(x)$ is the solution of:

$$\begin{cases} (-1)^m \frac{\partial^{2m} Q_y(x)}{\partial x^{2m}} = \delta(x-y), \\ \frac{\partial^i Q_y(a)}{\partial x^i} - (-1)^{m-i-1} \frac{\partial^{2m-i-1} Q_y(a)}{\partial x^{2m-i-1}} = 0, i = 0, 2, 3, ..., m-1, \\ \frac{\partial^{2m-i-1} Q_y(b)}{\partial x^{2m-i-1}} = 0, i = 1, 2, ..., m-1, \\ Q_y(b) = 0, \\ \frac{\partial^i Q_y(a)}{\partial x} = 0. \end{cases} \tag{67}$$

While $x \neq y$ is the easy to know that $Q_y(x)$ is the solution od the following linear homogenous differential equation with $2m$ orders,

$$(-1)^m \frac{\partial^{2m} Q_y(x)}{\partial x^{2m}} = 0, \tag{68}$$

with the boundary conditions:

$$\begin{cases} \frac{\partial^i Q_y(a)}{\partial x^i} - (-1)^{m-i-1} \frac{\partial^{2m-i-1} Q_y(a)}{\partial x^{2m-i-1}} = 0, i = 0, 2, 3, ..., m-1, \\ \frac{\partial^{2m-i-1} Q_y(b)}{\partial x^{2m-i-1}} = 0 \qquad i = 1, 2, ..., m-1, \\ Q_y(b) = 0, \\ \frac{\partial^i Q_y(a)}{\partial x} = 0. \end{cases} \tag{69}$$

We know that equation (68) has characteristic equation $\lambda^{2m} = 0$, and the eigenvalue $\lambda = 0$ is a root whose multiplicity is $2m$. Therefore, general solution of Eq.(67) is obtained as:

$$Q_y(x) = \begin{cases} \iota Q_y(x) = \sum_{i=1}^{2m} c_i(y) x^{i-1}, x < y, \\ r Q_y(x) = \sum_{i=1}^{2m} d_i(y) x^{i-1}, x > y. \end{cases} \tag{70}$$

The coefficients $c_i(y)$ and $d_i(y)$, $i = 1, 2, ..., 2m$ can be aobtained now. Since

$$(-1)^m \frac{\partial^{2m} Q_y(x)}{\partial x^{2m}} = \delta(x-y),$$

we have

$$\frac{\partial^i Q_y(y)}{\partial x^i} = \frac{\partial^i r Q_y(y)}{\partial xi}, i = 0, 1, ..., 2m-2, \tag{71}$$

$$(-1)^m(\frac{\partial^{2m-1}\iota Q_y(y^+)}{\partial x^{2m-1}} - \frac{\partial^{2m-1} r Q_y(y^-)}{\partial x^{2m-1}}) = 1. \quad (72)$$

The above equations in (71) and (72) gave $2m$ conditions for solving the coefficients $c_i(y)$ and $d_i(y)(i = 1, 2, ..., 2m)$ in Eq. (70) Note that (69) provided $2m$ boundary conditions, so we get $4m$ equations, (69),(71) and (72), it is easy to know these $4m$ equations are linear equations with the varibles $c_i(y)$ and $d_i(y)$, could be obtained by many techniques. As long as the coefficients $c_i(y)$ and $d_i(y)$ are obtained, the exact expression of the reproducing kernel function $Q_y(x)$ could be calculated from Eq. (70) (Mirzazadeh et al., 2014).

Theorem 4.5.2

The space

$$^oW_2^m[a,b] = \left\{ f(x)/f(x) \in^o W_2^m[a,b], \int_a^b \rho(x)f(x)dx = 0 \right\},$$

is a reproducing kernel space, where $\rho(x) > 0$ is a weighting function.

It is a key to construct the reproducing kernel space with different boundary conditions for solving different practical problems.

5.NEW APPLICATIONS

Reproducing kernel method and $SL(2,R)-$shooting method are investigated to get the approximate and exact solutions of nonlinear differential equation, i.e. Thomas-Fermi (Zhu and Zhu, 2012)

$$\begin{cases} u''(x) = \frac{1}{\sqrt{x}} u^{\frac{3}{2}}, \\ u(0) = 1, u(\infty) = 0. \end{cases} \tag{73}$$

5.1.Group preserving scheme

We suppose that $u(x) > -\infty$. We get a constant χ rendering in order to constitute an SL(2,R)-shooting method:

$$\theta(x) = u(x) + \chi > 0, x \in [x_0, x_f] = [0, \infty)$$

We take a truncation value η_∞ instead of ∞ in our calculations. Then $Eq.(67)$ takes the form:

$$\frac{d^2\theta}{dx^2} = \frac{1}{\sqrt{x}}(\theta - \chi)^{\frac{3}{2}}; BC^{'}s \rightarrow \begin{cases} \theta = 1 + \chi; x = 0, \\ \theta = \chi; x = \eta_\infty. \end{cases} \tag{75}$$

Let $\theta_1(x) = \theta(x)$ and $\theta_2(x) = \theta^{'}(x)$. Then, the equivalent first order system of $Eq.(69)$ takes the form:

$$\begin{cases} \frac{d\theta_1(x)}{dx} = \theta_2(x), \\ \frac{d\theta_2(x)}{dx} = \frac{1}{\sqrt{x}}(\theta_1 - \chi)^{\frac{3}{2}}, \end{cases} ;\ \theta_1(\eta_\infty) = \theta_1^f = \chi, \theta_1(0) = \theta_1^0 = 1 + \chi, \tag{76}$$

or equivalently:

$$\frac{d}{dx}\begin{pmatrix} \theta_1(x) \\ \theta_2(x) \end{pmatrix} = \begin{matrix} 0 & 1 \\ \Lambda(x, \theta_1, \theta_2) & 0 \end{matrix} \Bigg) \begin{pmatrix} \theta_1(x) \\ \theta_2(x) \end{pmatrix} \tag{77}$$

where

$$\Lambda(x, \theta_1, \theta_2) := \frac{(\theta_1 - \varkappa)^{\frac{3}{2}}}{\sqrt{x}\theta_1},$$

It is exotic that in spite of the sight of nonlinear term $\Lambda(x, \theta_1, \theta_2)$ in Eq. (77), we find the Lie symmetry of $SL(2,R)$ by:

$$\frac{d}{dx}\varsigma = A\varsigma, \varsigma(0) = I_{2\times 2,} \tag{78}$$

where det $(\varsigma) = 1$, trace $(A) = 0$ and

$$A = \begin{pmatrix} 0 & 1 \\ \Lambda & 0 \end{pmatrix}.$$

We establish an recurrent method. Namely GPS to solve Eq. (64) as:

$$\circleddash_{n+1=}\varsigma(n)\circleddash_{n,} \tag{79}$$

where $\varsigma(n) \in SL(2, R)$ and $\circleddash_n := |\circleddash|_{\varkappa=\varkappa_n}$. By investigating Eq. (64) and using Eq. (66) and with an initial condition $\circleddash(0) = \circleddash_{(0)}$ we get the value $\circleddash(\varkappa)$. Let $\Delta x = \frac{1}{N}$ be the utilised stepsize in GPS by:

$$\circleddash_{n+1=} = \circleddash_n + \frac{(\alpha_n - 1)F_n.\circleddash_n + \beta_n \|\circleddash_n\| \|F_n\|}{\|F_n\|^2}$$

where

$$\alpha_n = \cosh\left(\frac{\Delta x \|F_n\|}{\circleddash_n}\right), \ \beta_n = \sinh\left(\frac{\Delta x \|F_n\|}{\circleddash_n}\right),$$

which is an stable integrator of

$$\begin{cases} \circleddash' = F(x, \circleddash), \\ \circleddash(0) = \circleddash_{0.} \end{cases}$$

Therefore, we have

$$\circleddash_f = \varsigma_N(\triangle x)...\varsigma_1(\triangle x)\circleddash_0, \tag{80}$$

computes the value of θ at $x = 1$. Closure property of the Lie groups finalizes that if $\varsigma_I(\triangle x) \in SL(2, R), I = 1, ..., N$ then $\varsigma(\triangle x) := \varsigma_N(\triangle x)...\varsigma_1(\triangle x) \in SL(2, R)$. Thus, a one-step Lie group transformation from $\circleddash_0$ to $\circleddash_f$ can be established as:

$$\circleddash_f = \varsigma_{\circleddash_0}, \varsigma \in SL(2, R).$$

We have

$$\varsigma(x) = \exp\left(\int_0^x A(\eta)d\eta\right), \tag{82}$$

by the exponential in manifolds. We obtain

$$
\begin{aligned}
\overline{x} &= rx_0 + (1-r)x_f = (1-r)\eta_\infty, \\
\widetilde{\theta}_1 &= r\theta_1^0 + (1-r)\theta_1^f = r + \chi, \\
\widetilde{\theta}_2 &= r\theta_2^0 + (1-r)\theta_2^f,
\end{aligned}
$$

by a generalized mid-point rule, at appropriate mid-points. Where $r \in [0,1]$ is an unknow constant which we should define it.

$$
\varsigma(r) = \exp\left(A(\widetilde{x}, \widetilde{\theta}_1, \widetilde{\theta}_2)\right), \tag{83}
$$

which conforms with a constant matrix A:

$$
A(\widetilde{x}, \widetilde{\theta}_1, \widetilde{\theta}_2) =: \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ A(\widetilde{x}, \widetilde{\theta}_1, \widetilde{\theta}_2) & 0 \end{pmatrix}, \tag{84}
$$

where

$$
\lambda = A(\widetilde{x}, \widetilde{\theta}_1, \widetilde{\theta}_2) = \frac{(\widetilde{\theta}_1 - \chi)^{\frac{3}{2}}}{\sqrt{\widetilde{x}\widetilde{\theta}_1}}
$$

In the current work , $\theta_1^0 = 1 + \chi$, $\theta_1^f = \chi$ are known and θ_2^0, θ_2^f are unknown boundaries of the model. Determination of θ_2^0 as a missing initial values, converts the Eq.(67) into an initial value problem. Closed form of ς in Eq.(83), obtained from $A \in sl(2,R)$ is the form :

$$
\varsigma(r) = \begin{pmatrix} \cos(\sqrt{-\lambda}) & \frac{-\sin(\sqrt{-\lambda})}{\sqrt{-\lambda}} \\ -\sqrt{-\lambda}\sin\sqrt{-\lambda} & \cos(\sqrt{-\lambda}) \end{pmatrix}, \text{ if } \lambda < 0, \tag{85}
$$

$$
\varsigma(r) = \begin{pmatrix} \cosh(\sqrt{\lambda}) & \frac{\sinh(\sqrt{\lambda})}{\sqrt{\lambda}} \\ \sqrt{\lambda}\sinh(\sqrt{\lambda}) & \cosh(\sqrt{\lambda}) \end{pmatrix}, \text{ if } \lambda > 0, \tag{86}
$$

$$
\varsigma(r) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ if } \lambda = 0, \tag{87}
$$

from Eqs.(81) and (85)-(87) we obtain:

$$
\begin{pmatrix} \theta_1^f \\ \theta_2^f \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{-\lambda}) & \frac{-\sin(\sqrt{-\lambda})}{(\sqrt{-\lambda})} \\ -\sqrt{-\lambda}\sin(-\lambda) & \cos(\sqrt{-\lambda}) \end{pmatrix} \begin{pmatrix} \theta_1^0 \\ \theta_2^0 \end{pmatrix}, \text{ if } \lambda < 0, \tag{88}
$$

$$
\begin{pmatrix} \theta_1^f \\ \theta_2^f \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{\lambda}) & \frac{\sinh(\sqrt{\lambda})}{\cosh(\sqrt{\lambda})} \\ \sqrt{\lambda}\sinh(\lambda) & (\sqrt{\lambda}) \end{pmatrix} \begin{pmatrix} \theta_1^0 \\ \theta_2^0 \end{pmatrix}, \text{ if } \lambda > 0 \tag{89}
$$

$$\begin{pmatrix}\theta_1^f\\ \theta_2^f\end{pmatrix} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}\theta_1^0\\ \theta_2^0\end{pmatrix}, \text{ if } \lambda = 0 \tag{90}$$

thus from $\theta_1^0 = 1+\chi, \theta_1^f = \chi$ an for an endowed r,we find the unknown values $\theta_2^0$and θ_2^f as following :

$$\begin{cases} \theta_2^0 = \sqrt{-\lambda}((1+\chi)\cot(\sqrt{-\lambda}) - \chi\csc(\sqrt{-\lambda}), \\ \\ \theta_2^f = \sqrt{-\lambda}((1+\chi)\cos(2\sqrt{-\lambda})\csc(\sqrt{-\lambda}) - \chi\cot(\sqrt{-\lambda})), \end{cases} \tag{91}$$

when $\lambda < 0$, and

$$\begin{cases} \theta_2^0 = \sqrt{\lambda}(\chi\csc h(\sqrt{\lambda}) - (1+\chi)\coth(\sqrt{\lambda}), \\ \\ \theta_2^f = \sqrt{\lambda}(\chi\coth(\sqrt{\lambda}) - (1+\chi)\csc h(\sqrt{\lambda})), \end{cases} \tag{92}$$

when $\lambda > 0$ and finally for $\lambda = 0$ we obtain

$$\theta_2^0 = \theta_2^f = -1. \tag{93}$$

In this situation, we present a simple method to find the unknown initial and boundary values of θ_2^0 and θ_2^f by:

i) Define $r \in [0,1]$ and initial guesses θ_2^0 and θ_2^f given as $\theta_2^0(0)$ and $\theta_2^f(0)$, respectively

ii) Compute the mean values

$$\begin{cases} \overline{x} = (1-r)\eta_\infty, \\ \overline{\theta_1}(0)r + \chi, \\ \overline{\theta_2}(0) = r\theta_2^0(0) + (1-r)\theta_2^f(0) \end{cases}$$

iii) For $n = 1, 2, ..,$ do the the computations:

$$\begin{aligned} \tilde{\theta}_1(n) &= r + \chi, \\ \tilde{\theta}_2(n) &= r\theta_2^0(n-1) + (1-r)\theta_2^f(n-1), \\ \lambda(n) &= \frac{(\tilde{\theta}_1(n) - X)^{\frac{3}{2}}}{\sqrt{\tilde{X}\tilde{\theta}_1(n)}}, \end{aligned}$$

$$
= \begin{cases} \theta_2^0(n) = \sqrt{-\lambda(n)}(1+\chi)\cot(\sqrt{-\lambda(n)} \\ \quad -\chi \csc\sqrt{-\lambda(n)}), \\ & \text{if } \lambda(n) < 0, \\ \theta_2^f(n) = \sqrt{-\lambda(n)}((1+\chi)\cos(2\sqrt{-\lambda(n)})\csc(\sqrt{-\lambda(n)})) \\ \quad -\chi\cot(\sqrt{-\lambda(n)})) \end{cases}
$$

$$
= \begin{cases} \theta_2^0(n) = \sqrt{\lambda(n)}(\chi \csc h(\sqrt{\lambda(n)} \\ \quad -(1+\chi)\coth(\sqrt{\lambda(n)}, \\ & ; \text{if } \lambda(n) > 0 \\ \theta_2^f(n) = \sqrt{\lambda(n)}\coth(\sqrt{\lambda(n)}) \\ \quad -(1+\chi))\csc h(\sqrt{\lambda(n)})), \end{cases}
$$

$$
\theta_2^0(n) = \theta_2^f(n) = -1, \text{ if } \lambda(n) = 0
$$

If the stopping criterion :

$$
\sqrt{(\theta_2^0(n) - \theta_2^0(n-1))^2 + (\theta_2^f(n) - \theta_2^f(n-1))^2} \leq \epsilon \tag{94}
$$

holds, then stop; otherwise return to (iii).

For a trial r, we calculate θ_2^0 through the mentioned iterations and then approximately integrate (64) by the *GPS* from 0 to η_∞ and match the ending value of θ_1^f with the exact one $\theta(\eta_\infty) = \chi$. In the other word, we require the root of $\theta_1^f - \chi = 0$ or equivalently minimizing the problem $min_{r\in[0,1]} \left|\theta_1^f - \chi\right|$. The choice of $r \in [0,1]$ in our technique plays a critical role in finding the approximate value of $\theta^{'}(0)$. In Fig.1, we showed the mis-matching error. Best choice of $r = 0.2613$ is ocular from this figure. In Fig 2 acquired results from the $SL(2,R)$-shooting method at the range of $[0,20]$ are given. In our computations the values η_∞ and $\chi$are specified by 20 and 3, respectively(Akgül et al., 2017).

5.2. Reproducing kernel spaces

Definition 5.2.1

$W_2^1\,[0,1]$ is given as (Akgü let al., 2017):

$$W_2^1[0,1] = \left\{ u \in AC[0,1] : u^{'} \in L^2 \in [0,1] \right\},$$

where AC shows the space of absolutely continuous functions.

$$\langle u, g \rangle_{W_2^1} = \int_0^1 (u(\eta)g(\eta) + u^{'}(\eta)g^{'}(\eta))d\eta, u, g \in W_2^1[0,1] \quad (95)$$

and

$$\|u\|_{W_2^1} = \sqrt{\langle u, u \rangle_{W_2^1}}, u \in W_2^1[0,1], \quad (96)$$

are the inner product and the norm in $W_2^1[0,1]$ respectively. Reproducing kernel functions $\mathsf{T}_\eta(\varsigma)$ of $W_2^1[0,1]$ is given by [2]

$$\mathsf{T}_\eta(\varsigma) = \frac{1}{2\sinh(1)}[\cosh(\eta + \varsigma - 1) + \cosh(|\eta + \varsigma| - 1]. \quad (97)$$

Definition 5.2.2

The space $F_2^3[0,\infty)$ is given by (Akgü let al., 2017):

$$\begin{aligned} F_2^3[0,\infty) &= \left\{ \begin{array}{c} u \in AC[0,\infty) : u^{'}, u^{''} \in AC[0,\infty), u^{(3)} \in L^2[0,\infty), \\ u(0) = 0 = u(\infty) \end{array} \right\}. \\ \langle u, v \rangle_{{}^oF_2^3[0,\infty)} &= \sum_{i=0}^{2} u^{(i)}(0)v^{(i)}(0) + \int_0^\infty u^{(3)}(\eta)v^{(3)}(\eta)d\eta, \quad u, v \in^\circ F_2^3[0,\infty) \end{aligned}$$

and

$$\|u\|_{{}^oF_2^3[0,\infty)} = \sqrt{\langle u, u \rangle_{{}^oF_2^3[0,\infty)}}, \ u \in^o F_2^3[0,\infty),$$

are the inner product and the norm in ${}^oF_2^3[0,\infty)$ respectively.

Theorem 5.2.3

Reproducing kernel function $q_\varsigma \ of \ {}^oF_2^3[0,\infty)$ is given as (Akgü let al., 2017):

$$q_\varsigma(\eta) = \left\{ \begin{array}{l} \sum\limits_{i=0}^{5} c_i(\varsigma)\eta^i, 0 \le \eta \le \varsigma \le 1, \\ \\ \sum\limits_{i=0}^{5} d_i(\varsigma)\eta^i, 0 \le \varsigma < \eta \le 1. \end{array} \right. \quad (98)$$

Proof:

Let $u \in^o F_2^3\,[0,\infty)$ and $0 \le \varsigma \le 1$, Define q_ς by (98), we have

$$q_\varsigma^{'}(\eta) = \begin{cases} \sum_{i=0}^{4}(i+1)c_{i+1}(\varsigma)\eta^i, \ 0 \le \eta \le \varsigma \le 1, \\ \\ \sum_{i=0}^{4}(i+1)d_{i+1}(\varsigma)\eta^i, \ 0 \le \varsigma < \eta \le 1 \end{cases}$$

$$q_\varsigma^{''}(\eta) = \begin{cases} \sum_{i=0}^{3}(i+1)(i+2)\,c_{i+2}(\varsigma)\eta^i, \ 0 \le \eta \le \varsigma \le 1, \\ \\ \sum_{i=0}^{3}(i+1)(i+2)\,d_{i+2}(\varsigma)\eta^i, \ 0 \le \varsigma < \eta \le 1, \end{cases}$$

$$q_\varsigma^{(3)}(\eta) = \begin{cases} \sum_{i=0}^{2}(i+1)(i+2)(i+3)\,c_{i+3}(\varsigma)\eta^i, 0 \le \eta \le \varsigma \le 1 \\ \\ \sum_{i=0}^{2}(i+1)(i+2)(i+3)\,d_{i+3}(\varsigma)\eta^i, 0 \le \varsigma < \eta \le 1, \end{cases}$$

$$q^{(4)}(\eta) = \begin{cases} \sum_{i=0}^{1}(i+1)(i+2)(i+3)(i+4)c_{i+4}(\varsigma)\eta^i, \ 0 \le \eta \le \varsigma \le 1 \\ \\ \sum_{i=0}^{1}(i+1)(i+2)(i+3)(i+4)d_{i+4}(\varsigma)\eta^i, \ 0 \le \varsigma < \eta \le 1, \end{cases}$$

and

$$q_\varsigma^{(5)}(\eta) = \begin{cases} 120\ C_5(\eta), 0 \le \varsigma < \eta \le 1, \\ \\ 120\ d_5(\eta), 0 \le \varsigma < \eta \le 1, \end{cases}$$

we get

$$\begin{aligned} \langle u, q_\varsigma \rangle_{{}^oF_2^3} &= \sum_{i=0}^{2} u^{(i)}(0)R_\varsigma^{(i)}(0) + \int_0^\infty u^{(3)}(\eta)R_\varsigma^{(3)}(\eta)d\eta \\ &= u'(0)R_\varsigma'(0) + u^{''}(0)R_\varsigma^{''}(0) + u^{''}(\infty)R_\varsigma^{(3)}(1) - u^{''}(0)R_\varsigma^{(3)}(0) \\ &\quad -u^{'}(\infty)R_\varsigma^{(4)}(1) + u^{'}(0)R_\varsigma^{(4)}(0) + \int_0^\infty u^{'}(\eta)q_\varsigma^{(5)}(\eta)d\eta \\ &= u(\varsigma) \end{aligned}$$

Thus, we obtain the reproducing kernel function as:

$$q_{\varsigma}(\eta)=\begin{cases}\eta\varsigma+\frac{1}{4}\varsigma^2\eta^2+\frac{1}{12}\varsigma^2\eta^3-\frac{1}{24}\varsigma\eta^4+\frac{1}{120}\eta^5, 0\leq\eta\leq\varsigma\leq 1,\\ \\ \eta\varsigma+\frac{1}{4}\varsigma^2\eta^2+\frac{1}{12}\varsigma^3\eta^2-\frac{1}{24}\varsigma^4\eta+\frac{1}{120}\varsigma^5, 0\leq\varsigma<\eta\leq 1.\end{cases} \tag{99}$$

this completes the proof .

5.3. Solutions in ${}^{o}F_2^3\,[0,\infty)$

The solution of (66) is considered in the reproducing kernel space ${}^{o}F_2^3\,[0,\infty)$ in this section. On defining the linear operator

$$L :{}^{o} F_2^3\,[0,\infty)\rightarrow W_2^1\,[0,1]$$

as

$$Lu(\eta)=h(\eta)u^{(\frac{3}{2})}(\eta) \tag{100}$$

model problem (66) takes the form :

$$\begin{cases} Lu=M(\eta,u), \eta\in[0,1]\,,\\ \\ u(0)=0=u(\infty).\end{cases} \tag{101}$$

Theorem.5.3

L is a bounded linear operator (Akgül et al., 2017).

Proof:

We have to show $\|Lu\|^2_{W_2^1}\leq P\,\|u\|^2_{{}^{o}F_2^3}$, where $P>0$. By (82) and (83), we obtain

$$\|Lu\|^2_{W_2^1}=\langle Lu,Lu\rangle_{W_2^1}=\int_0^1 Lu(\eta)^2+Lu^{'}(\eta)^2 d\eta.$$

we obtain

$$u(\eta)=\langle u(.),q_{\eta}(.)\rangle_{{}^{o}F_2^3}\,,$$

by reproducing property, and

$$Lu(\eta)=\langle u(.),Lq_{\eta}(.)\rangle_{{}^{o}F_2^3}\,,$$

so

$$|Lu(\eta)| \leq \|u\|_{^oF_2^3} \|Lq_\eta\|_{^oF_2^3} = P_1 \|u\|_{^oF_2^3}$$

where $P_1 > 0$. Therefore, we obtain

$$\int_0^1 [(Lu)(\eta)]^2 \, d_\eta \leq P_1^2 \|u\|^2_{^oF_2^3} .$$

Since

$$(Lu)'(\eta) = \left\langle u(.), (Lq_\eta)'(.) \right\rangle_{^oF_2^3} ,$$

then

$$\left|(Lu)'(\eta)\right| \leq \|u\|_{^oF_2^3} \left\|(Lq_\eta)'\right\|_{^oF_2^3} = P_2 \|u\|_{^oF_2^3} ,$$

where $P_2 > 0$. Thus, we get

$$\left[(Lu)'(\eta)\right]^2 \leq P_2^2 \|u\|^2_{^oF_2^3}$$

and

$$\int_0^1 \left[(Lu)'(\eta)\right]^2 d\eta \leq P_2^2 \|u\|^2_{^oF_2^3} ,$$

that is

$$\|Lu\|^2_{W_2^1} \leq \int_0^1 \left([(Lu)(\eta)]^2 + [(Lu)'(\eta)]^2 \right) d\eta \leq (P_1^2 + P_2^2) \|u\|^2_{^oF_2^3} = P \|u\|^2_{^oF_2^3} ,$$

where

$$P = P_1^2 + P_2^2 > 0.$$

5.4. The fundamental results

Let $\varphi_i(\eta) = T_{\eta_i}(\eta)$ and $\psi_i(\eta) = L^*\varphi_i(x)$, L^* is adjoint operator of L. The orthonormal system$\left\{ \widehat{\psi}_i(\eta) \right\}_{i=1}^{\infty}$ of ${}^oF_2^3[0, \infty)$ can be achieved from Gramschmidt orthogonalization operation of $\{\psi_i(\eta)\}_{i=1}^{\infty}$,

$$\widehat{\psi}_i(\eta) = \sum_{k=1}^{i} \beta_{ik} \psi_k(\eta), (\beta_{ii} > 0, i = 1, 2, ..). \tag{102}$$

Theorem 5.4.1

Let $\{\eta_i\}_{i=1}^{\infty}$ be dense in $[0,1]$ and $\psi_i(\eta) = L_\varsigma q_\eta(\varsigma) \mid_{\varsigma=\eta_i}$. Then the sequence $\{\psi_i(\eta)\}_{i=1}^{\infty}$ is a complete system in $\circ F_2^3[0,\infty)$ (Akgül et al., 2017).

Theorem 5.4.2

If $u(\eta)$ is the exact solution of (94), then we have

$$u(\eta) = \sum_{i=1}^{\infty}\sum_{i=1}^{i} \beta_{ik} M(\eta_{k,} u_k)\widehat{\psi}_i(\eta), \tag{103}$$

where $\{(\eta_i)\}_{i=1}^{\infty}$ is dense in $[0,1]$.

Proof:

We obtain

$$\begin{aligned}
u(\eta) &= \sum_{i=1}^{\infty} \left\langle u(\eta), \widehat{\psi}_i(\eta) \right\rangle_{\circ F_2^3} \widehat{\psi}_i(\eta) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle u(\eta), \psi_k(\eta) \right\rangle_{\circ F_2^3} \widehat{\psi}_i(\eta), \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle u(\eta), L^* \varphi_k(\eta) \right\rangle_{\circ F_2^3} \widehat{\psi}_i(\eta) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle Lu(\eta), \varphi_k(\eta) \right\rangle_{W_2^1} \widehat{\psi}_i(\eta) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} Lu(\eta_k), \widehat{\psi}_i(\eta) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} M(\eta_k, u_k), \widehat{\psi}_i(\eta),
\end{aligned}$$

The approximate solution u_n can be obtained as:

$$u_{n=} \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} M(\eta_k, u_k)\widehat{\psi}_i(\eta).$$

Examples 5.5.

We applied the reproducing kernel method and the $SL(2,R)-$shooting method to investigate the Thomas- Fermi equation. Numerical results have been presented to prove the effectiveness and power of the techniques in this section.

Example 5.5.1

Let us consider:

$$\begin{cases} \varsigma''(x) = \frac{1}{\sqrt{\eta}}(\varsigma)^{\frac{3}{2}} \\ \\ \varsigma(0) = 1, \varsigma(\infty) = 0. \end{cases}$$

We use $u(\eta) = \varsigma(\eta) - \exp(-\eta)$ to homogenize the boundary conditions. After homogenizing the boundary conditions, we have

$$\begin{cases} u'' = \frac{1}{\sqrt{\eta}}(u + \exp(-\eta)^{(\frac{3}{2})} - \exp(-\eta), \\ u(0) = 0, u(\infty) = 0. \end{cases}$$

Numerical results of the techniques are shown in the Table 1. Results prove that the methods conclude close and confidential values.

x	RKM	$SL(2,R) - shooting$
0.0	1.0000000000	1.0000000000
0.25	0.7835167343	0.755903586
0.5	0.6267255540	0.605270335
0.75	0.5160495243	0.497822365
1.0	0.4346425510	0.420342365
1.25	0.372227888	0.362819542
1.5	0.3228774463	0.318730126
1.75	0.2830438479	0.284015236
2.0	0.2503563837	0.256010865
2.25	0.2232076622	0.232973956
2.5	0.2004338018	0.213706631
2.75	0.1811881098	0.197357785
3.0	0.1648317030	0.183313366
4.0	0.1199717397	0.142654023
5.0	0.09571524580	0.116720254
6.0	0.08130766388	0.098751985
7.0	0.07212283057	0.085572364
8.0	0.06645889993	0.075495213
9.0	0.06368974880	0.067537955
10.0	0.06358507593	0.061099850

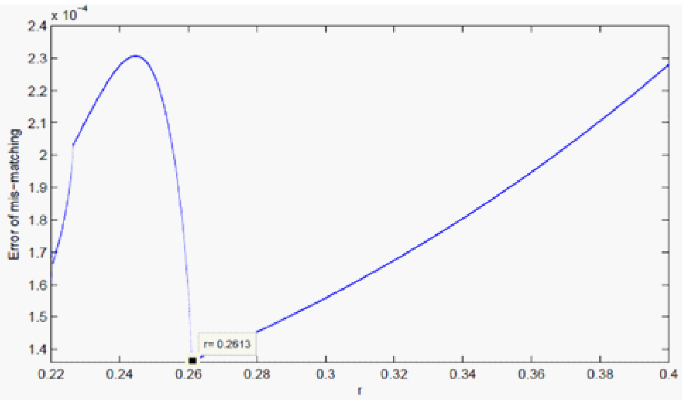

Fig. 1 Plot of mis-matching error in the specified interval

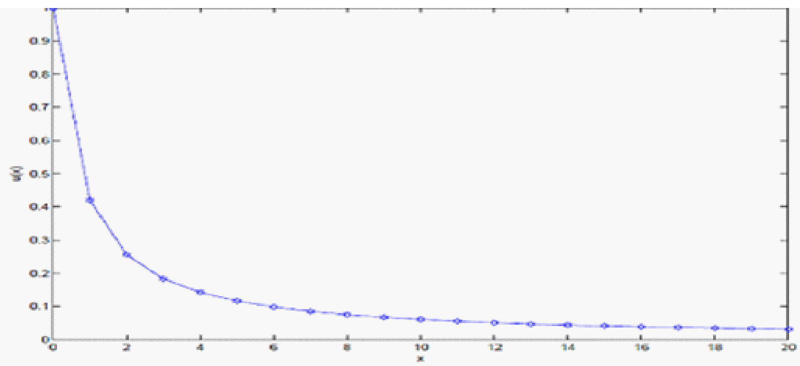

Fig. 2 Approximate solution by SL(2,R)-shooting method in the specified interval.

Table 1 Approximate Solutions by reproducing kernel method and SL(2,R)-method in Example 5.1.1.

6. CONCLUSION

The base goal of this work is to establish some numerical solutions of the nonlinear differential equations by the reproducing kernel method and $SL(2,R)$-shooting method. The acquired results are uniform convergent and the operator that was utilized in the reproduicing kernel method is a bounded linear operator. We proved that reproducing kernel method and group preserrving scheme are in good agreement and they are very accurate methods.

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